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CONEAT SUBMODULES AND CONEAT-FLAT MODULES

ENGIN BÜYÜKAŞIK AND YILMAZ DURĞUN

ABSTRACT. A submodule N of a right R -module M is called *coneat* if for every simple right R -module S , any homomorphism $N \rightarrow S$ can be extended to a homomorphism $M \rightarrow S$. M is called *coneat-flat* if the kernel of any epimorphism $Y \rightarrow M \rightarrow 0$ is *coneat* in Y . It is proven that (1) *coneat* submodules of any right R -module are coclosed if and only if R is right K -ring; (2) every right R -module is *coneat-flat* if and only if R is right V -ring; (3) *coneat* submodules of right injective modules are exactly the modules which have no maximal submodules if and only if R is right small ring. If R is commutative, then a module M is *coneat-flat* if and only if M^+ is m -injective. Every maximal left ideal of R is finitely generated if and only if every absolutely pure left R -module is m -injective. A commutative ring R is perfect if and only if every *coneat-flat* module is projective. We also study the rings over which *coneat-flat* and flat modules coincide.

1. Introduction

A subgroup A of an abelian group B is said to be *neat* in B if $pA = A \cap pB$ for every prime integer p . The notion of neat subgroup was generalized to modules by Renault (see, [12]). Namely, a submodule N of a right R -module M is called *neat* in M , if for every simple right R -module S , $\text{Hom}(S, M) \rightarrow \text{Hom}(S, M/N) \rightarrow 0$ is epic. Dually, in [8], a submodule N of a right R -module M is called *coneat* in M if $\text{Hom}(M, S) \rightarrow \text{Hom}(N, S) \rightarrow 0$ is epic for every simple right R -module S . The notions of neat and *coneat* are coincide over the ring of integers. By [8, Theorem], the commutative domains over which neat and *coneat* submodules coincide are exactly the domains with finitely generated maximal ideals (i.e., N -domains). This result was extended to certain commutative rings in [5]. Recently, modules related to neat and *coneat* submodules are considered by several authors. In [5], a right R -module M is called absolutely neat (resp. *coneat*) if M is a neat (resp. *coneat*) submodule of any module containing it. According to [16], a right R -module M is m -injective

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if for any maximal right ideal I of R , any homomorphism $I \rightarrow M$ can be extended to a homomorphism $R \rightarrow M$. By Theorem 3.4, a right R -module M is absolutely neat if and only if M is m -injective.

A ring R is called right C -ring if $\text{Soc}(R/I) \neq 0$ for each proper essential right ideal I of R . Left perfect rings, right semiartinian rings and almost perfect domains are right C -rings.

A dual notion of m -injective modules has been studied in [1] and [2]. A module M is called neat-flat if the kernel of any epimorphism $F \rightarrow M \rightarrow 0$ is a neat submodule of F . Closed submodules of any right R -module are neat, and neat submodules of any right R -module are closed if and only if R is a right C -ring (see, [9, Theorem 5]). In [21], a module M is called *weak-flat* if the kernel of any epimorphism $F \rightarrow M \rightarrow 0$ is a closed submodule of F . Hence, summing up we get, R is a right C -ring if and only if every neat-flat right R -module is weak-flat.

We call M *coneat-flat* if the kernel of any epimorphism $Y \rightarrow M \rightarrow 0$ is coneat in Y . In this paper, several characterizations of coneat submodules and coneat-flat modules are given. Some known results are generalized, and relations between coneat-flat modules and flat, m -injective, absolutely pure and projective modules are studied.

In Section 2, it is shown that a submodule N of a right R -module M is coneat if and only if for every maximal submodule K of N , N/K is a direct summand of M/K . A ring R is a right V -ring if and only if submodules of right R -modules are coneat. R is right small if and only if its absolutely coneat right modules are precisely those modules M such that $M = \text{Rad}(M)$.

In Section 3, we prove that, a module M is coneat-flat if and only if $M \cong P/N$ where P is a projective R -module and N is a coneat submodule of P . An R -module M is coneat-flat if and only if and only if M^+ is m -injective, over commutative rings. R is a right V -ring if and only if every right R -module is coneat-flat.

In Section 4, we prove that, if R is a left C -ring, then a right R -module M is flat if and only if $\text{Tor}_1^R(M, S) = 0$ for each simple left R -module S . If R is a commutative C -ring, then coneat-flat modules are only the flat modules, and the converse holds when R is noetherian. R is a left N -ring (i.e., maximal left ideals are finitely generated) if and only if every absolutely pure module is m -injective. A ring R is left artinian if and only if m -injective left R -modules are precisely those modules M with M^+ is projective.

In Section 5, we consider the projectivity of coneat-flat modules. We show that, if R is right perfect, then every coneat-flat R -module is projective, the converse holds if R is commutative. Finitely presented coneat-flat modules are projective, over semiperfect rings and over commutative rings.

Throughout, R is a ring with an identity element and all modules are unital right R -modules, unless otherwise stated. For an R -module M , the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ . We use the notation $E(M)$,

$\text{Soc}(M)$, $\text{Rad}(M)$, for the injective hull, socle, radical of M respectively. By $N \leq M$, we mean that N is a submodule of M .

2. Characterization and closure properties of coneat submodules

In this section, several characterizations and some properties of coneat submodules are given. Recall that a submodule K of M is called *small* in M (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule T of M . A submodule $L \leq M$ is called *coclosed in M* if $L/N \ll M/N$ implies $L = N$ for every $N \leq L$.

Proposition 2.1. *For a submodule $N \leq M$ the following are equivalent.*

- (1) N is coneat in M .
- (2) If $K \leq N$ with N/K finitely generated and $N/K \ll M/K$, then $K = N$.
- (3) For any maximal submodule K of N , N/K is a direct summand of M/K .
- (4) If K is a maximal submodule of N , then there exists a maximal submodule L of M such that $K = N \cap L$.

Proof. (1) \Rightarrow (4) Let K be a maximal submodule of N and $\pi : N \rightarrow N/K$ be the canonical epimorphism. By the hypothesis, there exists a homomorphism $f : M \rightarrow N/K$ such that $f|_N = \pi$. Then $\text{Ker } f$ is a maximal submodule of M and $N + \text{Ker } f = M$. So that $N \cap \text{Ker } f$ is a maximal submodule of N . Then $\pi(N \cap \text{Ker } f) = f(N \cap \text{Ker } f) = 0$. Therefore $K = N \cap \text{Ker } f$.

(3) \Rightarrow (1) Let S be a simple right R -module and $f : N \rightarrow S$ a nonzero homomorphism. Since f is an epimorphism, without loss of generality we may assume that $S = N/K$ for some maximal submodule K of N . So that $\text{Ker } f$ is a maximal submodule of N . Then, by (3), $M/\text{Ker } f = (N/\text{Ker } f) \oplus (L/\text{Ker } f)$ for some $L \leq M$. Let $\tilde{f} : N/\text{Ker } f \rightarrow N/K$ be the isomorphism induced by f . Consider the canonical epimorphisms $\pi : M \rightarrow M/\text{Ker } f$ and $\pi' : M/\text{Ker } f \rightarrow N/\text{Ker } f$. Then the homomorphism $g = \tilde{f}\pi'\pi$ is the extension of f .

(2) \Rightarrow (3) is clear.

(3) \Rightarrow (2) Suppose N/K is finitely generated and $N/K \ll M/K$ for some proper submodule $K \leq N$. Then there is a maximal submodule T of N such that $K \leq T$ and $N/T \ll M/T$, because N/T is the image of N/K under the canonical epimorphism $f : M/K \rightarrow M/T$, a contradiction.

(3) \Leftrightarrow (4) is straight forward. □

Properties of coclosed modules in [4, 3.7] are adapted to coneat submodules as follows. The proof is omitted.

Proposition 2.2. *Let $K \leq L \leq M$ be submodules. Then the following hold.*

- (1) If L is coneat in M , then L/K is coneat in M/K .
- (2) If $K \leq \text{Rad}(L)$ and L/K is coneat in M/K , then L is coneat in M .

- (3) If $L \leq M$ is coneat, then $K \leq \text{Rad}(M)$ implies $K \leq \text{Rad}(L)$; hence $\text{Rad}(L) = L \cap \text{Rad}(M)$.
- (4) If $f : M \rightarrow N$ is a small epimorphism and L is coneat in M , then $f(L)$ is coneat in N .
- (6) If K is coneat in M , then K is coneat in L and the converse is true if L is coneat in M .

The proof of [20, Lemma A.4] can be adapted to prove the following.

Proposition 2.3. *Let $K \leq L \leq M$ be submodules of M . If K is coneat in M and L/K is coneat in M/K , then L is coneat in M .*

Proof. Suppose X is a submodule of L such that L/X finitely generated and L/X is small in M/X . Firstly we will prove that $K/K \cap X$ is small in $M/K \cap X$.

Assume the contrary. Then there is an R -module W such that

$$(*) \quad K \cap X \leq W \text{ and } W + K = M.$$

Suppose $L/[K + (W \cap X)]$ is not small in $M/[K + (W \cap X)]$. Then there is an R -module Z such that $K + (W \cap X) \leq Z$ and $Z + L = M$. Since $K \leq Z$, $Z = Z \cap W + K$ by $(*)$, and so $M = Z \cap W + L$. By smallness of L/X is small in M/X , $Z \cap W + X = M$. Now $W = Z \cap W + X \cap W$, $W \leq Z$. Finally, since $Z + W = M$, $Z = M$. Recall that L/K is coneat in M/K and $L/[K + (W \cap X)]$ is epimorphic image of the finitely generated module L/X . Hence, $L = K + W \cap X$ by Proposition 2.1(2). By modular law, $X = K \cap X + W \cap X$, and $X \leq W$. Then $K + X = L$. Since L/X is small in M/X , $W = M$ by $(*)$. By our assumption K is coneat in M , hence $K = K \cap X$ and $K \leq X$. Since L/X is an epimorphic image of L/K and L/K is coneat in M/K , $L = X$ by Proposition 2.1(2), again. \square

Proposition 2.4 ([15, Lemma 6.1]). *Let A be a submodule of an R -module B and $i_A : A \hookrightarrow B$ be the inclusion map. For a right ideal I of R , $A \cap IB = IA$ if and only if $R/I \otimes A \xrightarrow{1_{R/I} \otimes i_A} R/I \otimes B$ is injective.*

An exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C$ is said to be *coneat exact* if $f(A)$ is a coneat submodule of B . A monomorphism $f : A \rightarrow B$ is said to be a coneat monomorphism, if the short exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow B/f(A) \rightarrow 0$ is coneat exact. Neat-exact sequences are defined in the same manner.

Theorem 2.5. *Let R be a commutative ring and $f : N \rightarrow M$ be a monomorphism. The following are equivalent.*

- (1) $f(N)$ is a coneat submodule of M .
- (2) $S \otimes_R N \xrightarrow{1_S \otimes f} S \otimes_R M$ is a monomorphism for each simple R -module S .
- (3) $mf(N) = f(N) \cap mM$ for each maximal ideal m of R .

Proof. (1) \Leftrightarrow (2) By [8, Proposition 3.1].

(2) \Leftrightarrow (3) Follows by Proposition 2.4. \square

Remark 2.6. If N is a pure submodule of M , then $NI = N \cap MI$ for every left ideal of R (see, [10, Corollary 4.92]). Therefore, over commutative rings, every pure submodule is coneat by Theorem 2.5(3). This fact will be used in the sequel.

Corollary 2.7. *Let R be a commutative ring. The following are equivalent.*

- (1) $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is coneat exact.
- (2) $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$ is neat exact.

Proof. By Theorem 2.5(2) and the adjoint isomorphism

$$(M \otimes N)^+ \cong \text{Hom}(M, N^+). \quad \square$$

Let M be an R -module with $\text{Rad } M = M$. It is easy to see that $\text{Hom}(M, S) = 0$ for each simple module. Hence,

Corollary 2.8. *Let M be a right R -module with $\text{Rad}(M) = M$. Then M is absolutely coneat.*

A ring R is said to be right *small* if $R_R \ll E(R_R)$. A ring R is small if and only if $E = \text{Rad}(E)$ for every injective R -module E (see, [11, Proposition 3.3]).

Proposition 2.9. *The following statements are equivalent for a ring R .*

- (1) R is a right small ring.
- (2) Absolutely coneat right R -modules are precisely those modules N such that $\text{Rad}(N) = N$.

Proof. (1) \Rightarrow (2) Let E be the injective hull of N . Then $\text{Rad}(E) = E$ as R is a small ring. Suppose N is coneat in E . So that $\text{Rad}(N) = N \cap \text{Rad}(E) = N$ by Proposition 2.2(3). The rest of (2) by Corollary 2.8.

(2) \Rightarrow (1) Every injective right R -module E is absolutely coneat. Then (2) implies $\text{Rad}(E) = E$, and so R is a small ring by [11, Proposition 3.3]. \square

Let R be a ring and M be a nonzero R -module. M is called *coatomic* if every proper submodule N of M is contained in a maximal submodule of M , i.e., $\text{Rad}(M/N) \neq 0$.

Proposition 2.10. *Let M be a module and N be a coatomic submodule of M . Then N is coneat in M if and only if it is coclosed in M .*

Proof. Suppose N is coneat and $N/X \ll M/X$ for some proper submodule $X \leq N$. Since N is coatomic, X is contained in a maximal submodule, say K , of N . Then $N/K \ll M/K$, and this contradicts with the fact that N is coneat. Hence N is coclosed. The converse implication is obvious. \square

In [19], a ring R is called right K -ring if every non-zero small right R -module is coatomic. Dedekind domains and right max rings (i.e., every nonzero right R -module has a maximal submodule) are right K -rings.

Theorem 2.11. *R is a right K -ring if and only if coneat submodules of any right R -module are coclosed.*

Proof. For the necessity, let M be a non-zero small module and suppose M/K has no maximal submodules, i.e., $\text{Rad}(M/K) = M/K$ for some proper submodule K of M . Then M/K is small and coneat submodule in $E(M/K)$. Hence M/K is coclosed in $E(M/K)$ by (1). This gives a contradiction, since coclosed submodules are not small. Consequently, K is contained in a maximal submodule of M , and so M is coatomic.

For the sufficiency, suppose the contrary that, there is a module M and a submodule N of M which is coneat but not coclosed. Then there is a proper submodule K of N such that $N/K \ll M/K$. By Proposition 2.2(1), N/K is a coneat submodule of M/K . Then N/K is coatomic by the hypothesis, and so N/K is coclosed by Proposition 2.10, a contradiction. \square

3. Coneat-flat modules

It is well known that, a right R -module M is flat if and only if any short exact sequence of the form $0 \rightarrow K \xrightarrow{f} N \rightarrow M \rightarrow 0$ is pure exact, i.e., $f(K)$ is a pure submodule of N . It is natural to ask for which right R -modules P any short exact sequence ending with P is coneat exact? In this section several characterizations of such modules are given.

A right R -module M is called coneat-flat if the kernel of any epimorphism $Y \rightarrow M \rightarrow 0$ is a coneat submodule of Y . Clearly, projective modules are coneat-flat but the converse need not be true in general (see, Theorem 5.1).

Theorem 3.1. *The following are equivalent for an R -module M :*

- (1) M is coneat-flat.
- (2) $\text{Ext}_R^1(M, S) = 0$ for each simple R -module S .
- (3) There is a coneat exact sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ with L projective.
- (4) There is a coneat exact sequence $0 \rightarrow K \rightarrow L \rightarrow M \rightarrow 0$ with L coneat-flat.

Proof. (1) \Rightarrow (2) Let $\mathbb{E} : 0 \rightarrow S \xrightarrow{\alpha} L \rightarrow M \rightarrow 0$ be a short exact sequence with S simple right R -module. Since M is coneat-flat, S is coneat in L , and there is a homomorphism $\beta : L \rightarrow S$ such that the following diagram is commutative.

$$(3.1) \quad \mathbb{E} : \begin{array}{ccccccc} 0 & \longrightarrow & S & \xrightarrow{\alpha} & L & \longrightarrow & P \longrightarrow 0 \\ & & \downarrow 1_S & \swarrow \beta & & & \\ & & S & & & & \end{array}$$

Then $1_S = \beta\alpha$, and so the sequence \mathbb{E} splits. Hence $\text{Ext}_R^1(M, S) = 0$.

(2) \Rightarrow (3) Assuming (2). There is a short exact sequence $\mathbb{E} : 0 \rightarrow C \rightarrow F \rightarrow M \rightarrow 0$ with F free R -module. Applying $\text{Hom}_R(-, S)$, we obtain the exact

sequence $0 \rightarrow \text{Hom}_R(M, S) \rightarrow \text{Hom}_R(F, S) \rightarrow \text{Hom}_R(C, S) \rightarrow \text{Ext}_R^1(M, S) = 0$.

That is, $\text{Hom}_R(\mathbb{E}, S)$ is exact for every simple R -module S , and so \mathbb{E} is coneat exact.

(3) \Rightarrow (4) is obvious.

(4) \Rightarrow (1) Let $s : B \rightarrow M$ be any epimorphism. Consider the following commutative diagram.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & K & \xlongequal{\quad} & K & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \text{Ker } s & \longrightarrow & X & \xrightarrow{\alpha} & L \longrightarrow 0 \\
 & & \parallel & & \downarrow t & & \downarrow \beta \\
 0 & \longrightarrow & \text{Ker } s & \longrightarrow & B & \xrightarrow{s} & M \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

$\beta\alpha = st$ is coneat epimorphism, i.e., $\text{Ker}(st)$ is a coneat submodule of X , by Proposition 2.3. Then s is coneat epimorphism by Proposition 2.2(1). This completes the proof. \square

By Theorem 3.1, we get the following.

Corollary 3.2. *The class of coneat-flat modules is closed under extensions, direct sums, direct summands and coneat quotients. In particular, coneat-flat modules are closed under pure quotients over commutative rings.*

Proof. Coneat-flat modules are closed under extensions, direct sums, direct summands and coneat quotients by Theorem 3.1, and under pure quotients by Remark 2.6 and Theorem 3.1. \square

Proposition 3.3. *Let R be a commutative ring and M be an R -module. Then M is coneat-flat if and only if $\text{Tor}_R(M, S) = 0$ for each simple R -module S .*

Proof. Let $0 \rightarrow K \xrightarrow{i} F \rightarrow M \rightarrow 0$ be a short exact sequence with F projective. Applying $- \otimes S$, we get

$$0 = \text{Tor}(F, S) \rightarrow \text{Tor}(M, S) \rightarrow K \otimes S \xrightarrow{i \otimes 1_S} F \otimes S \rightarrow M \otimes S \rightarrow 0.$$

Then $i \otimes 1_S$ is a monomorphism if and only if $\text{Tor}(M, S) = 0$. Now the proof is clear by Theorem 2.5 and Theorem 3.1. \square

Proposition 3.4. *The following are equivalent for a right R -module M .*

- (1) M is m -injective.
- (2) M is a neat submodule of an m -injective module.
- (3) M is a neat submodule of every module containing it.
- (4) $\text{Ext}_R^1(S, M) = 0$ for every simple right R -module S .

Proof. (1) \Leftrightarrow (4) Let I be a right ideal of R . Then applying $\text{Hom}(-, M)$ to the short exact sequence $0 \rightarrow I \xrightarrow{i} R \rightarrow R/I \rightarrow 0$, we get $0 \rightarrow \text{Hom}(R/I, M) \rightarrow \text{Hom}(R, M) \xrightarrow{i^*} \text{Hom}(I, M) \rightarrow \text{Ext}_R^1(R/I, M) \rightarrow \text{Ext}_R^1(R, M) = 0$. Then i^* is epic if and only if $\text{Ext}_R^1(R/I, M) = 0$.

(2) \Leftrightarrow (3) By [5, Theorem 3.3].

(3) \Leftrightarrow (4) By [5, Theorem 3.4. (i) \Leftrightarrow (ii)]. \square

Proposition 3.5. *Let R be a commutative ring. An R -module M is coneat-flat if and only if M^+ is m -injective.*

Proof. Let S be a simple R -module. We have the standard isomorphism

$$\text{Ext}_R^1(S, M^+) \cong \text{Tor}_1^R(M, S)^+.$$

Now, the proof is immediate by Proposition 3.3 and Proposition 3.4. \square

Corollary 3.6. *Let R be a commutative ring. The class of coneat-flat modules is closed under pure submodules.*

Proof. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a pure exact sequence of R -modules with B coneat-flat. Then the short exact sequence $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$ splits. By Proposition 3.5 the module B^+ is m -injective, and so A^+ is m -injective. Then A is coneat-flat by Proposition 3.5, again. \square

Proposition 3.7. *The following statements are equivalent for a ring R .*

- (1) R is a right V -ring.
- (2) for every right R -module M every submodule of M is coneat in M .
- (3) every right R -module is coneat-flat.

Proof. (1) \Rightarrow (2) is clear, since every simple right R -module is injective by (1).

(2) \Rightarrow (3) Let M be a right R -module. Consider an epimorphism $f : F \rightarrow M$ with F free right R -module. Then $\text{Ker } f$ is a coneat submodule of F by (2). Therefore M is coneat-flat by Theorem 3.1.

(3) \Rightarrow (1) Let S be a simple R -module and E be an injective module containing S . By the hypothesis E/S is coneat-flat. Hence the sequence $0 \rightarrow S \rightarrow E \rightarrow E/S \rightarrow 0$ splits by Theorem 3.1, and so S is injective. \square

4. When coneat-flat modules are flat

In this section, we study the flatness of coneat-flat modules, and the character of coneat-flat modules. We begin with the following. A module right R -module M is called *cotorsion* if $\text{Ext}_R^1(F, M) = 0$ for any flat R -module F .

Example 4.1. (1) Let R be a valuation domain with a non finitely generated maximal ideal P . Then $\text{Rad}(P) = P^2 = P$, and so P is a coneat submodule of R by Corollary 2.8. Hence R/P is coneat-flat by Theorem 3.1. On the other hand, R/P is not a flat R -module, since R/P is a torsion R -module.

(2) Let R be a regular ring that is not a right V -ring. Then there exists a flat module which is not coneat-flat by Proposition 3.7.

In light of Example 4.1, it is natural to consider the rings over which coneat-flat and flat modules coincide. We begin with the following lemma.

Lemma 4.2. *Let R be a ring and S be a simple R -module. If R is commutative or semilocal, then S is cotorsion.*

Proof. First suppose R is commutative and let $I = \text{Ann}_R(S)$. Then clearly S is an R/I -module. Since R/I is simple, S is cotorsion as an R/I -module. So that S is a cotorsion R -module by [18, Proposition 3.3.3]. If R is semilocal, then $J(R).S = 0$ and so S is an $R/J(R)$ -module. As R is semilocal, $R/J(R)$ is semisimple and so S is a cotorsion $R/J(R)$ -module. Now, S is a cotorsion R -module by [18, Proposition 3.3.3], again. \square

Corollary 4.3. *Suppose R is commutative or a semilocal ring. Then every flat module is coneat-flat.*

Proof. Let S be a simple R -module. Then S is a cotorsion module by Lemma 4.2. Therefore $\text{Ext}_R^1(M, S) = 0$, and so M is coneat-flat by Theorem 3.1. \square

Remark 4.4. A commutative domain R is called almost perfect if R/I is a perfect ring for each nonzero ideal I of R . It is clear that almost perfect domains are C -rings. In [14], the authors prove that, if R is an almost perfect domain, then an R -module M is injective if and only if $\text{Ext}_R^1(S, M) = 0$ (i.e., M is m -injective) for each simple module S . Actually, one of the characterization of right C -rings is the following: R is a right C -ring if and only if every m -injective right R -module is injective (see, [16, Lemma 4]).

Proposition 4.5. *Let R be a left C -ring. A right R -module M is flat if and only if $\text{Tor}_1^R(M, S) = 0$ for each simple left R -modules S .*

Proof. Necessity is clear. For the sufficiency assume that $\text{Tor}_1^R(M, S) = 0$ for each simple left R -modules S . Then $0 = \text{Tor}_1^R(M, S)^+ \cong \text{Ext}_R^1(S, M^+)$ implies M^+ is m -injective by Theorem 3.4. Therefore M^+ is injective, because R is a left C -ring. Hence M is flat by [7, Theorem 3.2.10]. \square

Proposition 4.6. *Let R be a commutative ring. Consider the following statements.*

- (1) R is a C -ring.
- (2) Coneat-flat R -modules are flat.

Then (1) \Rightarrow (2). If R is a noetherian, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2) By Corollary 3.3 and Proposition 4.5.

(2) \Rightarrow (1) Let M be an m -injective R -module. Then M^+ is flat by the hypothesis and Theorem 4.10. As R is noetherian, M is injective by [3, Theorem 2]. Hence R is a C -ring. \square

Theorem 4.7. *The following are equivalent for a commutative ring R .*

- (1) *Every coneat-flat module is flat.*
- (2) *Flat modules are precisely those modules M satisfying*

$$\text{Ext}^1(M, \prod_{i \in I} S_i) = 0,$$

where the S_i 's are all the non-isomorphic simple modules.

Proof. (1) \Rightarrow (2) By Lemma 4.2, simple modules are cotorsion. Then $\prod_{i \in I} S_i$ is cotorsion, since cotorsion modules are closed under direct products. Hence, if M is flat, then $\text{Ext}_R^1(M, \prod_{i \in I} S_i) = 0$. Conversely, suppose $\text{Ext}_R^1(M, \prod_{i \in I} S_i) = 0$. Then $\text{Ext}_R^1(M, S_i) = 0$ for each $i \in I$. So that M is coneat-flat by Theorem 3.1. Hence M is flat by (1).

(2) \Rightarrow (1) Suppose M is coneat-flat. Then $\text{Ext}_R^1(M, S) = 0$ for each simple R -module S . So that $\text{Ext}_R^1(M, \prod_{i \in I} S_i) = 0$ for any index set I and simple R -modules S_i . Hence M is flat by (2). \square

Proposition 4.8. *Let R be a commutative N -ring and M be an arbitrary R -module. Then the following hold.*

- (1) *M is m -injective if and only if M^+ is coneat-flat.*
- (2) *M is m -injective if and only if M^{++} is m -injective.*
- (3) *M is coneat-flat if and only if M^{++} is coneat-flat.*
- (4) *Any direct product of coneat-flat modules is coneat-flat.*
- (5) *Any direct product of copies of R is coneat-flat.*
- (6) *The class of m -injective modules is closed under pure quotients.*

Proof. (1) An R -module M is m -injective module if and only if M^+ is coneat-flat by [13, Theorem 9.51], since R is an N -ring

(2) M is m -injective if and only if M^+ is coneat-flat by (1), and M^+ is coneat-flat if and only if M^{++} is m -injective by Proposition 3.5.

(3) If M is coneat-flat, then M^+ is m -injective by Proposition 3.5. So M^{+++} is m -injective by (2), and hence M^{++} is coneat-flat. Conversely, if M^{++} is coneat-flat, then M is coneat-flat by Corollary 3.6, since M is a pure submodule of M^{++} .

(4) Let $(M_i)_{i \in J}$ be a family of coneat-flat R -modules. Since the class of coneat-flat modules is closed under direct sums, $\bigoplus_{i \in J} M_i$ is coneat-flat. So $(\bigoplus M_i)^{++} \cong (\prod M_i^+)^+$ is coneat-flat by (3). Since $\bigoplus_{i \in J} M_i^+$ is a pure submodule of $\prod_{i \in J} M_i^+$, $(\bigoplus_{i \in J} M_i^+)^+$ is a direct summand of $(\prod_{i \in J} M_i^+)^+$, and so $(\bigoplus_{i \in J} M_i^+)^+ \cong \prod_{i \in J} M_i^{++}$ is coneat-flat. Since coneat-flat modules are closed

under pure submodules and $\prod_{i \in J} M_i$ is a pure submodule of $\prod_{i \in J} M_i^{++}$, the module $\prod_{i \in J} M_i$ is coneat-flat.

(5) By (4).

(6) Take any pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with B m -injective. Then we have a split exact sequence $0 \rightarrow C^+ \rightarrow B^+ \rightarrow A^+ \rightarrow 0$. By (1), B^+ is coneat-flat, and so C^+ is coneat-flat. Then C is m -injective by (1), again. \square

An R -module M is called *absolutely pure* if it is pure in every module containing it as a submodule. It is well known that, a ring R is left noetherian if and only if every absolutely pure left R -module is injective.

Proposition 4.9. *R is a left N -ring if and only if every absolutely pure left R -module is m -injective.*

Proof. (\Rightarrow) Let M be an absolutely pure left R -module. Since R is a left N -ring, $\text{Ext}_R^1(S, M) = 0$ for each simple left R -module S . That is, M is m -injective.

(\Leftarrow) Let S be a simple left R -module. Then $\text{Ext}_R^1(S, M) = 0$ for each absolutely pure left R -module M by the assumption. Then S is finitely presented by [6, Proposition]. \square

Theorem 4.10. *Let R be a ring. The following statements are equivalent.*

- (1) (a) M is a flat right R -module if and only if $\text{Tor}_1^R(M, S) = 0$ for each simple left R -module S ,
- (b) R is a left N -ring.
- (2) M is an m -injective left R -module if and only if M^+ is flat.
- (3) M is an m -injective left R -module if and only if M is an absolutely pure left R -module.

Proof. (1) \Rightarrow (2) Let M be a left R -module and S be a simple left R -module. Suppose M is m -injective. Then $0 = \text{Ext}_R^1(S, M)^+ \cong \text{Tor}_1^R(M^+, S)$ by [13, Theorem 9.51], and so M^+ is flat by (1). Conversely suppose M^+ is flat. Then M^{++} is injective by [13, Theorem 3.52], and so M is absolutely pure, since M is pure in M^{++} . Therefore M is m -injective by Proposition 4.9.

(2) \Rightarrow (3) Firstly, we shall prove that a right R -module M is flat if and only if M^{++} is flat. Then R is left coherent by [3, Theorem 1]. Suppose M is a flat right R -module. Then M^+ is (m -)injective, and so M^{++} is flat by (2). Now, conversely suppose M^{++} is a flat right R -module. Then M is flat, since M is pure submodule of M^{++} and flat modules closed under pure submodules.

Let M be a left R -module. Then M^+ is flat if and only if M is absolutely pure by [3, Theorem 1], since R is left coherent. Hence the rest of (3) follows by (2).

(3) \Rightarrow (1) Suppose $\text{Tor}_1^R(M, S) = 0$ for each simple left R -module S . Then $\text{Ext}_R^1(S, M^+) = 0$, and so M^+ is m -injective. Then M^+ is absolutely pure by (3). Therefore M^+ is injective, since it is pure-injective. Thus M is flat. This proves (a), and (b) follows by Proposition 4.9. \square

Proposition 4.11. *Let R be a commutative ring. Consider the following statements.*

- (1) R is a C -ring.
- (2) *Coneat-flat R -modules are flat.*

Then (1) \Rightarrow (2). If R is a noetherian, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2) By Proposition 3.3 and Proposition 4.5.

(2) \Rightarrow (1) Let M be an m -injective R -module. Then M^+ is flat by the hypothesis and Theorem 4.10. As R is noetherian, M is injective by [3, Theorem 2]. Hence R is a C -ring. \square

It is easy to see that, a left N -ring and left semiartinian ring is left noetherian. The following is a slight generalization of this fact.

Corollary 4.12. *If R is a left N -ring and a left C -ring, then R is left noetherian.*

Proof. By Proposition 4.5 and Theorem 4.10, a left R -module M is m -injective if and only if it is absolutely pure. So that every absolutely pure left module is injective. Hence R is left noetherian. \square

Note that, Corollary 4.12, generalizes [5, Theorem 4.1 (ii) \Rightarrow (i)].

In [3, Theorem 4], the authors proves that, R is left artinian if and only if a left module M is injective exactly when M^+ is projective. We show that, this result still holds if we replace m -injective by injective.

Theorem 4.13. *Let R be a ring. The following are equivalent.*

- (1) R is left artinian.
- (2) *A left R -module M is m -injective if and only if M^+ is projective.*

Proof. (1) \Rightarrow (2) R is a left C -ring by (1), and so m -injective modules are injective. Now, (2) follows by [3, Theorem 4].

(2) \Rightarrow (1) Firstly, we show that a left R -module M is m -injective if and only if M is absolutely pure.

Let M be an absolutely pure left R -module. Consider the pure exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Then the short exact sequence $0 \rightarrow (E(M)/M)^+ \rightarrow E(M)^+ \rightarrow M^+ \rightarrow 0$ splits. Then $E(M)^+$ is projective, and hence M^+ is projective. By (2), M is m -injective. Conversely, let M be an m -injective left R -module. Since M is pure in M^{++} and M^{++} is injective, M is absolutely pure.

Then a left R -module M is m -injective if and only if M is absolutely pure if and only if M^+ is projective. By [3, Theorem 3], R is right perfect, and so it is a left C -ring, i.e., m -injective left R -modules are injective. Hence R is left artinian by [3, Theorem 4] and (2). \square

5. When coneat-flat modules are projective

In this section, we shall consider when coneat-flat modules are projective. We begin with the following result.

Theorem 5.1. *Consider the following statements.*

- (1) *R is a right perfect ring.*
- (2) *Every coneat-flat right R-module is projective.*

Then (1) \Rightarrow (2). If R is either commutative or semilocal, then (2) \Rightarrow (1).

Proof. (1) \Rightarrow (2) Let P be a coneat-flat module. Consider a short exact sequence $0 \rightarrow K \rightarrow F \rightarrow P \rightarrow 0$ with F free module. Since R is perfect, F is supplemented by [17, 43.9]. So K has a supplement in F , that is, $K + N = F$ and $A \cap N \ll N$ for some submodule N of F . On the other hand, K is coatomic, as R is a perfect ring. Then K is a coclosed submodule of F by Proposition 2.10. So that $K \cap N \ll K$. Hence K and N are mutual supplements, and so $K \oplus N = F$ by [17, 41.15]. Therefore $N \cong F/K \cong P$ is projective.

(2) \Rightarrow (1) Let M be a flat module. By Corollary 4.3, M is coneat-flat, and so M is projective by (2). Hence R is a perfect ring. □

The following is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let R be a perfect ring. Then an R-module P is projective if and only if $\text{Ext}_R^1(P, S) = 0$ for every simple R-module S.*

An epimorphism $f : N \rightarrow M$ is said to be a *small cover* of M if $\text{Ker } f \ll N$. Moreover, if N is projective, then f is called a *projective cover*.

Proposition 5.3. *Let R be a ring and M be a right R-module with a projective cover $f : P \rightarrow M$. Set $K = \text{Ker } f$. Then M is a coneat-flat module if and only if $\text{Rad}(K) = K$.*

Proof. (\Rightarrow) Assume $\text{Rad}(K) \neq K$. Then K has a maximal submodule, say A . By Proposition 2.1, there exists a maximal submodule L of P such that $A = K \cap L$. Then $K \leq \text{Rad } P$ implies $K = K \cap \text{Rad}(P) \leq K \cap L = A$. Contradiction. Hence (2) holds.

(\Leftarrow) By Corollary 2.8 and Theorem 3.1. □

Corollary 5.4. *Let R be a semiperfect ring. Then finitely presented coneat-flat modules are projective.*

Lemma 5.5. *Let R be a commutative ring and M be a coneat-flat R-module. Then, for all maximal ideals m of R, M_m is a coneat-flat R_m -module.*

Proof. Since M is a coneat-flat R -module, there is a short exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ where K is coneat submodule of F with F is a projective R -module by Theorem 3.1. By exactness of localization, for all maximal ideals m of R , the sequence $0 \rightarrow K_m \rightarrow F_m \rightarrow M_m \rightarrow 0$ is exact. Since $mK = K \cap mF$

for all maximal ideals m of R , we have $m_m K_m = K_m \cap m_m F_m$. Therefore M_m is a coneat-flat R_m -module by Theorem 2.5. \square

Corollary 5.6. *Let R be a commutative ring. Then a finitely presented R -module M is coneat-flat if and only if it is projective.*

Proof. Sufficiency is clear. For the necessity, suppose M is coneat-flat. Let m be a maximal ideal of R . Then M_m is a coneat-flat R_m -module by Lemma 5.5. So that M_m is projective (and so flat) over R_m by Corollary 5.4. Then M is flat by [10, page 160, Exercise 14]. Therefore M is projective by [10, Theorem 4.30]. \square

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ENGIN BÜYÜKAŞIK
İZMİR INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MATHEMATICS
35430, İZMİR, TURKEY
E-mail address: enginbuyukasik@iyte.edu.tr

YILMAZ DURĞUN
BİTLİS EREN UNIVERSITY
DEPARTMENT OF MATHEMATICS
13000, BİTLİS, TURKEY
E-mail address: ydurgun@beu.edu.tr