ASYMPTOTICS OF SPECTRAL GAPS OF HILL AND 1D DIRAC OPERATORS

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Abstract

Let L be the Hill operator or the one-dimensional Dirac operator with π-periodic potential considered on the real line $\mathbb{R}$. The spectrum of L has a band-gap structure, that is, the intervals of continuous spectrum alternate with spectral gaps. The endpoints of these gaps are eigenvalues of the same differential operator L but considered on the interval $[0, \pi]$ with periodic or antiperiodic boundary conditions.

In this thesis considering the Hill and the one-dimensional periodic Dirac operators, we provide precise asymptotics of the spectral gaps in case of specific potentials that are linear combinations of two exponential terms.

HILL VE 1 BOYUTLU DIRAC OPERATÖRLERİNİN SPEKTRAL BOŞLUKLARININ ASİMPTOTLARI

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Özet

Reel doğru $\mathbb{R}$ üzerinde düşünülen π-periyodik Hill operatörü ya da bir-boyutlu Dirac operatörü L olsun. L'nin spektrumu bant-aralıklı yapıdadır, yani sürekli spektrum aralıkları spektral boşluklarla birbirlerini izlerler. Bu boşlukların uç noktaları, aynı fakat $[0, \pi]$ aralığı üzerinde periyodik ve antiperiyodik sınır koşullarıyla düşünülen L diferansiyel operatörünün özdeğerleridir.

Bu tezde Hill ve bir-boyutlu periyodik Dirac operatörlerinin iki üssel terimin lineer kombinasyonu olan özgül potansiyeller ile düşünüldüğü durumunda spektral boşlukların kesin asimptotlarını temin ediyoruz.

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Chapter 1

HILL OPERATORS

1.1 Introduction

It is well-known (see Thm 2.3.1 in [16], or Thm 2.1 in [30]) that the Hill operator

$$L(v) = -\frac{d^2}{dx^2} + v, \quad x \in \mathbb{R}, \tag{1.1.1}$$

with π-periodic real-valued potential $v \in L^2(\mathbb{R})$ is self-adjoint and there exists a sequence of real numbers

$$-\infty < \lambda_0^+ < \lambda_1^- \leq \ldots \leq \lambda_{n-1}^+ < \lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \leq \ldots$$

such that the spectrum of L has a gap-band structure, i.e.,

$$Sp(L) = \bigcup_{n=1}^{\infty} [\lambda_{n-1}^+, \lambda_n^-],$$

and all intervals of the spectrum are separated by the spectral gaps

$$(-\infty, \lambda_0^+), (\lambda_1^-, \lambda_1^+), \ldots, (\lambda_n^-, \lambda_n^+), \ldots, \quad n \in \mathbb{N}.$$

Floquet theory shows that the endpoints λ_n^- and λ_n^+ of these gaps are eigenvalues of the same differential operator L defined in (1.1.1) but considered on the interval $[0, \pi]$ with periodic boundary conditions Per^+ (for even n) or antiperiodic boundary conditions Per^- (for odd n), where

$$Per^{\pm} : y(\pi) = \pm y(0); \quad y'(\pi) = \pm y'(0).$$

See [16, 30] for more details.

We study the behaviour of the lengths of spectral gaps

$$\gamma_n = \lambda_n^+ - \lambda_n^-, \quad n \in \mathbb{N}$$

of the Hill operator $L(v)$. Hochstadt [23, 24] discovered a direct connection between the smoothness of v and the rate of decay of the lenghts of spectral gaps (γ_n) as follows: If

(A) $v \in C^\infty$, i.e., v is infinitely differentiable, then

(B) (γ_n) decreases more rapidly than any power of $1/n$.

He also proved that if a continuous function v is a finite-zone potential, i.e., $\gamma_n = 0$ for large enough n, then $v \in C^\infty$. In the mid-70s (see [32, 34]) the latter statement was extended, namely, it was shown for real $L^2([0,\pi])$-potentials v that $(B) \Rightarrow (A)$. Moreover, Trubowitz [42] proved that an $L^2([0,\pi])$-potential v is analytic if and only if (γ_n) decays exponentially.

If v is a complex-valued potential then the operator (1.1.1) is non-self-adjoint, so one cannot talk about spectral gaps. Moreover, the periodic and antiperiodic eigenvalues $\lambda_n^\pm$ are well-defined for large n (see Lemma 1 below) but the asymptotics of $|\lambda_n^+ - \lambda_n^-|$ do not determine the smoothness of v. In [39] Tkachenko brought into discussion the Dirichlet b.v.p. $y(\pi) = y(0) = 0$. For large enough n, close to n^2 there is exactly one Dirichlet eigenvalue μ_n, so the *deviation*

$$\delta_n = \left| \mu_n - \frac{1}{2}(\lambda_n^+ + \lambda_n^-) \right| \tag{1.1.2}$$

is well defined. Using an adequate parametrization of potentials in spectral terms similar to Marchenko–Ostrovskii's ones [31, 32] for self-adjoint operators, V. Tkachenko [39, 40] (see also [38]) characterized C^∞-smoothness and analyticity in terms of δ_n and differences between critical values of Lyapunov functions. See further references and later results in [6, 7, 14].

In the case of specific potentials, like the Mathieu potential

$$v(x) = 2a \cos 2x, \quad a \in \mathbb{R} \setminus \{0\}, \tag{1.1.3}$$

or more general trigonometric polynomials

$$v(x) = \sum_{-N}^{N} c_k \exp(2ikx), \quad c_k = \overline{c_{-k}}, \quad 0 \le k \le N < \infty, \tag{1.1.4}$$

one comes to two classes of questions:

(i) Is the n-th spectral gap closed, i.e.,

$$\gamma_n = \lambda_n^+ - \lambda_n^- = 0, \tag{1.1.5}$$

or, equivalently, is the multiplicity of λ_n^+ equal to 2?

(ii) If $\gamma_n \neq 0$, could we tell more about the size of this gap, or, for large enough n, what is the asymptotic behavior of $\gamma_n = \gamma_n(v)$?

In [26] Ince proved that the Mathieu-Hill operator has only *simple* eigenvalues both for periodic and antiperiodic boundary conditions, i.e., $\gamma_n \neq 0$ for every $n \in \mathbb{N}$. His proof is presented in [16]; see other proofs of this fact in [22, 33, 35], and further references in [16, 29].

For fixed n and as $a \to 0$, Levy and Keller [28] gave asymptotics of the spectral gap $\gamma_n = \gamma_n(a)$ with $v \in$ (1.1.3); namely

$$\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8(|a|/4)^n}{[(n-1)!]^2} \left(1 + O(a)\right). \tag{1.1.6}$$

Almost 20 years later, Harrell [21] found, up to a constant factor, the asymptotics of the spectral gaps of the Mathieu operator for fixed a as $n \to \infty$. In [3] Avron and Simon gave an alternative proof of Harrell's asymptotics and found the exact value of the constant factor, which led to the formula

$$\gamma_n = \frac{8(|a|/4)^n}{[(n-1)!]^2} \left[1 + o(n^{-2})\right]. \tag{1.1.7}$$

Later, another proof of (1.1.7) was given by Hochstadt [25]. For general trigonometric polynomial potentials, Grigis [20] obtained a generic form of the main term in the gap asymptotics.

Recently, we [1] have refined the result of Harrell-Avron-Simon (1.1.7) by providing more precise asymptotics of the size of spectral gap for the Mathieu operator; namely, we proved for fixed $a \in \mathbb{C}$ and large enough $n \in \mathbb{N}$ that

$$\lambda_n^+ - \lambda_n^- = \pm \frac{8(a/4)^n}{[(n-1)!]^2} \left[1 - \frac{a^2}{4n^3} + O(n^{-4})\right]. \tag{1.1.8}$$

Our approach is based on the methods developed in [12, 13], where the gap asymptotics of the Hill operator with two term potential of the form

$$v(x) = A\cos 2x + B\cos 4x, \quad A \neq 0,\ B \neq 0,$$

was found.

In this thesis we use the same approach in order to find the asymptotics of $\gamma_n = \lambda_n^+ - \lambda_n^-$ in the case of potentials of the form

$$v(x) = ae^{-2ix} + be^{2ix}, \quad a, b \in \mathbb{C}. \tag{1.1.9}$$

and prove for fixed $a, b \in \mathbb{C}$ and large enough $n \in \mathbb{N}$ that

$$\gamma_n = \pm \frac{8(\sqrt{ab}/4)^n}{[(n-1)!]^2} \left[1 - \frac{ab}{4n^3} + O(n^{-4}) \right]. \tag{1.1.10}$$

Additionally, we provide asymptotics for the periodic (if n is even) and antiperiodic (if n is odd) eigenvalues for large enough $n \in \mathbb{N}$ that

$$\lambda_n^\pm = n^2 + \frac{a^2}{2n^2} + \frac{a^2}{2n^4} + O(n^{-6}).$$

Let $H_t(a, b)$ denotes the Hill operator (1.1.1) with a potential (1.1.9) subject to the boundary conditions

$$y(\pi) = e^{it} y(0), \quad y'(\pi) = e^{it} y'(0), \quad -\pi < t \leq \pi.$$

Veliev [43, Theorem 1] showed that the operators $H_t(a, b)$ have the following isospectral property:

$$Sp(H_t(a, b)) = Sp(H_t(c, d)) \quad \text{if } ab = cd,$$

where $Sp(H_t(a, b))$ denotes the spectrum of the operator $H_t(a, b)$. Therefore, (1.1.8) with $\sqrt{ab}$ instead of a implies directly (1.1.10).

1.2 Preliminaries

Let us consider the Hill operator

$$L(v) = -\frac{d^2}{dx^2} + v, \quad x \in [0, \pi], \tag{1.2.1}$$

with a potential $v \in L^2([0, \pi])$. Let

$$v(x) = \sum_{k \in \mathbb{Z}} v_k e^{2ikx}$$

be the Fourier series expansion of the function v. Throughout the paper we assume that

$$v_0 = \int_0^\pi v(x) dx = 0. \tag{1.2.2}$$

(The assumption that $v_0 = 0$ leads to no loss of generality because any shift of the potential by a constant shifts the spectrum by the same constant, and thus the spectral gaps remain the same.) For convenience we set

$$V(k) = \begin{cases} v_{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd ,} \end{cases} \qquad k \in \mathbb{Z}.$$

In this case,

$$\|v\|^2 = \frac{1}{\pi} \int_0^\pi |v(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} |v_k|^2 = \sum_{k \in 2\mathbb{Z}} |V(k)|^2.$$

We consider the periodic Per^+ and antiperiodic Per^- boundary conditions:

$$Per^\pm : y(0) = \pm y(\pi), \quad y'(0) = \pm y'(\pi). \tag{1.2.3}$$

We denote by $L_{Per^\pm}$ the closed operator defined on the domain

$$\Delta_{Per^\pm} = \{f \in H^1([0,\pi], \mathbb{C}) : f \in Per^\pm\}.$$

If $v = 0$, then we use the symbol $L^0_{Per^\pm}$ (or simply L^0). We can characterize the spectra and the eigenfunctions of $L^0_{Per^\pm}$. Namely;

(i) $Sp(L^0_{Per^+}) = \{n^2 : n = 0, 2, 4, \ldots\}$. The eigenspaces are $E^0_n = Span\{e^{\pm inx}\}$ for $n > 0$ and $E^0_0 = Span\{\text{const}\}$, where dim $E^0_n = 2$ for $n > 0$ and dim$E^0_0 = 1$.

(ii) $Sp(L^0_{Per^-}) = \{n^2 : n = 1, 3, 5, \ldots\}$. The eigenspaces are $E^0_n = Span\{e^{\pm inx}\}$ for $n > 0$ and dim $E^0_n = 2$ for $n > 0$.

Let $L_{Per^+}(v)$ and $L_{Per^-}(v)$ denote the operator (1.2.1) considered subject to the corresponding boundary conditions defined in (1.2.3). The following assertion is well-known (e.g., [12, Proposition 1]).

Lemma 1. *The spectra of $L_{Per^\pm}(v)$ are discrete. There is an $N_0 = N_0(v)$ such that the union $\bigcup_{n>N_0} D_n$ of the discs $D_n = \{z : |z - n^2| < 1\}$ contains all but finitely many of the eigenvalues of $L_{Per^\pm}$.*

Moreover, for $n > N_0$ the disc D_n contains exactly two (counted with algebraic multiplicity) periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ^-_n, λ^+_n (where $Re\,\lambda^-_n < Re\,\lambda^+_n$ or $Re\,\lambda^-_n = Re\,\lambda^+_n$ and $Im\,\lambda^-_n \le Im\,\lambda^+_n$).

Lemma 1 allows us to apply the Lyapunov–Schmidt projection method and reduce the eigenvalue equation $Ly = \lambda y$ for $\lambda \in D_n$ to an eigenvalue equation in the two-dimensional space $E^0_n = \{L^0 y = n^2 y\}$ (see [14, Section 2.2]).

This leads to the following (see the formulas (2.24)–(2.30) in [14]).

Lemma 2. *In the above notations, $\lambda = n^2 + z$, for $|z| < 1$, is an eigenvalue of $L_{Per^\pm}(v)$ if and only if z is a root of the equation*

$$\begin{vmatrix} z - S^{11} & S^{12} \\ S^{21} & z - S^{22} \end{vmatrix} = 0, \tag{1.2.4}$$

where $S^{11}, S^{12}, S^{21}, S^{22}$ can be represented as

$$S^{ij}(n,z) = \sum_{k=0}^{\infty} S_k^{ij}(n,z), \quad i,j = 1,2, \tag{1.2.5}$$

with

$$S_0^{11} = S_0^{22} = 0, \quad S_0^{12} = V(-2n), \quad S_0^{21} = V(2n), \tag{1.2.6}$$

and for each $k = 1, 2, ...,$

$$S_k^{11}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{1.2.7}$$

$$S_k^{22}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(-n-j_k)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{1.2.8}$$

$$S_k^{12}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(-n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(-n-j_k)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}, \tag{1.2.9}$$

$$S_k^{21}(n,z) = \sum_{j_1,\dots,j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1)\cdots V(j_k-j_{k-1})V(n-j_k)}{(n^2-j_1^2+z)\cdots(n^2-j_k^2+z)}. \tag{1.2.10}$$

The above series converge absolutely and uniformly for $|z| \leq 1$.

Moreover, (1.2.5)–(1.2.10) imply the following (see Lemma 23 in [14]).

Lemma 3. *For any (complex-valued) potential v*

$$S^{11}(n,z) = S^{22}(n,z). \tag{1.2.11}$$

Moreover, if $V(-m) = \overline{V(m)}\ \forall m$, then

$$S^{12}(n,z) = \overline{S^{21}(n,\bar{z})}, \tag{1.2.12}$$

and if $V(-m) = V(m)\ \forall m$, then

$$S^{12}(n,z) = S^{21}(n,z). \tag{1.2.13}$$

Proof. For each $k \in \mathbb{N}$, the change of summation indices $i_s = -j_{k+1-s}$, $s = 1, \ldots, k$ proves that $S_k^{11}(n,z) = S_k^{22}(n,z)$. In view of (1.2.5) and (1.2.6), (1.2.11) follows.

In a similar way, we obtain that (1.2.12) and (1.2.13) hold by using for each $k \in \mathbb{N}$ the change of indices $i_s = j_{k+1-s}$, $s = 1, 2, \ldots, k$. □

For convenience, we set

$$\alpha_n(z) := S^{11}(n,z) = S^{22}(n,z), \quad \beta_n^+(z) := S^{21}(n,z), \quad \beta_n^-(z) := S^{12}(n,z). \tag{1.2.14}$$

Under these notations the basic equation (1.2.4) becomes

$$(z - \alpha_n(z))^2 = \beta_n^-(z)\beta_n^+(z). \tag{1.2.15}$$

By Lemmas 1 and 2, for large enough $n \in \mathbb{N}$, this equation has in the unit disc exactly the following two roots (counted with multiplicity):

$$z_n^- = \lambda_n^- - n^2, \quad z_n^+ = \lambda_n^+ - n^2. \tag{1.2.16}$$

In the sequel we consider potentials of the form

$$v(x) = ae^{-2ix} + be^{2ix}$$

whose corresponding Fourier coefficients are

$$V(-2) = a, \quad V(2) = b, \quad V(k) = 0 \ \text{ if } k \neq \pm 2. \tag{1.2.17}$$

1.3 Asymptotic estimates for $z_n^\pm$ and $\alpha_n(z)$

In this section we use the basic equation (1.2.15) to derive asymptotic estimates for the deviations $z_n^\pm$. It turns out that $|\beta_n(z)|$, $|z| \leq 1$, is much smaller than $|\alpha_n(z)|$, so it is enough to analyze the asymptotics of $\alpha_n(z_n^\pm)$ in order to find asymptotic estimates for $z_n^\pm$.

The following inequality is well known (e.g., see Lemma 78 in [14]):

$$\sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|} < \frac{2\log 6n}{n}, \quad \text{for } n \in \mathbb{N}. \tag{1.3.1}$$

Lemma 4. *If $|z| \leq 1$, then*

$$\sum_{j_1,\ldots,j_\nu \neq \pm n} \frac{1}{|n^2 - j_1^2 + z| \cdots |n^2 - j_\nu^2 + z|} < \left(\frac{4\log 6n}{n}\right)^\nu. \tag{1.3.2}$$

Proof. If $|z| \leq 1$ and $j \neq \pm n$, then

$$|n^2 - j^2 + z| \geq |n^2 - j^2| - 1 \geq \frac{1}{2}|n^2 - j^2|.$$

Therefore,

$$\sum_{j_1,\ldots,j_\nu \neq \pm n} \frac{1}{|n^2 - j_1^2 + z| \cdots |n^2 - j_\nu^2 + z|} \leq 2^\nu \left(\sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|} \right)^\nu,$$

so (1.3.2) follows from (1.3.1). □

The next lemma gives a rough estimate for $\beta_n(z)$; we improve this estimate in the next section.

Lemma 5. *For $|z| \leq 1$ we have*

$$\beta_n(z) = O\left((4C \log n)^n / n^n\right), \tag{1.3.3}$$

where $C = \max\{|a|, |b|\}$.

Proof. If $\nu < n - 1$, then all terms of the sum $S_\nu^{21}(n,z)$ in (1.2.10) vanish. Indeed, each term of the sum $S_\nu^{21}(n,z)$ is a fraction which nominator has the form $V(x_1)V(x_2)\cdots V(x_{\nu+1})$ with $x_1 = n + j_1,\ x_2 = j_2 - j_1, \ldots, x_{\nu+1} = n - j_\nu$. Therefore, if $\nu < n-1$ then there are no $x_1, x_2, \ldots, x_{\nu+1} \in \{-2, 2\}$ satisfying $x_1 + x_2 + \cdots + x_{\nu+1} = 2n$, so every term of the sum $S_\nu^{21}(n,z)$ vanishes due to (1.2.17). Hence, by (1.2.17) we have

$$|\beta_n(z)| \leq \sum_{\nu=n-1}^{\infty} \sum_{j_1,\ldots,j_\nu \neq \pm n} \frac{|C|^{\nu+1}}{|n^2 - j_1^2 + z| \cdots |n^2 - j_\nu^2 + z|},$$

so (1.3.3) follows from (1.3.2). □

Lemma 6. *In the above notations, as $n \to \infty$,*

$$z_n^\pm = \frac{ab}{2n^2} + O(n^{-4}), \quad \alpha_n(z_n^\pm) = \frac{ab}{2n^2} + O(n^{-4}). \tag{1.3.4}$$

Proof. In view of (1.2.5), (1.2.7) and (1.2.14), we have

$$\alpha_n(z) = \sum_{p=1}^{\infty} A_p(n,z), \tag{1.3.5}$$

where

$$A_p(n,z) = \sum_{j_1,\ldots,j_p \neq \pm n} \frac{V(-n + j_1)V(j_2 - j_1)\cdots V(j_p - j_{p-1})V(n - j_p)}{(n^2 - j_1^2 + z)\cdots(n^2 - j_p^2 + z)}. \tag{1.3.6}$$

First we show that

$$A_{2k}(n,z) \equiv 0 \quad \forall k \in \mathbb{N}. \tag{1.3.7}$$

Indeed, for $p = 2k$ each term of the sum in (1.3.6) is a fraction which nominator has the form $V(x_1)V(x_2)\cdots V(x_{2k+1})$ with

$$x_1 = -n + j_1, \quad x_2 = j_2 - j_1, \quad \ldots, \quad x_{2k+1} = n - j_{2k}.$$

Since $x_1 + x_2 + \cdots + x_{2k+1} = 0$, it follows that there is i_0 such that $x_{i_0} \neq \pm 2$, so $V(x_{i_0}) = 0$ due to (1.2.17). Therefore, every term of the sum $A_{2k}(n,z)$ vanishes, hence (1.3.7) holds.

Next we estimate iteratively, in two steps, $\alpha_n(z)$ and $z_n^{\pm}$. The first step provides rough estimates which we improve in the second step.

Step 1. By (1.3.6), we have

$$A_1(n,z) = \sum_{j_1 \neq \pm n} \frac{V(-n+j_1)V(n-j_1)}{n^2 - j_1^2 + z}.$$

In view of (1.2.17), we get a non-zero term in the above sum if and only if $j_1 = n+2$, or $j_1 = n-2$. Therefore,

$$A_1(n,z) = \frac{ab}{n^2 - (n-2)^2 + z} + \frac{ab}{n^2 - (n+2)^2 + z} = ab\frac{8 - 2z}{(4n)^2 - (4-z)^2}, \tag{1.3.8}$$

which implies that

$$A_1(n,z) = O(n^{-2}) \quad \text{for } \ |z| \leq 1. \tag{1.3.9}$$

On the other hand, from (1.2.17), (1.3.2) and (1.3.6) it follows that

$$|A_{2k-1}(n,z)| \leq |C|^{2k} \left(\frac{4 \log 6n}{n} \right)^{2k-1}, \quad k = 2, 3, \ldots, \tag{1.3.10}$$

where $C = \max\{|a|, |b|\}$, which implies

$$\sum_{k=2}^{\infty} |A_{2k-1}(n,z)| \leq \sum_{k=2}^{\infty} |C|^{2k} \left(\frac{4 \log 6n}{n} \right)^{2k-1} = o(n^{-2}). \tag{1.3.11}$$

Hence, by (1.3.9) and (1.3.11) we obtain

$$\alpha_n(z) = O(n^{-2}) \quad \text{for } \ |z| \leq 1. \tag{1.3.12}$$

Furthermore, from (1.2.15), (1.2.16) and (1.3.3) it follows immediately that

$$z_n^{\pm} - \alpha_n(z_n^{\pm}) = O(n^{-k}), \quad \forall k \in \mathbb{N}. \tag{1.3.13}$$

Therefore, (1.3.12) implies that

$$z_n^{\pm} = O(n^{-2}). \tag{1.3.14}$$

Step 2. By (1.3.8) we have

$$A_1(n,z) = \frac{ab}{2n^2} + O(n^{-4}) \quad \text{if } \ z = O(n^{-2}). \tag{1.3.15}$$

Let us consider

$$A_3(n,z) = \sum_{j_1,j_2,j_3 \neq \pm n} \frac{V(-n+j_1)V(j_2-j_1)V(j_3-j_2)V(n-j_3)}{(n^2-j_1^2+z)(n^2-j_2^2+z)(n^2-j_3^2+z)}.$$

In view of (1.2.17), we get a non-zero term in the above sum if and only if

$$j_1 = n+2; \quad j_2 = n+4; \quad j_3 = n+2,$$

or

$$j_1 = n-2; \quad j_2 = n-4; \quad j_3 = n-2.$$

Hence,

$$\begin{aligned} A_3(n,z) &= \frac{a^2b^2}{[n^2-(n+2)^2+z][n^2-(n+4)^2+z][n^2-(n+2)^2+z]} \\ &\quad + \frac{a^2b^2}{[n^2-(n-2)^2+z][n^2-(n-4)^2+z][n^2-(n-2)^2+z]}, \end{aligned}$$

so it is easy to see that

$$A_3(n,z) = O(n^{-4}) \quad \text{if } \ |z| \leq 1. \tag{1.3.16}$$

On the other hand, by (1.3.10) we have

$$\sum_{k=3}^{\infty} |A_{2k-1}(n,z)| \leq \sum_{k=3}^{\infty} |C|^{2k} \left(\frac{4\log 6n}{n}\right)^{2k-1} = o(n^{-4}). \tag{1.3.17}$$

Therefore, by (1.3.15), (1.3.16) and (1.3.17) imply that

$$\alpha_n(z) = \frac{ab}{2n^2} + O(n^{-4}) \quad \text{if } \ z = O(n^{-2}). \tag{1.3.18}$$

Hence, from (1.3.13) it follows that

$$z_n^\pm = \frac{ab}{2n^2} + O(n^{-4}). \tag{1.3.19}$$

□

Remark. From (1.3.8) and (1.3.19) it follows that

$$A_1(n, z_n^\pm) = \frac{ab}{2n^2} + \frac{ab}{2n^4} - \frac{a^2b^2}{16n^4} + O(n^{-6}). \tag{1.3.20}$$

Similarly, it is easily seen that

$$A_3(n, z_n^\pm) = \frac{a^2b^2}{16n^4} + O(n^{-6}). \tag{1.3.21}$$

On the other hand, analyzing $A_5(n, z)$ one can show that

$$A_5(n, z) = O(n^{-6}) \quad \text{if } \ |z| \le 1. \tag{1.3.22}$$

Moreover, by (1.3.10) we have

$$\sum_{k=4}^{\infty} |A_{2k-1}(n, z)| = o(n^{-6}) \quad \text{if } \ |z| \le 1. \tag{1.3.23}$$

Hence, in view of (1.3.13), the estimates (1.3.20)–(1.3.23) lead to

$$z_n^\pm = \frac{ab}{2n^2} + \frac{ab}{2n^4} + O(n^{-6}). \tag{1.3.24}$$

This analysis could be extended in order to obtain more asymptotic terms of $z_n^\pm$, and even to explain that the corresponding asymptotic series along the powers of $1/n$ contains only even nontrivial terms. However, in this paper we need only the estimate (1.3.19).

The following assertion plays an essential role later.

Lemma 7. *With* $\gamma_n = \lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$, *we have*

$$d\alpha_n/dz = O(n^{-4}) \quad \textit{for} \;\; |z| \le 1/2, \tag{1.3.25}$$

and

$$\alpha_n(z_n^+) - \alpha_n(z_n^-) = \gamma_n \left[-\frac{ab}{8n^2} + O(n^{-4}) \right]. \tag{1.3.26}$$

Proof. By (1.3.5) and (1.3.7) we obtain

$$\alpha_n(z_n^+) - \alpha_n(z_n^-) = A_1(n, z_n^+) - A_1(n, z_n^-) + \int_{z_n^-}^{z_n^+} \frac{d}{dz} \tilde{\alpha}_n(z)\, dz, \tag{1.3.27}$$

where we integrate along the line segment between z_n^- and z_n^+, and

$$\tilde{\alpha}_n(z) = \alpha_n(z) - A_1(n, z) = A_3(n, z) + A_5(n, z) + \cdots .$$

In view of (1.3.16) and (1.3.17),

$$\tilde{\alpha}_n(z) = O(n^{-4}) \quad \text{for} \;\; |z| \le 1.$$

By the Cauchy formula for derivatives, this estimate implies

$$d\alpha_n/dz = O(n^{-4}) \quad \text{for} \;\; |z| \le 1/2.$$

Hence, we obtain

$$\int_{z_n^-}^{z_n^+} \frac{d}{dz} \tilde{\alpha}_n(z)\, dz = |\gamma_n| O(n^{-4}). \tag{1.3.28}$$

On the other hand, by (1.3.8) it follows that

$$\begin{aligned} A_1(n, z_n^+) - A_1(n, z_n^-) &= \left[\frac{8 - 2z_n^+}{(4n)^2 - (4 - z_n^+)^2} - \frac{8 - 2z_n^-}{(4n)^2 - (4 - z_n^-)^2} \right] ab \\ &= \gamma_n \left[\frac{-32n^2 - 32 + 8(z_n^+ + z_n^-) - 2z_n^+ z_n^-}{[(4n)^2 - (4 - z_n^+)^2][(4n)^2 - (4 - z_n^-)^2]} \right] ab. \end{aligned}$$

Therefore, taking into account (1.3.4), we obtain

$$A_1(n, z_n^+) - A_1(n, z_n^-) = \gamma_n \left[\frac{-ab}{8n^2} + O(n^{-4}) \right]. \tag{1.3.29}$$

In view of (1.3.27), the estimates (1.3.28) and (1.3.29) lead to (1.3.26). $\square$

1.4 Asymptotic formulas for $\beta_n^{\pm}(z)$ and γ_n

In this section we find more precise asymptotics of $\beta_n^{\pm}(z)$. These asymptotics, combined with the results of the previous section, lead to an asymptotics for γ_n.

In view of (1.2.17), each nonzero term in (1.2.10) corresponds to a k-tuple of indices $(j_1, ..., j_k)$ with $j_1, \dots, j_k \neq \pm n$ such that

$$(n + j_1) + (j_2 - j_1) + \dots + (j_k - j_{k-1}) + (n - j_k) = 2n \tag{1.4.1}$$

and

$$n + j_1, j_2 - j_1, \dots, j_k - j_{k-1}, n - j_k \in \{-2, 2\}. \tag{1.4.2}$$

Recall that a walk x on the integer grid $\mathbb{Z}$ from c to d (where $c, d \in \mathbb{Z}$) is a finite sequence of integers $x = (x_t)_{t=1}^{\mu}$ with $x_1 + x_2 + \ldots + x_\mu = d - c$. The numbers

$$j_k = c + \sum_{t=1}^{k} x_t, \quad 1 \leq k < \mu$$

are known as *vertices* of the walk x.

By (1.4.1) and (1.4.2), there is one-to-one correspondence between the nonzero terms in (1.2.10) and the *admissible* walks $x = (x(t))_{t=1}^{k+1}$ on $\mathbb{Z}$ from $-n$ to n with steps $x(t) = \pm 2$ and vertices $j_0 = -n,\ j_{k+1} = n$,

$$j_s = -n + \sum_{t=1}^{s} x(t) \neq \pm n, \quad s = 1, \dots, k. \tag{1.4.3}$$

Let $X_n(p),\ p = 0, 1, 2, \dots,$ denote set of all such walks with p negative steps. It is easy to see that every walk $x \in X_n(p)$ has totally $n + 2p$ steps because $\sum x(t) = 2n$. Therefore, every admissible walk has at least n steps.

In view of (1.2.5), (1.2.10), (1.2.17) and (1.2.14), we have

$$\beta_n^+(z) = \sum_{p=0}^{\infty} \sigma_p^+(n, z) \quad \text{with} \quad \sigma_p^+(n, z) = \sum_{x \in X_n(p)} h^+(n, z), \tag{1.4.4}$$

where, for $x = (x(t))_{t=1}^{k+1}$,

$$h^+(x, z) = \frac{b^{k+1}}{(n^2 - j_1^2 + z)(n^2 - j_2^2 + z) \cdots (n^2 - j_k^2 + z)} \tag{1.4.5}$$

with $j_1, \dots, j_k$ given by (1.4.3).

Of course, we can also write similar formulas for $\beta_n^-(z)$. Let $Y_n(p),\ p = 0, 1, 2, \ldots,$ denote the set of all admissible walks from n to $-n$ having p positive steps.

In view of (1.2.5), (1.2.9), (1.2.17) and (1.2.14), we have

$$\beta_n^-(z) = \sum_{p=0}^{\infty} \sigma_p^-(n,z) \quad \text{with} \quad \sigma_p^-(n,z) = \sum_{x \in X_n(p)} h^-(n,z), \tag{1.4.6}$$

where, for $x = (x(t))_{t=1}^{k+1}$,

$$h^-(x,z) = \frac{a^{k+1}}{(n^2 - j_1^2 + z)(n^2 - j_2^2 + z)\cdots(n^2 - j_k^2 + z)}. \tag{1.4.7}$$

We first analyze $\beta_n^+(z)$. The set $X_n(0)$ has only one element, namely the walk

$$\xi = (\xi(t))_{t=1}^{n}, \quad \xi(t) = 2 \ \ \forall t. \tag{1.4.8}$$

Therefore,

$$\sigma_0^+(n,z) = h^+(\xi,z) = \frac{b^n}{(n^2 - j_1^2 + z)\cdots(n^2 - j_{n-1}^2 + z)} \tag{1.4.9}$$

with $j_k = -n + 2k$, $k = 1, \cdots, n-1$. Moreover, since

$$\prod_{k=1}^{n-1} \left(n^2 - (-n+2k)^2\right) = 4^{n-1}[(n-1)!]^2,$$

the following holds.

Lemma 8. *In the above notations,*

$$\sigma_0^+(n,0) = h^+(\xi,0) = \frac{4(b/4)^n}{[(n-1)!]^2}. \tag{1.4.10}$$

It is well known (as a partial case of the Euler-Maclaurin sum formula, see [5, Sect. 3.6]) that

$$\sum_{k=1}^{m} \frac{1}{k} = \log m + g + \frac{1}{2m} + O(m^{-2}), \quad m \in \mathbb{N}, \tag{1.4.11}$$

where $g = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \log m\right)$ is the Euler constant.

Lemma 9. *In the above notations,*

$$\sigma_0^+(n, z_n^\pm) = \sigma_0^+(n, 0)\left[1 - \frac{ab\log n}{4n^3} - \frac{abg}{4n^3} + O(n^{-4})\right]. \tag{1.4.12}$$

Proof. By (1.4.9), we have

$$\sigma_0^+(n, z_n^\pm) = \sigma_0^+(n, 0)\prod_{k=1}^{n-1}\left(1 + \frac{z_n^\pm}{n^2 - (-n+2k)^2}\right)^{-1}. \tag{1.4.13}$$

For simplicity, we set $c_k(n) = \dfrac{z_n^\pm}{n^2 - (-n+2k)^2} = \dfrac{z_n^\pm}{4k(n-k)}$. Then,

$$\log\left(\prod_{k=1}^{n-1}(1 + c_k(n))^{-1}\right) = -\sum_{k=1}^{n-1}\log(1 + c_k(n)) = -\sum_{k=1}^{n-1}c_k(n) + O\left(\sum_{k=1}^{n-1}|c_k(n)|^2\right).$$

Using (1.3.4), we obtain

$$\begin{aligned}\sum_{k=1}^{n-1}c_k(n) &= \left(\sum_{k=1}^{n-1}\frac{1}{4k(n-k)}\right)\left[\frac{ab}{2n^2} + O(n^{-4})\right]\\ &= \frac{1}{2n}\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)\left[\frac{ab}{2n^2} + O(n^{-4})\right].\end{aligned}$$

By (1.4.11), it follows that

$$\sum_{k=1}^{n-1}c_k(n) = \frac{ab\log n}{4n^3} + \frac{abg}{4n^3} + O(n^{-4}).$$

On the other hand, by (1.3.4),

$$\sum_{k=1}^{n-1}|c_k(n)|^2 = \left(\sum_{k=1}^{n-1}\frac{1}{[4k(n-k)]^2}\right)O(n^{-4}) = O(n^{-4}).$$

Hence,

$$\log\left(\prod_{k=1}^{n-1}(1 + c_k(n))^{-1}\right) = -\frac{ab\log n}{4n^3} - \frac{abg}{4n^3} + O(n^{-4}),$$

which implies (1.4.12).

□

We also need the following modification of Lemma 9.

Lemma 10. *If* $z = O(n^{-2})$*, then*

$$\sigma_0^+(n,z) = \sigma_0^+(n,0)(1 + O((\log n)/n^3)). \tag{1.4.14}$$

Proof. We follow the proof of Lemma 9, replacing $z_n^\pm$ with z and using $z = O(n^{-2})$ instead of (1.3.4). □

Next we study the ratio $\sigma_1^+(n,z)/\sigma_0^+(n,z)$.

Lemma 11. *We have*

$$\sigma_1^+(n,z) = \sigma_0^+(n,z) \cdot \Phi(n,z), \tag{1.4.15}$$

where

$$\Phi(n,z) = \sum_{k=2}^{n-1} \varphi_k(n,z) \tag{1.4.16}$$

with

$$\varphi_k(n,z) = \frac{ab}{[n^2 - (-n+2k)^2 + z][n^2 - (-n+2k-2)^2 + z]}. \tag{1.4.17}$$

Proof. From the definition of $X_n(1)$ and (1.4.4) it follows that

$$\sigma_1^+(n,z) = \sum_{x \in X_n(1)} h^+(x,z) = \sum_{k=2}^{n-1} h^+(x_k,z), \tag{1.4.18}$$

where x_k denotes the walk with $(k+1)$'th step equal to -2, i.e.,

$$x_k(t) = \begin{cases} 2 & \text{if } t \neq k \\ -2 & \text{if } t = k \end{cases}, \quad 1 \le t \le n+2.$$

Now, we figure out the connection between vertices of ξ and x_k as follows:

$$j_\alpha(x_k) = \begin{cases} j_\alpha(\xi), & 1 \le \alpha \le k, \\ j_{k-1}(\xi) & \alpha = k+1, \\ j_{\alpha-2}(\xi) & k+2 \le \alpha \le n = 2. \end{cases}$$

Therefore, by (1.4.5)

$$h^+(x_k, z) = h^+(\xi, z)\frac{ab}{(n^2 - [j_{k-1}(\xi)]^2 + z)(n^2 - [j_k(\xi)]^2 + z)}. \tag{1.4.19}$$

Since $j_k(\xi) = -n + 2k,\ k = 2, \dots, n-1$, (1.4.18) and (1.4.19) imply (1.4.15). $\square$

Lemma 12. *In the above notations, if $z = O(n^{-2})$ then*

$$\Phi(n, z) = \Phi(n, 0) + O(n^{-4}) \tag{1.4.20}$$

and

$$\Phi^*(n, z) := \sum_{k=2}^{n-1} |\varphi_k(n, z)| = \Phi(n, 0) + O(n^{-4}). \tag{1.4.21}$$

Moreover,

$$\Phi(n, 0) = \frac{ab}{8n^2} + \frac{ab\log n}{4n^3} + \frac{ab(g-1)}{4n^3} + O(n^{-4}). \tag{1.4.22}$$

Proof. Since

$$\frac{\varphi_k(n, z)}{\varphi_k(n, 0)} = \left[1 + \frac{z}{n^2 - (-n+2k)^2}\right]^{-1}\left[1 + \frac{z}{n^2 - (-n+2k-2)^2}\right]^{-1},$$

it is easily seen that

$$\varphi_k(n, z)/\varphi_k(n, 0) = 1 + O(n^{-3}) \quad \text{if } \ z = O(n^{-2}).$$

On the other hand, $\varphi_k(n, 0) = O(n^{-2})$, so it follows that

$$\varphi_k(n, z) - \varphi_k(n, 0) = \varphi_k(n, 0)\, O(n^{-2}) = O(n^{-5}) \ \text{ if } \ z = O(n^{-2}).$$

Therefore, we obtain that

$$\sum_{k=2}^{n-1} |\varphi_k(n, z) - \varphi_k(n, 0)| = O(n^{-4}) \quad \text{if } \ z = O(n^{-2}).$$

The latter sum dominates both $|\Phi(n, z) - \Phi(n, 0)|$ and $|\Phi^*(n, z) - \Phi(n, 0)|$. Hence, (1.4.20) and (1.4.21) hold.

Next we prove (1.4.22). Since

$$\Phi(n, 0) = \sum_{k=2}^{n-1} \frac{ab}{16(k-1)k(n-k)(n+1-k)},$$

by using the identities

$$\frac{1}{k(n-k)} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right), \qquad \frac{1}{(k-1)(n+1-k)} = \frac{1}{n}\left(\frac{1}{k-1} + \frac{1}{n+1-k}\right)$$

we obtain

$$\Phi(n,0) = \frac{ab}{16n^2}\sum_{i=1}^{4} D_i(n), \tag{1.4.23}$$

where

$$D_1(n) = \sum_{k=2}^{n-1} \frac{1}{k(k-1)}, \quad D_2(n) = \sum_{k=2}^{n-1} \frac{1}{(n-k)(n+1-k)},$$

$$D_3(n) = \sum_{k=2}^{n-1} \frac{1}{k(n+1-k)}, \quad D_4(n) = \sum_{k=2}^{n-1} \frac{1}{(k-1)(n-k)}.$$

The change of summation index $m = n+1-k$ shows that $D_2(n) = D_1(n)$, and we have

$$D_1(n) = \sum_{k=2}^{n-1}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 1 - \frac{1}{n-1} = 1 - \frac{1}{n} + O(n^{-2}). \tag{1.4.24}$$

Moreover, since

$$D_3(n) = \frac{1}{n+1}\left(\sum_{k=2}^{n-1}\frac{1}{k} + \sum_{k=2}^{n-1}\frac{1}{n+1-k}\right) = \frac{2}{n+1}\sum_{k=2}^{n-1}\frac{1}{k},$$

by (1.4.11) we obtain that

$$D_3(n) = \frac{2\log n}{n} + \frac{2(g-1)}{n} - \frac{2\log n}{n^2} + O(n^{-2}). \tag{1.4.25}$$

Similarly,

$$D_4(n) = \frac{1}{n-1}\left(\sum_{m=1}^{n-2}\frac{1}{m} + \sum_{m=1}^{n-2}\frac{1}{n-m-1}\right) = \frac{2}{n-1}\sum_{m=1}^{n-2}\frac{1}{m},$$

and (1.4.11) leads to

$$D_4(n) = \frac{2\log n}{n} + \frac{2g}{n} + \frac{2\log n}{n^2} + O(n^{-2}). \tag{1.4.26}$$

Hence, in view of (2.2.12)–(1.4.26), we obtain (1.4.22). □

Proposition 13. *We have*

$$\beta_n^+(z) = \sigma_0^+(n,0)(1 + O((\log n)/n^3)), \quad \text{if } z = O(n^{-2}), \tag{1.4.27}$$

and

$$\beta_n^+(z_n^\pm) = \sigma_0^+(n,0)\left[1 + \frac{ab}{8n^2} - \frac{ab}{4n^3} + O(n^{-4})\right]. \tag{1.4.28}$$

Proof. From (1.4.12), (1.4.15), (1.4.20) and (1.4.22) it follows immediately that

$$\sigma_1^+(n,z_n^\pm) + \sigma_0^+(n,z_n^\pm) = \sigma_0^+(n,0)\left[1 + \frac{ab}{8n^2} - \frac{ab}{4n^3} + O\left(\frac{1}{n^4}\right)\right].$$

Since $\beta_n^+(z) = \sum_{p=0}^\infty \sigma_p^+(n,z)$, in view of (1.4.12) to complete the proof it is enough to show that

$$\sum_{p=2}^\infty \sigma_p^+(n,z_n^\pm) = \sigma_0^+(n,z_n^\pm)\, O(n^{-4}). \tag{1.4.29}$$

Next we prove (1.4.29). Recall that $\sigma_p^+(n,z) = \sum_{x\in X_n(p)} h^+(x,z)$. Now we set

$$\sigma_p^*(n,z) = \sum_{x\in X_n(p)} |h^+(x,z)|.$$

We are going to show that there is an absolute constant $C > 0$ such that

$$\sigma_p^*(n,z_n^\pm) \le \sigma_{p-1}^*(n,z_n^\pm)\cdot \frac{C}{n^2}, \quad p\in\mathbb{N}, \;\; n \ge N_0. \tag{1.4.30}$$

Since $\sigma_0^+(n,z)$ has one term only, we have $\sigma_0^*(n,z) = |\sigma_0^+(n,z)|$.

Let $p\in\mathbb{N}$. To every walk $x\in X_n(p)$ we assign a pair $(\tilde{x}, j)$, where $\tilde{x}\in X_n(p-1)$ is the walk that we obtain after dropping the first cycle $\{+2,-2\}$ from x, and j is the vertex of x where the first negative step of x is performed. In other words, we consider the map

$$\varphi : X_n(p) \longrightarrow X_n(p-1)\times I, \qquad I = \{-n+4, -n+6, \dots, n-2\},$$

defined by $\varphi(x) = (\tilde{x}, j)$, where

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } 1\le t\le k-1 \\ x(t+2) & \text{if } k\le t\le n+2p-2 \end{cases},$$

where $k = \min\{t : x(t) = 2,\; x(t+1) = -2\}$ and $j = -n + 2k$.

The map φ is clearly injective, and moreover, we have

$$h^+(x,z) = h^+(\tilde{x},z)\frac{ab}{(n^2-j^2+z)(n^2-(j-2)^2+z)}. \tag{1.4.31}$$

Since the mapping φ is injective, from (1.4.17), (1.4.21) and (2.4) it follows that

$$\sigma_p^*(n,z) \le \sigma_{p-1}^*(n,z)\cdot\Phi^*(n,z). \tag{1.4.32}$$

Hence, by (1.4.21) and (1.4.22), we obtain that (1.4.30) holds.

From (1.4.30) it follows (since $\sigma_0^*(n,z_n^\pm) = |\sigma_0^+(n,z_n^\pm)|$) that

$$\sigma_p^*(n,z_n^\pm) \le |\sigma_0^+(n,z_n^\pm)|\cdot\left(\frac{C}{n^2}\right)^p.$$

Hence, (1.4.29) holds, which completes the proof. □

The asymptotics of β_n^- could be found in a similar way. We have the following.

Proposition 14. *In the above notations,*

$$\beta_n^-(z) = \sigma_0^-(n,0)(1+O((\log n)/n^3)), \quad \text{if } z = O(n^{-2}), \tag{1.4.33}$$

and

$$\beta_n^-(z_n^\pm) = \sigma_0^-(n,0)\left[1+\frac{ab}{8n^2}-\frac{ab}{4n^3}+O(n^{-4})\right], \tag{1.4.34}$$

where

$$\sigma_0^-(n,0) = \frac{4(a/4)^n}{[(n-1)!]^2}. \tag{1.4.35}$$

Theorem 15. *The Hill operator*

$$Ly = -y'' + (ae^{-2ix}+be^{2ix})y, \quad a,b\in\mathbb{C},$$

considered with periodic or antiperiodic boundary conditions has, close to n^2 for large enough n, two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+. For fixed $a,b\in\mathbb{C}$, and as $n\to\infty$,

$$\lambda_n^+ - \lambda_n^- = \pm\frac{8(\sqrt{ab}/4)^n}{[(n-1)!]^2}\left[1-\frac{ab}{4n^3}+O(n^{-4})\right]. \tag{1.4.36}$$

Proof. Let

$$C = \max\{|a^2|,|b^2|\}, \quad \mathbb{D}_n = \{z:\ |z|<Cn^{-2}\}.$$

In view of (1.3.4), for large enough n we have

$$|z_n^\pm| < \frac{1}{2} C n^{-2}, \tag{1.4.37}$$

so $z_n^\pm \in \frac{1}{2}\mathbb{D}_n$.

On the other hand, from (1.4.27) and (1.4.33) it follows that for large enough n

$$\beta_n^\pm(z) = \sigma_0^\pm(n,0)(1 + r_n^\pm(z)) \quad \text{with } |r_n^\pm(z)| \le 1/2 \quad \text{for } z \in 2\mathbb{D}_n.$$

We set

$$\sqrt{\beta_n^-(z)\beta_n^+(z)} := \sqrt{\sigma_0^-(n,0)\sigma_0^+(n,0)}\,(1 + r_n^-(z))^{1/2}(1 + r_n^+(z))^{1/2},$$

where $\sqrt{\sigma_0^-(n,0)\sigma_0^+(n,0)}$ is a square root of $\sigma_0^-(n,0)\sigma_0^+(n,0)$ and $(1+w)^{1/2}$ is defined by its Taylor series about $w = 0$. Then $\sqrt{\beta_n^-(z)\beta_n^+(z)}$ is a well-defined analytic function on $2\mathbb{D}_n$, so the basic equation (1.2.4) splits into two equations

$$z - \alpha_n(z) - \sqrt{\beta_n^-(z)\beta_n^+(z)} = 0 \tag{1.4.38}$$

$$z - \alpha_n(z) + \sqrt{\beta_n^-(z)\beta_n^+(z)} = 0. \tag{1.4.39}$$

Next we show that for large enough n equation (1.4.38) has at most one root in the disc $\mathbb{D}_n$. Let

$$\varphi_n(z) = \alpha_n(z) + \sqrt{\beta_n^-(z)\beta_n^+(z)}, \quad f_n(z) = z - \varphi_n(z).$$

By (1.3.25) we have $\alpha_n'(z) = O(n^{-4})$ for $|z| \le 1/2$. On the other hand, Lemma 5 implies that

$$\sqrt{\beta_n^-(z)\beta_n^+(z)} = O(n^{-4}) \quad \text{for} \quad z \in 2\mathbb{D}_n,$$

so by the Cauchy formulas for the derivatives we have

$$\frac{d}{dz}\sqrt{\beta_n^-(z)\beta_n^+(z)} = O(n^{-2}) \quad \text{for} \quad z \in \mathbb{D}_n.$$

Therefore

$$\sup\{|\varphi_n'(z)| \,:\, z \in \mathbb{D}_n\} \le 1/2,$$

which implies

$$|\varphi_n(z_1) - \varphi_n(z_2)| = \left| \int_{z_1}^{z_2} \varphi_n'(z)dz \right| \le \frac{1}{2}\,|z_1 - z_2| \quad \text{for } z_1, z_2 \in \mathbb{D}_n.$$

Now we obtain, for $z_1, z_2 \in \mathbb{D}_n$, that

$$
\begin{aligned}
|f_n(z_1) - f_n(z_2)| &= |(z_1 + \varphi_n(z_1)) - (z_2 + \varphi_n(z_2))| \\
&\geq |z_1 - z_2| - |\varphi_n(z_1) - \varphi_n(z_2)| \geq \frac{1}{2}|z_1 - z_2|.
\end{aligned}
$$

Hence the equation $f_n(z) = 0$ (i.e., equation (1.4.38)) has at most one solution in the disc $\mathbb{D}_n$. Of course, the same argument gives that equation (1.4.39) also has at most one solution in the disc $\mathbb{D}_n$.

On the other hand, we know by Lemma 1 and (1.4.37) that for large enough n the equation (1.2.4) has exactly two roots z_n^-, z_n^+ in the disc $\mathbb{D}_n$, so either z_n^- is the root of (1.4.38) and z_n^+ is the root of (1.4.39), or vise versa z_n^+ is the root of (1.4.38) and z_n^- is the root of (1.4.39). Therefore, we obtain

$$
z_n^+ - z_n^- - [\alpha_n(z_n^+) - \alpha_n(z_n^-)] = \pm \left[\sqrt{\beta_n^-(z_n^+)\beta_n^+(z_n^+)} + \sqrt{\beta_n^-(z_n^-)\beta_n^+(z_n^-)} \right]. \quad (1.4.40)
$$

Now (1.3.26), (1.4.28) and (1.4.34) imply, with $\gamma_n = \lambda_n^+ - \lambda_n^-$,

$$
\gamma_n \left[1 + \frac{ab}{8n^2} + O(n^{-4}) \right] = \pm \frac{8(\sqrt{ab}/4)^n}{[(n-1)!]^2} \left[1 + \frac{ab}{8n^2} - \frac{ab}{4n^3} + O(n^{-4}) \right],
$$

which proves (1.4.36).

Finally, if at least one of the coefficients a, b becomes zero, then $\gamma_n = 0$ for all n. This follows from (1.4.40) where $\beta_n^-(z_n^\pm)\beta_n^+(z_n^\pm)$ becomes zero for all n, in consideration of (1.4.27), (1.4.10), (1.4.33) and (1.4.35).

□

Chapter 2

DIRAC OPERATORS

2.1 Introduction

The one-dimensional Dirac operator of the form

$$L(v) = iJ\frac{d}{dx} + v; \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad v(x+\pi) = v(x), \quad (2.1.1)$$

where $P, Q \in L^2(\mathbb{R})$ gives rise to a self-adjoint operator in $L^2(\mathbb{R}, \mathbb{C}^2)$ if $Q(x) = \overline{P(x)}$. It is well-known that the spectrum of $L(v)$ is absolutely continuous and has a band-gap structure, i.e.,

$$Sp(L) = \bigcup_{n=\infty}^{\infty} [\lambda_{n-1}^+, \lambda_n^-],$$

where

$$\ldots \leq \lambda_{n-1}^+ < \lambda_n^- \leq \lambda_n^+ < \lambda_{n+1}^- \leq \ldots.$$

Floquet theory shows that the endpoints λ_n^- and λ_n^+ of these gaps are eigenvalues of the same differential operator and (2.1.1) but considered on the interval $[0, \pi]$ with periodic boundary conditions for even n and antiperiodic boundary conditions for odd n, where

$$Per^{\pm} : y(\pi) = \pm y(0).$$

See [4, 27] for more details.

It is known that the potential smoothness determines the asymptotic behavior of the sequence of $\gamma_n = \lambda_n^+ - \lambda_n^-$. Moreover, in the self-adjoint case the asymptotic behavior of (γ_n) determines the potential smoothness as well. This phenomenon was first discovered and studied for Hill operators (see the discussion in section 1.1). The situation is similar for self-adjoint Dirac operators but the relationship between smoothness of the potential functions P, Q and decay rate of the spectral gaps (γ_n) were studied somewhat later (see [18, 19, 36, 37, 9, 8, 14]).

In the non-self-adjoint case, for both Hill and one-dimensional Dirac operators, the decay rate of $(|\gamma_n|)$ does not determine the potential smoothness as Gasymov's example [17] and its modifications in the Dirac case show. However, Tkachenko [39, 40, 41] discovered that the potential smoothness could be determined by the rate of decay of $(|\gamma_n| + |\delta_n|)$, where δ_n is the difference between λ_n^+ and the closest Dirichlet eigenvalue μ_n (see also [38, 7, 9, 14]).

Djakov and Mityagin [10, 11] provided an analogue of Ince's result [26] for the Mathieu-Hill operator. They studied the spectral gaps of Dirac operators (2.1.1) with potentials

$$v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}, \quad P(x) = Q(x) = 2a\cos(2x), \;\; a \in \mathbb{R} \setminus \{0\}, \tag{2.1.2}$$

and showed that $\gamma_{-n} = \gamma_n$ for all n, $\gamma_n = 0$ for even n, $\gamma_n > 0$ for odd n; and for large enough $m \in \mathbb{N}$,

$$\gamma_{2m+1} = 2\frac{|a|^{2m+1}}{4^{2m}(m!)^2}\left[1 + O\left(\frac{\log m}{m}\right)\right]. \tag{2.1.3}$$

Let us notice that here the operator L is considered on the interval $[0, \pi]$, whereas all operators in [10, 11] are considered on $[0, 1]$, and thus the coefficients in (2.1.3) are normalized correspondingly.

In this thesis, we study the same phenomenon for Dirac operators (2.1.1) with a four-parameter family of potentials

$$P(x) = ae^{-2ix} + Ae^{2ix}, \quad Q(x) = be^{-2ix} + Be^{2ix}, \quad a, A, b, B \in \mathbb{C}.$$

Our asymptotic formulas (2.4.28), (2.4.29) extend and refine (2.1.3), and show that $\gamma_n \neq 0$ for odd n with large enough $|n|$, so all but finitely many antiperiodic eigenvalues are simple (see also [2]). The main part of these asymptotics was given in [15, (8.5) in Theorem 29] but formula (8.5) is based on [15, Proposition 28] which is given without proof. We prove a refined version of that proposition in Section 4 (see Propositions 30 and 31).

Additionally, we provide asymptotics for the periodic (if n is even) and antiperiodic (if n is odd) eigenvalues for large enough $|n| \in \mathbb{Z}$ that

$$\lambda_n^{\pm} = n + \frac{Ab + aB}{2n} + \frac{aB - Ab}{2n^2} + O(|n|^{-3}).$$

2.2 Preliminaries

Let us consider the one-dimensional Dirac operator defined as

$$L(v) = iJ\frac{d}{dx} + v; \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}, \quad x \in [0,\pi], \tag{2.2.1}$$

with a potential $v \in L^2([0,\pi])$; that is $P, Q \in L^2([0,\pi])$. Let

$$P(x) = \sum_{m\in\mathbb{Z}} P_m e^{2imx} \quad \text{and} \quad Q(x) = \sum_{m\in\mathbb{Z}} Q_m e^{2imx}$$

be the Fourier series expansions of the functions P and Q, respectively. For convenience we set for $m \in \mathbb{Z}$

$$p(m) = \begin{cases} P_{m/2} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases} \quad \text{and} \quad q(m) = \begin{cases} Q_{m/2} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}.$$

Then

$$\|v\|^2 = \sum_{m\in\mathbb{Z}} (|P_m|^2 + |Q_m|^2) = \sum_{k\in 2\mathbb{Z}} (|p(k)|^2 + |q(k)|^2).$$

On the space $L^2([0,\pi],\mathbb{C}^2)$ we define the inner product as

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \frac{1}{\pi}\int_0^\pi (f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)})\, dx.$$

We consider the periodic (Per^+) and antiperiodic (Per^-) boundary conditions:

$$Per^\pm : y_1(0) = \pm y_1(\pi), \quad y_2(0) = \pm y_2(\pi) \tag{2.2.2}$$

We denote by $L_{Per^\pm}$ the closed operator defined on the domain

$$\Delta_{Per^\pm} = \{f \in H^1(([0,\pi]),\mathbb{C}^2) : f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Per^\pm\}.$$

If $v = 0$, that is, if $P \equiv 0$ and $Q \equiv 0$ then we use the symbol $L^0_{Per^\pm}$ (or simply L^0). We can characterize the spectra and the eigenfunctions of $L^0_{Per^\pm}$. Namely;

(i) $Sp(L^0_{Per^+}) = \{n \text{ even}\} = 2\mathbb{Z}$, each number $n \in 2\mathbb{Z}$ is a double eigenvalue, and the corresponding eigenspaces are

$$E_n^0 = Span\{e_n^1, e_n^2\}, \tag{2.2.3}$$

where

$$e_n^1(x) = \begin{pmatrix} e^{-inx} \\ 0 \end{pmatrix}, \quad e_n^2(x) = \begin{pmatrix} 0 \\ e^{inx} \end{pmatrix}. \tag{2.2.4}$$

(ii) $Sp(L^0_{Per^-}) = \{n \text{ odd}\} = 2\mathbb{Z}+1$, each number $n \in 2\mathbb{Z}+1$ is a double eigenvalue, and the corresponding eigenspaces E_n^0 are given by the same formulae (2.2.3) and (2.2.4) but with $n \in 2\mathbb{Z}+1$.

Let $L_{Per^+}(v)$ and $L_{Per^-}(v)$ denote the operator (2.2.1) subject to the corresponding boundary conditions defined in (2.2.2). The following is well-known (e.g., [14, Theorem 17]).

Lemma 16. *The spectra of $L_{Per^\pm}(v)$ are discrete. There is $N_0 = N_0(v)$ such that the union $\bigcup_{|n|>N_0} D_n$, where $D_n = \{z : |z-n| < \frac{1}{2}\}$, contains all but finitely many of the eigenvalues of $L_{Per^\pm}(v)$.*

Moreover each disc D_n, $|n| > N_0$, contains exactly two (counted with algebraic multiplicity) periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+ (where $Re\,\lambda_n^- < Re\,\lambda_n^+$ or $Re\,\lambda_n^- = Re\,\lambda_n^+$ and $Im\,\lambda_n^- \le Im\,\lambda_n^+$).

Remark. In the sequel we assume that $N_0 > 1$ and consider only integers $n \in \mathbb{Z}$ with $|n| > N_0$.

Technically, our approach is based on the following lemma (see [14, Section 2.4]).

Lemma 17. *Let $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$, and let $p(m)$ and $q(m)$, $m \in 2\mathbb{Z}$ be respectively the Fourier coefficients of P and Q about the system $\{e^{imx},\, m \in 2\mathbb{Z}\}$. Then, $\lambda = n + z$ with $|z| \le 1/2$ is an eigenvalue of $L_{Per^\pm}(v)$ if and only if z is an eigenvalue of a matrix $\begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}$ whose entries $S^{ij} = S^{ij}(n, z; v)$ are given by*

$$S^{ij}(n, z) = \sum_{k=0}^{\infty} S_k^{ij}(n, z), \tag{2.2.5}$$

where

$$S_0^{11} = S_0^{22} = 0, \quad S_0^{12} = p(-2n), \quad S_0^{21} = q(2n), \tag{2.2.6}$$

and for $\nu = 1, 2, \ldots$

$$S_{2\nu}^{11} = S_{2\nu}^{22} = 0, \quad S_{2\nu-1}^{12} = S_{2\nu-1}^{21} = 0, \tag{2.2.7}$$

$$S^{11}_{2\nu-1} = \sum_{j_1,\dots,j_{2\nu-1}\neq n} \frac{p(-n-j_1)q(j_1+j_2)\cdots p(-j_{2\nu-2}-j_{2\nu-1})q(j_{2\nu-1}+n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-2}+z)(n-j_{2\nu-1}+z)}, \tag{2.2.8}$$

$$S^{22}_{2\nu-1} = \sum_{j_1,\dots,j_{2\nu-1}\neq n} \frac{q(n+j_1)p(-j_1-j_2)\cdots q(j_{2\nu-2}+j_{2\nu-1})p(-j_{2\nu-1}-n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-2}+z)(n-j_{2\nu-1}+z)} \tag{2.2.9}$$

$$S^{12}_{2\nu} = \sum_{j_1,\dots,j_{2\nu}\neq n} \frac{p(-n-j_1)q(j_1+j_2)\cdots q(j_{2\nu-1}+j_{2\nu})p(-j_{2\nu}-n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-1}+z)(n-j_{2\nu}+z)}, \tag{2.2.10}$$

$$S^{21}_{2\nu} = \sum_{j_1,\dots,j_{2\nu}\neq n} \frac{q(n+j_1)p(-j_1-j_2)\cdots p(-j_{2\nu-1}-j_{2\nu})q(j_{2\nu}+n)}{(n-j_1+z)(n-j_2+z)\cdots(n-j_{2\nu-1}+z)(n-j_{2\nu}+z)}, \tag{2.2.11}$$

where in all sums $j_k \in n + 2\mathbb{Z}$.

For each $\nu \in \mathbb{Z}_+$ the change of summation indices $i_s = j_{2\nu+1-s}$, $s = 1, \dots, 2\nu+1$ shows that $S^{11}_{2\nu+1}(n,z) = S^{22}_{2\nu+1}(n,z)$; therefore,

$$S^{11}(n,z) = S^{22}(n,z). \tag{2.2.12}$$

For convenience we set

$$\alpha_n(z) := S^{11}(n,z), \quad \beta^+_n(z) := S^{21}(n,z), \quad \beta^-_n(z) := S^{12}(n,z). \tag{2.2.13}$$

In these notations the characteristic equation associated with the matrix (S^{ij}) becomes

$$(z - \alpha_n(z))^2 = \beta^-_n(z)\beta^+_n(z). \tag{2.2.14}$$

In view of Lemmas 16 and 17, for large enough $|n|$ equation (2.2.14) has in the disc $|z| \leq 1/2$ exactly the following two roots (counted with multiplicity):

$$z^-_n = \lambda^-_n - n, \quad z^+_n = \lambda^+_n - n. \tag{2.2.15}$$

In the sequel we consider potentials of the form $v(x) = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ with

$$P(x) = ae^{-2ix} + Ae^{2ix}, \quad Q(x) = be^{-2ix} + Be^{2ix}$$

whose corresponding Fourier coefficients are

$$p(-2) = a, \;\; p(2) = A; \quad q(-2) = b, \;\; q(2) = B, \tag{2.2.16}$$

and

$$p(m) = q(m) = 0 \;\text{ for } m \neq \pm 2. \tag{2.2.17}$$

Let us change in (2.2.11) the indices $j_2, j_4, \dots, j_{2\nu}$ by $-j_2, -j_4, \dots, -j_{2\nu}$. Then by (2.2.16) and (2.2.17) each nonzero term in the resulting sum comes from a 2ν-tuple of indices $(j_1, \dots, j_{2\nu})$ with $j_1, j_3, \dots, j_{2\nu-1} \neq n$ and $j_2, j_4, \dots, j_{2\nu} \neq -n$ such that

$$(n + j_1) + (j_2 - j_1) + \cdots + (j_{2\nu} - j_{2\nu-1}) + (n - j_{2\nu}) = 2n \tag{2.2.18}$$

and

$$n + j_1, j_2 - j_1, \dots, j_{2\nu} - j_{2\nu-1}, n - j_{2\nu} \in \{-2, 2\}. \tag{2.2.19}$$

So by (2.2.5), (2.2.6) and (2.2.13) we obtain that

$$\beta_n^+(z) = q(2n) + \sum_{\nu=1}^{\infty} \mathcal{B}_{2\nu}^+(n, z), \tag{2.2.20}$$

where

$$\mathcal{B}_{2\nu}^+ = \sum_{(j_l)_{l=1}^{2\nu} \in I_{2\nu}} \frac{q(n + j_1)p(j_2 - j_1) \cdots p(j_{2\nu} - j_{2\nu-1})q(n - j_{2\nu})}{(n - j_1 + z)(n + j_2 + z) \cdots (n - j_{2\nu-1} + z)(n + j_{2\nu} + z)}, \tag{2.2.21}$$

and

$$\begin{aligned} I_{2\nu} = \{(j_l)_{l=1}^{2\nu} : j_1, j_3, \dots, j_{2\nu-1} \neq n; j_2, j_4, \dots, j_{2\nu} \neq -n; \\ n + j_1, j_2 - j_1, \dots, j_{2\nu} - j_{2\nu-1}, n - j_{2\nu} \in \{-2, 2\}\}. \end{aligned} \tag{2.2.22}$$

In view of (2.2.18) and (2.2.19), there is one-to-one correspondence between the nonzero terms in (2.2.21) and the *admissible* walks $x = (x_t)_{t=1}^{2\nu+1}$ on $\mathbb{Z}$ from $-n$ to n with steps $x_t = \pm 2$ such that $j_1, j_3, \dots, j_{2\nu-1} \neq n$ and $j_2, j_4, \dots, j_{2\nu} \neq -n$. For every such walk $x = (x_t)_{t=1}^{2\nu+1}$ we set

$$h^+(x, z) = \frac{q(x_1)p(x_2)q(x_3) \cdots p(x_{2\nu})q(x_{2\nu+1})}{(n - j_1 + z)(n + j_2 + z) \cdots (n - j_{2\nu-1} + z)(n + j_{2\nu} + z)}. \tag{2.2.23}$$

Let $X_n(r), \; r = 0, 1, 2, \dots$ denote the set of all admissible walks from $-n$ to n, with r negative steps if $n > 0$ or with r positive steps if $n < 0$. It is easy to see that every walk $x \in X_n(r)$ has totally $|n| + 2r$ steps because $\sum x_t = 2n$. In these notations, we have

$$\beta_n^+(z) = \sum_{r=0}^{\infty} \sigma_r^+(n,z) \quad \text{with} \quad \sigma_r^+(n,z) = \sum_{x \in X_n(r)} h^+(x,z). \tag{2.2.24}$$

Of course, we may write similar formulas for $\beta_n^-(z)$ as well. A walk $y = (y_t)_{t=1}^{2\nu+1}$ from n to $-n$ is *admissible* if its steps are ± 2 and its vertices satisfy $j_1, j_3, \dots, j_{2\nu-1} \neq -n$ and $j_2, j_4, \dots, j_{2\nu} \neq n$. We set

$$h^-(y,z) = \frac{p(y_1)q(y_2)p(y_3)\cdots q(y_{2\nu})p(y_{2\nu+1})}{(n+j_1+z)(n-j_2+z)\cdots(n+j_{2\nu-1}+z)(n-j_{2\nu}+z)}, \tag{2.2.25}$$

and let $Y_n(r),\ r = 0,1,2,\dots$ denote the set of all admissible walks from n to $-n$ having r positive steps if $n > 0$ or r negative steps if $n < 0$. Then, changing in (2.2.10) the indices $j_1, \dots, j_{2\nu-1}$ by $-j_1, \dots, -j_{2\nu-1}$, we see that

$$\beta_n^-(z) = \sum_{r=0}^{\infty} \sigma_r^-(n,z) \quad \text{with} \quad \sigma_r^-(n,z) = \sum_{y \in Y_n(r)} h^-(y,z). \tag{2.2.26}$$

Finally, we consider $\alpha_n(z)$. A walk $(w_t)_{t=1}^{2\nu}$ from n to n is *admissible* if its steps are ± 2 and its vertices satisfy $j_1, \dots, j_{2\nu-1} \neq -n$ and $j_2, \dots, j_{2\nu-2} \neq n$. We set

$$h(w,z) = \frac{p(w_1)q(w_2)\cdots p(w_{2\nu-1})q(w_{2\nu})}{(n+j_1+z)(n-j_2+z)\cdots(n+j_{2\nu-2}+z)(n-j_{2\nu-1}+z)}, \tag{2.2.27}$$

and let $W_n(\nu),\ \nu = 1,2,\dots$ denote the set of all admissible walks from n to n having 2ν steps. In view of (2.2.5) and (2.2.13), changing in (2.2.8) the indices $j_1, \dots, j_{2\nu-1}$ by $-j_1, \dots, -j_{2\nu-1}$, we obtain that

$$\alpha_n(z) = \sum_{\nu=1}^{\infty} \tau_\nu(n,z) \quad \text{with} \quad \tau_\nu(n,z) = \sum_{w \in W_n(\nu)} h(w,z). \tag{2.2.28}$$

Of course, $\sigma_r^\pm$ and $\beta_n^\pm$ depend on the potential functions but in the above notations this dependence is suppressed. If we use instead the notations $\sigma_r^\pm(P,Q;n,z)$ and $\beta_n^\pm(P,Q;z)$ then the following holds.

Lemma 18. *In the above notations,*

$$\sigma_r^-(P,Q;n,z) = \sigma_r^+(Q,P;-n,-z), \quad r \in \mathbb{Z}_+, \tag{2.2.29}$$

and

$$\beta_n^-(P,Q;z) = \beta_{-n}^+(Q,P;-z). \tag{2.2.30}$$

Proof. Let us write also $h^{\pm}_{P,Q}(x,z)$. One can easily see that $Y_n(r) = X_{-n}(r)$ and if $y \in Y_n(r)$ then

$$h^-_{P,Q}(y,z) = h^+_{Q,P}(y,-z).$$

Now (2.2.29) follows and so (2.2.30) holds as well. $\square$

Proposition 19. *For $n \in 2\mathbb{Z}$ with large enough $|n|$ we have*

$$\beta^-_n(z) \equiv 0, \quad \beta^+_n(z) \equiv 0; \tag{2.2.31}$$

$$z^*_n = \alpha(z^*_n), \quad \textit{where } z^*_n = z^-_n = z^+_n; \tag{2.2.32}$$

$$\lambda^-_n = \lambda^+_n. \tag{2.2.33}$$

Proof. If n is even then there are no admissible walks from $-n$ to n. Indeed, since every admissible walk has odd number of steps (each equal to ± 2), the sum of all steps is not divisible by 4 while $2n$ is multiple to 4. Therefore, it follows that $\beta^+_n(z) \equiv 0$. The same argument shows that $\beta^-_n(z) \equiv 0$, so (2.2.31) is proved.

Now the equation (2.2.14) takes the form $(z - \alpha_n(z))^2 = 0$, so it has a double root, say z^*_n. Hence (2.2.32) and (2.2.33) hold. $\square$

2.3 Asymptotic estimates for $z^{\pm}_n$ and $\alpha_n(z)$

In this section we give the asymptotics of $\alpha_n(z)$ and derive asymptotic formulas for $z^{\pm}_n = \lambda^{\pm}_n - n$ using the basic equation (2.2.14).

The following lemma gives preliminary asymptotic estimates of $\beta^{\pm}_n(z)$ for odd $n \in \mathbb{Z}$; the precise asymptotics will be given in the next section.

Lemma 20. *If $n = \pm(2m+1),\ m \in \mathbb{N}$ then*

$$\beta^{\pm}_n(z) = O\left((8D^2)^m/m^m\right), \quad |z| \le 1/2, \tag{2.3.1}$$

where $D = \max\{|a|, |A|, |b|, |B|\}$.

Proof. We prove (2.3.1) for β^+_n only. The same argument could be used in the case of β^-_n as well, but by (2.2.30) the assertion for β^-_n follows if (2.3.1) is known for β^+_n.

Fix $r \in \mathbb{Z}_+$, and let $x \in X_n(r)$ be a walk from $-n$ to n having r negative (positive) steps if n is positive (respectively negative). If $(j_\ell)_{\ell=1}^{2\nu},\ \nu = m + r$, are the vertices of x, then

$$|n \pm j_\ell + z| \ge |n \pm j_\ell| - 2^{-1} \ge 2^{-1}|n \pm j_\ell|, \quad \ell = 1, \ldots, 2\nu. \tag{2.3.2}$$

On the other hand we have

$$|n - j_\ell| \cdot |n + j_{\ell+1}| \geq |n|, \quad \ell = 1, \ldots, 2\nu - 1. \tag{2.3.3}$$

Indeed, both $|n-j_\ell|$ and $|n+j_{\ell+1}|$ are even. If j_ℓ and $j_{\ell+1}$ have the same sign, then at least one of those numbers is greater than $|n|$, so (2.3.3) follows. Since $|j_{\ell+1}-j_\ell| = 2$, j_ℓ and $j_{\ell+1}$ could have opposite signs if, and only if, $|j_\ell| = |j_{\ell+1}| = 1$. But then

$$|n - j_\ell| \cdot |n + j_{\ell+1}| = n^2 - 1 > |n|,$$

so (2.3.3) holds. Now (2.3.2) and (2.3.3) imply, for $n = \pm(2m+1)$ and $|z| \leq 1/2$, that

$$\frac{1}{|n - j_1 + z||n + j_2 + z| \cdots |n - j_{2\nu-1} + z||n + j_{2\nu} + z|} \leq \frac{2^{2\nu}}{(2m)^\nu},$$

so in view of (2.2.23) we obtain

$$|h^+(x,z)| \leq D^{2\nu+1}(2/m)^\nu, \quad \nu = m + r.$$

Since the steps of every walk $x \in X_n(r)$ are equal to ± 2, we have $card\, X_n(r) \leq 2^{2\nu}$. Thus,

$$|\sigma_r^+(n,z)| \leq \sum_{x \in X_n(r)} |h^+(x,z)| \leq D\left(8D^2/m\right)^{m+r},$$

which implies (2.3.1). □

Proposition 21. *For odd $n \in \mathbb{Z}$ with large enough $|n|$*

$$z_n^\pm = \alpha_n(n, z_n^\pm) + O(|n|^{-p}) \ \ \forall p > 0. \tag{2.3.4}$$

Proof. Let $n = \pm(2m+1)$. We know that $z_n^\pm$ are roots of equation (2.2.14). Therefore, from (2.3.1) it follows that

$$|z_n^\pm - \alpha_n(n, z_n^\pm)| = O\left((8D^2/m)^m\right)$$

which implies (2.3.4). □

Lemma 22. *For $n \in \mathbb{Z}$ with large enough $|n|$*

$$\alpha_n(z) = \frac{Ab + aB}{2n} + O(n^{-2}), \quad |z| \leq 1/2, \tag{2.3.5}$$

and

$$z_n^\pm = \frac{Ab + aB}{2n} + O(n^{-2}). \tag{2.3.6}$$

Proof. We estimate $\alpha_n(z)$ by using (2.2.28). To evaluate $\tau_1(n,z)$ we consider the two-step walks from n to n. There are two such walks, respectively with steps $(2,-2)$ and $(-2,2)$, and the corresponding vertices are $j_1 = n+2$ and $j_1 = n-2$. Therefore, for $|z| \leq 1/2$ we have

$$\tau_1(n,z) = \frac{Ab}{2n+2+z} + \frac{aB}{2n-2+z}, \tag{2.3.7}$$

which implies

$$\tau_1(n,z) = \frac{Ab+aB}{2n} + O(n^{-2}), \quad |z| \leq 1/2. \tag{2.3.8}$$

Next we consider $\tau_2(n,z)$. The related set $W_n(2)$ of four-step walks from n to n has two elements: $(2,2,-2,-2)$ and $(-2,-2,2,2)$. The corresponding vertices are

$$j_1 = n+2, \quad j_2 = n+4, \quad j_3 = n+2$$

and

$$j_1 = n-2, \quad j_2 = n-4, \quad j_3 = n-2.$$

Therefore, in view of (2.2.27)

$$\begin{aligned} \tau_2(n,z) = &\frac{abAB}{[n+(n+2)+z][n-(n+4)+z][n+(n+2)+z]} \\ &+ \frac{abAB}{[n+(n-2)+z][n-(n-4)+z][n+(n-2)+z]}, \end{aligned} \tag{2.3.9}$$

so it follows that

$$\tau_2(n,z) = O(n^{-2}), \quad |z| \leq 1/2. \tag{2.3.10}$$

Further, if $w \in W_n(\nu)$, $\nu = 3,4,\ldots$ is a walk with 2ν steps from n to n, then $h(w,z)$ is a fraction whose denominator $d(w,z)$ has the form

$$d(w,z) = (2n \pm 2 + z) \prod_{k=1}^{\nu-1} (n - j_{2k} + z)(n + j_{2k+1} + z).$$

For $|z| \leq 1/2$, we have $|2n \pm 2 + z| \geq |n|/2$ and by (2.3.2) and (2.3.3) the absolute value of every factor of the product is greater than $|n|/2$, so

$$|d(w,z)| \geq (|n|/2)^\nu.$$

Now the same argument as in the proof of Lemma 20 leads to

$$|\tau_\nu(n,z)| \leq C^\nu/|n|^\nu, \quad \nu = 3,4,\ldots, \tag{2.3.11}$$

where C is a constant depending only on a, b, A, B. Therefore, it follows that

$$\sum_{\nu=3}^{\infty} |\tau_\nu(n,z)| \le \sum_{\nu=3}^{\infty} \frac{C^\nu}{|n|^\nu} = O(|n|^{-3}) \quad \text{for} \;\; |z| \le 1/2. \tag{2.3.12}$$

Now (2.3.8), (2.3.10) and (2.3.12) imply (2.3.5). In view of (2.2.32) and (2.3.4), (2.3.6) follows from (2.3.5). $\square$

Next we refine (2.3.6) by finding the next term in the asymptotic expansion of $z_n^\pm$ about the powers of $1/|n|$.

Proposition 23. *For large enough* $|n| \in \mathbb{Z}$

$$z_n^\pm = \frac{Ab + aB}{2n} + \frac{aB - Ab}{2n^2} + O(|n|^{-3}). \tag{2.3.13}$$

Proof. From (2.3.7) and (2.3.6) it follows that

$$\begin{aligned} \tau_1(n, z_n^\pm) &= \frac{Ab}{2n}(1 - 1/n + O(n^{-2})) + \frac{aB}{2n}(1 + 1/n + O(n^{-2})) \\ &= \frac{Ab + aB}{2n} + \frac{aB - Ab}{2n^2} + O(|n|^{-3}). \end{aligned}$$

On the other hand, (2.3.9) and (2.3.6) imply with $z = z_n^\pm$

$$\tau_2(n, z_n^\pm) = \frac{-abAB}{(2n+2+z)^2(4-z)} + \frac{abAB}{(2n-2+z)^2(4+z)} = O(|n|^{-3}).$$

Therefore, in view of (2.3.12) we obtain (2.3.13). $\square$

Remark. Of course, one can easily get more terms of the asymptotic expansion of $z_n^\pm$ by using (2.3.13) and refining further the asymptotic analysis of $\alpha_n(z_n^\pm)$.

In order to estimate $\gamma_n = \lambda_n^+ - \lambda_n^- = z_n^+ - z_n^-$ in the next section, we need the following.

Lemma 24. *If* $n = \pm(2m+1)$ *with* $m \in \mathbb{N}$*, then*

$$d\alpha_n(z)/dz = O(m^{-2}) \quad \textit{for} \quad |z| \le 1/4, \tag{2.3.14}$$

and

$$\alpha_n(z_n^+) - \alpha_n(z_n^-) = \gamma_n O(m^{-2}). \tag{2.3.15}$$

Proof. By (2.2.28) we have

$$\alpha_n(z) = \tau_1(n,z) + \tilde{\alpha}_n(z) \quad \text{with} \quad \tilde{\alpha}_n(z) = \sum_{\nu=2}^{\infty} \tau_\nu(n,z).$$

In view of (2.3.10) and (2.3.12),

$$\tilde{\alpha}_n(z) = O(m^{-2}) \quad \text{for } \; |z| \le 1/2.$$

Therefore, the Cauchy formula for derivatives implies that

$$d\tilde{\alpha}_n(z)/dz = O(m^{-2}) \quad \text{for } \; |z| \le 1/4.$$

On the other hand, by (2.3.7) we have

$$\partial_z \tau_1(n,z) = -\frac{Ab}{(2n+2+z)^2} - \frac{aB}{(2n-2+z)^2} = O(m^{-2}) \quad \text{for } \; |z| \le 1/2,$$

so (2.3.14) follows.

Further we have

$$\alpha_n(z_n^+) - \alpha_n(z_n^-) = \int_{z_n^-}^{z_n^+} \alpha_n'(z)\, dz, \tag{2.3.16}$$

where the integral is taken over the segment $[z_n^-, z_n^+]$ from z_n^- to z_n^+. Therefore, by 2.3.14 we obtain

$$|\alpha_n(z_n^+) - \alpha_n(z_n^-)| \le |z_n^+ - z_n^-| \sup_{[z_n^-, z_n^+]} |\alpha_n'(z)| = |z_n^+ - z_n^-|\, O(m^{-2}),$$

hence (2.3.15) holds. □

2.4 Asymptotic formulas for $\beta_n^{\pm}(z)$ and γ_n.

In this section only odd integers n with large enough $|n|$ are considered.

We use (2.2.24) to find precise asymptotics of $\beta_n^+(z)$. First we analyze $\sigma_0^+(n,z)$. If $n = 2m+1$ with $m \in \mathbb{N}$ then there is only one admissible walk from $-n$ to n with no negative steps. We denote this walk by ξ, so we have $X_n(0) = \{\xi\}$ and $\sigma_0(n,z) = h^+(\xi, z)$. Since

$$\xi(t) = 2, \quad 1 \le t \le 2m+1, \tag{2.4.1}$$

we obtain

$$\sigma_0^+(n,z) = \frac{A^m B^{m+1}}{(n-j_1+z)(n+j_2+z)\cdots(n-j_{2m-1}+z)(n+j_{2m}+z)} \tag{2.4.2}$$

with $j_\nu = -2m - 1 + 2\nu,\ \ \nu = 1, \ldots, 2m$.

If $n = -(2m+1)$, then again $X_n(0)$ has only one element, say

$$\bar{\xi} = (\bar{\xi}_t)_{t=1}^{2m+1}, \quad \bar{\xi}(t) = -2 \ \ \forall t. \tag{2.4.3}$$

Therefore $\sigma_0(n, z) = h^+(\bar{\xi}, z)$ and so it follows that

$$\sigma_0^+(n, z) = \frac{a^m b^{m+1}}{(n - j_1 + z)(n + j_2 + z) \cdots (n - j_{2m-1} + z)(n + j_{2m} + z)} \tag{2.4.4}$$

with $j_\nu = 2m + 1 - 2k,\ \ \nu = 1, \cdots, 2m$.

Lemma 25. *In the above notations,*

$$\sigma_0^+(n, 0) = \begin{cases} \frac{A^m B^{m+1}}{4^{2m}(m!)^2} & \text{for } \ n = 2m+1, \\ \frac{a^m b^{m+1}}{4^{2m}(m!)^2}, & \text{for } \ n = -2m-1. \end{cases} \tag{2.4.5}$$

Proof. In the case $n = 2m + 1$ we have

$$\prod_{k=1}^{m} [n - (-n + 2(2k-1))] \prod_{k=1}^{m} [n + (-n + 2(2k))] = 4^{2m}(m!)^2,$$

so (2.4.5) holds. The proof is similar for $n = -2m - 1$. □

Lemma 26. *For $n = \pm(2m+1)$,*

$$\frac{\sigma_0^+(n, z_n^\pm)}{\sigma_0^+(n, 0)} = \left[1 - \frac{(Ab + aB)\log m}{8m} - \frac{g(Ab + aB)}{8m} + O\left(\frac{\log^2 m}{m^2}\right)\right], \tag{2.4.6}$$

where g is the Euler constant.

Proof. From (2.4.2) and (2.4.4) it follows that

$$\frac{\sigma_0^+(n, z_n^\pm)}{\sigma_0^+(n, 0)} = \prod_{k=1}^{m} (1 + c_k(n))^{-1} \prod_{k=1}^{m} (1 + d_k(n))^{-1}$$

with

$$c_k(n) = sgn\,(n)\, \frac{z_n^\pm}{4k}, \quad d_k(n) = \frac{sgn\,(n)\, z_n^\pm}{4(m-k+1)}. \tag{2.4.7}$$

One can easily see that $\prod_{k=1}^m (1 + c_k(n))^{-1} = \prod_{k=1}^m (1 + d_k(n))^{-1}$ and

$$\log\left(\prod_{k=1}^{m} (1 + c_k(n))^{-1}\right) = -\sum_{k=1}^{m} \log(1 + c_k(n)) = -\sum_{k=1}^{m} c_k(n) + O\left(\sum_{k=1}^{m} |c_k(n)|^2\right).$$

In view of (2.3.6) and (2.4.7) we have

$$\sum_{k=1}^{m} c_k(n) = \left(\sum_{k=1}^{m} \frac{1}{4k}\right)\left[\frac{Ab+aB}{4m} + O(m^{-2})\right],$$

and

$$\sum_{k=1}^{m} |c_k(n)|^2 = \left(\sum_{k=1}^{m} \frac{1}{16k^2}\right) O(m^{-2}) = O(m^{-2}).$$

Therefore, by (1.4.11) we obtain

$$\log\left(\prod_{k=1}^{m} \frac{1}{1+c_k(n)}\right) = -\frac{(Ab+aB)\log m}{16m} - \frac{g(Ab+aB)}{16m} + O\left(\frac{\log m}{m^2}\right). \tag{2.4.8}$$

Hence,

$$\prod_{k=1}^{m} \frac{1}{1+c_k(n)} = 1 - \frac{(Ab+aB)\log m}{16m} - \frac{g(Ab+aB)}{16m} + O\left(\frac{\log^2 m}{m^2}\right),$$

which implies (2.4.6). □

We need also the following modification of Lemma 26.

Lemma 27. *For $n = \pm(2m+1)$, if $z = O(m^{-1})$ then*

$$\sigma_0^+(n,z) = \sigma_0^+(n,0)(1 + O((\log m)/m)). \tag{2.4.9}$$

Proof. We follow the proof of Lemma 26, replacing $z_n^\pm$ by z and using $z = O(m^{-1})$ instead of (2.3.6). □

Next we estimate the ratio $\sigma_1^+(n,z)/\sigma_0^+(n,z)$.

Lemma 28. *If $n = \pm(2m+1)$, then*

$$\sigma_1^+(n,z) = \sigma_0^+(n,z) \cdot \Phi(n,z), \tag{2.4.10}$$

where

$$\Phi(n,z) = \sum_{k=1}^{m} \varphi_k(n,z) + \sum_{k=2}^{m} \psi_k(n,z) \tag{2.4.11}$$

with

$$\varphi_k(n,z) = \frac{bA}{(4(m+1-k)\pm z)(4k\pm z)} \tag{2.4.12}$$

and

$$\psi_k(n,z) = \frac{aB}{(4(k-1)\pm z)(4(m+1-k)\pm z)}, \tag{2.4.13}$$

where we have + or − in front of z according as $n > 0$ or $n < 0$.

Proof. From the definition of $X_n(1)$ and (2.2.24) it follows that

$$\sigma_1^+(n,z) = \sum_{x\in X_n(1)} h^+(x,z) = \sum_{\nu=2}^{2m} h(x_\nu, z), \tag{2.4.14}$$

where x_ν denotes the walk with $(\nu+1)$-th step equal to -2 and all others equal to 2 if $n = 2m+1$ or the walk with $(\nu+1)$-th step equal to 2 and all others equal to -2 if $n = -(2m+1)$. The vertices of x_ν are given by

$$j_\alpha(x_\nu) = \begin{cases} i_\alpha, & 1 \le \alpha \le \nu \\ i_{\nu-1}, & \alpha = \nu+1 \\ i_{\alpha-2}, & \nu+2 \le \alpha \le |n|+2 \end{cases} \qquad \text{with } \; i_k = \begin{cases} -n+2k, & n > 0 \\ -n-2k, & n < 0 \end{cases}$$

Therefore, by (2.2.23)

$$h(x_{2k}, z) = h(\xi, z)\frac{bA}{(n - i_{2k-1} + z)(n + i_{2k} + z)}, \tag{2.4.15}$$

and

$$h(x_{2k-1}, z) = h(\xi, z)\frac{aB}{(n + i_{2k-2} + z)(n - i_{2k-1} + z)}. \tag{2.4.16}$$

Now (2.4.14))–(2.4.16) imply (2.4.10). □

Lemma 29. *If $n = \pm(2m+1)$ and $z = O(m^{-1})$, then*

$$\Phi(n,z) = \frac{(Ab+aB)\log m}{8m} + \frac{g(Ab+aB)}{8m} + O\left(\frac{\log m}{m^2}\right) \tag{2.4.17}$$

and

$$\Phi^*(n,z) := \sum_{k=1}^{m} |\varphi_k(n,z)| + \sum_{k=2}^{m} |\psi_k(n,z)| = O\left(\frac{\log m}{m}\right). \tag{2.4.18}$$

Proof. In view of (2.4.11)–(2.4.13),

$$\Phi(n,z) = \frac{bA}{2m+2\pm z}\sum_{k=1}^{m}\frac{1}{4k\pm z} + \frac{aB}{2m+2\pm z}\sum_{k=1}^{m-1}\frac{1}{4k\pm z}.$$

Since

$$\sum_{k=1}^{m}\frac{1}{4k+z} = \sum_{k=1}^{m}\frac{1}{4k} + O(m^{-1}) \quad \text{if } \; z = O(m^{-1}),$$

(2.4.17) follows immediately.

To obtain (2.4.18) we use that $|4k \pm z| \geq 2k$, and therefore,

$$\Phi_n^*(n,z) \leq 4\Phi_n^*(n,0) \leq \frac{4|bA|}{2m+2}\sum_{k=1}^{m}\frac{1}{4k} + \frac{4|aB|}{2m+2}\sum_{k=1}^{m-1}\frac{1}{4k} = O\left(\frac{\log m}{m}\right).$$

$\square$

Proposition 30. *For $n = \pm(2m+1)$ we have*

$$\beta_n^+(z) = \sigma_0^+(n,0)\left[1 + O\left(\frac{\log m}{m}\right)\right] \quad \textit{if } z = O(m^{-1}) \tag{2.4.19}$$

and

$$\beta_n^+(z_n^\pm) = \sigma_0^+(n,0)\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right], \tag{2.4.20}$$

with

$$\sigma_0^+(n,0) = \begin{cases} \frac{A^m B^{m+1}}{4^{2m}(m!)^2} & \textit{for } n = 2m+1, \\ \frac{a^m b^{m+1}}{4^{2m}(m!)^2}, & \textit{for } n = -2m-1. \end{cases} \tag{2.4.21}$$

Proof. From (2.4.9), (2.4.10) and (2.4.17) it follows that

$$\sigma_1^+(n,z) + \sigma_0^+(n,z) = \sigma_0^+(n,0)\left[1 + O\left(\frac{\log m}{m}\right)\right] \quad \text{if } z = O(m^{-1}).$$

Also, (2.4.6)), (2.4.10) and (2.4.17) imply that

$$\sigma_1^+(n,z_n^\pm) + \sigma_0^+(n,z_n^\pm) = \sigma_0^+(n,0)\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right].$$

Since $\beta_n^+(z) = \sum_{r=0}^\infty \sigma_r^+(n,z)$, to complete the proof it is enough to show that

$$\sum_{r=2}^\infty \sigma_r^+(n,z) = \sigma_0^+(n,z)\, O\left(\frac{\log^2 m}{m^2}\right) \quad \text{if } z = O(m^{-1}). \tag{2.4.22}$$

Next we prove (2.4.22). Recall that $\sigma_r^+(n,z) = \sum_{x \in X_n(r)} h^+(x,z)$. Now we set

$$\sigma_r^*(n,z) := \sum_{x \in X_n(r)} |h^+(x,z)|.$$

We are going to show that there is an absolute constant $C > 0$ such that for

$n = \pm(2m+1)$ with large enough m

$$\sigma_r^*(n,z) \le \sigma_{r-1}^*(n,z) \cdot \frac{C\log m}{m} \quad \text{if } \ z = O(m^{-1}), \ \ r = 1,2,\dots. \tag{2.4.23}$$

Since $\sigma_0^+(n,z)$ has one term only, we have $\sigma_0^*(n,z) = |\sigma_0^+(n,z)|$.

Let $r \in \mathbb{N}$. To every walk $x \in X_n(r)$ we assign a pair $(\tilde{x}, k)$, where k is such that $x(k+1)$ is the first negative (if $n > 0$) or positive (if $n < 0$) step of x and $\tilde{x} \in X_n(r-1)$ is the walk that we obtain after dropping from x the steps $x(k)$ and $x(k+1)$. In other words, we consider the map

$$\varphi : X_n(r) \longrightarrow X_n(r-1) \times I, \quad \varphi(x) = (\tilde{x}, k), \ \ k \in I = \{2,\dots,2m\},$$

where $k = \begin{cases} \min\{t : x(t) = 2, \ x(t+1) = -2\} & \text{if } \ n > 0, \\ \min\{t : x(t) = -2, \ x(t+1) = 2\} & \text{if } \ n < 0, \end{cases}$

$$\tilde{x}(t) = \begin{cases} x(t) & \text{if } 1 \le t \le k-1, \\ x(t+2) & \text{if } k \le t \le 2m+2r-1. \end{cases}$$

The map φ is clearly injective and we have

$$\frac{h(x,z)}{h(\tilde{x},z)} = \begin{cases} \frac{bA}{(n-j+2\pm z)(n+j\pm z)} & \text{if } k \text{ is even}, \\ \frac{aB}{(n+j-2\pm z)(n-j\pm z)} & \text{if } k \text{ is odd}, \end{cases} \qquad j = \begin{cases} -n+2k & \text{if } \ n > 0, \\ -n-2k & \text{if } \ n < 0, \end{cases}$$

where in front of z we have $+$ if $n > 0$ or $-$ if $n < 0$.

Since φ is injective, from (2.4.13), (2.4.18) it follows that

$$\sigma_r^*(n,z) \le \sigma_{r-1}^*(n,z) \cdot \Phi^*(n,z).$$

Hence, by (2.4.18) and (2.4.17), we obtain that (2.4.23) holds.

From (2.4.23) it follows (since $\sigma_0^*(n,z) = |\sigma_0^+(n,z)|$) that

$$\sigma_r^*(n,z) \le |\sigma_0^+(n,z)| \cdot \left(\frac{C\log m}{m}\right)^r.$$

Hence, (2.4.22) holds, which completes the proof. □

The asymptotics of β_n^- could be found in a similar way. We have the following.

Proposition 31. *If $n = \pm(2m+1)$ then*

$$\beta_n^-(z) = \sigma_0^-(n,0)\left[1 + O\left(\frac{\log m}{m}\right)\right] \quad \text{if } z = O(m^{-1}) \tag{2.4.24}$$

and

$$\beta_n^-(z_n^\pm) = \sigma_0^-(n,0)\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right], \tag{2.4.25}$$

with

$$\sigma_0^-(n,0) = \begin{cases} \frac{a^{m+1}b^m}{4^{2m}(m!)^2} & \text{for } n = 2m+1, \\ \frac{A^{m+1}B^m}{4^{2m}(m!)^2} & \text{for } n = -2m-1. \end{cases} \tag{2.4.26}$$

Proof. One could give a proof by following step by step the proof of Proposition 30 but analyzing the sums (2.2.26) instead of (2.2.24).

However, Lemma 18 provides an alternative approach. In view of (2.2.30), formula (2.4.24) follows from (2.4.19) immediately. □

Theorem 32. *The Dirac operator* (2.1.1) *considered with*

$$P(x) = ae^{-2ix} + Ae^{2ix}, \quad Q(x) = be^{-2ix} + Be^{2ix}, \quad a, A, b, B \in \mathbb{C},$$

has for large enough $|n| \in \mathbb{Z}$ two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues λ_n^-, λ_n^+ such that

$$\lambda_n^\pm = n + \frac{Ab + aB}{2n} + \frac{aB - Ab}{2n^2} + O(|n|^{-3}). \tag{2.4.27}$$

If n is even, then $\gamma_n = \lambda_n^+ - \lambda_n^- = 0$. For odd $n = \pm(2m+1)$ with $m \in \mathbb{N}$, we have

$$\gamma_{2m+1} = \pm 2\frac{\sqrt{(Ab)^m(aB)^{m+1}}}{4^{2m}(m!)^2}\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right], \tag{2.4.28}$$

and

$$\gamma_{-(2m+1)} = \pm 2\frac{\sqrt{(Ab)^{m+1}(aB)^m}}{4^{2m}(m!)^2}\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right]. \tag{2.4.29}$$

Proof. For even n with large enough $|n|$ we have $\lambda_n^+ = \lambda_n^-$ by Proposition 19, and (2.4.27) comes from (2.3.13).

Let $n = \pm(2m+1)$, and let

$$C = \max\{|a|^2, |b|^2, |A|^2, |B|^2\}, \qquad \mathbb{D}_m = \{z : \ |z| < Cm^{-1}\}.$$

In view of (2.3.6), for large enough m we have

$$|z_n^\pm| < \frac{1}{2}Cm^{-1}, \tag{2.4.30}$$

so $z_n^\pm \in \frac{1}{2}\mathbb{D}_m$.

On the other hand, from (2.4.19) and (2.4.24) it follows that for large enough m

$$\beta_n^\pm(z) = \sigma_0^\pm(n,0)(1 + r_n^\pm(z)) \quad \text{with} \ \ |r_n^\pm(z)| \le 1/2 \quad \text{for} \ \ z \in 2\mathbb{D}_m.$$

We set

$$\sqrt{\beta_n^-(z)\beta_n^+(z)} := \sqrt{\sigma_0^-(n,0)\sigma_0^+(n,0)}\,(1 + r_n^-(z))^{1/2}(1 + r_n^+(z))^{1/2},$$

where $\sqrt{\sigma_0^-(n,0)\sigma_0^+(n,0)}$ is a square root of $\sigma_0^-(n,0)\sigma_0^+(n,0)$ and $(1+w)^{1/2}$ is defined by its Taylor series about $w = 0$. Then $\sqrt{\beta_n^-(z)\beta_n^+(z)}$ is a well-defined analytic function on $2\mathbb{D}_m$, so the basic equation (2.2.14) splits into two equations

$$z - \alpha_n(z) - \sqrt{\beta_n^-(z)\beta_n^+(z)} = 0, \tag{2.4.31}$$

$$z - \alpha_n(z) + \sqrt{\beta_n^-(z)\beta_n^+(z)} = 0. \tag{2.4.32}$$

Next we show that for large enough m equation (2.4.31) has at most one root in the disc $\mathbb{D}_m$. Let

$$\varphi_n(z) = \alpha_n(z) + \sqrt{\beta_n^-(z)\beta_n^+(z)}, \quad f_n(z) = z - \varphi_n(z).$$

By (2.3.14) we have $\alpha_n'(z) = O(m^{-2})$ for $|z| \le 1/4$. On the other hand, Lemma 20 implies that

$$\sqrt{\beta_n^-(z)\beta_n^+(z)} = O(m^{-2}) \quad \text{for} \quad z \in 2\mathbb{D}_m,$$

so by the Cauchy formulas for the derivatives we have

$$\frac{d}{dz}\sqrt{\beta_n^-(z)\beta_n^+(z)} = O(m^{-1}) \quad \text{for} \quad z \in \mathbb{D}_m.$$

Therefore

$$\sup\{|\varphi_n'(z)| \,:\, z \in \mathbb{D}_m\} \le 1/2,$$

which implies

$$|\varphi_n(z_1) - \varphi_n(z_2)| = \left|\int_{z_1}^{z_2} \varphi_n'(z)dz\right| \le \frac{1}{2}|z_1 - z_2| \quad \text{for} \ \ z_1, z_2 \in \mathbb{D}_m.$$

Now we obtain, for $z_1, z_2 \in \mathbb{D}_m$, that

$$|f_n(z_1) - f_n(z_2)| = |(z_1 + \varphi_n(z_1)) - (z_2 + \varphi_n(z_2))|$$
$$\geq |z_1 - z_2| - |\varphi_n(z_1) - \varphi_n(z_2)| \geq \frac{1}{2}|z_1 - z_2|.$$

Hence the equation $f_n(z) = 0$ (i.e., equation (2.4.31)) has at most one solution in the disc $\mathbb{D}_m$. Of course, the same argument gives that equation (2.4.32) also has at most one solution in the disc $\mathbb{D}_m$.

On the other hand, we know by Lemma 16 and (2.4.30) that for large enough m equation (2.2.14) has exactly two roots z_n^-, z_n^+ in the disc $\mathbb{D}_m$, so either z_n^- is the root of (2.4.31) and z_n^+ is the root of (2.4.32), or vise versa z_n^+ is the root of (2.4.31) and z_n^- is the root of (2.4.32). Therefore, we obtain

$$z_n^+ - z_n^- - [\alpha_n(z_n^+) - \alpha_n(z_n^-)] = \pm \left[\sqrt{\beta_n^-(z_n^+)\beta_n^+(z_n^+)} + \sqrt{\beta_n^-(z_n^-)\beta_n^+(z_n^-)}\right]. \quad (2.4.33)$$

Now (2.3.15), (2.4.20), (2.4.21), (2.4.25) and (2.4.26) imply, for $n = 2m + 1$,

$$\gamma_n \left[1 + O(m^{-2})\right] = \pm 2\frac{\sqrt{(Ab)^m (aB)^{m+1}}}{4^{2m}(m!)^2}\left[1 + O\left(\frac{\log^2 m}{m^2}\right)\right],$$

which yields (2.4.28).

The same argument shows that (2.3.15), (2.4.20), (2.4.21), (2.4.25) and (2.4.26) imply (2.4.29).

Finally, if at least one of the coefficients a, A, b, B becomes zero, then $\gamma_n = 0$ for all n. This follows from (2.4.33) where $\beta_n^-(z_n^\pm)\beta_n^+(z_n^\pm)$ becomes zero for all n, in consideration of (2.4.19), (2.4.21), (2.4.24) and (2.4.26). □

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