On Dragilev's contribution to Mathematics

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Abstract

This article follows mainly author's talk on the workshop dedicated to the 90th anyversary of Professor Mikhail Mikhaylovich Dragilev.

I am going to talk mainly about some Dragilev's mathematical results and especially about his influence on further investigations of many mathematicians. And I stand myself on the first place among his followers. It happened that I was under great influence of Mikhail Mikhaylovich Dragilev beginning from 1959, when I was a student of the 5th year and we met regularly on Haplanov's seminar. Then he was a great example for me, a very beginner in mathematics, to follow.

Dragilev's self-exactingness in his mathematical investigations is a good example for anybody. He published not so many papers, but most of them are of the highest level. He began his mathematical career by hard working during almost 7 years on two difficult problems: regularity (absoluteness) and quasiequivalence of bases in the space A_1 of all analytic functions in the unit disk (with the locally convex topology of the local uniform convergence). He did not publish any intermediate results until he represented in 1957-58 a final solution of both problems. I remember, how his advisor Professor Mikhail Grigoryevich Haplanov told, with high enthusiasm, on his lectures in Complex Analysis about this Dragilev's achievement. Let me say shortly about the above mentioned problems.

First about the absoluteness of bases. In 1940-1950ies there was an avalanche of publications (Whittaker, Newns, Cannon, Makar, Mikhail, Mursi, Nassif, Eweida et all), where, side by side with nice concrete results about completeness, basisness, estimates of concrete systems of analytic functions, some general criteria were studied for a system of analytic functions to be a *conditional, unconditional, or absolute basis* in the space A_1 of all analytic functions in the open unit disk (with usual locally convex topology). And Dragilev had proved the following pioneering result, showing that all those kinds of bases do coincide!

Theorem 1 *(**[D1958]**)Let $\{x_k(z)\}$ be a basis in the space A_1 and $x(z) = \sum_{k=1}^{\infty} c_k\ x_k(z)$ be a basis expansion of $x \in A_1$. Then $\{x_k(z)\}$ is **absolute**, that is for every $r < 1$ we have*

$\|x\|_r := \sum_{k=1}^{\infty} |c_k|\ \max\{|x_k(z)| : |z| \le r\} < \infty$
and this system of norms determines the original topology on A_1.
Quite soon after, Dynin and Mityagin (1960) proved that the absoluteness of bases has a general nature, namely it is true for arbitrary nuclear Fréchet space.

Quasiequivalence property of bases in A_1. Mikhail Grigoryevich Haplanov introduced a notion of a power basis: a basis in A_1 is called *power* if it is equivalent to the Taylor basis, that is can be obtained from the latter as an image under an isomorphism of A_1 onto itself. By the way, he thought first that all bases in A_1 are power, this opinion was invalidated by a counterexample of Korobeinik, who showed in his student publication, that there are permutations and normalizations of the Taylor basis that result a non-power bases. Then Haplanov stated the following

Problem 2 *Is any basis $\{x_k(z)\}$ in A_1 quasipower, i. e. does there exist a permutation $\sigma : N \to N$ and a sequence of numbers t_k such that the basis $\{t_k\ x_{\sigma(k)}(z)\}$ becomes equivalent to the Taylor basis $e_k(z) = z^{k-1},\ k \in \mathbb{N}$, that is, there exists an isomorphism $T : A_1 \to A_1$ such that $T(e_k) = t_k\ x_{\sigma(k)},\ k \in \mathbb{N}$.?*

Finally Dragilev gave a positive answer to this question too.
Theorem.(Dragilev 1958, published in [**D1960**]) *All bases in A_1 are quasi-power.*

Mikhail Grigoryevich told me that Dynin and Mityagin, after Dragilev's talk on All-Union Conference in Moscow (1958), approached to him with a claim that similar (qusiequivalence) property must be also true for all nuclear Fréchet spaces with a basis. But, as we know now, this problem turned to be a much harder nut to crack, and till now their conjecture neither confirmed no disproved, in spite of intensive efforts of many mathematicians. The original Dragilev's proof was quite complicated. I remember, how we went through his proof in Haplanov's seminar and wondered, how M. M. succeeded to get this great result in such a tricky and miracle way. Let me remind some steps of his proof. Suppose that

$$x_k(z) = \sum_{n=0}^{\infty} c_{nk} z^n$$

be a basis in the space A_1, $0 < r < 1$, and $n_k(r)$ be the minimal number n such that $|c_{n,k}| r^n = \max\{|c_{jk}| r^j : j = 0, 1, \ldots\}$. Let $\omega_n(r)$ be the number of basis elements x_k such that $n_k(r) \leq n$. Then it was proved that the asymptotic estimates

$$1 \leq \liminf_{n\to\infty} \frac{\omega_n(r)}{n} \leq \limsup_{n\to\infty} \frac{\omega_n(r)}{n} < \infty, \ \mu < r < 1$$

hold for some positive $\mu < 1$.

While the right inequality is quite obvious, the proof of the left one, central in the proof, is quite tricky. A proper permutation of a basis $\{x_k\}$ was set by non-decreasing the numbers $n_k(r)$, the normalizing sequence t_k was also determined in terms of the numbers $n_k(r)$. But the proof, that after those permutation and normalization the basis becomes equivalent to the Taylor basis, is again very tricky, with some quite delicate estimates, which allowed, using the Riesz theory for compact operators, to reduce the problem to a linear algebraic one. It is worth to be noted that his first paper [D1958] contained some thin intermediate results, which were applied essentially in his proof of the quasiequivalence of bases in A_1.
Dragilev's result was an initial point for results of many mathematicians, including Dragilev himself and his students. First B. Mityagin ([M1961]) generalized this result onto the class of nuclear power series spases (centers of Riesz scales, see the definition in [M1961]) $E_\alpha(a)$, following step by step the way of Dragilev's proof. It is worth to be noted, that Mityagin's proof is much more detailed comparing with Dragilev's one and can be used for better understanding the original Dragilev's result. I think that Mityagin was always the most enthusiastic propagandist of Dragilev's achievements.
It seems that it was also Mityagin who used first the term "quasiequivalent bases" and stated the problem on the quasiequivalence of bases in a general

setting, for locally convex spaces (LCS). Namely, two bases $\{x_k\}$ and $\{y_k\}$ in LCS X are quasiequivalent if there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, a sequence of numbers t_k, and an isomorphism $T : X \to X$ such that

$$y_k = T\left(t_k\ x_{\sigma(k)}\right),\ k \in \mathbb{N}.$$

After Mityagin's result ([M1961]), it became clear that the original Dragilev's proof cannot be extended to wider classes of LCS, at least I do not know anybody's intention to do so. And the next key step, making more transparent the core of the problem, was again due to Dragilev, but about this a little bit later...

Dragilev's Ph. D. Thesis (kandidatskaya dissertaciya) The above results were included in Dragilev's Ph. D. Thesis, which was defended in Kharkov University in 1959. Professor Stechkin, one of his opponents, suggested to consider it as an outstanding candidate dissertation. In my opinion, this Dragilev's dissertation was of the Doctor Sci. level. There were some examples, when candidate dissertations were qualified as Doctor Sci. dissertation and Higher Qualification Committee (VAK) confirmed finally such a decision... But Dragilev's (and not only his) relation with VAK were quite unlucky, as it will be told in our story later.
By the way, I remember that M.M. invited us (me and A. L. Fuksman), as an active members of Haplanov's seminar, to the banquet, but we were young boys (5th year students) and gave up modestly .

Beginning of 1960th. It has to be noted that Mityagin's article ([M1961]), as well as his cycles of lectures in Rostov State University exert a great influence on mathematicians related to Haplanov's seminar (first of all on Dragilev and myself), especially by consistent using and propagating linear topological invariants (approximative and diametral dimensions), interpolation methods in operator theory and Grothendieck's, Pietsch's, Gelfand's and his own results on nuclear spaces. This relates also to the next great Dragilev's success, his work "On regular bases in nuclear spaces" ([D1965]). Before talking about this paper let me make a short deviation and speak about that time. I would like to notice that the beginning of 1960th was an especially good time in Rostov State University for doing Functional Analysis.
A seminar in Functional Analysis was functioning regularly with active role of I. I. Vorovich, M. G. Haplanov, V. I. Yudovich, I. B. Simonenko*).* Professor Haplanov organized a special seminar for studying Kantorovich-Akilov's monograph "Functional Analysis in Normed Spaces", especially Chapter XI,

where, contrary to the title of the book, the elements of the theory of linear topological spaces were considered. There was also the joint Scientific Council with Voronezh University, so prominent Voronezh mathematicians (M. A. Krasnoselskiy, S. G. Krein et al) were regular guests in Rostov State University.

By initiative of Professor Haplanov, some leading specialists in Functional Analysis (especially dealing with linear topological spaces) were invited to Rostov State University to give cycles of lectures: besides Mityagin, there were A. Pelczynski, Cz. Bessaga, S. Rolewicz, Z. Semadeni, W. Żelazko, E. Dubinsky). Not in the last instance, they were attracted by the outstanding figure of Dragilev. And in 1965 a further happy thing happened: Dragilev became a member of Mech.-Math. Faculty of Rostov State University. Till that time he was teaching in some engineering institutes first in Novocherkassk, then in Rostov-na-Donu).

"On regular bases in nuclear spaces" ([D1965]). In this paper M. M. suggested a fresh approach, which gave an essential simplification of the previous proofs in much more general settings. The central in this paper is the notion of a **regular basis**.

Definition *A basis* $\{x_k\}$ *in a (nuclear) Fréchet space* X *is called regular if there is a fundamental system of norms* $\left\{\|x\|_p\,,\ p\in\mathbb{N}\right\}$ *in* X *such that* $\frac{\|x_k\|_p}{\|x_k\|_q}\searrow 0$ *as* $k\to\infty$ *for every* p *and* $q>p$.

This notion turned to be extremely fruitful. First, the classical linear topological invariants (asymptotical and diametral dimensions) are easily computable for spaces with a regular bases (especially, nuclear ones), since the above ratio closely related with Kolmogorov diameters $d_k\left(U_q,U_p\right)$ of the corresponding balls.

Second, this notion helped to understand the limits of effectiveness of classical linear topological invariants (LTI), introduced in 1950s by Kolmogorov and Pelczynski: it became clear from further investigations (Dragilev himself, Mityagin, Zakharyuta, Kondakov, Chalov, Terzioğlu et al) that, for isomorphic distinguishing (even nuclear) spaces with non-regular basis, some new stronger invariants are needed, which can detect the non-regularity.

Dragilev proved in ([D1965]) the equivalence of all bases in Fréchet spaces with regular basis from two classes d_1 and d_2 (their definitions will be done below). Under immediate influence of this article, it was proved finally that *all bases are equivalent in all nuclear Fréchet spaces with regular basis.* It was done in 1974, independently, by L. Crone and N. Robinson ([CR]) and by V.

P. Kondakov ([Kon1974]); P. Djakov in 1975 ([Dj]) suggested even shorter and simpler proof. Kondakov in 1979 proved that the equivalence of bases take place in any Fréchet space with a regular absolute basis.

Weak quasiequivalence of bases In [D1965] Dragilev introduced another important notion. *A basis $\{x_k\}$ in a Fréchet space X is (quasidiagonaly) subordinated to a basis $\{y_k\}$ in X if there is a sequence of natural numbers $n(k) \to \infty$ (repetishions are allowed) such that $\{x_k\}$ is equivalent to the system $\left\{t_k \, y_{n(k)}\right\}$ with some normalizing sequence $\{t_k\}$. The bases are called weakly quasiequivalent if each of them is subordinated to another.*
That all bases in a nuclear Fréchet space are weakly quasiequivalent is an important step in Dragilev's proof of [D1965]. It was proved in 1982 by Kondakov-Zakharyuta [KZ]) (with the slightly changed original Dragilev's proof) that this fact is true in each Fréchet space with an absolute basis, that is in any Köthe space

$$\lambda(a_{n,p}) := \left\{ x = (\xi_n) : |x|_p := \sum_{n=1}^{\infty} |\xi_n| \ a_{n,p} < \infty \right\},$$

endowed with the topology determined by the system of (semi)norms $\left\{ |x|_p \right\}$; regularity of the canonical basis in $\lambda(a_{n,p})$ is equivalent to the condition: $\frac{a_{n.q}}{a_{n,p}} \nearrow \infty$ as $n \to \infty$ and $p < q$.
Dragilev showed in his monograph ([D2003]) that the equivalence of bases in an arbitrary Köthe space with a regular basis can be easily derived from this fact. So, Dragilev was in a small neighborhood of the general results of Crone-Robinson-Kondakov, mentioned above, already in 1965.

The weak quasiequivalence is also closely related with

Bessaga's conjecture. Let $\{x_k\}$ be a complemented basis system in a Köthe space X (that is a basis in a complemented subspace of X), then $\{x_k\}$ is quasiequivalent to a part of the canonical basis in X.

For an arbitrary Köthe space the following weaker result has been proved

Theorem 3 *(Bessaga 1968 for nuclear case [Bs]; Dragilev [D1983] and Kondakov [Kon1983] in general case) A complemented basis system in a Köthe space X is subordinated to the canonical basis of X.*

Interpolation classes D_1 and D_2. It seems that we are lucky that, by chance, Dragilev concentrated in [D1965] on the diametral dimension

$$\Gamma(X) := \left\{ (t_n) : \forall p \exists q : \frac{|t_n|}{d_n(U_q, U_p)} \text{ bounded} \right\},$$

but not on its counterpart

$$\Gamma'(X) := \{(t_n) : \exists p \forall q : |t_n| \, d_n(U_q, U_p) \text{ bounded}\},$$

where $\{U_q, \ q \in \mathbb{N}\}$ is a fundamental system of absolutely convex neiborhoods of the origin in a Fréchet space X and d_n are the Kolmogorov diameters (see, e.g., [M1961]). If he had used the latter, then the Crone-Robinson-Kondakov result of 1974 would be proved in 1965 by Dragilev, but the classes d_1 and d_2 would be not introduced at all in that time. The matter is that these classes are of high importance themselves. Dragilev introduced these classes in terms of Kolmogorov diameters, here we give his definitions in a slightly modified form.

Definition 4 *A Fréchet space X belongs to the class d_i, $i = 1, 2$, if*

$$\exists p \forall q \exists r \ : \ \lim_{n\to\infty} \frac{d_n(U_r, U_q)}{d_n(U_q, U_p)} = 0 \ \textit{for}\ i = 1;$$
$$\forall p \exists q \forall r \ : \ \lim_{n\to\infty} \frac{d_n(U_q, U_p)}{d_n(U_r, U_q)} = 0 \ \textit{for}\ i = 2.$$

So, these invariants are quite general, they are defined on the class of all Fréchet spaces and in a such generality they do not look as interpolation invariants at all. But if X is a regular Köthe space $\lambda(a_{n,p})$ then, the above conditions are equivalent to the following interpolation conditions (Dragilev [D1965], Bessaga 1968 [Bs]):

$$\exists p \forall q \exists r \exists C \ : \ a_{n,q}^2 \le C \ a_{n,p} \ a_{n,r} \text{ for } i = 1;$$
$$\forall p \exists q \forall r \exists C \ : \ a_{n,p} \ a_{n,r} \le C \ a_{n,q}^2 \text{ for } i = 2.$$

Later these interpolation conditions were considered for arbitrary Köthe spaces, sometimes with the same notation d_1 and d_2, although for non-regular Köthe spaces these conditions may be quite different from the original Dragilev's ones (see, e.g.,V. Zakharyuta [Z1970, Z1973], T. Terzioğlu [T1974], E. Dubinsky 1979 [Dub]). Dubinsky and Terzioğlu considered also several modifications of these classes. Finally these conditions were written (by Zakharyuta, Vogt and Wagner) in an invariant (basisless) interpolation form:

Definition 5 *A LCS X belongs to the class D_i if*

$$\exists U \forall V \exists W \exists C \ > \ 0 : (|x|_V)^2 \le C \ |x|_U \ \ |x|_W, \ \ \textit{if}\ i = 1,$$
$$\forall U \exists V \forall W \exists C \ > \ 0 : (|x'|_{V^\circ})^2 \le C \ |x'|_{U^\circ} \ \ |x'|_{W^\circ}, \ \ \textit{if}\ i = 2.$$

(Vogt's notation DN and $\overline{\Omega}$, respectively).
These and other interpolational classes, appeared by influence of the pioneering Dragilev's work of 1965, turned to be of great importance in many further investigations. It can be illustrated by the following result.

Theorem 6 *(Z. 1970,1973; Vogt 1982) Let X, Y be a pair of Fréchet spaces such that $X \in \mathcal{D}_2$ and $Y \in \mathcal{D}_1$. Then $L(X,Y) = LB(X,Y)$, where $LB(X,Y)$ is the space of all bounded linear operators from X to Y*

This result was important for an isomorphic classification on the class of all Cartesian products $E_0(a) \times E_\infty(b)$ (Z. 1970,1973; initial results were here also due to Dragilev [**D1969**]).

Another great application of those and related interpolation invariants lays in the structure theory (characterization of subspaces and quotient spaces) of power series spaces (D. Vogt, M. J. Wagner, A. Aytuna, J. Krone, T. Terzioğlu et al). More detailed information about this topic can be found in the survey of T. Terzioğlu [T2013], represented in this issue.

Classes L_f. Another important notion introduced in the same article [D1965] are classes of Köthe spaces determined by special Köthe matrices:

$$L_f(a,r) := \lambda(\exp f(r_p a_n)),$$

where the function $f : \mathbb{R} \to \mathbb{R}$ is continuous strictly increasing, odd and logarithmically convex on $(0,\infty)$, that is $\ln f(\exp x)$ is convex, $r_p \nearrow r \in (0,\infty]$, $a = (a_n)$, $a_n \nearrow \infty$.

This notion turned to be the most popular from Dragilev's ones. Many mathematicians worked under these spaces, including myself, E. Dubinsky. Those results are well reflected in the monographs of Dubinsky and Dragilev. So, I stop only here to talk about [D1965].

In the papers [**D1969, D1970**] Dragilev investigated the limits of dimension Γ in their ability to distinguish non-isomorphic spaces on the class $\mathfrak{E}_1$ of all Fréchet spaces with regular absolute basis. He showed that Γ is the strongest invariant on the its subclass $\mathfrak{E}_0 = d_1 \cup d_2 \cup (d_1 \times d_2)$. He introduced some stronger invariants distinguishing spaces from wider subclasses of $\mathfrak{E}_1$. It is worth to notice that in these papers Dragilev was the first who considered isomorphic classification of Cartesian products of "essentially different spaces" (later results [**Z1970, Z1973**], mentioned above, appeared under the great Dragilev's influence). I think the most exiting result in [**D1969**] is

the following, concerned with the class $\mathfrak{N}_1$ of ultranuclear Fréchet spaces (X is ultranuclear if $\exists s \forall p \exists q \forall r | \frac{d_n(U_q,U_p)}{d_{n-1}(U_r,U_s)} \to 0$ as $n \to \infty$):
Theorem *Every space $E \in \mathfrak{N}_1$ is nuclear and has a regular absolute basis.*

There are other brilliant results about the class $\mathfrak{N}_1$ and its modifications, published in [**DK**] (joint with Kondakov) and [**D1974**] (see also Dragilev's monograph 2003). For the sake of the simplicity I represent here only the simplest one.
Theorem Let $E \in \mathfrak{E}_1$. Then $E \in \mathfrak{N}_1$ if and only if every isomorphism $T : E \to E$ has a represetation $T = J(I - K)$, where J is a diagonal operator in some basis and K is compact.

Multiply regular bases ([D1970, D1976, AD, BD]). Absolute basis in a Schwartz Fréchet space X is *n-multiply regular* if it is repesentable as a disjoint union of n regular basis subsequences and the number n cannot be diminished, $n \in \mathbb{N} \cup \{\infty\}$. Notation $X \in \mathcal{R}^{(n)}$ means that X has a n-multiply regular basis. We cite here the most advanced result from [BD] (see also, [D2003]). It is worth to be noted that graph theory methods are crucial in its proof.

Theorem 7 *Every Schwartz Fréchet space X with an absolute basis belongs to one and only one class $\mathcal{R}^{(n)}$, and for each absolute basis in X there exists a permutation which makes it n-multiply regular.*

I think that this nice result still remains underestimated and might be an important step in attaking of the quasiequivalence problem.

Extendible (continuable) bases. In the beginning of his career, M. M. Dragilev was using mostly methods of Complex Analysis with some thin estimates, though the problems were formulated in terms of Functional Analysis. This relates also to his first papers mentioned above. There are several his papers ([D1961, D1962, D1963, D1997, D1999, D2000]) about common bases in a pairs of spaces of analytic functions, based on the application of the potential theory, in particular, three constant potentials. For illustration we represent here two nice results of this kind.

Theorem 8 *([D1961])Let $G_1 \Subset G$ be simply connected domains, φ is one-to-one analytic mapping of $G \setminus \overline{G_1}$ onto $\{1 < |z| < R\}$ and G_r be a domain confined by the curve $\Gamma_r = \{|\varphi(z)| = r\}$. Let $\{f_j(z)\}$ be a common basis in the spaces $A(G)$ and $A(G_1)$, then it is a basis in $A(G_r)$, $1 < r < R$, but cannot be a basis in any space $A(D)$ if $D \subset G$ has non-void intersection with $G \setminus \overline{G_1}$ and is distinct from any domain D_r.*

Theorem 9 *([D1962, D1999]) Let $G \subset D$ be simply connected domains and be a sequence of domains D_s such that all $\Gamma_s := \partial D_s$ are closed Jordan curves and $D_s \Subset D_{s+1},\ D = \bigcup\limits_{s=1}^{\infty} D_s$. If the harmonic measure of the set $\Gamma_s \cap (D \setminus G)$ relative to D_s and to some point in D_1 tends to 0 as $s \to \infty$, then the spaces $A(G)$ and $A(D)$ have no common basis.*

These results were developed and generalized in [Ng, ZK, Kad]. Dragilev investigated also extendible bases in a general context for pairs of Köthe spaces ([D1976, D1981, D1981a, D1990], see also [CDZ]).

Basisness and interpolation. Appeared in 1974 paper of three authors (Dragilev, Korobeinik and Zakharyuta) about a deep connection between basisness of a system of elements in a locally convex space and an a dual interpolation problem for values of linear functionals on this system was inspired by previous results of Dragilev (on interpolation problems for analytic functions, in particular, on the Abel-Goncharov problem [D1960a, DC]) and results of Korobeinik related to the theory of differential equations of infinite order. Later these results were applied and advanced in investigations of Korobeinik and his students (see, e.g., [1]). It is a pity that this "troyka" did not produce anything else together.

Dragilev's Doctor Sci. Thesis In 1973 Dragilev defended in Kharkov his Doctor Sci. Dissertation, which were laying in VAK (High Attesting Committee of USSR) several years without any decision. M. M. finally sent a letter to A. N. Kosygin (the premier of USSR in that time), in which he asked him, not about an approval of his dissertation, but at least about any definite decision. Fortunately the address has been chosen by M. M. correctly, and it was demanded from above to consider Dragilev's dissertation urgently. I remember that a draft of a positive report of an official VAK's opponent has been prepared by Mityagin and myself in Suhumi, on the Black See shore, where we met just for this purpose. Finally this distinguished dissertation has been confirmed by VAK. This story costed M. M. two hard attacks. Our teacher and supervisor Professor M. G. Haplanov, who worried hard about this story, unfortunately, died few month before its "happy end".

Dragilev as a theacher. Dragilev is an extremely responsible and accurate teacher. His courses are always prepared in good time, written by his specific tiny calligaraphic handwriting. It should be noted that he was the first in Rostov State University who prepared and realized a modern course of Probability and Mathematical Statistics: this course was tought before him in a quite old fashioned style, which became unconceivable after Dragilev's contribution. Although Mikhail Dragilev did not read any course to me, he was and is my true teacher, supervisor and, as a great honor for me, my close friend. Almost all my result has been discussed with him, his critics and remarks were very important for myself. Moreover, he thaught me not only mathematics, but he often shared his world wisdom, for example, he explained me thoroughly how much truth ("pravda" in Russian) was contained in the news paper "Pravda", an official organ of the Communist Party of USSR.

Conclusion. Ideas and methods suggested by Dragilev in 1950-1970th were applied, developed and generalized in works of himself and his students (O. P. Chuhlova, V. P. Kondakov, V. I. Baran, V. V. Kashirin, A. H. Oleynikov, E. O. Basangova, A. K. Rovickii, V. A. Grachev, T. I. Abanina, A. V.Vakulenko), as well as in the works of mathematicians in many countries (besides those mentioned above, P. Djakov, M. R. Ramanujan, N. De Grande-De Kimpe, A. Aytuna, J.Krone, T. Terzioğlu, M. J. Wagner, Ahonen, Lindström, K. Nyberg, M. Yurdakul, Z. Nurlu, M. Kocatepe (Alpseymen), P. A. Chalov, A. P. Goncharov, M. A. Shubarin, E. Karapınar et al) in various directions of Functional and Complex Analysis: linear topological invariants, quasiequivalence of bases, isomorphic classification of spaces of analytic functions, pairs of LCS such that $L(X, Y) = LB(X, Y)$; Cartesian and tensor products of "essentially different" spaces, multiply regular bases, quasinormable and asymptotically normable spaces and so on. All mentioned above mathematicians experienced an influence of Dragilev's ideas either directly (like B. S. Mityagin, E. Dubinsky, or myself, for instance), or induced by others.

Many of Dragilev's results have been reflected in monographs of S. Rolewicz [R], E. Dubinsky [Dub], R. Meise and D. Vogt [MV], A. I. Markushevich [Mar] and in surveys (B. Mityagin [M1961], V. Zakharyuta [Z1994]).

In 1983 Dragilev published the monograph "Bases in Köthe spaces" (the second, seriously elaborated edition was issued in 2003). Both editions (especially, the second one) contain also some important results never published before. The second edition is not just a reedition, but it is a new book with a new concept and a new vision of the whole.

Congratulations on Celebrating Anniversary! Best wishes, dear Mikhail Mikhaylovich!

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