ERATOSTHENES, HIPPARCHUS, AND THE OBLIQUITY OF THE ECLIPTIC

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It is well known that the precise value for the obliquity of the ecliptic that Ptolemy adopts in *Almagest* 1.14, namely 23°51′20″, is simply the sexagesimal representation in degrees, rounded to the second fractional place, of the value following from his statement that the ratio of the meridian arc between the tropic circles to the entire meridian circle is 11 : 83. In *Almagest* 1.12 Ptolemy had claimed to have established from observed meridian altitudes of the Sun at the equinoxes that the arc between the tropics is greater than $47\frac{2}{3}$° but less than $47\frac{3}{4}$°, adding that "from this results pretty well the same ratio as that of Eratosthenes, which Hipparchus also used; for the arc between the tropics comes to 11 of such units if the meridian is 83".[1]

Ptolemy's remark gives rise to a question of interpretation as well as a question, or conditionally two questions, of historical fact. First, does Ptolemy mean that Eratosthenes and Hipparchus assumed that the ratio of the arc between the tropics to the whole circle was 11 : 83, or merely that an arc between $47\frac{2}{3}$° and $47\frac{3}{4}$°, such as the arc resulting from the ratio 11 : 83, is close to some unstated value accepted by Eratosthenes and Hipparchus? Secondly, how was the ratio 11 : 83, which is obviously not expressed in conventional units of arc, obtained? And thirdly, if Ptolemy is indeed attributing the ratio 11 : 83 to his predecessors, is the attribution true? So many varied opinions have been expressed on all these questions that I would be reluctant to add one more voice to the hubbub.[2] But in fact clear evidence for Eratosthenes's value for the obliquity has been lying around, so far as I am aware unnoticed, and the evidence also reveals straightforwardly where this value came from.

We begin with another parameter that the discussion in *Almagest* 1.12 might suggest that Ptolemy measured simultaneously with the obliquity of the ecliptic: the latitude of his place of observation, Alexandria. In passing Ptolemy remarks that the latitude is obtained as a by-product of the observations that give the obliquity, although he only states his value for Alexandria's latitude, 30°58′, much later, in 5.12. It seems doubtful whether Ptolemy could have claimed a precision of one or two minutes for observed meridian altitudes, but this number in any case was originally obtained in a different way: though he does not say so, it is, to the nearest minute, the latitude for which the equinoctial noon shadow of a vertical gnomon has a ratio to the gnomon of exactly 3 : 5.[3] The calculation of latitude from equinoctial shadow ratio would have been feasible with the numerical methods available in the third century B.C. (*cf.* similar "pre-trigonometrical" calculations in Aristarchus's *On sizes and distances* and Archimedes's *Measurement of the circle*); the particular

0021-8286/02/3301-0015/$1.00

shadow ratio 3 : 5 for Alexandria is attested for Hipparchus (Strabo 2.5.38), and could certainly have been known earlier than Hipparchus's time.

Let us suppose (1) that an early geographer, say Eratosthenes himself, had taken the rounded latitude 31° derived from the equinoctial shadow,[4] and combined this with part of the information associated with Eratosthenes's famous determination of the circumference of the Earth: (2) that Syene (modern Aswan) is 5000 stades south of Alexandria and on the same meridian; (3) that at Syene the Sun crosses the meridian straight overhead on the summer solstice; and (4) that one degree of a meridian corresponds to 700 stades.[5] From these assumptions it follows that the latitude of Syene, or equivalently the obliquity of the ecliptic, is

$$31° - {}^{5000}\!/_{700}° \approx 23°51'26'',$$

which is extremely close to Ptolemy's value, and well within the narrow range of values for the ratio of which to a semicircle 11 : 83 can be considered a 'best' approximation in small whole numbers.[6]

Now consider the following scheme of latitudinal intervals, extending from the southern to the northern limit of the "inhabited world", which Strabo (1.4.2) ascribes to Eratosthenes:

Taprobane to Meroe	3400 stades
Meroe to Alexandria	10000 stades
Alexandria to Hellespont	8100 stades
Hellespont to Borysthenes	5000 stades
Borysthenes to Thule	11500 stades

Syene is absent from Strabo's list, but we already know that Eratosthenes situated it 5000 stades south of Alexandria, that is, exactly half-way between Meroe and Alexandria. We know that Eratosthenes accepted that Syene lies exactly on the summer tropic circle, and he also almost certainly accepted the report of Pytheas of Massilia (late fourth century B.C.) that Thule lies exactly on the arctic circle (Strabo 2.5.8). Hence taking 63000 stades (i.e., 90° times 700 stades/degree) for the distance from the equator to the north pole, we can assign to the interval from the equator to Syene exactly half the difference between 63000 stades and the total interval 29600 stades from Syene to Thule; the interval between the equator and the summer tropic comes to 16700 stades. Alexandria is 5000 stades further north, or 21700 stades from the equator, which is precisely equivalent to 31°. Thus the Eratosthenic intervals reported by Strabo confirm that Eratosthenes set Alexandria at the latitude implied by the 3 : 5 shadow ratio, as well as that he derived from this latitude a value for the obliquity very close to Ptolemy's. Eratosthenes may himself have expressed this parameter in the form of the 11 : 83 ratio cited by Ptolemy, though we cannot be sure of that.

In 2.5.34–42 (supplemented by 2.1.18) Strabo reports from Hipparchus's geographical treatise the latitudinal intervals in stades between a list of parallels, most of which are associated not only with particular localities but also with the duration in hours of the longest day. From the intervals it is possible to reconstruct stade distances from the equator, and hence (counting with Hipparchus 700 stades to the degree) latitudes in degrees for each parallel. The agreement with latitudes trigonometrically recomputed from the stated lengths of longest day is good enough to show that Hipparchus was able to make at least a reasonably good approximation to this computation.[7] In independent publications, A. Diller and D. Rawlins have derived a value for the obliquity, 23°40′, that yields a close fit to Strabo's stade figures (which are expressed in round hundreds of stades, thus to a precision of $\frac{1}{7}$°).[8] Unfortunately, there are some inconsistencies in the numbers reported by Strabo, and one may well suspect that one or two modest changes in the intervals, through either scribal error or deliberate tampering, could have introduced systematic errors which would affect the value of the obliquity best fitting the data.[9]

For three latitudes Strabo also cites shadow ratios from Hipparchus. Two of these are ratios of small whole numbers, presumably empirical in origin.[10] The third (2.5.41) is a shadow-to-gnomon ratio at the summer solstice of $41\frac{4}{5}$: 120 pertaining to the parallel where the longest day is $15\frac{1}{4}$ hours, which Hipparchus believed passed through both Byzantium (Istanbul) and Massilia (Marseilles). The extreme precision of this parameter shows it to be the result of calculation.[11] The correct determination of the ratio s is:

$$s = \tan(\varphi - \varepsilon)$$

where ε is the obliquity and φ the latitude, computed from the length in hours M of longest day as follows:

$$\tan \varphi = \frac{-\cos(15M/2)}{\tan \varepsilon}.$$

For $M = 15\frac{1}{4}$ hours we find for selected values of ε:

ε	φ	φ (stades)	s
23°40′	43°17′	30295	42.7592 : 120
23°51′20°	43°1′	30116	41.713 : 120
24°	42°50′	29980	40.9194 : 120

It is evident that s is sensitive to changes in ε, and that ε must have been within a couple of minutes of 23°51′20″ to yield approximately the ratio (41.8 : 120) reported by Strabo. On the other hand Strabo gives the distance of the parallel in question from the equator as 30300 stades, in agreement with Diller's and Rawlins's ε but not with Hipparchus's s. I believe we have to regard the shadow ratio as the more trustworthy datum; moreover, the closeness of the agreement between text and recomputation for ε = 23°51′20″ not only backs up Ptolemy's statement that Hipparchus used the same obliquity as Eratosthenes, but also that Hipparchus was in command of the correct trigonometrical relation between ε, φ, and M. Tiny errors in his calculation of s might result from imprecisions in Hipparchus's trigonometrical resources (i.e., his chord table, one supposes); Ptolemy calculated s more accurately as $41\frac{2}{3}$: 120 from the same parameters (*Almagest* 2.6).

To sum up, Strabo provides us with good evidence that in their geographical work both Eratosthenes and Hipparchus employed an obliquity very close to Ptolemy's 23°51′20″, verifying Ptolemy's assertion even if not entirely settling the question of whether Ptolemy meant to say that Eratosthenes and Hipparchus expressed this parameter by the ratio 11 : 83. The origin of the parameter turns out to be, not precise measurement, but crude round numbers associated with Eratosthenes's geodesy; and the wonder is not that it is more than ten minutes too great but that it is even this close to the truth. Both Eratosthenes and Hipparchus also sometimes set the obliquity at 24° (*cf.* Strabo 2.5.7 and Hipparchus, *Commentary on Aratus* 1.10), but apparently only in contexts where round figures, say in whole degrees, were appropriate.

REFERENCES

1. On Ptolemy's observations, see John P. Britton, *Models and precision: The quality of Ptolemy's observations and parameters* (New York, 1992), 1–11. The accurate value of the obliquity for Ptolemy's time was significantly lower, approximately 23°40′46″.
2. A selection from the more recent scholarship would include Dennis Rawlins, "Eratosthenes' geodesy unraveled: Was there a high-accuracy Hellenistic astronomy?", *Isis*, lxxiii (1982), 259–65; David H. Fowler, "Eratosthenes' ratio for the obliquity of the ecliptic", *Isis*, lxxiv (1983), 556–62 (with a reply by D. Rawlins); Bernard R. Goldstein, "The obliquity of the ecliptic in ancient Greek astronomy", *Archives internationales d'histoire des sciences*, xxxiii (1983), 3–14; C. M. Taisbak, "Eleven eighty-thirds: Ptolemy's reference to Eratosthenes in *Almagest* I.12", *Centaurus*, xxvii (1984), 165–7; and G. J. Toomer, *Ptolemy's Almagest* (London, 1984), 63 n. 75.
3. This is pointed out, for example, by O. Neugebauer, *A history of ancient mathematical astronomy* (3 vols, Berlin, 1975), i, 101 n. 1. Alexandria's correct latitude is 31°13′; Ptolemy should have been able to detect a discrepancy of a quarter degree.
4. Eratosthenes is not known to have divided the circle into 360°; Strabo (2.5.7) attributes to him a division into sixty units, which would of course translate easily into degrees.
5. The clearest source for (2) and (3) is Cleomedes 1.7; for (4), which involves the rounding up of Eratosthenes's estimate of the earth's circumference as 250000 stades to the nearest multiple of a thousand divisible by sixty, see Strabo 2.5.7.
6. For numbers within about half a minute of (23°51′20″/180) either way, approximation by continued fractions (or mathematically equivalent methods) yields $\frac{11}{83}$ after four steps. While the specific mathematical tools available in Antiquity for obtaining close fractional representations of given quantities are a matter for conjecture, there is no question that such tools did exist.
7. Neugebauer, *op. cit.* (ref. 3), ii, 304–6, attempted to explain the numbers as generated by an arithmetical sequence; but he failed to show how Hipparchus could have found a sequence matching so accurately the theoretically correct latitudes.
8. A. Diller, "Geographical latitudes in Eratosthenes, Hipparchus and Posidonius", *Klio*, xxvii (1934), 258–69; Dennis Rawlins, "An investigation of the ancient star catalog", *Publications of the Astronomical Society of the Pacific*, xciv (1982), 359–73, p. 368.
9. According to Strabo (2.5.35), Hipparchus stated that on his southernmost parallel, through the "Cinnamon-bearing country", the star α UMi grazes the horizon, and according to Ptolemy, *Geography* 1.7, Hipparchus placed α UMi $12\frac{2}{5}$° from the celestial pole. Hence Hipparchus should have situated this parallel 8680 stades from the equator, not 8800 as Strabo has it. In 2.5.36 Strabo reports Hipparchus as stating that Syene was both on the tropic circle and on the circle for which the longest day is $13\frac{1}{2}$ hours; both cannot be true if Syene is 16800 stades

(exactly 24°) north of the equator, whereas both could be approximately true if Syene was about 100 stades further south. In 2.5.38 the shadow ratio 3 : 5 is given for Alexandria, hence its latitude should be almost exactly 31°, or 21700 stades north of the equator; but Strabo's intervals yield 21800. Has someone added 100 stades to all Hipparchus's distances in order to put Syene at latitude 24°? Among other inconsistencies are two seriously faulty stade intervals near the northern end of the list, for which I believe the explanation of Neugebauer (*op. cit.* (ref. 3), ii, 305) is correct.

10. From 2.5.38: Alexandria, equinoctial shadow ratio 3 : 5 (an editorial emendation of 7 : 5 in the manuscripts); Carthage, 7 : 11. In 2.5.39 Strabo situates Syracuse 400 stades north of the parallel through Rhodes, which would make it 25600 stades, or 36°34′, north of the equator; this is certainly derived from an equinoctial shadow ratio of 3 : 4, though Strabo does not mention this.

11. Other attested shadow ratios for specific localities, e.g. those in Vitruvius 9.7 and Pliny 2.182–3, are always ratios of whole numbers (and all equinoctial). The only list of ratios (equinoctial and solstitial) that I know of that resemble this Hipparchian $41\frac{4}{5}$: 120 in refinement is Ptolemy's in *Almagest* 2.6, which is of course calculated. (So, at least in part, is the cruder list in Pliny 6.212–18.) It is true that Strabo (1.4.4, *cf.* 2.1.12 and 2.5.8) reports that Pytheas measured some shadow ratio at Massilia and that Hipparchus measured the same shadow ratio "at the same season" at Byzantium, but it is hardly to be believed that Pytheas's late fourth-century B.C. measurement was to a precision of $\frac{1}{500}$ of the gnomon's length. Nevertheless there is a long scholarly tradition of taking $41\frac{4}{5}$: 120 as an observation by Pytheas: see D. R. Dicks, *The geographical fragments of Hipparchus* (London, 1960), 178–9, and Árpád Szabó, "Strabon und Pytheas — die geographische Breite von Marseille. Zur Frühgeschichte der mathematischen Geographie", *Historia scientiarum*, xxix (1985), 3–15. Goldstein (*op. cit.* (ref. 2), 10–12) recognizes that the ratio must be the product of calculation, and infers as I do that Hipparchus must have had a value for the obliquity close to Ptolemy's.