# Illumination and Temperature on Rough Terrain: Fast Methods for Solving the Radiosity Equation

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## Thermal modeling on rough surfaces

- Airless bodies have strong horizontal temperature gradients due to shadows cast by rough topography
- Lunar cold traps exist because of terrain shadowing and are defined by surface temperature
- Thermal models must incorporate shadows, but also long- and short-wavelength radiation between surface elements [3]
- **Element-to-element radiation dominates runtime**

## Contribution

- We develop a fast algorithm for solving the equations governing scattering of long- and short-wavelength radiation
- Solve two discretized radiosity equations to compute temp.
- Offline, we precompute a compressed low-rank version of the discretized kernel matrix by compressing low-rank off-diagonal blocks using sparse SVDs
- **Online, multiplication requires nearly $O(N)$ time**, where $N$ is the number of triangular elements
- The matrix only depends on the geometry of the planet so can be used for simulations spanning a long time

## Physical model [5]

- **Energy balance on the surface**:

$$\epsilon\sigma T^4 = (1-\rho)(E+Q) + \epsilon Q_{\mathrm{IR}} \tag{1}$$

where:

$\rho :=$ albedo
$E :=$ incoming solar radiation (insolation) $[\mathrm{W\,m^{-2}}]$
$Q :=$ reflected sunlight $[\mathrm{W\,m^{-2}}]$
$Q_{\mathrm{IR}} :=$ thermal emission $[\mathrm{W\,m^{-2}}]$
$T :=$ temperature [K]
$\epsilon :=$ emissivity
$\sigma :=$ Stefan-Boltzmann constant $[\mathrm{W\,m^{-2}\,K^{-4}}]$

- **Governing equations for scattering**:

$$Q(\vec{x}) = \frac{1}{\pi}\int_S \rho(\vec{y})\left(E(\vec{y}) + Q(\vec{y})\right) F(\vec{x},\vec{y})dA(\vec{y}) \tag{2}$$

$$Q_{\mathrm{IR}}(\vec{x}) = \frac{1}{\pi}\int_S \left(\epsilon T(\vec{y})^4 + (1-\epsilon)Q_{\mathrm{IR}}(\vec{y})\right) F(\vec{x},\vec{y})dA(\vec{y}) \tag{3}$$

where:

$S :=$ surface of the planet or crater
$dA :=$ surface area element $[\mathrm{m}^2]$
$$F(\vec{x},\vec{y}) := \frac{\left[\vec{n}(\vec{x})\cdot(\vec{y}-\vec{x})\right]_+\left[\vec{n}(\vec{y})\cdot(\vec{x}-\vec{y})\right]_+}{\pi\|\vec{x}-\vec{y}\|^4}V(\vec{x},\vec{y})$$
$\vec{n} :=$ surface normal on $S$
$[x]_+ := \max(0,x) =$ positive part
$$V(\vec{x},\vec{y}) := \begin{cases} 1 & \text{if } \vec{y} \text{ is visible from } \vec{x} \\ 0 & \text{otherwise} \end{cases}$$

## The radiosity method

- **Equations (2) and (3) are *radiosity integral equations***:

$$B(\vec{x}) = E(\vec{x}) + \rho(\vec{x})\int_S G(\vec{x},\vec{y})B(\vec{y})dA(\vec{y}) \tag{4}$$

where:

$B :=$ radiosity $[\mathrm{W\,m^{-2}}]$
$E :=$ self-emitted radiosity $[\mathrm{W\,m^{-2}}]$
$\rho :=$ albedo
$F :=$ geometric kernel

- **Midpoint collocation discretization of (4) gives the system:**

$$\boldsymbol{K}\boldsymbol{B} = (\boldsymbol{I} - \rho\boldsymbol{F})\,\boldsymbol{B} = \boldsymbol{E} \tag{5}$$

$$\boldsymbol{F}_{ij} = \frac{\left[\boldsymbol{n}_i^\top(\boldsymbol{p}_j-\boldsymbol{p}_i)\right]_+\left[\boldsymbol{n}_j^\top(\boldsymbol{p}_i-\boldsymbol{p}_j)\right]_+}{\pi\|\boldsymbol{p}_i-\boldsymbol{p}_j\|^4}V_{ij}A_j \tag{6}$$

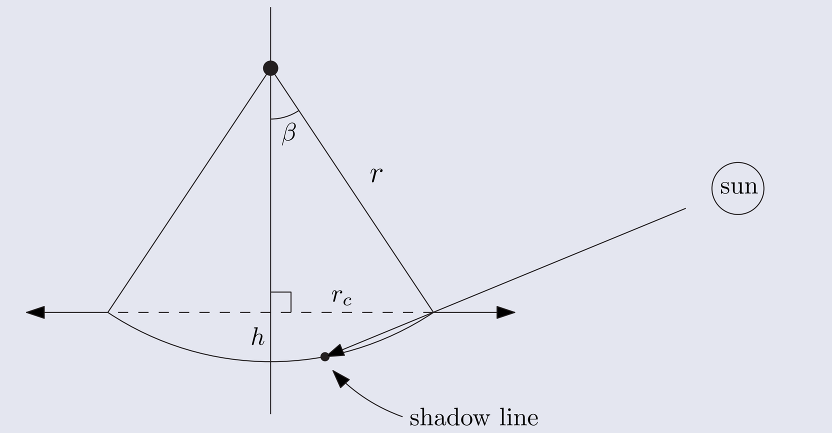
where $\boldsymbol{F}$ is the **view factor matrix** and:

$\boldsymbol{p}_i :=$ centroid of $i^{\mathrm{th}}$ triangle
$\boldsymbol{n}_i :=$ surface normal at $\boldsymbol{p}_i$
$A_i :=$ area of $i$th triangle
$V_{ij} :=$ visibility between $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$

- **Solving the discrete radiosity system:**
  - The system (5) can be solved in a small number of Neumann or Jacobi iterations (typically 2 to 5)
  - Nearly perfectly conditioned since it's a discretized BIE
  - **Main challenge: fast multiplication by $\boldsymbol{F}$**

## The Ingersoll crater test problem [2]

- **A crater formed from a spherical cap:**




- **The steady state temperature for the Ingersoll crater has an exact solution for Lambertian reflectance *inside the shadowed portion of the crater***:

$$T_{\mathrm{Ingersoll}} = \frac{F_0\sin(e_0)}{\sigma}\frac{1-A}{1-Af}\left(1 + \frac{A(1-f)}{\epsilon}\right)^{1/4} \tag{7}$$

where $\sigma$, $A$, and $\epsilon$ are as before, and:

$f := S_c/(4\pi r^2)$
$S_c :=$ crater surface area $[\mathrm{m}^2]$
$F_0 :=$ solar constant $[\mathrm{W\,m^{-2}}]$
$e_0 :=$ solar elevation relative to horizon

- **We use this test problem to validate our numerical method**

## Low rank compression of form factor matrix $\boldsymbol{F}$

- **Spatial partitioning**: use quadtree or octree to recursively partition triangular elements
- **Low-rank interactions**: blocks of $F$ that correspond to interactions between nonoverlapping cells in quadtree or octree are typically sparse with a dense low-rank subblock
- **SVD compression**: compute SVD to find dense subblock and compress it within a given tolerance $\epsilon$
- **Best low-rank approximation by SVD**:

$$\min_{\mathrm{rank}(\boldsymbol{F})\leq k}\|\boldsymbol{F}-\boldsymbol{F}_k\|_2 = \sigma_{k+1} \tag{8}$$

where $k$ is a fixed rank, $\boldsymbol{F}_k = \boldsymbol{U}_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^\top$ is computed from the rank $k$ truncated SVD of $\boldsymbol{F}$, and $\sigma_i$ is the $i$th singular value of $\boldsymbol{F}$

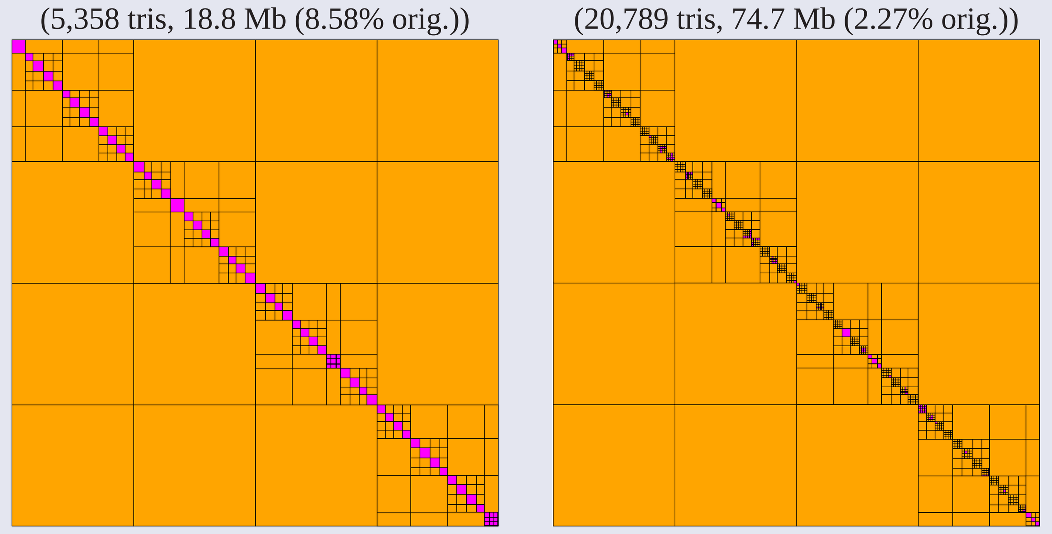
- This approach is similar to the $\mathcal{H}$-matrix format [1]

## Examples of compressed $\boldsymbol{F}$ matrices

■,■*: SVD block, ■: sparse block, ■: dense block, ■: zero block
(*: the cyan SVD blocks can be thought of as "especially sparse" SVD blocks)

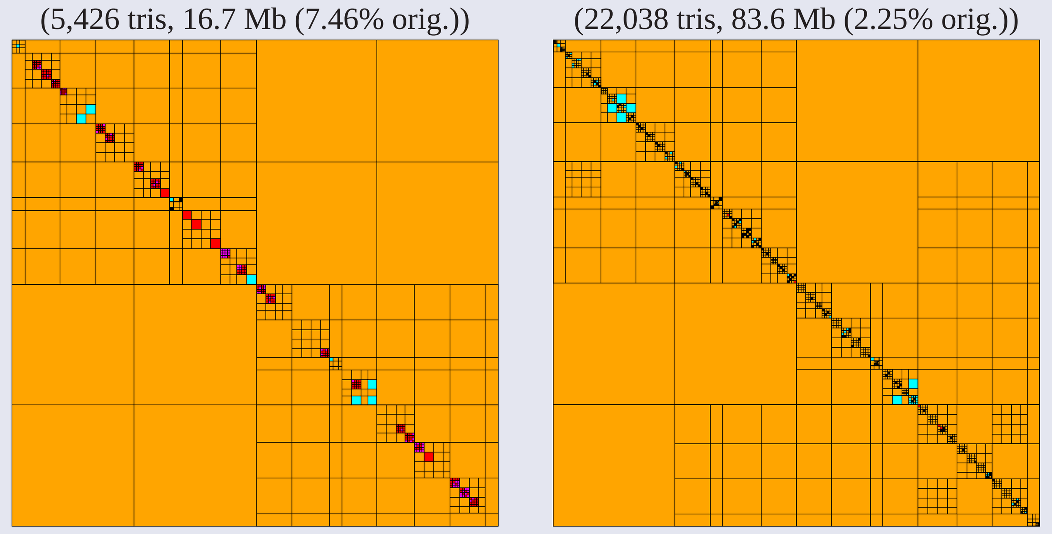
**Ingersoll crater**

(5,358 tris, 18.8 Mb (8.58% orig.)) (20,789 tris, 74.7 Mb (2.27% orig.))



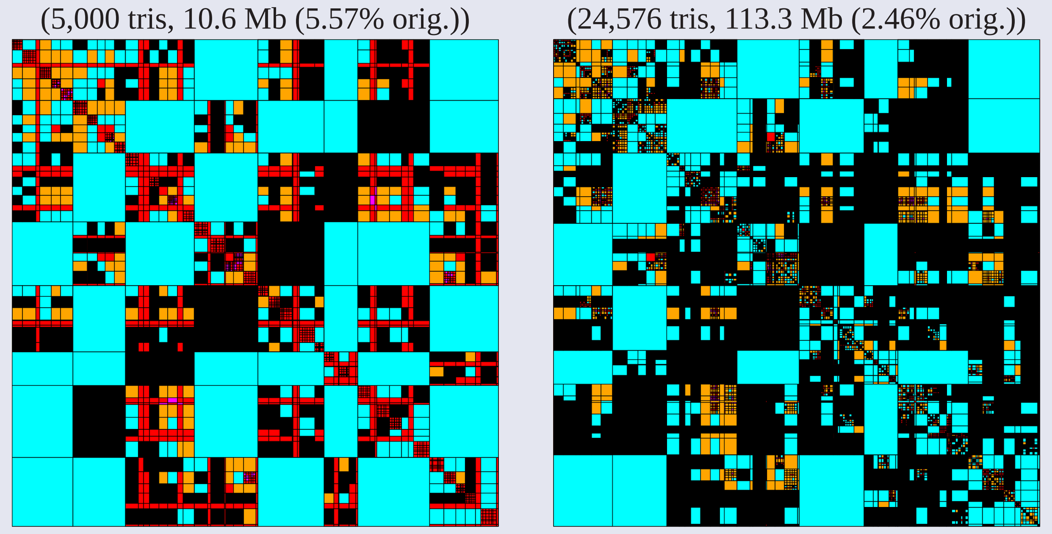
**Haworth crater (Lunar south pole)**

(5,426 tris, 16.7 Mb (7.46% orig.)) (22,038 tris, 83.6 Mb (2.25% orig.))



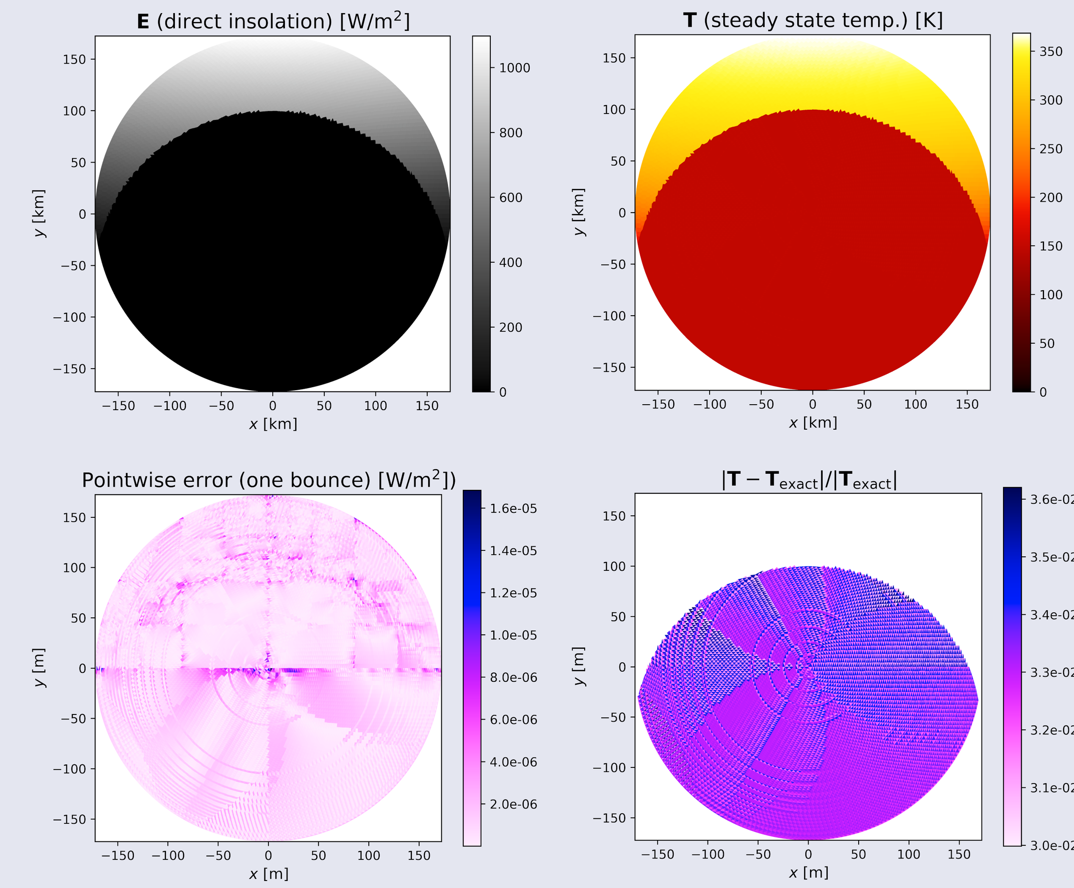
**67P/Churyumov-Gerasimenko (built using octree)**

(5,000 tris, 10.6 Mb (5.57% orig.)) (24,576 tris, 113.3 Mb (2.46% orig.))



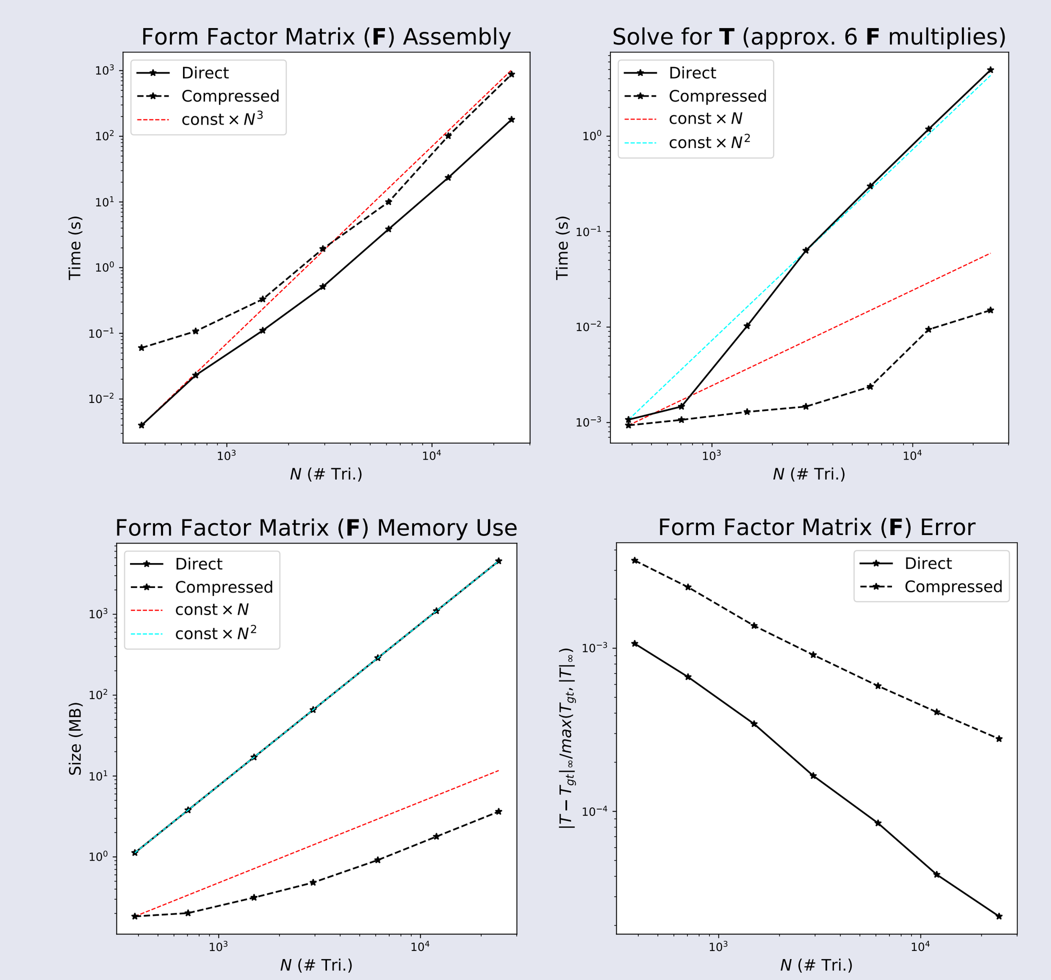
## Ingersoll test problem: numerical results

- Only part of the crater is illuminated (*top-left*)
- The steady state temperature in the shadow is constant (*top-right*)
- Pointwise error after one bounce is below the $\epsilon = 10^{-4}$ tolerance (*bottom-left*)
- Error between $T$ and exact Ingersoll temperature is uniform (*bottom-right*)
- Errors are sensitive to the triangulation (*bottom-left/right*)



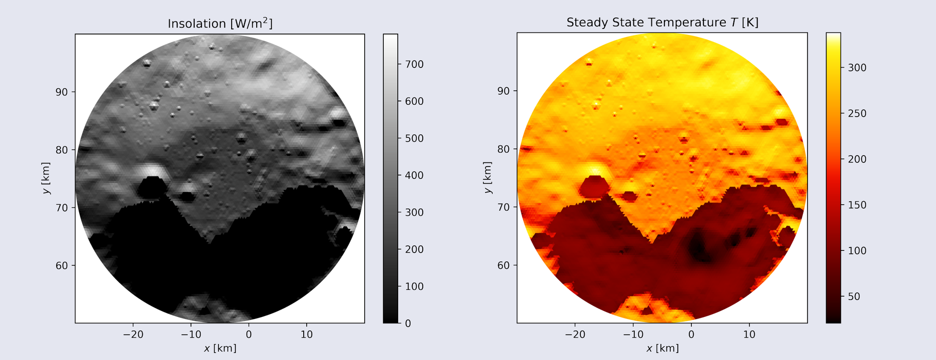

## Ingersoll test problem: **performance results**

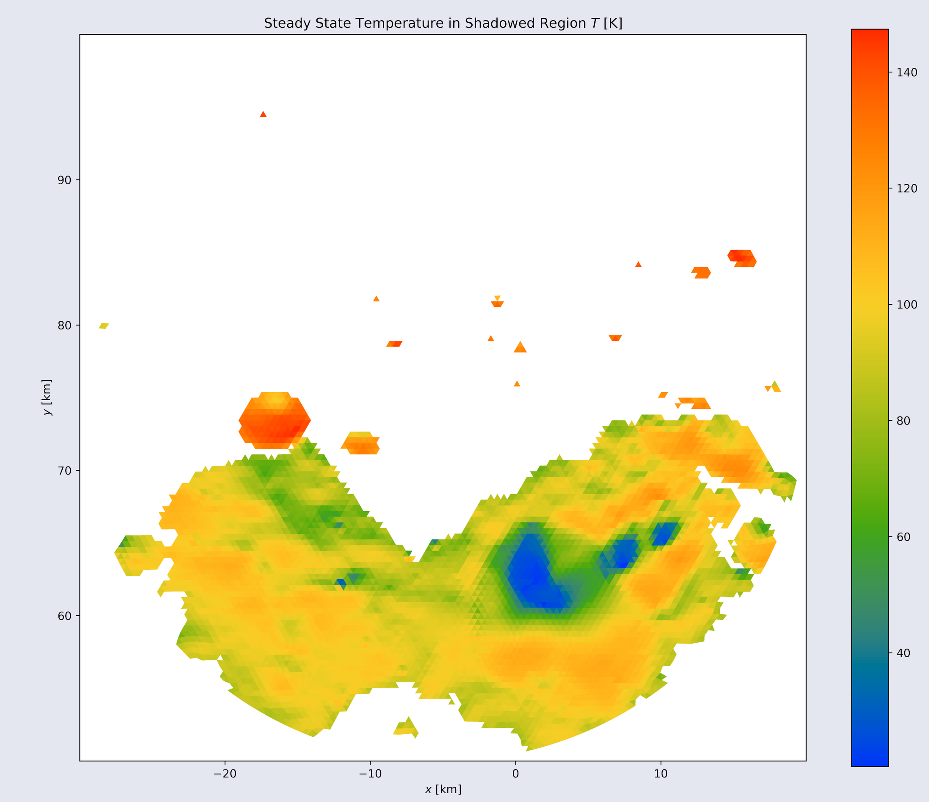
- We build the compressed $F$ matrix with a tolerance of $\epsilon = 10^{-4}$ and a maxmium SVD rank of $k = 60$
- We compare the exact steady state temperature in the shadow region given by (7) with the numerical steady temperature vs. problem size (*bottom-right*)




## Lunar south pole: Haworth crater test problem

- Lunar south pole terrain from DEM obtained by LOLA [6]
- High-quality triangle mesh constructed using Python version of `distmesh` [4]
- **Example here is physically unrealistic because of steady state assumption, but solving the steady state system is the necessary step for solving the time-dependent problem efficiently**
- The temperature is computed using the compressed form factor matrix
- The temperature plotted below approximates the temperature using the exact, dense form factor matrix *pointwise*, **validating the fast method**




## References

[1] Wolfgang Hackbusch. *Hierarchical matrices: algorithms and analysis*, volume 49. Springer, 2015.

[2] Andrew P Ingersoll, Tomas Svitek, and Bruce C Murray. Stability of polar frosts in spherical bowl-shaped craters on the Moon, Mercury, and Mars. *Icarus*, 100(1):40–47, 1992.

[3] David A Paige, Matthew A Siegler, Jo Ann Zhang, Paul O Hayne, Emily J Foote, Kristen A Bennett, Ashwin R Vasavada, Benjamin T Greenhagen, John T Schofield, Daniel J McCleese, et al. Diviner lunar radiometer observations of cold traps in the moon's south polar region. *science*, 330(6003):479–482, 2010.

[4] Per-Olof Persson and Gilbert Strang. A simple mesh generator in MATLAB. *SIAM review*, 46(2):329–345, 2004.

[5] Norbert Schörghofer. Planetary Code Collection. https://github.com/nschorgh/Planetary-Code-Collection, 2018.

[6] David E Smith, Maria T Zuber, Gregory A Neumann, Erwan Mazarico, Frank G Lemoine, James W Head III, Paul G Lucey, Oded Aharonson, Mark S Robinson, Xiaoli Sun, et al. Summary of the results from the lunar orbiter laser altimeter after seven years in lunar orbit. *Icarus*, 283:70–91, 2017.