THEODORSEN'S AND GARRICK'S COMPUTATIONAL AEROELASTICITY, REVISITED

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International Forum on Aeroelasticity and Structural Dynamics
Savannah, Georgia
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"Results of Theodorsen and Garrick Revisited"

by Thomas A. Zeiler

Journal of Aircraft

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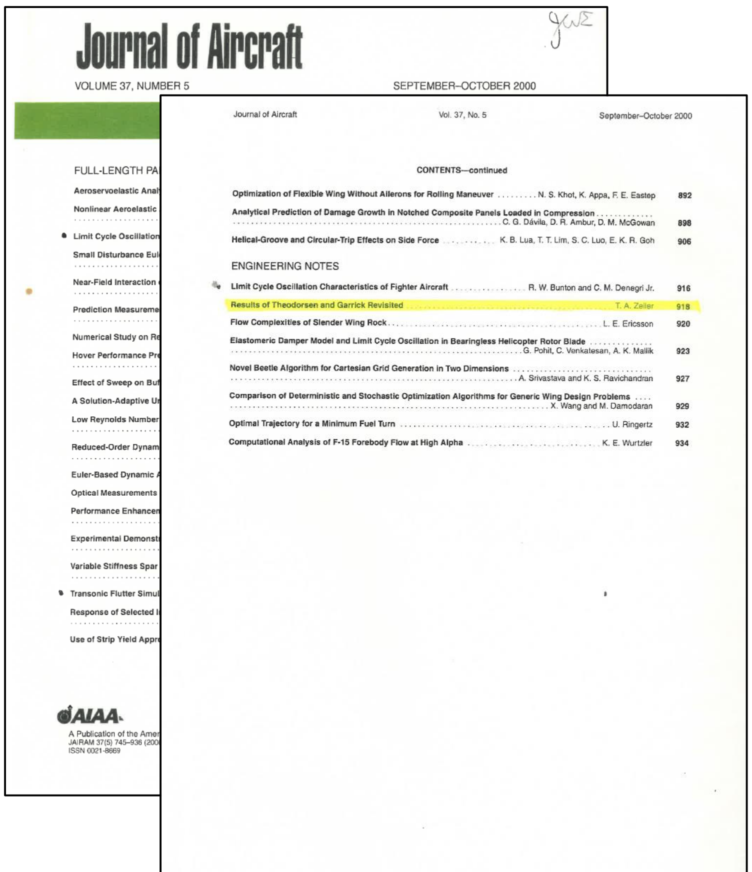
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Made known that –

- Some plots in the foundational trilogy of NACA reports on aeroelastic flutter by Theodore Theodorsen and I. E. Garrick are in error
- Some of these erroneous plots appear in classic texts on aeroelasticity

Recommended that –

- All of the plots in the foundational trilogy be recomputed and published

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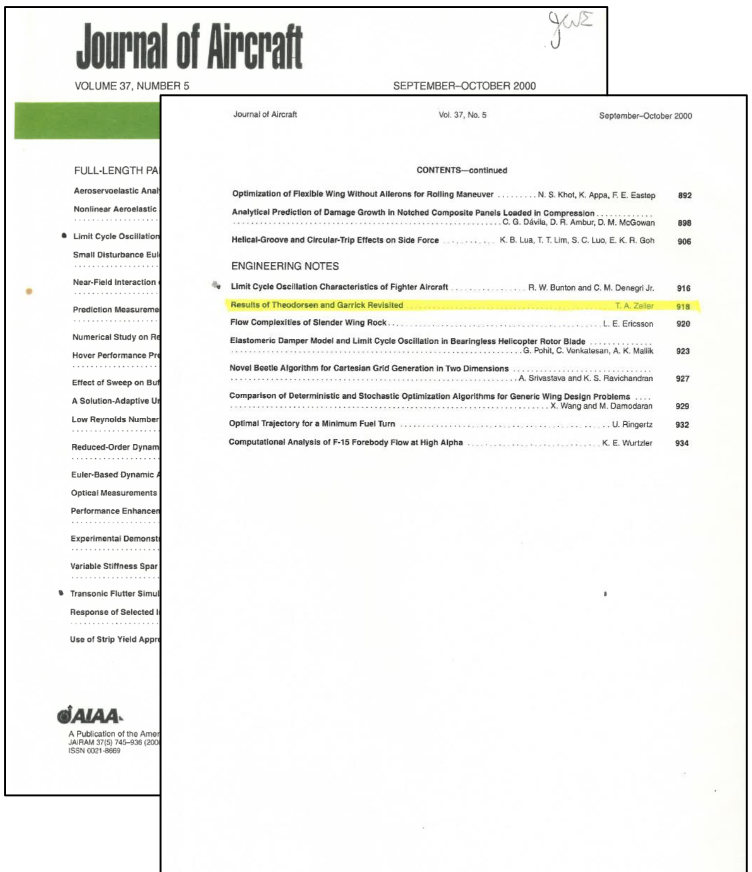
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Cautioned that –

- “One does not set about lightly to correct the masters.”

Works Containing Erroneous Plots

1. Theodorsen, T.: *General Theory of Aerodynamic Instability and the Mechanism of Flutter*. NACA Report No. 496, 1934.

2. Theodorsen, T. and Garrick, I. E.: *Mechanism of Flutter, a Theoretical and Experimental Investigation of the Flutter Problem*. NACA Report No. 685, 1940.

3. Theodorsen, T. and Garrick, I. E.: *Flutter Calculations in Three Degrees of Freedom*. NACA Report No. 741, 1942.

4. Bisplinghoff, R. L., Ashley, H., and Halfman, R. L.: *Aeroelasticity*, Addison-Wesley-Longman, Reading, MA, 1955, pp. 539-543.

5. Bisplinghoff, R. L. and Ashley, H.: *Principles of Aeroelasticity*, Dover, New York, 1975, pp. 247-249.

Purpose of This Presentation

- Make known a multi-year effort to re-compute all of the example problems in the foundational trilogy of NACA reports
  - Re-computations performed using the solution method specific to each NACA report
  - Re-computations checked and re-checked using modern flutter solution methods
    ("One does not set about lightly to correct the masters.")
  - NASA TP has been / will be published for each report in the trilogy
- Present outlines of Theodorsen's and Garrick's –
  - Equations of motion
  - Solution methods
- Present representative re-computations and comparisons with the originals

Publications of Re-Computed Results

NASA/TP–2015-218765

Re-Computation of Numerical Results
Contained in NACA Report No. 496

Boyd Perry, III
Langley Research Center, Hampton, Virginia

June 2015

NASA/TP-2019-xxxxxx

Re-Computation of Numerical Results
Contained in NACA Report No **685**

Boyd Perry, III
Langley Research Center, Hampton, Virginia

Summer or Fall 2019

Available on NASA Technical Report Server

https://ntrs.nasa.gov/

Outline

- Background and Purpose
- Brief History
- Equations of Motion
- Solution Methods
- Re-Computations and Comparisons
- Concluding Remarks

1930’s and 40’s NACA Computing Environment

1930's and 40's NACA Computing Environment

- "Computers"

1930's and 40's NACA Computing Environment

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  Employees whose job function was to perform computations

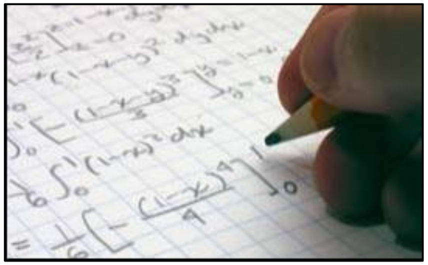
1930’s and 40’s NACA Computing Environment

- “Computers”
  Employees whose job function was to perform computations
- Tools

1930's and 40's NACA Computing Environment

- "Computers"
  Employees whose job function was to perform computations

- Tools
  - Pencil and paper



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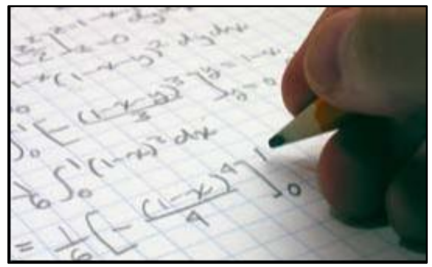
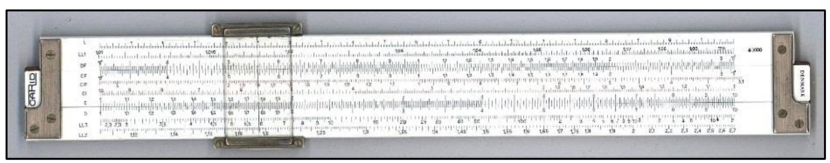


  - Slide rules



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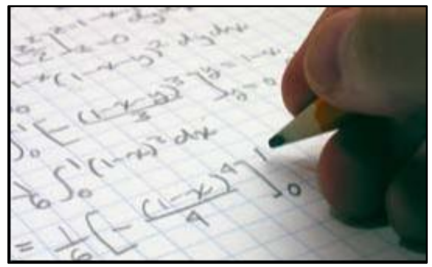

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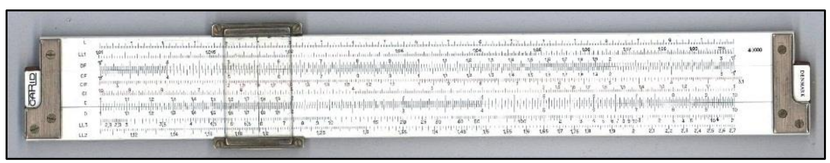
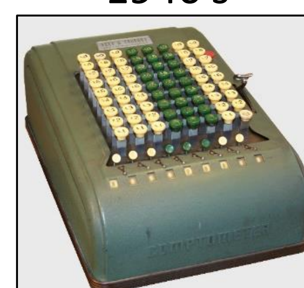


  - Mechanical calculators
    (comptometers – patented 1887)

1930's

1940's



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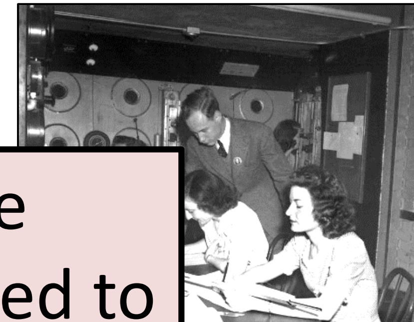
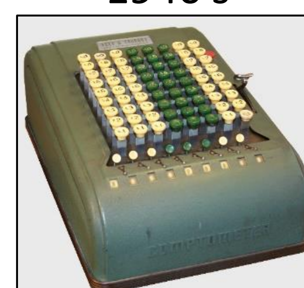


Strong motivation to minimize human time and effort required to solve equations:

- Recast equations to eliminate -
  - solution steps
  - complex arithmetic

1930's

1940's



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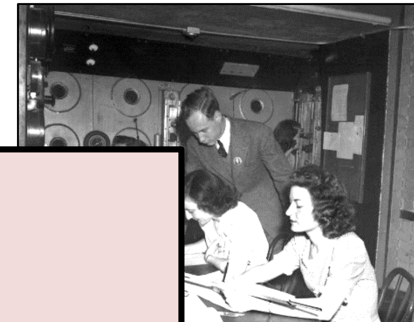
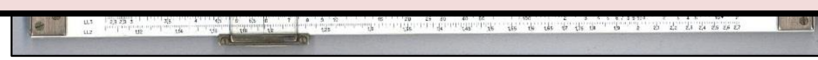
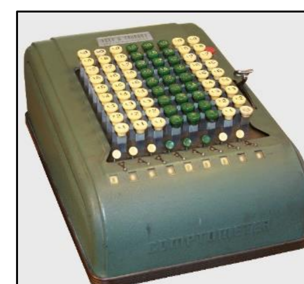


Unfortunately –
Human computers …
… are prone to error



1930's

1940's



Outline

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- Re-Computations and Comparisons
- Concluding Remarks

Assumptions Made in *NACA 496*

- Flow is potential, unsteady, incompressible
- "Wing" is two-dimensional typical section
- Three degrees of freedom
  - Torsion – α
  - Aileron deflection – β
  - Vertical deflection (flexure) – h
- Wing motions are sinusoidal and infinitesimal
- Wing has no internal or solid friction, resulting in no internal damping

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- Wing motions are sinusoidal and infinitesimal
- ~~Wing has no internal or solid friction, resulting in no internal damping~~

Assumption removed in *NACA 685* and *NACA 741*

Equations of Motion Collected

$$\text{(A)}\quad \ddot{\alpha}\left[r_\alpha{}^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta{}^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$

$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

$$\text{(B)}\quad \ddot{\alpha}\left[r_\beta{}^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta{}^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$

$$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot{h}\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

$$\text{(C)}\quad \ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$

$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

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(A) = Sum of moments about the elastic axis

Equations of Motion Collected

(A)
$$\ddot{\alpha}\left[r_\alpha^{\,2}+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^{\,2}+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$
$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

(B)
$$\ddot{\alpha}\left[r_\beta^{\,2}+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^{\,2}-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$
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(A) = Sum of moments about the elastic axis

(B) = Sum of moments about the aileron hinge

Equations of Motion Collected

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(A) = Sum of moments about the elastic axis

(B) = Sum of moments about the aileron hinge

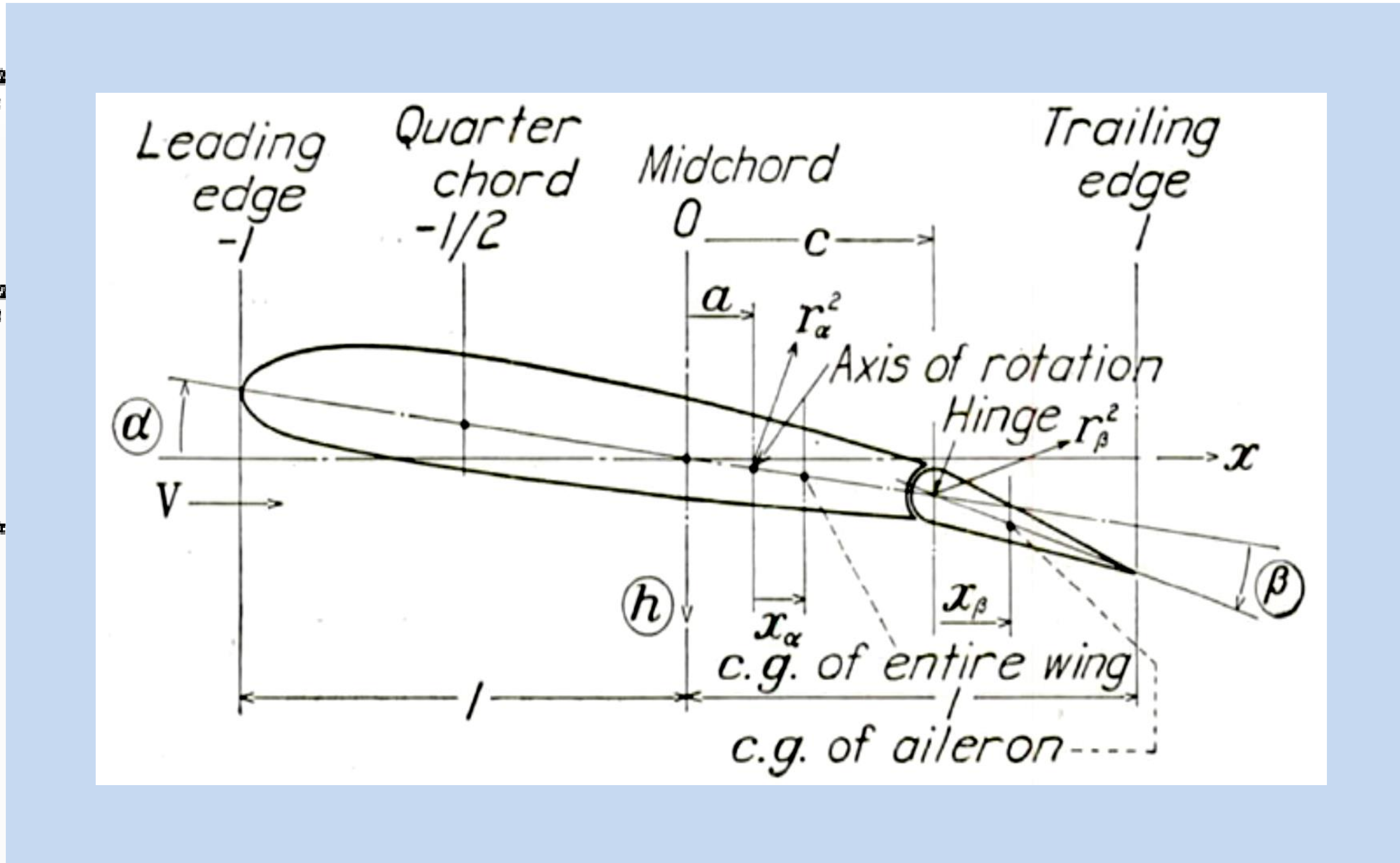
(C) = Sum of forces in the vertical direction

Equations of Motion Collected

(A) $\ddot{\alpha}\Big[r_\alpha^2$ … $\left(\frac{1}{2}-a\right)T_4\Big]$

(B) $\ddot{\alpha}\Big[r_\beta^2$ … $\frac{1}{\pi}\dot{\beta}\Big]=0$

(C) $\ddot{\alpha}\Big(x_\alpha$ …



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$$\text{(B)}\quad \ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$

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$$\text{(C)}\quad \ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$

$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

Equations of Motion Collected

(A) $$\ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$

$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

(B) $$\ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$

$$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot{h}\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

(C) $$\ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$

$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

Steps taken to obtain “final” equations of motion:

(1) Make substitutions

$$\alpha=\alpha_0 e^{ik\frac{v}{b}t}$$

$$\beta=\beta_0 e^{i(k\frac{v}{b}t+\varphi_1)}$$

$$h=h_0 e^{i(k\frac{v}{b}t+\varphi_2)}$$

Equations of Motion Collected

(A) $$\ddot\alpha\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot\alpha\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot\beta\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot\beta\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$
$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot h\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot h}{b}+\left(\frac{1}{2}-a\right)\dot\alpha+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot\beta\right]=0$$

(B) $$\ddot\alpha\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot\alpha\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot\beta\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot\beta T_4T_{11}\frac{v}{b}\kappa$$
$$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot h\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot h}{b}+\left(\frac{1}{2}-a\right)\dot\alpha+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot\beta\right]=0$$

(C) $$\ddot\alpha\left(x_\alpha-\kappa a\right)+\dot\alpha\frac{v}{b}\kappa+\ddot\beta\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot\beta\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot h(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$
$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot h}{b}+\left(\frac{1}{2}-a\right)\dot\alpha+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot\beta\right]=0$$

Steps taken to obtain “final” equations of motion:

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$$\left.\begin{aligned}\alpha &= \alpha_0 e^{ik\frac{v}{b}t}\\ \beta &= \beta_0 e^{i(k\frac{v}{b}t+\varphi_1)}\\ h &= h_0 e^{i(k\frac{v}{b}t+\varphi_2)}\end{aligned}\right\} \longrightarrow \begin{aligned}\dot\alpha &= ik\frac{v}{b}\alpha\\ \ddot\alpha &= -\left(k\frac{v}{b}\right)^2\alpha\\ &\text{etc.}\end{aligned}$$

Equations of Motion Collected

(A) $$\ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$
$$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

(B) $$\ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$
$$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot{h}\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

(C) $$\ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$
$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

After substitutions all terms, except these terms, contain v^2

Steps taken to obtain "final" equations of motion:

(1) Make substitutions

$$\left.\begin{aligned}\alpha &= \alpha_0 e^{ik\frac{v}{b}t}\\ \beta &= \beta_0 e^{i(k\frac{v}{b}t+\varphi_1)}\\ h &= h_0 e^{i(k\frac{v}{b}t+\varphi_2)}\end{aligned}\right\} \longrightarrow \quad \begin{aligned}\dot{\alpha} &= ik\frac{v}{b}\alpha\\ \ddot{\alpha} &= -\left(k\frac{v}{b}\right)^2\alpha\\ &\text{etc.}\end{aligned}$$

Equations of Motion Collected

(A) $\ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$

$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$

(B) $\ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$

$+\beta\left[\frac{C_\beta}{Mb^2}+\frac{1}{\pi^2}\frac{v^2}{b^2}\kappa(T_5-T_4T_{10})\right]+\ddot{h}\left(x_\beta-\frac{1}{\pi}\kappa T_1\right)\frac{1}{b}+\frac{T_{12}}{\pi}\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$

(C) $\ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$

$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$

Steps taken to obtain “final” equations of motion:

(1) Make substitutions

$$\left.\begin{aligned}\alpha &= \alpha_0 e^{ik\frac{v}{b}t}\\ \beta &= \beta_0 e^{i(k\frac{v}{b}t+\varphi_1)}\\ h &= h_0 e^{i(k\frac{v}{b}t+\varphi_2)}\end{aligned}\right\} \longrightarrow \begin{aligned}\dot{\alpha} &= ik\frac{v}{b}\alpha\\ \ddot{\alpha} &= -\left(k\frac{v}{b}\right)^2\alpha\end{aligned}$$

etc.

Equations of Motion Collected

(A) $\ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$

$+\beta\kappa\frac{v^2}{b^2}\frac{1}{\pi}(T_4+T_{10})+\ddot{h}\left(x_\alpha-a\kappa\right)\frac{1}{b}-2\kappa\left(a+\frac{1}{2}\right)\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$

(B) $\ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$

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(C) $\ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$

$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$

Steps taken to obtain “final” equations of motion:

(2) Normalize all equations by $\left(\frac{v}{b}k\right)^2\kappa$

Equations of Motion Collected

(A) $$\ddot{\alpha}\left[r_\alpha^2+\kappa\left(\frac{1}{8}+a^2\right)\right]+\dot{\alpha}\frac{v}{b}\kappa\left(\frac{1}{2}-a\right)+\alpha\frac{C_\alpha}{Mb^2}+\ddot{\beta}\left[r_\beta^2+(c-a)x_\beta-\frac{T_7}{\pi}\kappa-(c-a)\frac{T_1}{\pi}\kappa\right]+\frac{1}{\pi}\dot{\beta}\kappa\frac{v}{b}\left[-2p-\left(\frac{1}{2}-a\right)T_4\right]$$

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(B) $$\ddot{\alpha}\left[r_\beta^2+(c-a)x_\beta-\kappa\frac{T_7}{\pi}-(c-a)\frac{T_1}{\pi}\kappa\right]+\dot{\alpha}\left(p-T_1-\frac{1}{2}T_4\right)\frac{v}{b}\frac{\kappa}{\pi}+\ddot{\beta}\left(r_\beta^2-\frac{1}{\pi^2}\kappa T_3\right)-\frac{1}{2\pi^2}\dot{\beta}T_4T_{11}\frac{v}{b}\kappa$$

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(C) $$\ddot{\alpha}\left(x_\alpha-\kappa a\right)+\dot{\alpha}\frac{v}{b}\kappa+\ddot{\beta}\left(x_\beta-\frac{1}{\pi}T_1\kappa\right)-\dot{\beta}\frac{v}{b}T_4\kappa\frac{1}{\pi}+\ddot{h}(1+\kappa)\frac{1}{b}+h\frac{C_h}{M}\frac{1}{b}$$

$$+2\kappa\frac{vC(k)}{b}\left[\frac{v\alpha}{b}+\frac{\dot{h}}{b}+\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}\frac{v}{b}\beta+\frac{T_{11}}{2\pi}\dot{\beta}\right]=0$$

After normalization
only these terms
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Steps taken to obtain "final" equations of motion:

(2) Normalize all equations by $\left(\frac{v}{b}k\right)^2\kappa$

Final 3DOF Equations of Motion

(A) $$(A_{a\alpha} + \Omega_\alpha X)\alpha + A_{a\beta}\beta + A_{ah}h = 0$$

(B) $$A_{b\alpha}\alpha + (A_{b\beta} + \Omega_\beta X)\beta + A_{bh}h = 0$$

(C) $$A_{c\alpha}\alpha + A_{c\beta}\beta + (A_{ch} + \Omega_h X)h = 0$$

where $A_{a\alpha} = R_{a\alpha} + iI_{a\alpha}$, etc.

Final 3DOF Equations of Motion

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$$\underbrace{\left(\frac{\omega_\alpha r_\alpha}{\omega_r r_r}\right)^2}_{\Omega_\alpha} \underbrace{\frac{1}{\kappa}\left(\frac{b\omega_r r_r}{vk}\right)^2}_{X}$$

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$$\underbrace{\left(\frac{\omega_\alpha r_\alpha}{\omega_r r_r}\right)^2}_{\Omega_\alpha} \underbrace{\frac{1}{\kappa}\left(\frac{b\omega_r r_r}{vk}\right)^2}_{X}$$

The Ωs and X are central to Theodorsen's solution methods

Final 3DOF Equations of Motion

-- with addition of structural damping terms --

(A) $$(A_{a\alpha} + \Omega_\alpha(1 + ig_\alpha)X)\alpha + A_{a\beta}\beta + A_{ah}h = 0$$

(B) $$A_{b\alpha}\alpha + (A_{b\beta} + \Omega_\beta(1 + ig_\beta)X)\beta + A_{bh}h = 0$$

(C) $$A_{c\alpha}\alpha + A_{c\beta}\beta + (A_{ch} + \Omega_h(1 + ig_h)X)h = 0$$

Final 3DOF Equations of Motion

-- with addition of structural damping terms --

(A) $$(A_{\alpha\alpha} + \Omega_{\alpha}(1 + ig_{\alpha})X)\alpha + A_{\alpha\beta}\beta + A_{\alpha h}h = 0$$

(B) $$A_{b\alpha}\alpha + (A_{b\beta} + \Omega_{\beta}(1 + ig_{\beta})X)\beta + A_{bh}h = 0$$

(C) $$A_{c\alpha}\alpha + A_{c\beta}\beta + (A_{ch} + \Omega_{h}(1 + ig_{h})X)h = 0$$

For all equations of motion, 2DOF and 3DOF, solution is obtained when their determinant is zero

Outline

- Background and Purpose
- Brief History
- Equations of Motion
- Solution Methods
- Re-Computations and Comparisons
- Concluding Remarks

Solution Methods

Solution Method 1

Employed in *NACA 496*

2DOF only

No g, allows ξ

Solution Method 2

Employed in *NACA 685*

2DOF or 3DOF

Allows g and ξ

Solution Method 3

Employed in *NACA 741*

2DOF or 3DOF

Allows g and ξ

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← Expand complex determinant →
Separate into real and imaginary equations; set both to zero

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<u>2DOF Example – Torsion-Aileron (α, β)</u>

$$\begin{vmatrix} A_{a\alpha} + \Omega_\alpha X & A_{a\beta} \\ A_{b\alpha} & A_{b\beta} + \Omega_\beta X \end{vmatrix} = 0 \qquad \text{where} \quad A_{ij} = R_{ij} + iI_{ij}$$

Real equation

$$\Omega_\alpha \Omega_\beta X^2 + \left(\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha}\right)X + \left(R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}\right) = 0$$

Imaginary equation

$$\left(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha}\right)X + \left(R_{a\alpha}I_{b\beta} + R_{b\beta}I_{a\alpha} - R_{a\beta}I_{b\alpha} - I_{a\beta}R_{b\alpha}\right) = 0$$

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Separate into real and imaginary equations; set both to zero

2DOF Example – Torsion-Aileron (α, β)

Solution is obtained when real and imaginary equations are both satisfied for the same values of X and k

Real equation

$$\Omega_\alpha \Omega_\beta X^2 + \left(\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha}\right)X + \left(R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}\right) = 0$$

Imaginary equation

$$\left(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha}\right)X + \left(R_{a\alpha}I_{b\beta} + R_{b\beta}I_{a\alpha} - R_{a\beta}I_{b\alpha} - I_{a\beta}R_{b\alpha}\right) = 0$$

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Allows g and ξ

← Expand complex determinant
Separate into real and imaginary equations; set both to zero →

2DOF Example – Torsion-Aileron (α, β)

> Solution is obtained when real and imaginary equations are both satisfied for the same values of X and k

Do this for many values of k:

Real equation

$$\Omega_\alpha \Omega_\beta X^2 \; + \; \left(\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha}\right) X \; + \; \left(R_{a\alpha} R_{b\beta} - I_{a\alpha} I_{b\beta} - R_{a\beta} R_{b\alpha} + I_{a\beta} I_{b\alpha}\right) = 0$$

Imaginary equation

$$\left(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha}\right) X \; + \; \left(R_{a\alpha} I_{b\beta} + R_{b\beta} I_{a\alpha} - R_{a\beta} I_{b\alpha} - I_{a\beta} R_{b\alpha}\right) = 0$$

Solution Methods

<u>Solution Method 1</u>

Employed in *NACA 496*

2DOF only

No g, allows ξ

<u>Solution Method 2</u>

Employed in *NACA 685*

2DOF or 3DOF

Allows g and ξ

<u>Solution Method 3</u>

Employed in *NACA 741*

2DOF or 3DOF

Allows g and ξ

Straightforward

Expand complex determinant
Separate into real and imaginary equations; set both to zero

<u>2DOF Example – Torsion-Aileron (α, β)</u>

Do this for many values of k

Solution is obtained when real and imaginary equations are both satisfied <u>for the same values</u> of X and k

Real equation

$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

Imaginary equation

$$(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha})X + (R_{a\alpha}I_{b\beta} + R_{b\beta}I_{a\alpha} - R_{a\beta}I_{b\alpha} - I_{a\beta}R_{b\alpha}) = 0$$

Solution Methods

Solution Method 1

Employed in *NACA 496*

2DOF only

No g, allows ξ

Ingenious,
but complicated

Solution Method 2

Employed in *NACA 685*

F or 3DOF

ws g and ξ

Straightforward

Solution Method 3

Employed in *NACA 741*

F or 3DOF

Allows g and ξ

Expand complex determinant
Separate into real and imaginary equations; set both to zero

2DOF Example – Torsion-Aileron (α, β)

Do this for many values of k

Solution is obtained when real and imaginary equations are both satisfied for the same values of X and k

Real equation

$$\Omega_\alpha \Omega_\beta X^2 + \left(\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha}\right) X + \left(R_{a\alpha} R_{b\beta} - I_{a\alpha} I_{b\beta} - R_{a\beta} R_{b\alpha} + I_{a\beta} I_{b\alpha}\right) = 0$$

Imaginary equation

$$\left(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha}\right) X + \left(R_{a\alpha} I_{b\beta} + R_{b\beta} I_{a\alpha} - R_{a\beta} I_{b\alpha} - I_{a\beta} R_{b\alpha}\right) = 0$$

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 \ + \ (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X \ + \ (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

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Solution Method 1

- Treat Ω_α and X as parameters
- Solve 2 equations in 2 unknowns, Ω_α and X

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

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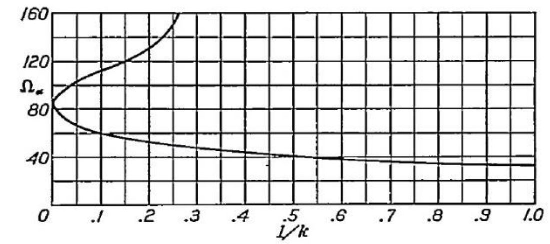
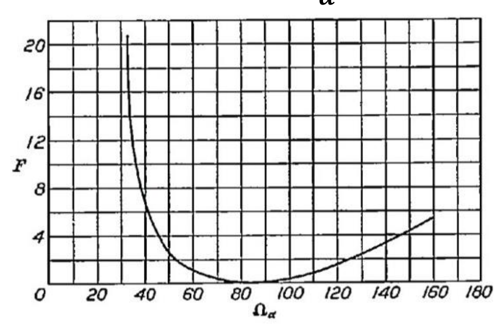
Ω_α vs $1/k$



<u>Solution Method 1</u>

- Treat Ω_α and X as parameters
- Solve 2 equations in 2 unknowns, Ω_α and X

F vs Ω_α



Solution Methods

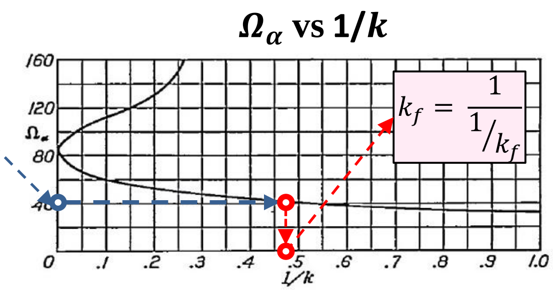
$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

$$(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha})X + (R_{a\alpha}I_{b\beta} + R_{b\beta}I_{a\alpha} - R_{a\beta}I_{b\alpha} - I_{a\beta}R_{b\alpha}) = 0$$

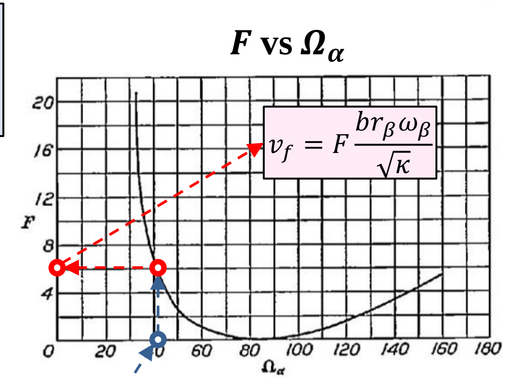
Solution Method 1

- Treat Ω_α and X as parameters
- Solve 2 equations in 2 unknowns, Ω_α and X

Identify problem-specific value of Ω_α



$$\boldsymbol{\Omega_\alpha} = \left(\frac{\boldsymbol{\omega_\alpha r_\alpha}}{\boldsymbol{\omega_\beta r_\beta}}\right)^2$$



Identify problem-specific value of Ω_α

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 \; + \; (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X \; + \; (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

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Solution Method 1

- Treat Ω_α and X as parameters
- Solve 2 equations in 2 unknowns, Ω_α and X

Solution Method 2

- Define Ω_α and Ω_β
- Treat X as a parameter
- Solve polynomial equations for X_1 and X_2

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

$$(\Omega_\alpha I_{b\beta} + \Omega_\beta I_{a\alpha})X + (R_{a\alpha}I_{b\beta} + R_{b\beta}I_{a\alpha} - R_{a\beta}I_{b\alpha} - I_{a\beta}R_{b\alpha}) = 0$$

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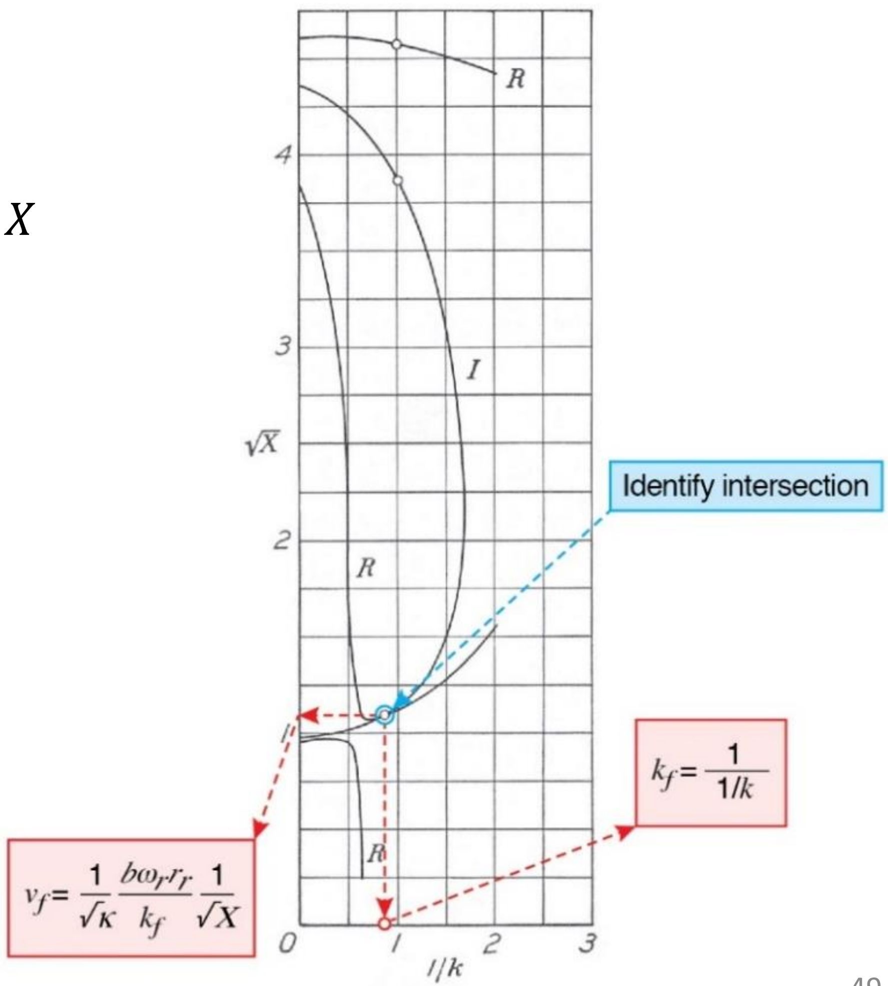

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

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- Define Ω_α and Ω_β
- Treat X as a parameter
- Solve polynomial equations for X_1 and X_2

Solution Method 3

- Define Ω_α and Ω_β
- Treat X as a parameter
- Employ method of elimination → $\begin{cases} a_1 X + a_0 = 0 \\ b_1 X + b_0 = 0 \end{cases}$
- Solve linear equations for X_1 and X_2

Solution Methods

$$\Omega_\alpha \Omega_\beta X^2 + (\Omega_\alpha R_{b\beta} + \Omega_\beta R_{a\alpha})X + (R_{a\alpha}R_{b\beta} - I_{a\alpha}I_{b\beta} - R_{a\beta}R_{b\alpha} + I_{a\beta}I_{b\alpha}) = 0$$

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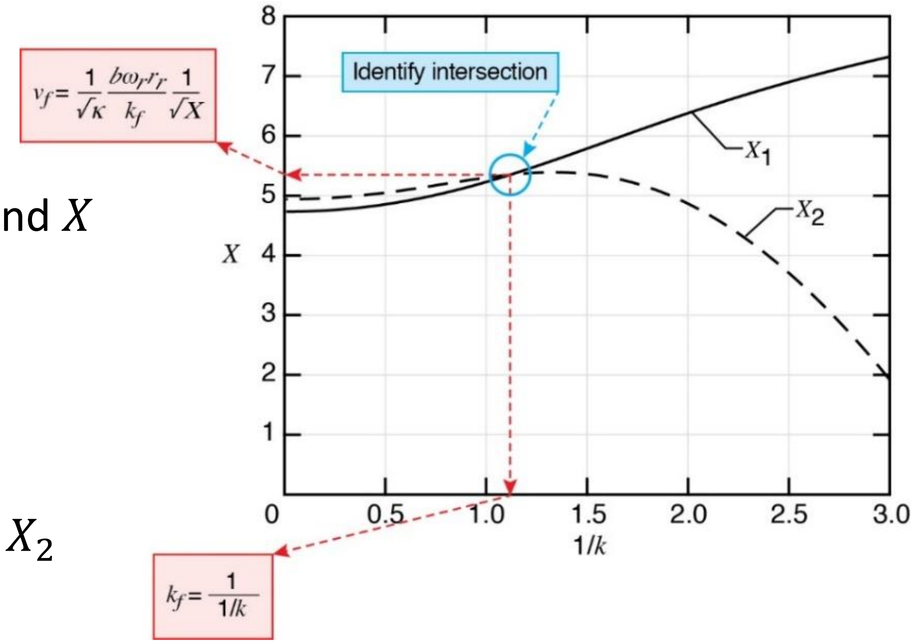

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- Treat X as a parameter
- Employ method of elimination
- Solve linear equations for X_1 and X_2

$$\begin{cases} a_1 X + a_0 = 0 \\ b_1 X + b_0 = 0 \end{cases}$$

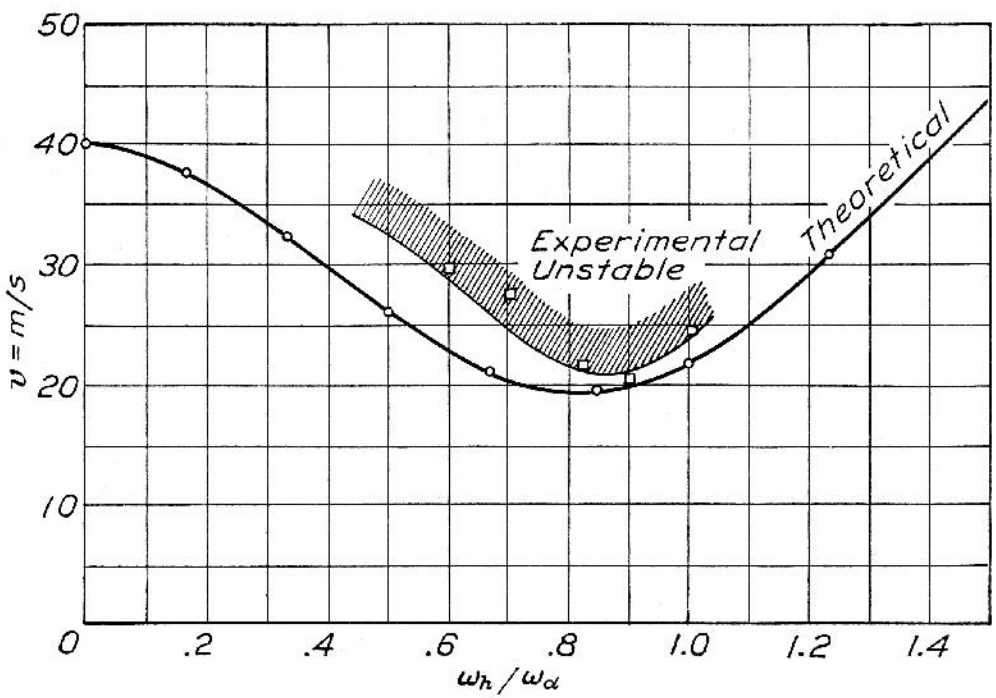
Outline

- Background and Purpose
- Brief History
- Equations of Motion
- Solution Methods
- Re-Computations and Comparisons
- Concluding Remarks

Comparisons for Solution Method 1

From *NACA 496;* Case 1

Effect of $\frac{\omega_h}{\omega_\alpha}$ on Flutter Velocity, v



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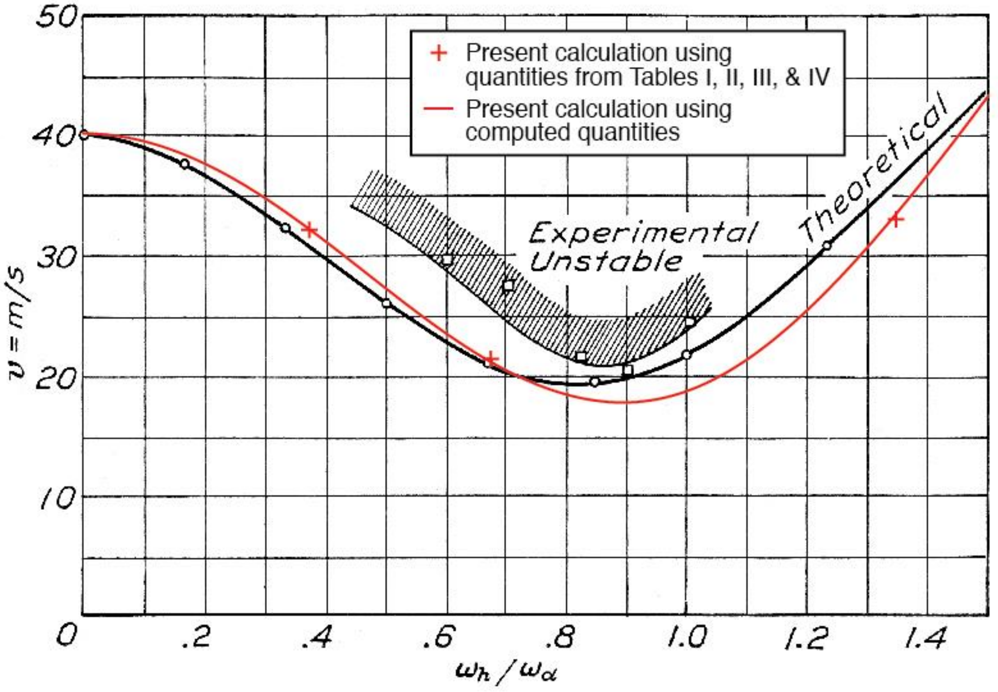
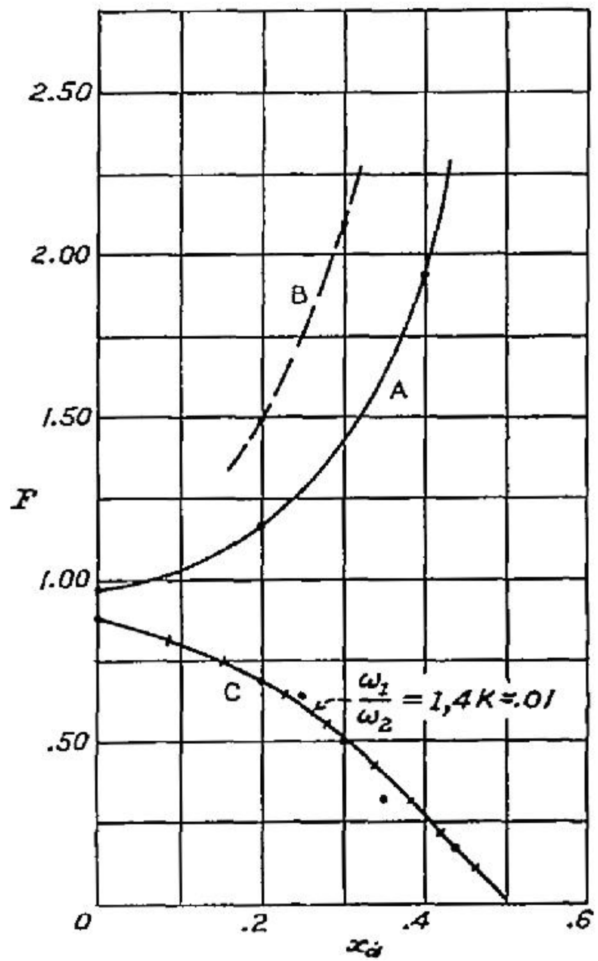


Comparisons for Solution Method 1

From *NACA 496;* Case 1

Effect of x_α on F



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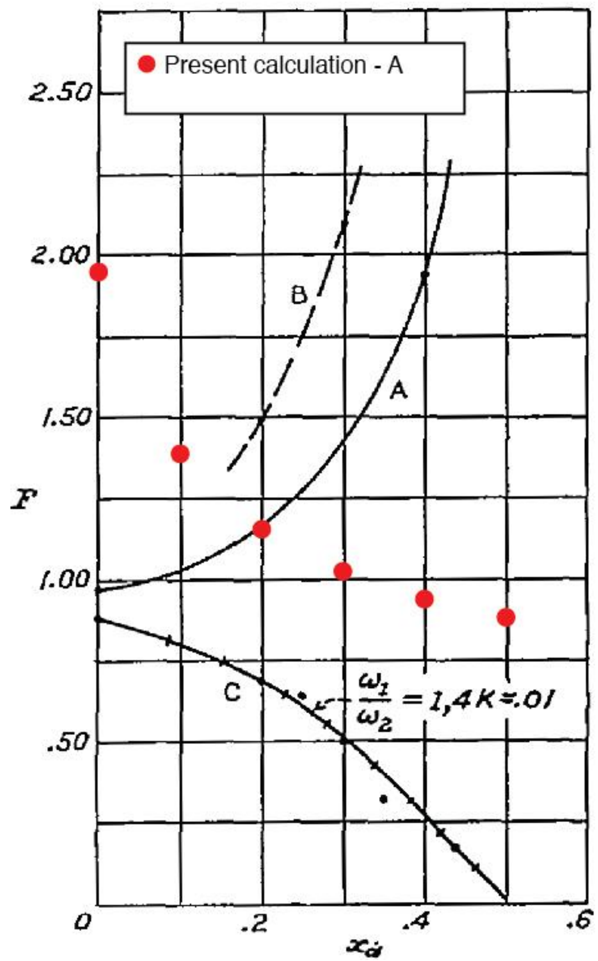
Effect of x_α on F



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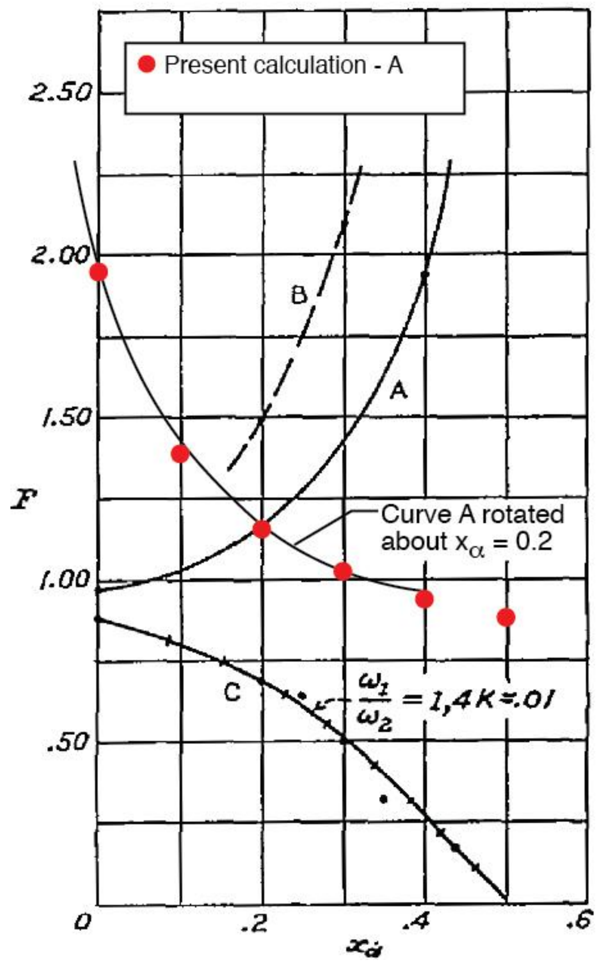
Effect of x_α on F



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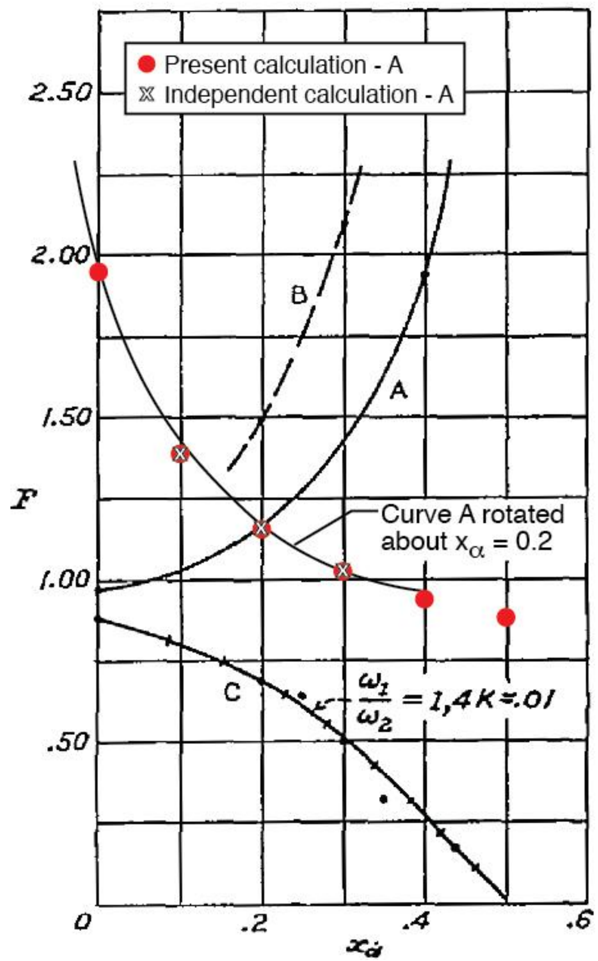
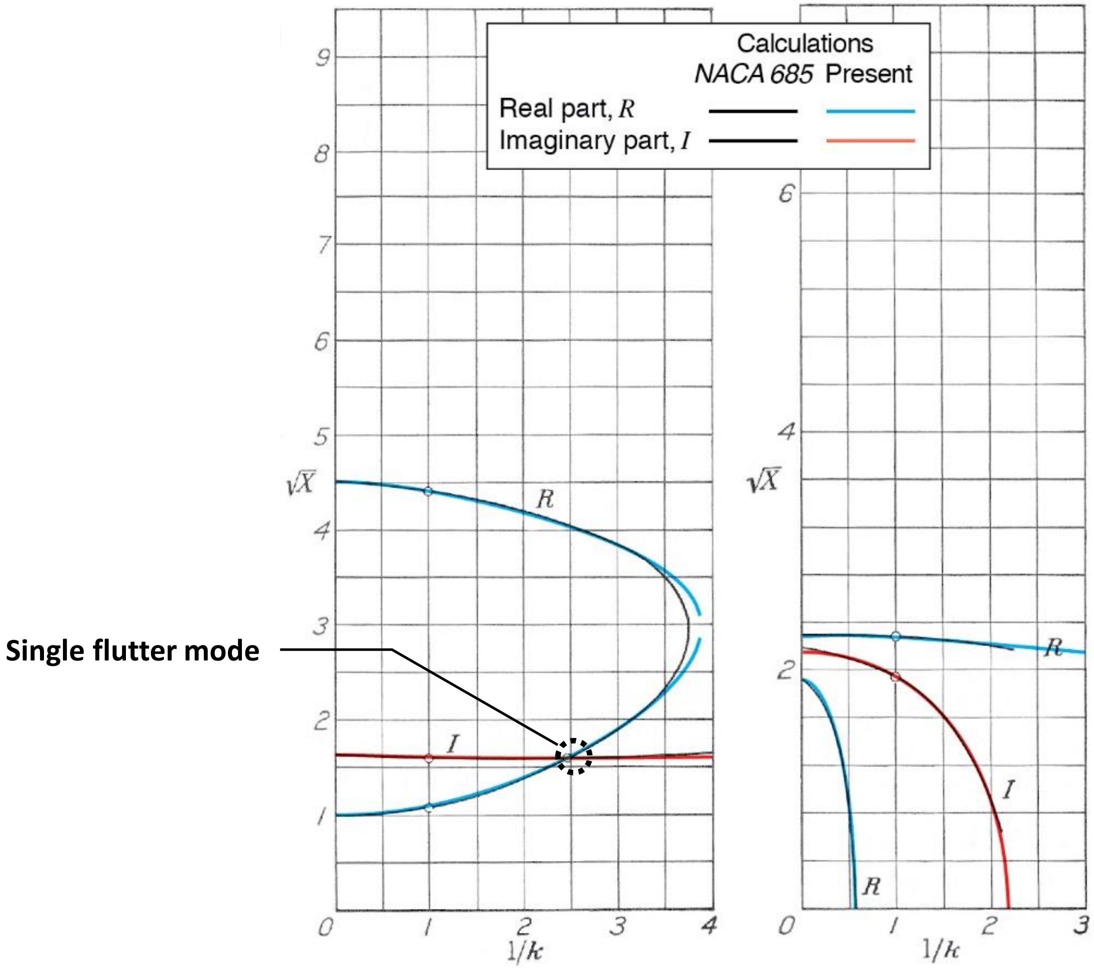


Comparisons for Solution Method 2

From *NACA 685;* 2DOF

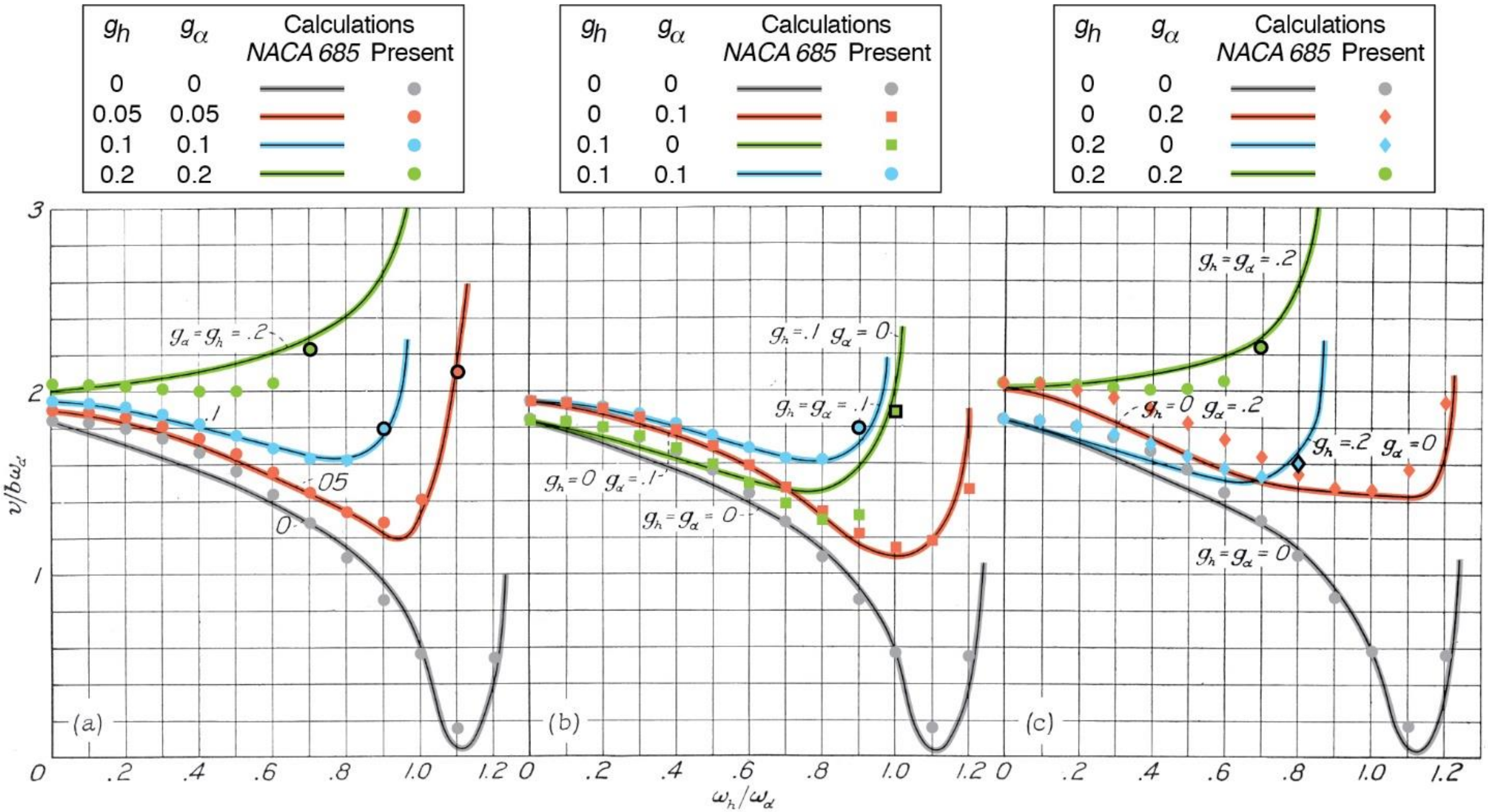
Case 1 Case 2



Comparisons for Solution Method 2

From *NACA 685;* Case 1

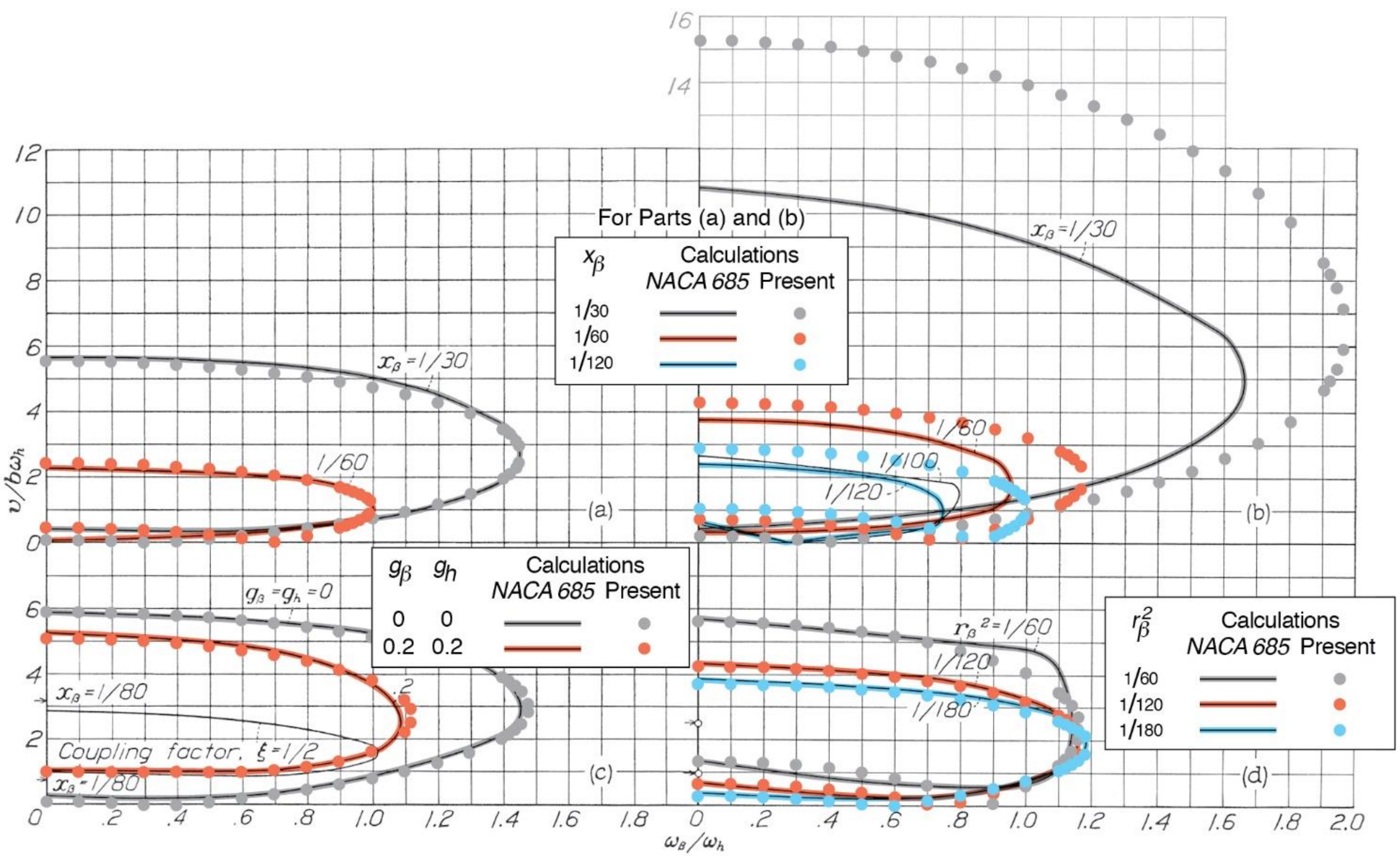
Effect of $\frac{\omega_h}{\omega_\alpha}$ on $\frac{v}{b\omega_\alpha}$ for various g_h and g_α



Comparisons for Solution Method 2

From *NACA 685;* Case 2

Effect of $\frac{\omega_\beta}{\omega_h}$ on $\frac{v}{b\omega_h}$ for various quantities



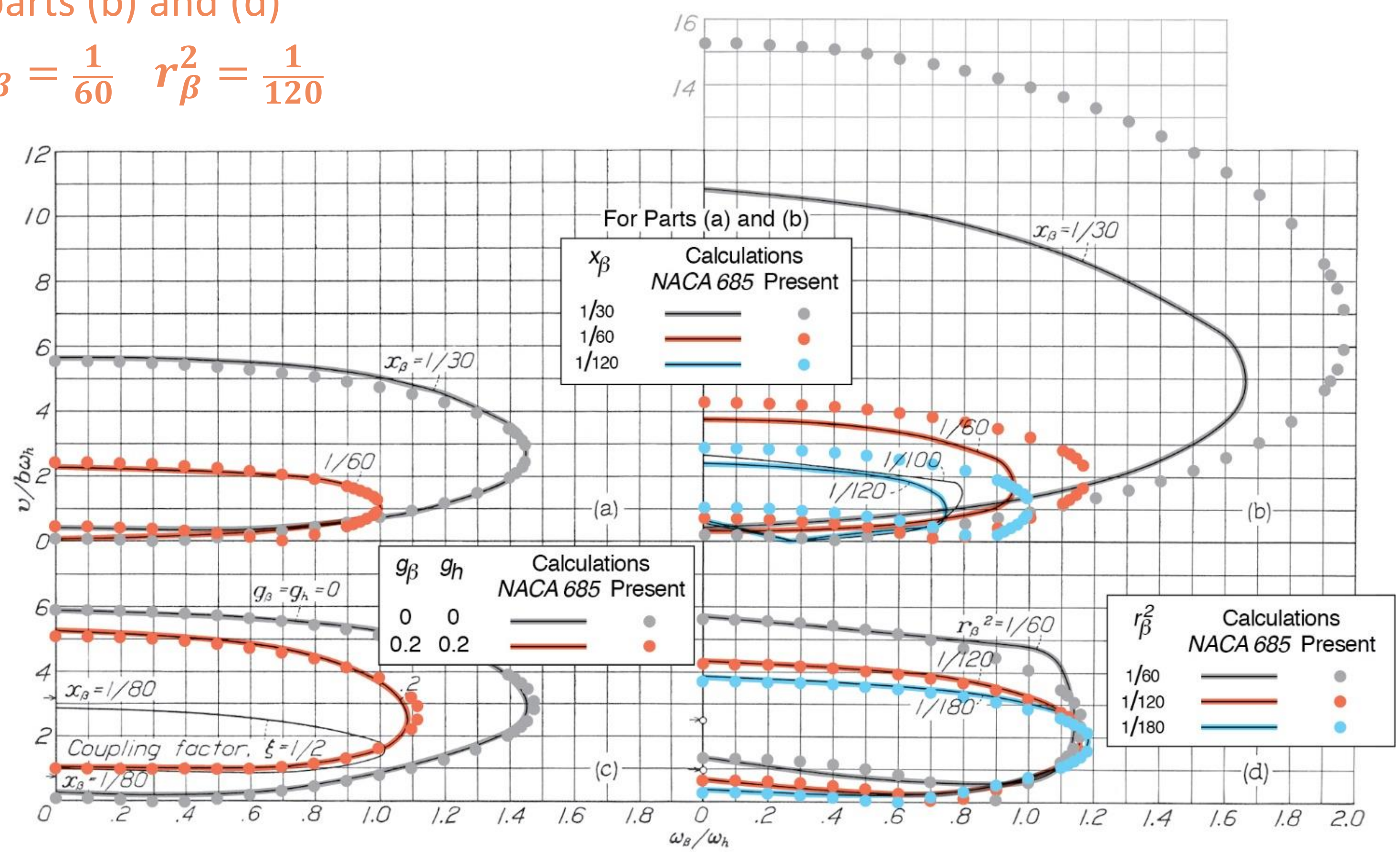
Comparisons for Solution Method 2

From *NACA 685;* Case 2

Effect of $\frac{\omega_\beta}{\omega_h}$ on $\frac{v}{b\omega_h}$ for various quantities

In parts (b) and (d)

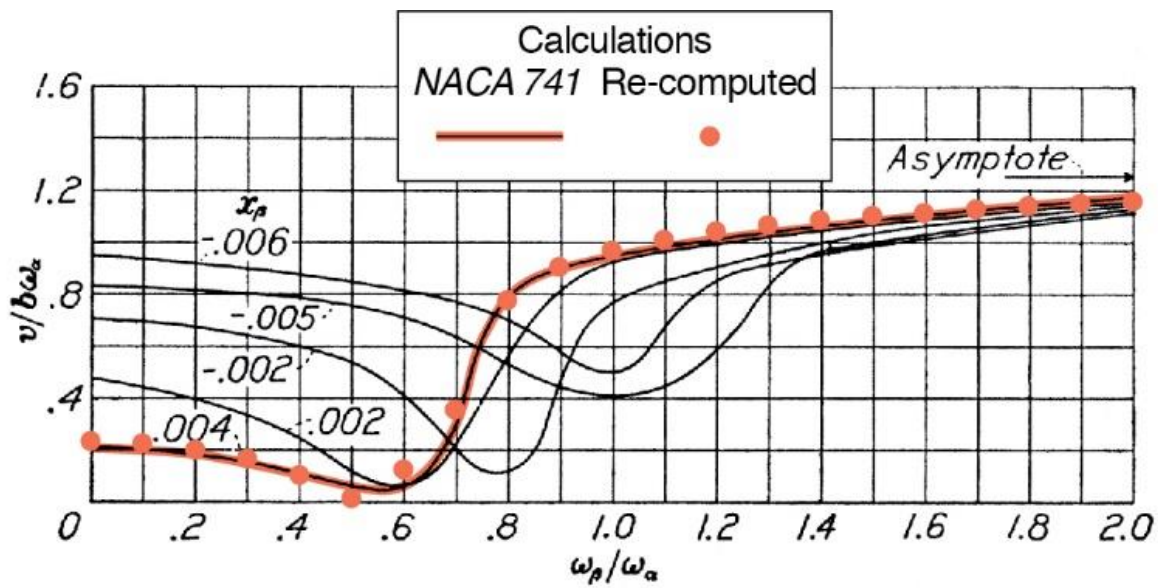
- $x_\beta = \frac{1}{60}$ $r_\beta^2 = \frac{1}{120}$



Comparisons for Solution Method 3

From *NACA 741;* Case 2

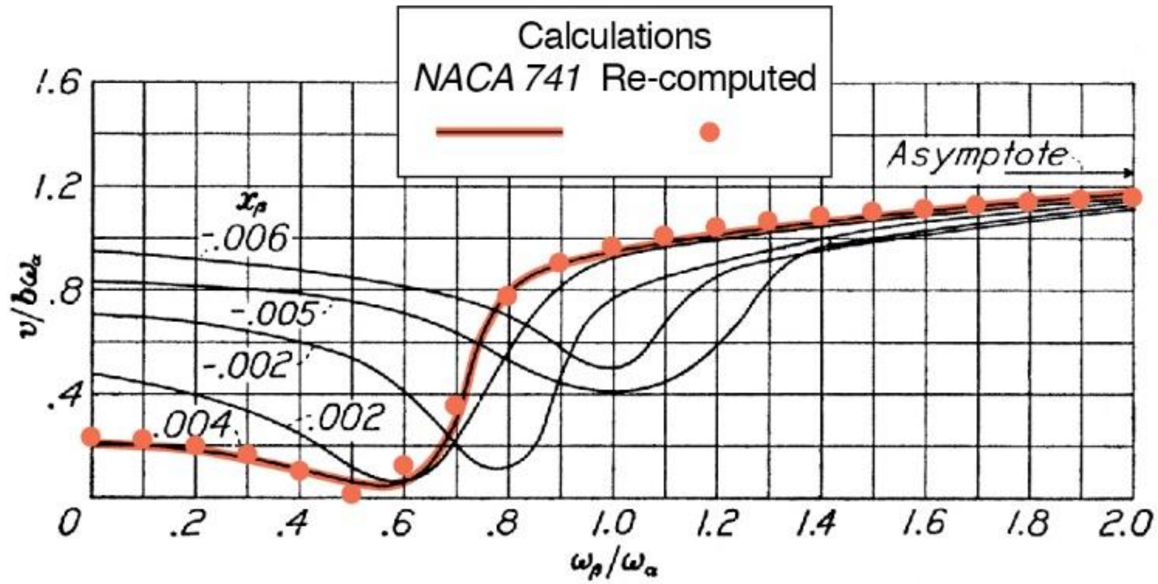
Effect of $\frac{\omega_\beta}{\omega_\alpha}$ on $\frac{v}{b\omega_a}$ for various x_β

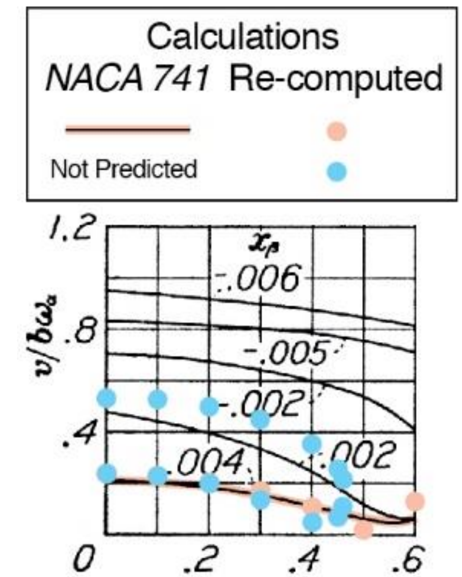


Comparisons for Solution Method 3

From *NACA 741;* Case 2

Effect of $\frac{\omega_\beta}{\omega_\alpha}$ on $\frac{v}{b\omega_a}$ for various x_β





Outline

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- Concluding Remarks

Concluding Remarks

- In an AIAA Engineering Note Thomas A. Zeiler –
  - Made known that numerical errors exist in three foundational reports on aeroelastic flutter and on early aeroelasticity texts
  - Recommended that all of the plots in *NACA 496*, *NACA 685*, and *NACA 741* be re-computed and published
- Current work is following Zeiler's recommendation by –
  - Re-computing and checking all numerical examples in these foundational reports
  - Comparing original and re-computed results
  - Publishing and making known the existence of the re-computations
- This paper has presented –
  - Theodorsen's and Garrick's equations and solution methods
  - Representative examples of re-computations and comparisons
  - Overall good agreement between original and re-computed results (with some notable discrepancies)