Assembly Maps and Pseudoisotopy Functors

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Abstract

We show the existence of a stable, smooth pseudoisotopyfunctor and construct in the topological, piecewise linear, and smooth category a zig-zag of natural weak equivalences between the stable pseudoisotopyfunctor and the corresponding functor of Whitehead spectra.

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Introduction

The automorphism spaces of manifolds have been of major interest to topologists for at least eighty years. The study of automorphism spaces of high-dimensional manifolds as its own field of research came into fruition a little later:

Following Stephen Smale's proof of the h–cobordism theorem [42] in 1960, topologists established a flurry of results on the classification of high-dimensional manifolds. In particular, the notion of block bundle automorphisms was introduced by Rourke and Sanderson [40] which gave the homotopy fibre sequence of simplicial sets

$$\mathrm{Aut}(M) \to \widetilde{\mathrm{Aut}}(M) \to \widetilde{\mathrm{Aut}}(M)/\,\mathrm{Aut}(M)$$

where $\mathrm{Aut}(M)$ and $\widetilde{\mathrm{Aut}}(M)$ refers to automorphisms and block bundle automorphisms, respectively.

The calculation of $\widetilde{\mathrm{Aut}}(M)$ via surgery theory proceeded in the following decades. We focus our attention on the quotient $\widetilde{\mathrm{Aut}}(M)/\,\mathrm{Aut}(M)$.

Let us explain the geometric problem encoded in this quotient: Given some automorphism, we want to decide whether it is isotopic to the Identity. If we follow a strategy similar to the one to figure out whether two given manifolds are isomorphic, then the tools of surgery theory present us with a 1–parameter family of h–cobordisms.

The key difference between the two questions is now, that the classification of parametrised families of h–cobordisms is far more subtle than that of h–cobordisms.

Parametrised families of h–cobordisms are just bundles of h–cobordisms with nice clutching maps, hence the problem can be rephrased in understanding the space of possible clutching maps. This is what pseudoisotopies are and how pseudoisotopy spaces enter our story.

The study of pseudoisotopies began in earnest in 1970, when Jean Paul Cerf showed his pseudoisotopy theorem [8] which raised the methods of Morse theory, key in the proof of the h–cobordism theorem, to a parametrised setting. In the following years, the study of pseudoisotopy spaces was advanced greatly by, among others, Allen Hatcher, Dan Burghelea, Kyoshi Igusa, Richard Lashof and Friedhelm Waldhausen.

In the 1970s, Hatcher [22] described an approach to reduce the geometric questions about pseudoisotopies to a more combinatorial setting. Although his arguments contained several flaws, the general direction of research in the next decades followed along the lines of his ideas.

In particular, he claimed that the pseudoisotopy space, intrinsically linked to the automorphism spaces of manifolds, could be connected to the *stable* pseudoisotopy space which seemed much more accessible via homotopy theory and thus provided a chance for actual computation.

Starting in the late 1970s, Waldhausen showed in a series of papers, beginning with [49], that this hope was not ill-founded. He introduced A–theory, a functor with good homotopy theoretic properties, and showed that computations about homotopy invariants of stable pseudoisotopy could essentially be reduced to computations on A–theory.

In 1987, the same year Waldhausen finished his program (for the most part - we come back to this in a moment), Michael Weiss and Bruce Williams [53] answered precisely how the difference between automorphisms and block bundle automorphisms relates to pseudoisotopies.

Let $\mathrm{Cat} = \mathrm{Top}, \mathrm{PL}$ or Diff and let M be a compact Cat–manifold, possibly with boundary. Let $\mathrm{Aut}_\partial(M) \subseteq \mathrm{Aut}(M)$ denote the subspace of automorphisms which restrict to the Identity on the boundary, similarly for $\widetilde{\mathrm{Aut}}_\partial(M)$. Also, let $E\mathbb{Z}/2$ denote a contractible, free $\mathbb{Z}/2$–CW–complex and $\mathbb{P}(M)$ the stable space of Cat–pseudoisotopies relative boundary.

Theorem 0.0.0.1 ([53, Theorem A]). *There exists a map*

$$\Phi^s\colon \widetilde{\mathrm{Aut}}_\partial(M)/\,\mathrm{Aut}_\partial(M) \to \Omega^\infty(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \Omega^2\mathbb{P}(M))$$

which is $(k+1)$*–connected if* k *is in the* Cat*–concordance stable range for* M*.*

This was only the first article in their ongoing program on combining all of the previous material into a single algebro-topological object to compute the automorphism spaces.

Also in 1987, Thomas Farrell and Lowell Jones [15] showed that their methods to simplify h–cobordisms on hyperbolic manifolds via carefully applied geodesic flows and controlled topology [14] could be transferred to pseudoisotopies. Roughly speaking, their result stated that (for a hyperbolic manifold) each pseudoisotopy was determined by its behaviour on a set of representatives of the isotopy classes of the geodesics. This marked the beginning of what is now known as the Farrell-Jones conjecture for pseudoisotopies.

Igusa [30] connected the pseudoisotopy space, intrinsically linked to the automorphism spaces of manifolds, to the *stable* pseudoisotopy space which by the work of Waldhausen now seemed less mysterious. It also empowered [53, Theorem A]. We denote the unstable pseudoisotopy space by $P(M)$.

Theorem 0.0.0.2. *The suspension map*

$$\sigma\colon P^{\mathrm{Diff}}(M) \to P^{\mathrm{Diff}}(M \times [0,1])$$

is k*–connected, if* $\dim(M) \geq \max(2k+7, 3k+4)$*, i.e. the concordance stable range of* M *is at least* k*.*

By now, we also have result for the concordance stable range in the other cases, see [31, Corollary 1.4.2].

As a brief tangent, we note that Goodwillie [18] introduced his calculus of functors in 1990, providing a computation of the differential of pseudoisotopy and thus another tool for actual calculations.

After almost two decades, Bjørn Jahren, John Rognes and Friedhelm Waldhausen [31] gave a proof for the stable parametrised h–cobordism theorem in

2005, originally announced by Waldhausen but not published as a part of his original series. In the following theorem, the functors are defined on codimension zero embeddings.

Theorem 0.0.0.3 ([31, Theorem 0.1])**.** *There is a natural homotopy equivalence*

$$\mathbb{H}^{\mathrm{Cat}}(M) \simeq \Omega\,\mathrm{Wh}^{\mathrm{Cat}}(M)$$

between the stable h–cobordism space and the one fold loops of the Whitehead space for each compact Cat*–manifold M.*

At this point, the results finally provide us with a clear connection between the homotopy groups of the automorphism spaces of high-dimensional manifolds in a certain range and the homotopy groups of A– and L–theory.

Starting with Hatcher, the notion of a pseudoisotopy functor has been part of the discussion and played a rather curious role. Hatcher [23] sketched the definition of a functor on homotopy categories, Burghelea and Lashof [6] provided some necessary details and Waldhausen [50] briefly mentioned it - but did not, in contrast to later references, claim the existence of a strict pseudoisotopy functor on all continuous maps.

In this work, we show that a smooth pseudoisotopy functor does indeed exist. This is Theorem 1.4.2.4.

Theorem 0.0.0.4. *Let* Top *and* Spectra *denote the category of compactly generated weak Hausdorff spaces and prespectra, respectively. There is a functor* $\mathcal{P}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Spectra}$ *with the following properties:*

1. *It descends to a functor of homotopy categories*

$$\mathrm{ho}\,\mathcal{P}^{\mathrm{Diff}}\colon\ \mathrm{ho}(\mathrm{Top}) \to \mathrm{ho}(\mathrm{Spectra}).$$

2. *There is a natural weak equivalence*

$$\Omega^\infty \mathcal{P}^{\mathrm{Diff,\ Spectra}} \to \mathcal{P}^{\mathrm{Diff,\ Spaces}}.$$

3. *The subset inclusion $\mathcal{P}^{\mathrm{Diff}} \subseteq \mathcal{P}^{\mathrm{Top}}$ extends to a natural transformation of functors of quasicategories. The construction of $\mathcal{P}^{\mathrm{Top}}$ is given in [12].*

As these properties do not uniquely determine the functor, the obvious next step is to compare it with the well-established functor structure on the computational side. It turns out that the functors are compatible. This is Corollary 2.1.5.18, Theorem 2.2.3.13 and Theorem 2.2.4.2.

Theorem 0.0.0.5. *Let* $\mathrm{Cat} = \mathrm{Top}, \mathrm{PL}$ *or* Diff*. There is a natural weak equivalence of $(\infty,1)$–functors*

$$\Psi\colon \mathcal{P}^{\mathrm{Cat}} \Rightarrow \Omega^2\,\mathrm{Wh}^{\mathrm{Cat},-\infty}$$

from the $(\infty,1)$–functor $\mathcal{P}^{\mathrm{Cat}}\colon \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Spectra}_\Delta$ of pseudoisotopies to the twofold loops of the functor given by the Whitehead spectrum.

In particular, there is a zig-zag of natural weak equivalences between the strict functors $\mathcal{P}^{\mathrm{Cat}}\colon\ \mathrm{Top} \to \mathrm{Spectra}$ and $\Omega^2\,\mathrm{Wh}^{\mathrm{Cat},-\infty}$.

A similar statement holds for the space level versions $\mathbb{P}^{\mathrm{Cat}}$ and $\Omega^2\,\mathrm{Wh}^{\mathrm{Cat}}$.

One application of our results is a proof of the Farell-Jones conjecture for a far larger class of groups than was previously established. This result has been published by Nils-Edvin Enkelmann, Wolfgang Lück, Mark Ullmann, Christoph Winges and the author [13]. By the time this thesis was submitted, the article has been accepted for publication.

This result permits us to reduce a wide range of computations to a select few special cases, for example the following holds:

Theorem 0.0.0.6 ([13, Theorem 1.3])**.** *Let M be a smoothable aspherical closed manifold of dimension ≥ 10, whose fundamental group π is hyperbolic.*

Then there is a $\mathbb{Z}/2$–action on $\mathrm{Wh}^{\mathrm{Top}}(B\pi)$, the Whitehead space of the classifying space of π, such that we obtain isomorphisms

$$\pi_n(\mathrm{Top}(M)) \cong \pi_{n+2}\Big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathrm{Wh}^{\mathrm{Top}}(BC)\big)\Big)$$

for $1 \leq n \leq \min\{(\dim M - 7)/2, (\dim M - 4)/3\}$ and an exact sequence

$$1 \to \pi_2\Big(E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \big(\bigvee_C \mathrm{Wh}^{\mathrm{Top}}(BC)\big)\Big) \to \pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi) \to 1$$

where C ranges over the conjugacy classes of maximal infinite cyclic subgroups of π, and $\mathrm{Out}(\pi)$ denotes the outer automorphisms.

Although the results of this thesis were used in the article they are not truly necessary. The point-wise connection between the homotopy groups of pseudoisotopies and the Whitehead spectrum is enough. Our results might prove useful to find geometric representatives of non-trivial classes in the homotopy groups of automorphism spaces. So far, however, no such applications are known.

The above theorem suggests that understanding the homotopy groups of the Whitehead space is a possible approach to calculate homotopy groups of automorphism spaces. Unfortunately, explicit calculations of the homotopy groups of the Whitehead space are known only in very few cases due to Lars Hesselholt [25], building upon his earlier work with Ib Madsen, and Joachim Grunewald, John Klein and Tibor Macko [19].

Further, it has so far not been shown that the $\mathbb{Z}/2$–action on $\mathbb{P}(M)$ used in Theorem 0.0.0.3 (which actually induces the action on $\mathrm{Wh}^{\mathrm{Top}}(BC)$ in Theorem 0.0.0.6) is compatible with the homotopically well-behaved $\mathbb{Z}/2$–action on $\mathrm{Wh}^{\mathrm{Cat}}(M)$ induced by the action on A–theory given in [29].

These two obstacles have so far prevented a complete description of the homotopy groups in terms of (geometric) representatives and relations in any but the most elementary of cases.

The study of automorphism spaces in a stable range has recently garnered new interest by methods of homological stability, as part of the program by Søren Galatius and Oscar Randal-Williams started in [16], and some connections to the classic approach have been established, e.g. a bound on the concordance stable range by Randal-Williams [39] and finiteness results for certain cases due to Alexander Kupers [32]. We hope that our results help to further the study of the still mysterious objects that are automorphism spaces of manifolds.

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I owe many thanks and an apology to Christoph Winges. The former for fruitful collaborations and his interest in this thesis. The latter for not believing him immediately when he pointed out a serious flaw in a previous version of the second chapter.

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Notation and Convention

Simplicial sets are functors from the category Δ to sets. We denote the category of simplicially enriched categories by Cat_Δ and the full subcategory on small categories by cat_Δ. Similarly, we denote the category of simplicial categories by sCat and its small counterpart by scat. Finally, we obtain the category of quasicategories Qcat and use qcat for the small ones.

We write $\mathcal{S}_\bullet\colon \mathrm{Top} \to \mathrm{sSet}$ for the singular sets functor, right adjoint to geometric realization $X_\bullet \mapsto |X_\bullet|$. If spaces or simplicial sets appear as lower or upper case, we write X instead of $X_\bullet$ or $|X_\bullet|$.

Let Top_Δ denote the simplicially enriched category of compactly generated weak Hausdorff spaces with $\mathrm{Top}_\Delta(X,Y) = \mathcal{S}_\bullet C^0(X,Y)$ where X and Y are topological spaces in $\mathrm{ob}\,\mathrm{Top}_\Delta$ and the space of continuous maps $C^0(X,Y)$ carries the compact-open topology. Note that every mapping space is actually a Kan-complex.

We denote simplicially enriched categories by a lower case Δ and, for categories enriched in compactly generated weak Hausdorff spaces, we abuse notation and denote them with a lower case Δ as well.

This should not lead to confusion, since there is an equivalence between simplicially and topologically enriched categories $\mathcal{S}_*\colon \mathrm{Cat}_{\mathrm{Top}} \to \mathrm{Cat}_\Delta$ given by applying the singular simplicial sets functor $\mathcal{S}\colon \mathrm{Top} \to \mathrm{sSet}$ to the mapping spaces, i.e. $\mathcal{S}_*(\mathcal{C}_{\mathrm{Top}})(c,d) = \mathcal{S}_\bullet(\mathcal{C}_{\mathrm{Top}}(c,d))$, see Lurie [33, Chapter 1.1.4].

Every argument in this thesis which is carried out in topologically enriched categories could just as well be formulated in terms of the simplicially enriched categories obtained via $\mathcal{S}_*$.

Manifolds are always submanifolds of $\mathbb{R}^\infty$ to avoid smallness issues.

Reader's Guide

The two main chapters of this thesis can be mostly read separately, aside from two exceptions:

First, the comparison of the smooth and topological pseudoisotopyfunctor relies on the definition of both functors given in Chapter 1 and Section 2.1.3 for the smooth and topological case, respectively.

Second, the strictification of homotopy coherent diagrams following the work of Cordier and Porter explained in Section 1.2.1 is used in both chapters.

A detailed account of the content of the various sections and their interdependencies is included in each chapter's introduction.

Chapter 1

The Smooth Pseudoisotopy Functor

In [23, Proposition 1.3], Hatcher defined a homotopy functor $\mathbb{P}^{PL}$ which sends a PL manifold to the space of "stable PL pseudoisotopies". Later on, Burghelea and Lashof [6] refined his work, showing the existence of homotopy functors $\mathbb{P}^{\mathrm{Diff},\mathrm{BL}}\colon \mathrm{Mfd}^{\mathrm{Diff}} \to \mathrm{Top}$ and $\underline{\mathbb{P}}^{\mathrm{Diff},\mathrm{BL}}\colon \mathrm{Mfd}^{\mathrm{Diff}} \to \mathrm{Spectra}$ and constructing the space, respectively connective spectrum, of "stable smooth pseudoisotopies". We note that $\mathbb{P}^{\mathrm{Diff},\mathrm{BL}}$ and $\Omega^\infty \underline{\mathbb{P}}^{\mathrm{Diff},\mathrm{BL}}$ coincide up to natural weak equivalence.

The literature on pseudoisotopies at some points suggests that a strict functor has already been established. To the author's knowledge, the sources of confusion are a remark due to Waldhausen [50, p. 152] and a construction due to Quinn [38, Definition 5.3].

Unfortunately, in the first case no details are given, while in the second case it was not shown that the resulting functor evaluates to the right homotopy type on objects. Neither of the authors claimed otherwise. Moreover, it bears mentioning that Waldhausen did suggest the use of homotopy limits.

In this chapter we establish a strict functor for smooth pseudoisotopies which is compatible with the constructions of Hatcher, Burghelea and Lashof. The last condition stated in each of the following theorems enables us to carry over the main results of Chapter 2 from the topological to the smooth case. The theorems are shown as Theorem 1.4.1.2 and Theorem 1.4.2.4, respectively.

Theorem 1.0.0.1. *There is a functor $\mathbb{P}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Top}$ with the following properties:*

1. *It descends to a functor of homotopy categories*

$$\mathrm{ho}\,\mathbb{P}^{\mathrm{Diff}}\colon \mathrm{ho}(\mathrm{Top}) \to \mathrm{ho}(\mathrm{Top}).$$

2. *There is a natural isomorphism of functors*

$$\alpha\colon \mathrm{ho}\,\mathbb{P}^{\mathrm{Diff}} \to \mathrm{ho}\,\mathbb{P}^{\mathrm{Diff},\mathrm{BL}}.$$

3. *The subset inclusion $\mathbb{P}^{\mathrm{Diff}} \subseteq \mathbb{P}^{\mathrm{Top}}$ extends to a natural transformation of functors of quasicategories. The construction of $\mathbb{P}^{\mathrm{Top}}$ is given in [12].*

The non-connective pseudoistopy spectrum given in [23, Appendix II] is the point-wise value of our functor to spectra, see Remark 1.4.2.2.

Theorem 1.0.0.2. *There is a functor* $\mathcal{P}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Spectra}$ *with the following properties:*

1. *It descends to a functor of homotopy categories*
$$\mathrm{ho}\,\mathcal{P}^{\mathrm{Diff}}\colon \mathrm{ho}(\mathrm{Top}) \to \mathrm{ho}(\mathrm{Spectra}).$$
2. *There is a natural weak equivalence*
$$\mathbb{P}^{\mathrm{Diff}} \to \Omega^{\infty}\mathcal{P}^{\mathrm{Diff},\mathrm{Spectra}}.$$
3. *The subset inclusion* $\mathcal{P}^{\mathrm{Diff}} \subseteq \mathcal{P}^{\mathrm{Top}}$ *extends to a natural transformation of functors of quasicategories. The construction of* $\mathcal{P}^{\mathrm{Top}}$ *is given in [12].*

Here, Top and Spectra denote the usual 1–categories of compactly generated weak Hausdorff spaces and of (naive) spectra, respectively. The homotopy categories are formed with respect to π_*–isomorphisms. Since every object in the image of these functors is cofibrant and fibrant, this implies that weak equivalences are sent to homotopy equivalences. The $(\infty, 1)$–categories Top and Spectra are the simplicial nerves obtained from the usual topological enrichments of Top and Spectra, see Section 1.2.1.

Our definition coincides with the homotopy type of Igusa's construction [30]. This, and the natural weak equivalence to the Whitehead spectrum given in Chapter 2, justify our claim that the functor defined here is indeed the appropriate notion of a smooth pseudoisotopy functor.

Let us outline the proof. We are going to see that it is enough to construct a homotopy coherent diagram due to a strictification result by Cordier and Porter [10]. A homotopy coherent diagram is just a map between quasicategories. For spaces, it is going to be a map $\mathbb{P}\colon \mathcal{N}^{h.c.}_{\bullet}(\mathrm{Mfd}, \mathrm{cts})_{\Delta} \to \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Top}_{\Delta}$ starting in the quasicategory of smooth, compact manifolds with corners and continuous maps between them and ending in the quasicategory of topological spaces. Despite this general framework, the theory of $(\infty, 1)$–categories only plays a minor role, as the main ideas are geometric in nature.

Roughly speaking, the unstable space of pseudoisotopies $P(M)$ is a subspace of $Diff(M \times I)$ containing those diffeomorphisms which are given by the Identity on $M \times \{0\} \cup \partial M \times I$.

There are two constructions for maps between pseudoisotopy spaces. First, let $i\colon M \to N$ be a codimension zero embedding. Then $i_*\colon P(M) \to P(N)$ extends some $F \in P(M)$ to $N \times I$ by the Identity on $(N - i(M)) \times I$. Second, consider a vectorbundle $p\colon E \to M$. There is a transfer map $p_!\colon P(M) \to P(DE)$, where DE denotes the disk bundle of E. Morally speaking, we send F to a map which looks like $F \times \mathrm{Id}_{D^n}$ in local coordinates, and bend a neighbourhood of $\partial DE \times I$ into $DE \times \{0\}$, to make sure that we still end up with a map given by the Identity on $\partial DE \times I$. Unfortunately, bending around corners and smooth maps do not mesh well, so let us be a little more honest.

Let $\mathrm{St}_m\colon S^m - \{e_{m+1}\} \to \mathbb{R}^m$ denote the stereographic projection with e_{m+1} the $m+1$–th unit vector in $\mathbb{R}^{m+1}$. It restricts to a map $\mathrm{St}_m\colon S^m_- \to D^m$ where $S^m_- = S^m \cap \mathbb{R}^m \times (-\infty, 0]$ is the lower half of S^m. We identify it with D^m via the canonical diffeomorphism.

To describe the actual transfer in local coordinates, we need a parametrised and slightly deformed version $\underline{\mathrm{St}}_m\colon D^m \times I \to D^m \times I$. Fibre-wise, the map $p_!(F)$ can be understood as a diffeomorphism of $D^m \times I$. On $\underline{\mathrm{St}}_m^{-1}(D^m \times I)$ it is given by $\underline{\mathrm{St}}_m^{-1} \circ F \times \mathrm{Id}_{D^m} \circ \underline{\mathrm{St}}_m$. Up to isotopy of topological pseudoisotopies, this means we are "bending around corners".

The stabilised space of pseudoisotopies is $\mathbb{P} := \mathrm{hocolim}_{k\in\mathbb{N}} P(M \times (D^1)^k)$ with the transfers of the trivial bundles $M \times (D^1)^{k+j} \to M \times (D^1)^k$ as structure maps. This definition concludes Section 1.1.

Let us sketch how to pass from a continuous map $f\colon M \to N$ to a zig-zag of a vectorbundle and a codimension zero embedding. For some $k >> 0$ we have a homotopy $\mathrm{incl} \circ f \simeq \iota\colon M \to N \times (D^1)^k$ to a smooth embedding ι. Let $\nu\iota$ denote a tubular neighbourhood of ι. We obtain $M \leftarrow \nu\iota \to N \times (D^1)^k$, where the first map is a disk bundle and the second map is a codimension zero embedding. Section 1.2 is devoted to passing, in a coherent fashion, from continuous maps to such decompositions.

The last step is to pass from the zig-zags to induced maps between stable pseudoisotopy spaces. We divide it into two parts. In Section 1.3 we introduce most constructions and proof several technical results, while Section 1.4 contains the construction of the homotopy coherent diagram.

The main task is to relate a composition of transfers $p_{k!} \circ \ldots \circ p_{1!}(F)$ to the transfer of the composed bundle $(p_1 \circ \ldots \circ p_k)_!(F)$ for a sequence of vectorbundles p_i. Essentially, we compare the subsets of every fibre on which these maps boil down to applying $F(-, t)$.

We can understand these "level sets" as a tubular neighbourhood of a fixed level set corresponding to, say $t = 1/2$. The level sets $t = 1/2$ for different compositions of transfers turn out to be isotopic up to higher coherences and all other data can be carried along the isotopy via parallel transport.

The generalisation to spectra is fairly easy, if one is a bit careful in the construction of homotopy coherent diagrams for the various levels of the spectrum.

Reader's guide

Section 1.2 only requires the basic notions of manifolds with corners, introduced right at the beginning of Section 1.1.

Section 1.3 is devoted to the main geometric content of the proof. We use most of Section 1.1. As this part's purpose lies in the applications in Section 1.4, the first-time reader might wish to skip ahead.

Section 1.4 uses most of the definitions and results from Section 1.1, some definitions introduced at the beginning of Section 1.2, the detailed construction of homotopy coherent diagrams in Section 1.2.1 and the main results of Section 1.3.

Throughout this chapter we are going to choose quite a few parameters denoted as δ with various decorations. An overview with further references can be found in Section 1.4.3.

1.1 The Space of Stable Pseudoisotopies

Our objects of interest are certain diffeomorphisms of smooth, compact manifolds with corners. Let $0 \leq k \leq n$ be natural numbers. Let $\mathbb{H}^k \subseteq \mathbb{R}^k$ be the subspace of $\mathbb{R}^k$ consisting of those vectors $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ fulfilling $x_i \geq 0$ for $1 \leq i \leq k$. A *smooth manifold with corners* M is a paracompact Hausdorff space with an atlas containing charts $\alpha\colon \mathbb{H}^k \times \mathbb{R}^{n-k} \to U$ (where k may vary for a fixed manifold), such that we may extend each change of coordinates $\beta^{-1} \circ \alpha\colon \alpha^{-1}(U \cap V) \to \beta^{-1}(U \cap V)$ to a diffeomorphism on some open neighbourhoods of $\alpha^{-1}(U \cap V)$ and $\beta^{-1}(U \cap V)$ in $\mathbb{R}^n$. Here U and V denote open subsets of M and $\beta\colon \mathbb{H}^l \times \mathbb{R}^{n-l} \to V$ is another chart.

Given a chart $\alpha\colon \mathbb{H}^k \times \mathbb{R}^{n-k} \to U$, the point $\alpha(0)$ is a *corner of degree k*. Manifolds with boundary are the same as manifolds with corners of degree at most 1. A *higher corner* is a corner of degree two or more.

The diffeomorphisms under consideration are going to restrict to the Identity on an open neighbourhood of the boundary.

Before we turn to pseudoisotopies, we introduce some language for manifolds with corners. Note that even the notion of a smooth map between manifolds with corners is not uniform within the literature. For our purposes, the general idea is to define all notions by requiring extensions from $\mathbb{H}^k \times \mathbb{R}^{m-k}$ to $\mathbb{R}^m$, where we may then use the usual definitions.

Definition 1.1.0.1. A *smooth map between manifolds with corners* is a continuous map $f\colon M \to N$, such that for every point $x \in M$ there are charts $\alpha\colon \mathbb{H}^k \times \mathbb{R}^{m-k} \to V_x$ and $\beta\colon \mathbb{H}^l \times \mathbb{R}^{n-l} \to U_{f(x)}$, such that

$$\begin{array}{ccccccc}
\mathbb{H}^k \times \mathbb{R}^{m-k} & \xrightarrow{\alpha} & V_x & \xrightarrow{f} & U_{f(x)} & \xleftarrow{\beta} & \mathbb{H}^l \times \mathbb{R}^{n-l} \\
\big\downarrow{\scriptstyle \mathrm{incl}} & & & & & & \big\downarrow{\scriptstyle \mathrm{incl}} \\
\mathbb{R}^m & & & \xrightarrow{\quad\tilde{f}\quad} & & & \mathbb{R}^n
\end{array}$$

commutes, where $\tilde{f}\colon \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map.

Definition 1.1.0.2. An *embedding of a manifold with corners* is a continuous and injective map $\iota\colon M \hookrightarrow N$, such that for every point $x \in M$ there are charts $\alpha\colon \mathbb{H}^k \times \mathbb{R}^{m-k} \to V_x$ and $\beta\colon \mathbb{H}^l \times \mathbb{R}^{n-l} \to U_{\iota(x)}$, such that

$$\begin{array}{ccccccc}
\mathbb{H}^k \times \mathbb{R}^{m-k} & \xrightarrow{\alpha} & V_x & \xrightarrow{\iota} & U_{\iota(x)} & \xleftarrow{\beta} & \mathbb{H}^l \times \mathbb{R}^{n-l} \\
\big\downarrow{\scriptstyle \mathrm{incl}} & & & & & & \big\downarrow{\scriptstyle \mathrm{incl}} \\
\mathbb{R}^k \times \mathbb{R}^{m-k} & & & \xrightarrow{\quad\tilde{\iota}\quad} & & & \mathbb{R}^l \times \mathbb{R}^{n-l}
\end{array}$$

commutes, where $\tilde{\iota}\colon \mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$ is the embedding in the first m coordinates and we have $0 \leq l \leq k \leq m \leq n$.

Definition 1.1.0.3. A *submanifold with corners* is the image of an embedding of a manifold with corners. Naturally, the embedding is not part of the data.

Definition 1.1.0.4. The *tangent space of an embedded manifold with corners* $\iota\colon M \hookrightarrow \mathbb{R}^n$ at a point $x \in M$ is given as follows.

For every $x \in M - \partial M$ it is simply $\iota_*(T_x(M - \partial M))$. For $x \in \partial M$, choose a path $\gamma\colon [0,1] \to M$ with $\gamma([0,1)) \subseteq M - \partial M$ and $\gamma(1) = x$. Further, choose a basis B of $\iota_*(T_{\gamma(0)}(M - \partial M))$. Then $T_xM \subseteq T_{\iota(x)}\mathbb{R}^n$ is the subvectorspace spanned by the image of B under the parallel transport along γ with respect to the standard Riemannian metric on $\mathbb{R}^n$.

These assemble into a tangent bundle $\tilde{p}\colon TM \to M$, which is a subbundle of $T\mathbb{R}^n$, since the transition functions restrict appropriately. Finally, a *Riemannian metric* $R \in \mathrm{Riem}(M)$ on M is a smooth map $R\colon TM \otimes TM \to \mathbb{R}$ which admits an extension to a Riemannian metric on an open neighbourhood of $\iota(M)$.

If not explicitly stated otherwise, "manifold" refers to a smooth, compact submanifold of $\mathbb{R}^\infty$ with corners.

Now, we begin the discussion of pseudoisotopies. Let $I = [0,1]$ denote the standard interval. A *pseudoisotopy* of a manifold M relative boundary of M is a diffeomorphism $F\colon M \times I \to M \times I$, such that it restricts to the Identity on $M \times \{0\} \cup \partial M \times I$. The pseudoisotopies relative boundary form a subspace $P_\partial(M)$ of $C^\infty(M \times I, M \times I)$ equipped with the Whitney C^∞–topology. Igusa [30, Proposition 1.3] has shown that the space of pseudoisotopies is weakly homotopy equivalent to the following subspace of diffeomorphisms of $M \times I$.

Definition 1.1.0.5. The *unstable pseudoisotopy space* $P(M)$ is the subspace of $C^\infty(M \times I, M \times I)$ which contains the diffeomorphisms $F\colon M \times I \to M \times I$ that satisfy:

1. There is an open neighbourhood U of $M \times \{0\} \cup \partial M \times I$ in $M \times I$ such that $F|_U = \mathrm{Id}_U$.
2. There is an open neighbourhood V of $M \times \{1\}$ in $M \times I$ and a diffeomorphism $g\colon M \to M$ such that $F|_V = g \times \mathrm{Id}_I$.

We also need the simplicial set obtained by applying the singular simplicial sets functor $\mathcal{S}$ to this space. We refer to it by $\mathcal{S}_\bullet P(M)$.

Our interest lies in the stable pseudoisotopy space. As stated in the introduction, the stabilisation maps are special cases of the transfers. Thus we first discuss the two basic types of induced maps between pseudoisotopy spaces. The first one can be found, for example, in [23] but its origins are unclear.

Definition 1.1.0.6. Let $i\colon M \to N$ be a smooth embedding of codimension zero. Then we obtain an induced continuous map

$$\begin{aligned} i_*\colon P(M) &\to P(N) \\ f &\mapsto i \circ f \circ i^{-1} \cup \mathrm{Id}\colon N \times I = (i(M) \cup (N - i(M))) \times I \to N \times I. \end{aligned}$$

Remark 1.1.0.7. *We only have to assume our manifolds to be compact because the above definition works for every codimension zero embedding with closed image. Every other construction works without the compactness assumption. This is important for the case of a spectrum valued functor.*

Remark 1.1.0.8. *This construction actually yields a continuous map*

$$\begin{aligned} P(M) \times \mathrm{Emb}^0(M,N) &\to P(N) \\ (F,i) &\mapsto i_*(F) \end{aligned}$$

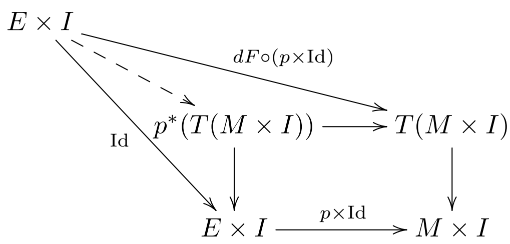
where all spaces carry the C^∞*–topology. Here* $\mathrm{Emb}^0(M,N)$ *denotes the space of smooth codimension zero embeddings.*

The description of the transfer is due to Burghelea and Lashof [6]. It is a little more involved, but still elementary.

Definition 1.1.0.9. Let M be a smooth manifold and $F \in P(M)$ be a pseudoisotopy on M. The differential of F gives a section $dF\colon M\times I \to T(M\times I)$ by $(x,t) \mapsto dF_{(x,t)}(e_I(t))$, where e_I is the canonical unit basis vectorfield $I \to TI$.

Let $p\colon E \to M$ be a smooth vector bundle over M together with a Riemannian metric on E. We restrict to the disk bundle $p\colon DE \to M$ and obtain a subbundle $\ker(dp) \subseteq TDE$. This bundle admits an orthogonal complement $\ker(dp)^\perp$ with respect to the Riemannian metric. Since dp is surjective, it induces an isomorphism $\ker(dp)^\perp \cong p^*TM$ and a split $TE \cong p^*TM \oplus \ker(dp)$.

We have a map



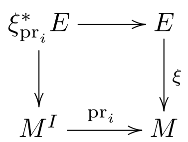
which, via the split $TE \cong p^*TM \oplus \ker(dp)$, yields a vector field on $E\times I$. We restrict to the disk bundle DE in E. The *transfer* $tr(F) \in C^\infty(DE\times I, DE\times I)$ of F along p is the unique solution of the differential equation determined by the vector field and the initial values $tr(F)|_{DE\times\{0\}} = \mathrm{Id}$.

Remark 1.1.0.10. *The transfer* $tr\colon P(M) \to C^\infty(DE\times I, DE\times I)$ *preserves the Identity and, by the chain rule, compositions. Therefore, it descends to a group homomorphism* $tr\colon P(M) \to \mathrm{Diff}(DE\times I)$*.*

In local coordinates, which respect $TDE \cong \ker(dp)^\perp \oplus \ker(dp)$*, the map* $tr(F)$ *is given by* $F\times \mathrm{Id}_{D^k}$ *where* k *is the dimension of the fibre of* p*.*

We briefly explain the relation to the geometric transfer in the topological category, constructed in [12], which also relies on [6]. This is going to be important once we compare the topological and smooth pseudoisotopy functors.

Remark 1.1.0.11. *Let* $\xi\colon E \to M$ *be a topological fibre bundle with compact fibre. Then we have pullbacks*



for $i = 0, 1$*. A parallel transport in the topological category is an isomorphism of fibre bundles* $\nu\colon \xi^*_{\mathrm{pr}_0}E \to \xi^*_{\mathrm{pr}_1}E$ *over* M^I *with* $\nu\circ s_0 = s_1$*, where* s_i *is the canonical section* $s_i\colon E \to \xi^*_{\mathrm{pr}_i}E$*.*

In the special case of a smooth vector bundle with a Riemannian metric on the total space, we obtain a parallel transport in the sense of Riemannian *geometry. This is an isomorphism* $\nu^{\text{smooth}}\colon \xi^*_{\text{pr}_0} E^{\text{smooth}} \to \xi^*_{\text{pr}_1} E^{\text{smooth}}$*, where* $\xi^*_{\text{pr}_i} E^{\text{smooth}} \subseteq \xi^*_{\text{pr}_i} E$ *is the subbundle over the subspace of smooth paths* $I \to M$*.*

Since the smooth paths are dense in the locally compact space M^I*, we obtain a unique continuous extension of* ν^{smooth} *to a parallel transport in the topological category.*

Now we finish the construction of the transfer in the topological category. Let ν' *be a parallel transport as above and* ν *the parallel transport of the bundle* $p \times \text{Id}_I$ *given by* $(\omega, (e,t)) \mapsto (\omega, \nu'(\text{pr}_M \circ \omega, e), \text{pr}_I \circ \omega(1))$*. For* $(m,t) \in M \times I$*, let* $\omega_{(m,t)}\colon [0,1] \to M \times I$ *be given by* $s \mapsto (m, ts)$*. Let* F *be a pseudoisotopy on* M*. The geometric transfer of* F *along* p *with respect to* ν' *is given by the formula* $\text{Tr}_{\nu'}(F)(e,t) = \text{pr}_{E\times I} \circ \nu(F \circ \omega_{(p(e),t)}, (e,0))$ *for* $(e,t) \in E \times I$*.*

The same construction makes sense in the smooth category, if we use a smooth pseudoisotopy F *and it yields the same map* $tr(F)$ *we defined in Definition 1.1.0.9.*

To sum up our discussion, given a smooth vector bundle $p\colon E \to M$ *with a Riemannian metric, there is a transfer map* $\text{Tr}_{\nu'}\colon P^{\text{Top}}(M) \to P^{\text{Top}}(E)$*, which restricts to the smooth transfer map* $tr\colon P^{\text{Diff}}(M) \to P^{\text{Diff}}(E)$*.*

It follows from Remark 1.1.0.10 above that $tr(F)$ fulfils the second condition of an element of $P(DE)$. However, it does not satisfy the first condition. The Jacobian of $tr(F)$ is the Identity in a neighbourhood of $p^{-1}(\partial M) \times I \cup DE \times \{0\}$, but the boundary of the disk bundle also contains the sphere subbundle of DE. For later reference we introduce a space of pseudoisotopies as the target of these transfer maps.

Definition 1.1.0.12. Let $P^\dagger(M) \subseteq C^\infty(M \times I, M \times I)$ be the subspace of diffeomorphisms $F\colon M \times I \to M \times I$, which fulfil

1. There is an open neighbourhood U of $M \times \{0\}$ such that $F|_U = \text{Id}_U$.
2. There is an open neighbourhood V of $M \times \{1\}$ and a diffeomorphism $g\colon M \to M$ such that $F|_V = g \times \text{Id}_I$.

In order to obtain an element of $P(DE)$, we are going to fibre-wise bend our map $tr(F)$. We are going to give a lot of details, because the construction of our homotopy coherent diagram very much depends on the precise notion of transfer we introduce.

1.1.1 Stereographic projections

We define an embedding, reminiscent of the stereographic projection. Its inverse is going to serve as the main tool to construct a coherent generalisation of the classical constructions of a bending map to higher dimensions.

Definition 1.1.1.1. Let $m \in \mathbb{N}$ be a natural number, let $1/4 > \tilde{\delta} > \delta > 0$ and let $\lambda\colon \mathbb{R} \to \mathbb{R}$ be given by $t \mapsto t(\delta - 3/4) + 3/4$. Note that $\lambda(0) = 3/4$ and $\lambda(1) = \delta$. Let e_{m+1} be the $(m+1)$–th unit vector. The *inverse of the parametrised stereographic projection* in m coordinates is

$$\begin{aligned}\text{St}^{-1}_{m,\delta,\tilde{\delta}}\colon \mathbb{R}^m \times I &\to (\mathbb{R}^m \times \mathbb{R}) - (\{0\} \times \mathbb{R}_{<1-\tilde{\delta}}) \\ (x,t) &\mapsto \lambda(t)\left(\frac{2x}{\|x\|^2+1}, \frac{\|x\|^2-1}{\|x\|^2+1}\right) + (1-\tilde{\delta})e_{m+1}.\end{aligned}$$

We set $s' := s-(1-\tilde{\delta})$ for $s \in \mathbb{R}$. The *stereographic projection* is an appropriate restriction of

$$\begin{aligned}\mathrm{St}_{m,\delta,\tilde{\delta}}\colon (\mathbb{R}^m \times \mathbb{R}) - (\{0\} \times \mathbb{R}_{<1-\tilde{\delta}}) &\to \mathbb{R}^m \times I \\ (y,s) &\mapsto \left(\frac{y}{\|(y,s')\| - s'}, \lambda^{-1}(\|(y,s')\|) \right).\end{aligned}$$

We usually suppress the δ–indices of the stereographic projection to improve readability. We mostly work with the inverse and refer to it as the stereographic inverse.

We are interested in the restriction of St_m^{-1} to $D^m \times I$ which parametrises a tubular neighbourhood of the lower half of the sphere. We give a sketch in the context of our application in Picture 1.1.1.11.

Given a decomposition $\mathbb{R}^m = \prod_{i=1}^k \mathbb{R}^{m_i}$ we define a parametrised stereographic inverse which only applies to some of the coordinates

$$\begin{aligned}\mathrm{St}_{m_j}^{-1}\colon \mathbb{R}^m \times I &\to \mathbb{R}^m \times \mathbb{R} \\ (x_{<m_j}, x_{m_j}, x_{>m_j}, t) &\mapsto (x_{<m_j}, \mathrm{pr}_{\mathbb{R}^{m_j}} \circ \mathrm{St}_{m_j}^{-1}(x_{m_j}, t), x_{>m_j}, \mathrm{pr}_{\mathbb{R}} \circ \mathrm{St}_{m_j}^{-1}(x_{m_j}, t)).\end{aligned}$$

We are going to define various extensions of the parametrised stereographic inverse, for each of which we implicitly assume a version which only refers to some coordinates.

Remark 1.1.1.2. *The choice of* $1/4$ *and* $3/4$ *is to ensure that compositions of parametrised stereographic inverses make sense on certain subspaces (namely, products of disks). There are, however, many other possibilities which would work just as well.*

Since the image of the stereographic inverse is contained in $\mathbb{R}^m \times I$*, we often refer to the last coordinate as* I *instead of* $\mathbb{R}_{>0}$*.*

Remark 1.1.1.3. *The choices of* δ *and* $\tilde{\delta}$ *induce a continuous map*

$$\begin{aligned}\{(\tilde{\delta},\delta) \in (0,1/4))^2 | \tilde{\delta} > \delta\} &\to C^\infty(\mathbb{R}^m \times I, (\mathbb{R}^m \times \mathbb{R}) - (\{0\} \times \mathbb{R}_{<1-\tilde{\delta}})) \\ (\delta,\tilde{\delta}) &\mapsto \mathrm{St}_{m,\delta,\tilde{\delta}}^{-1}\end{aligned}$$

with respect to the Whitney topology. The analogous statement holds for $\mathrm{St}_{m,\delta,\tilde{\delta}}$*.*

If we restrict the parametrised stereographic inverse to the unit disks $D^m \times I$ in $\mathbb{R}^m \times I$, the image is a family of lower hemispheres. We wish to extend them smoothly by a cylinder in the I–direction. To fit the pieces together we adjust the parametrised stereographic inverse near $S^m \times I$.

Definition 1.1.1.4. Let

$$\begin{aligned}\beta\colon \{(\bar{\delta},\delta^\dagger) \in (0,1/4)^2 | 2\delta^\dagger > \bar{\delta} > \delta^\dagger\} &\to C^\infty(\mathbb{R},[0,1]) \\ (\bar{\delta},\delta^\dagger) &\mapsto \beta_{\bar{\delta},\delta^\dagger}\end{aligned}$$

be a continuous map, such that

$$\beta_{\bar{\delta},\delta^\dagger}\colon \mathbb{R} \to [0,1]$$

satisfies the following properties for each pair $\bar{\delta},\delta^\dagger$.

1. $\beta_{\bar{\delta},\delta^\dagger}$ is smooth and monotonously increasing.
2. $\beta_{\bar{\delta},\delta^\dagger}(t) = 0$ for $t \leq 1 - 2\delta^\dagger - \bar{\delta}$
3. $\beta_{\bar{\delta},\delta^\dagger}(t) = 1$ for $t \geq 1 - \bar{\delta}$
4. $\lim_{\bar{\delta},\delta^\dagger \to 0} \sup_{t\in\mathbb{R}} \delta^{\dagger 2} \| \frac{d}{dt} \beta_{\bar{\delta},\delta^\dagger}(t) \| = 0$

Lemma 1.1.1.5. *There is a map β as defined above.*

Proof. Let $1/4 > \bar{\delta} > \delta^\dagger > 0$. We define the smooth map

$$\begin{aligned} \phi \colon \mathbb{R} \to & [0,1] \\ x \mapsto & \begin{cases} 0 \text{ for } \|x\| \geq 1 \\ exp(-\frac{1}{1-x^2}) \text{ for } \|x\| < 1. \end{cases} \end{aligned}$$

The map

$$\begin{aligned} \beta_{\bar{\delta},\delta^\dagger} \colon \mathbb{R} \to & [0,1] \\ t \mapsto & \frac{1}{\int_{-\infty}^{\infty} \phi(\frac{x}{\delta^\dagger}) dx} \int_{-\infty}^{t-1+\bar{\delta}+\delta^\dagger} \phi(\frac{x}{\delta^\dagger}) dx \end{aligned}$$

has the desired properties. We only give an argument for the fourth one.

Let $c_{\delta^\dagger} := \int_{-\infty}^{\infty} \phi(\frac{x}{\delta^\dagger}) dx$. We substitute via $\rho \colon [-\delta^\dagger, \delta^\dagger] \to [-1,1]$, $s \mapsto s/\delta^\dagger$ and obtain

$$\begin{aligned} \frac{1}{\delta^\dagger} c_{\delta^\dagger} =& \frac{1}{\delta^\dagger} \int_{-\infty}^{\infty} \phi(\frac{x}{\delta^\dagger}) dx \\ =& \int_{-\delta^\dagger}^{\delta^\dagger} exp(-\frac{1}{1-(\frac{s}{\delta^\dagger})^2}) \frac{1}{\delta^\dagger} ds = \int_{-1}^{1} exp(-\frac{1}{1-x^2}) dx = c_1 \end{aligned}$$

Now we may calculate

$$\sup_t \| \frac{d}{dt} \beta_{\bar{\delta},\delta^\dagger}(t) \| = \sup_t \frac{1}{c_{\delta^\dagger}} \phi(\frac{t}{\delta^\dagger}) = \frac{1}{c_{\delta^\dagger}} = \frac{1}{\delta^\dagger} \frac{1}{c_1}.$$

and the result follows. □

As with the stereographic projection St we abuse notation and suppress the variables $\bar{\delta}$ and $\delta^\dagger$ when we refer to some $\beta_{\bar{\delta},\delta^\dagger}$.

Definition 1.1.1.6. The *extendable parametrised stereographic inverse* is the map

$$\begin{aligned} \underline{\mathrm{St}}_m^{-1} \colon \mathbb{R}^m \times I \to & (\mathbb{R}^m \times \mathbb{R}) - (\{0\} \times \mathbb{R}_{<1-\tilde{\delta}}) \\ (x,t) \mapsto & \left(x \left(\beta(\|x\|) \frac{1}{\|x\|} + (1 - \beta(\|x\|)) \frac{2}{\|x\|^2+1} \right), \frac{\|x\|^2 - 1}{\|x\|^2+1} \right) \lambda(t) \\ & + (1-\tilde{\delta}) e_{m+1}. \end{aligned}$$

We show below that for every sufficiently small choice of $\bar{\delta}$ and $\delta^\dagger$ the restriction to $D^m \times I$ is still a smooth embedding.

We abuse notation and set $\beta(\|x\|)(1/\|x\|) = 0$ for $\|x\| = 0$. Since this is the unique continuous extension of the given formula to $\{0\} \times I$, this should not be a source of confusion.

Remark 1.1.1.7. *The map*

$$\{(\bar\delta,\tilde\delta,\delta,\delta^\dagger)\in(0,1/4)^4 | 2\delta^\dagger>\bar\delta>\tilde\delta>\delta>\delta^\dagger\}\to C^\infty(\mathbb{R}^m\times I,\mathbb{R}^m\times\mathbb{R})$$
$$(\bar\delta,\tilde\delta,\delta,\delta^\dagger)\mapsto \underline{\mathrm{St}}^{-1}_{m,\bar\delta,\tilde\delta,\delta,\delta^\dagger}$$

is continuous.

Lemma 1.1.1.8. *For sufficiently small choices of $\bar\delta$ and $\delta^\dagger$, there is an open neighbourhood U of $D^m\times I$, such that the restriction of $\underline{\mathrm{St}}_m^{-1}$ to U is a smooth embedding.*

Proof. We first show that $\underline{\mathrm{St}}_m^{-1}$ is an immersion for sufficiently small choices of $\bar\delta$ and $\delta^\dagger$. Then we show that the local embeddings assemble into a global one.

There is a neighbourhood of the origin where $\underline{\mathrm{St}}_m^{-1}$ and St_m^{-1} coincide. Away from $\{0\}\times I$ we parametrise the source of our map $\mathbb{R}^m\times I$ as $\mathbb{R}_{>0}\times S^{m-1}\times I$. Then the stereographic inverses are maps of the form

$$(r,v_1,\dots,v_m,t)\mapsto(\lambda(t)\mu(r)r,v_1,\dots,v_m,\nu(r)\lambda(t)+1-\tilde\delta)$$

with, in the case of $\underline{\mathrm{St}}_m^{-1}$, smooth maps

$$\tilde\mu(r)=\beta_{\bar\delta,\delta^\dagger}(r)\frac{1}{r}+(1-\beta_{\bar\delta,\delta^\dagger}(r))\frac{2}{r^2+1}$$

and

$$\nu(r)=\frac{r^2-1}{r^2+1}.$$

For St_m^{-1}, we have

$$\mu(r)=\frac{2}{r^2+1}$$

and the same formula for $\nu(r)$ as above.

From this description a straightforward computation shows that

$$|\det(d\underline{\mathrm{St}}^{-1}_{m,\bar\delta,\delta^\dagger})-\det(d\,\mathrm{St}_m^{-1})|\le c\cdot\sup_r(\tilde\mu_{\bar\delta,\delta^\dagger}(r)-\mu(r),\frac{d}{dr}\tilde\mu_{\bar\delta,\delta^\dagger}(r)-\frac{d}{dr}\mu(r))$$

for some constant $c>0$ which is independent of $r,v_1,\dots,v_m$ and t, as well as $\bar\delta,\delta^\dagger$ and $\tilde\delta$. It does depend on δ, though.

This estimate implies that

$$\lim_{\bar\delta,\delta^\dagger\to0}|\det(d\underline{\mathrm{St}}_{m,\bar\delta^{-1},\delta^\dagger})-\det(d\,\mathrm{St}_m^{-1})|=0$$

holds. We indicate the main step. Since $(\bar\delta,\delta^\dagger)\to\beta_{\bar\delta,\delta^\dagger}$ is a C^∞–continuous map, it is enough to show that

$$\frac{d}{dr}\beta_{\bar\delta,\delta^\dagger}(r)\left\|\frac{2}{r^2+1}-\frac{1}{r}\right\|=d\beta_{\bar\delta,\delta^\dagger}(r)\frac{(r-1)^2}{(r^2+1)r}+\beta_{\bar\delta,\delta^\dagger}(r)\frac{(r^2-1)^2}{(r^2+1)^2r}$$

converges to zero for $\bar\delta$ and $\delta^\dagger$ going to zero. Note that $\underline{\mathrm{St}}_m$ and St_m agree for $r\le1-2\delta^\dagger-\bar\delta$. Thus, since $\beta_{\bar\delta,\delta^\dagger}$ is bounded, the second summand converges to zero.

Further, as $2\delta^\dagger > \bar{\delta}$ they in particular agree for $r \leq 1 - 4\delta^\dagger$. For the first summand we set $\epsilon = 1 - r$ and obtain

$$\left\| d\beta_{\bar{\delta},\delta^\dagger}(r)\frac{(r-1)^2}{(r^2+1)r} \right\| = \left\| d\beta_{\bar{\delta},\delta^\dagger}(r)\epsilon^2 \frac{1}{2 - 4\epsilon + 3\epsilon^2 - \epsilon^3} \right\| \leq c'|d\beta_{\bar{\delta},\delta^\dagger}(r)\epsilon^2|$$

where $c' > 0$ denotes some constant. The fourth property of $\beta_{\bar{\delta},\delta^\dagger}$ ensures that this expression converges to zero, as desired.

We have learned that the differential of $\underline{\mathrm{St}}_m^{-1}$ is invertible in all points of $D^m \times I$ for sufficiently small choices of $\bar{\delta}$ and $\delta^\dagger$. Hence the map $\underline{\mathrm{St}}_m^{-1}$ is an immersion by invariance of domain.

It is a straightforward calculation to check that our map is injective. □

Now we are finally in position to properly describe the 'fibre-wise bending'.

Definition 1.1.1.9. Let $p\colon E \to M$ be an m–dimensional vector bundle together with a Riemannian metric on E. Then we define the *fibre-wise extendable parametrised stereographic inverse* $\underline{\mathbf{St}}_m^{-1}\colon DE \times I \to DE \times I$ by the formula

$$\begin{aligned}\underline{\mathbf{St}}_m^{-1}(e,t) := &\left(e\left(\beta(\|e\|)\frac{1}{\|e\|} + (1-\beta(\|e\|))\frac{2}{\|e\|^2+1}\right), \frac{\|e\|^2-1}{\|e\|^2+1}\right)\lambda(t) \\ &+ (1-\tilde{\delta})e_{m+1}\end{aligned}$$

where $\|e\|$ denotes the norm of e and e_{m+1} is again the unit vector in the I–coordinate. Similarly, we obtain the *fibre-wise parametrised stereographic inverse* $\mathbf{St}_m^{-1}\colon DE \times I \to DE \times I$.

Corollary 1.1.1.10. *The fibre-wise stereographic inverse is a smooth embedding* $\underline{\mathbf{St}}_m^{-1}\colon DE \times I \to DE \times I$ *and the map*

$$\begin{aligned}\{(\bar{\delta},\tilde{\delta},\delta,\delta^\dagger) \in (0,1/4)^4 | 2\delta^\dagger > \bar{\delta} > \tilde{\delta} > \delta > \delta^\dagger\} &\to C_p^\infty(DE \times I, DE \times I) \\ (\bar{\delta},\tilde{\delta},\delta,\delta^\dagger) &\mapsto \underline{\mathbf{St}}_{m,\bar{\delta},\tilde{\delta},\delta,\delta^\dagger}^{-1}\end{aligned}$$

is continuous where $C_p^\infty(DE \times I, DE \times I) \subseteq C^\infty(DE \times I, DE \times I)$ *denotes the subspace of fibre preserving smooth maps.*

Proof. It is enough to check this property on a covering $\{U_i\}_{i\in I}$ by local trivialisations since the restrictions to local neighbourhoods yield a continuous, open and injective map $C_p^\infty(DE \times I, DE \times I) \to \prod_{i\in I} C_p^\infty(p^{-1}(U_i) \times I, p^{-1}(U_i) \times I)$.

There is a covering by local trivialisations $\phi\colon p^{-1}(U) \xrightarrow{\cong} U \times F$ for p, such that we obtain a commutative square

$$\begin{array}{ccc} p^{-1}(U) \times I & \xrightarrow{\underline{\mathbf{St}}_m^{-1}} & p^{-1}(U) \times I \\ \downarrow \phi & & \downarrow \phi \\ U \times F \times I & \xrightarrow{\mathrm{Id} \times \underline{\mathrm{St}}_m^{-1}} & U \times F \times I \end{array}$$

where $\underline{\mathrm{St}}_m^{-1}$ denotes the extendable parametrised stereographic inverse. The result now follows from Lemma 1.1.1.8 and Remark 1.1.1.7. □

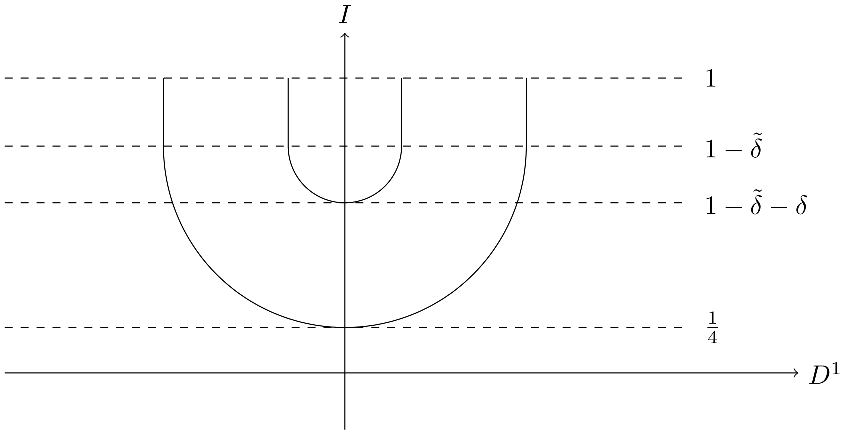
Next we describe the bending of a transferred pseudoisotopy. Each fibre Φ of $p\colon E \to M$ is an inner product space. Hence so is $\Phi \times I$ with the usual norm on I. Let $x = (x_\Phi, x_I) \in \Phi \times I$. We denote by $B_r(f)$ the closed ball of radius r around f in $\Phi = \Phi \times \{1\} \subseteq \Phi \times I$. Similarly $\mathring{B}_r(f)$ is the open ball. Further, $B_r^-(f,t)$ is the set of all $x = (x_\Phi, x_I) \in \Phi \times I$ with $\|(f,t) - x\| \leq r$ and $x_I \leq t$, i.e. the lower half of the ball and we write $\mathring{B}_r^-(f,t)$ for the lower half of the open ball.

On each fibre, we obtain a decomposition

$$\begin{aligned} F \times I = & B_\delta^-(0, 1-\tilde{\delta}) \\ & \cup B_\delta(0) \times [1-\tilde{\delta}, 1] \\ & \cup B_{3/4}^-(0, 1-\tilde{\delta}) - \mathring{B}_\delta^-(0, 1-\tilde{\delta}) \\ & \cup (B_{3/4}(0) - \mathring{B}_\delta(0)) \times [1-\tilde{\delta}, 1] \\ & \cup \Phi \times I - (B_{3/4}^-(0, 1-\tilde{\delta}) \cup B_{3/4}(0) \times [1-\tilde{\delta}, 1]) \end{aligned}$$

into the lower half of the ball of radius δ around $(0, 1-\tilde{\delta}) \in \Phi \times I$, the cylinder above its boundary, the image of $\mathrm{St}_m(D\Phi \times I) = B_{3/4}^-(0, 1-\tilde{\delta}) - \mathring{B}_\delta^-(0, 1-\tilde{\delta})$, its cylinder $\mathrm{cyl}(\mathrm{St}_m^{-1}(DF \times I))$ and the complement of all these which we denote by Φ^c.

Picture 1.1.1.11. *We sketch the decomposition of a single fibre for $m = 1$ in local coordinates:*



Since our inner product stems from a Riemannian metric, these fibre-wise decompositions assemble into a decomposition of the whole bundle

$$\begin{aligned} E \times I = & E_\delta^-(0, 1-\tilde{\delta}) \\ & \cup E_\delta(0) \times [1-\tilde{\delta}, 1] \\ & \cup E_{3/4}^-(0, 1-\tilde{\delta}) - \mathring{E}_\delta^-(0, 1-\tilde{\delta}) \\ & \cup (E_{3/4}(0) - \mathring{E}_\delta(0)) \times [1-\tilde{\delta}, 1] \\ & \cup E \times I - (E_{3/4}^-(0, 1-\tilde{\delta}) \cup E_{3/4}(0) \times [1-\tilde{\delta}, 1]) \end{aligned}$$

where a decorated E denotes the bundle which is fibre-wise given by 'B' with the same decorations. Also, we denote the last component, complement of the other four, by E^c.

Given a pseudoisotopy $F \in P(M)$, we now define a diffeomorphism on each of these components separately. It follows from the definitions that they assemble into a diffeomorphism on all of $E \times I$.

Definition 1.1.1.12. Let $F \in P(M)$ be a pseudoisotopy. We have a diffeomorphism

$$\begin{aligned} E_\delta((0, 1-\tilde{\delta})) &\to E_\delta((0, 1-\tilde{\delta})) \\ (e,t) &\mapsto (\mathrm{pr}_E \circ tr(F)(e,1), t) \end{aligned}$$

given by $g \times \mathrm{Id}_{D^m} \times \mathrm{Id}_I$ in local coordinates. Similarly, we obtain

$$\begin{aligned} B_\delta(0) \times [1-\tilde{\delta}, 1] &\to B_\delta(0) \times [1-\tilde{\delta}, 1] \\ (e,t) &\mapsto (\mathrm{pr}_E \circ tr(F)(e,1), t). \end{aligned}$$

Next is the most interesting part

$$\begin{aligned} \mathbf{St}_m^{-1}(DF \times I) &\to \mathbf{St}_m^{-1}(DF \times I) \\ (e,t) &\mapsto \mathbf{St}_m^{-1} \circ tr(F) \circ \mathbf{St}_m(e,t) \end{aligned}$$

where the full pseudoisotopy is employed. On the cylinder we have

$$\begin{aligned} \mathrm{cyl}(\mathbf{St}_m^{-1}(DF \times I)) &\to \mathrm{cyl}(\mathbf{St}_m^{-1}(DF \times I)) \\ (e,t) &\mapsto (\mathrm{pr}_E \circ \mathbf{St}_m^{-1} \circ tr(F) \circ \mathbf{St}_m(e, 1-\tilde{\delta}), t) \end{aligned}$$

which copies the behaviour on height $1-\tilde{\delta}$ to $[1-\tilde{\delta}, 1]$. Finally we use

$$\mathrm{Id}\colon E^c \to E^c.$$

On each of these areas, the map is smooth. It is straightforward to check that applying the same procedure to F^{-1} yields a set-theoretic inverse. So our newly constructed map is a bijection. However, it is in general only smooth away from $(E_{3/4}(0) - \mathring{E}_\delta(0)) \times \{1-\tilde{\delta}\}$. To remedy this issue we replace $\mathbf{St}_m^{-1}$ by $\underline{\mathbf{St}}_m^{-1}$ and use the analogous construction.

We denote by $V = \mathrm{pr}_E(\mathrm{im}(\underline{\mathbf{St}}_m^{-1}) \cap (E \times \{(1-\tilde{\delta})\}))$ the analogue of the non-smooth area. The set

$$(F \times I) - ((F \times I) \cap (\mathrm{im}(\underline{\mathbf{St}}_m^{-1}) \cup (V \times [1-\tilde{\delta}, 1])))$$

consists of two connected components, corresponding to the areas where $g \times \mathrm{Id}$ and Id were applied. Let us describe them in a little more detail.

Definition 1.1.1.13. Recall from Definition 1.1.1.1 that $\lambda(1) = \delta$. Consider the curve

$$\begin{aligned} \gamma\colon [0,1] &\to D^{m+1} \subseteq D^1 \times D^{m-1} \times I \\ s &\mapsto \left((1-\beta(s)) \frac{2s}{\|s\|^2+1} + \beta(s) \frac{s}{\|s\|}, 0, \frac{\|s\|^2-1}{\|s\|^2+1} \right) \delta + (1-\tilde{\delta}) e_{m+1}. \end{aligned}$$

It is the image of $D^1 \times \{0\} \times \{1\} \subseteq D^m \times I$ under $\underline{St}_m^{-1}$. The orthogonal group $O(m)$ acts on $D^m \times I$ via $A.(x,t) := (Ax, t)$. Let $O(m)\,\mathrm{im}(\gamma)$ denote the orbit of $\mathrm{im}(\gamma)$ under this action. Since $\underline{St}_m^{-1}$ is an $O(m)$–equivariant map, this orbit is the image of $D^m \times \{1\}$ under $\underline{St}_m^{-1}$.

The space $O(m)\,\mathrm{im}(\gamma) \cup ((B_\delta(0)) \times \{1-\tilde{\delta}\})$ bounds a codimension zero submanifold (with corners) of $D^m \times I$: The submanifold consists of all points which lie on a line from $(0, 1-\tilde{\delta})$ to a point on the boundary. We call this submanifold T.

To see that T is indeed a submanifold, we first note that the smooth embedding

$$\begin{aligned} D^m \times (0,1] &\to D^{m+1} \\ (x,r) &\mapsto r(\underline{\mathrm{St}}_m^{-1}(x,1) - (1-\tilde{\delta})e_{m+1}) + (1-\tilde{\delta})e_{m+1} \end{aligned}$$

yields local coordinates away from $(0, 1-\tilde{\delta}) \in T$. Since there is some $\epsilon > 0$ with $B_\epsilon^-(0, 1-\tilde{\delta}) \cap T = B_\epsilon^-(0, 1-\tilde{\delta})$ we also obtain local coordinates around the single point left. We do not make use of the fact that T is a submanifold before Section 1.3.

The connected component containing $(0,1) \in F \times I$ is T.

Definition 1.1.1.14. Let $3/4 > \delta > 0$. We set

$$\begin{aligned} \gamma\colon [0,1] \times [0, 3/(3-4\delta)] &\to D^{m_j+1} = D^1 \times D^{m_j-1} \times I \\ (s,t) &\mapsto \left((1-\beta(s))\frac{2s}{\|s\|^2+1} + \beta(s)\frac{s}{\|s\|}, 0, \frac{\|s\|^1 - 1}{\|s\|^2+1} \right) \lambda(t) \\ &\quad + (1-\tilde{\delta})e_{m+1}. \end{aligned}$$

The non-compact component of $\mathbb{R}^m \times I - O(m)\gamma([0,1] \times \{0\})$ is the other connected component.

Definition 1.1.1.15. Let $p\colon E \to M$ be an m–dimensional vector bundle together with a Riemannian metric on E. Then the *transfer map* is

$$\begin{aligned} p_!\colon P(M) &\to P(DE) \\ F \mapsto p_!(F) &= \begin{cases} (\mathrm{pr}_E \circ tr(F) \circ (\mathrm{pr}_E \times \{1\})) \times \mathrm{Id}_I \text{ on } T \\ \underline{\mathbf{St}}_m^{-1} \circ tr(F) \circ \underline{\mathbf{St}}_m \text{ on } \mathrm{im}(\underline{\mathbf{St}}_m^{-1}) \\ (\mathrm{pr}_E \circ \underline{\mathbf{St}}_m^{-1} \circ tr(F) \circ \underline{\mathbf{St}}_m \circ (\mathrm{pr}_E \times \{1-\tilde{\delta}\})) \times \mathrm{Id}_I \\ \quad \text{on } V \times [1-\tilde{\delta}, 1] \\ \mathrm{Id} \text{ on } D^m \times I - O(m)\gamma([0,1] \times [0, 3/(3-4\delta)]). \end{cases} \end{aligned}$$

In local coordinates $(\mathrm{pr}_E \circ tr(F) \circ (\mathrm{pr}_E \times \{1\})) \times \mathrm{Id}_I = g \times \mathrm{Id}_{D^k} \times \mathrm{Id}_I$ holds. In contrast to Definition 1.1.1.12 we describe the area where this map is employed as a single space.

Remark 1.1.1.16. *One can easily generalise this definition to the case where we only bend with respect to the coordinates of a subbundle in each fibre, because* $\mathrm{im}(\underline{\mathbf{St}}_{m_j}^{-1}) \cup (\mathrm{pr}_E(\mathrm{im}(\underline{\mathbf{St}}_m^{-1}) \cap (E \times \{(1-\tilde{\delta})\})) \times [1-\tilde{\delta}, 1])$ *divides each fibre into two connected components as well.*

We give a little more detail. Consider a vectorbundle E, a subvectorbundle $\hat{E} \subseteq E$ and a Riemannian metric on E. Then we obtain a split $\hat{E} \oplus \hat{E}^{\perp} \cong E$. Hence we may reduce the structure group of E to $O(\dim(\hat{E})) \times O(\dim(\hat{E}^{\perp}))$. The map $\underline{\mathbf{St}}^{-1}_{\dim(\hat{E})} \times \mathrm{Id}$ is $O(\dim(\hat{E})) \times O(\dim(\hat{E}^{\perp}))$–equivariant and thus invariant under a change of trivialisations. Thus we obtain a fibre-wise stereographic inverse with respect to the coordinates of the subbundle, say $\mathbb{R}^{m_j} \subseteq \mathbb{R}^m$ in local coordinates.

Now we proceed similar to the absolute case. The analogue of T is the submanifold $\tau(\mathbb{R}^{m^+_{j-1}} \times T_j \times \mathbb{R}^{m-m^+_j})$, where τ shuffles the I–coordinate to the end, while the analogue of the other component is given via the decomposition $\mathbb{R}^m \times I - \tau(\mathbb{R}^{m^+_{j-1}} \times O(m)\gamma([0,1] \times \{0\}) \times \mathbb{R}^{m-m^+_j})$.

Remark 1.1.1.17. *We define bending maps $bd\colon P^{\dagger}(E) \to P^{\dagger}(E)$ by replacing $tr(F)$ with $G \in P^{\dagger}(E)$ in the above definition.*

Remark 1.1.1.18. *Composition and Cartesian product with a smooth map induce continuous maps with respect to the Whitney C^{∞}–topology. Since tr is in local coordinates given by taking the Cartesian product with the Identity on the fibre, $p_!$ is thus a continuous map.*

Moreover, we obtain a continuous map

$$\{(\bar{\delta}, \tilde{\delta}, \delta, \delta^{\dagger}) \in (0,1/4)^4 | 2\delta^{\dagger} > \bar{\delta} > \tilde{\delta} > \delta > \delta^{\dagger}\} \times \mathrm{Riem}(E) \times P(M) \to P(DE)$$
$$((\bar{\delta}, \tilde{\delta}, \delta, \delta^{\dagger}), R, F) \mapsto (p_!)_{\bar{\delta}, \tilde{\delta}, \delta, \delta^{\dagger}, R}(F)$$

where $\mathrm{Riem}(E) \subseteq \mathrm{Hom}(TE \otimes TE, \mathbb{R})$ is the space of Riemannian metrics on the tangent space of E. In particular, such a metric induces one on E. It can be checked in local coordinates that the map is indeed continuous.

For later use we briefly discuss the compatibility of the transfer and bending operations. Here we need the Riemannian metric on TE.

Lemma 1.1.1.19. *Let $p_1\colon E_1 \to M$ and $p_2\colon E_2 \to E_1$ be two disk bundles and R_1, R_2 Riemannian metrics on E_1 and E_2, respectively. Then we obtain a diagram*

$$\begin{array}{ccccc}
P(M) & \xrightarrow{tr} & P^{\dagger}(E_1) & \xrightarrow{tr} & P^{\dagger}(E_2) \\
 & & \big\downarrow{\scriptstyle bd_1} & & \big\downarrow{\scriptstyle bd_1} \\
 & & P^{\dagger}(E_1) & \xrightarrow{tr} & P^{\dagger}(E_2) \\
 & & & & \big\downarrow{\scriptstyle bd_2} \\
 & & & & P(E_2)
\end{array}$$

of transfer and bending maps, where bd_1 and bd_2 refer to the bending operations with respect to the boundary originating from the fibre coordinates of E_1 and E_2, respectively. The square and hence the whole diagram commutes.

Proof. The structure group of the bundle $p_1 \circ p_2\colon E_2 \to M$ is $O(m_1) \times O(m_2)$. Let $F \in P^{\dagger}(E_1)$ be a pseudoisotopy of E_1. We compare $tr \circ bd_1(F)$ and $bd_1 \circ tr(F)$ in $P^{\dagger}(E_2)$.

In local coordinates both maps are given by $F(-,1)$ on the fibre-wise subspace $D^{m_1} \times D^{m_2} \times I - \tau(O(m_1)\gamma([0,1] \times [0, 3/(3-4\delta)]) \times D^{m_2})$ and the Identity on $\tau(T_1 \times D^{m_2})$.

These are the two connected components which form the complement to $\mathrm{im}(\underline{\mathbf{St}}_{m_1}^{-1}) \cup V \times [1-\tilde{\delta}, 1]$, the image of the parametrised stereographic inverse, with respect to the fibre coordinates of E_1, together with its cylinder. So we restrict to the subspace $\mathrm{im}(\underline{\mathbf{St}}_1^{-1})$ of E_1 on which bd_1 is given by conjugation with $\underline{\mathbf{St}}_1^{-1}$. Consequently, we henceforth consider $p_2^{-1}(\mathrm{im}(\underline{\mathbf{St}}_1^{-1}))$ instead of E_2. Also, we write $\underline{\mathbf{St}}_1^{-1}$ instead of $\underline{\mathbf{St}}_{m_1}^{-1}$ to ease up notation.

By definition $tr \circ bd_1(F)$ is the solution of a certain differential equation with initial values and the latter are fulfilled by any pseudoisotopy of E_2. Hence we study the differentials of our maps.

The tangent space of E_2 splits as $TE_2 \cong TE_1 \oplus \ker(dp_2)$. On our subspace the differential of $tr \circ bd_1(f)$ with respect to this split is, in local coordinates,

$$\begin{pmatrix} d(\underline{\mathrm{St}}_1 \circ F \circ (\underline{\mathrm{St}}_1)^{-1}) & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

by definition of the transfer. The parametrised stereographic inverse with respect to the fibre coordinates of E_1 on E_2 is in some local trivialisation given by $\underline{\mathrm{St}}_1 \times \mathrm{Id}$. So the differential of $bd_1 \circ tr(F)$ on the subspace $p_2^{-1}(\mathrm{im}(\underline{\mathbf{St}}_1))$ is

$$\begin{pmatrix} d(\underline{\mathrm{St}}_1) & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \circ \begin{pmatrix} dF & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \circ \begin{pmatrix} d(\underline{\mathrm{St}}_1)^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

and thus the differentials of the pseudoisotopies agree everywhere. As they also have the same initial values they have to coincide and the diagram commutes. □

With the transfer maps at hand, we are in position to define the spaces of stable pseudoisotopy. It is easy to check that our transfer, applied to the trivial 1–dimensional disk bundle, yields a map which represents the homotopy class of the stabilisation map used by Igusa to define the space of stable smooth pseudoisotopy [30, Conditions 1.3, p. 44].

Definition 1.1.1.20. The *stable pseudoisotopy space* of a smooth manifold M is

$$\mathbb{P}(M) := \mathrm{hocolim}_{k\in\mathbb{N}}\, P(M \times (D^1)^k)$$

with the transfer $\mathrm{pr}_!\colon P(M \times (D^1)^k) \to P(M \times (D^1)^{k+1})$ with respect to the trivial 1–dimensional disk bundle $\mathrm{pr}\colon M \times (D^1)^{k+1} \to M \times (D^1)^k$ as its k–th structure map.

1.2 From Continuous Maps to Smooth Embeddings

To define a homotopy coherent diagram we are going to give a parametrised version of the following construction. Let $f\colon M \to N$ be a continuous map between manifolds. We smoothly approximate f up to homotopy by some $\tilde{f}\colon M \to N$. Then we compose with an embedding $j\colon N \hookrightarrow N \times (D^1)^k$ for some large $k \in \mathbb{N}$. Now we find a smooth homotopy $j \circ \tilde{f} \simeq \iota$ to a smooth embedding $\iota\colon M \hookrightarrow N \times (D^1)^k$. We choose a tubular neighbourhood $p\colon \nu\iota \to \iota(M)$ via a Riemannian metric on N and obtain a zig-zag $M \leftarrow \nu\iota \hookrightarrow N \times (D^1)^k$ of a disk bundle, as the restriction of a vectorbundle, and a codimension zero embedding $i\colon \nu\iota \hookrightarrow N \times (D^1)^k$. We define $P(f)\colon P(M) \to P(N)$ as $i_* \circ p_!$.

This chapter takes care of every step but the passage from disk bundles and embeddings to transfers and induced maps We repeatedly employ the obstruction theory explained in Section 1.2.1. In particular, the results presented here are independent of our intended application to pseudoisotopies. We need some definitions to formulate the key result.

Definition 1.2.0.1. Let $(\mathrm{Mfd}, \mathrm{cts})_\Delta$ be the topologically enriched full subcategory of Top_Δ with smooth, compact manifolds with corners as objects.

The objects of the topologically enriched category $(\mathrm{Mfd}, \mathrm{smooth})_\Delta$ are compact smooth manifolds with corners and its mapping spaces are smooth maps equipped with the Whitney C^∞–topology.

Let $(\mathrm{Mfd}, \mathrm{emb})_\Delta$ be the topologically enriched category which has smooth manifolds with corners as objects and as mapping spaces the spaces of embeddings $\mathrm{Emb}(M,N) \subseteq C^\infty(M,N)$ equipped with the Whitney topology.

Definition 1.2.0.2. Let $\mathcal{D}_\Delta$ be the simplicially enriched category with manifolds with corners as objects and the following mapping spaces:

Let $M, N \in \mathcal{D}_\Delta$. An n–simplex in $\mathcal{D}_\Delta(M,N)$ is a 3–tuple $(\iota, \nu\iota, p)$, where

- $\iota\colon |\Delta^n| \to \mathrm{Emb}(M,N)$ is a C^∞–continuous family of embeddings of manifolds with corners.
- $\nu\iota \subseteq N \times |\Delta^n|$ is a submanifold and $p\colon \nu\iota \to M \times |\Delta^n|$ is a smooth map, such that $(\nu\iota \cap N \times \{t\}, p\colon \nu\iota \cap N \times \{t\} \to M)$ is an embedded tubular neighbourhood of $\iota(t)$ for each $t \in |\Delta^n|$.

The composition

$$(\iota_2\colon M_1 \hookrightarrow M_2, \nu M_1, p_2) \circ (\iota_1\colon M_0 \hookrightarrow M_1, \nu M_0, p_1)$$

is given by

$$(\iota_2 \circ \iota_1\colon M_0 \hookrightarrow M_2, p_2^*(\nu M_0), p_1 \circ p_2)$$

where $p_2^*(\nu M_0)$ is the disk subbundle of dimension $m_1 + m_2$ of the composed bundle $p_1 \circ p_2\colon p_2^{-1}(\nu M_0) \to M_0$ with fibre $D^{m_1} \times D^{m_2}$.

There is a projection functor $\mathrm{pr}\colon \mathcal{D}_\Delta \to (\mathrm{Mfd}, \mathrm{emb}) \subseteq (\mathrm{Mfd}, \mathrm{smooth})$, given by $(\iota, \nu\iota, p) \mapsto \iota$.

We define tubular neighbourhoods of manifolds with corners in Section 1.2.2 below.

In order to make room for the stabilisation along $M\times(D^1)^k \hookrightarrow M\times(D^1)^{k+1}$ in our quasicategories we rely on Ind–completion. A detailed explanation of Ind–objects in the framework of quasicategories is given in [33, Chapter 5.3.5]. We state all the properties necessary for our applications:

Proposition 1.2.0.3. *Let $\mathcal{C}$ be a small quasicategory. There is a small quasicategory $\mathrm{Ind}(\mathcal{C})$ with the following properties:*

1. *There is a Yoneda map $j\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$, which is a fully faithful embedding, see [33, Remark 5.3.5.2]. We often identify $\mathcal{C}$ with its image under j.*

2. *The quasicategory $\mathrm{Ind}(\mathcal{C})$ contains all countably filtered homotopy colimits, see [33, Proposition 5.3.5.3].*[1]

3. *The objects in the image of j are compact in $\mathrm{Ind}(\mathcal{C})$, see [33, Proposition 5.3.5.5].*

4. *The Ind–construction extends to an endofunctor of the quasicategory of small quasicategories $\mathrm{Ind}\colon \mathrm{qcat} \to \mathrm{qcat}$, see the argument following [33, Proposition 5.3.5.12].*

5. *The map $j^*\colon \mathrm{Map}_\omega(\mathrm{Ind}(\mathcal{C}), \mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta) \to \mathrm{Map}(\mathcal{C}, \mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta)$ is an equivalence, where the left hand side denotes the ∞–category of all countably continuous functors from $\mathrm{Ind}(\mathcal{C})$ to $\mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta$, see [33, Proposition 5.3.5.10]. Here Kan_Δ is the simplicially enriched category of Kan-complexes.*

Remark 1.2.0.4. *A reader not too keen on ∞–categorical techniques may use Ind–completions of simplicially enriched categories instead. Note that this leads to differences in technical details.*

Now we can state this section's main result, a "parametrised version" of the construction explained in the beginning of the chapter.

Theorem 1.2.0.5. *There is a map $\mathrm{F}_{\mathrm{ch}}\colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta \to \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta)$ with the following properties:*

1. *Let $i\colon (\mathrm{Mfd}, \mathrm{smooth})_\Delta \hookrightarrow (\mathrm{Mfd}, \mathrm{cts})_\Delta$ be the inclusion functor. There is a natural transformation of $(\infty,1)$–functors $\alpha\colon j \Rightarrow \mathrm{Ind}(\mathcal{N}^{h.c.}(i\circ \mathrm{pr}))\circ \mathrm{F}_{\mathrm{ch}}$.*

2. *Let $M \in (\mathrm{Mfd}, \mathrm{cts})_\Delta$. Then $\mathrm{F}_{\mathrm{ch}}(M) = \mathrm{hocolim}_{n\in\mathbb{N}}\, M\times(D^1)^n$ with structure maps $M\times(D^1)^n\times\{0\} \hookrightarrow M\times(D^1)^{n+1}$ (with some choice of a tubular neighbourhood). In particular, α is a point-wise homotopy equivalence.*

3. *The two properties have the following consequence: Let $f\colon M \to N$ be a continuous map and $i_M\colon M \to \mathrm{F}_{\mathrm{ch}}(M)$ the structure map of the homotopy colimit.*

   We have $\mathrm{Ind}(\mathcal{N}^{h.c.}(i\circ\mathrm{pr}))(\mathrm{F}_{\mathrm{ch}}(f))\circ i_M \simeq i_N\circ f$ in $\mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta)$.

[1] Note that Lurie refers to the cones in quasicategories which correspond to homotopy colimits in, for example, simplicially enriched categories, as colimits.

1.2.1 Strictification of homotopy coherent diagrams

In order to improve the constructions explained so far to an actual functor we rely on strictification results due to Cordier and Porter [10]. Their work is a variation of Vogt's [48] strictification of homotopy coherent diagrams of topological spaces. To apply these results in our case we have to recall some basic facts about quasicategories, which can be found in e.g. Lurie [33, Chapter 1.1.4 and Chapter 1.1.5].

In principle, some smallness issues are going to occur in this part. But in our applications we may consider the submanifolds of $\mathbb{R}^\infty$ to ensure everything stays in the realm of sets.

Let $\mathcal{A}$ be a small category. Let $\mathcal{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ denote the *nerve functor* and τ its left adjoint, the *fundamental category functor*. Let $\iota\colon \mathrm{Cat} \to \mathrm{Cat}_\Delta$ denote the functor which sends a small category $\mathcal{A}$ to the small simplicially enriched category with $\mathrm{ob}(\iota(\mathcal{A})) = \mathrm{ob}(\mathcal{A})$ and discrete mapping spaces $\iota(\mathcal{A})(a,b) = \mathcal{A}(a,b)$. Let $\mathfrak{S}\colon \mathrm{sSet} \to \mathrm{Cat}_\Delta$ denote the *rigidification functor* from simplicial sets into simplicially enriched categories and $\mathcal{N}^{h.c.}$ denote its right adjoint, the *simplicial nerve*. We write $\mathfrak{S}(\mathcal{A})$ for $\mathfrak{S}(\mathcal{N}_\bullet\mathcal{A})$.

The simplicial nerve restricts to a functor from Kan-enriched categories to quasicategories $\mathcal{N}^{h.c.}_\bullet\colon \mathrm{Cat}_{\mathrm{Kan}} \to \mathrm{Qcat}$. The simplicial nerve of a topologically enriched category is obtained by applying $\mathcal{N}^{h.c.}_\bullet \circ \mathcal{S}_*$. We typically suppress $\mathcal{S}_*$ in this context.

There is a natural isomorphism $\tau(\mathcal{N}^{h.c.}_\bullet\mathcal{A}) \to \mathcal{A}$ of categories. We obtain an isomorphism $\mathcal{N}^{h.c.}_\bullet\iota(\mathcal{A}) \to \mathcal{N}_\bullet\mathcal{A}$ via the $\tau - \mathcal{N}$–adjunction. Thus we obtain a natural transformation $\epsilon\colon \mathfrak{S}(\mathcal{N}_\bullet\mathcal{A}) \to \iota(\mathcal{A})$ by the composition

$$\mathfrak{S}(\mathcal{N}_\bullet\mathcal{A}) \xleftarrow{\cong} \mathfrak{S}(\mathcal{N}^{h.c.}_\bullet\iota(\mathcal{A})) \xrightarrow{\cong} \iota(\mathcal{A}), \qquad \epsilon\colon \mathfrak{S}(\mathcal{N}_\bullet\mathcal{A}) \to \iota(\mathcal{A})$$

with the latter map the counit of the $\mathfrak{S} - \mathcal{N}^{h.c.}$–adjunction. We identify $\mathcal{A}$ with $\iota(\mathcal{A})$ from here on out.

Before we proceed, let us recall the rigidification functor.

Definition 1.2.1.1. Let $i \leq j$ be natural numbers. The category $P_{i,j}$ is the partially ordered set which contains the subsets of $[i,j]_\mathbb{N} = \{i, i{+}1, \dots, j\}$ which contain i and j.

For $i \leq j \leq k$ we obtain a "union" functor $P_{i,j} \times P_{j,k} \to P_{i,k}$ given by $(U,V) \mapsto U \cup V$.

A non-decreasing morphism $\alpha\colon [i,j]_\mathbb{N} \to [\alpha(i),\alpha(j)]_\mathbb{N}$ induces a "change of sets" functor $P(\alpha)\colon P_{i,j} \to P_{\alpha(i),\alpha(j)}$ given by $U \mapsto \alpha(U)$.

Definition 1.2.1.2. The rigidification functor is given on the n–simplex Δ^n as follows:

- The objects of the simplicially enriched category $\mathfrak{S}(\Delta^n)$ are the objects of the category $[n]$.
- The morphism space $\mathfrak{S}(\Delta^n)(i,j)$ is $\mathcal{N}_\bullet P_{i,j}$ the nerve of the category $P_{i,j}$.
- The composition law $\mathfrak{S}(\Delta^n)(i,j) \times \mathfrak{S}(\Delta^n)(j,k) \to \mathfrak{S}(\Delta^n)(i,k)$ is induced by the union functor $P_{i,j} \times P_{j,k} \to P_{i,k}$.

A simplicial map $\alpha\colon \Delta^m \to \Delta^n$ between simplices is just an order-preserving map of ordered sets. We obtain an induced map via the change of sets functors.

Since every simplicial set is the colimit over its simplices and the rigidification functor preserves colimits (as it is a left adjoint), this is enough.

Let Kan_Δ be the *simplicially enriched category of Kan-complexes* and Kan its underlying category. Consider the homotopy category $Ho(\mathrm{Kan}^{\mathcal{A}})$, formed by localisation at the natural transformations which are point-wise homotopy equivalences. In the simplicially enriched category of simplicially enriched categories we obtain the mapping space $\mathrm{Cat}_\Delta(\mathfrak{S}(\mathcal{A}), \mathrm{Kan}_\Delta)$ and we denote its homotopy category by $\mathrm{Coh}(\mathcal{A}, \mathrm{Kan}_\Delta)$. The following result is due to Cordier and Porter [10, Theorem 4.7]. As mentioned in their article's introduction the result applies to the simplicially enriched category Kan_Δ.

Theorem 1.2.1.3. *The functor*

$$\begin{aligned}\gamma := \epsilon^*\colon\ \mathrm{Kan}^{\mathcal{A}} &\to \mathrm{Coh}(\mathcal{A}, \mathrm{Kan}_\Delta)\\ (X\colon \mathcal{A} \to \mathrm{Kan}) &\mapsto (X \circ \epsilon\colon \mathfrak{S}(\mathcal{A}) \to \mathrm{Kan})\\ (\alpha\colon \mathcal{A} \times [1] \to \mathrm{Kan}) &\mapsto [\alpha \circ \epsilon\colon \mathfrak{S}(\mathcal{A} \times [1]) \to \mathrm{Kan}]\end{aligned}$$

preserves weak equivalences and thus induces a functor

$$\gamma_*\colon\ \mathrm{ho}(\mathrm{Kan}^{\mathcal{A}}) \to \mathrm{Coh}(\mathcal{A}, \mathrm{Kan}_\Delta)$$

by the universal property of the localisation. Moreover, the induced functor γ_* is an equivalence of categories. The same holds if we replace Kan_Δ with any other simplicially enriched category which contains all homotopy limits.

Remark 1.2.1.4. *It follows from the proof by Cordier and Porter that an inverse equivalence is given by sending $X\colon \mathcal{A} \to \mathrm{Kan}_\Delta$ to*

$$\gamma_*^{-1}(X)(a) := \mathrm{holim}_{a\downarrow\mathcal{A}}\, X$$

for a functorial model of the homotopy limit.

By the theorem, we only have to construct a homotopy coherent pseudoisotopy functor to obtain an actual one. In order to construct an enriched functor $X\colon \mathfrak{S}(\mathcal{N}_\bullet\mathcal{A}) \to \mathrm{Kan}_\Delta$ it is by adjunction enough to describe a map of simplicial sets $X\colon \mathcal{N}_\bullet\mathcal{A} \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$. Let us assume that $\mathcal{A}$ is itself the truncation of a simplicially enriched category, i.e. there is $\mathcal{A}_\Delta \in \mathrm{ob}(\mathrm{Cat}_\Delta)$ with $\mathrm{ob}\,\mathcal{A}_\Delta = \mathrm{ob}\,\mathcal{A}$ and for two objects $a, b \in \mathrm{ob}\,\mathcal{A}$ we have $\mathcal{A}_\Delta(a,b)_0 = \mathcal{A}(a,b)$.

There is an inclusion map $\mathcal{N}_\bullet\mathcal{A} \hookrightarrow \mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta$. Therefore, a morphism of simplicial sets $X\colon \mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$ induces a map $X\colon \mathcal{N}_\bullet\mathcal{A} \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$ via restriction along the inclusion. Such a map always descends to an object in $\mathrm{Coh}(\mathcal{A}, \mathrm{Kan}_\Delta)$. Thus, in order to obtain a functor $X\colon \mathcal{A} \to \mathrm{Kan}_\Delta$, it suffices to construct a morphism $X\colon \mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$.

We reformulate the construction of a map $X\colon \mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$ in terms of an almost tautological obstruction theory. The argument follows easily from [33, Chapter 1.1.5] and is similar to arguments given in [10] and [48], albeit the latter worked in the topological realm.

Let us assume that X has already been defined on the $n-1$–skeleton of $\mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta$ for $n \geq 1$. Let $G\colon \mathfrak{S}(\Delta^n) \to \mathcal{A}_\Delta$ be a simplicially enriched functor, i.e. an n–simplex of $\mathcal{N}_\bullet^{h.c.}\mathcal{A}_\Delta$. If G is non-degenerate, we need a functor

$X(G)\colon \mathfrak{S}(\Delta^n) \to \mathrm{Kan}_\Delta$ such that the diagram

$$\begin{array}{ccc} \mathfrak{S}(\partial\Delta^k) & \xrightarrow{X(G\circ\mathfrak{S}(i))} & \mathrm{Kan}_\Delta \\ \downarrow{\scriptstyle \mathfrak{S}(i)} & \nearrow{\scriptstyle X(G)} & \\ \mathfrak{S}(\Delta^k) & & \end{array}$$

commutes where $i\colon \partial\Delta^k \hookrightarrow \Delta^k$ is the subspace inclusion.

By an induction argument, using the $\mathfrak{S}-\mathcal{N}^{h.c.}$–adjunction, the construction of $X(G)$ reduces to solving an extension problem

$$\begin{array}{ccc} \mathfrak{S}(\partial\Delta^n)(0,n) & \xrightarrow{X(\partial G)} & \mathrm{Kan}(G(0),G(n)) \\ \downarrow & \nearrow & \\ \mathfrak{S}(\Delta^n)(0,n) & & \end{array}$$

which can be identified with

$$\begin{array}{ccc} \partial(\Delta^1)^{n-1} & \longrightarrow & \mathrm{Kan}(G(0),G(n)) \\ \downarrow & \nearrow & \\ (\Delta^1)^{n-1} & & \end{array}$$

via $(\mathfrak{S}(\Delta^n)(0,n), \mathfrak{S}(\partial\Delta^n)(0,n)) \cong ((\Delta^1)^{n-1}, \partial(\Delta^1)^{n-1})$, an isomorphism of pairs. Obviously, the lifting problem depends highly on the solutions chosen in previous steps of the induction.

As explained in e.g. [10, pp. 70], maps from the rigidification of a simplex admit an explicit description. This is, however, also an easy consequence of the definition.

Definition 1.2.1.5. A simplicially enriched functor $G\colon \mathfrak{S}(\Delta^k) \to \mathcal{A}_\Delta$ consists of

- A map $\mathrm{ob}(G)\colon \{0,\dots,k\} \to \mathrm{ob}(\mathcal{A}_\Delta)$
- for each $n-1$–simplex $\sigma = (\sigma_0 \xrightarrow{f_0} \sigma_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} \sigma_n) \in \Delta^k$ a simplicial map $G(\sigma)\colon (\Delta^1)^{n-1} \to \mathcal{A}_\Delta(G(\sigma_0), G(\sigma_n))$, such that:
- Degeneracies are preserved:
  - if $f_0 = \mathrm{Id}$, $G(\sigma)(t_1,\dots,t_{n-1}) = G(d_0^*\sigma)(t_2,\dots,t_{n-1})$
  - for $0 < i < n-1$, if $f_i = \mathrm{Id}$,
    $G(\sigma)(t_1,\dots,t_{n-1}) = G(d_i^*\sigma)(t_1,\dots,t_i,t_{i+2},\dots,t_{n-1})$
  - if $f_{n-1} = \mathrm{Id}$, $G(\sigma)(t_1,\dots,t_{n-1}) = G(d_{n-1}^*\sigma)(t_1,\dots,t_{n-2})$
- Boundaries are preserved: for $1 \le i \le n-2$ we have

$$\begin{aligned} &G(\sigma)(t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_{n-1}) \\ &= G(d_i^*(\sigma))(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n-1}) \end{aligned}$$

- Compositions of maps are preserved: for $1 \leq i \leq n-2$ we have

$$\begin{aligned} &G(\sigma)(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_{n-1}) \\ &= G(\sigma_{\geq i})(t_{i+1}, \ldots, t_{n-1}) G(\sigma_{\leq i})(t_1, \ldots, t_{i-1}) \end{aligned}$$

  where $\sigma_{\leq i} = ((\sigma_0 \xrightarrow{f_0} \ldots \xrightarrow{f_{i-1}} \sigma_i))$, similar for $\sigma_{\geq i}$.

Here, t_i is a simplex of the i–th copy of Δ^1 in the source of $G(\sigma)$.

We return to our extension problem. It follows from the definition of the rigidification $\mathfrak{S}$, that $X(G)$ is determined on all objects and all mapping spaces except for $\mathfrak{S}(\Delta^k)(0,k)$ by its restriction $X(G \circ \mathfrak{S}(i))$.

Also by definition, the restriction of $X(G)$ to $\partial((\Delta^1)^{k-1})$ is given by various compositions of elements contained in $\mathfrak{S}(d_i^* \Delta^k)$ for $0 \leq i \leq k$. So one has to solve an extension problem

$$\begin{array}{ccc} \partial((\Delta^1)^{k-1}) & \longrightarrow & \mathrm{Kan}_\Delta(G(F(0)), G(F(k))) \\ \big\downarrow & \nearrow & \\ (\Delta^1)^{k-1} & & \end{array}$$

and this going to be the strategy throughout the argument.

Corollary 1.2.1.6. *Let $r \geq 1$. Let $\mathcal{N}_{\leq r}\mathcal{A}_\Delta$ denote the r–skeleton of $\mathcal{N}_\bullet\mathcal{A}_\Delta$. Let $\mathcal{B}_\Delta$ be a Kan-enriched category.*

A map of simplicial sets $G\colon \mathcal{N}^{h.c.}_{\leq r}\mathcal{A}_\Delta \to \mathcal{N}^{h.c.}_\bullet\mathcal{B}_\Delta$ extends to a homotopy coherent diagram $G \in \mathrm{Coh}(\mathcal{A}, \mathcal{B}_\Delta)$, if and only if certain obstructions in the groups $\pi_n(\mathcal{B}_\Delta(G(A), G(B)))$ for $A, B \in \mathcal{A}$ and $n \geq r$ vanish.

The applications of this theory in the next section rely on Corollary 1.2.1.6. We also use that the obstructions come about by gluing together compositions of previous solutions.

Similar to the above argument we obtain a criterion for the existence of coherently natural transformations. This is, for example, applied in the proof of Proposition 1.2.3.1, leading up to the comparison of the smooth and topological pseudoisotopy functor, as well as the main results of the second chapter Corollary 2.1.5.18, Theorem 2.2.3.13 and Theorem 2.2.4.2.

Corollary 1.2.1.7. *Let $r \in \mathbb{N}$. Let $\mathcal{N}_{\leq r}\mathcal{A}_\Delta$ denote the r–skeleton of $\mathcal{N}_\bullet\mathcal{A}_\Delta$. Let $\mathcal{B}_\Delta$ be a Kan-enriched category.*

A simplicial homotopy $\alpha\colon \mathcal{N}^{h.c.}_{\leq r}\mathcal{A}_\Delta \times \Delta^1 \cup \mathcal{N}^{h.c.}_\bullet\mathcal{A}_\Delta \times \partial\Delta^1 \to \mathcal{N}^{h.c.}_\bullet\mathcal{B}_\Delta$ extends to a homotopy coherent diagram $\alpha \in \mathrm{Coh}(\mathcal{A} \times [1], \mathcal{B}_\Delta)$, if and only if certain obstructions in the groups $\pi_{n+1}(\mathcal{B}_\Delta(\alpha(A), \alpha(B)))$ for $A, B \in \mathcal{A}$ and $n+1 \geq r$ vanish.

1.2.2 Basic results on manifolds with corners

Before we turn to the main theorem, we have to establish a few facts about manifolds with corners so that we can generalise standard results about ordinary manifolds.

Lemma 1.2.2.1. *Every smooth compact manifold with corners embeds as a smooth neighbourhood deformation retract into the interior of a smooth compact manifold with boundary but without higher corners.*

A smooth neighbourhood deformation retraction consists of a smooth inclusion i and retraction map p which admit a continuous homotopy $i \circ p \simeq \mathrm{Id}$

Proof. We only sketch the proof. Let M be an n–dimensional manifold with corners.

We fix a Riemannian metric on M and define for some small enough $\epsilon > 0$, the submanifold $M' = M - \{x \in M | d(x, \partial M) < \epsilon\} \cong M$.

In suitable local coordinates for M it is given by $\mathbb{R}^k_{\geq 1} \times \mathbb{R}^{n-k} \subseteq \mathbb{H}^k \times \mathbb{R}^{n-k}$. We obtain a diffeomorphism $M' \cong M$ of manifolds with corners.

Now, we may find a smooth manifold N with boundary - but without higher corners - in M, which contains M' such that $N - \mathring{M}'$ is a collar of M' as a topological manifold and with a smooth retraction map $r' \colon N \to M'$.

This can be done via a partition of unity, since $M - M'$ locally looks like $(\mathbb{H}^k - \mathbb{R}^k_{>1}) \times \mathbb{R}^{m-k}$. Similarly, we obtain a smooth retraction map. □

Corollary 1.2.2.2. *Every compact manifold with corners embeds into $\mathbb{R}^n$ for some $n \geq 0$.*

Corollary 1.2.2.3. *Every manifold with corners is a smooth euclidean neighbourhood retract.*

Definition 1.2.2.4. A *tubular neighbourhood of a manifold with corners* with respect to an embedding $\iota \colon M \subseteq N$ is a submanifold with corners $\nu\iota \subseteq N$ together with a smooth disk bundle $p \colon \nu\iota \to M$ such that there is a codimension zero embedding $\tilde{\iota} \colon \nu\iota \hookrightarrow N$ with $\tilde{\iota}(x, 0) = \iota(x)$, for every point $x \in M$, and we obtain, in the local coordinates of an embedding given in Definition 1.1.0.2, a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{H}^k \times \mathbb{R}^{m-k} \times D^{n-m} & \xrightarrow[\cong]{\beta} & p^{-1}(U_x) & \xrightarrow{p} & U_x & \xleftarrow[\cong]{\alpha} & \mathbb{H}^k \times \mathbb{R}^{m-k} \\
\downarrow{\scriptstyle \mathrm{incl}} & & & & & & \downarrow{\scriptstyle \mathrm{incl}} \\
\mathbb{R}^n & & & \xrightarrow{\quad\mathrm{pr}\quad} & & & \mathbb{R}^m
\end{array}$$

where $\mathrm{pr} \colon \mathbb{R}^n \to \mathbb{R}^m$ denotes the standard projection on the first m coordinates.

Similarly, we define a *collar of a manifold with corners.*

Lemma 1.2.2.5. *Let $\iota \colon M \hookrightarrow N$ be an embedding with codimension m. Let R be a Riemannian metric on TN. Let $\tilde{\phi} \colon M \times S^{m-1} \to \mathbb{R}_{>0}$ be a smooth map and $\phi \colon M \to C^\infty(S^{m-1}, \mathbb{R}_{>0})$ its adjoint.*

Let $B_\phi TM$ denote the submanifold of the tangent bundle $\tilde{p} \colon TM \to M$ which is given by the ball of radius $\phi(x)$ around the origin in every fiber of TM. Finally, exp denotes the exponential map.

The tuple $(B_\phi TM, exp \colon B_\phi TM \to N, \tilde{p} \colon B_\phi TM \to M)$ is a tubular neighbourhood of $\iota \colon M \to N$ for some $\tilde{\phi} \colon M \to C^\infty(S^{m-1}, \mathbb{R}_{>0})$.

Suppose that the above tuple is a tubular neighbourhood. Then, for every smooth map $\tilde{\psi} \colon M \times S^{m-1} \to \mathbb{R}_{>0}$, which satisfies $\tilde{\psi}(x, v) \leq \tilde{\phi}(x, v)$ for every $(x, v) \in M \times S^{m-1}$, the corresponding tuple determines a tubular neighbourhood as well.

For every tubular neighbourhood $(\nu\iota, \tilde{\iota}, p)$ *there are a Riemannian metric* R *and a smooth map* $\tilde{\phi}$*, such that* $\mathrm{im}(\tilde{\iota}) = exp(B_\phi TM)$ *and the diffeomorphism* $exp^{-1} \circ \tilde{\iota} \colon \nu\iota \to B_\phi TM$ *is an isomorphism of smooth disk bundles over* M.

Proof. The classical proof carries over. □

Definition 1.2.2.6. Let $\iota\colon M \hookrightarrow N$ be an embedding. An *embedded tubular neighbourhood* of ι is a tubular neighbourhood $(\nu\iota, \tilde{\iota}, p)$, such that $\nu\iota \subseteq N$ is a codimension zero submanifold and $\tilde{\iota}$ is the subspace inclusion. Note that $\tilde{\iota}$ is redundant in this case. We denote the simplicial set of embedded tubular neighbourhoods by $\mathrm{Tub}(\iota)$.

1.2.3 Smooth approximation

In this part we describe a homotopy coherent version of replacing a continuous map by a homotopic smooth map. The methods are close to arguments given in [26]. The relevant statement for our functor $\mathrm{F}_{\mathrm{ch}}\colon (\mathrm{Mfd}, \mathrm{cts})_\Delta \to \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta)$ is the following proposition.

Proposition 1.2.3.1. *Let* $i\colon (\mathrm{Mfd}, \mathrm{smooth})_\Delta \hookrightarrow (\mathrm{Mfd}, \mathrm{cts})_\Delta$ *denote the inclusion functor given by the Identity on objects and* $C^\infty(M, N) \subseteq C^0(M, N)$ *on mapping spaces.*

There is a map $\tau_1\colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta \to \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{smooth})_\Delta$*, such that:*

1. *There is a natural transformation of* $(\infty, 1)$*–functors, i.e. a simplicial homotopy* $\alpha_1\colon \mathrm{Id} \Rightarrow \mathcal{N}^{h.c.}(i) \circ \tau_1$.
2. *The map* $\mathcal{N}^{h.c.}(i)$ *is an equivalence of quasicategories and* τ_1 *is an inverse.*
3. *The map* τ_1 *is given by the Identity on objects.*
4. *For a continuous map* $f\colon M \to N$*, i.e. a* 1*–simplex of* $\mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta$*, we have* $\mathcal{N}^{h.c.}(i) \circ \tau_1(f) \simeq f$.

Later on, we wish to compare the smooth and topological pseudoisotopy functors, see [12] for the latter. There we are going to use the second part of the proposition.

Let N be a smooth manifold with corners. By standard arguments from controlled topology it follows that there is a continuous (even smooth) map $\epsilon\colon N \to (0, 1]$ such that the ϵ–ball around a continuous map $f\colon X \to N$ for any topological space X is contractible. We make this precise.

Definition 1.2.3.2. Let $C^0(X, N)$ be the space of all continuous maps from X to N and $f \in C^0(X, N)$. The ϵ–ball $C^0(X, N, f; \epsilon)$ around f is the subspace of all $g \in C^0(X, N)$ which fulfil the following property. For each $x \in X$ there is a $y \in N$ such that $f(x)$ and $g(x)$ are contained in $B_{\epsilon(y)}(y)$.

Lemma 1.2.3.3. *Let* N *be a smooth manifold with corners. Then there is a map* $\epsilon\colon N \to (0, 1]$*, such that* $C^0(X, N, f; \epsilon')$ *is contractible for every topological space* X*, every* $\epsilon' \leq \epsilon$ *and every* $f \in C^0(X, N)$.

Proof. By Corollary 1.2.2.3 there is a smooth euclidean neighbourhood retract $\tilde{N} \subseteq \mathbb{R}^k$ of N for some $k \in \mathbb{N}$ with projection p and inclusion i. Choose

$\epsilon\colon N \to [0,1]$, such that for each $y \in N$ the ball $B_{\epsilon(y)}(y)$ is contained in $\tilde{N}$. Then the map

$$\begin{aligned} C^0(X,N,f;\epsilon) \times I \to & C^0(X,N,f;\epsilon) \\ (g,t) \mapsto & p(t(i \circ f) + (1-t)(i \circ g)) \end{aligned}$$

is a homotopy between the Identity and the projection to f. □

Now let X be a manifold M. Then we also obtain a contractible subset $C^\infty(M,N,f;\epsilon) := C^0(M,N,f;\epsilon) \cap C^\infty(M,N)$. The homotopy simply restricts to this subspace, because i and p are smooth.

Note that we do not need to assume that f is smooth to make sense of the smooth ϵ–ball around f and, if it is not empty, it is still contractible for ϵ small enough. We now show that a smooth ϵ–approximation exists.

Lemma 1.2.3.4. *Let $f \in C^0(M,N)$ and $\epsilon\colon N \to (0,1]$. Then $C^\infty(M,N,f;\epsilon)$ is not empty.*

Proof. Let $\epsilon'\colon N \to (0,1]$ be a continuous map, such that $B_{\epsilon'(y)}(y) \subseteq \tilde{N}$ and $p(B_{\epsilon'(y)}(y)) \subseteq B_{\epsilon(y)}(y)$ hold for each $y \in N$.

Let $\delta\colon N \to (0,1]$ be a continuous map with $2\delta < \epsilon'$. Then $\{B_{\delta(y)}(y)\}_{y\in N}$ is a covering of an (open) subset of $\tilde{N}$, such that for each $y \in N$ the union of all covering sets which contain y, i.e. the set $\bigcup_{y'\in N, y\in B_{\delta(y')}(y')} B_{\delta(y')}(y')$, is contained in $B_{\epsilon'(y)}(y)$.

Now we choose a covering $\{V_x\}_{x\in M\times|\Delta^1|^n}$ of $M \times |\Delta^1|^n$ such that for each $x \in M \times |\Delta^1|^n$ there is some $y \in N$ with $f(V_x) \subseteq B_{\delta(y)}(y)$. Choose a locally finite sub-cover $\{V_\alpha\}_{\alpha\in I}$ and an associated partition of unity $\{\phi_\alpha\}_{\alpha\in I}$ consisting of smooth maps. The map

$$\begin{aligned} \tilde{f}\colon M &\to N \\ x &\mapsto p\left(\sum_{\alpha\in I} \phi_\alpha(x) f(x_\alpha)\right) \end{aligned}$$

is an appropriate approximation. In order to show that this is well-defined, i.e. $\sum_{\alpha\in I}\phi_\alpha(x)f(x_\alpha) \in \tilde{N}$ holds, and it is indeed an ϵ–approximation of f, we proof that $d(f(x), \sum_{\alpha\in I}\phi_\alpha(x)f(x_\alpha)) < \epsilon'(f(x))$ for each $x \in M$.

If for $\alpha \in I$ we have $\phi_\alpha(x) \neq 0$, then it follows that $x \in V_\alpha$. This implies $f(x), f(x_\alpha) \in B_{\delta(y_\alpha)}(y_\alpha)$ and all of the $B_{\delta(y_\alpha)}(y_\alpha)$ occurring for various α intersect in $f(x)$. Hence all of these $B_{\delta(y_\alpha)}(y_\alpha)$ are contained in $B_{\epsilon'(f(x))}(f(x))$. We calculate

$$\begin{aligned} d(f(x), \sum_{\alpha\in I}\phi_\alpha(x)f(x_\alpha)) =& \| \sum_{\alpha\in I}\phi_\alpha(x)(f(x) - f(x_\alpha))\| \\ \leq& \sum_{\alpha\in I}\phi_\alpha(x)\|f(x) - f(x_\alpha)\| \\ <& \epsilon'(f(x))\sum_{\alpha\in I}\phi_\alpha(x) \\ =& \epsilon'(f(x)) \end{aligned}$$

and obtain the estimate. □

Proof of Proposition 1.2.3.1. We apply Corollary 1.2.1.6 of the obstruction theory explained in Section 1.2.1.

We fix for each N a small $\epsilon_N^{(0)} \colon N \to (0,1]$ in the sense of Lemma 1.2.3.3. There is some $\epsilon_N^{(1)} \le \epsilon_N^{(0)}$, such that the composition map

$$C^\infty(N, P, g, \frac{\epsilon_P^{(0)}}{2}) \times C^\infty(M, N, f, \epsilon_N^{(1)}) \to C^\infty(M, P)$$

factors as

$$C^\infty(N, P, g, \frac{\epsilon_P^{(0)}}{2}) \times C^\infty(M, N, f, \epsilon_N^{(1)}) \to C^\infty(M, P, gf, \epsilon_P^{(0)}).$$

Further, the inclusion $C^\infty(M, N, f, \epsilon_N^{(1)}) \subseteq C^\infty(M, N, f, \epsilon_N^{(0)})$ is a strong deformation retraction.

An induction argument now shows that we can choose all obstructions in $\pi_k(C^\infty(M, N, f, \epsilon_N^{(k+1)}))$ for various smooth manifolds with corners M, N, continuous maps f and $k \ge 0$. Since these groups are all trivial, we obtain a functor.

All properties are either obvious or follow from the second one so we have to show that τ_1 is an equivalence. Since $\tau_1 \circ \mathcal{N}^{h.c.} i = \mathrm{Id}$ by definition, we only have to consider the other composition.

We apply Corollary 1.2.1.7. The approximation arguments used to define τ_1 carry over since we also have contractible neighbourhoods in spaces of continuous maps. □

1.2.4 Stable embeddings

We construct the parametrised version of replacing a smooth map $f \colon M \to N$ by a smooth embedding $\iota \colon M \hookrightarrow N \times (D^1)^k$ with $\mathrm{pr}_N \circ \iota = f$. The main technique is transversality, see [26] for an introduction.

Proposition 1.2.4.1. *There is a map*

$$\tau_2 \colon \mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}, \mathrm{smooth})_\Delta \to \mathrm{Ind}(\mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}, \mathrm{emb})_\Delta)$$

with the following properties:

1. *Let* $i' \colon (\mathrm{Mfd}, \mathrm{emb})_\Delta \to (\mathrm{Mfd}, \mathrm{smooth})_\Delta$ *denote the inclusion. There is a natural transformation of* $(\infty, 1)$*–functors* $\alpha_2 \colon j \Rightarrow \mathcal{N}^{h.c.}(i') \circ \tau_2$.
2. *Let* $M \in (\mathrm{Mfd}, \mathrm{cts})_\Delta$. *Then* $\tau_2(M) = \mathrm{hocolim}_{n \in \mathbb{N}} M \times (D^1)^n$ *with structure maps* $M \times (D^1)^n \times \{0\} \hookrightarrow M \times (D^1)^{n+1}$.
3. *Let* $f \colon M \to N$ *be a smooth map and* $i_M \colon M \to \tau_2(M)$ *be the structure map of the homotopy colimit. Then* $i_N \circ f \simeq \tau_2(f) \circ i_M$ *holds in* $\mathrm{Ind}(\mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}, \mathrm{smooth})_\Delta)$.

Proof. Via Lemma 1.2.2.1, we fix, for every smooth and compact manifold with corners M, a smooth neighbourhood retraction $i \colon M \hookrightarrow \tilde{M}$ into a smooth, compact manifold with boundary but without higher corners.

Let $\mathrm{Emb}_{\tilde{M}}(M, (D^1)^l) \subseteq \mathrm{Emb}(M, (D^1)^l)$ denote the subspace of all smooth embeddings which admit an extension to a smooth embedding of $\tilde{M}$.

There is a map

$$C^\infty(M,N) \times \mathrm{Emb}_{\tilde{M}}(M,(D^1)^l) \to \mathrm{Emb}(M, N \times (D^1)^l)$$

given by $(f,\iota) \mapsto f \times \iota$ and a map

$$\mathrm{Emb}(M, N \times (D^1)^l) \to \mathrm{Emb}(\mathrm{hocolim}_{k\in\mathbb{N}} M \times (D^1)^k, \mathrm{hocolim}_{m\in\mathbb{N}} N \times (D^1)^m)$$

induced by $\tilde{\iota} \mapsto (\tilde{\iota} \times \mathrm{Id}_{(D^1)^k})_{k\in\mathbb{N}}$.

By obstruction theory, see Corollary 1.2.1.6, we have to show that certain elements in $\pi_n(\mathrm{Emb}(\mathrm{hocolim}_{k\in\mathbb{N}} M \times (D^1)^k, \mathrm{hocolim}_{l\in\mathbb{N}} N \times (D^1)^l))$ vanish.

Since $\pi_n(\mathrm{Emb}_{\tilde{M}}(M,(D^1)^l))$ is trivial for $l = l(n) >> 0$ by transversality, we can extend to $C^\infty(M,N) \times \mathrm{Emb}_{\tilde{M}}(M,(D^1)^l)$ where we make sure to send $f \in C^\infty(M,N)$ to some pair (f,ι). Then we use the above maps to obtain the required families of embeddings. □

1.2.5 Disk bundles and codimension zero embeddings

In this part we finish the proof of Theorem 1.2.0.5 with the next proposition.

Proposition 1.2.5.1. *There is a map* $\tau_3 \colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd},\mathrm{emb})_\Delta \to \mathcal{D}_\Delta$ *with the following properties:*

1. *The map* τ_3 *is given by the Identity on objects.*
2. *The map* $\mathrm{pr}\colon \mathcal{D}_\Delta \to \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd},\mathrm{emb})_\Delta$*, induced by* $(\iota,\nu\iota,p) \mapsto \iota$ *on mapping spaces, is a left-inverse, i.e.* $\mathrm{pr} \circ \tau_3 \simeq \mathrm{Id}$.
3. *The map* pr *is an equivalence of quasicategories and* τ_3 *is an inverse.*

Proof. We show that the functor $\mathrm{pr}\colon \mathcal{D}_\Delta \to (\mathrm{Mfd},\mathrm{emb})_\Delta$ is an acyclic fibration in the model structure on simplicially enriched categories, see [3]. Since the simplicial nerve $\mathcal{N}^{h.c.}$ is a Quillen right adjoint (even a Quillen equivalence) and every simplicial set is cofibrant in the Joyal model structure, this implies that pr is an equivalence between fibrant and cofibrant objects in the Joyal model structure. In particular, we obtain an inverse map τ_3.

A simplicially enriched category is fibrant in this model structure, if it is Kan-enriched, thus $(\mathrm{Mfd},\mathrm{emb})_\Delta$ is fibrant.

A map between fibrant objects is an acyclic fibration, if

- The induced map on the homotopy categories is an equivalence of categories and surjective on objects.
- The induced map $\mathcal{D}_\Delta(M,N) \to \mathcal{S}_\bullet(\mathrm{Mfd},\mathrm{emb})_\Delta(M,N)$ is a trivial fibration for every pair of manifolds M,N.

The first condition follows easily once we have shown the latter, so we have to solve a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\tau_3(\partial\iota)} & \mathcal{D}_\Delta(M,N) \\ \downarrow & \overset{\tau_3(\iota)}{\nearrow} & \downarrow \mathrm{pr} \\ \Delta^n & \xrightarrow{\iota} & \mathcal{S}_\bullet(\mathrm{Mfd},\mathrm{emb})_\Delta(M,N). \end{array}$$

The characterisation of tubular neighbourhoods given in Lemma 1.2.2.5 shows that the space of embedded tubular neighbourhoods is contractible. □

Proof of Theorem 1.2.0.5. We set $\mathrm{F}_{\mathrm{ch}} := \mathrm{Ind}(\tau_3) \circ \tau_2 \circ \tau_1$. The desired transformation is a straightforward consequence of the transformations α_1 and α_2 as well as the fact that τ_3 is an equivalence. The remaining properties follow easily. □

Remark 1.2.5.2. *For later use we note that the proof of Proposition 1.2.5.1 in particular shows that $\mathcal{D}_\Delta$ is a fibrant object in the model structure on simplicially enriched categories.*

Remark 1.2.5.3. *To compare the differentiable and the topological pseudoisotopy functor we have to connect the respective categories of choices. We recall the topological case in Section 2.1.3.*

There is a functor $j\colon \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta) \to \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathrm{Ch}_\Delta)$ which sends a tuple $(\iota, \nu\iota, p)$ to $(\nu\iota, p, \iota, \nu_p, H_p)$ where ν_p is a topological parallel transport in the sense of Definition 2.1.3.5. It is induced by a parallel transport in the sense of differential geometry (depending on the contractible choice of a Riemannian metric) as explained in Remark 1.1.0.11. Further, H_p is a bending isotopy in the sense of Definition 2.1.3.4 with respect to ν_p.

By [12, Theorem 5.21] the forgetful map $\mathrm{Ind}(\mathrm{Ch}_\Delta) \to \mathrm{Ind}(\mathrm{Mfd}, \mathrm{cts})_\Delta$ is an acyclic fibration over the objects of the form $\mathrm{hocolim}_{n \in \mathbb{N}} M \times (D^1)^n$ for M in $(\mathrm{Mfd}, \mathrm{cts})_\Delta$. We apply the tautological obstruction theory of Section 1.2.1 to obtain the desired functor as a lift of $\mathrm{Ind}(\mathcal{N}^{h.c.}(i \circ \mathrm{pr}))$.

1.3 Unique Points of Intersection

In this part we take care of the main geometric constructions. However, without the context provided by the construction in Definition 1.4.1.11 (and the discussion leading up to it), the arguments might seem somewhat unmotivated. Thus the reader might want to skip ahead to Section 1.4 first and return once a result presented here is required. Regardless, we try to explain this chapter's role in the proof. We use some definitions introduced in Section 1.1.1 and Section 1.2.2. Also, the reader might find Picture 1.4.3.1 helpful to understand the various geometric arguments presented here.

Let $m = \sum_{i=1}^{k} m_i$ be an additive partition. We set $m_j^+ := \sum_{i=1}^{j} m_i$, too. The main argument in Section 1.4.1 is the construction of a homotopy coherent diagram in the sense of Section 1.2.1. After some preliminaries, we reduce the problem to a geometric question asked in local trivialisations with structure group $\prod_{i=1}^{k} O(m_i)$. Hence we have to work equivariantly in this chapter, but this turns out to be a mere formality.

The local problem we have to solve can be seen in the following example. Given a sequence of trivial vector bundles $M \times \mathbb{R}^m \to M \times \mathbb{R}^{m_{k-1}^+} \to \ldots \to M$, we are going to compare iterated transfer maps $p_{k!} \circ \ldots \circ p_{1!}$ with the transfer map of the composition $p_! := (p_1 \circ \ldots \circ p_k)_!$. For some pseudo-isotopy $F \in P(M)$ we obtain a map

$$p_{k!} \circ \ldots \circ p_{1!}(F) \colon M \times \prod_{i=1}^{k} D^{m_i} \times I \to M \times \prod_{i=1}^{k} D^{m_i} \times I$$

and $i_*(p_!(F))$, where i is the embedding $i \colon M \times D^m \subseteq M \times \prod_{i=1}^{k} D^{m_i}$. We have to deform $\underline{\mathrm{St}}_{m_k}^{-1} \circ \ldots \circ \underline{\mathrm{St}}_{m_1}^{-1} \circ tr(F) \circ \underline{\mathrm{St}}_{m_1} \circ \ldots \circ \underline{\mathrm{St}}_{m_k}$ into $\underline{\mathrm{St}}_{m}^{-1} \circ tr(F) \circ \underline{\mathrm{St}}_{m}$.

At least in the $M \times I$–coordinates this map is essentially given by applying the original F, evaluated at certain points. First we describe the area in which values of F with a fixed I–coordinate are used. It turns out that such an area is up to isotopy (in some contractible space of diffeomorphisms) given by the lower half of the sphere $S_-^n := S^n \cap (\mathbb{R}^n \times (-\infty, 0])$.

We are going to force our different transfer maps to use values of the form $F(-, t)$ on the same (subspace isotopic to the) lower sphere. This type of subset is going to be called a level set.

Throughout this chapter we sometimes drop the lower index of $\underline{\mathrm{St}}_n^{-1}$ if n is clear from the context. Let us begin with some notation.

Let $1/4 > \tilde{\delta} > \delta > 0$. Let $m = \sum_{i=1}^{k} m_i$ and $m_j^+ := \sum_{i=1}^{j} m_i$. Let $D_-^n := D^n \cap (\mathbb{R}^{n-1} \times (-\infty, 0])$ and $S_-^n := S^n \cap (\mathbb{R}^n \times (-\infty, 0])$.

Definition 1.3.0.1. Given $X \subseteq \left(\prod_{i=1}^{j} D^{m_i}\right) \times I$, we set

$$X^{\langle 1-\tilde{\delta} \rangle} := \mathrm{pr}_{\mathbb{R}^{m_j^+}} \left(X \cap \left(\left(\prod_{i=1}^{j} D^{m_i} \right) \times \{1 - \tilde{\delta}\} \right) \right).$$

Definition 1.3.0.2. Let $\tau \colon \left(\prod_{i=1}^{j-1} D^{m_i}\right) \times I \times D^{m_j} \to \left(\prod_{i=1}^{j} D^{m_i}\right) \times I$ be the canonical diffeomorphism.

It is immediate from Definition 1.1.1.15 that the level sets of the iterated transfer maps are given by the subsets A_k we define below. The collection of these hypersurfaces is given by the manifold $\mathbf{A}$. We are going to show in a moment that these are indeed manifolds. The submanifold T_j was defined in Definition 1.1.1.13.

Definition 1.3.0.3. We define subsets of $\left(\prod_{i=1}^{j} D^{m_i}\right) \times I$ for $0 \leq j \leq k$. Let $\mathbf{A}_0 := I$ and $\mathbf{B}_0 := \emptyset$. For $1 \leq j \leq k$ we set

$$\begin{aligned}\mathbf{A}_j :=&\mathbf{B}_j \cup \mathbf{C}_j\\ \mathbf{B}_j :=&\underline{\mathrm{St}}_{m_j}^{-1}(\tau(\mathbf{A}_{j-1} \times D^{m_j})) \cup \mathbf{B}_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j\\ \mathbf{C}_j :=&\mathbf{B}_j^{\langle 1-\tilde{\delta}\rangle} \times [1-\tilde{\delta}, 1].\end{aligned}$$

Definition 1.3.0.4. Fix $t \in I$. We define subsets of $\left(\prod_{i=1}^{j} D^{m_i}\right) \times I$ for $1 \leq j \leq k$. Let $A_0 := \{t\} \subseteq I$ and $B_0 := \emptyset$. For $1 \leq j \leq k$ we set

$$\begin{aligned}A_j :=&B_j \cup C_j\\ B_j :=&\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j})) \cup B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j\\ C_j :=&B_j^{\langle 1-\tilde{\delta}\rangle} \times [1-\tilde{\delta}, 1].\end{aligned}$$

The key idea to identify A_j with half a sphere is to see that there is a unique point of intersection between A_j and every line through the origin. In order to do this, we are going to approximate A_j by the orbit of A_{j-1} under the standard $O(m_j)$–action on $\mathbb{R}^{m_j^+ +1}$.

In particular, this requires us to make precise how A_{j-1} is contained in A_j.

Lemma 1.3.0.5. *Let $\iota\colon \mathbb{R}^{m_{j-1}^+} \hookrightarrow \mathbb{R}^{m_j^+}$ be the inclusion given by $v \mapsto (v, 0)$. Recall that $\lambda(t) = t(\delta - 3/4) + 3/4$ and that e_{m+1} denotes the unit vector in the I–coordinate from Definition 1.1.1.1. We have an equality of sets:*

$$\begin{aligned}A_j \cap \left(\prod_{i=1}^{j-1} D^{m_i}\right) \times \{0\} \times I =&\iota \times (-\lambda)(A_{j-1}) + (1-\tilde{\delta})e_{m+1}\\ &\cup A_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times \{0\} \times [1-\tilde{\delta}-\delta, 1-\tilde{\delta}]\\ &\cup A_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times \{0\} \times [1-\tilde{\delta}, 1]\end{aligned}$$

Also, the restriction of $\underline{\mathrm{St}}_{m_j}^{-1} \circ \tau$ to A_{j-1} is given by $\iota \times (-\lambda) + (1-\tilde{\delta})e_{m+1}$.

Proof. We have $A_j = \underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j})) \cup B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j \cup B_j^{\langle 1-\tilde{\delta}\rangle} \times [1-\tilde{\delta}, 1]$ by definition. We treat these three sets separately.

First we study the intersection $\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j})) \cap \left(\prod_{i=1}^{j-1} D^{m_i}\right) \times \{0\} \times I$. By definition of $\underline{\mathrm{St}}_{m_j}^{-1}$, we have

$$\begin{aligned}\underline{\mathrm{St}}_{m_j}^{-1}(x, x_j, t) = \Bigg(x, \lambda(t)\bigg((1-\beta(\|x_j\|))\frac{2x_j}{\|x\|^2+1} + \beta(\|x_j\|)\frac{x_j}{\|x_j\|}\bigg),\\ \lambda(t)\frac{\|x_j\|^2-1}{\|x_j\|^2+1}\Bigg) + (1-\tilde{\delta})e_{m+1}.\end{aligned}$$

Since

$$\lambda(t)\left((1-\beta(\|x_j\|))\frac{2x_j}{\|x\|^2+1}+\beta(\|x_j\|)\frac{x_j}{\|x_j\|}\right)=0$$

holds precisely if $x_j = 0$, we restrict $\underline{\mathrm{St}}_{m_j}^{-1}$ to $\tau(A_{j-1}\times\{0\})$. A straightforward computation yields

$$\underline{\mathrm{St}}_{m_j}^{-1}(x,0,t)=(x,0,1-\tilde{\delta}-\lambda(t))$$

and we obtain

$$\begin{aligned}&\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1}\times D^{m_j}))\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\\ =&\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1}\times\{0\}))\\ =&\iota\times(-\lambda)(A_{j-1})+(1-\tilde{\delta})e_{m+1}.\end{aligned}$$

For the second part, we first observe that

$$\begin{aligned}B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I=&B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{(x,t)\in T_j|x=0\}\\ =&B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta}-\delta,1-\tilde{\delta}]\end{aligned}$$

holds by definition. We have $A_{j-1}^{\langle 1-\tilde{\delta}\rangle}=B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\cup C_{j-1}^{\langle 1-\tilde{\delta}\rangle}=B_{j-1}^{\langle 1-\tilde{\delta}\rangle}$ by definition of C_{j-1}, so we obtain

$$B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I=A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta}-\delta,1-\tilde{\delta}].$$

For the third part, we start with

$$\begin{aligned}&B_j^{\langle 1-\tilde{\delta}\rangle}\times[1-\tilde{\delta},1]\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\\ =&\operatorname{pr}_{\mathbb{R}^{m_j^+}}(\{(x,x_j,t)\in B_j|x_j=0,t=1-\tilde{\delta}\})\times[1-\delta,\tilde{1}]\\ =&\left(B_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\right)^{\langle 1-\tilde{\delta}\rangle}\times[1-\tilde{\delta},1].\end{aligned}$$

Now we obtain

$$\begin{aligned}B_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I=&\iota\times(-\lambda)(A_j)+(1-\tilde{\delta})e_{m+1}\\ &\cup A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta}-\delta,1-\tilde{\delta}]\end{aligned}$$

from the previous two steps. Since $1-\tilde{\delta}-\lambda(t)<1-\tilde{\delta}$ for $t\in I$, we get

$$\begin{aligned}&\left(B_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\right)^{\langle 1-\tilde{\delta}\rangle}\\ =&(\iota\times(-\lambda)(A_j)+(1-\tilde{\delta})e_{m+1}\cup A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta}-\delta,1-\tilde{\delta}])^{\langle 1-\tilde{\delta}\rangle}\\ =&(\iota\times(-\lambda)(A_j)+(1-\tilde{\delta})e_{m+1})^{\langle 1-\tilde{\delta}\rangle}\cup(A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta}-\delta,1-\tilde{\delta}])^{\langle 1-\tilde{\delta}\rangle}\\ =&A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\end{aligned}$$

and thus obtain

$$\begin{aligned}&B_j^{\langle 1-\tilde{\delta}\rangle}\times[1-\tilde{\delta},1]\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\\ =&\left(B_j\cap\left(\prod_{i=1}^{j-1}D^{m_i}\right)\times\{0\}\times I\right)^{\langle 1-\tilde{\delta}\rangle}\times[1-\tilde{\delta},1]\\ =&A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times\{0\}\times[1-\tilde{\delta},1]\end{aligned}$$

as desired. ☐

In order to obtain a smooth isotopy from the level sets of an iterated transfer to those of the transfer in all coordinates we have to show that these subsets are indeed manifolds. Essentially, we observe that γ, see Definition 1.1.1.14, parametrises a neighbourhood of the intersection of $\underline{\mathrm{St}}^{-1}(\tau(A_{j-1}\times D^{m_j}))$ and $B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j$.

Lemma 1.3.0.6. *For every $0\le j\le k$ the set A_j is a smooth submanifold (with boundary but no higher corners) of $\mathbb{R}^{m_j^+}\times I$. Moreover, $\partial A_j=A_j\cap\mathbb{R}^{m_j^+}\times\{1\}$*

Proof. We proceed by induction to show that A_j and B_j are manifolds for all $0\le j\le k$. The statement is clear for $j=0$.

First we show that B_j is a smooth submanifold with boundary but without corners.

By induction $\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1}\times D^{m_j}))$ is a smooth submanifold with corners. So is $B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j=A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j$, since $A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times[1-\tilde{\delta},1]\subseteq A_{j-1}$ and thus the intersection $A_{j-1}\cap\prod_{i=1}^{j-1}D^{m_i}\times\{1-\tilde{\delta}\}$ is transverse.

Since $\underline{\mathrm{St}}_{m_j}^{-1}(D^{m_j}\times\{1\})=O(m_j)\,\mathrm{im}(\gamma)$ and λ is decreasing, we have

$$\underline{\mathrm{St}}_{m_j}^{-1}(D^{m_j}\times I)\cap T_j=O(m_j)\,\mathrm{im}(\gamma).$$

Therefore

$$\begin{aligned}&\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1}\times D^{m_j}))\cap B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j\\ =&\underline{\mathrm{St}}_{m_j}^{-1}\left(\tau(A_{j-1}\times D^{m_j})\cap\prod_{i=1}^{j}D^{m_i}\times\{1\}\right)\cap B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j\\ =&\underline{\mathrm{St}}_{m_j}^{-1}(A_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times D^{m_j}\times\{1\})\cap B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times T_j\\ =&B_{j-1}^{\langle 1-\tilde{\delta}\rangle}\times O(m_j)\,\mathrm{im}(\gamma)\end{aligned}$$

holds, where we used $A_{j-1}^{\langle 1-\tilde{\delta}\rangle} = B_{j-1}^{\langle 1-\tilde{\delta}\rangle}$ in the last line again.

Moreover, the corners of $\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j}))$ and $B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j$ coincide, since the corners of the former are given by

$$\begin{aligned}\underline{\mathrm{St}}_{m_j}^{-1}(\tau(\partial A_{j-1} \times S^{m_j-1})) =& \underline{\mathrm{St}}_{m_j}^{-1}(A_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times S^{m_j-1} \times \{1\}) \\ =& B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times O(m_j)\gamma(1)\end{aligned}$$

which are exactly the corners of $B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j$.

The intersection has local neighbourhoods $\partial A_{j-1} \times O(m_j)\gamma([0,1] \times (1-\bar{\delta}, 1])$ in $\mathrm{St}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j}))$ and $B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times O(m_j)\gamma([0,1] \times [1, 3/(3-4\delta))$ in $B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j$.

Thus the map γ on $[0,1] \times (1-\tilde{\delta}, 3/(3-4\delta))$ gives an embedding in a neighbourhood of the intersection of the two smooth submanifolds with corners. So B_j is a smooth submanifold with boundary. The parametrisation by γ shows that all previous corners vanish. Hence B_j does not have any corners.

The above description of $\underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j})) \cap B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j$ also implies that $\partial B_j = \underline{\mathrm{St}}_{m_j}^{-1}(\tau(A_{j-1} \times \partial D^{m_j})) \cup B_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times \delta D^{m_j} \times \{1-\tilde{\delta}\}$ holds.

In order to see that $A_j = B_j \cup C_j$ is again a manifold, we show that ∂B_j admits a neighbourhood of the form $\partial B_j^{\langle 1-\tilde{\delta}\rangle} \times [1-\tilde{\delta}-\rho, 1-\tilde{\delta}]$ in B_j for some small enough $\rho > 0$.

We choose a collar of ∂D^{m_j} given by $c\colon \partial D^{m_j} \times [1-\bar{\delta}, 1] \to \mathbb{R}^{m_j^+}$, $(v,t) \mapsto tv$. Then we obtain

$$\begin{aligned}&\underline{\mathrm{St}}^{-1}(\tau(A_{j-1} \times \mathrm{im}(c))) \\ =&\{(v,w) \in \mathbb{R}^{m_{j-1}^+} \times \mathbb{R}^{m_j} | (v, \lambda^{-1}(\|w\|)) \in A_{j-1}\} \times [1-\tilde{\delta}-\bar{\delta}, 1-\tilde{\delta}].\end{aligned}$$

Now we restrict γ to an interval $[1-\rho', 1]$ with $\beta(1-\rho') = 1$ which yields an image of the form $\gamma([1-\rho', 1]) = \{s/\|s\|, 0\} \times [1-\tilde{\delta}-\rho, 1-\tilde{\delta}]$. Since $\mathrm{pr}_I \circ \gamma$ is strictly increasing this implies that T_j contains $\delta D^{m_j} \times [1-\tilde{\delta}-\rho, 1-\tilde{\delta}]$.

Since $A_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times T_j$ is contained in B_j we get

$$A_{j-1}^{\langle 1-\tilde{\delta}\rangle} \times \delta D^{m_j} \times [1-\tilde{\delta}-\rho, 1-\tilde{\delta}] \subseteq B_j$$

for some $\bar{\delta} > \rho > 0$ (e.g. $\rho = \delta\bar{\delta}/3$ works). Hence, $\partial B_j \times [1-\tilde{\delta}-\rho] \subseteq B_j$. Since $C_j = \partial B_j \times [1-\tilde{\delta}, 1]$ holds by definition, $A_j = B_j \cup C_j$ is a smooth manifold with boundary but without corners and $\partial A_j = A_j \cap \mathbb{R}^{m_j^+} \times \{1\}$. □

The following lemma is the key step to identify our level sets as desired. Its length is mostly owed to the point-set topology involved, not the difficulty of the argument.

Lemma 1.3.0.7. *For sufficiently small δ, $\tilde{\delta}$, δ^* and $\bar{\delta}$ the set A_j intersects $L_\zeta = \{r\zeta + e_{m+1} | r \in \mathbb{R}_{\geq 0}\}$, i.e. the ray from the origin in the direction of ζ, in exactly one point for every $1 \leq j \leq k$ and for every $\zeta \in S_-^{m_j^+}$.*

Proof. We use induction. We assume inductively that for every $v \in S^{m_{j-1}^+-1}$ there is a smooth curve $\xi\colon [0,1] \to \mathbb{R}^{m_{j-1}^+} \times I$ which parametrises the intersection $A_{j-1} \cap (\mathbb{R}_{\geq 0}(v,0) \oplus \mathbb{R}e_{m+1})$ and satisfies:

1. The map $\mathrm{pr}_I \circ \xi$ is strictly increasing.

2. There is a neighbourhood U of 1 in $[0,1]$, such that $\xi(U)$ is parallel to e_{m+1}.

3. We have $\xi(1) = (\mu v, 1)$ for some $\mu \geq 0$.

4. The norm map $t \mapsto \|\xi(t)\|$ is increasing.

This has the following consequence. Let $x \in A_{j-1} \cap (\mathbb{R}(v,0) \oplus \mathbb{R}e_{m+1})$ and denote by $u \in T_x\mathbb{R}^{m_j^+ + 1}$ the unique tangent vector with the following properties:

1. It is contained in the 1–dimensional tangent space of the line through x and $(0,1) \in \mathbb{R}^{m_j^+} \times \mathbb{R}$, the "origin of A_j".

2. It is a unit vector, i.e. $\|u\| = 1$.

3. It is pointing outwards, i.e. $(0,1) \notin x + \mathbb{R}_{\geq 0} u$.

The vector u is not contained in $T_x A_j$. Either it has negative slope in I and thus cannot lie in the tangent space of our curve, or we are at height 1 where $T_{\gamma(1)} \operatorname{im}(\gamma) = \langle e_{m+1} \rangle$ holds.

The case A_0 is clear. Throughout this proof we write $\underline{\mathrm{St}}^{-1}$ instead of $\underline{\mathrm{St}}_{m_j}^{-1}$.

For the induction step, we assume the assumption holds for $j-1$.

We first show the statement for a vector $(v,0) \in \mathbb{R}^{m_{j-1}^+} \times \{0\} \subseteq \mathbb{R}^{m_j^+}$. We obtain a 2–dimensional subspace $\mathbb{R}\langle (v,0), e_{m+1} \rangle \subseteq \mathbb{R}^{m_j^+} \times \mathbb{R}$.

By induction the intersection $A_{j-1} \cap (\mathbb{R}_{\geq 0} v \oplus \mathbb{R}e_{m+1})$ is parametrised by a smooth curve $\xi \colon [0,1] \to \mathbb{R}^{m_{j-1}^+} \times I$ such that $\mathrm{pr}_I \circ \xi$ is a strictly increasing map. Since $\underline{\mathrm{St}}^{-1}$ restricts to $(\iota \times -\lambda + (1-\tilde{\delta}))$ by Lemma 1.3.0.5, the smooth curve $\underline{\mathrm{St}}^{-1} \circ \tau \circ \xi$ also yields a strictly increasing map $\mathrm{pr}_I \circ \underline{\mathrm{St}}^{-1} \circ \tau \circ \xi$.

By induction we also know that there is an open neighbourhood U of 1 such that $\xi(U)$ is parallel to e_{m+1} and $\xi(1) = (\mu v, 1)$. Since it is also strictly increasing, this determines ξ on U uniquely, if we parametrise by arc length. We also note that $\underline{\mathrm{St}}^{-1} \circ \tau \circ \xi(1) = (\mu' v, 0, 1 - \tilde{\delta} - \delta)$ for $\mu' \geq 0$.

From this we see that $\underline{\mathrm{St}}^{-1} \circ \tau \circ \xi$ can be smoothly extended to a smooth curve $\bar{\xi}$ which parametrises $\operatorname{im}(\underline{\mathrm{St}}^{-1} \circ \tau \circ \xi)$ as well as $\{(\mu' v, 0)\} \times [1 - \tilde{\delta} - \delta, 1]$.

Moreover we can choose $\bar{\xi}$ such that $\mathrm{pr}_I \circ \bar{\xi}$ is again strictly increasing (e.g. via parametrisation by arc length). Note that near 1 this curve is given by (up to parametrisation) $t \in [1 - \tilde{\delta} - \delta, 1] \mapsto (\mu' v, 0, t)$.

It is a straightforward computation using $\underline{\mathrm{St}}^{-1} = (\iota \times -\lambda + (1 - \tilde{\delta}))$ that the norm map yields an increasing map.

This shows the statement of the induction step for $(v, 0)$.

Now consider $(v, w) \in \mathbb{R}^{m_{j-1}^+} \times \mathbb{R}^{m_j}$ with $\|(v,w)\| = 1$. We are going to show that

1. The set $A_j \cap \mathbb{R}\langle (v,w,0), e_{m+1} \rangle$ is a 1–dimensional submanifold of $\mathbb{R}^{m_j^+ + 1}$.

2. The projection $\mathrm{pr}_I \colon A_j \cap (\mathbb{R}_{\geq 0}(v,w,0) \oplus \mathbb{R}e_{m+1}) \to I$ to the last coordinate is injective.

3. The image $\operatorname{im}(\mathrm{pr}_I)$ of this map is a closed interval.

Once this is shown, the rest of the proof is not difficult. The classification of compact 1–dimensional manifolds (compactness follows from the second and third claim) shows that $A_j \cap (\mathbb{R}(v,w,0) \oplus \mathbb{R}e_{m+1})$ has to be a smooth curve (as the second and third claim imply that it has only one connected component) which we denote by ξ.

If we parametrise ξ by arc-length, it necessarily yields a strictly increasing map $\mathrm{pr}_I \circ \xi$ by the second claim and the collar given by C_j ensures the desired behaviour of the curve near 1.

For the last property, we show that if (av, bw, s) and $(a'v, b'w, s')$ are contained in $A_j \cap (\mathbb{R}_{\geq 0}(v,w,0) \oplus \mathbb{R}e_{m+1})$ and fulfil $s' > s$, then $\|(a',b')\| \geq \|(a,b)\|$. By induction we know $a' \geq a$. Since $a'/b' = a/b$ holds the map $t \mapsto \|\xi(t)\|$ is increasing. We turn to the claims in order.

Claim 1: The first claim follows by transversality, once we have shown that $T_x A_j \oplus T_x\mathbb{R}\langle (v,w,0), e_{m+1}\rangle = T_x\mathbb{R}^{m_j^+ +1}$.

We first consider $x \in B_j$. Let us replace $\underline{\mathrm{St}}^{-1}$ by $\mathrm{St}_{m_j}^{-1}$ for the moment. In this case we obtain $B_j' = \mathrm{St}_{m_j}^{-1}(\tau(A_{j-1} \times D^{m_j})) \cup \delta D^{m_j} + (1-\tilde{\delta})e_{m+1}$.

We have a smooth action $A\colon S_-^{m_j+1} \times \mathbb{R}^{m_j^+ +1} \to \mathbb{R}^{m_j^+ +1}$ given by the formula $A(w,v) = A(w)(v - (0, 1-\tilde{\delta})) + (0, 1-\tilde{\delta})$, where $A\colon S_-^{m_j+1} \to O(m_j+1)$ is the canonical section of $A \mapsto Ae_{m+1}$. It restricts to B_j' and $A.u(x) = u(A.x)$ holds by definition of the orthogonal group. Now, since every element $x \in B_j'$ is contained in the $S_-^{m_j+1}$–orbit of an element $(v,0,t) \in B_j' \cap \mathbb{R}^{m_{j-1}^+} \times \{0\} \times \mathbb{R}$, we get $u \notin T_x B_j'$ for all $x \in B_j'$ and thus $T_x B_j' \oplus T_x\mathbb{R}\langle (v,w,0), e_{m+1}\rangle = T_x\mathbb{R}^{m_j^+ +1}$.

There is a smooth $S_-^{m_j+1}$–action on B_j given by $w.(v,0,t) \mapsto \underline{\mathrm{St}}^{-1}(v,w,t)$, too. Unfortunately it is not compatible with u so the argument does not carry over immediately.

Nevertheless, we can use it. First we note that there is an isotopy of embeddings from the embedding $\iota\colon B_j \subseteq \mathbb{R}^{m_j^+ +1}$ to an embedding ι' of B_j' which is induced by pre-composing the map β in the definition of $\underline{\mathrm{St}}^{-1}$ with a smooth homotopy $\phi\colon [0,1] \times [0,1] \to [0,1]$ with $\phi_0 = \mathrm{const}_0$ and $\phi_1 = \mathrm{Id}$.

Now, since there is a vector $(v,0,t') \in T_{(v,0,s')}\mathbb{R}^{m_j^+ +1}$ which is orthogonal to $T_{(v,0,s')}B_j'$ for every $(v,0,s') \in B_j'$, we obtain a vector (v,w,t) orthogonal to $T_{(v,w,s)}B_j'$ for every $\iota'(x) = (v,w,s) \in B_j'$. The scalar product is continuous, hence we can choose the parameter $\bar{\delta}$ of β sufficiently small to ensure that (v,w,t) associated to $\iota'(x)$ understood as a vector $(v,w,t) \in T_{\iota(x)}$, (i.e. transported with respect to the flat connection) is linearly independent of $T_{\iota(x)}B_j$.

The argument for C_j is now immediate by parallel transport to ∂B_j and the first claim follows.

Claim 2: We fix $v \in \mathbb{R}^{m_{j-1}^+}$, $w \in \mathbb{R}^{m_j}$ and $t \in I$ with $\|v\| = \|w\| = 1$ and study the set $A_j \cap ((\mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\})$. As explained above we may understand B_j as the orbit of $B_j \cap \mathbb{R}^{m_{j-1}^+} \times \{0\} \times \mathbb{R}$ under $S_-^{m_j}$. Therefore (and by definition of the action) $B_j \cap \mathbb{R}\langle (v,0,0),(0,w,0),e_{m+1}\rangle$ may be obtained as the orbit of $B_j \cap \mathbb{R}\langle (v,0,0), e_{m+1}\rangle$ under the action of $D^1 w = S_-^{m_j} \cap \mathbb{R}\langle (0,w,0), e_{m+1}\rangle$.

By induction $B_j \cap (\mathbb{R}_{\geq 0}(v,0,0) \oplus \mathbb{R}e_{m+1})$ is a smooth curve which is strictly increasing after projection to I. By definition of $\underline{\mathrm{St}}^{-1}$ we also obtain a smooth curve $D_{\geq 0}^1 w.(av,0,s)$, which is strictly increasing after projection to I, where $(av,0,s) \in B_j \cap (\mathbb{R}_{\geq 0}(v,0,0) \oplus \mathbb{R}e_{m+1})$. We claim that there is at most one element (av,bw,t) contained in the intersection $B_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\}$ for every $a/b \in [0,\infty]$. The other directions follow by using $\pm v$ and $\pm w$.

Let us assume that (av, bw, t) is such an element. Then there is a unique element $(av, 0, s)$ in its orbit. Now let $s' > s$. Then there is at most one element $(a'v, 0, s')$ in the curve $B_j \cap (\mathbb{R}_{\geq 0}(v,0,0) \oplus \mathbb{R}e_{m+1})$. By induction it has to satisfy $a' \geq a$. Let us assume that, if there was an element of the form $(a'v, b'w, t)$ in the orbit of $(a'v, 0, s')$, then it would satisfy $b' < b$. But then $a/b < a'/b'$ and thus there is at most one element (av, bw, t) in the intersection $B_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\}$ for every $a/b \in [0,\infty]$.

We have to show that $b' < b$ does indeed hold in the above situation. There are $r, r' \in [0,1]$ with $(av, bw, t) = rw.(av, 0, s)$ and $(a'v, b'w, t) = r'w.(a'v, 0, s')$. By definition of $\underline{\mathrm{St}}_{m_j}^{-1}$, they satisfy

$$\lambda(s)\frac{r^2-1}{r^2+1} = \lambda(s')\frac{r'^2-1}{r'^2+1}.$$

Since λ is strictly decreasing, we have $\lambda(s') < \lambda(s)$, hence

$$\frac{r^2-1}{r^2+1} > \frac{r'^2-1}{r'^2+1}$$

which yields $r > r'$. Again by definition of $\underline{\mathrm{St}}^{-1}$ we have

$$b = \lambda(s)\beta(r) + \lambda(s)(1-\beta(r))\frac{2r}{r^2+1}$$

and similarly for b'. Since $r > r'$ and $s < s'$ we obtain $\beta(r) \geq \beta(r')$ and $\lambda(s) > \lambda(s')$. A straightforward calculation now shows $b > b'$ as desired.

We have for $0 \leq t \leq 1 - \tilde{\delta}$

$$A_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\} = B_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\})$$

and for $1 - \tilde{\delta} \leq t \leq 1$

$$A_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{t\}) = (B_j \cap \mathbb{R}\langle (v,0),(0,w)\rangle \times \{1-\tilde{\delta}\}))^{\langle 1-\tilde{\delta}\rangle} \times \{t\}$$

holds. Therefore, the map $\mathrm{pr}_I \colon A_j \cap (\mathbb{R}_{\geq 0}(v,w,0) \oplus \mathbb{R}e_{m+1}) \to I$ is injective.

<u>Claim 3</u>: We also have to show that if (av, bw, s) and $(a'v, b'w, s')$ are contained in $A_j \cap (\mathbb{R}_{\geq 0}(v,w,0) \oplus \mathbb{R}e_{m+1})$ and fulfil $s < s'$, then for every $s < s'' < s'$ there is some $(a''v, b''w, s'')$ in $A_j \cap (\mathbb{R}_{\geq 0}(v,w,0) \oplus \mathbb{R}e_{m+1})$. This is enough to show that $\mathrm{im}(\mathrm{pr}_I)$ is an interval. It is closed because it is the image of a closed set.

We know that $(a'v, b'w, s') = (rav, rbw, s')$ for some $r \in \mathbb{R}_{\geq 0}$ and $b' \in [0,1]$ holds by definition of $\underline{\mathrm{St}}^{-1}$. By induction there is some element $(\tau av, 0, h)$ in A_{j-1} for every $\tau \in [1, r]$ as $A_{j-1} \cap (\mathbb{R}_{\geq 0}v \oplus \mathbb{R}e_{m+1})$ is a curve. Now the orbit $D^1_{\geq 0}w.(\tau av, 0, h)$ contains $(\tau av, 0, h)$ and $(\tau av, rbw, h')$. Hence the intermediate value theorem implies that it contains $(\tau av, \tau bw, h'')$ as well.

Since we have $A_j = S_-^{m_j}.A_{j-1}$ and $S^{m_j^+} = S_-^{m_j}.S^{m_{j-1}^+}$ it follows by induction that, as a topological manifold, A_j is isotopic (via locally flat embeddings) to $S_-^{m_j^+} + e_{m+1}$ relative boundary (i.e. the isotopy restricts to an isotopy of the boundary) and relative to the point e_{m+1}. Therefore the complement $\mathbb{R}^{m_j^+} \times \mathbb{R}_{\leq 1} - A_j$ consists of precisely two connected components, one of which is compact and contains the vector e_{m+1}.

If $\mathrm{im}(\mathrm{pr}_I)$ was not an interval, it would be a disjoint union of closed subintervals $I_1, \dots, I_t = \mathrm{im}(\mathrm{pr}_I)$, which we assume ascendingly ordered with respect to their distance to $0 \in I$, for some $t \in \mathbb{N}$. If $1 \notin I_t$, the curve $s \mapsto se_1 + e_{m+1}$ for $s \in \mathbb{R}_{\geq 0}$ contradicts the decomposition into two connected components. So suppose that $1 \in I_t = [1-t', 1]$. Write $I_{t-1} = [1-t'', 1-t'-\epsilon]$ for some $\epsilon > 0$ and $t', t'' \in [0,1]$. Let $\sigma_1(s) = (1-(t'+\epsilon/2)s)e_{m+1}$ for $s \in [0,1]$ and $\sigma_2(r) = re_1 + (1-t'-\epsilon/2)e_{m+1}$. Then the concatenation $\sigma_1 * \sigma_2$ yields a contradiction, since $A_j \cap \mathbb{R}e_{m+1}$ consists of a single point by induction, which is not the maximum of pr_I on A_j by induction for the case $w = 0$, and by definition of $\underline{\mathrm{St}}^{-1}$ for the case $w \neq 0$. □

Remark 1.3.0.8. *Let $H_{m_i^+} \colon \mathbb{R}^{m_i^+} \times I \to \mathbb{R}^{m_i^+} \times I$ denote a diffeomorphism.*

Note that the same proof would apply to every construction A'_j similar to that of A_j, in which we replace the map $\underline{\mathrm{St}}_{m_i}^{-1}$ by $\underline{\mathrm{St}}_{m_i}^{-1} \circ H_{m_i^+}$, as long as the inductive assumptions on $A_i \cap (\mathbb{R}_{\geq 0}(v,0) \oplus \mathbb{R}e_{m+1})$, specified at the beginning of the proof, are satisfied if we replace A_i by $H_{m_i^+}(A'_i)$.

In particular, if H was a linear isotopy of diffeomorphisms such that H_0 and H_1 preserved the properties, then the same holds for every H_s, $s \in [0,1]$.

Our discussion so far was concerned with the case of a fixed value $t \in I$. The next lemma states that the different values do not interfere with each other.

Lemma 1.3.0.9. *The subsets A_k are disjoint for different $t \in I$ and the subspace inclusions $A_k \subseteq \mathbb{R}^{m+1}$ assemble to a smooth embedding of a subspace $\mathbf{A}_k \subseteq \mathbb{R}^{m+1}$.*

Proof. Showing that the A_k are disjoint is a straightforward induction, employing that $\underline{\mathrm{St}}_n^{-1}$ is a smooth embedding. The second part follows by copying the proof of Lemma 1.3.0.6 and replacing A, B and C by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ throughout. □

Finally, we are in position to describe the desired isotopies which identify the level sets of the different transfers.

Lemma 1.3.0.10. *There is a smooth map $\chi \colon \mathbb{R}^{m+1} \to \mathbb{R}_{>0}$ such that the multiplication $m_\chi \colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ given by $x \mapsto \chi(x)x$ is a diffeomorphism which maps $A_1(t) = \underline{\mathrm{St}}_m^{-1}(D^m \times \{t\}) \cup (\underline{\mathrm{St}}_m^{-1}(\partial D^m \times \{t\}) \times [1-\tilde{\delta}, 1])$ to $A_k(t)$ for every $t \in I$. The linear homotopy H_χ between m_χ and the Identity is an isotopy of diffeomorphisms.*

Moreover, the multiplication m_χ is $\prod_{i=1}^k O(m_i)$–equivariant, hence so is H_χ.

Proof. This is just a change of coordinates. Since $S^m \times \mathbb{R} \cong \mathbb{R}^{m+1} - \{0\} \subseteq \mathbb{R}^{m+1}$ is a smooth embedding and transverse to A_k, the implicit function theorem and Lemma 1.3.0.7 yield the existence of χ. Now m_χ is given by a family of orientation-preserving diffeomorphisms of $\mathbb{R}$ parametrised over S^m. Hence we obtain the desired isotopy. The maps are equivariant since $A_1(t)$ and $A_k(t)$ are. □

Remark 1.3.0.11. *The map*

$$\{(\bar{\delta}, \tilde{\delta}, \delta, \delta^\dagger) \in (0, 1/4) | \bar{\delta} > \tilde{\delta} > \delta > \delta^\dagger\} \times \mathrm{Riem}(R) \to C^\infty(\mathbb{R}^{m+1} \times [0,1], \mathbb{R}^{m+1})$$
$$((\delta, \tilde{\delta}, \delta^*, \bar{\delta}), R) \mapsto H_{\chi, \delta, \tilde{\delta}, \delta^\dagger, \bar{\delta}, R}$$

is continuous.

The only step left is to describe a contractible space of isotopies.

Definition 1.3.0.12. Let $m = \sum_{i=1}^{k} m_i$ be a partition. By Lemma 1.3.0.10 we obtain a map χ associated to it. The space $I^1(m)$ is the convex hull of the associated maps m_χ for sufficiently small $\bar{\bar{\delta}}, \tilde{\delta}, \delta, \delta^\dagger$ and all partitions.

Definition 1.3.0.13. There is a composition $I^1(m_2) \times I^1(m_1) \to \mathrm{Diff}(\mathbb{R}^m \times I)$ as follows: We send a pair of partitions $m_1 = \sum_{i=1}^{k_1} m_i^{(1)}$ and $m_2 = \sum_{i=1}^{k_2} m_i^{(2)}$ to $m_1 + m_2 = \sum_{i=1}^{k_1+k_2} m_i^{(1+2)}$ with $m_i^{(1+2)} = m_i^{(1)}$ for $1 \leq i \leq k$ and, otherwise, $m_i^{(1+2)} = m_{i-k_1}^{(2)}$.

Accordingly, we send the associated pair of maps $m_\chi^{(1)}, m_\chi^{(2)}$ to the map associated to the above partition and extend linearly, i.e we define spaces A'_j where we use linear isotopies of diffeomorphisms (contained in the convex hull of the maps m_χ) to modify our construction. By Remark 1.3.0.8 we obtain a map associated to the resulting space.

We define $I^2(m)$ as the convex hull of all maps in the image of some composition map with respect to the partition. We iterate the construction to obtain $I^n(m)$ for every $n \in \mathbb{N}$.

Although we do not need it, we observe that if our partition has k summands, then $I^n(m) = I^k(m)$ for $n \geq k - 1$.

Definition 1.3.0.14. We set $I(m) = \bigcup_{n \in \mathbb{N}} I^n(m) = I^{k-1}(m)$ with respect to a partition $m = \sum_{i=1}^{k} m_i$.

Remark 1.3.0.15. *In Section 1.4, we could use a far larger class of embeddings, determined by the properties required for Lemma 1.3.0.7 and some minor conditions to ensure that we obtain an induced pseudoisotopy, similar to the one defined in Definition 1.1.1.15. However, since there is no apparent advantage to this, we are going to work with a set determined by $I(m)$.*

1.4 The Homotopy Coherent Diagram

The main proof is contained in Section 1.4.1. The case of spectra is taken care of in Section 1.4.2. We give a brief summary of the parameters chosen throughout this thesis for the variations of the stereographic inverse in Section 1.4.3.

1.4.1 Proof of the main theorem

In this section we pass from the quasicategory $\mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta$, see Definition 1.2.0.2, to spaces of pseudoisotopies. First, we show that it is enough to construct an unstable pseudoisotopyfunctor (in the sense of quasicategories) with source $\mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta$.

The real work, however, is to construct the unstable coherent diagram. We use the obstruction theory established in Section 1.2.1.

The main idea is as follows: Given a simplex of $\mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta$ we consider the iterated pullback of the tubular neighbourhoods to obtain a bundle with a product of disks as fibre. Outside of this subspace, the pseudoisotopy functor is given by extending with the Identity.

To define the functor on the subspace we pass to local trivialisations and give a fibre-wise definition which is invariant under change of trivialisations, i.e. equivariant under the action of the structure group of the bundle. This requires the local results explained in Section 1.3.

Since the geometric constructions we have established so far depend C^∞–continuously on certain contractible choices of parameters we can adjust all choices as required (in particular we may assume all of the δ to be small enough).

Lemma 1.4.1.1. *Let*

$$P\colon \mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$$

be a functor of quasicategories (i.e. a map of simplicial sets) such that

1. *The space $P(M)$ is the (singular simplicial set of the) unstable pseudoisotopy space of M, see Definition 1.1.0.5.*
2. *A 1–simplex $(\iota, M, \mathrm{Id})\colon M \to N$ with ι a codimension zero embedding is sent to the induced map $\iota_*\colon P(M) \to P(N)$, see Definition 1.1.0.6.*
3. *A 1–simplex of the form $(\iota, \nu\iota, p)\colon M \to \nu\iota$ is sent to the transfer map $p_!\colon P(M) \to P(\nu\iota)$, see Definition 1.1.1.15.*
4. *Let $G\colon \mathfrak{S}(\Delta^k) \to \mathcal{D}_\Delta$ be a k–simplex with $G(0) = M$ and $G(k) = N$, and σ an m–simplex in $\mathfrak{S}(\Delta^k)(0,k)$ with $G(\sigma) = (\iota, \nu\iota, p)$ in $\mathcal{D}_\Delta(M,N)$. Let $i_M\colon P^{\mathrm{Diff}}(M) \to P^{\mathrm{Top}}(M)$ denote the inclusion*

   There are a topological parallel transport ν_p induced by a geometric transfer, see Definition 2.1.3.5, a bending isotopy H_p in the sense of Definition 2.1.3.4 with respect to ν_p and an m–simplex $(\iota, \nu\iota, p, \nu_p, H_p)$ in $\mathrm{Ch}_\Delta(M,N)$ such that $P^{\mathrm{Top}}(\iota, \nu\iota, p, \nu_p, H_p) \circ i_M = i_N \circ P^{\mathrm{Diff}}(G)(\sigma)$ holds.

Then there is a functor

$$\mathbb{P}\colon \mathrm{Ind}(\mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta) \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Kan}_\Delta$$

with

1. *The space* $\mathbb{P}(M)$ *is the (singular simplicial set of the) stable pseudoisotopy space* $\operatorname{hocolim}_{n\in\mathbb{N}} P(M \times (D^1)^n)$ *of* M, *see Definition 1.1.1.20.*

2. *A* 1*–simplex* $(\iota\times\mathrm{Id}, M\times(D^1)^n, \mathrm{Id})_{n\in\mathbb{N}} \colon (M\times(D^1)^n)_{n\in\mathbb{N}} \to (N\times(D^1)^n)_{n\in\mathbb{N}}$ *with* ι *a codimension zero embedding is sent to*

$$\operatorname{hocolim}_{n\in\mathbb{N}}(\iota \times \mathrm{Id}_{(D^1)^n})_* \colon \mathbb{P}(M) \to \mathbb{P}(N),$$

see Definition 1.1.0.6.

3. *A map* $(\iota \times \mathrm{Id}, \nu(\iota \times \mathrm{Id}), p \times \mathrm{Id})_{n\in\mathbb{N}} \colon (M \times (D^1)^n)_{n\in\mathbb{N}} \to (\nu\iota \times (D^1)^n)_{n\in\mathbb{N}}$ *is sent to the transfer map*

$$\operatorname{hocolim}_{n\in\mathbb{N}}(p \times \mathrm{Id}_{(D^1)^n})_! \colon \mathbb{P}(M) \to \mathbb{P}(\nu\iota),$$

see Definition 1.1.1.15.

4. *We have* $\mathbb{P}^{Top} \circ j \simeq \mathbb{P}^{\mathrm{Diff}}$ *with* $j\colon \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta) \to \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet)\,\mathrm{Ch}_\Delta$ *as defined in Remark 1.2.5.3.*

Proof. We obtain a functor between Ind–quasicategories

$$\mathrm{Ind}(P)\colon \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta) \to \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta).$$

Given a quasicategory $\mathcal{C}$ with countably filtered homotopy colimits there is an equivalence $\mathrm{Ind}(\mathcal{C}) \simeq \mathcal{C}$. Since all homotopy colimits exist in $\mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta$, we get a functor

$$\mathbb{P}\colon \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta) \to \mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta$$

and the first three properties are an immediate consequence of the conditions on P. For the comparison with $\mathbf{P}^{\mathrm{Top}}$ it is unfortunately necessary to understand its definition. We refer to Section 2.1.3, specifically statement 6 of Theorem 2.1.3.21. □

Theorem 1.4.1.2. *Suppose that the assumptions of Lemma 1.4.1.1 hold. Then there is a functor* $\mathbb{P}^{\mathrm{Diff}} \colon \mathrm{Top} \to \mathrm{Top}$ *with the following properties:*

1. *It descends to a functor of homotopy categories*

$$\mathrm{ho}\,\mathbb{P}^{\mathrm{Diff}} \colon \mathrm{ho}(\mathrm{Top}) \to \mathrm{ho}(\mathrm{Top}).$$

2. *There is a natural isomorphism of functors*

$$\alpha\colon \mathrm{ho}\,\mathbb{P}^{\mathrm{Diff}} \to \mathrm{ho}\,\mathbb{P}^{\mathrm{Diff},\mathrm{BL}}.$$

3. *The subset inclusion* $\mathbb{P}^{\mathrm{Diff}} \subseteq \mathbb{P}^{\mathrm{Top}}$ *extends to a natural transformation of functors of quasicategories. The construction of* $\mathbb{P}^{\mathrm{Top}}$ *is given in [12] and recalled in Section 2.1.3.*

Proof. Recall from Theorem 1.2.0.5 the map $F_{\mathrm{ch}} \colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta$. By Lemma 1.4.1.1 we obtain a map $\mathbb{P} \circ F_{\mathrm{ch}} \colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}, \mathrm{cts})_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta$ which, according to Theorem 1.2.1.3, admits a strictification to a simplicially enriched functor $\mathbb{P}^{\mathrm{Diff}} \colon \mathrm{Mfd}_\Delta \to \mathrm{Kan}_\Delta$ where Mfd_Δ has smooth manifolds with corners as objects and the singular sets of the spaces of continuous maps as mapping spaces.

This is not quite enough, since we originally set out to define a functor from topological spaces to Kan-complexes. However, we obtain a diagram

$$\begin{array}{ccc} \mathrm{Mfd}_\Delta & \xrightarrow{\mathbb{P}} & \mathrm{Kan}_\Delta \\ \big\downarrow & \nearrow & \\ \mathrm{Top}_\Delta & & \end{array}$$

of simplicially enriched categories. We extend $\mathbb{P}$ as a homotopy left Kan extension to $\mathbb{P}\colon \mathrm{Top}_\Delta \to \mathrm{Kan}_\Delta$. Its value on a space X is $\mathbb{P}(X) = \mathrm{hocolim}_{\mathrm{Mfd}\downarrow X}\,\mathbb{P}$. Here $\mathrm{Mfd} \downarrow X$ is the category whose objects are pairs (M, f), where M is a smooth, compact manifold with corners and $f\colon M \to X$ is a continuous map, and morphisms are continuous maps $g\colon M \to M'$ compatible with the reference maps, i.e. $f' \circ g = f$ holds. We define the desired 1–functor as the restriction of this simplicially enriched functor.

The first property is immediate, since our functor is the restriction of a simplicially enriched functor. The stable pseudoisotopy functor on the homotopy category defined by Burghelea and Lashof is constructed analogously to the 1–skeleton of our pseudoisotopy functor, hence the second claim follows.

We show the last property as part of Theorem 2.1.3.21. □

In contrast to the case of the topological pseudoisotopyfunctor given in [12] we are not going to define a simplicially enriched pseudoisotopy functor on a category of choices.

Let us assemble all choices relevant for the definition of our diagram which are not concerned with the construction in local coordinates in a single fibre.

Definition 1.4.1.3. Let $G\colon \mathfrak{S}(\Delta^k) \to \mathcal{D}_\Delta$ denote a simplicially enriched functor.

Let $\mathcal{D}_\Delta^G$ denote the simplicially enriched category with the same objects as $\mathfrak{S}(\Delta^k)$ and the following morphism spaces: For $j > i$ the space $\mathfrak{S}(\Delta^k)(i, j)$ is empty. Otherwise, an n–simplex consists of

1. an ordered subsequence $\mathcal{I} = \{i = i_1, i_2, \ldots, i_t = j\}$ of $i, i+1, \ldots, j$.

2. an n–simplex $(\iota_r, \nu\iota_r, p_r)$ in $\mathcal{D}_\Delta(G(i_r), G(i_{r+1}))$ for every $1 \le r \le t-1$.

3. a Riemannian metric on $\nu\iota_r$, i.e. a smooth map $T\nu\iota_r \otimes T\nu\iota_r \to \mathbb{R}_{\ge 0}$, which restricts to a Riemannian metric over each $t \in |\Delta^n|$, such that the tubular neighbourhood is a disk bundle with respect to the metric for every $1 \le r \le t-1$.

We also obtain a partition

$$\dim(G(j)) - \dim(G(i)) = \sum_{r=1}^{t-1} \dim(G(i_{r+1})) - \dim(G(i_r))$$

associated to the subsequence $\mathcal{I}$.

Now we introduce our local parameters. Essentially, they amount to a level set in the sense of Section 1.3, a tubular neighbourhood of our level set, and a choice of coordinates for our tubular neighbourhood.

Definition 1.4.1.4. Let $m = \sum_{i=1}^{k} m_i$ be a partition. Recall from Definition 1.3.0.4 that $\underline{\mathrm{St}}_m^{-1}(D^m \times \{1/2\}) = B_1(1/2)$. There is an embedding emb_1 defined analogously to $\underline{\mathrm{St}}_m^{-1}\colon D^m \times \{1/2\} \hookrightarrow D^m \times I$ with image $A_1(1/2)$, obtained by smoothly stretching the I–coordinate.

Let $\theta\colon (D^m, \partial D^m) \hookrightarrow \left(\prod_{i=1}^{k} D^{m_i} \times I, \prod_{i=1}^{k} D^{m_i} \times \{1\}\right)$ be an embedding of the form $\sigma \circ \mathrm{emb}_1$ for $\sigma \in I(m)$, see Definition 1.3.0.14.

It is a smooth and $\prod_{i=1}^{k} O(m_i)$–equivariant embedding such that

1. $\mathrm{im}(\theta)$ intersects $L_\zeta = \{r\zeta + e_{m+1} | r \in \mathbb{R}_{\geq 0}\}$, i.e. the ray from the "origin" in the direction of ζ, in exactly one point for every $\zeta \in S_-^{m_j^+}$.
2. θ is parallel to e_{m+1} on a neighbourhood of the boundary, i.e. there is a neighbourhood U of ∂D^m in D^m such that $\theta(x) = (\theta_1(x/\|x\|), \theta_2(x))$ holds for every $x \in U$.

We call $\mathrm{im}(\theta)$ a *good image* and denote the space of good images by $E(m)$. An n–simplex in $E(m)$ is a fibre preserving smooth family of good images over $|\Delta^n|$.

Definition 1.4.1.5. Let θ be an embedding with good image. We define $\tilde{p}\colon \nu\theta \to \mathrm{im}(\theta)$ as a 1–dimensional $\prod_{i=1}^{k} O(m_i)$–equivariant embedded tubular neighbourhood of θ such that

1. there is a neighbourhood U of $\partial\,\mathrm{im}(\theta)$ in $\mathrm{im}(\theta)$ such that on U we have $\tilde{p} = \tilde{q} \times \mathrm{Id}_I$ for a 1–dimensional embedded tubular neighbourhood $\tilde{q}$ of $\partial\,\mathrm{im}(\theta)$ in $\prod_{i=1}^{k} D^{m_i} \times \{1\}$, i.e. the tubular neighbourhood is orthogonal to e_{m+1} on a neighbourhood of the boundary.
2. the intersection of $\nu\theta$ and the boundary of $\prod_{i=1}^{k} D^{m_i} \times I$ is $\tilde{p}^{-1}(\partial\,\mathrm{im}(\theta))$. Further, neither the boundary of $\prod_{i=1}^{k} D^{m_i} \times \{1\}$ nor e_{m+1} are contained in the intersection.

We call $\tilde{p}$ a *good tubular neighbourhood* and denote the space of these by $\mathrm{Tub}(\mathrm{im}(\theta))$. An n–simplex in $\mathrm{Tub}(\mathrm{im}(\theta))$ is a fibre preserving smooth family of good tubular neighbourhoods over $|\Delta^n|$.

Definition 1.4.1.6. The complement of $\nu\theta$ in $\prod_{i=1}^{k} D^{m_i} \times I$ consists of two connected components.

Let $\phi\colon D^m \times I \to \nu\theta$ denote an $\prod_{i=1}^{k} O(m_i)$–equivariant trivialisation of the disk bundle such that $\phi(D^m \times \{1\})$ bounds the connected component which contains e_{m+1}.

We call ϕ a *good trivialisation* and denote the space of these by $\mathrm{Triv}(\nu\theta)$. An n–simplex in $\mathrm{Triv}(\nu\theta)$ is a fibre preserving smooth family of good trivialisations over $|\Delta^n|$.

The trivialisations are in bijection with the $\prod_{i=1}^{k} O(m_i)$–equivariant maps $\phi'\colon D^m \to \mathrm{Diff}_+^0(I)$ such that the adjoint $D^m \times I \to I$ is smooth. Here $\mathrm{Diff}_+^0(I)$ denotes the space of orientation preserving diffeomorphisms from I to itself which preserve $1/2$ (which corresponds to the zero section).

Remark 1.4.1.7. *The space of good images $E(m)$ is contractible since $I(m)$ is convex by definition. The space of good tubular neighbourhoods* $\mathrm{Tub}(\mathrm{im}(\theta))$ *is contractible via Lemma 1.2.2.5. The space of good trivialisations* $\mathrm{Triv}(\nu\theta)$ *is contractible by the characterisation in terms of* $\mathrm{Diff}_0(I)$.

Definition 1.4.1.8. We denote the space of compatible triples $(\mathrm{im}(\theta), \tilde{p}, \phi)$ by $G(m)$. We note that it depends on the choice of a partition of m.

Lemma 1.4.1.9. *The space $G(m)$ is contractible.*

Proof. We fix some 0–simplex. Let $(\mathrm{im}(\theta), \tilde{p}, \phi)$ be an n–simplex of $G(m)$.

We characterise $\tilde{p}$ as the image under the exponential map of the normal vector field orthogonal to the tangent space and change ϕ to the exponential map, see Lemma 1.2.2.5. Then an isotopy of θ induces isotopies of $\tilde{p}$ and ϕ.

Since $\mathrm{im}(\theta)$ has unique points of intersection, we find an isotopy from $\mathrm{im}(\theta)$ to the first entry of our fixed 0–simplex.

These constructions are compatible with the simplicial structure maps, hence $G(m)$ is a contractible Kan-complex. □

This concludes our choices. Before we define how our choices induce a map between spaces of pseudoisotopies we have to set up appropriate coordinates.

Remark 1.4.1.10. *Given an enriched functor $G\colon \mathfrak{S}(\Delta^k) \to \mathcal{D}_\Delta$, we obtain a sequence $G(0) \leftarrow \nu\iota_0 \hookrightarrow G(1) \leftarrow \nu\iota_1 \hookrightarrow \ldots \leftarrow \nu\iota_{k-1} \hookrightarrow G(k)$ of vectorbundles $p_j\colon \nu\iota_j \to G(j)$ and codimension zero embeddings $\iota_j\colon \nu\iota_{j-1} \hookrightarrow G(j)$. Note that we abuse notation and identify $G(j)$ with its image $\iota_j(G(j))$. Let m_i be the dimension of a fibre of $p_i\colon \nu\iota_{i-1} \to G(i-1)$. We define the tubular neighbourhood $\nu\iota = (\iota_0^* \circ \iota_1^* \circ \ldots \circ \iota_{k-1}^*)(\nu\iota_{k-1})$ inductively via the pullbacks*

$$\begin{array}{ccc} \iota_n^*(\nu\iota_n) & \xrightarrow{\iota_n} & \nu\iota_n \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_n} \\ \nu\iota_{n-1} & \xrightarrow{\iota_n} & G(n) \end{array}$$

and

$$\begin{array}{ccc} (\iota_j^* \circ \iota_{j+1}^* \circ \ldots \circ \iota_n^*)(\nu\iota_n) & \xrightarrow{\iota_j} & (\iota_{j+1}^* \circ \ldots \circ \iota_n^*)(\nu\iota_n) \\ \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle p_n} \\ (\iota_j^* \circ \iota_{j+1}^* \circ \ldots \circ \iota_{n-1}^*)(\nu\iota_{n-1}) & \xrightarrow{\iota_j} & (\iota_{j+1}^* \circ \ldots \circ \iota_{n-1}^*)(\nu\iota_{n-1}). \end{array}$$

We obtain an embedding $\iota\colon \nu\iota \hookrightarrow G(k)$ given by $\iota_k \circ \iota_{k-1} \circ \ldots \circ \iota_0$ and a bundle $p\colon \nu\iota \to G(0)$ with fibre $\prod_{i=1}^k D^{m_i}$ via $p = p_0 \circ p_1 \circ \ldots \circ p_k$. The composed tubular neighbourhood of the p_i is the maximal disk subbundle of the bundle $p\colon \nu\iota \to G(0)$.

We get an analogue for every subsequence $\mathcal{I} = \{i_0 = i < i_1 < \ldots < i_r = j\}$ of $\{0, 1, \ldots, k\}$. We denote the composition $(\iota_{i_{q+1}-1}, p_{i_{q+1}-1}) \circ \ldots \circ (\iota_{i_q}, p_{i_q})$ with $0 \le q \le r-1$ by $(\iota_q^{\mathcal{I}}, p_q^{\mathcal{I}})$. Then we obtain a $\prod_{q=0}^{r-1} D^{m_{i_q}}$–bundle $p^{\mathcal{I}}\colon \nu\iota^{\mathcal{I}} \to G(i)$ given by $p^{\mathcal{I}} = p_0^{\mathcal{I}} \circ p_1^{\mathcal{I}} \circ \ldots \circ p_{r-1}^{\mathcal{I}}$ with $p_q^{\mathcal{I}}$ the appropriate pullback. There is also a codimension zero embedding $\iota^{\mathcal{I}}\colon \nu\iota^{\mathcal{I}} \hookrightarrow G(j)$.

Finally, we can define the key to the construction of our homotopy coherent diagrams. The idea is as follows: Outside of the bundle just introduced, all compositions of transfers are given by the Identity.

In local coordinates, the complement of our tubular neighbourhood $\nu\theta$ of a level set in each fibre consists of two pieces. We use Id and $F(-,1)$ on those, similar to Definition 1.1.1.15. Finally, we use the trivialisation ϕ of our tubular neighbourhood $\nu\theta$ to define our map as $\phi \circ (F \times \mathrm{Id}) \circ \phi^{-1}$ on $\nu\theta$.

Definition 1.4.1.11. Let $G\colon \mathfrak{S}(\Delta^n) \to \mathcal{D}_\Delta$ be a simplicially enriched functor. We define a map

$$\Phi\colon \mathcal{D}^G_\Delta(i,j) \times_{\mathcal{I}} G(m_j^+ - m_i^+) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(i)), \mathcal{S}_\bullet P(G(j))$$

where $\mathcal{D}^G_\Delta(i,j) \times_{\mathcal{I}} G(m_j^+ - m_i^+)$ is the subspace of $\mathcal{D}^G_\Delta(i,j) \times G(m_j^+ - m_i^+)$ consisting of elements (d, G) with G in the space $G(m_j^+ - m_i^+)$ with respect to the partition associated to d.

By adjunction we obtain an n-simplex of $\mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(i)), \mathcal{S}_\bullet P(G(j))$ by constructing a map $\mathcal{S}_\bullet P(G(i)) \times \Delta^n \to \mathcal{S}_\bullet P(G(j))$.

We define the desired map via two maps $\mathcal{S}_\bullet P(G(i)) \times \Delta^n \to \mathcal{S}_\bullet P(\nu\iota^{\mathcal{I}})$ and $\mathcal{S}_\bullet P(\nu\iota^{\mathcal{I}}) \times \Delta^n \to \mathcal{S}_\bullet P(G(j))$ where the latter map is induced by the codimension zero embedding incl: $\nu\iota^{\mathcal{I}} \subseteq G(j)$.

We now turn to the construction of the first map. We reduce the structure group of our bundle $p^{\mathcal{I}}\colon \nu\iota^{\mathcal{I}} \to G(i)$ to $\prod_{q=i}^{r-1} O(m_{i_q}) \subseteq O(m_j^+ - m_i^+)$. So there is a cover $\{(U_x, \phi_x)\}_{x\in\mathcal{X}}$ of $G(i)$ consisting of local trivialisations such that a change of trivialisations is given by a map $g_{xy}\colon U_x \cap U_y \to \prod_{q=i}^{r-1} O(m_{i_q})$. From here on out, we are going to work with constructions which are equivariant with respect to the $\prod_{q=i}^{r-1} O(m_{i_q})$–action.

Let d be an element in $\mathcal{D}_\Delta(i,j)$, $G = (\mathrm{im}(\theta), \tilde{p}, \phi)$ an element in $G(m_j^+ - m_i^+)$, and let $F \in \mathcal{S}_{n'} P(G(i))$ denote an n'–simplex of the pseudoisotopy space. We write $F_{v'} \in P(G(i))$ for the restriction of F to $G(i) \times I \times \{v'\}$ for $v' \in |\Delta^{n'}|$.

Let U, V be open sets in $G(i)$ with $F(U \times I \times |\Delta|^{n'}) \subseteq V \times I \times |\Delta|^{n'}$. Let $v \in |\Delta^n|$. The set $\nu\theta_v$ divides $\mathbb{R}^m \times I$ into two connected components, say C_1 and C_2, out of which exactly one, say C_1, is bounded. We apply our map Φ at v to $F_{v'}$.

The image $\Phi(d,G)_v(F_{v'})$ is in these local coordinates given by

$$\Phi(d,G)_v(F_{v'})\colon U \times \prod_{q=0}^{r-1} D^{m_q} \times I \to V \times \prod_{q=0}^{r-1} D^{m_q} \times I$$

$$(x,t) \mapsto \begin{cases} (F_{v'}(u,1), x, t) \text{ for } (u,x,t) \in U \times C_1 \\ (\mathrm{Id} \times \phi) \circ (\mathrm{F}_{v'} \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi^{-1})(u,x,t) \\ \text{ for } (u,x,t) \in U \times \nu\theta_v \\ (u,x,t) \text{ for } (u,x,t) \in U \times C_2. \end{cases}$$

The following observation is essential to compare the smooth and topological pseudoisotopyfunctor.

Remark 1.4.1.12. *A bending map is a homeomorphism from $\prod_{q=0}^{r-1} D^{m_q} \times I$ to itself which identifies $\prod_{q=0}^{r-1} D^{m_q} \times \{0\}$ with $\prod_{q=0}^{r-1} D^{m_q} \times \{0\} \cup \partial(\prod_{q=0}^{r-1} D^{m_q}) \times I$. We note that the map $\Phi(d,G)_v(F_{v'})$ is in local coordinates given by conjugation* *of $F_{v'} \times \mathrm{Id}$ with a bending map. This follows from the isotopy extension theorem in the topological category.*

Since the transfer relative boundary in the topological case is defined by conjugating the absolute transfer with a bending map, there is a topological transfer relative boundary which restricts to the map $\Phi(d, G)$ on the subspace of smooth pseudoisotopies. Here we use Remark 1.1.0.11.

We also have to relate our construction to the maps for pseudoisotopies introduced in Section 1.1.

Remark 1.4.1.13. *By an induction argument, using Definition 1.1.1.15 of the transfer map, Definition 1.3.0.4 of A_r and Lemma 1.1.1.19 we see that the map $\iota_* \circ (p_j)_! \circ \ldots \circ (p_{i+1})_! \colon P(G(i)) \to P(G(j))$ is contained in the image of Φ.*

We proceed with the main argument. To link $\mathcal{D}_\Delta(G(i), G(j)$ to Φ we define a quotient of our choices $\mathcal{D}^G_\Delta(i,j) \times G(m^+_j - m^+_i)$.

Definition 1.4.1.14. There is a composition map on $\mathcal{D}^G_\Delta(i,j)$ which sends an element $(\mathcal{I}, \tilde{d}_1, \ldots, \tilde{d}_t, R_1, \ldots, R_t)$ to the tuple $(\{i,j\}, \tilde{d}_t \circ \ldots \circ \tilde{d}_1, R_t \circ \ldots \circ R_1)$ where $\tilde{d}_t \circ \ldots \circ \tilde{d}_1$ is the composition in $\mathcal{D}_\Delta$ and the composition of metrics R_1 and R_2 is defined as $R_{1+2}((x_1, x_2), (y_1, y_2)) = \sqrt{R_1(x_1, y_1)^2 + R_2(x_2, y_2)^2}$.

We call two n–simplices (d, G) and (d', G') in $\mathcal{D}^G_\Delta(i,j) \times G(m^+_j - m^+_i)$ equivalent, if d' and d have the same composition and $G = G'$. This is an equivalence relation.

We denote the quotient by $Q(\mathcal{D}^G_\Delta(i,j) \times G(m^+_j - m^+_i))$.

Remark 1.4.1.15. *The map Φ is compatible with the equivalence relation and we obtain $\Phi \colon Q(\mathcal{D}^G_\Delta(i,j) \times G(m^+_j - m^+_i)) \to \mathrm{Kan}_\Delta(P(G(i), P(G(j))$.*

Lemma 1.4.1.16. *The map $Q(\mathcal{D}^G_\Delta(i,j) \times G(m^+_j - m^+_i)) \to \mathcal{D}_\Delta(G(i), G(j))$, induced by the composition map, is a trivial fibration.*

Proof. By pulling back along some retraction $|\Delta^n| \to |\Lambda^n_k|$ we see that the map is a Kan fibration.

Let (d, G) be in the fibre over some d'. Since $G(m^+_j - m^+_i)$ is contractible, every G admits an isotopy to a fixed $O(m^+_j - m^+_i)$–equivariant G'. Together with the fact that the space of Riemannian metrics is convex this yields a homotopy to (d', G'). □

Since the obstructions to the construction of a homotopy coherent diagram are obtained by gluing together compositions of previous extension problems, we have to make sure that compositions of maps are in fact contained in the desired subspace.

Lemma 1.4.1.17. *Let G_1 in $(G(m^{(1)}))_n$ and G_2 in $G(m^{(2)})_n$ be good tuples.*

Let M be a compact manifold and $d_j \colon M \to M \times \prod_{i=1}^{k^{(i)}} D^{m_i^{(j)}}$ the canonical map in $\mathcal{D}_\Delta$ for $j = 1, 2$.

Let us assume that $\mathrm{im}(\theta_1)$ restricts on the boundary $\partial\Delta^n$ to a family of submanifolds $A'(t)$ constructed with respect to elements of $I(m^{(1)})$. Moreover, the tubular neighbourhood $\nu\theta_1$ is, over $\partial\Delta^n$, given by subsets of the line through the origin and x for every $x \in \mathrm{im}(\theta)$, i.e. it is given by a union of $A'(s)$ for $s \in [a,b] \subseteq [0,1]$. Finally, the trivialisation ϕ_1 is given by a fixed parametrisation of $A'(t)$ for every $s \in [a,b]$. Similarly for G_2.

In particular, there is a good tuple $\partial G = (\partial\theta, \nu\partial\theta, \partial\phi) \in G(m^{(1)} + m^{(2)})$ over $\partial\Delta^n$ such that $\phi(d_2, G_2)(\phi(d_1, G_1)(F)) = \phi(d_2 \circ d_1, \partial G)(F)$ holds upon restriction to $\partial\Delta^n$ for every pseudoisotopy F in $P(M)$.

Then G_1 and G_2 are isotopic relative boundary to good tuples G'_1 and G'_2 which satisfy the above condition over the whole simplex.

In particular, there is a good tuple $G' = (\theta, \nu\theta, \phi) \in G(m^{(1)} + m^{(2)})_n$, which restricts to ∂G over $\partial\Delta^n$, such that $\phi(d_2, G'_2)(\phi(d_1, G'_1)(F)) = \phi(d_2 \circ d_1, G')(F)$ holds for every pseudoisotopy F in $P(M)$.

Proof. The sets $\mathrm{im}(\theta_1)$ and $\mathrm{im}(\theta_2)$ are of the right form by definition. We obtain the desired isotopy from the characterisation of tubular neighbourhoods in Lemma 1.2.2.5. Adjusting the trivialisations is easy. □

Corollary 1.4.1.18. *Let G_1 in $(G(m^{(1)}))_n$ and G_2 in $G(m^{(2)})_n$ be good tuples which satisfy the same assumptions as in the above lemma.*

Let d_1 in $\mathcal{D}^G_\Delta(i,j)$ and d_2 in $\mathcal{D}^G_\Delta(j,l)$ be arbitrary with codimension $m^{(1)}$ and $m^{(2)}$, respectively.

Then the analogue of the above lemma holds. In particular, there is a good tuple $G' = (\theta, \nu\theta, \phi) \in G(m^{(1)} + m^{(2)})_n$ such that $\phi(d_2, G'_2)(\phi(d_1, G'_1)(F)) = \phi(d_2 \circ d_1, G')(F)$ holds for every pseudoisotopy F in $P(G(i))$ and $\partial G' = \partial G$.

Proof. Both maps are the identity outside an iterated tubular neighbourhood of $G(i)$. Via local coordinates the previous lemma shows the result. □

The final step is to assemble all constructions in this chapter to construct a functor of quasicategories from choices to Kan-complexes.

Proposition 1.4.1.19. *A functor $P\colon \mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Kan}_\Delta$ with the properties specified in Lemma 1.4.1.1 exists.*

Proof. The conditions stated in Lemma 1.4.1.1 on spaces (i.e. 0–simplices) and 1–simplices determine our functor $\mathcal{S}_\bullet P$ on the 1–skeleton of $\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta$.

As explained in Section 1.2.1 we have to solve certain lifting problems to obtain the desired functor. Let $G\colon \mathfrak{S}(\Delta^k) \to \mathcal{D}_\Delta$ be a k–simplex of $\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta$ and $\mathcal{S}_\bullet P(G|_{\partial\Delta^k})\colon \mathfrak{S}(\partial\Delta^k) \to \mathrm{Kan}_\Delta$ the image of its boundary under the pseudoisotopy functor $\mathcal{S}_\bullet P$. We have to solve

$$\begin{array}{ccc} \mathfrak{S}(\partial\Delta^k) & \xrightarrow{\mathcal{S}_\bullet P(G|_{\partial\Delta^k})} & \mathrm{Kan}_\Delta \\ \downarrow & \overset{\mathcal{S}_\bullet P(G)}{\nearrow} & \\ \mathfrak{S}(\Delta^k) & & \end{array}$$

to extend our functor $\mathcal{S}_\bullet P$ from the boundary $G|_{\partial\Delta^k}$ to G. For degenerate simplices, we use the obvious extension. The non-degenerate case is more interesting.

The lifting problem is equivalent to

$$\begin{array}{ccc} \mathfrak{S}(\partial\Delta^k)(0,k) & \xrightarrow{\mathcal{S}_\bullet P(G|_{\partial\Delta^k})} & \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(0)), \mathcal{S}_\bullet P(G(k)) \\ \downarrow & \overset{\mathcal{S}_\bullet P(G)}{\nearrow} & \\ \mathfrak{S}(\Delta^k)(0,k) & & \end{array}$$

and we are going to show by induction that each of these lifting problems admits a solution given by a lift

$$\begin{array}{ccc} \mathfrak{S}(\partial\Delta^k)(0,k) & \longrightarrow & Q(\mathcal{D}^G_\Delta(0,k) \times G(m)) \\ \downarrow & \nearrow & \downarrow \\ \mathfrak{S}(\Delta^k)(0,k) & \longrightarrow & \mathcal{D}_\Delta(G(0), G(k)) \end{array}$$

composed with $\Phi\colon Q(\mathcal{D}^G_\Delta(0,k) \times G(m)) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(0)), \mathcal{S}_\bullet P(G(k)))$.

The composition of

$$\phi_1\colon Q(\mathcal{D}^G_\Delta(0,i) \times G(m_i^+)) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(0)), \mathcal{S}_\bullet P(G(i)))$$

and

$$\phi_2\colon Q(\mathcal{D}^G_\Delta(i,k) \times G(m - m_i^+)) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(i)), \mathcal{S}_\bullet P(G(k)))$$

admits, after a homotopy, a factorisation over

$$\phi\colon Q(\mathcal{D}^G_\Delta(0,k) \times G(m)) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(0)), \mathcal{S}_\bullet P(G(k)))$$

by Corollary 1.4.1.18 for every $0 \le i \le k$.

The map $\mathcal{S}_\bullet P(G|_{\partial\Delta^k})$ is given by gluing together compositions of this form, hence we obtain a map $\mathfrak{S}(\partial\Delta^k)(0,k) \to Q(\mathcal{D}^G_\Delta(0,k) \times G(m))$.

Since the map $Q(\mathcal{D}^G_\Delta(0,k) \times G(m)) \to \mathcal{D}_\Delta(G(0), G(k))$ is a trivial fibration by Lemma 1.4.1.16, we obtain a lift $\mathfrak{S}(\Delta^k)(0,k) \to Q(\mathcal{D}^G_\Delta(0,k) \times G(m))$ which we compose with $\phi\colon Q(\mathcal{D}^G_\Delta(0,k) \times G(m)) \to \mathrm{Kan}_\Delta(\mathcal{S}_\bullet P(G(0)), \mathcal{S}_\bullet P(G(k)))$ to obtain the solution of our lifting problem. □

1.4.2 Spectra

To improve our construction to a functor from topological spaces to spectra we extend our previous constructions a little. We first recall the definition of the pseudoisotopy spectrum.

Definition 1.4.2.1. Let $k \in \mathbb{N}$ be a natural number. Let M be a manifold. Then $P(M; \mathbb{R}^k)$ denotes the *space of bounded pseudoisotopies*. It is the subspace of $P(M \times \mathbb{R}^k)$ containing those pseudoisotopies which fulfil the following property. Let $F \in P(M; \mathbb{R}^k)$ be a pseudoisotopy. Then there is $\alpha > 0$ such that for each $(x, v, t) \in M \times \mathbb{R}^k \times I$ the inequality $\|v - (\mathrm{pr}_{\mathbb{R}^k} \circ F)(x, v, t)\| < \alpha$ holds. Similarly, we define $P(M; \mathbb{R}^k \times \mathbb{R}_{\ge 0})$ and $P(M; \mathbb{R}^k \times \mathbb{R}_{\le 0})$.

One can check that we still obtain maps induced by codimension zero embeddings and transfer maps via the same definitions as before. In fact, for an α–controlled pseudoisotopy F the maps $tr(F)$ and $p_!(F)$ are still α–controlled.

The constructions in Section 1.3 and Section 1.4 are local in nature. Thus the definition of Φ in Definition 1.4.1.11 carries over verbatim to the controlled setting. We also have maps Φ_k^+ and Φ_k^- which correspond to $P(M; \mathbb{R}^k \times \mathbb{R}_{\ge 0})$ and $P(M; \mathbb{R}^k \times \mathbb{R}_{\le 0})$, respectively.

To obtain an $(\infty,1)$–functor $P\colon \mathcal{D}^G_\Delta \to \mathrm{Spectra}_\Delta$ which is level-wise given by $P(-;\mathbb{R}^k)$ we have to define structure maps. Let us consider the square

$$\begin{array}{ccc} P(M;\mathbb{R}^k) & \longrightarrow & P(M;\mathbb{R}^k\times\mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ P(M;\mathbb{R}^k\times\mathbb{R}_{\leq 0}) & \longrightarrow & P(M;\mathbb{R}^{k+1}) \end{array}$$

where $(i_0)_*\circ \mathrm{pr}_!\colon P(M;\mathbb{R}^k)\to P(M;\mathbb{R}^k\times\mathbb{R}_{\geq 0})$ is the composition of the transfer $\mathrm{pr}_!\colon P(M;\mathbb{R}^k)\to P(M\times[0,1];\mathbb{R}^k)$ associated to $\mathrm{pr}\colon M\times[0,1]\times\mathbb{R}^k\to M\times\mathbb{R}^k$ and the map $(i_0)_*\colon P(M\times[0,1];\mathbb{R}^k)\to P(M;\mathbb{R}^k\times\mathbb{R}_{\geq 0})$ which is induced by the codimension zero embedding $i_0\colon M\times\mathbb{R}^k\times[0,1]\subseteq M\times\mathbb{R}^k\times\mathbb{R}_{\geq 0}$.

From the definition of Φ_k and Φ_k^+ one sees that $\mathrm{pr}_!\colon \Phi_k\Rightarrow\Phi_k\circ(-\times[0,1])$ and $(i_0)_*\colon \Phi_k\circ(-\times[0,1])\Rightarrow\Phi_k^+$ determine natural transformations of $(\infty,1)$–functors.

Moreover, the upper right and lower left corner are contractible by a natural Eilenberg Swindle. This is enough to obtain the desired structure maps.

Remark 1.4.2.2. *For an appropriate choice of the Eilenberg Swindle, the structure map from the k–th to the $k+n$–th level arises from a homotopy coherent diagram which is given in degree 0 by*

$$\begin{aligned} P(M;\mathbb{R}^k)\wedge(\mathbb{R}^n\cup\{\infty\}) &\to P(M;\mathbb{R}^{k+n}) \\ (F,v) &\mapsto (i_v)_*\circ\mathrm{pr}_!(F) \end{aligned}$$

where pr *is as above and $i_v\colon M\times\mathbb{R}^k\times([-1,1]^n+v)\subseteq M\times\mathbb{R}^{k+n}$ the codimension zero embedding. This is the definition of the pseudoisotopy spectrum given by Hatcher in [23, Appendix II].*

Analogous to Lemma 1.4.1.1 we obtain

Lemma 1.4.2.3. *There is a functor*

$$\mathcal{P}\colon\ \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet\mathcal{D}_\Delta)\to\mathcal{N}^{h.c.}_\bullet\,\mathrm{Spectra}_\Delta$$

with

1. *The spectrum $\mathcal{P}(M)$ is the stable pseudoisotopy spectrum of M which is given by $\mathbb{P}(M;\mathbb{R}^k)=\mathrm{hocolim}_{n\in\mathbb{N}}P(M\times[0,1]^n;\mathbb{R}^k)$ in level k and uses the structure maps specified above.*

2. *A map $(\iota\times\mathrm{Id},M\times[0,1]^n,\mathrm{Id})_{n\in\mathbb{N}}\colon(M\times[0,1]^n)_{n\in\mathbb{N}}\to(N\times[0,1]^n)_{n\in\mathbb{N}}$ with ι a codimension zero embedding is sent to*

$$\mathrm{hocolim}_{n\in\mathbb{N}}(\iota\times\mathrm{Id}_{[0,1]^n})_*\colon\mathbb{P}(M;\mathbb{R}^k)\to\mathbb{P}(N;\mathbb{R}^k)$$

in level k.

3. *A map $(\iota\times\mathrm{Id},\nu(\iota\times\mathrm{Id}),p\times\mathrm{Id})_{n\in\mathbb{N}}\colon(M\times[0,1]^n)_{n\in\mathbb{N}}\to(\nu\iota\times[0,1]^n)_{n\in\mathbb{N}}$ is sent to the transfer*

$$\mathrm{hocolim}_{n\in\mathbb{N}}(p\times\mathrm{Id}_{[0,1]^n})_!\colon\mathbb{P}(M;\mathbb{R}^k)\to\mathbb{P}(\nu\iota;\mathbb{R}^k)$$

in level k.

4. *We have* $\mathbb{P}^{Top}(-;\mathbb{R}^k)\circ j \simeq \mathbb{P}^{\mathrm{Diff}}(-;\mathbb{R}^k)$ *where we use the Yoneda embedding* $j\colon \operatorname{Ind}(\mathcal{N}^{h.c.}_\bullet \mathcal{D}_\Delta) \to \operatorname{Ind}(\mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta)$ *as defined in Remark 1.2.5.3.*

Now we note that $\mathcal{P}(M)$ is level-wise Kan and an Ω–spectrum for every manifold M, see [53, Proposition 1.10]. Therefore, it is a cofibrant and fibrant object in the stable model structure on sequential prespectra, see [5, Theorem 2.3].

This is enough to ensure that the argument to deduce Theorem 1.4.1.2 from Lemma 1.4.1.1 admits a direct analogue to show that Lemma 1.4.2.3 implies our main theorem.

Theorem 1.4.2.4. *There is a functor* $\mathcal{P}^{\mathrm{Diff}}\colon \mathrm{Top} \to \mathrm{Spectra}$ *with the following properties:*

1. *It descends to a functor of homotopy categories*

$$\operatorname{ho}\mathcal{P}^{\mathrm{Diff}}\colon \operatorname{ho}(\mathrm{Top}) \to \operatorname{ho}(\mathrm{Spectra}).$$

2. *There is a natural weak equivalence*

$$\mathbb{P}^{\mathrm{Diff}} \to \Omega^\infty \mathcal{P}^{\mathrm{Diff},\mathrm{Spectra}}.$$

3. *The subset inclusion* $\mathcal{P}^{\mathrm{Diff}} \subseteq \mathcal{P}^{\mathrm{Top}}$ *extends to a natural transformation of functors of quasicategories. The construction of* $\mathcal{P}^{\mathrm{Top}}$ *is given in [12].*

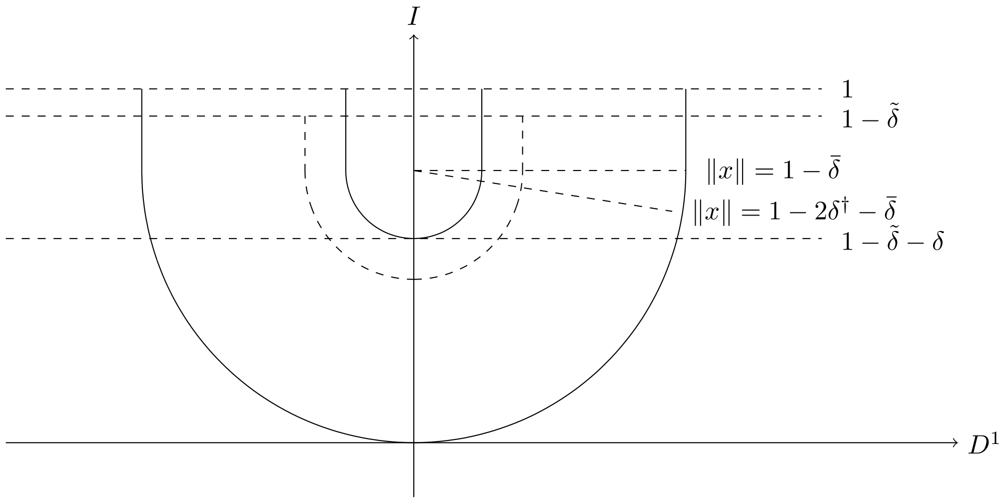
We show the third property later on as statement 6 of Theorem 2.2.1.16.

1.4.3 All of the δ

We give an overview over the δ chosen throughout this chapter. For a given pseudoisotopy $F \in P(M)$ and a natural number $m > 0$ there are four smooth maps from M to $(0, 1/4)$, namely $\bar{\delta} > \tilde{\delta} > \delta > \delta^\dagger$. They fulfil certain properties stated below. But first, let us give an example.

Picture 1.4.3.1. *We sketch the distorted extendable parametrised inverse for $m = 1$. We start with a parametrised family of half-circles whose radius ranges from δ to $\frac{3}{4}$, centred around $(0, 1 - \tilde{\delta})$. In the range $1 - 2\delta^\dagger - \bar{\delta} \le \|x\| \le 1 - \bar{\delta}$ we use β to deform each circle into a straight line, parallel to the I–axis.*

Now assume that $m = 2$ and $m_1 = m_2 = 1$. Let us consider the slice $\{\frac{3}{4}\} \times D^1 \times I \subseteq D^{m_1} \times D^{m_2} \times I$. There is some $\delta' > 0$, such that the set $\underline{\mathrm{St}}_{m_2}^{-1} \circ \underline{\mathrm{St}}_{m_1}^{-1}([1-\delta', 1] \times D^1 \times \{0\})$ contains all distorted half circles with sufficiently small radius, but at least radius δ, in the slice $\{\frac{3}{4}\} \times D^1 \times I$. The distorted half circles of radius less than δ are precisely the holes we fill with $B_i(0)$ in Section 1.3.



The precise properties are as follows:

- $\delta > 0$ to ensure that St_m^{-1} is an embedding, see Definition 1.1.1.1.
- $1/4 > \tilde{\delta}$ to make sure that the composition $\mathrm{St}_{m_k}^{-1} \circ \ldots \circ \mathrm{St}_{m_1}^{-1}$ is well-defined.
- $2\delta^\dagger > \bar{\delta} > \delta^\dagger$ makes the definition of β sensible, see Definition 1.1.1.4.
- $\bar{\delta}$ and $\delta^\dagger$ are sufficiently small to ensure that $\underline{\mathrm{St}}_n^{-1}$ is an embedding for each $1 \le n \le m$, see Lemma 1.1.1.8. They are also small enough to make sure that each level set can be deformed to the lower half of a sphere, see Lemma 1.3.0.7.

Chapter 2

A natural h–cobordism theorem

In this chapter we show that our notion of a pseudoisotopy functor is compatible with the Whitehead spectrum functor.

Theorem 2.0.0.1. *Let* Cat = Top, PL *or* Diff. *There is a natural weak equivalence of* $(\infty, 1)$*–functors*

$$\Psi\colon \mathcal{P}^{\mathrm{Cat}} \Rightarrow \Omega^2 \mathrm{Wh}^{\mathrm{Cat},-\infty}$$

from the $(\infty, 1)$*–functor* $\mathcal{P}^{\mathrm{Cat}}\colon \mathcal{N}_\bullet^{h.c.}\,\mathrm{Top}_\Delta \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Spectra}_\Delta$ *of pseudoisotopies to the twofold loops of the* $(\infty, 1)$*–functor given by the Whitehead spectrum.*

In particular, there is a zig-zag of natural weak equivalences between the strict functors $\mathcal{P}^{\mathrm{Cat}}\colon \mathrm{Top} \to \mathrm{Spectra}$ *and* $\Omega^2 \mathrm{Wh}^{\mathrm{Cat},-\infty}$.

A similar statement holds for the space level versions $\mathbb{P}^{\mathrm{Cat}}$ *and* $\Omega^2 \mathrm{Wh}^{\mathrm{Cat}}$.

The underlying idea of this result is - as in the first chapter - geometric in nature, but fortunately not as subtle.

A pseudoisotopy on a PL manifold M is a PL automorphism of $M \times I$ which is given by the Identity on $M \times \{0\}$. It is relative boundary, if it is also given by the Identity on $\partial M \times I$. The space of pseudoisotopies $P(M)$ and the subspace of those relative boundary $P_\partial(M)$ are homotopy equivalent.

There is a model for $\Omega\,\mathrm{Wh}(M \times [0, 1])$ which is given by the classifying space of a simplicial category $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(M \times I)$. Roughly speaking, an object in degree n is a bundle of polyhedra over $|\Delta^n|$ and a morphism is a morphism of bundles which is also a simple map (i.e. the preimage of each point is contractible). All of this is relative to a trivial subbundle with fibre $M \times [0, 1]$.

We can understand $\mathcal{S}_\bullet P(M)$ as a discrete simplicial category. Let F in $\mathcal{S}_n P(M)$ be a piecewise linear pseudoisotopy on $M \times |\Delta^n|$. We send F to a bundle $\mathrm{c}(F)$ over $|\Delta^n| \times S^1$ with fibre $M \times I$ and classifying map F. We also choose a collar $\tilde{c}\colon M \times [0, 1] \to M \times I$ with $F|_{\mathrm{im}(\tilde{c})} = \mathrm{Id}$. This yields a trivial subbundle with fibre $M \times [0, 1]$.

We have to explain what both functors do on morphisms. There are two interesting cases. A zero section $i\colon M \to \tau_N(M)$ of a piecewise linear tubular neighbourhood $p\colon \tau_N(M) \to M$ and a piecewise linear embedding $\iota\colon M \to N$ of codimension zero.

The map $P(i)\colon P(M) \to P(\tau_N(M))$ sends F to its transfer $p_!(F)$ which is of the form $F \times \mathrm{Id}\colon M \times |\Delta^n| \times I \times D^k \to M \times |\Delta^n| \times I \times D^k$ in local trivialisations.

On the subspace $P_\partial(M)$ the second map $\iota_* = P_\partial(\iota)\colon P_\partial(M) \to P_\partial(N)$ extends F with the Identity on $(N - M) \times |\Delta^n| \times I$.

The map $\Omega\,\mathrm{Wh}(i)\colon \Omega\,\mathrm{Wh}(M \times [0,1]) \to \Omega\,\mathrm{Wh}(N \times [0,1])$ is just given by pushout, so $\mathrm{c}(F)$ is sent to $\mathrm{c}(F) \cup_{M \times [0,1] \times S^1} N \times [0,1] \times S^1$, similarly for ι.

There is a bundle map $\mathrm{c}(p_!(F)) \to \Omega^2\,\mathrm{Wh}(i)(\mathrm{c}(F))$ which is also a retraction and a simple map. It is thus a morphism in the category associated to $\Omega^2\,\mathrm{Wh}$, similarly for $\mathrm{c}(\iota_*(F)) \to \Omega^2\,\mathrm{Wh}(\iota)(\mathrm{c}(F))$. We are going to construct a space of “admissible retractions” which turns out to be contractible by an Alexander trick.

The non-connective case is similar. The main issue is that we do not have as good a model for $\Omega\,\mathrm{Wh}$. Instead, we have to work with a simplicial category of retractive simplicial sets. Once we have constructed an appropriate model we triangulate the solution used in the connective case to obtain the desired natural transformation.

Reader’s guide

In Section 2.1.1 we recall results due to Jahren, Rognes and Waldhausen [31] and Waldhausen [51]. Since we need an explicit map, we show that pseudoisotopies classify bundles of h–cobordisms in Section 2.1.2. A detailed discussion of the pseudoisotopy functor in the topological and piecewise linear category occupies all of Section 2.1.3 and is crucial for the main argument. We use the Ind–completion of $(\infty, 1)$–categories recalled earlier in Proposition 1.2.0.3.

Afterwards we define an $(\infty, 1)$–functor version of the Whitehead space in Section 2.1.4 which relies on Section 2.1.1. In the final part of the first half of the chapter, Section 2.1.5, we construct the natural transformation. We make use of the $(\infty, 1)$–functor models for pseudoisotopies and the Whitehead space introduced in Section 2.1.3 and Section 2.1.4, respectively. We also need the classifying map given in Section 2.1.2. In order to obtain a zig-zag of strict natural transformations, the results of Section 1.2.1 are required.

We generalise the content of Section 2.1.2 and Section 2.1.3 in Section 2.2.1 to a controlled setting. Section 2.2.2 introduces the controlled analogue of $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$ and we discuss its relation to the Whitehead spectrum. This section draws heavily from Waldhausen’s work [51]. To show the existence of a coherent natural transformation in Section 2.2.3 we rely on the constructions given in Section 2.2.1 and use the controlled analogue of $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$, defined in Section 2.2.2, in the crucial step of the proof. Further, we rely on Section 1.2.1 for strictification.

The smooth case of our main result is shown in Section 2.2.4 by reduction to the topological case. We also use the main statement of the first chapter, given in Theorem 0.0.0.4, and some basic results on manifolds with corners from Section 1.2.2.

2.1 Connective Naturality

In this section we show that the stable parametrised h–cobordism theorem shown by Jahren, Rognes and Waldhausen in [31] is compatible with the full functoriality of pseudoisotopies and the common functor structure of A–theory. Our main result is the following theorem:

Theorem 2.1.0.1. *Let* Cat = Top *or* PL. *There is a zig-zag of natural weak equivalences between the stable* Cat*–pseudoisotopy functor* $\mathbb{P}_\partial^{\mathrm{Cat}}\colon \mathrm{Top} \to \mathrm{Top}$ *as defined in [12] and the* 2*–fold loop space of the* Cat*–Whitehead space functor* $\Omega^2 \operatorname{Wh}^{\mathrm{Cat}}\colon \mathrm{Top} \to \mathrm{Top}$.

We recall the pseudoisotopy functor in Section 2.1.3 below. Concerning the Whitehead space, we are content to define its onefold loops.

Definition 2.1.0.2. Let Cat = Top or PL. The loop space of the Cat–Whitehead space is defined as the homotopy fibre

$$\Omega \operatorname{Wh}^{\mathrm{Cat}} \to \Omega^\infty((-)_+ \wedge A(*)) \to A$$

of the assembly map $\Omega^\infty((-)_+ \wedge A(*)) \to A$ of A–theory where $A(*)$ denotes the connective A–theory spectrum of the point [51, p. 13].

Note that the A–theoretical assembly map $\Omega^\infty((-)_+ \wedge A(*)) \to A$ does not depend on the category Cat. Hence $\Omega \operatorname{Wh}^{\mathrm{Top}} = \Omega \operatorname{Wh}^{\mathrm{PL}}$ holds. See [54] for an introduction of the assembly homology theory.

In Section 2.1.1 we recall the point-wise zig-zag along weak equivalences between pseudoisotopies and the twofold loops of the Whitehead space as well as the known degrees of naturality for each of the maps involved. It turns out that we may restrict our attention to the construction in [31]. We end the section with a brief reminder of the definitions and results relevant for our purposes.

In Section 2.1.2 and Section 2.1.3 we recall, respectively, the h–cobordism space and the pseudoisotopy functor, their relation, and various properties required throughout the proof.

We finish our preparations in Section 2.1.4. Here, we extend a model for the piecewise linear Whitehead space to a simplicially enriched functor. We also lift the zig-zag of [31] to one of functors of simplicial sets and natural transformations between them.

Finally, in Section 2.1.5, we construct an enriched natural transformation of functors enriched in simplicial categories between bundles of polyhedra and pseudoisotopy spaces. This is the key step of the proof and, once finished, the desired zig-zag of natural weak equivalences becomes an easy consequence.

2.1.1 The stable parametrised h-cobordism theorem

We analyse the functoriality of the connection between pseudoisotopies and the Whitehead space. The reader familiar with the work of Waldhausen [51] and Jahren, Rognes and Waldhausen [31] can safely skip this section.

Waldhausen [51] gave a functorial model for the homotopy fibre of the assembly map in terms of the K–theory of a Waldhausen category.

Theorem 2.1.1.1 ([51, Theorem 3.3.1]). *Let $X_\bullet$ be a simplicial set. There is a homotopy fibre sequence*

$$K(R_f^h(X_\bullet), s) \to K(R_f(X_\bullet), s) \to K(R_f(X_\bullet), h)$$

where $K(R_f(X_\bullet), h)$ is a model for $A(X_\bullet)$ and $K(R_f(X_\bullet), s)$ is a model for the assembly homology theory of A evaluated at $X_\bullet$.

Consequently, $K(R_f^h(X_\bullet), s)$ is a model for $\Omega\,\mathrm{Wh}(X_\bullet)$, the one-fold loop space of the Whitehead space of $X_\bullet$.

Further, he gave a zig-zag of weak equivalences to obtain a different model. The functoriality for A–theory given in [51] allows obvious analogues for each of the spaces in this zig-zag and one obtains a zig-zag of natural weak equivalences.

Theorem 2.1.1.2 ([51, Theorem 3.1.7]). *Let $X_\bullet$ be a simplicial set. There are natural homotopy equivalences*

$$sN_\bullet \mathcal{C}_f^h(X_\bullet) \to sN_\bullet \mathcal{C}_f^h(X_\bullet^{\Delta^\bullet}) \leftarrow sN_\bullet R_f^h(X_\bullet^{\Delta^\bullet}) \to s\mathbf{S}_\bullet R_f^h(X_\bullet^{\Delta^\bullet}).$$

Proposition 2.1.1.3 ([51, Proposition 3.1.1]). *There is a natural homotopy equivalence*

$$|s\mathcal{C}_f^h(X_\bullet)| \simeq \Omega|sN_\bullet \mathcal{C}_f^h(X_\bullet)|.$$

This new model $s\mathcal{C}_f^h(X_\bullet)$ (of $\mathrm{Wh}(X_\bullet)$) was, in turn, the final step in the zig-zag of [31]. There it was denoted by $s\mathcal{C}^h(X_\bullet)$.

Theorem 2.1.1.4. *Let M be a compact* PL *manifold. Let $(X_\bullet, t\colon |X_\bullet| \xrightarrow{\cong} M)$ be a triangulation of M. There is a zig-zag of weak equivalences*

$$\mathcal{H}_\bullet^{\mathrm{PL}}(M) \xrightarrow{u} \mathrm{s}\tilde{\mathcal{E}}_\bullet^{\mathrm{h}}(M) \xleftarrow{\tilde{n}r} s\mathcal{D}^h(X_\bullet) \xrightarrow{i} s\mathcal{C}^h(X_\bullet).$$

The diagram is explained in the introduction of [31]. We briefly recall these constructions below.

So far, we obtained a connection between h–cobordisms and the Whitehead space. As the last step we note that pseudoisotopies are the structure group of bundles of h–cobordisms which yields a weak equivalence

$$\mathrm{c}\colon \mathbb{P}^{\mathrm{PL}}(M) \xrightarrow{\simeq} \Omega|\mathbb{H}_\bullet^{\mathrm{PL}}(M)|.$$

We give a careful treatment of this result in Section 2.1.2.

In the zig-zag of Theorem 2.1.1.4 functoriality and naturality become more subtle issues:

As before $s\mathcal{C}^h(X_\bullet)$ yields a functor of simplicial sets, while $s\mathcal{D}^h(X_\bullet)$ is a functor only on non-singular simplicial sets and cofibrations, and for $s\tilde{\mathcal{E}}_\bullet^h(M)$ we have to use polyhedra and piecewise linear embeddings.

There is a similarly restricted variation for $\mathbb{H}_\bullet^{\mathrm{PL}}(M)$ which uses manifolds and codimension zero piecewise linear embeddings but we do not need it.

The functor $i\colon s\mathcal{D}^h(X_\bullet) \to s\mathcal{C}^h(X_\bullet)$ is natural in non-singular simplicial sets.

Our actual task is thus to

1. improve the natural weak equivalences i and $\tilde{n}r$ of functors from non-singular simplicial sets $\mathrm{sSet}^{\text{non-sing}}$ to Top to zig-zags of natural weak equivalences between functors from Top to Top and
2. construct a zig-zag of natural weak equivalences between $\Omega s\tilde{\mathcal{E}}^h_\bullet$ and $\mathbb{P}$.

To proceed, we have to recall some definitions from the work of Jahren, Rognes and Waldhausen [31].

Definition 2.1.1.5 ([31, Definition 1.1.1(a)])**.** Let $M \subseteq \mathbb{R}^\infty$ be a compact PL manifold, possibly with boundary. The *h–cobordism space* $H_\bullet(M)$ is the simplicial set with $H_n(M)$ the set of all diagrams

$$\begin{array}{ccc} M \times |\Delta^n| & \overset{\iota}{\hookrightarrow} & W \\ & \underset{\mathrm{pr}}{\searrow} & \downarrow p \\ & & |\Delta^n| \end{array}$$

where $p\colon W \to |\Delta^n|$ is a PL subbundle of $\mathbb{R}^\infty \times |\Delta^n| \to |\Delta^n|$, i.e. there is an open cover $\{U_\alpha\}$ of $|\Delta^n|$ and a PL isomorphism over U_α from $p^{-1}(U_\alpha) \to U_\alpha$ to a product bundle for each α. Each local trivialisation restricts to the Identity on the product subbundle specified by ι. Finally, the fibre $W_x := p^{-1}(x)$ is a piecewise linear h–cobordism with $M \cong \iota(M \times \{x\})$ as one boundary component for every $x \in |\Delta^n|$.

The simplicial structure maps are induced by pullback along the structure maps of the cosimplicial space $[n] \mapsto |\Delta^n|$.

Definition 2.1.1.6 ([31, Definition 1.1.1(b)])**.** Let $H^c_\bullet(M)$ be the simplicial set with $H^c_n(M)$ containing the tuples which consist of an element of $H_n(M)$ and a collar $c\colon M \times |\Delta^n| \times [0,1] \to W$ which restricts to a fibre-wise collar $c\colon M \times \{x\} \times [0,1] \to W_x$ for each $x \in |\Delta^n|$.

Remark 2.1.1.7 ([31, Definition 1.1.1(b)])**.** *We obtain an acyclic fibration $H^c_\bullet(M) \to H_\bullet(M)$ given by the forgetful map.*

Definition 2.1.1.8 ([31, Definition 1.1.3])**.** There is a *stabilisation map*

$$\sigma\colon H_\bullet(M) \to H_\bullet(M \times [0,1])$$

given by

$$(W, p\colon W \to |\Delta^n|, \iota\colon M \times |\Delta^n| \hookrightarrow W) \mapsto (W \times [0,1], p \circ \mathrm{pr}_W, \iota \times \mathrm{Id}_{[0,1]})$$

where we mildly abuse notation since the subbundle is $M \times [0,1] \times |\Delta^n|$, not $M \times |\Delta^n| \times [0,1]$.

In the decorated case, we send a collar $c\colon M \times |\Delta^n| \times [0,1] \to W$ to $c \times \mathrm{Id}_{[0,1]}$.

Definition 2.1.1.9 ([31, Definition 1.1.3])**.** The *stable h–cobordism space* is

$$\mathbb{H}_\bullet(M) := \mathrm{colim}_{k\in\mathbb{N}}\, H_\bullet(M \times [0,1]^k)$$

and similar for $\mathbb{H}^c_\bullet(M)$. We denote the geometric realisation by $\mathbb{H}(M)$.

Definition 2.1.1.10 ([31, Definition 1.1.5])**.** A PL map $f\colon K \to L$ of compact polyhedra is called a *simple map*, if $f^{-1}(p)$ is contractible for every $p \in L$.

A map of finite simplicial sets $f\colon X_\bullet \to Y_\bullet$ is called a *simple map* if for its geometric realisation the preimage $|f|^{-1}(p)$ is contractible for every $p \in |Y_\bullet|$.

Definition 2.1.1.11 ([31, Definition 1.1.6])**.** Let K be a compact polyhedron. The simplicial category $s\tilde{\mathcal{E}}^h_\bullet(K)$ consists of fibrations of compact polyhedra containing K as a deformation retract, and simple PL maps. Precisely:

In simplicial degree q the objects of $s\tilde{\mathcal{E}}^h_q(K)$ are diagrams

$$\begin{array}{ccc} K \times |\Delta^q| & \xrightarrow{s} & E \\ & \searrow^{\text{pr}} & \downarrow{\scriptstyle \pi} \\ & & |\Delta^q| \end{array}$$

where π is a PL Serre fibration (i.e. a PL map whose underlying map of spaces is a Serre fibration) of compact polyhedra and s is a PL embedding and homotopy equivalence. We only consider polyhedra in $\mathbb{R}^\infty \times |\Delta^n|$ for smallness.

A morphism $f\colon (\pi, s) \to (\pi', s')$ is a simple PL map of relative fibrations $f\colon E \to E'$, i.e. we have $\pi = f \circ \pi'$ and $s' = s \circ f$.

Given a PL embedding $\iota\colon K \to K'$, the construction $E \mapsto E \cup_K K'$ induces a functor of simplicial categories $\iota_*\colon s\tilde{\mathcal{E}}^h_\bullet(K) \to s\tilde{\mathcal{E}}^h_\bullet(K')$, and further a functor $s\tilde{\mathcal{E}}^h_\bullet$ from compact polyhedra and PL embeddings to simplicial categories.

There is a stabilisation map $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(K) \to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(K \times [0,1])$ which is an acyclic cofibration.

Definition 2.1.1.12 ([31, Definition 1.1.7])**.** Let M be a compact PL manifold. Let $u\colon H^{\mathrm{PL}}_\bullet(M)^c \to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(M \times [0,1])$ be given by sending the pair consisting of a diagram

$$\begin{array}{ccc} M \times |\Delta^n| & \xhookrightarrow{\iota} & W \\ & \searrow^{\text{pr}} & \downarrow{\scriptstyle p} \\ & & |\Delta^n| \end{array}$$

and a parametrised family of collars $c\colon M \times [0,1] \times |\Delta^n| \to W$ over $|\Delta^n|$ to

$$\begin{array}{ccc} M \times [0,1] \times |\Delta^n| & \xhookrightarrow{c} & W \\ & \searrow^{\text{pr}} & \downarrow{\scriptstyle p} \\ & & |\Delta^n|. \end{array}$$

The map is compatible with stabilisation.

Theorem 2.1.1.13 ([31, Theorem 1.1.8])**.** *The stabilised map*

$$\operatorname{colim}_{k\in\mathbb{N}} H^{\mathrm{PL}}(M \times [0,1]^k)^c \to \operatorname{colim}_{k\in\mathbb{N}} \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(M \times I \times [0,1]^k)$$

induced by u is a homotopy equivalence.

Definition 2.1.1.14 ([31, Definition 1.2.3]). Let $X_\bullet$ be a finite simplicial set. Let $s\mathcal{C}^h(X_\bullet)$ denote the category with objects $y\colon X_\bullet \rightarrowtail Y_\bullet$ where y is an acyclic cofibration and $Y_\bullet$ is generated by $X_\bullet$ and finitely many other simplices. A morphism is a simple map of simplicial sets $f\colon Y_\bullet \to Y'_\bullet$ with $y' = f \circ y$.

A map of simplicial sets $f\colon X_\bullet \to X'_\bullet$ induces a functor $s\mathcal{C}^h(X_\bullet) \to s\mathcal{C}^h(X'_\bullet)$ given by $Y_\bullet \mapsto Y_\bullet \cup_{X_\bullet} X'_\bullet$. Furthermore, this makes $s\mathcal{C}^h$ into a functor from simplicial sets to categories.

Let $s\mathcal{D}^h(X_\bullet)$ denote the full subcategory of $s\mathcal{C}^h(X_\bullet)$ on those objects for which $Y_\bullet$ is non-singular. We obtain a functor from non-singular simplicial sets and cofibrations to categories.

Let $i\colon s\mathcal{D}^h(X_\bullet) \hookrightarrow s\mathcal{C}^h(X_\bullet)$ denote the inclusion functor. It induces a natural transformation of functors from non-singular simplicial sets and cofibrations to categories.

Let $X_\bullet$ be an arbitrary simplicial set. We define $s\mathcal{C}^h(X_\bullet)$ as the colimit of $s\mathcal{C}^h$ over the finite simplicial subsets of $X_\bullet$. Similarly for the other constructions.

Theorem 2.1.1.15 ([31, Theorem 1.2.5]). *Let $X_\bullet$ be a finite, non-singular simplicial set. The inclusion functor $i\colon s\mathcal{D}^h(X_\bullet) \hookrightarrow s\mathcal{C}^h(X_\bullet)$ is a homotopy equivalence.*

Definition 2.1.1.16 ([31, Definition 1.2.4]). Let $X_\bullet$ be a finite, non-singular simplicial set. Its geometric realisation is canonically a polyhedron. The polyhedral structure is uniquely determined by requiring that the characteristic map $\bar{x}\colon |\Delta^q| \to |X_\bullet|$ is a PL map for each simplex of $X_\bullet$. Simplicial maps yield PL maps with respect to these polyhedral structures.

We obtain a *polyhedral realisation functor*

$$r\colon \text{sSet}^{\text{fin,non-sing}} \to (\text{Poly}, \text{pl})$$

and an induced natural transformation

$$r\colon s\tilde{\mathcal{D}}^h \Rightarrow s\tilde{\mathcal{E}}^h_0 \circ r.$$

Let $\tilde{n}\colon s\tilde{\mathcal{E}}^h_0 \Rightarrow s\tilde{\mathcal{E}}^h_\bullet$ denote the natural transformation of simplicial categories which includes the category of degree 0.

Theorem 2.1.1.17 ([31, Theorem 1.2.6]). *Let $X_\bullet$ be a finite, non-singular simplicial set. The composed functor $\tilde{n} \circ r\colon s\tilde{\mathcal{D}}^h(X_\bullet) \to s\tilde{\mathcal{E}}^h(r(X_\bullet))$ is a homotopy equivalence.*

2.1.2 The H–cobordism map

In this part we explain how h–cobordism spaces classify bundles with pseudoisotopies as structure group. This yields a map $\mathrm{c}\colon \mathbb{P}(M) \to \Omega\mathbb{H}^{\mathrm{PL}}(M)$, see Definition 2.1.2.3 and Lemma 2.1.2.10, and allows us to study the naturality of $\mathrm{u} \circ \mathrm{c}\colon \mathbb{P}(M) \to \operatorname{colim}_{k\in\mathbb{N}} \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(M \times I \times [0,1]^k)$ later on. Since most of our work is carried out in the PL category, we typically drop the upper index “PL” from our notation.

Let $M \subseteq \mathbb{R}^\infty$ be a compact PL manifold, possibly with boundary. To describe the map c, we first recall the stable pseudoisotopy space.

Definition 2.1.2.1. Let M be a PL manifold. A *piecewise linear pseudoisotopy* on M is a PL isomorphism $F\colon M \times I \to M \times I$, such that $F|_{M\times\{0\}} = \mathrm{Id}$.

The *pseudoisotopy space* $P(M)$ is the simplicial set whose n–simplices are the pseudoisotopies on $M \times |\Delta^n|$ which are compatible with the projections to $|\Delta^n|$.

The *stabilisation map* $P(M) \to P(M \times [0,1])$, $F \mapsto F \times \mathrm{Id}$ allows us to define the stable space $\mathbb{P}(M) := \mathrm{colim}_{k\in\mathbb{N}}\, P(M \times [0,1]^k)$.

We abuse notation and do not distinguish between $P(M)$ and its geometric realisation.

Next, we explain how to "glue along a pseudoisotopy" to obtain an h–cobordism bundle.

Definition 2.1.2.2. Let $F\colon M \times |\Delta^n| \times I \to M \times |\Delta^n| \times I$ be an n–simplex of the space of piecewise linear (unstable) pseudoisotopies. Let $e_M\colon M \hookrightarrow \mathbb{R}^\infty$ be the subspace inclusion and $e_n\colon M \times \mathbb{R} \times |\Delta^n| \hookrightarrow \mathbb{R}^\infty \times |\Delta^n|$ the standard embedding given by $(x, r, v) \mapsto (r, e_M(x), v)$.

We define the embedding

$$\begin{aligned} F_s\colon M \times I \times |\Delta^n| &\hookrightarrow M \times \mathbb{R} \times |\Delta^n| \\ (x, t, v) &\mapsto F(x, t - s, v) + (0, s, 0);\ \text{if } t - s \geq 0 \\ &\qquad (x, t, v);\ \text{otherwise} \end{aligned}$$

for $s \in [0, 1]$, where we use $M \subseteq \mathbb{R}^\infty$ to define addition. We obtain an embedding

$$\begin{aligned} e(F_s)\colon M \times I \times |\Delta^n \times \Delta^1| &\hookrightarrow \mathbb{R}^\infty \times |\Delta^n \times \Delta^1| \\ (x, t, v, s) &\mapsto (e_n \circ F_s(x, t, v), s). \end{aligned}$$

This, in turn, yields an $n \times 1$–simplex in $H_\bullet(M)$:

$$\begin{array}{ccc} M \times \{0\} \times |\Delta^n \times \Delta^1| & \xrightarrow{e(F_s)\circ i_0} & \mathrm{im}(e(F_s)) \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle \mathrm{pr}\,\circ(e(F_s))^{-1}} \\ & & |\Delta^n \times \Delta^1| \end{array}$$

where $i_0\colon M \times \{0\} \times |\Delta^n \times \Delta^1| \hookrightarrow M \times I \times |\Delta^n \times \Delta^1|$ is the subspace inclusion.

We obtain a loop $\phi_F\colon (\Delta^1, \partial\Delta^1) \to (H_\bullet(M), (\mathrm{pr}\colon M \times I \to |\Delta^0|))$ since the embeddings $d_0^*(e(F_s))$ and $d_1^*(e(F_s))$ have the same image.

The bundle $e(F_s)$ is just a model for the S^1–bundle classified by F. Now we are in position to precisely state the connection between pseudoisotopies and h–cobordisms as a weak equivalence induced by the classifying map.

Definition 2.1.2.3. We define a map of simplicial sets

$$\begin{aligned} \mathrm{c}\colon P(M) &\to \Omega H_\bullet(M) \\ F &\mapsto \phi_F. \end{aligned}$$

With the definition in place, we show that the classifying map is a weak equivalence and induces a weak equivalence after stabilisation.

Definition 2.1.2.4. Let $E_0(M)$ be the simplicial set with n–simplices the piecewise linear embeddings $M \times I \times |\Delta^n| \hookrightarrow \mathbb{R}^\infty \times |\Delta^n|$ which are compatible with the projection to $|\Delta^n|$ and restrict to the standard embedding on $M \times \{0\} \times |\Delta^n|$.

Lemma 2.1.2.5. *The simplicial set $E_0(M)$ is contractible.*

Proof. This is analogous to the usual proof that the space of collars is contractible. Alternatively it follows from transversality, see [41, Chapter 5]. □

Lemma 2.1.2.6. *We have a Kan fibration $P(M) \to E_0(M) \to E_0(M)/P(M)$ where we form the quotient under the group action $P(M) \times E_0(M) \to E_0(M)$, $(F, \iota) \mapsto \iota \circ F$.*

Proof. Since the action of $P(M)_n$ on $E_0(M)_n$ is free for each $n \in \mathbb{N}$ this follows directly from [17, Corollary 2.7]. □

Definition 2.1.2.7. Let $H^0_\bullet(M) \subseteq H_\bullet(M)$ denote the connected component of the cylinder.

Lemma 2.1.2.8. *The map $E_0(M)/P(M) \to H^0_\bullet(M)$, which in degree n is given by $[\iota] \mapsto (\mathrm{pr} \circ \iota^{-1} \colon \mathrm{im}(\iota) \to |\Delta^n|)$, is a trivial fibration.*

Proof. Consider a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{[\partial\iota_\psi]} & E_0(M)/P(M) \\ \downarrow & \overset{[\iota]}{\nearrow} & \downarrow \\ \Delta^n & \xrightarrow{\psi} & H^0_\bullet(M). \end{array}$$

Then there is an embedding $\iota'_\psi \colon M \times I \times |\Delta^n| \to \mathbb{R}^\infty \times |\Delta^n|$ in $E_0(M)$ which lifts ψ.

Since the composition $\partial\iota_\psi \circ (\partial\iota'_\psi)^{-1}$ is a pseudoisotopy, we have $[\partial\iota_\psi] = [\partial\iota'_\psi]$ in $E_0(M)/P(M)$. So $[\iota'_\psi]$ solves the lifting problem. This is enough. □

Lemma 2.1.2.9. *The map* c *is a weak equivalence.*

Proof. Consider the map of fibrations

$$\begin{array}{ccc} |P(M)| & \xrightarrow{|c|} & \Omega|H_\bullet(M)| \\ \downarrow & & \downarrow \\ |E_0(M)| & \xrightarrow{H_*} & P|H_\bullet(M)| \\ \downarrow p & & \downarrow \\ |H^0_\bullet(M)| & \xrightarrow{\mathrm{Id}} & |H^0_\bullet(M)| \end{array}$$

where $P|H_\bullet(M)|$ denotes the based path space of $|H_\bullet(M)|$ and H_* is the map $\iota \mapsto (t \mapsto p \circ H(\iota, t))$ with H an appropriately chosen contraction of $|E_0(M)|$, i.e. one which makes the upper square commute (the lower commutes for every contraction).

We only have to explain how to obtain a homotopy H which is given on the simplices of the form $e \circ F$ by the isotopy $s \mapsto e \circ F_s$. Given any contraction this follows from the homotopy extension property. □

Lemma 2.1.2.10. *The map* c *commutes with the stabilisation maps and thus induces a weak equivalence*

$$\mathrm{c}\colon \mathbb{P}^{\mathrm{PL}}(M) \to \Omega\mathbb{H}_\bullet^{\mathrm{PL}}(M).$$

Proof. Since both objects are just models for the respective homotopy colimits, the latter claim follows once we have shown that the stabilisations are compatible. It is a computation from the definitions that $\phi_{F\times\mathrm{Id}} = \phi_F \times \mathrm{Id}_{[0,1]^k}$ holds for every $k \in \mathbb{N}$. □

2.1.3 The pseudoisotopy functor

We recall the topological and piecewise linear pseudoisotopy functors introduced in [12]. Each of them is the composition of a homotopy coherent diagram $\mathrm{F_{ch}}\colon \mathcal{N}_\bullet \mathrm{Top} \to \mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta$ and a simplicially enriched functor $r\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$.

To construct the natural weak equivalence in Section 2.1.5, we slightly adjust the category of contractible choices Ch_Δ, see Definition 2.1.3.8, to $\mathrm{Ch}_\Delta^{c,s,+}$ starting with Definition 2.1.3.11 and up to Lemma 2.1.3.17.

To avoid cumbersome notation, we introduce some abbreviations in Notation 2.1.3.20. The properties of our adjusted pseudoisotopy functor are summarized in Theorem 2.1.3.21. We begin by repeating the definition of pseudoisotopies.

Definition 2.1.3.1. Let $(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{pl})$ denote the category of PL manifolds with PL maps as morphisms. Similarly, we define $(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{cts})$ and $(\mathrm{Mfd}^{\mathrm{Top}}, \mathrm{cts})$ where $\mathrm{Mfd}^{\mathrm{Top}}$ refers to topological manifolds and cts to continuous maps.

Definition 2.1.3.2. Let M be a PL manifold. A *piecewise linear pseudoisotopy* on M is a PL isomorphism $F\colon M \times I \to M \times I$, such that $F|_{M\times\{0\}} = \mathrm{Id}$.

The *pseudoisotopy space* $P(M)$ is the simplicial set whose n–simplices are the pseudoisotopies on $M \times |\Delta^n|$ commuting with the projections to $|\Delta^n|$.

The *stabilisation map* $P(M) \to P(M \times [0,1])$, $F \mapsto F \times \mathrm{Id}$ allows us to define the stable space $\mathbb{P}(M) := \mathrm{colim}_{k\in\mathbb{N}} P(M \times [0,1]^k)$.

We abuse notation and do not distinguish between $P(M)$ and its geometric realisation.

We obtain the analogous notions for the topological case if we replace "PL manifold" by "topological manifold" and "PL isomorphism" by "homeomorphism".

We also need pseudoisotopies relative boundary.

Definition 2.1.3.3. Let M be a PL manifold. A *piecewise linear pseudoisotopy relative boundary* on M is a PL isomorphism $F\colon M \times I \to M \times I$ such that $F|_{M\times\{0\}\cup\partial M\times I} = \mathrm{Id}$.

Again, there is a topological analogue.

We need notation from [12], starting with some categories. Let $\mathrm{Mfd}_\Delta^{\mathrm{Top}}$ be the full simplicially enriched subcategory of Top_Δ on the compact topological manifolds (possibly with boundary). Let Ch_Δ be the *simplicially enriched category of choices* with compact submanifolds of $\mathbb{R}^\infty$ as objects, see [12, Proof of Theorem 6.3], and the *unstable choice spaces* as mapping spaces, see [12, Definition 5.4] and [12, Definition 5.9]. Let us recall their construction.

Definition 2.1.3.4. A *continuous map over a space* X is a morphism of fibre bundles over X, see [12, Definition 2.1]. Let $p\colon E \to M$ be a fibre bundle of manifolds over X and $H\colon E \times I \times [0,1] \to E \times I \times [0,1]$ an isotopy of homeomorphisms over X such that:

- H starts with the Identity on $E \times I$ and
- the set $E \times \{0\}$ is contained in $H_t(E \times \{0\})$ for every $t \in [0,1]$.

We call H a *bending isotopy* and $h = H_1$ a *partial bending map*. If H also satisfies the condition

- H is fibre preserving in the sense that $H_t(p^{-1}(M) \times I) = p^{-1}(m) \times I$ for all $m \in M$ and $t \in [0,1]$

we call H a *fibre-wise bending isotopy* and h a *fibre-wise partial bending map*.

If H is a fibre-wise bending isotopy and

- for each fibre $F = p^{-1}(m)$ the set $F \times \{0\} \cup \partial F \times I$ is contained in $h(F \times \{0\})$

then h is called a *fibre-wise bending map*.

Consider a disk bundle $p\colon E \to M$ with zero section $s\colon M \to E$. A fibre-wise bending isotopy H *preserves the zero section* if

- $H\colon E \times I \times [0,1] \to E \times I \times [0,1]$ restricts to the Identity on the zero section $H = \mathrm{Id}\colon s(M) \times I \times [0,1] \to s(M) \times I \times [0,1]$.

Analogously, a *PL–map over a polyhedron* X is a morphism of fibre bundles over X. For a fibre bundle $p\colon E \to M$ of PL manifolds over X and a piecewise linear isotopy of PL isomorphisms $H\colon E \times I \times [0,1] \to E \times I \times [0,1]$ over X, i.e. a PL isomorphism over $X \times [0,1]$, we obtain analogues of the other definitions.

Definition 2.1.3.5. Let $\xi\colon E \to M$ be a topological fibre bundle with compact fibre. Then we have pullbacks

$$\begin{array}{ccc} \xi^*_{\mathrm{pr}_i} E & \longrightarrow & E \\ \downarrow & & \downarrow \xi \\ M^{[0,1]} & \xrightarrow{\mathrm{pr}_i} & M \end{array}$$

for $i = 0, 1$. A *parallel transport in the topological category* is an isomorphism of fibre bundles $\nu\colon \xi^*_{\mathrm{pr}_0} E \to \xi^*_{\mathrm{pr}_1} E$ over M^I with $\nu \circ s_0 = s_1$ where s_i is the canonical section $s_i\colon E \to \xi^*_{\mathrm{pr}_i} E$.

Suppose that $\xi\colon E \to M$ is a disk bundle with zero section $s\colon M \to E$. A parallel transport *preserves the zero section* if $\nu(\omega, s(\omega(0))) = (\omega, s(\omega(1)))$ holds.

In the special case of a PL bundle we obtain the pullback diagram

$$\begin{array}{ccc} \xi^*_{\mathrm{pr}_i} E_{\mathrm{PL}} & \longrightarrow & \mathcal{S}^{\mathrm{PL}}_\bullet E \\ \downarrow & & \downarrow \xi \\ M^{[0,1]}_{\mathrm{PL}} & \xrightarrow{\mathrm{pr}_i} & \mathcal{S}^{\mathrm{PL}}_\bullet M \end{array}$$

where $M_{\mathrm{PL}}^{[0,1]} \subseteq \mathcal{S}_\bullet M^{[0,1]}$ is the simplicial subset, which contains simplices whose adjoint $\phi\colon |\Delta^n| \times I \to M$ is a piecewise linear map. Further, $\mathcal{S}_\bullet^{\mathrm{PL}}(X) \subseteq \mathcal{S}_\bullet(X)$ denotes the simplicial subset of piecewise linear maps.

A *parallel transport in the* PL *category* $\nu\colon \xi^*_{\mathrm{pr}_0} E_{\mathrm{PL}} \to \xi^*_{\mathrm{pr}_1} E_{\mathrm{PL}}$ is an isomorphism over M^I_{PL} with $\nu \circ s_0 = s_1$ where $s_i\colon \mathcal{S}_\bullet E \to \xi^*_{\mathrm{pr}_i} E_{\mathrm{PL}}$ is the canonical section. We obtain a similar notion of zero section preserving.

Now we introduce the *transfer in the topological category.* Let ν' be a parallel transport as above and ν the parallel transport of the bundle $p \times \mathrm{Id}_I$ given by $(\omega, (e,t)) \mapsto (\omega, \mathrm{pr}_E \circ \nu'(\mathrm{pr}_M \circ \omega, e), \mathrm{pr}_I \circ \omega(1))$. For $(m,t) \in M \times I$, let $\omega_{(m,t)}\colon [0,1] \to M \times I$ be given by $s \mapsto (m, ts)$. Let F be a pseudoisotopy on M. The *geometric transfer of F along p with respect to ν'* is given by the formula $\mathrm{Tr}_{\nu'}(F)(e,t) = \mathrm{pr}_{E\times I} \circ \nu(F \circ \omega_{(p(e),t)}, (e,0))$ for $(e,t) \in E \times I$.

The same construction makes sense in the PL category, if we use a PL pseudoisotopy F. Hence we obtain the *transfer in the* PL *category.*

Given a fibre-wise bending map h we define $\mathrm{Tr}_{\nu,h}\colon P(M) \to P(E)$, the *transfer with respect to ν and h*, which sends a pseudoisotopy F to the pseudoisotopy $(h \times \mathrm{Id}_{|\Delta^k|}) \circ \mathrm{Tr}_\nu(F) \circ (h \times \mathrm{Id}_{|\Delta^k|})^{-1}$.

It restricts to $\mathrm{Tr}_{\nu,h}\colon P_\partial(M) \to P_\partial(E)$, the *transfer relative boundary with respect to ν and h.* As before, the construction has a topological and a piecewise linear case.

In [12] the constructions did not have to preserve the zero sections. The point of these additional conditions is the following lemma which enters the construction of the coherent natural transformation between $\Omega s \tilde{\mathcal{E}}^h_\bullet$ and P^{PL}.

Lemma 2.1.3.6. *Let $p\colon E \to M$ be a disk bundle with zero section $s\colon M \to E$. Let ν' be a parallel transport and h a fibre-wise bending map, both of which preserve the zero section. Then the square*

$$\begin{array}{ccc} M \times I \times |\Delta^m| & \xrightarrow{F} & M \times I \times |\Delta^m| \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ E \times I \times |\Delta^m| & \xrightarrow{\mathrm{Tr}_{\nu',h}(F)} & E \times I \times |\Delta^m| \end{array}$$

commutes for every $F \in P(M)$.

Proof. This is a straightforward calculation. □

In the proof of Lemma 2.1.5.10 we need the following description of the transfer in local coordinates.

Lemma 2.1.3.7. *Let $p\colon E \to M$ be a disk bundle with zero section $s\colon M \to E$. Let ν' be a parallel transport preserving the zero section. We choose a metric to reduce the structure group of the bundle to* $\mathrm{Aut}(S^{d-1})$.

Then there is a covering $(U_i, \omega_i)_{i\in\mathcal{I}}$ of M by local trivialisations of p for some indexing set $\mathcal{I}$ with respect to the reduced structure group.

Further, let (U, ω) and (U', ω') be trivialisations of the covering and let F be a pseudoisotopy on M. Then the geometric transfer without bending the *boundary* $\mathrm{Tr}_\nu(F)$ *fits into a commutative square*

$$\begin{array}{ccc} N \times I \times |\Delta^k| \times |\Delta^m| & \xrightarrow[\mathrm{Tr}_\nu(F)]{\cong} & N \times I \times |\Delta^k| \times |\Delta^m| \\ \uparrow{\scriptstyle \omega\times\mathrm{Id}} & & \uparrow{\scriptstyle \omega'\times\mathrm{Id}} \\ U \times D^d \times I \times |\Delta^k| \times |\Delta^m| & \xhookrightarrow[F\times\rho]{} & U' \times D^d \times I \times |\Delta^k| \times |\Delta^m| \end{array}$$

where $\rho\colon U \times D^d \times |\Delta^k| \times |\Delta^m| \to D^d$ *is adjoint to a family of automorphisms of the fibre* $\tilde{\rho}\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(D^d)$ *which, in turn, is induced by a transition map for the reduced structure group* $\tilde{\rho}'\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(S^{d-1})$.

Proof. The first part is clear. The second part follows from a computation in local coordinates. □

We are in position to state the definition of the unstable choice space.

Definition 2.1.3.8. Let $\tilde{\mathrm{Ch}}^{\mathrm{Top}}_\Delta$ denote the simplicially enriched category of preliminary choices with compact topological manifolds as objects and continuous choices as morphisms, see [12, Definition 5.4]. Precisely, an m–simplex in $\tilde{\mathrm{Ch}}^{\mathrm{Top}}_\Delta(M, N)$ consists of:

- a subset $E \subseteq N \times |\Delta^m|$ which is a family of codimension zero submanifolds over $|\Delta^m|$ of N,
- a sequence of disk bundles

$$E = E_n \xrightarrow{p_n} E_{n-1} \xrightarrow{p_{n-1}} \dots \xrightarrow{p_1} E_0 = M$$

  where all E_i are families of manifolds over $|\Delta^m|$ (and the bundles are also all over $|\Delta^m|$),
- a zero section $s\colon M \to E$ of the composed bundle $p_1 \circ \dots \circ p_n$,
- a parallel transport ν_i and a fibre-wise bending isotopy $H^{(i)}$ over p_i, for each p_i, such that $H^{(i)}$ is fibre-wise with respect to the bundle $p_1 \circ \dots \circ p_i$ and the composition $\mathrm{Tr}_{\nu_n, h^{(n)}} \circ \dots \circ \mathrm{Tr}_{\nu_1, h^{(1)}} : P_\partial(M) \to P(E)$ has image in $P_\partial(E)$.

The *simplicially enriched category of choices* $\mathrm{Ch}^{\mathrm{Top}}_\Delta$, see [12, Definition 5.9], has compact manifolds as objects and the morphism space $\mathrm{Ch}^{\mathrm{Top}}_\Delta(M, N)$ is the quotient of $\tilde{\mathrm{Ch}}^{\mathrm{Top}}_\Delta(M, N)$ by the equivalence relation generated by the following relations:

- Let $\mathrm{ch} = (E_m \subseteq N, p_m, \dots, p_1, s, \nu_m, \dots, \nu_1, H^{(m)}, \dots, H^{(1)})$ be an n–simplex with $H^{(i)}$ the constant isotopy with value the Identity on $E_i \times I$ for some $1 \le i \le m$. Then we identify ch with the tuple without E_i and $H^{(i)}$,

$$\begin{aligned}(E_m \subseteq N, p_m, \dots, p_i \circ p_{i+1}, \dots, p_1, s,\\ \nu_m, \dots, \nu_{i+1} \cdot \nu_i, \dots, \nu_1, H^{(m)}, \dots, H^{(1)}).\end{aligned}$$

  The composition $\nu_{i+1} \cdot \nu_i$ of parallel transports is [12, Definition 3.12]. Its only property relevant to us is that it satisfies $\mathrm{Tr}_{\nu_{i+1}\cdot\nu_i} = \mathrm{Tr}_{\nu_{i+1}} \circ \mathrm{Tr}_{\nu_i}$ by [12, Proposition 3.13].

- Let $\mathrm{ch} = (E_m \subseteq N, p_m, \ldots, p_1, s, \nu_m, \ldots, \nu_1, H^{(m)}, \ldots, H^{(1)})$ be an n–simplex with $p_i \colon E_i \to E_{i-1}$ a homeomorphism for some $1 < i \leq n$. Then we identify ch with the tuple, which omits E_{i-1} and uses the fibre-wise bending isotopy $\bar{H}^{(i-1)}(-, t) = (p_i \times \mathrm{Id}_I)^{-1} \circ H^{(i-1)}(-, t) \circ (p_i \times \mathrm{Id}_I)(-)$ for $t \in [0, 1]$,

$$(E_m \subseteq N, p_m, \ldots, p_i \circ p_{i-1}, \ldots, p_1, s,$$
$$\nu_m, \ldots, \nu_{i+1} \cdot \nu_i, \ldots, \nu_1, H^{(m)}, \ldots, \bar{H}^{(i-1)}, \ldots, H^{(1)}).$$

  Note that, since p_i is a homeomorphism, $H^{(i)}$ is a constant isotopy with value the Identity on $E_i \times I$ and Tr_{ν_i} acts by conjugation with $p_i \times \mathrm{Id}_I$.

We also state the piecewise linear version:

Definition 2.1.3.9. Let $\mathrm{Ch}_\Delta^{\mathrm{PL}}$ denote the simplicially enriched category of choices with compact piecewise linear manifolds as objects and piecewise linear choices as morphisms. Precisely, an m–simplex in $\mathrm{Ch}_\Delta(M, N)$ consists of:

- a subset $E \subseteq N$ which is a family of codimension zero PL submanifolds over $|\Delta^m|$ of N,
- a composition of piecewise linear disk bundles

$$E = E_n \xrightarrow{p_n} E_{n-1} \xrightarrow{p_{n-1}} \ldots \xrightarrow{p_1} E_0 = M$$

  where all E_i are families of PL manifolds over $|\Delta^m|$ (and the bundles are also all over $|\Delta^m|$),
- a zero section $s \colon M \to E$ of the composition bundle $p_1 \circ \ldots \circ p_n$,
- a piecewise linear parallel transport ν_i and a fibre-wise bending isotopy $H^{(i)}$ over p_i, for each p_i, such that $H^{(i)}$ is fibre-wise with respect to the bundle $p_1 \circ \ldots \circ p_i$ and the map $\mathrm{Tr}_{\nu_n, h^{(n)}} \circ \ldots \circ \mathrm{Tr}_{\nu_1, h^{(1)}} \colon P_\partial(M) \to P(E)$ has image in $P_\partial(E)$.

The equivalence relation is analogous, we just replace "$p_i \colon E_i \to E_{i-1}$ is a homeomorphism" by " ... is a PL isomorphism".

At this point, we have to note that the mapping spaces of Ch_Δ are in general not Kan-complexes. Thus its homotopy coherent nerve $\mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta$ is not a quasicategory. Therefore, whenever a quasicategory is needed, we are going to fibrantly replace it in the Joyal model structure without further comment. Note that this is only necessary when we wish to form its Ind–completion. We recall all properties of the Ind–completion of a quasicategory relevant to our purposes in Proposition 1.2.0.3.

Finally, let Ch_Δ^{st} denote the *simplicially enriched category of stable choices* which was denoted by $\mathbf{Ch}^{\mathcal{P}}$ in [12]. There is a faithful inclusion into the Ind–completion $\iota \colon \mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta^{st} \hookrightarrow \mathrm{Ind}(\mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta)$ with image the full subcategory on the objects of the form $\mathrm{hocolim}_{k \in \mathbb{N}} M \times (D^1)^k$. To see this one observes that the definition of $\mathrm{Ch}_\Delta^{st}(M, N)$ is an explicit model for $\mathrm{Ind}(\mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta)(\iota(M), \iota(N))$.

We turn to functors. There is a forgetful functor $\mathfrak{f} \colon \mathrm{Ch}_\Delta \to \mathrm{Mfd}_\Delta^{\mathrm{Top}}$ given by $\mathfrak{f}(\iota \colon E_m \subseteq N, s, \ldots) = \iota \circ s$.

Let $\mathrm{F}_{\mathrm{ch}}^{\mathrm{Top}} \colon \mathcal{N}_\bullet \mathrm{Mfd}_\Delta^{\mathrm{Top}} \to \mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta^{st}$ denote the functor of contractible choices constructed in the proof of [12, Theorem 6.4]. It satisfies $\mathfrak{f} \circ \mathrm{F}_{\mathrm{ch}}^{\mathrm{Top}} \simeq \mathrm{Id}$ with $\mathfrak{f}$ the induced forgetful functor.

Definition 2.1.3.10 ([12, Definition 5.10])**.** Let $r\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$ denote the *realisation functor* of simplicially enriched categories. On an m–simplex it is given by the factorisation of

$$\mathrm{Tr}_{\nu_n,h^{(n)}} \circ \ldots \circ \mathrm{Tr}_{\nu_1,h^{(1)}} \colon P_\partial(M) \to P(E)$$

over $P_\partial(E)$ composed with the map $P_\partial(E) \to P_\partial(N)$ which extends by the Identity.

The stable realisation functor $r^{st}\colon \mathrm{Ch}^{st}_\Delta \to \mathrm{Top}_\Delta$, see [12, Definition 5.24], is the restriction of the unstable one's Ind–completion, i.e. the diagram

$$\begin{array}{ccc} \mathcal{N}^{h.c.}_\bullet \mathrm{Ch}^{st}_\Delta & \xrightarrow{\;r^{st}\;} & \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta \\ \big\downarrow \iota & \nearrow_{\mathrm{Ind}(r)} & \\ \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet \mathrm{Ch}_\Delta) & & \end{array}$$

commutes.

The *stable pseudoisotopy functor* constructed in [12] is $r^{st} \circ \mathrm{F}_{\mathrm{ch}}$ and we may identify it as $\mathrm{Ind}(r) \circ \iota \circ \mathrm{F}_{\mathrm{ch}}$. Analogously we define a stable pseudoisotopy functor in the piecewise linear case. To obtain functors on Top one applies homotopy left Kan extension along the inclusion $\mathrm{Mfd}^{\mathrm{Cat}} \subseteq \mathrm{Top}$.

We have to slightly adjust the simplicial category of unstable choices to suit our purposes.

Definition 2.1.3.11. The enriched submonoid $\tilde{\mathrm{Ch}}^{c,m}_\Delta$ of $\tilde{\mathrm{Ch}}_\Delta$ contains all objects and its mapping spaces contain those n–simplices $\mathrm{ch} = (E_m \subseteq N, \ldots)$ in $\tilde{\mathrm{Ch}}_\Delta(M,N)$ which satisfy $\partial(E_m^{(t)}) \cap \partial N^{(t)} = \emptyset$ for every $t \in |\Delta^n|$, where $E_m^{(t)}$ denotes the fibre of $E_m \to |\Delta^n|$ over t, similarly for $N^{(t)}$. We make it into an enriched subcategory $\tilde{\mathrm{Ch}}^c_\Delta$ by formally adjoining the Identity.

We note that every inclusion $E_m^{(t)} \subseteq N^{(t)}$ in our new category admits a collar, i.e. there is a collar of $\partial E_m^{(t)}$ in $\overline{N^{(t)} - E_m^{(t)}}$.

We define Ch^c_Δ as the quotient of $\tilde{\mathrm{Ch}}^c_\Delta$, analogously to Ch_Δ.

Lemma 2.1.3.12. *The induced map $i\colon \mathrm{Ch}^c_\Delta \to \mathrm{Ch}_\Delta$ admits a section on simplicial nerves $\mathcal{N}^{h.c.}_\bullet \mathrm{Ch}_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Ch}^c_\Delta$.*

Proof. By the obstruction theory of Section 1.2.1 it is enough to show that the map $i\colon \mathrm{Ch}^{c,m}_\Delta(M,N) \to \mathrm{Ch}_\Delta(M,N)$ is an acyclic cofibration for every pair of compact manifolds M and N.

To see this, fix a collar $c\colon \partial N \times J \to N$ of ∂N in N where J denotes the interval. We obtain an embedding $\Phi^c\colon N \times [0,1] \to N \times [0,1]$ over $[0,1]$ given by Id on $(N - \mathrm{im}(c)) \times [0,1]$ and $\rho\colon \partial N \times J \times [0,1] \to \partial N \times J \times [0,1]$, $\rho(x,s,t) = (x, \alpha(s,t), t)$ on the collar with $\alpha\colon J \times [0,1] \to J$ some map with $\alpha(-,0) = \mathrm{Id}$ and $\alpha(-,t)\colon J \to [t/2,1] \subseteq J$ a PL isomorphism onto $[t/2,1]$ for every $t \in [0,1]$.

We define $\Phi^c(-,1)_*\colon \mathrm{Ch}_\Delta(M,N) \to \mathrm{Ch}^c_\Delta(M,N)$ by composing $E_n \subseteq N$ with $\Phi^c(-,1)$. The isotopy Φ^c induces a homotopy $\Phi^c(-,1)_* \circ i \simeq \mathrm{Id}$ as well as $i \circ \Phi^c(-,1)_* \simeq \mathrm{Id}$. □

Definition 2.1.3.13. Let $\tilde{\mathrm{Ch}}^{c,s}_{\Delta}$ denote the enriched subcategory of $\tilde{\mathrm{Ch}}^{c}_{\Delta}$ with all objects and mapping spaces containing those n–simplices $\mathrm{ch} \in \tilde{\mathrm{Ch}}^{c}_{\Delta}(M,N)$ for which the induced parallel transport $\nu_m \cdot \ldots \cdot \nu_1$ and the bending isotopy $H^{(i)}$ preserve the zero sections s and $p_{i+1} \circ \ldots \circ p_m \circ s$ of the composed bundle $p_1 \circ \ldots \circ p_i$, respectively, for $1 \leq i \leq m$.

We define $\mathrm{Ch}^{c,s}_{\Delta}$ as the quotient of $\tilde{\mathrm{Ch}}^{c,s}_{\Delta}$, analogously to Ch_{Δ}.

Lemma 2.1.3.14. *The induced map* $\mathrm{Ch}^{c,s}_{\Delta} \to \mathrm{Ch}^{c}_{\Delta}$ *is a categorical equivalence.*

Proof. It is enough to show that the map $i\colon \mathrm{Ch}^{c,s}_{\Delta}(M,N) \to \mathrm{Ch}^{c}_{\Delta}(M,N)$ is a weak equivalence for every pair of compact manifolds M and N.

We consider the parallel transport first. Note that we can reduce the structure group of $p_1 \circ \ldots \circ p_m$ to $\mathrm{Aut}(S^{d-1})$ for d the dimension of the fibre. The result now follows by a fibre-wise Alexander trick.

For the space of fibre-wise bending isotopies, this is analogous to the proof that the space of fibre-wise bending isotopies is contractible, see [12, Proposition 4.4]. ☐

To reduce technical complications later on, we include some contractible choices in the definition of the pseudoisotopy functor.

Definition 2.1.3.15. Let M be a compact manifold. Let $P^+(M)$ denote the simplicial set whose n–simplices are of the form (F,s), where F is an n–simplex of $P(M)$ and s is an S^1–family of collars of $M \times \{0\} \times |\Delta^n| \times S^1$ in $c(F)$, i.e. $(c(F),s)$ is a lift of $c(F)$ to $\Omega H^c_{\bullet}(M)$.

In order to still obtain a functor we have to include a map of collars into our contractible choice category.

Definition 2.1.3.16. Let $\mathrm{Ch}^{c,s,+}_{\Delta}(M,N)$ consist of pairs (ch,e) where ch is an n–simplex of $\mathrm{Ch}^{c,s}_{\Delta}(M,N)$ and $e(F,s)$ is a family of collars of $N \times \{0\} \times |\Delta^n| \times S^1$ in $c(r(\mathrm{ch})(F))$ parametrised over S^1 and restricting to s over $M \times \{0\} \times |\Delta^n| \times S^1$.

Lemma 2.1.3.17. *There is a simplicially enriched functor* $r^+\colon \mathrm{Ch}^{c,s,+}_{\Delta} \to \mathrm{Top}$ *given by* $r^+(M) = P^+(M)$ *on objects and* $r^+(\mathrm{ch},e)(F,c) = (r(\mathrm{ch})(F), e(c))$ *on the mapping spaces.*

We obtain an $(\infty,1)$*–functor* $P^+\colon \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Ch}^+_{\Delta} \to \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Kan}_{\Delta}$ *and an equivalence of* $(\infty,1)$*–functors* $r \circ \mathrm{pr} \simeq P^+$*, where* $\mathrm{pr}\colon \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Ch}^{c,s,+}_{\Delta} \to \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Ch}_{\Delta}$ *is the forgetful functor.*

Proof. Since the projection map $P^+(M) \to P(M)$ is a trivial fibration and the space of collars extending a fixed collar over a submanifold is contractible, this is an application of Corollary 1.2.1.7 and the functoriality of r. ☐

Corollary 2.1.3.18. *The map* $\mathrm{c}^+\colon P^+(M) \to H^c_{\bullet}(M)$ *given by sending* (F,s) *to* $(c(F),s)$ *makes the square*

$$\begin{array}{ccc} P^+(M) & \xrightarrow{\mathrm{c}^+} & H^c_{\bullet}(M) \\ \downarrow & & \downarrow \\ P(M) & \xrightarrow{\mathrm{c}} & H_{\bullet}(M) \end{array}$$

commute. Since c *as well as both vertical maps are weak equivalences, so is* c^+.

Remark 2.1.3.19. *There are choice functors F^{c}_{ch}, $F^{c,s}_{\mathrm{ch}}$ and $F^{c,s,+}_{\mathrm{ch}}$ compatible with each other and F_{ch} via the maps constructed above. In particular, we obtain a choice functor $F^{c,s,+}_{\mathrm{ch}} \colon \mathcal{N}_\bullet(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{pl}) \to \mathcal{N}^{h.c.}_\bullet \mathrm{Ch}^{c,s,+}_\Delta$.*

Notation 2.1.3.20. *We adjust some notation of [12] to be consistent within this thesis. Starting with the next section we write P^{Cat} for the unstable realisation r^+ where $\mathrm{Cat} = \mathrm{Top}, \mathrm{PL}$. We drop that we work with pseudoisotopies relative boundary from the notation. Further, we denote the unstable and stable pseudoisotopy functor P^+ and $\mathrm{Ind}(P^{\mathrm{Cat}}) \circ \iota \circ \mathrm{F}^{\mathrm{Cat}}_{\mathrm{ch}}$ by P and $\mathbb{P}^{\mathrm{Cat}}$, respectively. Also, we are going to refer to $\mathrm{Ch}^{c,s,+}_\Delta$ by Ch_Δ and to c^+ by c. Finally, we may drop the collar extension map e from the notation for some morphism (ch, e) in $\mathrm{Ch}^{c,s,+}_\Delta$.*

We conclude this section by collecting all properties of the pseudoisotopy functor necessary throughout the proof in the following theorem.

Theorem 2.1.3.21. *Let $Cat = \mathrm{PL}$ or Top.*

1. *The pseudoisotopy functor $\mathbb{P}^{\mathrm{Cat}} \colon \mathrm{Top} \to \mathrm{Top}$ is given by homotopy left Kan extension of its restriction to $\mathrm{Mfd}^{\mathrm{Cat}}$.*

2. *The functor $\mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet(r^+)) \circ \iota \circ F^{c,s,+}_{\mathrm{ch}}$ coincides - up to natural weak equivalence of $(\infty, 1)$–functors - with the stable pseudoisotopy functor defined in [12].*

3. *Let $i \colon (\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{cts}) \subseteq (\mathrm{Mfd}^{\mathrm{Top}}, \mathrm{cts})$ denote the inclusion. The point-wise inclusion maps $\mathbb{P}^{\mathrm{PL}}_\partial(M) \to \mathbb{P}^{\mathrm{Top}}_\partial(i(M))$ extend to a natural weak equivalence of functors.*

4. *Let $(i \colon E \subseteq N \times |\Delta^m|, (p_i)^n_{i=1}, s, (\nu_i)^n_{i=1}, (H^{(i)})^n_{i=1}, e)$ be a choice which factors over $\phi_0 = (E = E, (p_i)^n_{i=1}, s, (\nu_i)^n_{i=1}, (H^{(i)})^n_{i=1}, e_0)$ for some collar extension map e_0. Then the square*

$$\begin{array}{ccc} M \times I \times |\Delta^m| & \xrightarrow{F} & M \times I \times |\Delta^m| \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ E \times I \times |\Delta^m| & \xrightarrow{P(\phi_0)(F)} & E \times I \times |\Delta^m| \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ N \times I \times |\Delta^m| & \xrightarrow{P(\phi)(F)} & N \times I \times |\Delta^m| \end{array}$$

commutes for every $F \in P(M)$, i.e. the induced pseudoisotopy restricts to the original pseudoisotopy on the zero section of the disk bundle.

We reduce the structure group of the bundle to $\mathrm{Aut}(S^{d-1})$ by choosing a metric. Then there is a covering $(U_i, \omega_i)_{i \in I}$ of M by local trivialisations of $p = p_n \circ \cdots \circ p_1$ for some indexing set I with respect to the reduced structure group.

Further, let (U, ω) and (U', ω') be trivialisations of the covering. Then the geometric transfer without bending the boundary $P(\phi_0)(F)$ fits into a *commutative square*

$$\begin{array}{ccc}
N \times I \times |\Delta^k| \times |\Delta^m| & \xrightarrow[P(\phi_0)(F)]{\cong} & N \times I \times |\Delta^k| \times |\Delta^m| \\
\uparrow{\scriptstyle \omega\times\mathrm{Id}} & & \uparrow{\scriptstyle \omega'\times\mathrm{Id}} \\
U \times D^d \times I \times |\Delta^k| \times |\Delta^m| & \xhookrightarrow[F\times\rho]{} & U' \times D^d \times I \times |\Delta^k| \times |\Delta^m|
\end{array}$$

where $\rho\colon U \times D^d \times |\Delta^k| \times |\Delta^m| \to D^d$ *is adjoint to a family of automorphisms of the fibre* $\tilde{\rho}\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(D^d)$ *which, in turn, is induced by the reduced structure group* $\tilde{\rho}'\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(S^{d-1})$.

5. *Let* $p\colon E \to M$ *be a disk bundle,* ν *a transfer map and* H *a fibre-wise bending isotopy, everything parametrised over some compact manifold* X*. Let* F *in* $P(M)$ *be a pseudoisotopy. We obtain an isotopy of pseudoisotopies* $\mathrm{Tr}_{\nu,h}(F)\colon E \times I \to E \times I$ *over* X *where* $h = H(-,1)$ *is the associated bending map.*

6. *The point-wise inclusion maps* $\mathbb{P}^{\mathrm{Diff}}_{\partial}(M) \to \mathbb{P}^{\mathrm{Top}}_{\partial}(M)$ *extend to a natural transformation of quasicategories* $\mathbb{P}^{\mathrm{Diff}}_{\partial} \Rightarrow \mathbb{P}^{\mathrm{Top}}_{\partial}$ *between endofunctors on* $\mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Top}_{\Delta}$.

7. *The inclusion* $P^{\mathrm{Cat}}_{\partial}(M \times [0,1]^k) \subseteq P^{\mathrm{Cat}}(M \times [0,1]^k)$ *is a weak equivalence for every* $k \in \mathbb{N}$ *and they induce a weak equivalence* $\mathbb{P}^{\mathrm{Cat}}_{\partial}(M) \to \mathbb{P}^{\mathrm{Cat}}(M)$.

Proof. The first statement is just the definition of P. The second property was explained in the discussion preceding the theorem. The third claim follows from the second property and [12, Remark 6.1].

The first part of the fourth statement follows via an induction argument and Lemma 2.1.3.6 for the upper square. The lower one commutes by definition. The second part is Lemma 2.1.3.7. The fifth part follows immediately from the definitions.

To show the sixth part, we construct a homotopy commutative diagram

$$\begin{array}{ccccc}
\mathcal{N}_{\bullet}\,\mathrm{Mfd}^{\mathrm{Diff}}_{\Delta} & \xrightarrow{\mathrm{F_{ch}}} & \mathrm{Ind}(\mathcal{N}_{\bullet}\mathcal{D}_{\Delta}) & \xrightarrow{P^{\mathrm{Diff}}} & \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Top}_{\Delta} \\
\downarrow & & \downarrow{\scriptstyle j} & \nearrow{\scriptstyle P^{\mathrm{Top}}} & \\
\mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Mfd}^{\mathrm{Top}}_{\Delta} & \xrightarrow{\mathrm{F_{ch}}} & \mathrm{Ind}(\mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Ch}_{\Delta}) & &
\end{array}$$

which is enough by definition of the pseudoisotopy functors as homotopy left Kan extensions.

We first recall the map $j\colon \mathcal{N}^{h.c.}_{\bullet}\mathcal{D}_{\Delta} \to \mathcal{N}^{h.c.}_{\bullet}\,\mathrm{Ch}_{\Delta}$ from Remark 1.2.5.3. On objects it is given by forgetting the smooth structure. On simplices it is given by sending a triple $(\iota, \nu\iota, p)$ to $(\nu\iota, p, \iota, \nu_p, H_p)$ where ν_p is a topological parallel transport in the sense of Definition 2.1.3.5. It is induced by a parallel transport in the sense of differential geometry (depending on the contractible choice of a Riemannian metric) as explained in Remark 1.1.0.11. Further, H_p is a fibre-wise bending isotopy in the sense of Definition 2.1.3.4 with respect to p.

The commutativity of the square on the left hand side follows from three observations.

First, by Proposition 1.2.3.1 the triangle

$$\begin{array}{ccc} \mathcal{N}_\bullet^{h.c.}\,\mathrm{Mfd}_\Delta^{\mathrm{Diff}} & \xrightarrow{\mathrm{Id}} & \mathcal{N}_\bullet^{h.c.}\,\mathrm{Mfd}_\Delta^{\mathrm{Diff}} \\ \downarrow{\scriptstyle \tau_1} & \nearrow{\scriptstyle \mathrm{incl}} & \\ \mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Diff}}, \mathrm{smooth})_\Delta^{\mathrm{Diff}} & & \end{array}$$

commutes up to homotopy.

Further, the square

$$\begin{array}{ccc} \mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Diff}}, \mathrm{smooth})_\Delta & \xrightarrow{\mathfrak{f}\circ \mathrm{F}_{\mathrm{ch}}^{\mathrm{Diff}}} & \mathrm{Ind}(\mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Diff}}, \mathrm{emb})_\Delta) \\ \downarrow & & \downarrow \\ \mathcal{N}_\bullet^{h.c.}\,\mathrm{Mfd}_\Delta^{\mathrm{Top}} & \xrightarrow{\mathfrak{f}\circ \mathrm{F}_{\mathrm{ch}}} & \mathrm{Ind}(\mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Top}}, \mathrm{emb})_\Delta) \end{array}$$

commutes up to homotopy, since both compositions are up to homotopy given by the identity. Here $(\mathrm{Mfd}^{\mathrm{Top}}, \mathrm{emb})_\Delta$ is the simplicially enriched category of locally flat embeddings, $\mathrm{F}_{\mathrm{ch}}^{\mathrm{Diff}}$ denotes the restriction of the smooth choice functor, see Theorem 1.2.0.5, and $\mathfrak{f}$ is the forgetful functor.

Finally, the square

$$\begin{array}{ccc} \mathcal{N}_\bullet^{h.c.}\mathcal{D}_\Delta & \xrightarrow{\mathfrak{f}} & \mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Diff}}, \mathrm{emb})_\Delta \\ \downarrow{\scriptstyle j} & & \downarrow \\ \mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta & \xrightarrow{\mathfrak{f}} & \mathcal{N}_\bullet^{h.c.}(\mathrm{Mfd}^{\mathrm{Top}}, \mathrm{emb})_\Delta \end{array}$$

commutes strictly.

Plugging these results together, the homotopy commutativity of the square on the left hand side follows since the forgetful functor $\mathfrak{f}$ is an equivalence of quasicategories in the differentiable case by Proposition 1.2.5.1 and similarly so in the topological case after stabilisation [12, Theorem 5.22].

The triangle on the right hand side is the last part of Lemma 1.4.1.1 which remained unproven till now. It is a direct consequence of the last property stated in Lemma 1.4.1.1 of the unstable smooth pseudoisotopyfunctor and the definition of the topological pseudoisotopyfunctor.

We turn our attention to the last property. The unstable inclusion maps are weak equivalences by the existence of bending isotopies, see [12, Chapter 4]. Let $\mathrm{F}_{\mathrm{ch}}(\mathrm{incl}) = (\mathrm{Id}, \mathrm{pr}\colon M \times [0,1]^{k+1} \to M \times [0,1]^k, s_0, \nu_{\mathrm{pr}}, H_{\mathrm{pr}})$ be the choice for the inclusion $M \times [0,1]^k \hookrightarrow M \times [0,1]^{k+1}$. Then the square

$$\begin{array}{ccc} P_\partial^{\mathrm{Cat}}(M \times [0,1]^k) & \longrightarrow & P^{\mathrm{Cat}}(M \times [0,1]^k) \\ \downarrow{\scriptstyle P(\mathrm{F}_{\mathrm{ch}}(\mathrm{incl}))} & & \downarrow{\scriptstyle -\times \mathrm{Id}} \\ P_\partial^{\mathrm{Cat}}(M \times [0,1]^{k+1}) & \longrightarrow & P^{\mathrm{Cat}}(M \times [0,1]^{k+1}) \end{array}$$

commutes up to a homotopy induced by H_{pr}. □

2.1.4 The Whitehead space $(\infty, 1)$–functor

We wish to extend the 1–functoriality of $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|$ to obtain a map of quasicategories

$$|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|\colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb})_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta$$

where $(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb})_\Delta$ denotes the simplicially enriched category with compact PL manifolds as objects and, as morphisms of degree n from M to N, piecewise linear embeddings $e\colon M \times |\Delta^n| \to N \times |\Delta^n|$ over $|\Delta^n|$.

To compare this functor to $|s\tilde{C}^h|$ we also need

$$|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|\colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Poly}, \mathrm{emb})_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta$$

where $(\mathrm{Poly}, \mathrm{emb})_\Delta$ denotes the simplicially enriched category with compact polyhedra as objects and, as morphisms of degree n from M to N, piecewise linear embeddings $e\colon M \times |\Delta^n| \to N \times |\Delta^n|$ over $|\Delta^n|$.

The construction of the extended functor spans from Definition 2.1.4.1 to Proposition 2.1.4.3. The analogue of the zig-zag, the "non-manifold part", of [31] is given in Proposition 2.1.4.5.

We first note that it makes sense to glue bundles along piecewise linear families of embeddings.

Definition 2.1.4.1. Let $e\colon X \times |\Delta^n| \to Y \times |\Delta^n|$ denote a morphism in $(\mathrm{Poly}, \mathrm{emb})_\Delta(X, Y)_n$. We obtain a functor of bisimplicial categories

$$\begin{aligned} e_*\colon \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\mathrm{m}}(X) \times \Delta^n &\to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(Y) \\ (W \to |\Delta^m|) &\mapsto (W \times |\Delta^n| \cup_{X \times |\Delta^m| \times |\Delta^n|} Y \times |\Delta^m| \times |\Delta^n| \to |\Delta^m| \times |\Delta^n|) \end{aligned}$$

where the additional structure is constructed in the obvious fashion and we glue along the map $e \times \mathrm{Id}_{|\Delta^m|}$.

The simplicial category $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(Y)$ is understood as the bisimplicial category given by $(p, q) \mapsto \mathrm{Hom}_{\mathrm{scat}}(\Delta^p \times \Delta^q, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(Y))$.

Lemma 2.1.4.2. *Given a functor of bisimplicial categories $\phi\colon C_\bullet \times \Delta^n \to D_\bullet$ we obtain a functor of degree n of simplicial categories enriched in simplicial sets by the formula*

$$\begin{aligned} \tilde{\phi}\colon C_\bullet \times \Delta^n &\to D_\bullet \\ (c, \alpha) &\mapsto \mathrm{diag}^* \circ (\mathrm{Id} \times \alpha)^* \circ \phi(c, \mathrm{Id}) \\ (f, \mathrm{Id}_\alpha) &\mapsto \mathrm{diag}^* \circ (\mathrm{Id} \times \alpha)^* \circ \phi(f, \mathrm{Id}_{\mathrm{Id}}) \end{aligned}$$

where $\mathrm{diag}\colon \Delta^q \to \Delta^q \times \Delta^q$ *is the diagonal map.*

Proof. This is a straightforward calculation. □

Proposition 2.1.4.3. *We obtain a functor of simplicial categories enriched in simplicial sets* $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\colon (\mathrm{Poly}, \mathrm{emb})_\Delta \to \mathrm{scat}_\Delta$ *given by*

$$\begin{aligned} K &\mapsto \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(K) \\ (e\colon K \times |\Delta^n| \to L \times |\Delta^n|) &\mapsto \tilde{e}_*\colon \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(K) \times \Delta^n \to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(L). \end{aligned}$$

We denote its restriction to compact PL *manifolds by*

$$|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'\colon (\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb})_\Delta \to \mathrm{Top}_\Delta\,.$$

Proof. This is, yet again, a straightforward calculation. □

Remark 2.1.4.4. *A similar approach allow us to extend* Wh *itself to a functor of quasicategories. This is going to be relevant for the non-connective case.*

The zig-zag of weak equivalences explained in [31] between $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$ and $s\mathcal{C}^h$ can be extended to a zig-zag of natural weak equivalences between endofunctors of Top by homotopy left Kan extensions. We could avoid this part completely via the arguments used in the non-connective case. However, we obtain a shorter proof in the connected setting without any real difficulties so it seems acceptable to explain this alternative.

Let $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1 \colon (\mathrm{Poly}, \mathrm{emb}) \to \mathrm{Top}$ and $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'_1 \colon (\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb}) \to \mathrm{Top}$ denote the underlying functors of 1–categories.

By homotopy left Kan extension we may extend the functor $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'_1$ to a functor $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|''_1$ with source category $(\mathrm{Poly}, \mathrm{emb})_\Delta$.

Since every compact polyhedron has the homotopy type of a compact PL manifold and the natural transformation $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'(M) \to |\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'(M \times [0,1])$ induced by the inclusion is a weak equivalence [31, Lemma 4.1.12], it is not hard to see that the natural transformation $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|''_1 \Rightarrow |\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1$ of functors from $(\mathrm{Poly}, \mathrm{emb})$ to Top is a weak equivalence.

Let $r \colon (\mathrm{sSet}^{\mathrm{fin,non\text{-}sing}}, \mathrm{cof}) \to (\mathrm{Poly}, \mathrm{emb})$ denote the polyhedral realisation functor, see Definition 2.1.1.16. We extend the functor $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1 \circ r$ by homotopy left Kan extension along r to a functor $(|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1 \circ r)' \colon (\mathrm{Poly}, \mathrm{emb}) \to \mathrm{Top}$.

The natural transformation $(|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1 \circ r)'(K) \to |\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1(K)$ is a weak equivalence. This follows from triangulation theory and, again, the fact that the natural transformation $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'(M) \to |\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'(M \times [0,1])$ is a weak equivalence.

We have functors $s\mathcal{C}^h \colon \mathrm{sSet}^{\mathrm{fin}} \to \mathrm{Top}$ and $s\mathcal{D}^h \colon (\mathrm{sSet}^{\mathrm{fin,non\text{-}sing}}, \mathrm{cof}) \to \mathrm{Top}$, see Definition 2.1.1.14.

Further, the natural transformations of functors from $(\mathrm{sSet}^{\mathrm{fin,non\text{-}sing}}, \mathrm{cof})$ to Top given by polyhedral realisation $\tilde{n} \circ r \colon s\tilde{\mathcal{D}}^h \Rightarrow s\tilde{\mathcal{E}}^h_\bullet \circ r$ and inclusion $\iota \colon s\tilde{\mathcal{D}}^h \Rightarrow s\tilde{\mathcal{C}}^h$ induce natural weak equivalences.

Finally, we explain how to pass from $s\tilde{\mathcal{C}}^h \colon (\mathrm{sSet}^{\mathrm{fin,non\text{-}sing}}, \mathrm{cof}) \to \mathrm{Top}$ to the 1–functor $s\tilde{\mathcal{C}}^h \colon \mathrm{sSet} \to \mathrm{Top}$ which is related to A–theory by Waldhausen's zig-zag of weak equivalences. For this purpose, we denote the latter by $(s\tilde{\mathcal{C}}^h)_1$.

By homotopy left Kan extension, $s\tilde{\mathcal{C}}^h \colon (\mathrm{sSet}^{\mathrm{fin,non\text{-}sing}}, \mathrm{cof}) \to \mathrm{Top}$ yields a functor $\mathrm{hlKe}(s\tilde{\mathcal{C}}^h) \colon \mathrm{sSet}^{\mathrm{fin,non\text{-}sing}} \to \mathrm{Top}$. By restriction along the inclusion $i \colon \mathrm{sSet}^{\mathrm{fin,non\text{-}sing}} \subseteq \mathrm{sSet}$, we obtain $(s\tilde{\mathcal{C}}^h)_2 = (s\tilde{\mathcal{C}}^h)_1 \circ i \colon \mathrm{sSet}^{\mathrm{fin,non\text{-}sing}} \to \mathrm{Top}$.

The natural transformation $\mathrm{hlKe}(s\tilde{\mathcal{C}}^h) \Rightarrow (s\tilde{\mathcal{C}}^h)_2$ is a weak equivalence, because $s\tilde{\mathcal{C}}^h$ is a restriction of $(s\tilde{\mathcal{C}}^h)_1$.

The homotopy left Kan extension of $(s\tilde{\mathcal{C}}^h)_2 \colon \mathrm{sSet}^{\mathrm{fin,non\text{-}sing}} \to \mathrm{Top}$ to sSet coincides with the functor $(s\tilde{\mathcal{C}}^h)_1 \colon \mathrm{sSet} \to \mathrm{Top}$ of [31] and [51] up to weak homotopy equivalence. For this, we note first that every finite simplicial set has the weak homotopy type of a non-singular simplicial set by [31, Theorem 2.5.2]. Since $|s\tilde{\mathcal{C}}^h|_1$ commutes with directed colimits, it coincides with the homotopy left Kan extension of its restriction to $\mathrm{sSet}^{\mathrm{fin}}$.

In total, we obtain a zig-zag of natural weak equivalences of functors from $(\mathrm{Poly}, \mathrm{emb})$ to Top

$$|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|''_1 \to |\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|_1 \leftarrow (|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,| \circ r)',$$

a zig-zag of natural weak equivalences of functors from $(\mathrm{sSet}^{\text{non-sing,fin}}, \mathrm{cof})$ to Top

$$|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}\,| \circ r \leftarrow |s\tilde{\mathcal{D}}^{h}| \rightarrow |s\tilde{\mathcal{C}}^{h}|,$$

a natural weak equivalence of functors from $\mathrm{sSet}^{\text{non-sing,fin}}$ to Top

$$|\,\mathrm{hlKe}(s\tilde{\mathcal{C}}^{h})| \rightarrow |s\tilde{\mathcal{C}}^{h}|_{(2)}$$

and a natural weak equivalences of functors from sSet to Top

$$|\,\mathrm{hlKe}(s\tilde{\mathcal{C}}^{h})|_{2} \rightarrow |s\tilde{\mathcal{C}}^{h}|_{1}.$$

All of these may be extended by homotopy left Kan extensions to functors from sSet to Top. One uses the functors specified in the various zig-zags and the singular simplicial sets functor $\mathcal{S}$, right adjoint to geometric realisation.

We sum up our results in the following proposition.

Proposition 2.1.4.5. *There is a zig-zag of natural weak equivalences of functors between*

$$|s\tilde{\mathcal{C}}^{h}|\colon\ \mathrm{Top} \rightarrow \mathrm{Top}$$

defined in Definition 2.1.1.14 and the homotopy left Kan extension

$$\mathrm{hlKe}(|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}\,|'_{1})\colon\ \mathrm{Top} \rightarrow \mathrm{Top}$$

of the 1*–functor underlying* $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}\,|'\colon (\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb})_{\Delta} \rightarrow \mathrm{Top}_{\Delta}$ *which is given in Proposition 2.1.4.3.*

2.1.5 The natural transformation

The key to obtain our natural transformation is a natural weak equivalence between $P\colon \mathrm{Ch}_{\Delta} \rightarrow \mathrm{sSet}$ and $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}\colon (\mathrm{Mfd}, \mathrm{pl})_{\Delta} \rightarrow \mathrm{sCat}$ as 2–functors (with appropriate identifications of source and target). The main tool to get these transformations is the fact that the simplicial functors $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}(M) \rightarrow \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}(N)$ form the objects of a simplicial functor category. The morphisms of this category, i.e. the simplicial natural transformations, induce the required homotopies to obtain a 2–functorial transformation.

These simplicial natural transformations are formally introduced as "admissible retractions" in Definition 2.1.5.6 and Definition 2.1.5.7. The geometric idea stems from the explicit construction in the proof of Lemma 2.1.5.10 that such transformations exist.

Since admissible retractions depend on certain choices we have to thicken up our choice category Ch_{Δ} to Ch^{R}_{Δ} in Definition 2.1.5.11. Then, we have to show that these further choices are indeed contractible which is the content of Proposition 2.1.5.14. The desired 2–functor becomes an easy Corollary 2.1.5.17.

Aside from the abbreviations given in Notation 2.1.3.20 we introduce one more in Notation 2.1.5.2.

We begin with a reduction to functors from the choice category. By statement 1 of Theorem 2.1.3.21 the functor $\mathbb{P}^{\mathrm{PL}}\colon \mathrm{Top} \rightarrow \mathrm{Top}$ is the homotopy left Kan extension of its restriction to $\mathcal{N}^{h.c.}_{\bullet}(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{pl})_{\Delta}$. Upon restriction we obtain $\mathbb{P}^{\mathrm{PL}} = \mathrm{Ind}(\mathcal{N}^{h.c.}_{\bullet}(r^{+})) \circ \iota \circ F^{c,s,+}_{\mathrm{ch}}$ by statement 2 in Theorem 2.1.3.21.

Since $\mathrm{hlKe}(|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'_1)\colon \mathrm{Top} \to \mathrm{Top}$ is a homotopy left Kan extension by definition, it is enough to find a zig-zag between the restricted functors.

We want to compare the $(\infty,1)$–functors $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|'\colon (\mathrm{Mfd}^{\mathrm{PL}},\mathrm{pl})_\Delta \to \mathrm{Top}_\Delta$ and $P^{\mathrm{Cat}}_\partial = r^+\colon \mathrm{Ch}^{c,s,+}_\Delta \to \mathrm{Top}_\Delta$. We first get rid of the Ind–completion.

As mentioned in Notation 2.1.3.20 we write Ch_Δ instead of $\mathrm{Ch}^{c,s,+}_\Delta$ to refer to the thickened up category of choices. All conventions of Notation 2.1.3.20 are in effect.

Lemma 2.1.5.1. *Let* Cat = Top *or* PL. *Let* $\mathrm{st}\colon \mathrm{Mfd}^{\mathrm{Cat}}_\Delta \to \mathrm{Top}_\Delta$ *denote the simplicially enriched functor given by* $M \mapsto \mathrm{hocolim}_{k\in\mathbb{N}}\, M \times [0,1]^k$. *Further, let* $\mathfrak{f}\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$ *be the forgetful functor given by the Identity on objects and* $(i\colon E \subseteq N, s, \dots) \mapsto i \circ s$ *on mapping spaces.*

The square

$$\begin{array}{ccc}
\mathcal{N}^{h.c.}_\bullet\,\mathrm{Mfd}^{\mathrm{Cat}}_\Delta & \xrightarrow{\iota\circ \mathrm{F}_{\mathrm{ch}}} & \mathrm{Ind}(\mathcal{N}^{h.c.}_\bullet\,\mathrm{Ch}_\Delta) \\
\Big\downarrow{\scriptstyle \mathrm{Id}} & \searrow^{\mathrm{st}} & \Big\downarrow{\scriptstyle \mathrm{Ind}(\mathfrak{f})} \\
\mathcal{N}^{h.c.}_\bullet\,\mathrm{Mfd}^{\mathrm{Cat}}_\Delta & \xrightarrow{\ j\ } & \mathcal{N}^{h.c.}_\bullet\,\mathrm{Top}_\Delta
\end{array}$$

commutes up to natural weak equivalence of $(\infty,1)$*–functors, where* j *denotes the Yoneda embedding into the* Ind*–completion, see Proposition 1.2.0.3.*

Proof. This follows from the fact that $\mathfrak{f} \circ \mathrm{F}_{\mathrm{ch}} \simeq \mathrm{Id}$ holds. □

The lemma implies that it is enough to find a zig-zag between the functors $\Omega|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,| \circ \mathfrak{f}$ and P^{PL}.

Notation 2.1.5.2. *Going forward, we do not need any other variations of* $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|' \circ \mathfrak{f}\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$. *Hence we refer to it by* $|\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet\,|\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$ *from here on out.*

As stated in the introduction we are going to use the morphisms of the simplicial category $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$ to obtain homotopies which induce the desired naturality up to coherent homotopy. Formally, we work in the category of simplicial categories enriched in simplicial categories.

Definition 2.1.5.3. We understand the simplicial set Δ^n as a discrete simplicial category. Let $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ denote simplicial categories. We define the *simplicial mapping space category* $\mathrm{map}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$ as follows: It is given in degree n by $\mathrm{scat}(\mathcal{C}_\bullet \times \Delta^n, \mathcal{D}_\bullet)$, i.e. the simplicial functors from $\mathcal{C}_\bullet \times \Delta^n$ to $\mathcal{D}_\bullet$. The morphisms in degree n are $\mathrm{scat}(\mathcal{C}_\bullet \times [1] \times \Delta^n, \mathcal{D}_\bullet)$, i.e. the simplicial natural transformations between simplicial functors.

Let $\mathcal{C}'_\bullet \subseteq \mathcal{C}_\bullet$ and $\mathcal{D}'_\bullet \subseteq \mathcal{D}_\bullet$ be pairs of simplicial categories. The *relative simplicial mapping space category* $\mathrm{map}((\mathcal{C}_\bullet, \mathcal{C}'_\bullet), (\mathcal{D}_\bullet, \mathcal{D}_\bullet)')$ is the simplicial subcategory of $\mathrm{map}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$ consisting of those objects and morphisms which restrict to $\mathrm{map}(\mathcal{C}'_\bullet, \mathcal{D}'_\bullet)$.

There is an inclusion from the category of simplicial categories enriched in simplicial sets scat_Δ to the category of simplicial categories enriched in simplicial categories $\mathrm{scat}_{\Delta^{\mathrm{cat}}}$. It is induced by the inclusion of simplicial sets into simplicial categories as discrete simplicial categories.

Definition 2.1.5.4. There is an inclusion $i\colon \mathrm{scat}_\Delta \to \mathrm{scat}_{\Delta^{\mathrm{cat}}}$ which is given by $i(C_\bullet) = C_\bullet$ on objects and the Identity on morphisms.

Next, we introduce a certain class of simplicial natural transformations, namely "admissible retractions". They are going to form the 2–cells of our desired coherent transformation.

Remark 2.1.5.5. *Let* $\mathrm{ch} = (E_m \subseteq N, p_m, \ldots, p_1, s, \nu_m, \ldots, \nu_1, H^{(m)}, \ldots, H^{(1)})$ *be an* n*–simplex in* $\mathrm{Ch}_\Delta(M, N)$*.*

By Definition 2.1.3.11 the embedding $\iota\colon E_m \subseteq N$ *admits a family of collars* $c\colon \partial E_m^0 \times |\Delta^n| \times [0,1] \to \overline{N^0 - E_m^0} \times |\Delta^n|$*. Here we use some trivialisations of* E_m *and* N *over* $|\Delta^n|$ *with fibres* E_m^0 *and* N^0*.*

Definition 2.1.5.6. Let ch be an n–simplex in $\mathrm{Ch}_\Delta(M, N)$. It is of the form $\mathrm{ch} = (E_m \subseteq N, p_m, \ldots, p_1, s, \nu_m, \ldots, \nu_1, H^{(m)}, \ldots, H^{(1)})$.

Let $M = M_0, M_1, \ldots, M_m = N$ be a sequence of manifolds over $|\Delta^n|$. All of the following constructions are over $|\Delta^n|$. Let $\iota_i\colon M_i \to M_{i+1}$ be an embedding, $\tilde{p}_i\colon \tau_{M_{i+1}} M_i \to M_i$ a tubular neighbourhood with respect to ι_i and $c_i\colon \partial\tau_{M_{i+1}} M_i \times [0,1] \to M_{i+1}$ a collar for every $0 \le i \le m-1$.

Let $\mathrm{incl}_i\colon \tau_{M_i} M_{i-1} \subseteq M_i$ denote the embedding for every $0 \le i \le m-1$. We call a tuple of sequences $(M_i, \iota_i, \tilde{p}_i, \mathrm{im}(c_i))_{0 \le i \le m-1}$ *admissible* if it satisfies $p_{i+1} = \mathrm{incl}_1^* \circ \mathrm{incl}_2^* \circ \ldots \circ \mathrm{incl}_i^*(\tilde{p}_i)$ for every $0 \le i \le m-1$.

An *admissible retraction over* ch *with respect to* $(M_i, \iota_i, \tilde{p}_i, \mathrm{im}(c_i))_{0 \le i \le m-1}$ is a natural transformation $\alpha\colon \mathrm{u} \circ \mathrm{c} \circ P(\mathrm{ch}) \Rightarrow \Omega\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(\mathrm{ch}) \circ \mathrm{u} \circ \mathrm{c}$ along simple retraction maps such that every map $\alpha\colon \mathrm{u} \circ \mathrm{c} \circ P(\mathrm{ch})(F) \to \Omega\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(\mathrm{ch}) \circ \mathrm{u} \circ \mathrm{c}(F)$ satisfies the following two properties:

1. The retraction α is given by the standard simple retraction map pr (i.e. the projection) on $(M_{i+1} - \mathrm{im}(c_i) - \tau_{M_{i+1}} M_i) \times I$ for every $0 \le i \le m-1$.
2. Upon the choice of a parametrisation of $\mathrm{im}(c_i)$ as a collar we obtain a tubular neighbourhood $p'\colon \mathrm{im}(c_i) \cup \tau_{M_{i+1}} M_i \to M_i$ of M_i in M_{i+1} given by $p' = \tilde{p}_i \circ \mathrm{pr}_{\partial\tau_{M_{i+1}} M_i} \circ c_i^{-1} \cup \tilde{p}_i$.

   We set $U = c_{i-1}(\partial\tau_{M_i} M_{i-1} \times [0,1)) \cup \tau_{M_i} M_{i-1}$ for a choice of c_{i-1}. Let p'' denote the pullback of p' along the subspace inclusion $U \subseteq M_i$. Let T'' denote the total space of this pulled back tubular neighbourhood.

   The condition is as follows: There is a parametrisation of $\mathrm{im}(c_i)$ such that the map $\mathrm{pr} \circ (p' \times \mathrm{Id}_I)\colon (\mathrm{im}(c_i) \cup \tau_{M_{i+1}} M_i - T'') \times I \to (M_i - U) \times [0,1]$ and the restriction of $(p' \times \mathrm{Id}_{[0,1]}) \circ \alpha$ to the same subspace coincide, i.e. the restriction of the retraction map is fibre-preserving with respect to p'.

Here $\Omega\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$ is a simplicial diagram category $\mathrm{map}((\Delta^1, \partial\Delta^1), (\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(-), *(-))$ for a certain contractible discrete simplicial subcategory. The homotopy type of $\mathrm{map}((\Delta^1, \partial\Delta^1), (\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(-), *(-))$ is independent of $*$. In Lemma 2.1.5.10 we are going to specify an explicit choice of $*$ for all cases we care about.

In order to obtain a contractible space of further choices over each simplex ch, we have to allow shorter sequences. We follow the same ideas as underlying the choice space $\mathrm{Ch}_\Delta(M, N)$.

Definition 2.1.5.7. Let ch be an n–simplex in $\mathrm{Ch}_\Delta(M, N)$. As before, we have $\mathrm{ch} = (E_m \subseteq N, p_m, \ldots, p_1, s, \nu_m, \ldots, \nu_1, H^{(m)}, \ldots, H^{(1)})$. All constructions are over $|\Delta^n|$.

Let $0 = i_0, i_1, \ldots, i_k = m$ be an ordered subset of $0, 1, \ldots, m$. As above, let $M = M_0, M_{i_1}, \ldots, M_m = N$ be a sequence of manifolds, $\iota_{i_t} \colon M_{i_t} \to M_{i_t+1}$ an embedding, $\tilde{p}_{i_t} \colon \tau_{M_{i_t+1}} M_{i_t} \to M_{i_t}$ a tubular neighbourhood, and finally $c_{i_t} \colon \partial\tau_{M_{i_t+1}} M_{i_t} \times [0,1] \to M_{i_t+1}$ a collar for every $0 \leq t \leq k-1$.

We obtain a sequence of bundles $p'_{i_t} \colon E_{i_t} \to E_{i_{t-1}}$ for $1 \leq t \leq k$ via composition.

A collection $(M_{i_t}, \iota_{i_t}, \tilde{p}_{i_t}, \mathrm{im}(c_{i_t}))_{0 \leq t \leq k-1}$ is called *admissible* if it satisfies $p_{i_t+1} = \mathrm{incl}_1^* \circ \mathrm{incl}_2^* \circ \ldots \circ \mathrm{incl}_{i_t}^*(\tilde{p}_{i_t})$ for every $0 \leq t \leq k-1$.

Note that the notion of an admissible retraction over ch only depends on the underlying sequence of bundles $(E_m \subseteq N, p_m, \ldots, p_1, s)$ of ch. Hence we may define an *admissible retraction over* ch *with respect to* $(M_{i_t}, \iota_{i_t}, \tilde{p}_{i_t}, \mathrm{im}(c_{i_t}))_{0 \leq t \leq k-1}$ to be an admissible retraction over $(E_m \subseteq N, p'_m, p'_{i_{k-1}} \ldots, p'_1, s)$ with respect to $(M_{i_t}, \iota_{i_t}, \tilde{p}_{i_t}, \mathrm{im}(c_{i_t}))_{0 \leq t \leq k-1}$.

Remark 2.1.5.8. *We note that admissible retractions are closed under composition. More precisely, given admissible retractions α and α' with respect to collections A and A' and composable choices* ch *and* ch$'$*, the composed map $\alpha' \circ \alpha$ is an admissible retraction with respect to the concatenation of the collections $A' * A$ and the choice* ch $\circ$ ch$'$.

Now we make sure that admissible retractions actually exist. The construction given in the proof is the main geometric insight and motivates the notion of admissible retractions as a small extension of these maps which grants us just enough flexibility to obtain contractible choices.

Definition 2.1.5.9. Let $i \colon X \hookrightarrow Y$ be an inclusion. A *simple retraction map* $r \colon Y \to X$ with respect to i is a retraction, i.e. $r \circ i = \mathrm{Id}$, and a simple map.

Lemma 2.1.5.10. *Let* $\mathrm{ch} = (f, e)$ *be an n–simplex in* $\mathrm{Ch}_\Delta(M, N)$ *with first entry* $f = (E_m \subseteq N, p_m, \ldots, p_1, s, \nu_m, \ldots, \nu_1, H^{(m)}, \ldots, H^{(1)})$.

Let s_i denote the zero section of p_i (induced by choosing a metric). Then $M_i = E_i$ for $0 \leq i \leq m$ and $M_{m+1} = N$, $\iota_i = s_i$ for $0 \leq i \leq m$ and $i_f \colon E_m \subseteq N$, $\tilde{p}_{i+1} = p_i$ for $0 \leq i \leq m-1$ and some collar for $i_f(E_m)$ in N form an admissible sequence associated to ch. *We write s_f instead of s.*

There is an admissible retraction with respect to this sequence.

Proof. It follows from the definition that we have indeed described an admissible sequence. We have to show the existence of an admissible retraction.

We are going to construct a natural transformation from $\mathrm{u} \circ \mathrm{c} \circ P^{\mathrm{PL}}(f, e)$ to $\Omega\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(f, e) \circ \mathrm{u} \circ \mathrm{c}$ in $\mathrm{map}(P^{\mathrm{PL}}(M), \mathrm{map}((\Delta^1, \partial\Delta^1), (\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(N \times [0,1]), *)))$ where $*$ denotes the smallest simplicial subcategory which contains

$$\begin{array}{ccc} N \times [0,1] \times |\Delta^k| & \xrightarrow{\ s_N\ } & N \times I \times |\Delta^k| \\ & \searrow^{\mathrm{pr}} & \big\downarrow \mathrm{pr} \\ & & |\Delta^k| \end{array}$$

for $k \in \mathbb{N}$ and s_N a family of collars of $N \times \{0\}$ in $N \times I$ parametrised over $|\Delta^k|$. Note that $*$ is a contractible Kan complex. We assume for now that ch factors as $\iota \circ (f_0, e_0)$ with $\iota = (E_m \subseteq N, e')$ in $\mathrm{Ch}_\Delta(M, E_m)$.

Consider a k–simplex (F, s_F) of $P^{\mathrm{PL}}(M)$ with $F\colon M\times I\times|\Delta^k| \to M\times I\times|\Delta^k|$ and s_F a collar in $\mathrm{c}(F)$. We compute $\Omega\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(f,e)\circ \mathrm{u}\circ \mathrm{c}(F,s_F)$. The k–simplex $\mathrm{c}(F,s_F)$ in $\Omega H^{\mathrm{PL}}_\bullet(M)$ is given by the PL bundle

$$\begin{array}{ccc} M\times\{0\}\times S^1\times|\Delta^k| & \xrightarrow{e(F)\circ i_0} & W \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle \mathrm{pr}\circ(e(F))^{-1}} \\ & & S^1\times|\Delta^k| \end{array}$$

with $S^1 = [0,1]/0\sim 1$ for a certain $W \cong M\times I\times|\Delta^k|\times[0,1]/\sim$ with $(m,t,v,1)\sim(F(m,t,v),0)$, see Definition 2.1.2.2. Applying u yields a relative bundle

$$\begin{array}{ccc} \tilde{M} = M\times[0,1]\times S^1\times|\Delta^k| & \xrightarrow{s_W} & W \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle p_W} \\ & & S^1\times|\Delta^k| \end{array}$$

where the map $s_W = s_F\colon M\times[0,1]\times S^1\times|\Delta^k| \to W$ is given by the choice of the collar of $M\times\{0\}\times S^1\times|\Delta^k|$. We form the pushout along $i_f\circ s_f$ which yields

$$\begin{array}{ccc} \tilde{N} = N\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n| & \longrightarrow & W\times|\Delta^n| \cup_{M\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n|} \tilde{N} \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle (p_W\times \mathrm{pr}_{\Delta^n})\cup \mathrm{pr}} \\ & & S^1\times|\Delta^k|\times|\Delta^n| \end{array}$$

and the image of $((F,s_F),\alpha\colon[k]\to[n])$ under $\Omega\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(f)\circ\mathrm{u}\circ\mathrm{c}$ is given by the pullback along $((\mathrm{Id}\times\alpha)\circ\mathrm{diag})_*\colon|\Delta^k|\to|\Delta^k|\times|\Delta^n|$.

We compute $\mathrm{u}\circ\mathrm{c}\circ P(f,e)(F,s_F)$ next. There is an induced pseudoisotopy $P(f)(F)'\colon N\times I\times|\Delta^k|\times|\Delta^n| \to N\times I\times|\Delta^k|\times|\Delta^n|$ which pulls back to $P(f)(F) = (\alpha\circ\mathrm{diag})^*(P(f)(F)')$ and via c we obtain a PL bundle

$$\begin{array}{ccc} N\times\{0\}\times S^1\times|\Delta^k|\times|\Delta^n| & \xrightarrow{e(F)\circ i_0} & V \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle p_V=\mathrm{pr}\circ(e(F))^{-1}} \\ & & S^1\times|\Delta^k|\times|\Delta^n| \end{array}$$

with gluing map $P(f)(F)'$ for some $V\cong N\times I\times|\Delta^k|\times|\Delta^n|\times[0,1]/\sim$ with $(m,t,v,w,1)\sim(P(f)(F)'(m,t,v,w),0)$. Now u grants us a relative bundle

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{s_V} & V \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle p_V} \\ & & S^1\times|\Delta^k|\times|\Delta^n| \end{array}$$

as above.

Next, we construct a natural subbundle inclusion of $\Omega\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}(f,e)\circ\mathrm{u}\circ\mathrm{c}(F,s_F)$ into $\mathrm{u}\circ\mathrm{c}\circ P(f,e)(F,s_F)$. We define f_0 in $\mathrm{Ch}_\Delta(M,N)$ as the n–simplex which is the Identity on E_m in the first entry and coincides with f otherwise. The pseudoisotopy

$$P(f_0)(F)\colon E_m\times I\times|\Delta^k|\to E_m\times I\times|\Delta^k|$$

restricts to a pseudoisotopy on the zero sections which is given by F, see statement 4 of Theorem 2.1.3.21.

Thus, $p_W\times\mathrm{pr}_{\Delta^n}\colon W\times|\Delta^n|\to S^1\times|\Delta^k|\times|\Delta^n|$ may be thought of as a subbundle of $p_V\colon V\to S^1\times|\Delta^k|\times|\Delta^n|$ and we write $i\colon W\times|\Delta^n|\hookrightarrow V$ for the inclusion. It is induced by

$$(i_f\circ s_f)\times\mathrm{Id}\colon M\times I\times[0,1]\times|\Delta^k|\times|\Delta^n|\hookrightarrow N\times I\times[0,1]\times|\Delta^k|\times|\Delta^n|.$$

By definition of (f,e) the collar used to construct s_W is the restriction of the collar used for s_V to $M\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n|$ along i, i.e. the square

$$\begin{array}{ccc}\tilde{M}=M\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n| & \xrightarrow{s_W} & W\\ \downarrow{\scriptstyle i_f\circ\iota_f} & & \downarrow{\scriptstyle i}\\ \tilde{N}=N\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n| & \xrightarrow{s_V} & V\end{array}$$

commutes. Hence we can define an injective bundle map

$$i\cup s_V\colon W\cup_{M\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n|}\tilde{N}\to V.$$

This concludes our preparations. Now, we construct a natural transformation from $\mathrm{u}\circ\mathrm{c}\circ P(f,e)(F,s_F)$ to $\Omega\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}(f,e)\circ\mathrm{u}\circ\mathrm{c}(F,s_F)$ along simple retraction maps where each retraction is relative to the inclusion $i\cup s_V$.

The natural transformation consists of maps $V_F\to W_F$ which satisfy the following conditions:

1. They are compatible with sections and projections.
2. They are compatible with the simplicial structure maps.

Note that $P^{\mathrm{PL}}(M)$ is a discrete simplicial category, hence naturality imposes no condition.

We restrict our attention to the two special cases where f is either a transfer or a codimension zero embedding, i.e.

- either the family of codimension zero submanifolds $E\subseteq N\times|\Delta^m|$ is $N\times|\Delta^m|$ itself
- or the disk bundles p_i are all given by the Identity, forcing the zero section and the parallel transports to be given by Identity maps as well, and further implies that the fibre-wise bending isotopies are given by constant isotopies at the Identity.

Suppose that f is a codimension zero embedding. We have

$$W\cup_{\tilde{M}}\tilde{N}=\mathrm{u}\circ\mathrm{c}(F)\cup_{\partial M\times[0,1]\times S^1\times|\Delta^k|\times\Delta^n}\overline{(N-M)}\times[0,1]\times S^1\times|\Delta^k|\times|\Delta^n|$$

and

$$V = \mathrm{u} \circ \mathrm{c}(F) \cup_{\partial M \times I \times S^1 \times |\Delta^k| \times \Delta^n} \overline{(N - M)} \times I \times S^1 \times |\Delta^k| \times |\Delta^n|.$$

Let $a \in (0,1)$. There is a simple retraction map

$$r\colon [0,1] \times I \to \{0\} \times I \cup_{\{0\} \times [0,a]} [0,1] \times [0,a].$$

which restricts to the standard projection $\tilde{r}\colon \{1\} \times I \to \{1\} \times [0,a]$. We abuse notation and suppress the $\{1\}$–factor.

We define a simple map $\rho\colon V \to W \cup_{\tilde{M}} \tilde{N}$. To do so, let us choose a collar $c\colon \partial M \times J \to \overline{N - M}$, where $J = [0,1]$ denotes the standard interval, and write $W \cup_{\tilde{M}} \tilde{N}$ and V as

$$\begin{aligned} W \cup_{\tilde{M}} \tilde{N} = & \,\mathrm{u} \circ \mathrm{c}(F) \\ & \cup_{\partial M \times \{0\} \times [0,1] \times S^1 \times |\Delta^k| \times \Delta^n} \\ & \quad \partial M \times J \times [0,1] \times S^1 \times |\Delta^k| \times |\Delta^n| \\ & \cup_{\partial M \times \{1\} \times [0,1r] \times S^1 \times |\Delta^k| \times |\Delta^n|} \\ & \quad \overline{N - \mathrm{im}(c) - M} \times [0,1] \times S^1 \times |\Delta^k| \times |\Delta^n| \end{aligned}$$

and

$$\begin{aligned} V = & \,\mathrm{u} \circ \mathrm{c}(F) \\ & \cup_{\partial M \times \{0\} \times I \times S^1 \times |\Delta^k| \times \Delta^n} \\ & \quad \partial M \times J \times I \times S^1 \times |\Delta^k| \times |\Delta^n| \\ & \cup_{\partial M \times \{1\} \times I \times S^1 \times |\Delta^k| \times |\Delta^n|} \\ & \quad \overline{N - \mathrm{im}(c) - M} \times I \times S^1 \times |\Delta^k| \times |\Delta^n| \end{aligned}$$

respectively.

The simple map ρ is given by Id on $\mathrm{u} \circ \mathrm{c}(F)$,

$$\mathrm{Id}_{\partial M} \times r \times \mathrm{Id}_{S^1 \times |\Delta^k| \times |\Delta^n|}$$

on

$$\begin{array}{c} \partial M \times J \times I \times S^1 \times |\Delta^k| \times |\Delta^n| \\ \big\downarrow \\ \partial M \times (J \times [0,1] \cup \{0\} \times I) \times S^1 \times |\Delta^k| \times |\Delta^n| \end{array}$$

and

$$\mathrm{Id}_{\overline{N - \mathrm{im}(c) - M}} \times \tilde{r} \times \mathrm{Id}_{S^1 \times |\Delta^k| \times |\Delta^n|}$$

on

$$\begin{array}{c} \overline{N - \mathrm{im}(c) - M} \times I \times S^1 \times |\Delta^k| \times |\Delta^n| \\ \big\downarrow \\ \overline{N - \mathrm{im}(c) - M} \times [0,1] \times S^1 \times |\Delta^k| \times |\Delta^n|. \end{array}$$

We turn our attention to the second case. First, we denote our transfer map by $f = (N = N, s, (p_i^f, i_f, \nu_i^f, H_f^{(i)})_{1 \leq i \leq m})$. Let d denote the dimension of the disk bundle $p^f = p_m^f \circ \ldots \circ p_1^f$. We reduce the structure group of the bundle p^f to $\mathrm{Aut}(S^{d-1})$ by choosing a metric.

By the second part of statement 4 of Theorem 2.1.3.21 we obtain a covering $(U_i, \omega_i)_{i \in \mathcal{I}}$ of M by local trivialisations of p^f for some indexing set $\mathcal{I}$ with respect to the reduced structure group.

Further, let (U, ω) and (U', ω') be trivialisations of the covering. Then the geometric transfer without bending the boundary $\mathrm{Tr}(F) = \mathrm{Tr}_{\nu_m} \circ \ldots \circ \mathrm{Tr}_{\nu_1}(F)$ fits into a commutative square

$$\begin{array}{ccc}
N \times I \times |\Delta^k| \times |\Delta^m| & \xrightarrow[\mathrm{Tr}(F)]{\cong} & N \times I \times |\Delta^k| \times |\Delta^m| \\
\uparrow{\scriptstyle \omega \times \mathrm{Id}} & & \uparrow{\scriptstyle \omega' \times \mathrm{Id}} \\
U \times D^d \times I \times |\Delta^k| \times |\Delta^m| & \xhookrightarrow[F \times \rho]{} & U' \times D^d \times I \times |\Delta^k| \times |\Delta^m|
\end{array}$$

where $\rho\colon U \times D^d \times |\Delta^k| \times |\Delta^m| \to D^d$ is adjoint to a family of automorphisms of the fibre $\tilde{\rho}\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(D^d)$ which, in turn, is induced by the reduced structure group $\tilde{\rho}'\colon U \times |\Delta^k| \times |\Delta^m| \to \mathrm{Aut}(S^{d-1})$. We have bundles V and W as above and a new bundle $\tilde{V} = \mathrm{u} \circ \mathrm{c}(\mathrm{Tr}(F))$.

The fibre-wise bending isotopies $H_f^{(i)}$ induce an isotopy

$$\mathrm{Tr}(F) \simeq P(f)(F) = \mathrm{Tr}_{\nu_m^f, h_f^{(m)}} \circ \ldots \circ \mathrm{Tr}_{\nu_1^f, h_f^{(1)}}(F)$$

which is constant on $(M \times I \cup N \times [0,1]) \times |\Delta^k| \times |\Delta^n|$ and induces a PL isomorphism $\tilde{H}\colon V \to \tilde{V}$. It is constant on the subspace since each of the $H_f^{(i)}$ preserves the zero section, see Definition 2.1.3.4.

Via the trivialisations above we can locally write $\tilde{V}$ as

$$\mathrm{u} \circ \mathrm{c}(F) \times D^d$$

and W as

$$\mathrm{u} \circ \mathrm{c}(F) \times \{0\} \cup_{M \times [0,1] \times S^1 \times |\Delta^k| \times |\Delta^n| \times \{0\}} M \times [0,1] \times S^1 \times |\Delta^k| \times |\Delta^n| \times D^d.$$

Let $r\colon D^d \times I \to D^d \times [0,1] \cup_{\{0\} \times [0,1]} \{0\} \times I$ be an $\mathrm{Aut}(S^{d-1})$–equivariant simple map. Here, $\mathrm{Aut}(S^{d-1})$ acts on D^d only. Similarly to the previous case we obtain a simple map $\rho'\colon \tilde{V} \to W$. We define the desired simple map ρ as $\rho = \rho' \circ \tilde{H}$.

In contrast to the first case we do not have to choose an additional collar because the space admits the structure of a disk bundle. The argument of the previous case does not carry over because our map is given by $F \times \rho$, not the Identity, outside of the zero section.

For a general map (f, e) we define the associated simple map by composition of the two constructions. If the collar extension map does not factor over the disk bundle, an isotopy of collar induces the desired retraction map. The properties of an admissible retraction are fulfilled by definition of the retraction maps. □

The last step is to thicken up our category of choices Ch_Δ to a category Ch_Δ^R which contains the required choices for an admissible retraction.

Recall that the choice space $\mathrm{Ch}_\Delta(M,N)$ is the quotient of the "preliminary choice space" $\tilde{\mathrm{Ch}}_\Delta(M,N)$ along certain equivalence relations, see Definition 2.1.3.16 and Definition 2.1.3.9. While referring there, keep Notation 2.1.3.20 in mind.

Definition 2.1.5.11. Let $\tilde{\mathrm{Ch}}_\Delta(M,N)^R$ denote the simplicial set whose n-simplices are tuples (ch,α,A), where $\mathrm{ch} \in \tilde{\mathrm{Ch}}_\Delta(M,N)$ and α is an admissible retraction over ch with respect to the admissible collection A.

Let $\mathrm{Ch}_\Delta(M,N)^R$ be the quotient of $\tilde{\mathrm{Ch}}_\Delta(M,N)^R$ under the equivalence relation generated by the following relations:

Let $A = (M_{i_t}, \iota_{i_t}, \tilde{p}_{i_t}, \mathrm{im}(c_{i_t}))_{0\le t\le k-1}$ be an admissible collection over ch. Then so is $A' = (M'_{i_t}, \iota'_{i_t}, \tilde{p}'_{i_t}, \mathrm{im}(c_{i_t})')_{0\le t\le k-2}$ which is given by *omitting* M_{i_t} for some $1 \le t \le k-1$. Let us explain:

We define A' via the sequence

$$M_{i_0} \xrightarrow{\iota_{i_0}} \dots \xrightarrow{\iota_{i_{t-1}}} M_{i_t} \xrightarrow{\iota_{i_{t+1}}\circ\iota_{i_t}} M_{i_{t+2}} \dots \xrightarrow{\iota_{i_{k-1}}} M_{i_k}$$

of embeddings and use the appropriate tubular neighbourhoods and images of collars for every embedding but $\iota'_{i_t} = \iota_{i_{t+1}} \circ \iota_{i_t}$. We fix a parametrisation of the collar space $\mathrm{im}(c_{i_{t+1}})$. We abuse notation and call it $c_{i_{t+1}}$.

The new tubular neighbourhood is

$$\tilde{p}_{i_t} \circ \iota_{i_t}^*(\tilde{p}_{i_{t+1}})\colon \iota_{i_t}^*(\tau_{M_{i_{t+2}}} M_{i_{t+1}}) \to M_{i_t}$$

and the collar space is given by

$$\tilde{p}_{i_t}^{-1}(\mathrm{im}(c_{i_t})) \cup c_{i_{t+1}}(((\tilde{p}_{i_t}^{-1}(\mathrm{im}(c_{i_t})) \cup \iota_{i_t}^*(\tau_{M_{i_{t+2}}} M_{i_{t+1}})) \cap \partial\tau_{M_{i_{t+1}}} M_{i_t}) \times [0,1]).$$

While the tubular neighbourhood is familiar from the composition of choices in Ch_Δ, the collar space might seem less clear. The idea goes as follows: We start with the collar of M_{i_t} in $M_{i_{t+1}}$. Then we take its preimage under the bundle $\tilde{p_{i_t}}$ to get a collar space in all directions orthogonal to the fibre direction of the bundle. Finally, we extend this to a collar space in all directions via the collar $c_{i_{t+1}}$ of $\tau_{M_{i_{t+1}}} M_{i_t}$ in $M_{i_{t+1}}$.

We note that the construction of A' depends on t and the parametrisation $c_{i_{t+1}}$.

Let α be an admissible retraction over ch with respect to A which is also an admissible retraction over ch with respect to A'. Further, let α be given by the standard projection map on $(\mathrm{im}(c_{i_t+1}) \cup \tau_{M_{i_{t+2}}} M_{i_{t+1}} - T'') \times I$ instead of just fibre preserving. The tuples (ch,α,A) and (ch,α,A') are equivalent.

Let ch' be an n–simplex in the preliminary choice space with $[\mathrm{ch}] = [\mathrm{ch}']$ in $\mathrm{Ch}_\Delta(M,N)$. Let α be an admissible retraction over ch with respect to A, which is also an admissible retraction over ch' with respect to A. The tuples (ch,α,A) and (ch',α,A) are equivalent.

Definition 2.1.5.12. The composition

$$\mathrm{Ch}_\Delta(N,N')^R \times \mathrm{Ch}_\Delta(M,N)^R \to \mathrm{Ch}_\Delta(M,N')^R$$

is induced by sending $((\mathrm{ch}_2,\alpha_2,A_2),(\mathrm{ch}_1,\alpha_1,A_1))$ to $(\mathrm{ch}_2\circ\mathrm{ch}_1, \alpha_1\circ\alpha_2, A_1 * A_2)$ where $A_1 * A_2$ is the concatenation of A_1 and A_2.

The appropriate choice category established, we show that the additional choices are contractible, i.e. the forgetful map from Ch_Δ^R to Ch_Δ is a categorical equivalence.

Proposition 2.1.5.13. *The forgetful map* $\mathrm{Ch}_\Delta(M,N)^R \to \mathrm{Ch}_\Delta(M,N)$ *is a Kan fibration.*

Proof. We are going to consider the special case $i_t = t$ for $0 \leq t \leq m = k$ to ease up notation. The general case is analogous.

We have to solve lifting problems

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathrm{Ch}_\Delta(M,N)^R \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Ch}_\Delta(M,N). \end{array}$$

We are going to construct a lift of a representative ch of the n–simplex [ch] in $\mathrm{Ch}_\Delta(M,N)$, relative to a lift (ch,α,A) over Λ^n_k which represents $[\mathrm{ch},\alpha,A]$. The construction is going to be compatible with the equivalence relations.

Since $|\Lambda^n_k| \subseteq |\Delta^n|$ is up to isomorphism $|\Delta^{n-1} \times \{0\}| \subseteq |\Delta^{n-1} \times \Delta^1|$ we may consider a sequence of bundles $p'_i \times \mathrm{Id}\colon E'_i \times |\Delta^1| \to E'_{i-1} \times |\Delta^1|$ with zero section $s' \times \mathrm{Id}$.

By the isotopy extension theorem (in the piecewise linear version) we may assume without loss of generality that the subspace inclusion $E_m \subseteq N$ is of the form $\iota \times \mathrm{Id}\colon E'_m \times |\Delta^1| \subseteq N' \times |\Delta^1|$ with ι independent of the $|\Delta^1|$–coordinate.

Accordingly, we may extend the embeddings ι_i for $0 \leq i \leq m-1$ to $\iota_i \times \mathrm{Id}$ and similarly for $\tilde{p}_i$ and c_i. We call this collection of admissible data $A \times \mathrm{Id}$.

To construct the lift we have to find an admissible retraction α' over ch with respect to $A \times \mathrm{Id}$ which restricts to α over $|\Delta^{n-1}| \times \{0\}$.

By statement 5 of Theorem 2.1.3.21 the families of transfer maps $(\nu_i)_{1\leq i\leq m}$ and fibre-wise bending isotopies $(H^{(i)})_{1\leq i\leq m}$ parametrised over $|\Delta^{n-1}| \times |\Delta^1|$ induce, for every pseudoisotopy F in $P(M)$, an isomorphism of pairs

$$\begin{aligned} \mathrm{u}\circ\mathrm{c}\circ P(\mathrm{ch})(F) &\cong \mathrm{u}\circ\mathrm{c}\circ P(\mathrm{ch}')(F)) \times |\Delta^1| \\ \mathrm{u}\circ\mathrm{c}\circ P(\mathrm{ch}')(F)) &\cong \mathrm{u}\circ\mathrm{c}\circ P(\mathrm{ch}')(F) \times \{0\} \end{aligned}$$

where ch' denotes the restriction of ch to $|\Delta^{n-1}| \times \{0\}$.

It is not hard to see that this isomorphism of pairs may be chosen compatibly with the simplicial structure maps of $P(M)$.

We extend the retraction to $\mathrm{u}\circ\mathrm{c}\circ P(\mathrm{ch}')(F)) \times |\Delta^1| \times I$ by $\alpha \times \mathrm{Id}_{|\Delta^1|}$. Upon conjugation with the isomorphism of pairs, we obtain the desired retraction. □

Proposition 2.1.5.14. *The forgetful map* $F\colon \mathrm{Ch}_\Delta(M,N)^R \to \mathrm{Ch}_\Delta(M,N)$ *is a trivial fibration.*

Proof. By Remark 2.1.5.8 the space of admissible retractions over some ch in $\mathrm{Ch}_\Delta(M,N)$ is not empty.

Let $R(\mathrm{ch})$ denote the fibre over ch. Let $R(\mathrm{ch})_0$ denote the simplicial subset containing those equivalence classes of tuples which contain a representative (ch,α,A) such that the subsequence $0 = i_0, i_1, \ldots, i_k = m$ of $0,1,\ldots,m$ associated to A is of the form $i_0 = 0, i_1 = m$.

By Proposition 2.1.5.13, it is enough to show that

1. the inclusion $R(\mathrm{ch})_0 \subseteq R(\mathrm{ch})$ is an acyclic cofibration and
2. the space $R(\mathrm{ch})_0$ is contractible.

We show these results in the subsequent lemmata. □

Lemma 2.1.5.15. *The inclusion $R(\mathrm{ch})_0 \subseteq R(\mathrm{ch})$ is a weak equivalence.*

Proof. Since $R(\mathrm{ch})$ is a Kan complex, it is enough to show that the relative simplicial homotopy groups $\pi_n(R(\mathrm{ch}), R(\mathrm{ch})_0)$ vanish for $n \geq 0$.

Let (ch, α, A) be a representative of an n–simplex of $R(\mathrm{ch})$, which restricts to an element of $R(\mathrm{ch}')_0$ on the boundary where ch' is the restriction of ch. We may assume without loss of generality that (ch, α, A) restricts to an element of $R(\mathrm{ch}'')_0$ over a neighbourhood of the boundary where ch'' denotes the restriction of ch to said neighbourhood.

Let $0 = i_0, i_1, \dots, i_k = m$ denote the sequence associated to A. We are going to construct a sequence of homotopies.

Let $A_{\leq t}$ be given by omitting $M_{i_{t+1}}$ from $A_{\leq t+1}$ for every $0 \leq t \leq k-2$ and $A_{\leq k-1} = A$. We are going to define homotopies $H^{(t)}$ of simple retraction maps such that

1. We have $H_0^{(k-1)} = \alpha$ and $H_0^{(t)} = H_1^{(t+1)}$ for $0 \leq t \leq k-2$.
2. The map $H_s^{(t)}$ is an admissible retraction over ch with respect to $A_{\leq t}$ for every $0 \leq s \leq 1$.
3. The map $H_1^{(t)}$ is admissible over ch with respect to $A_{\leq t-1}$ for $0 \leq t \leq k-1$.
4. The homotopies are compatible with simplicial structure maps.
5. If $H_0^{(t)}$ is already admissible with respect to $A_{\leq t-1}$, then the homotopy is constant.

Let us define the homotopies $H^{(t)}$. Recall that an admissible retraction has to satisfy the following conditions:

1. The retraction α is given by the standard simple retraction map (i.e. the projection) on $(M_{i_{t+1}} - \mathrm{im}(c_{i_t}) - \tau_{M_{i_{t+1}}} M_{i_t}) \times I$ for every $0 \leq t \leq k-1$.
2. Upon the choice of a parametrisation of $\mathrm{im}(c_{i_t})$ as a collar we obtain a tubular neighbourhood $p'\colon \mathrm{im}(c_{i_t}) \cup \tau_{M_{i_{t+1}}} M_{i_t} \to M_{i_t}$ of M_{i_t} in $M_{i_{t+1}}$ given by $p' = \tilde{p}_{i_t} \circ \mathrm{pr}_{\partial\tau_{M_{i_{t+1}}} M_{i_t}} \circ c_{i_t}^{-1} \cup \tilde{p}_{i_t}$.

   We set $U = c_{i_{t-1}}(\partial\tau_{M_{i_t}} M_{i_{t-1}} \times [0,1)) \cup \tau_{M_{i_t}} M_{i_{t-1}}$ for a choice of $c_{i_{t-1}}$. Let p'' denote the pullback of p' along the subspace inclusion $U \subseteq M_{i_t}$. Let T'' denote the total space of this pulled back tubular neighbourhood.

   The map $\mathrm{pr} \circ (p' \times \mathrm{Id}_I)\colon (\mathrm{im}(c_{i_t}) \cup \tau_{M_{i_{t+1}}} M_{i_t} - T'') \times I \to (M_{i_t} - U) \times \{0\}$ and the restriction of $(p' \times \mathrm{Id}_{[0,1]}) \circ \alpha$ to the same subspace coincide, i.e. the restriction of the retraction map is fibre-preserving with respect to p'.

Since $A_{\leq t}$ mostly coincides with $A_{\leq t+1}$, an admissible retraction β with respect to $A_{\leq t+1}$ is almost admissible with respect to $A_{\leq t}$ as well. It can only fail on the restriction of $p' \colon \tau_{M_m}(M_{i_{t+1}}) \cup \operatorname{im}(c_{i_{t+1}}) \to M_{i_{t+1}}$ to

$$M_{i_{t+1}} - c_{i_t}(\partial\tau_{M_{i_{t+1}}}(M_{i_t}) \times [0,1)) - \tau_{M_{i_{t+1}}}(M_{i_t}).$$

To ease up notation, we assume that $i_t = t$ for every $0 \leq t \leq m = k$ and consider the case $A_{\leq t+1} = A$. The general case is analogous.

The admissible retraction $H_1^{(k-1)}$ has to be given by the standard simple retraction map (i.e. the projection) on

$$M_m \times I - (p'_m)^{-1}(\tau_{M_{m-1}}(M_{m-2} \cup c_{m-2}(\partial\tau_{M_{m-1}}(M_{m-2} \times [0,1))) \times I$$

with respect to some choice of collars c_{m-2} and c_{m-1} parametrising $\operatorname{im}(c_{m-2})$ and $\operatorname{im}(c_{m-1})$, respectively.

We assume that α is given by the standard projection map on a neighbourhood of $\partial(M_m - \operatorname{im}(c_{m-1}) - \tau_{M_m}M_{m-1}) \times I$ in $M_m \times I$. We can achieve this, for example, by slightly extending $\operatorname{im}(c_{m-1})$.

We are going to describe the desired homotopy H after decomposing M_m into various subspaces. First, let us choose a parametrisation of $\operatorname{im}(c_{m-1})$ and $\operatorname{im}(c_{m-2})$ each. We again abuse notation and call them c_{m-1} and c_{m-2}.

Recall that (ch, α, A) restricts to an element of $R(\mathrm{ch}'')_0$ over a neighbourhood of the boundary. Hence there is a choice of c_{m-1}, such that the following holds: Over the boundary of the n–simplex, α is given by the standard projection on the complement of $(p'_m)^{-1}(\tau_{M_{m-1}}M_{m-2} \cup \operatorname{im}(c_{m-2})) \times I$. This is going to guarantee that the homotopy H is constant over the boundary.

Let V be the complement of $\tau_{M_{m-1}}M_{m-2} \cup c_{m-1}(\partial\tau_{M_{m-1}}M_{m-2} \times [0,1])$ in M_m. Let $\iota_V \colon V \subseteq M_{m-1}$ and $\iota_c \colon \operatorname{im}(c_{m-2}) \subseteq M_{m-1}$ denote the subspace inclusions. We obtain

$$\begin{aligned} M_m = & M_m - \operatorname{im}(c_{m-1}) - \tau_{M_m}M_{m-1} \\ & \cup \iota_{m-2}^*(\tau_{M_m}M_{m-1}) \\ & \cup c_{m-1}(\partial\iota_{m-2}^*(\tau_{M_m}M_{m-1}) \times [0,1]) \\ & \cup \iota_c^*(\tau_{M_m}M_{m-1}) \\ & \cup c_{m-1}(\partial\iota_c^*(\tau_{M_m}M_{m-1}) \times [0,1]) \\ & \cup \iota_V^*(\tau_{M_m}M_{m-1}) \\ & \cup c_{m-1}(\partial\iota_V^*(\tau_{M_m}M_{m-1}) \times [0,1]). \end{aligned}$$

On $(M_m - \operatorname{im}(c_{m-1}) - \tau_{M_m}M_{m-1}) \times I$ the homotopy is constantly the standard projection map. This takes care of the first condition of an admissible retraction for the index $i = m-1$ with respect to A.

On $\iota_{m-2}^*(\tau_{M_m}M_{m-1}) \times I$ the homotopy is constant, too. This guarantees both conditions for $0 \leq i \leq m-3$ with respect to A.

On $c_{m-1}(\partial\iota_{m-2}^*(\tau_{M_m}M_{m-1}) \times [0,1]) \times I$ the homotopy is constant once more. Now we turn to the interesting part.

The remaining subspace carries the structure of a tubular neighbourhood T of $\iota_c^*(\tau_{M_m}M_{m-1}) \cup \iota_V^*(\tau_{M_m}M_{m-1})$. We are going to apply a fibre-wise Alexander trick to take care of the remaining conditions.

We note that there is a piecewise linear isotopy of piecewise linear automorphisms $\rho_s\colon M_m \times [0,1) \to M_m$ with the following properties:

1. We have $\rho_0 = \mathrm{Id}$.
2. On the subspaces $M_m - \mathrm{im}(c_{m-1}) - \tau_{M_m} M_{m-1}$, $\iota^*_{m-2}(\tau_{M_m} M_{m-1})$ as well as $c_{m-1}(\partial\iota^*_{m-2}(\tau_{M_m} M_{m-1}) \times [0,1])$ the isotopy ρ_s is constant. In particular, this is important to make sure that H is constant on the boundary.
3. The complement of these three carries the structure of a tubular neighbourhood T of $\iota^*_c(\tau_{M_m} M_{m-1}) \cup \iota^*_V(\tau_{M_m} M_{m-1})$. The isotopy ρ preserves the fibres of this bundle.
4. The bundle given by the fibre-wise boundary of T has, by assumption, a neighbourhood W on which α is given by the standard projection. The isotopy ρ_s satisfies $\mathrm{diam}(\rho_s(T - W)) \leq 1 - s$ with respect to some fixed metric.

On $\iota^*_V(\tau_{M_m} M_{m-1}) \times I \cup c_{m-1}(\partial\iota^*_V(\tau_{M_m} M_{m-1}) \times [0,1]) \times I$ the homotopy is given by $(\alpha, s) \mapsto \rho_s \circ \alpha \circ (\rho_s^{-1} \times \mathrm{Id}_I)$ for $s \in [0,1)$ and the standard projection for $s = 1$.

On $\iota^*_c(\tau_{M_m} M_{m-1}) \times I \cup c_{m-1}(\partial\iota^*_c(\tau_{M_m} M_{m-1}) \times [0,1]) \times I$ we use local coordinates. Let (x, r, v, w) be an element of $\partial\tau_{M_{m-1}} M_{m-2} \times [0,1] \times D^z \times I$ where z is the dimension of the fibre of the tubular neighbourhood.

The homotopy is $(\alpha, s)(x, r, v, w) = \rho_{\max(r,s)} \circ \alpha \circ (\rho^{-1}_{\max(r,s)} \times \mathrm{Id}_I)(x, r, v, w)$ in local coordinates for $(r, s) \in [0,1] \times [0,1] - \{(1,1)\}$. For $(1,1)$ we use the standard projection map.

This construction preserves fibres and thus the second condition is satisfied for $i = m-2$ with respect to A. If α happens to be the projection map on a fibre already, then the homotopy is constant. This ensures the first condition for $i = m-1$.

By definition, the collar c_{m-2} for $A_{\leq m-2}$ contains the space where H_1 is non-trivial, namely $\iota^*_c(\tau_{M_m} M_{m-1}) \cup c_{m-1}(\partial\iota^*_c(\tau_{M_m} M_{m-1}) \times [0,1])$. It is then straightforward that H_1 satisfies the first condition for $i = m-2$ with respect to $A_{\leq m-2}$. □

Lemma 2.1.5.16. *Let $(E \subseteq N, p, s, \nu, H, \alpha)$ be a 0–simplex in $\mathrm{Ch}_\Delta(M, N)$. The space $R(E \subseteq N, p, s, \nu, H)_0$ is contractible.*

Proof. This is similar to the proof that $R(\mathrm{ch})_0 \subseteq R(\mathrm{ch})$ is a weak equivalence. On the tubular neighbourhood $E \cup \mathrm{im}(c)$, we fix a fibre-wise simple retraction map $E \cup \mathrm{im}(c) \times I \to E \cup \mathrm{im}(c) \times [0,1] \cup M \times I$ which is the standard projection outside of $E \times I$.

For an admissible retraction $(\alpha, (\iota\colon M \hookrightarrow N, p\colon E \to M, c\colon \partial E \times [0,1] \to N))$ we use the isotopy ρ to pass from α to the fixed retraction. □

We obtain the desired 2–functor.

Corollary 2.1.5.17. *There is a simplicial 2–functor $\Psi\colon \mathrm{Ch}^R_\Delta \times [1] \to \mathrm{scat}_{\Delta^{\mathrm{cat}}}$ given by:*

- $P^{\mathrm{PL}}\colon \mathrm{Ch}^R_\Delta \times \{0\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$
- $\Omega s \tilde{\mathcal{E}}^h_\bullet \circ (- \times [0,1])\colon \mathrm{Ch}^R_\Delta \times \{1\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$

- $\mathrm{u} \circ \mathrm{c}\colon \operatorname{ob} \mathrm{Ch}^R_\Delta \times \{0 \leq 1\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$
- $\Psi\colon \operatorname{mor} \mathrm{Ch}^R_\Delta \times \{0 \leq 1\} \to \mathrm{scat}_{\Delta^{\mathrm{cat}}}$ *which sends* $[(\mathrm{ch}, \alpha, A)]$ *to* α.

It induces an $(\infty, 1)$*–functor* $\mathcal{N}^{h.c.}_\bullet \mathrm{Ch}^R_\Delta \times [1] \to \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta$ *upon geometric realisation since* Top_Δ *is an* $(\infty, 1)$*–category.*

The 2–functor makes the desired natural zig-zag between pseudoisotopies and polyhedra a fairly straightforward consequence.

Corollary 2.1.5.18. *There is a zig-zag of natural weak equivalences between* $\mathbb{P}^{\mathrm{PL}}_\partial\colon \mathrm{Top} \to \mathrm{Top}$ *and* $\Omega s\tilde{\mathcal{E}}^h_\bullet\colon \mathrm{Top} \to \mathrm{Top}$.

Proof. We consider the unstable pseudoisotopies first. As explained in the beginning of this section both functors are defined by homotopy left Kan extensions of their restrictions to $(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{pl})_\Delta$.

A zig-zag between $\Omega|\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet \,| \circ \mathfrak{f}\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$ and $P^{\mathrm{PL}}\colon \mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$ is enough by Lemma 2.1.5.1.

By the tautological obstruction theory, see Corollary 1.2.1.7, the above corollary yields a zig-zag of natural transformations between $\Omega|\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet \,| \circ \mathfrak{f}$ and P^{PL}.

By Lemma 2.1.2.10 and statement 7 of Theorem 2.1.3.21 we obtain a zig-zag of natural weak equivalences after stabilisation. □

2.2 Non-Connective Naturality

In this part, we construct a zig-zag of natural weak equivalences between the pseudoisotopy spectrum and the Whitehead spectrum.

Definition 2.2.0.1. The topological Whitehead spectrum is the homotopy cofiber of the assembly map in A–theory

$$(-)_+ \wedge A(*) \to A \to \mathrm{Wh}^{\mathrm{Top},-\infty}$$

where $A\colon \mathrm{Top} \to \mathrm{Spectra}$ is the non-connective A–theory functor. The piecewise linear Whitehead spectrum is the same as the topological one.

The Whitehead spectrum in the smooth category is defined as the homotopy cofiber of

$$(-)_+ \wedge \mathbb{S} \to A \to \mathrm{Wh}^{\mathrm{Diff},-\infty}$$

where $\mathbb{S}$ is the sphere spectrum and the map is the composition of the unit map $(-)_+ \wedge \mathbb{S} \to (-)_+ \wedge A(*)$ and assembly.

Let us restate our main objective.

Theorem 2.2.0.2. *Let* $\mathrm{Cat} = \mathrm{Top}, \mathrm{PL}$ *or* Diff. *There is a natural weak equivalence of* $(\infty, 1)$*–functors*

$$\Psi\colon \mathcal{P}^{\mathrm{Cat}} \Rightarrow \Omega^2 \, \mathrm{Wh}^{\mathrm{Cat},-\infty}$$

from the $(\infty, 1)$*–functor* $\mathcal{P}^{\mathrm{Cat}}\colon \mathcal{N}_\bullet^{h.c.} \mathrm{Top}_\Delta \to \mathcal{N}_\bullet^{h.c.} \mathrm{Spectra}_\Delta$ *of pseudoisotopies to the twofold loops of the* $(\infty, 1)$*–functor given by the Whitehead spectrum.*

In particular, there is a zig-zag of natural weak equivalences between the strict functors $\mathcal{P}^{\mathrm{Cat}}\colon \mathrm{Top} \to \mathrm{Spectra}$ *and* $\Omega^2 \, \mathrm{Wh}^{\mathrm{Cat},-\infty}$.

The ideas are similar to the case of spaces. Unfortunately, there is no known polyhedra model for the one fold loops of the Whitehead spectrum similar to $\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet$ for the loops of the Whitehead space.

Since the standard models for the non-connective pseudoisotopy spectrum as well as the non-connective A–theory spectrum utilize controlled categories, it seems feasible that one could translate the zig-zags by Waldhausen and Jahren, Rognes and Waldhausen recalled in Theorem 2.1.1.2 and Theorem 2.1.1.4 into a controlled setting.

Indeed, all spaces and maps admit analogues (with quite a few variations). However, the author is unable to show that any such zig-zag consists of weak equivalences. The main problem is as follows: To show that polyhedral realisation $\mathrm{s}\tilde{\mathcal{D}}^{\mathrm{h}}_\bullet(X_\bullet) \to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(|X_\bullet|)$ is a weak equivalence Jahren, Rognes and Waldhausen used triangulations of fibrations of polyhedra, i.e. for $W \to |\Delta^n|$ a PL Serre fibration we need a triangulation $W_\bullet \to B_\bullet$ where $B_\bullet$ is a *finite* triangulation of $|\Delta^n|$.

In a controlled setting the polyhedron W is only locally compact over a typically non-compact control space, e.g. $\mathbb{R}^n$. Indeed, there are polyhedra W such that there is no triangulation of W which admits a compatible finite triangulation of $|\Delta^n|$. If one tries to relax the conditions on the triangulation of $|\Delta^n|$, the issues cascade into other parts of the zig-zag.

Instead we define an analogue of the 2–functor $\Psi\colon P \Rightarrow \Omega\,\mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}(- \times [0,1])$ in the controlled setting and then construct a lift

$$\begin{array}{ccc} & & \Omega\,\mathrm{Wh}^{?}(- \times [0,1,B) \\ & \overset{\Psi^D}{\nearrow} & \downarrow \\ P(-;B) & \xrightarrow{\Psi} & \Omega|\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_{\bullet}(- \times [0,1],B)| \end{array}$$

for a control space B and a space which very much looks as if it is a model for "one fold loops of a controlled Whitehead space" (and close enough that we obtain a natural transformation $\Omega\,\mathrm{Wh}^{?}(-,B) \Rightarrow \Omega\,\mathrm{Wh}(-,B)$). This map bypasses the zig-zags almost completely. Finding a lift then amounts to finding "triangulations" of $W \to |\Delta^n|$ of the form $W_\bullet \to \mathcal{S}_\bullet(|\Delta^n|)$.

In Section 2.2.1 we show that bounded pseudoisotopies classify bundles of bounded h–cobordisms, similar to Section 2.1.2, and give a detailed description of the non-connective pseudoisotopy functor, akin to Section 2.1.3. We also proof that bounded h–cobordisms yield an Ω–spectrum and identify its non-positive stable homotopy groups.

We construct the suspected model for the one fold loops of the controlled Whitehead space in Section 2.2.2 and show that different deloopings via controlled categories give equivalent spectra. In particular, our results apply in the context of the non-connective A–theory functor of Ullmann and Winges [44].

Finally we translate the 2–functor Ψ to the non-connective setting, construct the desired lift via triangulations, and show that we indeed obtain a zig-zag of weak equivalences. This is the content of Section 2.2.3.

We take care of the smooth case in Section 2.2.4. The argument is an adaption of the deduction of the smooth case of the stable parametrised h–cobordism theorem in [31].

2.2.1 Non-connective pseudoisotopies and h–cobordisms

We know that the negative homotopy groups of $\mathcal{P}$ and $\Omega^2\,\mathrm{Wh}$ are abstractly isomorphic to negative K–groups. This is a consequence of [47] for $\Omega^2\,\mathrm{Wh}$, while Anderson and Hsiang [1] showed this for $\mathcal{P}$ via comparison to bounded h–cobordisms.

Unfortunately the construction in [1] is not easily seen to be compatible with our eventual map $\mathcal{H}(M) \to \Omega\,\mathrm{Wh}(M)$ from the h–cobordism to the Whitehead spectrum. Instead we rely on the bounded h–cobordism theorem by Pedersen [37] to identify bounded h–cobordisms with negative K–groups.

We define the non-connective spectrum of bounded h–cobordisms, then we discuss bounded pseudoisotopies, starting with Definition 2.2.1.11. To pass from pseudoisotopies to h–cobordisms, we again use a classifying map for bundles, see Definition 2.2.1.19 and Definition 2.2.1.20. We use [37] to identify the negative homotopy groups of the h–cobordisms, see Corollary 2.2.1.30. However, since the h–cobordism theorem only applies to the connected components of the spaces of bounded h–cobordisms, we also have to show that our spectrum is an Ω–spectrum in Proposition 2.2.1.24.

Bounded h–cobordisms

The following definitions are adapted from [37] to our situation.

Definition 2.2.1.1. A *manifold W parametrised by $\mathbb{R}^k$* consists of a manifold W together with a proper and surjective map $q\colon W \to \mathbb{R}^k$.

Definition 2.2.1.2. Let $K \subseteq W$ be a subset of a manifold (W, p) parametrised over $\mathbb{R}^k$. The *size* of K is $S(K) = \inf\{r|\exists y \in \mathbb{R}^k\colon q(K) \subseteq B(y, r/2)\}$ where $B(y, r/2)$ is the closed ball in $\mathbb{R}^k$ with radius $r/2$.

Definition 2.2.1.3. An h–cobordism $(W, \partial_0 W, \partial_1 W)$ parametrised by $\mathbb{R}^k$ is a *bounded h–cobordism* (bounded by t) if it is an ordinary h–cobordism and there are deformations $D_i\colon W \times I \to W$ from W to $\partial_i W$, such that $S(D_i(\{w\} \times I)) < t$ for each $w \in W$.

Now we are in position to describe the spaces of bounded h–cobordisms.

Definition 2.2.1.4. Let M be a compact PL manifold, possibly with boundary. We define the *simplicial set $H_\bullet(M; \mathbb{R}^k)$ of bounded h–cobordisms over M and parametrised by $\mathbb{R}^k$*. An n–simplex is a map $q\colon W \to \mathbb{R}^k$ and a diagram

$$\begin{array}{ccc} M \times |\Delta^n| \times \mathbb{R}^k & \xhookrightarrow{\iota} & W \\ & \overset{\mathrm{pr}}{\searrow} & \downarrow p \\ & & |\Delta^n| \end{array}$$

where $p\colon W \to |\Delta^n|$ is a PL bundle, each local trivialisation restricts to the identity on the product subbundle specified by ι, and each fibre $W_x := p^{-1}(x)$ is a piecewise linear h–cobordism on $M \times \mathbb{R}^k \cong M \times \{x\} \times \mathbb{R}^k$ for every $x \in |\Delta^n|$.

Moreover, $(W, M \times |\Delta^n| \times \mathbb{R}^k, \partial_1 W)$ and q form a bounded h–cobordism and there are deformations $D_i\colon W \times I \to W$ with $S(D_i(\{w\} \times I)) < t$ for each $w \in W$ over $|\Delta^n|$, i.e. $p \circ D_i = p \circ \mathrm{pr}_W$ holds.

The simplicial structure maps are induced by pullback along the structure maps of the cosimplicial space $[n] \mapsto |\Delta^n|$.

Definition 2.2.1.5. Let $H^c_\bullet(M; \mathbb{R}^k)$ be the simplicial set with an n–simplex consisting of an element of $H_n(M; \mathbb{R}^k)$ and a collar $c\colon M \times |\Delta^n| \times \mathbb{R}^k \times [0, 1] \to W$ which restricts to a fibre-wise collar $c\colon M \times \{x\} \times \mathbb{R}^k \times [0, 1] \to W_x$ for each $x \in |\Delta^n|$. Moreover, we require c to be a bounded map, i.e. there is some $R > 0$ such that $||q(y) - q(c(y))|| < R$ holds for all $y \in M \times |\Delta^n| \times \mathbb{R}^k \times [0, 1]$. We call c a *bounded collar.*

An n–simplex of $H^{c,r}_\bullet(M; \mathbb{R}^k)$ is an element of $H^c_n(M; \mathbb{R}^k)$ together with a fibre-wise retraction $r\colon W \to M \times |\Delta^n| \times [0, 1]$ onto the collar such that r is a bounded map.

Lemma 2.2.1.6. *The forgetful maps $H^{c,r}_\bullet(M; \mathbb{R}^k) \to H^c_\bullet(M; \mathbb{R}^k) \to H_\bullet(M; \mathbb{R}^k)$ are acyclic fibrations.*

Proof. For the map $H^c_\bullet(M; \mathbb{R}^k) \to H_\bullet(M; \mathbb{R}^k)$ we use that the space of bounded collars is contractible. For the map $H^{c,r}_\bullet(M; \mathbb{R}^k) \to H^c_\bullet(M; \mathbb{R}^k)$ this follows from the fact that the inclusion $p^{-1}(|\partial\Delta^n|) \cup M \times [0, 1] \times |\Delta^n| \times \mathbb{R}^k \subseteq W$ has the bounded homotopy extension property for every $n \in \mathbb{N}$. $\square$

Definition 2.2.1.7. There is a *stabilisation map*, see [31, Definition 1.1.3], $\sigma\colon H_\bullet(M;\mathbb{R}^k) \to H_\bullet(M \times [0,1];\mathbb{R}^k)$ given by

$$(W,p,\iota) \mapsto (W \times [0,1], p \circ \mathrm{pr}_W, \iota \times \mathrm{Id}_{[0,1]})$$

where we mildly abuse notation, since the subbundle is $M \times [0,1] \times |\Delta^n| \times \mathbb{R}^k$, not $M \times |\Delta^n| \times \mathbb{R}^k \times [0,1]$.

In the decorated cases we send a collar $c\colon M \times |\Delta^n| \times \mathbb{R}^k \times [0,1] \to W$ to $c \times \mathrm{Id}_{[0,1]}$ and a retraction to $r \times \mathrm{Id}_{[0,1]}$, again abusing notation.

Definition 2.2.1.8. The *stable h–cobordism space* is

$$\mathbb{H}_\bullet(M;\mathbb{R}^k) := \mathrm{colim}_{n\in\mathbb{N}}\, H_\bullet(M \times [0,1]^n;\mathbb{R}^k)$$

similar for $\mathbb{H}^c_\bullet(M;\mathbb{R}^k)$ and $\mathbb{H}^{c,r}_\bullet(M;\mathbb{R}^k)$. We denote the geometric realisation by $\mathbb{H}(M;\mathbb{R}^k)$.

To finish the definition of the h–cobordism spectrum we require structure maps.

Definition 2.2.1.9. Let $\iota_+\colon [0,1] \hookrightarrow \mathbb{R}_{\geq 0}$ and $j_+\colon \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}$ denote the subspace inclusions, similar for ι_- and j_- with respect to $\mathbb{R}_{\leq 0}$. Let (W,q,p,ι,c) be an n–simplex of $H^c_\bullet(M;\mathbb{R}^k)$. We obtain a pushout

$$\begin{array}{ccc}
M \times |\Delta^n| \times \mathbb{R}^k \times [0,1] \times [0,1] & \xrightarrow{\mathrm{Id} \times \iota_+} & M \times |\Delta^n| \times \mathbb{R}^k \times [0,1] \times \mathbb{R}_{\geq 0} \\
\downarrow{\scriptstyle c \times \mathrm{Id}} & & \downarrow \\
W \times [0,1] & \longrightarrow & (\iota_+)_*(W)
\end{array}$$

and similar for ι_-, $j_\pm$. In total we obtain a square

$$\begin{array}{ccc}
H^c_\bullet(M;\mathbb{R}^k) & \xrightarrow{(\iota_+)_*} & H^c_\bullet(M;\mathbb{R}^k \times \mathbb{R}_{\geq 0}) \\
\downarrow{\scriptstyle (\iota_-)_*} & & \downarrow{\scriptstyle (j_-)_*} \\
H^c_\bullet(M;\mathbb{R}^k \times \mathbb{R}_{\leq 0}) & \xrightarrow{(j_+)_*} & H^c_\bullet(M;\mathbb{R}^{k+1})
\end{array}$$

where the upper right and lower left corner are contractible by an Eilenberg Swindle. Since these constructions are compatible with stabilisation, we get a map $\sigma_H\colon \mathbb{H}^c_\bullet(M;\mathbb{R}^k) \to \Omega\mathbb{H}^c_\bullet(M;\mathbb{R}^{k+1})$. The universal property of the pushout gives an extension for $\mathbb{H}^{c,r}_\bullet(M;\mathbb{R}^k)$.

Definition 2.2.1.10. The *h–cobordism spectrum* has $\mathbb{H}^c_\bullet(M;\mathbb{R}^k)$ as its k–th level and structure maps $\sigma_H\colon \mathbb{H}^c_\bullet(M;\mathbb{R}^k) \to \Omega\mathbb{H}^c_\bullet(M;\mathbb{R}^{k+1})$.

Bounded pseudoisotopies

We introduce the non-connective pseudoisotopy spectrum.

Definition 2.2.1.11. Let M be a PL manifold. A *bounded* PL *pseudoisotopy* on M over $\mathbb{R}^k$ is a PL–isomorphism $F\colon M \times \mathbb{R}^k \times I \to M \times \mathbb{R}^k \times I$ such that $F|_{M\times\{0\}\times\mathbb{R}^k} = \mathrm{Id}$ and there is some $R > 0$ such that $\|\,\mathrm{pr}_{\mathbb{R}^k} \circ F(x) - \mathrm{pr}_{\mathbb{R}^k}(x)\| < R$

for every $x \in M \times \mathbb{R}^k \times I$. A *bounded piecewise linear pseudoisotopy F relative boundary* additionally satisfies $F|_{\partial M \times I \times \mathbb{R}^k} = \mathrm{Id}$.

An n–simplex of the *bounded pseudoisotopy space* $P(M;\mathbb{R}^k)$ is a bounded pseudoisotopy on $M \times |\Delta^n|$ over $\mathbb{R}^k$ which is compatible with the projections to $|\Delta^n|$. Similarly we define $P_\partial(M;\mathbb{R}^k)$ for pseudoisotopies relative boundary.

The *stabilisation map* $P(M;\mathbb{R}^k) \to P(M\times[0,1],\mathbb{R}^k)$, $F \mapsto F\times\mathrm{Id}$ allows us to define the *stable pseudoisotopy space* $\mathbb{P}(M;\mathbb{R}^k) := \mathrm{colim}_{n\in\mathbb{N}}\, P(M \times [0,1]^n;\mathbb{R}^k)$.

We abuse notation and do not distinguish between $P(M;\mathbb{R}^k)$ and its geometric realisation.

Definition 2.2.1.12 ([12, Definition 7.6 and Remark 7.7])**.** Let M, N be families of manifolds over X and let $B = \mathbb{R}^k$ or $B = \mathbb{R}^{k-1} \times \mathbb{R}_{\geq 0}$. We define a map

$$- \times \mathrm{Id}_K \colon \mathrm{Ch}_\Delta(M,N) \to \mathrm{Ch}_\Delta(M \times B, N \times B)$$

by taking the cross product component-wise, i.e. we cross each disk bundle, each fibre-wise bending isotopy, and the zero section of some choice simplex ch with Id_K and we cross each parallel transport with the trivial parallel transport on B.

We obtain

$$r(\mathrm{ch}\times\mathrm{Id}_K)\colon \mathbb{P}(M \times B) \to \mathbb{P}(N \times B)$$

which restricts to a map on bounded pseudoisotopies

$$r(\mathrm{ch}\times\mathrm{Id}_K)\colon \mathbb{P}(M;B) \to \mathbb{P}(N;B).$$

Definition 2.2.1.13. We define $P^+(M;B)$ analogously to $P^+(M)$, but we require a bounded collar. We define $\mathrm{Ch}_\Delta^{c,s,B}$ analogously to $\mathrm{Ch}_\Delta^{c,s,+}$.

Definition 2.2.1.14. Let $F_{\mathrm{ch}}^{c,s,B}\colon \mathcal{N}_\bullet(\mathrm{Mfd}^{\mathrm{PL}},\mathrm{pl}) \to \mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta^{c,s,B}$ denote the associated choice functor.

Notation 2.2.1.15. *We write $P^{\mathrm{Cat}}(-;B)$ for the realisation $r^B \circ (- \times \mathrm{Id}_B)$, denote the stable pseudoisotopy functor by $\mathbb{P}^{\mathrm{Cat}}(-;B)$, refer to $\mathrm{Ch}_\Delta^{c,s,B}$ by Ch_Δ and to c^+ by c. Finally, we may drop the collar extension map e from the notation for some morphism (ch, e) in $\mathrm{Ch}_\Delta^{c,s,B}$.*

Theorem 2.2.1.16. *Let $Cat = \mathrm{PL}$ or Top.*

1. *The pseudoisotopy functor $\mathbb{P}^{\mathrm{Cat}}(-;B)\colon \mathrm{Top} \to \mathrm{Top}$ is given by homotopy left Kan extension of its restriction to $\mathrm{Mfd}^{\mathrm{Cat}}$.*

2. *The functor $\mathrm{Ind}(P^{\mathrm{Cat}}(-;B)) \circ \iota \circ F_{\mathrm{ch}}^{c,s,+}$ coincides - up to natural weak equivalence of $(\infty,1)$–functors - with the piecewise linear pseudoisotopy functor defined in [12].*

3. *Let $i\colon (\mathrm{Mfd}^{\mathrm{PL}},\mathrm{cts}) \subseteq (\mathrm{Mfd}^{\mathrm{Top}},\mathrm{cts})$ denote the inclusion. The point-wise inclusion maps $\mathbb{P}_\partial^{\mathrm{PL}}(M;K) \to \mathbb{P}_\partial^{\mathrm{Top}}(i(M);K)$ extend to a natural weak equivalence of functors.*

4. *Let $(i\colon E \subseteq N \times |\Delta^m|, (p_i)_{i=1}^n, s, (\nu_i)_{i=1}^n, (H^{(i)})_{i=1}^n)$ be a choice which factors over $\phi_0 = (E = E, (p_i)_{i=1}^n, s, (\nu_i)_{i=1}^n, (H^{(i)})_{i=1}^n, e_0)$ for some collar*

extension map e_0. Then the square

$$\begin{array}{ccc}
M \times I \times |\Delta^m| \times B & \xrightarrow{F} & M \times I \times |\Delta^m| \times B \\
\downarrow s & & \downarrow s \\
E \times I \times |\Delta^m| \times B & \xrightarrow{P(\phi_0)(F)} & E \times I \times |\Delta^m| \times B \\
\downarrow i & & \downarrow i \\
N \times I \times |\Delta^m| \times B & \xrightarrow{P(\phi)(F)} & N \times I \times |\Delta^m| \times B
\end{array}$$

commutes for every $F \in P(M;B)$, i.e. the induced pseudoisotopy restricts to the original pseudoisotopy on the zero section of the disk bundle.

We reduce the structure group of the bundle to $\mathrm{Aut}(S^{d-1})$ by choosing a metric. Then there is a covering $(U_i, \omega_i)_{i\in I}$ of M by local trivialisations of $p = p_n \circ \cdots \circ p_1$ for some indexing set I with respect to the reduced structure group.

Now let (U,ω) and (U',ω') be trivialisations of the covering. Then the geometric transfer without bending the boundary $P(\phi_0)(F)$ fits into a commutative square

$$\begin{array}{ccc}
N \times I \times |\Delta^k| \times |\Delta^m| \times B & \xrightarrow[P(\phi_0)(F)]{\cong} & N \times I \times |\Delta^k| \times |\Delta^m| \times B \\
\uparrow \omega\times\mathrm{Id} & & \uparrow \omega'\times\mathrm{Id} \\
U \times D^d \times I \times |\Delta^k| \times |\Delta^m| \times B & \xhookrightarrow[F\times\rho]{} & U' \times D^d \times I \times |\Delta^k| \times |\Delta^m| \times B
\end{array}$$

where $\rho\colon U \times D^d \times |\Delta^k| \times |\Delta^m| \times B \to D^d$ is adjoint to a family of automorphisms of the fibre $\tilde{\rho}\colon U \times |\Delta^k| \times |\Delta^m| \times B \to \mathrm{Aut}(D^d)$, induced by a map $\tilde{\rho}'\colon U \times |\Delta^k| \times |\Delta^m| \times B \to \mathrm{Aut}(S^{d-1})$ into the reduced structure group.

5. *Let $p\colon E \to M$ be a disk bundle, ν a transfer map and H a fibre-wise bending isotopy, everything parametrised over some compact manifold X. Let F in $P(M;B)$ be a pseudoisotopy. We obtain an isotopy of pseudoisotopies $\mathrm{Tr}_{\nu,h}(F)\colon E \times I \times B \to E \times I \times B$ over X where $h = H(-,1)$ is the associated bending map.*

6. *The point-wise inclusion maps $\mathbb{P}_\partial^{\mathrm{Diff}}(M;B) \to \mathbb{P}_\partial^{\mathrm{Top}}(M;B)$ extend to a natural transformation of quasicategories $\mathbb{P}_\partial^{\mathrm{Diff}} \Rightarrow \mathbb{P}_\partial^{\mathrm{Top}}$ between endofunctors on $\mathcal{N}_\bullet^{h.c.}\,\mathrm{Top}_\Delta$.*

7. *The inclusion $P_\partial^{\mathrm{Cat}}(M \times [0,1]^k; B) \subseteq P^{\mathrm{Cat}}(M \times [0,1]^k; B)$ is a weak equivalence for every $k \in \mathbb{N}$ and they induce a weak equivalence on stable pseudoisotopies $\mathbb{P}_\partial^{\mathrm{Cat}}(M;B) \to \mathbb{P}^{\mathrm{Cat}}(M;B)$.*

Proof. The proof of the connective case, Theorem 2.1.3.21, carries over. For the third property we use [12, Remark 7.1] instead of [12, Remark 6.1], and for the sixth one we use the analogue of Lemma 1.4.1.1 for bounded pseudoisotopies. □

Let $i\colon M \hookrightarrow M \times [0,1]$ denote the inclusion. We obtain an induced map $P(i;\mathbb{R}^k)\colon P(M;\mathbb{R}^k) \to P(M\times[0,1];\mathbb{R}^k)$. By extending with the identity we also get a pseudoisotopy on $M \times I \times |\Delta^n| \times \mathbb{R}^k \times \mathbb{R}_{\geq 0}$, namely

$$\begin{aligned}(x,t,v,w,w_0) \mapsto & P(i;\mathbb{R}^k)(F)(x,t,v,w,w_0) \text{ if } w_0 \in [0,1]\\ & (x,t,v,w,w_0) \qquad\qquad \text{otherwise.}\end{aligned}$$

Moreover, we have an extension map of bounded collars. This determines a natural transformation of $(\infty,1)$–functors which makes the diagram

$$\begin{array}{ccc}\mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta & \xrightarrow{P(-;\mathbb{R}^k)} & \mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta^{\mathbb{R}^k} \\ \downarrow & & \downarrow \\ \mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta^{\mathbb{R}^k\times\mathbb{R}_{\geq 0}} & \xrightarrow{P(-;\mathbb{R}^k\times\mathbb{R}_{\geq 0})} & \mathcal{N}^{h.c.}_\bullet \operatorname{Top}_\Delta\end{array}$$

commutative for a suitable choice of the section $\mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta \to \mathcal{N}^{h.c.}_\bullet \operatorname{Ch}_\Delta^{\mathbb{R}^k\times\mathbb{R}_{\geq 0}}$.

Definition 2.2.1.17. The *pseudoisotopy spectrum* $\mathcal{P}$ is level wise given by $\mathcal{P}_k(M) = \mathbb{P}(M;\mathbb{R}^k)$. The *k–th structure map of the pseudoisotopy spectrum* is induced by the homotopy Cartesian square, see [53, Proposition 1.10],

$$\begin{array}{ccc}P(M;\mathbb{R}^k) & \longrightarrow & P(M;\mathbb{R}^k\times\mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ P(M;\mathbb{R}^k\times\mathbb{R}_{\leq 0}) & \longrightarrow & P(M;\mathbb{R}^{k+1})\end{array}$$

where each map is a composition similar to the above construction.

Corollary 2.2.1.18. *The level wise natural transformations $\mathbb{P}_\partial^{\mathrm{Diff}} \Rightarrow \mathbb{P}_\partial^{\mathrm{Top}}$ assemble into a natural transformation $\mathcal{P}^{\mathrm{Diff}} \Rightarrow \mathcal{P}^{\mathrm{Top}}$ of $(\infty,1)$–functors from $\mathcal{N}^{h.c.}_\bullet \operatorname{Top}_\Delta$ to $\mathcal{N}^{h.c.}_\bullet \operatorname{Spectra}_\Delta$.*

Proof. Since the structure maps for the smooth and topological pseudoisotopy spectrum are constructed in the same way, the level wise natural transformations are compatible. ☐

The bounded classifying map

As in the connective case we have to define a classifying map $\mathrm{c}\colon \mathcal{P} \to \Omega\mathcal{H}$.

Definition 2.2.1.19. Let $F\colon M \times |\Delta^n| \times \mathbb{R}^k \times I \to M \times |\Delta^n| \times \mathbb{R}^k \times I$ be an n–simplex of the space of piecewise linear pseudoisotopies. Let $e_M\colon M \hookrightarrow \mathbb{R}^\infty$ be the subspace inclusion and $e_n\colon M \times \mathbb{R} \times |\Delta^n| \times \mathbb{R}^k \hookrightarrow \mathbb{R}^\infty \times |\Delta^n| \times \mathbb{R}^k$ the standard embedding given by $(x,r,v,w) \mapsto (r, e_M(x), v, w)$.

We define the embedding

$$\begin{aligned}F_s\colon M \times I \times |\Delta^n| \times \mathbb{R}^k \hookrightarrow & M \times \mathbb{R} \times |\Delta^n| \times \mathbb{R}^k\\ (x,t,v,w) \mapsto & F(x,t-s,v,w) + (0,s,0,0); \text{ if } t-s \geq 0\\ & (x,t,v,w); \qquad\qquad \text{otherwise}\end{aligned}$$

for $s \in [0,1]$, where we use $M \subseteq \mathbb{R}^\infty$ to define addition. We obtain an embedding

$$\begin{aligned} e(F_s)\colon M \times I \times |\Delta^n \times \Delta^1| \times \mathbb{R}^k &\hookrightarrow \mathbb{R}^\infty \times |\Delta^n \times \Delta^1| \times \mathbb{R}^k \\ (x,t,v,s,w) &\mapsto (e_n \circ F_s(x,t,v), s, w). \end{aligned}$$

This, in turn, yields an $n \times 1$–simplex in $H_\bullet(M;\mathbb{R}^k)$

$$\begin{array}{ccc} M \times \{0\} \times |\Delta^n \times \Delta^1| \times \mathbb{R}^k & \xrightarrow{e(F_s)\circ i_0} & \operatorname{im}(e(F_s)) \\ & \overset{\mathrm{pr}}{\searrow} & \downarrow{\scriptstyle \mathrm{pr}\circ(e(F_s))^{-1}} \\ & & |\Delta^n \times \Delta^1| \end{array}$$

where $i_0\colon M \times \{0\} \times |\Delta^n \times \Delta^1| \times \mathbb{R}^k \hookrightarrow M \times I \times |\Delta^n \times \Delta^1| \times \mathbb{R}^k$ is the subspace inclusion. The proper and surjective map is given by projection to $\mathbb{R}^k$. This is well-defined because F is a bounded pseudoisotopy.

We obtain a loop $\phi_F\colon (\Delta^1, \partial\Delta^1) \to (H_\bullet(M;\mathbb{R}^k), (\mathrm{pr}\colon M \times I \times \mathbb{R}^k \to |\Delta^0|))$ since the embeddings $d_0^*(e(F_s))$ and $d_1^*(e(F_s))$ have the same image.

As before the bundle $e(F_s)$ is just a model for the S^1–bundle classified by F and we obtain a classifying map.

Definition 2.2.1.20. We define a map of simplicial sets

$$\begin{aligned} \mathrm{c}\colon P(M;\mathbb{R}^k) &\to \Omega H_\bullet(M;\mathbb{R}^k) \\ F &\mapsto \phi_F. \end{aligned}$$

The arguments used in Section 2.1.2 carry over to the bounded case. We omit most details and only define the space of bounded embeddings necessary for the fibre sequences.

Definition 2.2.1.21. Let $E_0(M;\mathbb{R}^k)$ be the simplicial set with n–simplices the *bounded piecewise linear embeddings* $M \times I \times |\Delta^n| \times \mathbb{R}^k \hookrightarrow \mathbb{R}^\infty \times |\Delta^n|$ which are compatible with the projection to $|\Delta^n|$ and restrict to the standard embedding on $M \times \{0\} \times |\Delta^n|$. An embedding ι is *bounded* if $\| \mathrm{pr}_{\mathbb{R}^k} \circ \iota(x) - \mathrm{pr}_{\mathbb{R}^k}(x)\|$ admits a common bound for all x.

Lemma 2.2.1.22. *The map* c *is a weak equivalence and commutes with the stabilisation maps. It thus induces a weak equivalence*

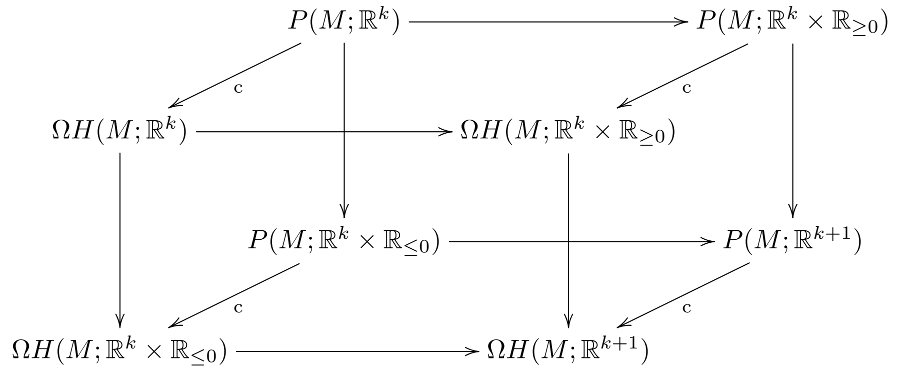
$$\mathrm{c}\colon \mathbb{P}^{\mathrm{PL}}(M) \to \Omega \mathbb{H}_\bullet^{\mathrm{PL}}(M).$$

Proof. This is analogous to the proof in Section 2.1.2. □

Lemma 2.2.1.23. *The map* c *commutes with the structure maps up to homotopy and thus induces a weak equivalence of spectra*

$$\mathrm{c}\colon \mathcal{P}^{\mathrm{PL}}(M) \to \Omega \mathcal{H}_\bullet^{\mathrm{PL}}(M).$$

Proof. Using that spaces of collars as well as spaces of embeddings of a compact manifold into $\mathbb{R}^\infty$ are contractible, it is straightforward to show that the cube



is homotopy commutative up to higher coherences, i.e. it extends to a map $(\Delta^1)^3 \to \mathcal{N}_\bullet^{h.c.} \operatorname{Top}_\Delta$. This implies that c is compatible with the structure maps up to homotopy.

Now, a mapping telescope construction allows one to strictify this collection into a map of spectra. Since the mapping telescope is a model for the homotopy colimit, this is enough. Equivalently our spectra are cofibrant and fibrant objects in the model category of prespectra [5, Theorem 2.3]. □

The negative homotopy groups of the h–cobordism spectrum

In the rest of this section we show that the parametrised torsion of Pedersen [37] induces an isomorphism between the negative homotopy groups of the spectrum of bounded h–cobordisms and negative K–groups. The key step is the following proposition.

Proposition 2.2.1.24. *The (non-connective) h–cobordism spectrum $\mathcal{H}(M)$ is an Ω–spectrum.*

Remark 2.2.1.25. *The spectrum $\mathcal{P}(M)$ is an Ω–spectrum for every M by [53, Proposition 1.10]. The proposition follows except for the statement about the map $\pi_0(\mathbb{H}(M;\mathbb{R}^k)) \to \pi_1(\mathbb{H}(M;\mathbb{R}^{k+1}))$ for every $k \in \mathbb{N}$.*

The following arguments are adapted from the case of pseudoisotopies explained in [53, Proposition 1.5], [53, Lemma 1.6] and their respective proofs, see also [41, Chapter 5] for transversality in the PL category. Since we do not have a strict composition law on h–cobordisms, some adjustments are necessary. Let H^0 be the connected component of the trivial h–cobordism.

Definition 2.2.1.26. Let $E^{(k)}(M)$ be the simplicial set given in degree n by all pairs (p, q) where p, q are bundles over $|\Delta^n|$ of bounded h–cobordisms over $M \times \mathbb{R}^{k+1}$ such that q restricted to a bundle of h–cobordisms over $M \times (-\infty, -1)$ is given by the cylinder $M \times (-\infty, -1) \times [0, 1]$ and p and q restricted to bundles of cobordisms over $M \times [z, \infty)$ coincide for some $z \geq 0$.

Lemma 2.2.1.27. *The diagram $E_0^{(k)}(M) \to E^{(k)}(M) \to H_\bullet^0(M;\mathbb{R}^{k+1})$ is a fibration for every $n \in \mathbb{N}$.*

We note that $E_0^{(k)}(M)$ and $E_0(M)$ are not inherently linked. This is just an unfortunate clash of notation.

Proof. The map $E^{(k)}(M) \to H_\bullet^0(M;\mathbb{R}^{k+1})$ is surjective on 0–simplices because $H_\bullet^0(M;\mathbb{R}^{k+1})$ only contains h–cobordisms which are isotopic to the trivial one. Essentially, one lifts an h–cobordism p to a bounded embedding e_p into $\mathbb{R}^\infty$, chooses an isotopy to the trivial h–cobordism, and uses an intermediate level of this isotopy to find an embedding e_q which yields a lift (p,q) of p in $E^{(k)}(M)$.

For higher simplices we use a relative version of the same argument. Let $\tilde{p}$ be an h–cobordism bundle over $|\Delta^n|$ and (p,q) a lift of its restriction to $|\Lambda^n_m|$. Let e_q be an embedding of q into $\mathbb{R}^\infty \times |\Lambda^n_m|$ and i_q an isotopy of embeddings to an embedding e_p of p.

We extend e_q to an embedding e'_q into $\mathbb{R}^\infty \times |\Delta^n|$ along a piecewise linear retraction $|\Delta^n| \to |\Lambda^n_m|$ and similarly extend i_q to an isotopy i'_q from e'_q to the extension along pullback e'_p of e_p. By general position we find an isotopy i_p from e'_p to an embedding $e_{\tilde{p}}$ of $\tilde{p}$ relative e_p. Choosing appropriate intermediate levels of the isotopy $i'_q * i_p$ yields an embedding $\tilde{q}$ which yields a lift $(\tilde{p},\tilde{q})$ of $\tilde{p}$. □

Lemma 2.2.1.28. *The space* $\operatorname{colim}_{n\in\mathbb{N}} E^{(k)}(M \times [0,1]^n)$ *is contractible.*

Proof. We first note that every $E^{(k)}(M \times [0,1]^n)$ is a Kan-complex by pulling back a pair of h–cobordism bundles along a retraction $|\Delta^m| \hookrightarrow |\Lambda^m_p|$. Since the structure maps are cofibrations, $\operatorname{colim}_{n\in\mathbb{N}} E^{(k)}(M \times [0,1]^n)$ is Kan as well.

Let $[(p,q)]$ be an element in $\pi_r(\operatorname{colim}_{n\in\mathbb{N}} E^{(k)}(M\times[0,1]^n))$. It is represented by an r–simplex of some $E^{(k)}(M \times [0,1]^n))$, relative boundary. Since p and q are r–simplices of $H_\bullet^0(M \times [0,1]^n \times \mathbb{R}^{k+1})$ we may lift them (e.g. via the construction explained in the proof of Lemma 2.2.1.27) to bounded embeddings relative boundary, i.e. r–simplices in $E_0(M \times [0,1]^n; \mathbb{R}^{k+1})$, with the following properties

1. On $M \times I \times [0,1]^n \times \mathbb{R}^k \times \mathbb{R}_{\leq 0}$ the lift e_q of q restricts to the standard embedding.
2. The lift e_p of p coincides with e_q on $M \times I \times [0,1]^k \times \mathbb{R}^k \times \mathbb{R}_{\geq z}$ for some $z \in \mathbb{R}_{>0}$.
3. Both lifts are given by the standard embedding for each point on the boundary of $|\Delta^r|$.

By transversality (and possibly stabilising the target of our embeddings) we find an isotopy of embeddings relative $M \times I \times [0,1]^n \times \mathbb{R}^k \times \mathbb{R}_{\geq z+1}$ from e_p to an embedding e'_p (which restricts to the constant isotopy on the boundary), such that e'_p is a lift of q.

The image of the isotopy yields $[(p,q)] = [(q,q)]$ in $\pi_r(E^{(k)}(M \times [0,1]^n))$. But by an Eilenberg Swindle, every element of the form $[(q,q)]$ is trivial. Hence $[(p,q)] = 0$ and we are done. □

Proof of Proposition 2.2.1.24. The fibre $E_0^{(k)}(M) \subset E^{(k)}(M)$ is the subspace consisting of pairs (p,q) with p the trivial h–cobordism. The inclusion map $(i_0)_*\colon H_\bullet(M \times D^1;\mathbb{R}^k) \to E_0^{(k)}(M)$ is an acyclic cofibration.

The result now follows from Lemma 2.2.1.27 and Lemma 2.2.1.28 since there is a commutative diagram

$$\begin{array}{ccc} H_\bullet(M \times D^1; \mathbb{R}^k) & \longrightarrow & E_0^{(k)}(M) \\ \downarrow & & \downarrow \\ CH_\bullet(M \times D^1; \mathbb{R}^k) & \dashrightarrow & E^{(k)}(M) \\ \downarrow & & \downarrow \\ \Sigma H^0_\bullet(M \times D^1; \mathbb{R}^k) & \xrightarrow{\sigma} & H_\bullet(M; \mathbb{R}^{k+1}) \end{array}$$

where the middle horizontal map sends a simplex $|\Delta|^1 \times W$ to the pair (σ, h) defined as follows:

We have an isotopy of embeddings $\rho\colon \mathbb{R} \to \mathrm{Emb}(M \times D^1, M \times \mathbb{R})$ given by $z \mapsto i_z$ where i_z is the canonical isometric embedding with $i_z(0) = z$. We obtain an induced map $\rho_*\colon H_\bullet(M \times D^1; \mathbb{R}^k) \to \Omega H_\bullet(M; \mathbb{R}^{k+1})$ on the one point compactification $\mathbb{R} \cup \infty$. This is a model for the structure map σ of the h–cobordism spectrum.

Similarly, consider the isotopy of embeddings $\rho'\colon \mathbb{R} \to \mathrm{Emb}(M \times D^1, M \times \mathbb{R})$ given by i_0 for $z \in \mathbb{R}_{\leq 0}$ and i_z for $z \in \mathbb{R}_{\geq 0}$. The pair (ρ, ρ') yields the map (σ, h). □

Since we have now shown that the h–cobordism spectrum is an Ω–spectrum, we can use one of the main results of [37] to calculate its negative homotopy groups.

Theorem 2.2.1.29 (Bounded h–cobordism theorem)**.** *Let $(W, \partial_0 W, \partial_1 W)$ be a bounded h–cobordism of dimension at least 6, parametrized by $\mathbb{R}^k$ with bounded fundamental group π. Then there is an obstruction in $\tilde{K}_{-k+1}(\mathbb{Z}\pi)$ which vanishes if and only if W admits a bounded product structure. All such invariants are realized by bounded h-cobordisms.*

The invariant under consideration is in fact the bounded Whitehead torsion of the inclusion $\partial_0 W \subseteq W$, see [37, Definition 3.2] and [36, Theorem 2.5]. Also, for every compact manifold M, the fundamental group of $M \times \mathbb{R}^k$ is bounded for every $k \in \mathbb{N}$, see [37, Definition 1.3].

Corollary 2.2.1.30. *The map $\tau\colon \pi_0(\mathbb{H}(M; \mathbb{R}^k)) \to \tilde{K}_{-k+1}(\mathbb{Z}\pi_1(M))$, which associates to a bounded h–cobordism its bounded Whitehead torsion, is a bijection.*

Proof. There is a monoidal structure on $\pi_0(\mathbb{H}(M; \mathbb{R}^k))$ given as follows: Let W, W' be two h–cobordisms over M and choose some collars $c\colon [0,1] \to W$ and c' of M. Then the sum of their classes is represented by $W \times D^1 \cup_{M \times [0,1]} W' \times D^1$ where we stabilise once with D^1 and glue along the collars.

The result is a consequence of the theorem, the monoidal structure and the sum formula for the bounded Whitehead torsion. The sum formula for the bounded Whitehead torsion can be shown analogously to the classical case, see [9, Theorem 23.1]. □

2.2.2 A suspected model for the Whitehead spectrum

We show that the non-connective deloopings of A–theory due to Ullmann and Winges [44] and Vogell [46] agree up to level equivalence of spectra, see Corollary 2.2.2.19. Next, we introduce a simplicial variation of Vogell's model which allows for triangulations of bounded h–cobordisms as objects. The construction is finished in Definition 2.2.2.29.

The simplicial variation for the one fold loops of the Whitehead spectrum requires us to revisit quite a few arguments of Waldhausen [51] which originally identified the homotopy fibre of the A–theory assembly map as the K–theory of a Waldhausen category, see [54] for an introduction of assembly homology theories.

We introduce simple maps in the controlled setting in Definition 2.2.2.35 and conclude this part with the suspected (point wise) model of $\Omega\,\mathrm{Wh}$ in Definition 2.2.2.46. The problem with this candidate is explained in Remark 2.2.2.44.

Most of Waldhausen's arguments allow straightforward adaptions to our setting of K–theory of bounded retractive spaces. However, the degree to which the functors under consideration preserve weak equivalences is always subtle. On a technical level one finds similarities between these questions and the triangulation problems which make the adaption of the connective zig-zags difficult.

The non-connective deloopings of A–theory

We begin by recalling the setting of Ullmann and Winges. The reader familiar with [44] can safely skip ahead to Definition 2.2.2.9.

Definition 2.2.2.1 ([44, Definition 2.1])**.** Let a *coarse structure* denote a triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ such that Z is a Hausdorff space, $\mathfrak{C}$ is a collection of reflexive and symmetric relations on Z which is closed under taking finite unions and compositions, and $\mathfrak{S}$ is a collection of subsets which is closed under taking finite unions.

Definition 2.2.2.2 ([44, Definition 3.23])**.** A *morphism of coarse structures* $z\colon \mathfrak{Z}_1 \to \mathfrak{Z}_2$ is a map of sets $z\colon Z_1 \to Z_2$ satisfying the following properties:

1. For every $S_1 \in \mathfrak{S}_1$, there is some $S_2 \in \mathfrak{S}_2$ such that $z(S_1) \subseteq S_2$ holds.
2. For every $S_1 \in \mathfrak{S}_1$ and $C_1 \in \mathfrak{C}_1$, there is some $C_2 \in \mathfrak{C}_2$ such that we have $(z \times z)((S \times S) \cap C_1) \subseteq C_2$.
3. For every $S \in \mathfrak{S}_1$ and all subsets $A \subseteq S$ which are locally finite in Z_1, the set $z(A)$ is locally finite in Z_2 and for all $x \in z(A)$ the set $z^{-1}(x) \cap A$ is finite.

Example 2.2.2.3. *Let (B, d) be a metric space. The bounded morphism control condition is*

$$\mathfrak{C}_{bdd}(B) = \{P \subseteq B \times B | \exists\, R > 0 \text{ such that } d(p_1, p_2) \leq R \text{ for all } (p_1, p_2) \in P\}.$$

We obtain the bounded coarse structure $\mathfrak{B}(B) = (B, \{B\}, \mathfrak{C}_{bdd}(B))$ on B.

Fix a coarse structure $\mathfrak{Z}$ and a topological space W. For a CW-complex Y relative W denote by $\diamond Y$ the (discrete) set of relative cells of Y and by $\diamond_k Y$ the subset of all relative k–cells in Y.

Definition 2.2.2.4 ([44, Definition 2.3])**.** A *labelled CW-complex relative* W is a pair (Y, κ) where Y is a CW-complex relative W and $\kappa\colon \diamond Y \to Z$ is a map of sets.

Let $e \in Y$ be a cell. We denote by $\langle e \rangle$ the smallest CW-subcomplex of Y which contains e. A $\mathfrak{Z}$*–controlled map* $f\colon (Y, \kappa) \to (Y', \kappa')$ is a cellular map $f\colon Y \to Y'$ relative W such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa' \times \kappa)(\{(e', e) | e \in \diamond_k Y, e' \in \diamond Y', \langle f(e) \rangle \cap e' \neq \emptyset\}) \subseteq C\}$$

holds.

A $\mathfrak{Z}$*–controlled CW-complex relative* W is a labelled CW-complex (Y, κ) relative W such that the Identity map on Y is a $\mathfrak{Z}$–controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S$$

holds.

Definition 2.2.2.5 ([44, Definition 3.1])**.** A $\mathfrak{Z}$*–controlled retractive CW-complex relative* W is a $\mathfrak{Z}$–controlled CW-complex (Y, κ) relative W together with a retraction $r\colon Y \to W$, i.e. a left inverse r to the structural inclusion $s\colon W \hookrightarrow Y$.

The $\mathfrak{Z}$–controlled retractive spaces relative W form a category $\mathcal{R}(W, \mathfrak{Z})$ in which morphisms are $\mathfrak{Z}$–controlled maps compatible with the retractions.

Definition 2.2.2.6 ([44, Definition 2.5])**.** Let (Y, κ, r, s) and (Y', κ', r', s') denote two $\mathfrak{Z}$–controlled retractive spaces relative W. A $\mathfrak{Z}$*–controlled homotopy equivalence* from (Y, κ, r, s) to (Y', κ', r', s') is a morphism $f\colon Y \to Y'$ in $\mathcal{R}(W, \mathfrak{Z})$ such that there is a $\mathfrak{Z}$–controlled map $\bar{f}\colon Y' \to Y$ together with $\mathfrak{Z}$–controlled homotopies $f \circ \bar{f} \simeq \mathrm{Id}$ and $\bar{f} \circ f \simeq \mathrm{Id}$.

Definition 2.2.2.7 ([44, Definition 3.1 and Definition 3.3])**.** A $\mathfrak{Z}$–controlled retractive space (Y, κ, r, s) is called *finite*, if it is finite dimensional, the image of $Y - W$ under the retraction meets only finitely many connected components of W, and for all $z \in Z$ there is some open neighbourhood U of z such that $\kappa^{-1}(U)$ is finite.

We denote the full subcategories of finite $\mathfrak{Z}$–controlled retractive spaces by $\mathcal{R}_f(W, \mathfrak{Z}) \subseteq \mathcal{R}(W, \mathfrak{Z})$.

Lemma 2.2.2.8 ([44, Corollary 3.22])**.** *The category $\mathcal{R}_f(W, \mathfrak{Z})$ carries a Waldhausen structure with inclusions of subcomplexes up to isomorphism as cofibrations and $\mathfrak{Z}$–controlled homotopy equivalences as weak equivalences.*

There is a cylinder functor on $\mathcal{R}_f(W, \mathfrak{Z})$ and the cylinder and saturation axiom hold.

We summarise controlled retractive CW-complexes and bounded homotopy equivalences in the sense of Vogell, see [46, pp. 164] for further details. No new material is presented before the comparison, beginning with Definition 2.2.2.15.

Let B be a closed subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. Let W be a topological space.

Definition 2.2.2.9. A *controlled topological space* is a space Y together with a map $q\colon Y \to B$.

Definition 2.2.2.10. A *bounded n–cell* (of diameter c) is a pair $(J \times D^n, q)$ where J is a discrete index set and $q\colon J \times D^n \to W \times B$ is a map satisfying the following:

1. The set $\mathrm{pr}_B \circ q(\{j\} \times D)$ has diameter at most c for each $j \in J$.
2. For $K \subseteq B$ compact, the set $\{j \in J | \mathrm{pr}_B \circ q(\{j\} \times D^n) \cap K \neq \emptyset\}$ is finite.

Definition 2.2.2.11. A *boundedly finite CW-complex relative W* is a bounded CW-complex (Y, r, s) where Y is a retractive CW-complex relative W which is built from W by attaching finitely many bounded cells in order of increasing dimension, i.e. $r\colon W \to Y$ and s are a retraction and a section, respectively, and r restricts to the control map q on each bounded n–cell.

The category $\mathfrak{R}_f(W, B)$ of *boundedly finite retractive CW-complexes* has as morphisms cellular maps which respect sections and retractions.

Definition 2.2.2.12. A *bounded homotopy equivalence* $f\colon (Y, q) \to (Y', q')$ between controlled spaces is a bounded map $f\colon Y \to Y'$ such that there is a bounded map $g\colon Y' \to Y$ together with bounded homotopies $H\colon g \circ f \simeq \mathrm{Id}$ and $G\colon f \circ g \simeq \mathrm{Id}$, i.e. they satisfy $sup_{x \in W} diam(H(\{x\} \times I)) \in \mathbb{R}_{\geq 0}$, similar for G.

Definition 2.2.2.13. A *boundedly homotopy finite space relative W* is a controlled space (Y, q) which is connected by a zig-zag of bounded homotopy equivalences to a boundedly finite CW-complex relative W.

We obtain the category $\mathfrak{R}_{hf}(W, B)$ of *boundedly homotopy finite retractive spaces* with morphisms continuous, controlled maps which are compatible with sections and retractions.

Lemma 2.2.2.14 ([46, Lemma 1.1]). *The category $\mathfrak{R}_f(W, B)$ carries a Waldhausen structure with maps which are inclusions of subcomplexes up to isomorphism as cofibrations and bounded homotopy equivalences as weak equivalences.*

The category $\mathfrak{R}_{hf}(W, B)$ carries a Waldhausen structure with maps which satisfy the bounded homotopy extension property as cofibrations and bounded homotopy equivalences as weak equivalences.

There is a cylinder functor on each of these categories and they satisfy the saturation as well as the cylinder axiom.

Proof. Although not explicitly stated, Vogell uses the cylinder functor as well as the saturation and cylinder axiom. It is not hard to check that the usual construction of the cylinder functor works and to show the two axioms. □

Since the applications to the Farrell-Jones conjecture in [13] use the model by Ullmann and Winges, while our arguments are related to Vogell's, we show that they yield equivalent K–theory.

Definition 2.2.2.15. Let $F\colon \mathfrak{R}_f(W; B) \to \mathcal{R}_f(W, \zeta)$ denote the functor which is given by $(Y, r, s) \mapsto (Y \cup_{W \times B} W, \mathrm{pr}_W \circ (r \cup \mathrm{Id}), \iota_W, \mathrm{pr}_B \circ r \circ \mathrm{bary})$ on objects, where $\mathrm{bary}\colon \diamond Y \to Y$ sends every cell to its barycentre, and uses the obvious functorial extensions for morphisms.

Definition 2.2.2.16. Let $\mathfrak{Z}$ be the bounded coarse structure over a vector space B with a Riemannian metric. Let $(Y, r, s, \kappa) \in \mathcal{R}_f(W, \mathfrak{Z})$ be an object. Let $Y' := \bigcup_{e \in \diamond Y} < e > \subseteq Y$ be the subset of all cells. We inductively define the map $r_B\colon Y' \to B$ as follows. On a 0–simplex e, we set $r_B(0,e) := \kappa(e)$. For the general case, we have to find extensions

$$\begin{array}{ccc} \partial < e > & \xrightarrow{r_B} & B \\ \downarrow & \nearrow_{r_B} & \\ < e > & & \end{array}$$

and we set

$$v \mapsto \begin{cases} |v| r_B(\frac{v}{|v|}) + (1-|z|)\kappa(e) & |v| \neq 0 \in D^m \\ \kappa(e) & |v| = 0 \in D^m. \end{cases}$$

Proposition 2.2.2.17. *The functor* $F\colon \mathfrak{R}_f(W;B) \to \mathcal{R}_f(W,\mathfrak{Z})$ *induces an equivalence on algebraic* K*–theory.*

Proof. The functor F is exact. The statement follows from the approximation theorem, see [51, Theorem 1.6.7]. All of the theorem's prerequisites but the second approximation property are easy to check.

We are given (Y,r,s), (Y',r',s',κ') and a morphism $f\colon Y \cup_{W\times B} W \to Y'$. We construct an object $\tilde{Y}$ in $\mathfrak{R}_f(W;B)$ whose image under F is weakly equivalent to Y'. For that, we choose attaching maps for Y'. We define an object in $\mathfrak{R}_f(W;B)$ inductively, thus we start out with $W \times B$. Let e be a cell in degree m and $\alpha_e\colon S^{m-1} \to (Y')^{(m-1)}$ its gluing map. We attach a cell to $\tilde{Y}^{(m-1)}$ along the following map:

$$\tilde{\alpha}_e(v) := \begin{cases} (\alpha_e(v), r_B(v)) \text{ on } \alpha_e^{-1}(W) \\ \alpha_e(v) \text{ on } \bigcup_{e'\in\diamond_{\leq m-1}Z} \alpha_e^{-1}(< e' >). \end{cases}$$

We obtain $(\tilde{Y}, r\times r_B, \iota_{W\times B})$ in $\mathfrak{R}_f(W;B)$. Since morphisms in $\mathcal{R}_f(W,\zeta)$ are bounded and B is convex, there is a homotopy of retractions $H\colon Y\times[0,1] \to B$ from $p_B \circ r$ to $p_B \circ r_B \circ f$.

Now we can define a reduced mapping cylinder with a slightly altered retraction $M(\tilde{f}) := ((Y\times[0,1]\cup_{Y\times\{1\}}\tilde{Y})\cup_{W\times B\times[0,1]} W\times B, ((p_W\circ r)\times H)\cup r_B, \iota_{W\times B})$ where we glue along the map $\tilde{f}\colon Y \to \tilde{Y}$ which is obtained from f by induction and satisfies $F(\tilde{f}) = f$.

Finally, the square

$$\begin{array}{ccc} F(Y,r,s) & \xrightarrow{f} & (Y,r,s,\kappa) \\ {\scriptstyle F(i_0)}\downarrow & \nearrow_{p} & \\ F(M(\tilde{f})) & & \end{array}$$

shows that the second approximation property is satisfied. Here, p denotes the mapping cylinder projection. □

For Vogell's definition of delooping, see [46, § 1] up to, but not including, [46, Lemma 1.2], the Definition on p. 169 and the final two Remarks on p. 186 and

p. 187. The structure map in degree k is induced by the homotopy Cartesian square

$$\begin{array}{ccc} A^V(W;\mathbb{R}^k) & \longrightarrow & A^V(W;\mathbb{R}^k \times \mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ A^V(W;\mathbb{R}^k \times \mathbb{R}_{\leq 0}) & \longrightarrow & A^V(W;\mathbb{R}^{k+1}) \end{array}$$

where the upper right and lower left terms are contractible.

The definition of the non-connective algebraic K–theory spectrum in the sense of Ullmann and Winges is given at the beginning of [44, Chapter 5]. The structure maps are again defined in terms of homotopy Cartesian diagrams and it is immediate from the definitions that they are compatible.

Lemma 2.2.2.18. *The functors* $F\colon \mathfrak{R}_f(W,\mathbb{R}^k) \to \mathcal{R}_f(W,\mathfrak{Z}(k))$ *for every* $k \in \mathbb{N}$ *are compatible with the structure maps of the non-connective algebraic* K*–theory spectrum in the sense of Vogell, respectively Ullmann and Winges.*

Hence we obtain a natural transformation $F\colon A^V \Rightarrow A^{UW}$ *between the non-connective algebraic K-theory functors.*

Corollary 2.2.2.19. *The natural transformation* $F\colon A^V \Rightarrow A^{UW}$ *is a point wise level equivalence.*

Proof. This is immediate from Proposition 2.2.2.17. □

From here on we only work with Vogell's setting. Unfortunately, there is no reasonable notion of "higher homotopy theory of simple maps" for CW-complexes. Hence, in order to mix simple maps with Waldhausen categories, we have to introduce a simplicial version of Vogell's approach.

Definition 2.2.2.20. Let B be a closed subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$. Let W be a compact polyhedron.

Let $T_\bullet$ denote a simplicial set and $t\colon (|T_\bullet|, \mathrm{pr}_B \circ t) \to (W \times B, \mathrm{pr}_B)$ a bounded homotopy equivalence.

Example 2.2.2.21. *Let* $T_\bullet$ *denote a simplicial set and* $t\colon |T_\bullet| \to W \times B$ *a homeomorphism, i.e.* $(T_\bullet, t)$ *is a triangulation of* $W \times B$*.*

We fix a collection of B, W and $(T_\bullet, t)$.

Definition 2.2.2.22. A *bounded simplicial set* is a simplicial set $Y_\bullet$ together with a map $q\colon Y_\bullet \to T_\bullet$ such that there is some constant $c > 0$ such that for every simplex $\sigma\colon \Delta^n \to Y_\bullet$ the set $\mathrm{pr}_B \circ t \circ |q| \circ |\sigma|(|\Delta^n|)$ has diameter at most c.

Definition 2.2.2.23. We define the category of retractive simplicial sets. An object is a triple $(Y_\bullet, r, s)$ with $Y_\bullet$ a simplicial set, $r\colon Y_\bullet \to T_\bullet$ a retraction, and $s\colon T_\bullet \to Y_\bullet$ a section of simplicial sets, i.e. $r \circ s = \mathrm{Id}$ holds. A morphism $f\colon (Y_\bullet, r, s) \to (Y'_\bullet, r', s')$ is a map of simplicial sets $f\colon Y_\bullet \to Y'_\bullet$ satisfying $r = r' \circ f$ and $f \circ s = s'$.

The category of *bounded retractive simplicial sets* $\mathfrak{R}(W, B, T_\bullet, t)$ is the full subcategory of the category of retractive simplicial sets on those objects $(Y_\bullet, r, s)$ for which the tuple $(Y_\bullet, r)$ is a bounded simplicial set.

For every $c > 0$ we obtain the full subcategory of *c–bounded retractive simplicial sets* $\mathfrak{R}(W, B, T_\bullet, t)_c$.

Definition 2.2.2.24. A *bounded n–simplex* (of diameter c) is a pair $(J \times \Delta^n, q)$, where J is a discrete index set and $q\colon J \times \Delta^n \to T_\bullet$ is a map satisfying the following:

1. The set $\mathrm{pr}_B \circ t \circ |q|(\{j\} \times |\Delta^n|)$ has diameter at most c for each $j \in J$.

2. For $K \subseteq B$ compact, the set $\{j \in J| \, \mathrm{pr}_B \circ t \circ |q|(\{j\} \times |\Delta^n|) \cap K \neq \emptyset\}$ is finite.

Definition 2.2.2.25. A *boundedly finite simplicial set relative* $W_\bullet$ is a bounded simplicial set $(Y_\bullet, q)$ where $Y_\bullet$ is a simplicial set relative $W_\bullet$ which is generated by $W_\bullet$ and finitely many bounded simplices.

We obtain the category $\mathfrak{R}_f(W, B, T_\bullet, t)$ of *boundedly finite retractive simplicial sets* as the full subcategory of $\mathfrak{R}(W, B, T_\bullet, t)$.

Analogously we obtain $\mathfrak{R}_f(W, B, T_\bullet, t)_c$ for every $c > 0$.

Definition 2.2.2.26. A *bounded weak equivalence* $f\colon (Y_\bullet, q) \to (Y'_\bullet, q')$ between bounded simplicial sets is a map of simplicial sets $f\colon Y_\bullet \to Y'_\bullet$ such that $|f|$ is a bounded homotopy equivalence between controlled spaces.

Definition 2.2.2.27. A *boundedly homotopy finite simplicial set relative* $W_\bullet$ is a bounded simplicial set $(Y_\bullet, q)$ which is connected by a zig-zag of bounded weak equivalences to a boundedly finite simplicial set relative $W_\bullet$.

We obtain the category of *boundedly homotopy finite retractive simplicial sets* $\mathfrak{R}_{hf}(W, B, T_\bullet, t)$ as the full subcategory of $\mathfrak{R}(W, B, T_\bullet, t)$.

We obtain $\mathfrak{R}_{hf}(W, B, T_\bullet, t)_c$ as the full subcategory of $\mathfrak{R}(W, B, T_\bullet, t)_c$ for every $c > 0$ as well.

Lemma 2.2.2.28. *The categories $\mathfrak{R}_f(W, B, T_\bullet, t)$ and $\mathfrak{R}_{hf}(W, B, T_\bullet, t)$ carry Waldhausen structures with injective morphisms as cofibrations and bounded weak equivalences as weak equivalences. They satisfy the saturation axiom.*

There is a cylinder functor on each of these categories and they satisfy the cylinder axiom.

The analogous statements hold for $\mathfrak{R}_f(W, B, T_\bullet, t)_c$ and $\mathfrak{R}_{hf}(W, B, T_\bullet, t)_c$ for every $c > 0$.

Proof. The Waldhausen structures are a direct consequence of the analogous statement for bounded retractive CW-complexes [46, Lemma 1.1]. The saturation axiom and the cylinder axiom are easy to show. □

Definition 2.2.2.29. We choose triangulations $(T^k_\bullet, t^k)$ of $W \times \mathbb{R}^k$ for every $k \in \mathbb{N}$, such that $(T^k_\bullet, t^k)$ restricts to $(T^{k-1}_\bullet, t^{k-1})$ on $W \times \mathbb{R}^{k-1} \times \{0\} \subseteq W \times \mathbb{R}^k$.

We define the simplicial version of the A–theory spectrum in the sense of Vogell as the spectrum with

$$A^{V,s}_k(W) = K(\mathfrak{R}_f(W, \mathbb{R}^k, T^k_\bullet, t^k))$$

as its k–th degree and the structure maps induced by the squares

$$\begin{array}{ccc} A^{V,s}(W;\mathbb{R}^k) & \longrightarrow & A^{V,s}(W;\mathbb{R}^k \times \mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ A^{V,s}(W;\mathbb{R}^k \times \mathbb{R}_{\leq 0}) & \longrightarrow & A^{V,s}(W;\mathbb{R}^{k+1}). \end{array}$$

This concludes the definition of our simplicial model for A–theory. We are going to see in a moment that simple maps form a Waldhausen category over locally finite retractive simplicial sets. The same is unclear if we only require local finiteness up to homotopy. On the other hand, we require the larger category to show that the simplicial model gives the same K–theory as the topological one. So we have to show that the two simplicial models agree on K–theory.

Lemma 2.2.2.30 (compare [51, Proposition 2.1.1])**.** *The inclusion functor* $\mathfrak{R}_f(W, B, T_\bullet, t) \to \mathfrak{R}_{hf}(W, B, T_\bullet, t)$ *induces an equivalence on algebraic K–theory.*

Proof. We adapt Waldhausen's original proof of the non-controlled statement [51, Proposition 2.1.1].

The argument proceeds via the approximation theorem [51, Theorem 1.6.7]. All conditions but the second part of the approximation property are easy to check. As in the classical case we may ignore retractions and it is enough to show the following:

Let $(Y_\bullet, r, s)$ be an object in $\mathfrak{R}_f(W, B, T_\bullet, t)$ and $f\colon (Y_\bullet, r, s) \to (Y', r', s')$ a morphism in $\mathfrak{R}_{hf}(W, B, T_\bullet, t)$. Then there are maps

$$(Y_\bullet, r, s) \xrightarrow{i} (Y_\bullet^1, r^1, s^1) \xrightarrow{f} (Y_\bullet', r', s')$$

with i a cofibration and g a weak equivalence such that $f = g \circ i$.

The corresponding statement in the controlled topological setting was shown in the proof of [46, Proposition 1.4].

In the simplicial case, we know from the topological case that there is a factorisation $(|Y_\bullet|, |r|, |s|) \to (|Y_\bullet^1|, |r^1|, |s^1|) \to (|Y_\bullet'|, |r'|, |s'|)$ after geometric realisation. We construct the desired factorisation by induction over the degree of the cells of Y^1 not contained in Y. Again, we ignore retractions.

Let $\mathcal{I}$ denote an index set for the cells $\{\sigma_i\}_{i\in\mathcal{I}}$ of degree n. For each $i \in I$ we use the construction given in [51, Proposition 2.1.1] which uses simplicial subdivision to find simplicial maps approximating the topological solution up to homotopy. □

We introduce a mild generalisation of Vogell's topological model. It is only used to simplify the proof that the simplicial and topological model agree.

Definition 2.2.2.31. Let T be a space and $t\colon (T, \mathrm{pr}_B \circ t) \to W \times B$ a bounded homotopy equivalence. We define $\mathfrak{R}(W, B, T, t)$ analogously to $\mathfrak{R}(W, B, T_\bullet, t)$.

Lemma 2.2.2.32. *The exact functor* $t_*\colon \mathfrak{R}(W, B, T, t) \to \mathfrak{R}(W, B)$ *induces an equivalence on K–theory.*

Proof. This is analogous to [46, Lemma 1.3]. Essentially, bounded homotopies induce natural transformations. □

The functor relating the simplicial and topological setting is induced by geometric realisation.

Definition 2.2.2.33. There is a functor $\mathfrak{R}(W, B, T_\bullet, t) \to \mathfrak{R}(W, B, |T_\bullet|, |t|)$ given by geometric realisation $(Y_\bullet, r, s) \mapsto (|Y_\bullet|, |r|, |s|)$.

It restricts to an exact functor of Waldhausen categories on $\mathfrak{R}_f(W, B, T_\bullet, t)$ and $\mathfrak{R}_{hf}(W, B, T_\bullet, t)$.

Proposition 2.2.2.34. *Let $T_\bullet$ be a Kan-complex. Then the geometric realisation functor $\mathfrak{R}_{hf}(W,B,T_\bullet,t) \to \mathfrak{R}_{hf}(W,B,|T_\bullet|,|t|)$ induces a weak equivalence on K–theory.*

The proof of this proposition proceeds similarly to (the corrected[1] version of) Waldhausen's proof of the classical statement [51, Proposition 2.1.2].

Proof. Since $T_\bullet$ is Kan, there is a retraction map $r\colon \mathcal{S}_\bullet(|T_\bullet|) \to T_\bullet$. For each $n \in \mathbb{N}$, the functor $\mathbf{F}_n$ is the $\mathbf{S}_n$–construction without the choice of quotients, see [51, Definition on page 7]. They satisfy $|\mathbf{F}_n| \simeq |\mathbf{S}_n|$.

Hence it is enough to show that the geometric realisation map

$$h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,T_\bullet,t) \to h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,|T_\bullet|,|t|)$$

admits a homotopy inverse. In the classical case the inverse map is given by composition of $\mathcal{S}_\bullet\colon h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,|T_\bullet|,|t|) \to h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,\mathcal{S}_\bullet(|T_\bullet|),t\circ|r|)$ and $r_*\colon h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,\mathcal{S}_\bullet(|T_\bullet|),t\circ|r|) \to h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,T_\bullet,t)$.

The unit and counit of the adjunction between geometric realisation and singular simplicial sets induce natural transformations between the compositions of these functors and the identity functors on the respective categories.

For the controlled case, we cannot use the singular simplicial sets functor directly, since there is no good control map on $\mathcal{S}_\bullet(Y)$ such that we obtain controlled homotopy equivalences $|\mathcal{S}_\bullet(Y)| \simeq Y$ for a controlled space Y.

We use an argument analogous to one we learned from Arthur Bartels and Paul Bubenzer. Similar to [13, Proposition 7.2] we have natural weak equivalences $\operatorname{hocolim}_{c\in\mathbb{N}} K(\mathfrak{R}_{hf}(W,B,T_\bullet,t))_c \to K(\mathfrak{R}_{hf}(W,B,T_\bullet,t))$, analogous for the topological case.

Let $c>0$. The c–controlled singular simplicial set functor $\mathcal{S}^c_\bullet$ is given on Y in degree n by those maps $\sigma\colon |\Delta^n| \to Y$ which satisfy $\operatorname{diam}(\sigma) \le c$.

The map $h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,T_\bullet,t)_c \to h\mathbf{F}_n\mathfrak{R}_{hf}(W,B,|T_\bullet|,|t|)_c$, given by geometric realisation, admits a homotopy inverse for every $c>0$. It is given by $r_* \circ \mathcal{S}^c_\bullet$.

The restrictions of the unit and counit of the adjunction as well as the usual homotopies show that we obtain controlled homotopy equivalences between the composed functors and the identity. □

The A–theory fibre sequence for bounded simple maps

The next step is to translate Waldhausen's model for the homotopy fibre sequence of the A–theory assembly map (whose homotopy fibre is the Whitehead spectrum) to bounded simplicial sets. The arguments closely follow Waldhausen's original train of thought and only at a few points we have to support it with slightly varied arguments to account for the generalisation to bounded simplicial sets.

First we introduce simple maps and show that they can be used as the weak equivalences of a Waldhausen category.

Definition 2.2.2.35. A bounded map $f\colon (X_\bullet,p) \to (Y_\bullet,q)$ is *simple*, if the preimage $|f|^{-1}(y)$ of every point y in $Y_\bullet$ is contractible.

[1] Originally Waldhausen used the approximation theorem. But it is not clear that his construction for the non-trivial part is compatible with retraction maps.

Remark 2.2.2.36. *Since every contractible preimage is necessarily controlled contractible, there is no additional condition in the controlled setting.*

Proposition 2.2.2.37 (compare [31, Proposition 2.1.8])**.** *For a map of boundedly finite simplicial sets $f\colon (X_\bullet, p) \to (Y_\bullet, q)$ the following are equivalent.*

1. *The map f is simple.*
2. *The preimage $|f|^{-1}(p)$ has the Čech homotopy type of a point for every element $p \in |Y_\bullet|$.*
3. *The map $|f|$ is cell-like.*
4. *The map $|f|$ is a hereditary homotopy equivalence, i.e. the restricted map $|f|^{-1}(U) \to U$ is a homotopy equivalence for each open subset $U \subseteq |Y_\bullet|$.*
5. *The map $|f|$ is a hereditary weak homotopy equivalence.*

Proof. Let $p \in |Y_\bullet|$. There is a finite simplicial subset $Y_\bullet(p)$ in $Y_\bullet$ with p contained in the interior of $|Y_\bullet(p)|$.

By [31, Lemma 2.1.4] applied to the restricted map $f\colon f^{-1}(Y_\bullet(p)) \to Y_\bullet(p)$, each preimage $|f|^{-1}(p)$ is a finite CW-complex, hence a compact ENR. So by [4, (8.6)], $|f|^{-1}(p)$ is contractible if and only if it has the Čech homotopy type of a point. Thus (1.) ⇔ (2.).

As explained following [31, Definition 2.1.5] every ENR is a finite dimensional, separable metric space. Thus $|f|^{-1}(p)$ is a finite-dimensional, compact, separable metric space. Thus (2.) ⇔ (3.) by the equivalence of (a) and (b) in [31, Theorem 2.1.6].

The map $|f|\colon |X_\bullet| \to |Y_\bullet|$ is proper, since every compact subset $K \subseteq |Y_\bullet|$ is contained in the geometric realisation of a finite simplicial set $B_\bullet(K)$ and the restricted map $|f^{-1}(B_\bullet(K))| \to |B_\bullet(K)|$ certainly is, as $|f^{-1}(B_\bullet(K))|$ is compact and $|B_\bullet(K)|$ is Hausdorff.

Let $U \subseteq |Y_\bullet|$ be an open subset which is contained in the geometric realisation of some finite simplicial subset $Y'_\bullet$ of $Y_\bullet$. Then the restricted map $f^{-1}(U) \to U$ is a homotopy equivalence (even a proper one) by [31, Theorem 2.1.7].

Now, consider an arbitrary open subset $U \subseteq |Y_\bullet|$. We obtain a filtration $U_i = U \cap q^{-1}(B \cap (-i, i)^n)$ of U with $i \in \mathbb{N}$. This yields a map

$$\mathrm{hocolim}_{i\in\mathbb{N}} |f|_i \colon \mathrm{hocolim}_{i\in\mathbb{N}} |f|^{-1}(U_i) \to \mathrm{hocolim}_{i\in\mathbb{N}} U_i$$

which is a weak homotopy equivalence, because each of the maps $|f|_i$ is.

By [35, Theorem 1(a) and (d)], each of the spaces $|f|^{-1}(U_i)$ and U_i has the homotopy type of a countable CW-complex. Hence the same holds for the homotopy colimits. Thus (3.) ⇒ (4.).

Clearly, (4.) ⇒ (5.).

Finally, we show (5.) ⇒ (3.). So consider a hereditary weak homotopy equivalence $f\colon X_\bullet \to Y_\bullet$ such that the preimage of every simplex of $Y_\bullet$ is a finite simplicial set.

Let $p \in |Y_\bullet|$. Let $U_p \subseteq B$ be an open neighbourhood of $q(p)$ with $q^{-1}(U_p)$ contained in a finite CW-complex $Y_\bullet(U_p)$.

We shall demonstrate below that $|f|$ is surjective and that for every p in $|Y_\bullet|$ the inclusion $|f|^{-1}(p) \subseteq Y_\bullet(U_p)$ has the property UV^∞. This is going to complete the proof, since $Y_\bullet(U_p)$ is an ENR, so by the equivalence of [31, Theorem 2.1.6(a) and (c)] each point inverse $|f|^{-1}(p)$ is cell-like which implies (3.).

The image $L = |f|(|X_\bullet|) \subseteq |Y_\bullet|$ is closed, since $|f|$ is closed, so $U = |Y_\bullet| - L$ is open. Its preimage $|f|^{-1}(U)$ is empty, so the restricted map $|f|^{-1}(U) \to U$ can only be a weak homotopy equivalence when U is empty, i.e. when $|f|$ is surjective.

It remains to verify the property UV^∞. Let $|f|^{-1}(p) \subseteq U \subseteq Y_\bullet(U_p)$ with U open. The complement $K = Y_\bullet(U_p) - U$ is closed, so its image $|f|(K) = |Y_\bullet|$ is closed and does not contain p. Each point in a CW-complex has arbitrarily small contractible open neighbourhoods [20, Proposition A.4], so $p \in |Y_\bullet|$ has a contractible open neighbourhood N that does not meet $|f|(K)$. By assumption, the restricted map $|f|^{-1}(N) \to N$ is a weak homotopy equivalence, so by defining $V = |f|^{-1}(N)$ we have obtained an open, weakly contractible neighbourhood of A that is contained in U.

Now $Y_\bullet(U_p)$ is an ENR, and thus an ANR, so its subset V is also an ANR. Thus V has the homotopy type of a CW-complex, by [35, Theorem 1(a) and (d)]. Hence the weakly contractible space V is in fact contractible, so the inclusion $V \subseteq U$ is in fact null-homotopic. □

Lemma 2.2.2.38. *The category $\mathfrak{R}_f(W, B, T_\bullet, t)$ with injective maps as cofibrations and simple maps as weak equivalences is a Waldhausen category.*

Proof. The two non-trivial statements are that simple maps are closed under composition and the gluing lemma. These can be shown analogously to [31, Proposition 2.1.3 (a) and (d)], respectively. This argument uses the different characterisations of simple maps given in Proposition 2.2.2.37. □

Eventually, the functors $s\mathfrak{R}_f(W, B, T_\bullet, t) \to h\mathfrak{R}_f(W, B, T_\bullet, t)$ shall induce homotopy fibre sequences, so we have to show exactness.

Lemma 2.2.2.39. *Every simple map $f\colon (X_\bullet, p) \to (Y_\bullet, q)$ is a bounded weak equivalence.*

Proof. Let $(Y_\bullet, q)$ be c–bounded. Let $\sigma\colon \Delta^n \to Y_\bullet$ be a simplex of $Y_\bullet$. Then the image of $|\sigma|$ is contained in $q^{-1}(U)$ for some open set $U \subseteq B$ with $\operatorname{diam}(U) \leq c$.

Since f is a hereditary homotopy equivalence, we can use induction over the simplicial degree to construct a homotopy inverse g of f such that $g \circ \sigma$ maps to $(q \circ f)^{-1}(U)$ which is c–bounded for every simplex σ. □

Now everything is ready to define the desired homotopy fibre sequence. Due to Waldhausen's work this requires little effort from our part.

Lemma 2.2.2.40. *There is a homotopy fibre sequence*

$$\begin{array}{c} s\mathbf{S}_\bullet\mathfrak{R}^h_f(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id}) \\ \big\downarrow \\ s\mathbf{S}_\bullet\mathfrak{R}_f(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id}) \\ \big\downarrow \\ h\mathbf{S}_\bullet\mathfrak{R}_f(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id}). \end{array}$$

Proof. The proof of [51, Theorem 3.3.1] carries over verbatim save for one technical point:

We denote the full subcategory of $\mathfrak{R}_f(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id})$ on the objects $(Y_\bullet, \kappa, r, s)$ with $s\colon W_\bullet^{\Delta^\bullet} \times B_\bullet \to Y_\bullet$ a 1–connected map with an upper index (2). We have to show that the the extension axiom holds for the weak equivalences in $\mathfrak{R}_f^{(2)}(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id})$.

This follows from the boundedly controlled Whitehead theorem of Anderson and Munkholm [2] and the long exact sequence of fragmented homology [2, (1.8)]. Via their example of a "boundedness control structure" [2, Example 1.1], "bounded homotopy equivalences" in the sense of Vogell correspond to "boundedly controlled homotopy equivalences" in the sense of [2]. □

Next we explain how $s\mathbf{S}_\bullet\mathcal{R}_f(|W_\bullet^{\Delta^\bullet}|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, b \times \mathrm{Id})$ is related to the assembly homology theory. Again, most of the work has already been taken care of.

Definition 2.2.2.41 ([51, Chapter 3.2])**.** Let $F\colon \mathrm{sSet} \to \mathrm{Top}$ be a functor. It is called *excisive*, if it satisfies the following properties:

1. The functor F commutes with colimits.
2. If $W_\bullet^0 \to W_\bullet^1$ is a cofibration and $W_\bullet^0 \to W_\bullet^2$ any map, then we obtain a homotopy Cartesian square

$$\begin{array}{ccc} F(W_\bullet^0) & \longrightarrow & F(W_\bullet^1) \\ \downarrow & & \downarrow \\ F(W_\bullet^2) & \longrightarrow & F(W_\bullet^1) \cup_{F(W_\bullet^0)} F(W_\bullet^2). \end{array}$$

An excisive functor is a *homology theory*, if it preserves weak equivalences.

Proposition 2.2.2.42 ([51, Proposition 3.2.4])**.** *Let $F\colon \mathrm{sSet} \to \mathrm{Top}$ be an excisive functor and suppose that $F(W_\bullet)$ is connected for every $W_\bullet$. Then the associated functor $W_\bullet \mapsto ([n] \mapsto F(W_\bullet^{\Delta^n}))$ is a homology theory.*

We fix a pair $(B_\bullet, b\colon (|B_\bullet|, b) \to B)$ of a simplicial set and a bounded homotopy equivalence.

Lemma 2.2.2.43. *The functor $W_\bullet \mapsto s\mathbf{S}_\bullet\mathfrak{R}_f(|W_\bullet|, B, W_\bullet \times B_\bullet, \mathrm{Id} \times b)$ is excisive.*

Proof. This is analogous to [51, Proposition 3.2.3]. □

Remark 2.2.2.44. *If we had a full analogy with Waldhausen's work, then we would now show that $W_\bullet \mapsto s\mathbf{S}_\bullet\mathfrak{R}_f(|W_\bullet|, B, W_\bullet^{\Delta^\bullet} \times B_\bullet, \mathrm{Id} \times b)$ is a model for the assembly homology theory.*

But the homotopy fibre $s\mathbf{S}_\bullet\mathfrak{R}_f^h(\Delta^0, B, (\Delta^0)^{\Delta^\bullet} \times B_\bullet, \mathrm{Id} \times b)$ of Lemma 2.2.2.40 for $W_\bullet = \Delta^0$ does not admit an obvious contraction. We strongly suspect that its K–theory vanishes. This is the case if and only if our suspected model $\Omega\,\mathrm{Wh}^?(-, B)$ is actually a model for $\Omega\,\mathrm{Wh}(-, B)$.

Remark 2.2.2.45. *Let* $\epsilon\colon |-|\circ \mathcal{S}_\bullet \Rightarrow \mathrm{Id}$ *denote the counit of the adjunction.*

Since the functor $W_\bullet \mapsto s\mathbf{S}_\bullet\mathfrak{R}_f(|W_\bullet|, B, \mathcal{S}_\bullet|W_\bullet| \times B_\bullet, \epsilon \times b)$ *preserves weak equivalences, the natural transformation which sends*

$$W_\bullet \mapsto s\mathbf{S}_\bullet\mathfrak{R}^h_f(|W_\bullet|, B, \mathcal{S}_\bullet|W_\bullet| \times B_\bullet, \epsilon \times b)$$

to the homology theory

$$W_\bullet \mapsto s\mathbf{S}_\bullet\mathfrak{R}^h_f(|W_\bullet^{\Delta_\bullet}|, B, \mathcal{S}_\bullet|W_\bullet^{\Delta^\bullet}| \times B_\bullet, \epsilon \times b)$$

is a weak equivalence by [51, Lemma 3.1.2].

Borrowing from the map $u\colon H^c_\bullet(M) \to \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_\bullet(M \times [0,1])$, we want to be able to interpret bundles of h–cobordisms over $|\Delta^n|$ as objects of our category for every $n \in \mathbb{N}$. We introduce an additional simplicial direction to be able to send an object in $H^c_n(M;\mathbb{R}^k)$ to a retractive space over $M \times [0,1] \times |\Delta^n| \times \mathbb{R}^k$.

Definition 2.2.2.46. We have a simplicial object $s\mathbf{S}_\bullet\mathfrak{R}^h_f(W, B, \mathcal{S})_\bullet$ given by $[n] \mapsto s\mathbf{S}_\bullet\mathfrak{R}^h_f(W \times |\Delta^n|, B, \mathcal{S}_\bullet(W \times |\Delta^n| \times B), \epsilon)$.

We introduce the suspected model we are going to use to define our natural transformation.

Definition 2.2.2.47. We set

$$\Omega\,\mathrm{Wh}^?(W,B) = \mathrm{hocolim}_{[n]\in\Delta}\, K(\mathfrak{R}^h_f(W \times |\Delta^n|, B, \mathcal{S}_\bullet(W \times |\Delta^n| \times B), \epsilon)).$$

Lemma 2.2.2.48. *By the universal property of the assembly homology theory we obtain a map* $\Omega\,\mathrm{Wh}^?(W,B) \to \Omega\,\mathrm{Wh}(W,B)$*. It is a weak equivalence for* $B = *$.

Proof. For $B = *$ we can use Waldhausen's original result [51, Addendum 3.2.2]. □

2.2.3 The natural transformation

With the recipient of the natural transformation defined we finally turn to the actual construction of the transformation. After generalising some results from the connected case to the controlled situation, in particular the 2–functor Ψ in Corollary 2.2.3.6, we can restate our task as a lifting problem. We show the existence of a lift in Proposition 2.2.3.9.

In the connected case we worked with relative polyhedra. In contrast to that we are now working with retractive simplicial sets. The spaces of retractions turn out to be contractible and we use triangulations, provided by Lemma 2.2.3.11, to construct the required lifts.

In order to see that we have constructed a natural transformation along weak equivalences we compare with the connected case for non-negative homotopy groups and otherwise observe that the isomorphisms to identify the negative homotopy groups are induced by the bounded Whitehead torsion on both the h–cobordism and the Whitehead spectrum. This is Lemma 2.2.3.12. Finally, we obtain our main theorem as Theorem 2.2.3.13.

We remind the reader that we introduced several abbreviations in Notation 2.2.1.15 which we are going to use throughout this part of the thesis.

We have an $(\infty,1)$–functor $|s\tilde{\mathcal{E}}^h_\bullet|\colon \mathcal{N}^{h.c.}_\bullet(\mathrm{Mfd}^{\mathrm{PL}}, \mathrm{emb})_\Delta \to \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta$ by Section 2.1.4. We define a controlled version $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(M,B)$ similar to the classical case Definition 2.1.1.11, but only work with fibre bundles instead of Serre fibrations because it makes control conditions easier to formulate.

Definition 2.2.3.1. Let (B,d) be a metric space. Let K be a compact polyhedron. The simplicial category $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K,B)$ consists of fibre bundles of polyhedra containing K as a bounded deformation retract and bounded simple PL maps. Precisely:

In simplicial degree q the objects of $s\mathcal{E}^h_q(K,B)$ are diagrams

$$\begin{array}{ccc} K\times|\Delta^q|\times B & \xrightarrow{\;s\;} & E \\ & \searrow^{\mathrm{pr}} & \downarrow{\scriptstyle \pi} \\ & & |\Delta^q| \end{array}$$

together with a proper control map $\tilde{q}\colon E\to B$. Here π is a PL bounded fibre bundle (i.e. a PL map whose underlying map of spaces is a fibre bundle such that there is a bounded PL isomorphism to the trivial bundle) of polyhedra and s is a PL embedding and a bounded homotopy equivalence.

A morphism $f\colon(\pi,s)\to(\pi',s')$ is a bounded simple PL map of relative fibrations $f\colon E\to E'$, i.e. we have $\pi = f\circ\pi'$ and $s' = s\circ f$.

Let $\iota\colon K\to K'$ be a bounded PL embedding. The construction $E\mapsto E\cup_K K'$ then induces a functor of simplicial categories $\iota_*\colon\ \mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K,B)\to \mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K',B)$ and further a functor $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(-,B)$ from compact polyhedra and PL embeddings to simplicial categories.

There is a stabilisation map $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K,B)\to \mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K\times[0,1],B)$.

Definition 2.2.3.2. We define admissible tuples of sequences as in the connective case. An *admissible retraction* over a choice simplex $\mathrm{ch}\in\mathrm{Ch}_\Delta(M,N)$ with respect to an admissible tuple $(M_i,\iota_i,\tilde{p}_i,\mathrm{im}(c_i))_{0\le i\le m-1}$ is a natural transformation $\alpha\colon u\circ\mathrm{c}\circ P(\mathrm{ch};B)\Rightarrow\Omega\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(\mathrm{ch},B)\circ u\circ\mathrm{c}$ along bounded simple retraction maps such that every map $\alpha\colon u\circ\mathrm{c}\circ P(\mathrm{ch};B)(F)\to\Omega\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(\mathrm{ch},B)\circ u\circ\mathrm{c}(F)$ satisfies the following two properties:

1. The retraction α is given by the standard simple retraction map pr (i.e. the projection) on $(M_{i+1}-\mathrm{im}(c_i)-\tau_{M_{i+1}}M_i)\times I\times B$ for every $0\le i\le m-1$.

2. Upon the choice of a parametrisation of $\mathrm{im}(c_i)$ as a collar we obtain a tubular neighbourhood $p'\colon\ \mathrm{im}(c_i)\cup\tau_{M_{i+1}}M_i\to M_i$ of M_i in M_{i+1} given by $p'=\tilde{p}_i\circ\mathrm{pr}_{\partial\tau_{M_{i+1}}M_i}\circ c_i^{-1}\cup\tilde{p}_i$.

   We set $U=c_{i-1}(\tau_{M_i}M_{i-1}\times[0,1))\cup\tau_{M_i}M_{i-1}$. Let p'' denote the pullback of p' along the subspace inclusion $U\subseteq M_i$. Let T'' denote the total space of this pulled back tubular neighbourhood.

   The restriction of the retraction map is fibre-preserving with respect to p', i.e. $\mathrm{pr}\circ(p'\times\mathrm{Id}_I)\colon(\mathrm{im}(c_i)\cup\tau_{M_{i+1}}M_i-T'')\times I\times B\to(M_i-U)\times[0,1]\times B$ and the restriction of $(p'\times\mathrm{Id}_{[0,1]})\circ\alpha$ to the same subspace coincide.

Lemma 2.2.3.3. *The map* $\mathrm{Ch}^R_\Delta(M,N)\to\mathrm{Ch}_\Delta(M,N)$ *is surjective for every pair of* PL *manifolds* M *and* N *in* Ch_Δ.

Proof. The proof of Lemma 2.1.5.10 carries over. □

Proposition 2.2.3.4. *The forgetful map* $\mathrm{Ch}^R_\Delta(M,N) \to \mathrm{Ch}_\Delta(M,N)$ *is a Kan fibration.*

Proof. This is analogous to Proposition 2.1.5.13. □

Proposition 2.2.3.5. *The forgetful map* $\mathrm{Ch}^R_\Delta(M,N) \to \mathrm{Ch}_\Delta(M,N)$ *is a trivial fibration.*

Proof. The proof of Proposition 2.1.5.14 carries over verbatim. □

Corollary 2.2.3.6. *There is a simplicial* 2*–functor* $\Psi\colon \mathrm{Ch}^R_\Delta \times [1] \to \mathrm{scat}_{\Delta^{\mathrm{cat}}}$, *given by:*

- $P(-;B)\colon \mathrm{Ch}^R_\Delta \times \{0\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$
- $\Omega\, \mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(-,B) \circ (- \times [0,1])\colon \mathrm{Ch}^R_\Delta \times \{1\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$
- $\mathrm{u} \circ \mathrm{c}\colon \mathrm{ob}\,\mathrm{Ch}^R_\Delta \times \{0 \leq 1\} \to \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$
- $\Psi\colon \mathrm{mor}\,\mathrm{Ch}^R_\Delta \times \{0 \leq 1\} \to \mathrm{scat}_{\Delta^{\mathrm{cat}}}$ *which sends* $[(\mathrm{ch}, \alpha, A)]$ *to* α.

It induces an $(\infty,1)$*–functor* $\mathcal{N}^{h.c.}_\bullet \mathrm{Ch}^R_\Delta \times [1] \to \mathcal{N}^{h.c.}_\bullet \mathrm{Top}_\Delta$ *upon geometric realisation, since* Top_Δ *is an* $(\infty,1)$*–category.*

We observe that the suspected model for the Whitehead spectrum

$$\Omega\,\mathrm{Wh}^?(W,B) = \mathrm{hocolim}_{[n]\in\Delta} K(\mathfrak{R}^h_f(M \times |\Delta^n|, B, \mathcal{S}_\bullet(M \times |\Delta^n| \times B), \epsilon))$$

yields a simplicially enriched functor $\Omega\,\mathrm{Wh}^?(-,B)\colon \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$, analogously to $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(-,B)\colon \mathrm{Ch}_\Delta \to \mathrm{scat}_\Delta$. The last preparation we need is to define a subspace of $\mathrm{Wh}^?(K,B)$ which we can compare to $\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(K,B)$.

Definition 2.2.3.7. We denote by $\tilde{\mathfrak{R}}^h_f(M \times |\Delta^n|, B, \mathcal{S}_\bullet(M \times |\Delta^n| \times B), \epsilon)$ a full subcategory of $\mathfrak{R}^h_f(M \times |\Delta^n|, B, \mathcal{S}_\bullet(M \times |\Delta^n| \times B), \epsilon)$. An object $(Y_\bullet, r, s)$ is in the subcategory if its geometric realisation is an object of $s\mathcal{E}^h_n(|M\times, B|)$, i.e. $(\mathrm{pr}_{|\Delta^n|} \circ \epsilon \circ |r|\colon |Y_\bullet| \cup_{|\mathcal{S}(M\times|\Delta^n|\times B)|} M \times |\Delta^n| \times B, |s|)$ is a relative bounded fibre bundle with respect to the control map induced by $\epsilon \circ \mathrm{pr}_{\mathcal{S}(B)} \circ |r|\colon |Y_\bullet| \to B$.

We write $\Omega|\mathfrak{R}(-,B)| = \mathrm{hocolim}_{[n]\in\Delta}\, \Omega|s\tilde{\mathfrak{R}}^h_f(-\times|\Delta^n|, B, \mathcal{S}_\bullet(-\times|\Delta^n|\times B), \epsilon)|$ to ease up notation.

Definition 2.2.3.8. There is a natural transformation $\mathcal{N}(w\mathcal{C}) \to w_\bullet \mathbf{S}_1(\mathcal{C})$ of functors from Waldhausen categories to simplicial sets. Together with the inclusion $w_\bullet \mathbf{S}_1(\mathcal{C}) \to w_\bullet \mathbf{S}_\bullet(\mathcal{C})$, we obtain a natural transformation of $(\infty,1)$–functors

$$\Omega|\mathfrak{R}(-,B)|| \Rightarrow \Omega\,\mathrm{Wh}^?(-,B)$$

and another one by geometric realisation

$$\Omega|\mathfrak{R}(-,B)| \Rightarrow |\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(-,B)|.$$

The next proposition yields the desired natural transformation.

Proposition 2.2.3.9. *The lifting problem in $(\infty, 1)$–functors from $\mathcal{N}_\bullet^{h.c.} \operatorname{Ch}_\Delta^R$ to $\mathcal{N}_\bullet^{h.c.} \operatorname{Top}_\Delta$*

$$\begin{array}{ccc} & & \Omega|\mathfrak{R}(- \times [0,1], B)| \\ & \overset{\Psi^D}{\nearrow} & \downarrow \\ P(-;B) & \xrightarrow{\Psi} & \Omega|\operatorname{s}\mathcal{E}_\bullet^{\mathrm{h}}(- \times [0,1], B)| \end{array}$$

admits a lift which factors, for every PL *manifold $M \in \mathcal{N}_0^{h.c.} \operatorname{Ch}_\Delta^R$, as a composition $u^D \circ c^+ \colon P(M;B) \to \Omega H^c(M;\mathbb{R}^k) \to \Omega|\mathfrak{R}(M \times [0,1], B)|$ over the classifying map $c^+ \colon P(M;B) \to \Omega H^c(M;\mathbb{R}^k)$.*

Before we show this result we give a criterion for triangulations of locally finite polyhedra. Essentially, we exhaust these polyhedra with compact subspaces and use relative triangulation results for pairs of compact polyhedra to construct a triangulation of the full space.

Definition 2.2.3.10. Let $\mathcal{E}_\bullet(W,B) \subseteq \mathfrak{R}(W,B,T_\bullet,t)$ denote the subcategory with objects (Y,r,s), where $Y \subseteq \mathbb{R}^\infty$ is a piecewise linearly retractive polyhedron over $W \times B$, and piecewise linear maps compatible with sections and retractions as morphisms.

Lemma 2.2.3.11. *Let $k \in \mathbb{N}$. Let $E \colon [k] \to \mathcal{E}_\bullet(W,B)$ denote a functor and $W' \subseteq W$ a sub-polyhedron.*

Let $E' \colon [k] \to \mathcal{E}_\bullet(W',B)$ denote a sub-functor of E, i.e. a functor, such that $E'(i) \subseteq E(i)$ for every $0 \leq i \leq k$ and every morphism in $[k]$ is mapped to the restriction of its image under E.

Let $(T'_\bullet, t')$ be a triangulation of $W' \times B$ and $(T_{E'}, t_{E'})$ a triangulation of E' over $T'_\bullet$, i.e. a functor $T_{E'} \colon [k] \to \mathfrak{R}(W', B, T'_\bullet, t')$ and a natural transformation $t_{E'} \colon |-| \circ T_{E'} \Rightarrow E'$ along isomorphisms.

Then there is a triangulation $(T_\bullet, t)$ of $W \times B$ and a triangulation (T_E, t_E) of E over $T_\bullet$ such that $(T_\bullet, t)$ restricts to a subdivision of $(T'_\bullet, t')$ over $W' \times B$ and $(T_E(i), t_E(i))$ restricts to a subdivision of $(T_{E'}(i), t_{E'}(i))$ for every $0 \leq i \leq k$.

Moreover, if we have $(E, E') \cong (E' \times |\Delta^1| \cup_{E' \times \{1\}} E, E' \times \{0\})$, a natural isomorphism of pairs, then we may assume that $(T_\bullet, t)$ and $(T_E(i), t_E(i))$ restrict to $(T'_\bullet, t')$ and $(T_{E'}(i), t_{E'}(i))$, respectively.

Proof. If B is a compact polyhedron this follows from [27, Theorem 1.11], compare [31, Lemma 3.4.8]. We note that our polyhedra inherit from $\mathbb{R}^\infty$ a preferred PL structure, see also [43].

For the general case we consider the sequence of polyhedra B_n for $n \in \mathbb{N}$ with $B_n = B \cap [-n,n]^\infty$. We have $B = \bigcup_{n \in \mathbb{N}} B_n$. Let K_n denote the closure of $B_{n+1} - B_n$.

We show by induction that there is a triangulation of $E|_{B_{n+1}}$ which leaves the triangulation of $E|_{B_{n-1}}$ unchanged and restricts to a triangulation of $E|_{\partial B_n}$ for every $n \in \mathbb{N}$.

We use the given triangulation $T(n)$ of $E|_{B_n}$, which leaves $E|_{B_{n-2}}$ unchanged, on $E|_{B_{n-1}}$ and choose a triangulation $T_K(n)$ of $E|_{K_n}$ via the compact case such that it restricts to a triangulation of both $E|_{\partial B_{n+1}}$ and $E|_{\partial B_n}$. Moreover the restriction of $T_K(n)$ to $E|_{\partial B_n}$ subdivides the restriction of $T(n)$.

All that is left is to triangulate $E|_{K_{n-1}}$ relative to the fixed triangulations on $K_{n-1} \cap B_{n-1}$ and $K_n \cap B_n$. We note that $T(n)$ is compatible with the fixed triangulation on K_{n-1}. The following argument is similar to the one after the proof of [31, Lemma 3.4.8].

Let $E_i(n)$ denote the polyhedron $E|_{K_{n-1}}(i)$. Its retraction and section maps are $r_i(n)$ and $s_i(n)$, respectively.

Consider $E_k(n)$ with the given triangulation $T(n)$. Let $E_k(n,K) \subseteq E_k(n)$ denote the sub-polyhedron generated by the simplices which have non-empty intersection with $r_k(n)^{-1}(K_n)$ but are not fully contained in $r_k(n)^{-1}(K_n)$. We use induction over the simplicial degree of the simplices to find a subdivision of the triangulation of each simplex which extends the triangulation on its boundary.

A triangulation of the desired form exists by starring each simplex, i.e. we form the cone over the triangulation of its boundary with cone point some point in the interior of the simplex, see [31, Definition 3.2.11]. The new cone vertex is always numbered as the last vertex of each simplex.

Now we proceed by induction over the elements of $[k]$, starting with k and descending. Since every morphism is compatible with the retraction maps we have $E_{i-1}(n,K) = E|_{B_n}(i-1 \leq i)^{-1}(E_i(n,K))$. We can choose each cone point for the starring of a simplex in $E_{i-1}(n,K)$ such that their image under $E|_{B_n}(i-1 \leq i)$ is a cone point in $(E_i(n,K))$, because for every surjective map $|\Delta^p| \to |\Delta^q|$ the interior of $|\Delta^p|$ maps onto the interior of $|\Delta^q|$.

For the addendum we use the subdividing triangulation on $E = E \cup E' \times \{1\}$ and the given triangulation on $E' \times \{0\}$. One obtains the desired triangulation of $E' \times |\Delta^1|$ by inductively starring simplices, analogously to the argument following the proof of [31, Lemma 3.4.8] (and similar to the above argument). □

Proof of Proposition 2.2.3.9. We use induction over the simplicial degree of $\mathcal{N}_\bullet^{h.c.} \mathrm{Ch}_\Delta^R$. Let $G \colon \mathfrak{S}(\Delta^k) \to \mathrm{Ch}_\Delta^R$ be a k–simplex. We are going to refer to restrictions of Ψ and G by the same notation. To extend our functor to G we have to solve lifting problems

$$\begin{array}{ccc}
\mathfrak{S}(\Delta^k \times \partial\Delta^1 \cup \partial\Delta^k \times \Delta^1) & \xrightarrow{\Psi^D(\partial G)} & \mathrm{Top}_\Delta \\
\downarrow & \overset{\Psi^D(G)}{\nearrow} & \downarrow \mathrm{Id} \\
\mathfrak{S}(\Delta^k \times \Delta^1) & \xrightarrow{\Psi(G)} & \mathrm{Top}_\Delta
\end{array}$$

where the lower triangle commutes up to natural transformation. By the usual reduction argument this is equivalent to a lifting problem on the largest mapping spaces

$$\begin{array}{ccc}
\mathfrak{S}(\partial(\Delta^k \times \Delta^1))((0,0),(k,1)) & \longrightarrow & \mathrm{Top}_\Delta(P(G(0);B), \Omega|\mathfrak{R}(G(k) \times [0,1], B)|) \\
\downarrow & \overset{\Psi^D(G)}{\nearrow} & \downarrow \\
\mathfrak{S}(\Delta^k \times \Delta^1)((0,0),(k,1)) & \xrightarrow{\Psi(G)} & \mathrm{Top}_\Delta(P(G(0);B), \Omega|\,\mathrm{s}\mathcal{E}_\bullet^{\mathrm{h}}(G(k) \times [0,1], B)|)
\end{array}$$

with the upper horizontal map given by $\Psi^D(G \times \{0,1\}) \cup \Psi^D(\partial G \times \Delta^1)$.

We can reduce further by the mapping space adjunction and induction over the simplicial degree of $P_\bullet(G(0);B)$. Let $F \in P_n(G(0);B)$ be an n–simplex.

We denote $|\Delta^k \times \Delta^1 \times \Delta^n|$ by Q and its boundary

$$|\Delta^k \times \Delta^1 \times \partial\Delta^n \cup \Delta^k \times \partial\Delta^1 \times \Delta^n \cup \partial\Delta^k \times \Delta^1 \times \Delta^n|$$

by ∂Q. We also observe that they satisfy $(Q, \partial Q) \cong (Q \cup_{\partial Q \times \{1\}} \partial Q \times |\Delta^1|, \partial Q \times \{0\})$. Now we have to find a lift

$$\begin{array}{ccc} \partial Q & \longrightarrow & \Omega\mathfrak{R}(G(k) \times [0,1], B) \\ \big\downarrow & \overset{\Psi^D(G)(F)}{\nearrow} & \big\downarrow \\ Q & \xrightarrow[\Psi(G)(F)]{} & \Omega|\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(G(k) \times [0,1], B)| \end{array}$$

with the upper horizontal map given by

$$\Psi^D(G \times [1])(\partial F) \cup \Psi^D(G \times \{0,1\})(F) \cup \Psi^D(\partial G \times \Delta^1)(F).$$

We first consider the case $k = 0$. We have to find a map

$$\mathrm{u}^D \circ \mathrm{c}\colon P(G(0), B) \to \Omega|\mathfrak{R}(G(0) \times [0,1], B)|$$

which lifts $\mathrm{u} \circ \mathrm{c}\colon P(G(0), B) \to \Omega|\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(G(0) \times [0,1], B)|$.

By definition of $\mathrm{u} \circ \mathrm{c}$ as the geometric realisation of a functor of simplicial categories it is enough to find a triangulation of $\mathrm{u} \circ \mathrm{c}(F)$, i.e. a triangulation of the collar $G(0) \times [0,1] \times |\Delta^n| \times B \to \mathrm{u} \circ \mathrm{c}(F)$, together with a retraction map $\mathrm{u} \circ \mathrm{c}(F) \to \mathcal{S}_\bullet(G(0) \times [0,1] \times |\Delta^n| \times B)$, both relative to the boundary.

By adjunction, the desired retraction is equivalent to a retraction map $|\,\mathrm{u} \circ \mathrm{c}(F)| \to G(0) \times [0,1] \times |\Delta^n| \times B$ which restricts to a given retraction $|\,\mathrm{u} \circ \mathrm{c}(\partial F)| \to G(0) \times [0,1] \times |\partial\Delta^n| \times B$. The retraction exists by Lemma 2.2.1.6.

The retractive fibre bundle of h–cobordisms admits a triangulation relative to the triangulation of the boundary by Lemma 2.2.3.11.

Now we construct the lift in the case $k > 0$. By definition of Ψ as a 2–functor on simplicial categories enriched over simplicial categories it is enough to find the following two diagrams: We set $N(G, B) = G(k) \times [0,1] \times |\Delta^k \times \Delta^n| \times B$. First, we need a simple map

$$\alpha^D\colon\ \mathrm{u}^D \circ \mathrm{c}(P(G; B)(F)) \to \Omega\mathfrak{R}(G, B)(\mathrm{u}^D \circ \mathrm{c}(F))$$

relative to

$$\mathrm{u}^D \circ \mathrm{c}(P(\partial G; B)(F)) \to \Omega\mathfrak{R}(\partial G, B)(\mathrm{u}^D \circ \mathrm{c}(F))$$

and

$$\mathrm{u}^D \circ \mathrm{c}(P(G; B)(\partial F)) \to \Omega\mathfrak{R}(G, B)(\mathrm{u}^D \circ \mathrm{c}(\partial F))$$

such that its realisation

$$\begin{array}{c} |\,\mathrm{u}^D \circ \mathrm{c}(P(G; B)(F))| \cup_{|\mathcal{S}(N(G,B))|} N(G, B) \\ \big\downarrow \\ |\Omega\,\mathrm{s}\mathcal{E}^{\mathrm{h}}_\bullet(G, B)(\mathrm{u}^D \circ \mathrm{c}(F))| \cup_{|\mathcal{S}(N(G,B))|} N(G, B) \end{array}$$

is the simple retraction map

$$\Psi(G \times [1])(F)\colon\ \mathrm{u} \circ \mathrm{c}(P(G; B)(F)) \to \Omega\, \mathrm{s}\mathcal{E}^{\mathrm{h}}_{\bullet}(G, B)(\mathrm{u} \circ \mathrm{c}(F)).$$

Here, the simplicial loop category $\Omega\mathfrak{R}(G, B)$ is defined analogously to the simplicial diagram category $\Omega\, \mathrm{s}\tilde{\mathcal{E}}^{\mathrm{h}}_{\bullet}$ given right before Definition 2.1.5.7.

Second, we require a zig-zag

$$\begin{array}{c} \mathrm{u}^D \circ \mathrm{c}(P(G; B)(F)) \\ \downarrow \\ (\mathrm{u}^D \circ \mathrm{c}(P(G; B)(F))) \times \Delta^1 \\ \uparrow \\ \mathrm{u}^D \circ \mathrm{c}(P(G; B)(F)) \end{array}$$

where the upper line is equipped with a retraction map which makes α^D into a morphism of retractive simplicial sets over $\mathcal{S}_\bullet(G(k) \times [0, 1] \times |\Delta^k \times \Delta^n| \times B)$ while the lower line carries the retraction map obtained from $\mathrm{u}^D \circ \mathrm{c}$.

To achieve this we have to take care of retractions and the simplicial structure. For the former we again apply the adjunction between singular simplicial sets and geometric realisation and use Lemma 2.2.1.6 to obtain the desired retractions.

For the existence of a map of simplicial sets, which lifts the given piecewise linear admissible retraction map, we once again defer to the general result on the existence of triangulations given in Lemma 2.2.3.11. □

We also have to see that our construction is compatible with the structure maps of the spectra. Recall that the k–th structure map of the pseudoisotopy spectrum is induced by the homotopy Cartesian square

$$\begin{array}{ccc} P(M; \mathbb{R}^k) & \longrightarrow & P(M; \mathbb{R}^k \times \mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ P(M; \mathbb{R}^k \times \mathbb{R}_{\leq 0}) & \longrightarrow & P(M; \mathbb{R}^{k+1}) \end{array}$$

where each map is a composition of the map $P(M; \mathbb{R}^k) \to P(M \times [0, 1]; \mathbb{R}^k)$ and a map of the form $i_*\colon P(M \times [0, 1]; \mathbb{R}^k) \to P(M; \mathbb{R}^k \times \mathbb{R}_{\geq 0})$ which extends with the identity.

Similarly, the k–th structure map of the suspected loops of the Whitehead spectrum is induced by the homotopy Cartesian square

$$\begin{array}{ccc} \Omega\, \mathrm{Wh}^?(M, \mathbb{R}^k) & \longrightarrow & \Omega\, \mathrm{Wh}^?(M, \mathbb{R}^k \times \mathbb{R}_{\geq 0}) \\ \downarrow & & \downarrow \\ \Omega\, \mathrm{Wh}^?(M, \mathbb{R}^k \times \mathbb{R}_{\leq 0}) & \longrightarrow & \Omega\, \mathrm{Wh}^?(M, \mathbb{R}^{k+1}) \end{array}$$

where each map is a composition of $\Omega\, \mathrm{Wh}^?(M, \mathbb{R}^k) \to \Omega\, \mathrm{Wh}^?(M \times [0, 1], \mathbb{R}^k)$ and a map of the form $i_*\colon \Omega\, \mathrm{Wh}^?(M \times [0, 1], \mathbb{R}^k) \to \Omega\, \mathrm{Wh}^?(M, \mathbb{R}^k \times \mathbb{R}_{\geq 0})$ which is induced by pushout along some inclusion $[0, 1] \subseteq \mathbb{R}_{\geq 0}$.

Since the first map of each of these compositions is part of the functorial structure of $P(-;\mathbb{R}^k)\colon \mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta \to \mathrm{Top}_\Delta$, respectively $\Omega\,\mathrm{Wh}^?(-,\mathbb{R}^k)$, they are certainly compatible with our natural transformation.

For the latter, consider some $F \in P(M \times [0,1];\mathbb{R}^k)$. Then we can use admissible retractions $\mathrm{u} \circ \mathrm{c}(i_*(F)) \to i_*(\mathrm{u} \circ \mathrm{c}(F))$ which are given by the canonical retraction map on the trivial bundle over $\mathbb{R}_{\geq 0} - [0,1)$. We obtain a commutative square

$$\begin{array}{ccc}
\mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta & \xrightarrow{\Psi^{(D,\mathbb{R}^k)}} & \mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta^{\mathbb{R}^k,R} \\
\downarrow & & \downarrow \\
\mathcal{N}_\bullet^{h.c.}\,\mathrm{Ch}_\Delta^{\mathbb{R}^k\times\mathbb{R}_{\geq 0},R} & \xrightarrow{\Psi^{(D,\mathbb{R}^k\times\mathbb{R}_{\geq 0})}} & \mathcal{N}_\bullet^{h.c.}\,\mathrm{Top}_\Delta
\end{array}$$

and with respect to these choices the level wise natural transformations of $(\infty,1)$–functors commute.

By the universal property of the assembly homology theory we obtain a commutative diagram of $(\infty,1)$–functors into $\mathcal{N}_\bullet^{h.c.}\,\mathrm{Spectra}_\Delta$

$$\begin{array}{ccc}
\Omega\,\mathrm{Wh}^{?,-\infty} & \longrightarrow & \Omega\,\mathrm{Wh}^{-\infty} \\
\downarrow & & \downarrow \\
h & \longrightarrow & (-)_+ \wedge A(*) \\
\downarrow & & \downarrow \\
A & \longrightarrow & A
\end{array}$$

where we made every object into an $(\infty,1)$–functor analogously to $\mathrm{s}\mathcal{E}_\bullet^{\mathrm{h}}(-,B)$. Also, h denotes the homology theory of Proposition 2.2.2.42 which makes the left hand side into a homotopy fibre sequence. In particular, we obtain a natural transformation $\mu\colon \Omega\,\mathrm{Wh}^{?,-\infty} \Rightarrow \Omega\,\mathrm{Wh}^{-\infty}$.

The last result we need is to show that the map $\mu \circ \mathrm{u}^D \circ c^+$ is actually a weak equivalence. Basically, this result is already contained within the literature and we only have to assemble the pieces.

Lemma 2.2.3.12. *The map $\mu \circ \mathrm{u}^D \circ \mathrm{c}\colon \mathcal{P}(M) \to \Omega\,\mathrm{Wh}^{-\infty}(M)$ of spectra induces an isomorphism on stable homotopy groups.*

Proof. We only have to show that $\mu \circ \mathrm{u}^D\colon \mathcal{H}(M) \to \Omega\,\mathrm{Wh}^{-\infty}(M)$ is a π_*–isomorphism by Lemma 2.2.1.23. Furthermore, it is enough to consider u^D for non-negative homotopy groups, since μ is a weak equivalence in the connected case by Lemma 2.2.2.48. We first note that both spectra are Ω–spectra by Proposition 2.2.1.24 and [46, p. 168], respectively.

For non-negative homotopy groups we note that our triangulation lemma yields a map $t\colon \mathbb{H}_\bullet(M) \to \mathrm{colim}_{n\in\mathbb{N}}\, \mathrm{s}\widetilde{\mathcal{D}}_\bullet^{\mathrm{h}}(X_\bullet(n))$ where $X_\bullet(n)$ is a triangulation of $M \times [0,1] \times (D^1)^n$ such that $r \circ t = u$ holds, so t is a weak equivalence.

The map $\mathbb{H}(M) \to \mathrm{hocolim}_{[n]\in\Delta}\, K(\mathfrak{R}_f^h(M\times|\Delta^n|,*,\mathcal{S}_\bullet(M\times|\Delta^n|),\epsilon))$ factors over $\mathrm{hocolim}_{[n]\in\Delta}\, \Omega|\mathbf{N}_\bullet(\mathfrak{R}_f^h(M\times|\Delta^n|,*,\mathcal{S}_\bullet(M\times|\Delta^n|),\epsilon))|$ which we abbreviate to $\mathrm{hocolim}_{[n]\in\Delta}\, \Omega|\mathbf{N}_\bullet(\mathfrak{R}_f^h(M(n)))|$. It is not hard to define analogues of the remainder of Waldhausen's zig-zag recalled in Theorem 2.1.1.2 such that the square

$$\begin{array}{ccc} \mathbb{H}(M) & \longrightarrow & \operatorname{colim}_{n\in\mathbb{N}} |\, \mathrm{s}\tilde{\mathcal{D}}^{\mathrm{h}}_\bullet(X_\bullet(n))| \\ \downarrow & & \downarrow \\ \operatorname{hocolim}_{[n]\in\Delta} \Omega|\mathbf{N}_\bullet(\mathfrak{R}^h_f(M(n)))| & \longrightarrow & \operatorname{colim}_{n\in\mathbb{N}} \Omega|\mathbf{N}_\bullet(\mathcal{C}^h_\bullet(X_\bullet(n)))| \end{array}$$

commutes. Due to the connected case this is enough.

For non-positive homotopy groups we note that there is a commutative diagram

$$\begin{array}{ccc} \pi_0(\mathbb{H}(M;\mathbb{R}^k)) & \xrightarrow{(\mu\circ \mathrm{u}^D)_*} & \pi_0(\Omega\,\mathrm{Wh}(M\times[0,1],\mathbb{R}^k)) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\lambda} \\ \tilde{K}_{-k+1}(\mathbb{Z}\pi_1(M)) & \xrightarrow{\cong} & \tilde{K}_{-k+1}(\mathbb{Z}\pi_1(M\times I)) \end{array}$$

where the map $\lambda\colon \pi_0(\underline{\mathrm{Wh}}^{\mathrm{PL}}(M\times I,\mathbb{R}^k)) \to \tilde{K}_{-k+1}(\mathbb{Z}\pi_1(M))$ is the induced linearisation map. It is defined as follows. The linearisation defined by Vogell [46] is a natural transformation $\lambda\colon A \to K\circ\mathbb{Z}\pi_1$. By the universal property of assembly, we obtain an induced linearisation on assembly and thus on the homotopy fibre, i.e. on the loops of the Whitehead spectrum.

To see that the diagram commutes we note that sending a bounded retractive CW-complex to its relative cellular chain complex describes a functor on $\mathcal{R}_f(M;\mathbb{R}^k)$. As was pointed out by John Klein [11, p. 43] this functor is not exact, but close enough to obtain an induced map regardless if one works with Thomason's variation of the $\mathbf{S}_\bullet$–construction defined at the end of [51, § 1.3]. Using this model of the linearisation map, checking commutativity is easy.

Since Vogell's linearisation induces an isomorphism on negative homotopy groups [46], the same holds for the induced linearisation. By Corollary 2.2.1.30, the map τ is an isomorphism. Therefore, the induced map $(\mathrm{u}^D\circ\mathrm{c})_*$ is an isomorphism. □

Theorem 2.2.3.13. *Let* Cat = Top, PL. *There is a natural weak equivalence of* $(\infty,1)$*–functors*

$$\Psi\colon \mathcal{P}^{\mathrm{Cat}} \Rightarrow \Omega^2\,\mathrm{Wh}^{\mathrm{Cat},-\infty}$$

from the $(\infty,1)$*–functor* $\mathcal{P}^{\mathrm{Cat}}\colon \mathcal{N}^{h.c.}_\bullet\,\mathrm{Top}_\Delta \to \mathcal{N}^{h.c.}_\bullet\,\mathrm{Spectra}_\Delta$ *of pseudoisotopies to the twofold loops of the* $(\infty,1)$*–functor given by the Whitehead spectrum.*

In particular, there is a zig-zag of natural weak equivalences between the strict functors $\mathcal{P}^{\mathrm{Cat}}\colon \mathrm{Top}\to\mathrm{Spectra}$ *and* $\Omega^2\,\mathrm{Wh}^{\mathrm{Cat},-\infty}$.

Proof. The argument is, for the most part, analogous to the connective case. We use Proposition 2.2.3.9 and the natural transformations preceding it to obtain a natural transformation $\Psi^{-\infty}\colon \mathcal{P}\Rightarrow\Omega\,\mathrm{Wh}^{-\infty}$ of $(\infty,1)$–functors from $\mathcal{N}^{h.c.}_\bullet\,\mathrm{Ch}_\Delta$ to $\mathrm{Spectra}_\Delta$.

By Lemma 2.2.3.12 it is a natural transformation along weak equivalences. The zig-zag of strict natural transformations follows from Theorem 1.2.1.3, since all spectra involved are cofibrant and fibrant in the model structure on prespectra, see [5, Theorem 2.3]. □

2.2.4 The smooth case

We follow [31, Proof of the DIFF case of Theorem 0.1, and Theorem 0.3] in which the stable parametrised h–cobordism theorem in the smooth category is shown. We could have given a similar statement in the connective setting, but doing so here avoids issues concerning π_0. The connective case follows by comparison with the non-connective case.

Remark 2.2.4.1. *In the following arguments we use results formulated for smooth manifolds with boundary as if they applied to manifolds with corners. Whenever this is the case one can use Lemma 1.2.2.1 to generalise. The key point is that $\pi_*\mathcal{P}$ sends homotopy equivalences of spaces to isomorphisms.*

Theorem 2.2.4.2. *Let $i\colon \mathcal{P}^{\mathrm{Diff}} \to \mathcal{P}^{\mathrm{Top}}$ be a natural transformation which extends the object-wise inclusion $\mathcal{P}^{\mathrm{Diff}}(X) \to \mathcal{P}^{\mathrm{Top}}(X)$, see Corollary 2.2.1.18, and let $\mathrm{Wh}^{\mathrm{Diff},-\infty} \to \mathrm{Wh}^{\mathrm{Top},-\infty}$ be a natural transformation (unique up to contractible homotopy) induced by the universal property of the assembly homology theory $X \mapsto \Sigma^\infty X_+ \wedge A(*)$.*

There is a natural equivalence of $(\infty,1)$–functors $\mathcal{P}^{\mathrm{Diff}} \to \Omega^2\,\mathrm{Wh}^{\mathrm{Diff},-\infty}$ such that the square

$$\begin{array}{ccc}
\mathcal{P}^{\mathrm{Diff}} & \longrightarrow & \Omega^2\,\mathrm{Wh}^{\mathrm{Diff},-\infty} \\
\Big\downarrow{\scriptstyle i} & & \Big\downarrow \\
\mathcal{P}^{\mathrm{Top}} & \xrightarrow{\mu\circ\Psi^D} & \Omega^2\,\mathrm{Wh}^{\mathrm{Top},-\infty}
\end{array}$$

commutes up to homotopy.

Proof. We only consider the restrictions of $\mathcal{P}^{\mathrm{Diff}}$ and $\mathrm{Wh}^{\mathrm{Diff}}$ to smooth manifolds with corners since both functors coincide with the homotopy left Kan extensions of their restrictions.

Via Goodwillie Calculus for quasicategories, see [34, Chapter 7], we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathrm{hofib}(i) & \longrightarrow & \mathcal{P}^{\mathrm{Diff}} & \xrightarrow{i} & \mathcal{P}^{\mathrm{Top}} \\
\Big\downarrow{\scriptstyle \simeq} & & \Big\downarrow & & \Big\downarrow \\
\mathrm{hofib}(i^S) & \longrightarrow & \mathcal{P}^{S,\mathrm{Diff}} \simeq * & \xrightarrow{i^S} & \mathcal{P}^{S,\mathrm{Top}}.
\end{array}$$

We have $\mathrm{hofib}(F(a))^S \simeq \mathrm{hofib}(F^S(a))$ for every natural transformation $a\colon F \to G$. Moreover, for any functor F there is a natural transformation $F \to F^S$, which is an equivalence if F is homological.

In the connective case the fibre $\mathrm{hofib}(i)$ is homological with Top/O as coefficients by [6, §5]. Further, the negative homotopy groups of $\mathcal{P}^{\mathrm{Diff}}$ and $\mathcal{P}^{\mathrm{Top}}$ coincide by [53, Corollary 5.3]. Since the negative homotopy groups of Top/O are trivial, this is enough to show that $\mathrm{hofib}(i)$ is homological in the non-connective case as well.

The functor $\mathcal{P}^{S,\mathrm{Diff}}$ is contractible by Morlet's disjunction lemma [7, §1] as explained in [23, Lemma 5.4]. The argument given there also implies that $\mathcal{P}^{S,\mathrm{Diff}}$ is contractible as a non-connective spectrum. Hence, the homotopy fibre of $\mathcal{P}^{\mathrm{Top}} \to \mathcal{P}^{S,\mathrm{Top}}$ is given by $\mathcal{P}^{\mathrm{Diff}}$. Using the topological case and yet again Goodwillie Calculus we obtain a commutative diagram

$$\begin{array}{ccccc}
\Omega^2 \operatorname{fib} & \xrightarrow{\simeq} & \operatorname{hofib} & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 A & \longrightarrow & \mathcal{P}^{\mathrm{Top}} & \longrightarrow & \Sigma\Omega^2 h(-, A(*)) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 A^S & \longrightarrow & \mathcal{P}^{S,\mathrm{Top}} & \longrightarrow & \Sigma\Omega^2 h(-, A(*))
\end{array}$$

where fib denotes the homotopy fibre of $A \mapsto A^S$. One last application of Goodwillie Calculus yields the commutative diagram

$$\begin{array}{ccccc}
* & \longrightarrow & \Omega^2 \operatorname{fib} & \longrightarrow & \Omega^2 \mathrm{Wh}^{\mathrm{Diff},-\infty} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 h(-, \Sigma^\infty S^0) & \longrightarrow & \Omega^2 A & \longrightarrow & \Omega^2 \mathrm{Wh}^{\mathrm{Diff},-\infty} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^2 h(-, \Sigma^\infty S^0) & \longrightarrow & \Omega^2 A^S & \longrightarrow & \Omega^2 \mathrm{Wh}^{S,\mathrm{Diff},-\infty}\,.
\end{array}$$

The composition $h(-, \Sigma^\infty S^0) \to A \to A^S$ is a weak equivalence, because it is an isomorphism on non-negative homotopy groups by [52] and A^S is a connective homology theory, since A preserves n–connected maps [49, Proposition 2.3]. Hence $\mathrm{Wh}^{S,\mathrm{Diff},-\infty}$ is contractible.

Since the space of homotopy fibres is a contractible Kan-complex we obtain the natural transformation via the obstruction theory of Section 1.2.1. This gives the result in the case of a smooth manifold. □

Conclusion and Outlook

The problems addressed throughout this thesis are formulated in the language of algebraic topology while concerning objects which are geometric in nature.

We have constructed homotopy coherent diagrams in the language of quasicategories to reduce these problems to purely geometric questions. Although their solutions are often stated in fairly algebraic terms, the underlying key arguments are anything but. In fact, the steps the author deems most important are best described as part of point set topology. In every case, the difficulty arises from the fact that we have to find a “parametrised” or “coherent” version of an otherwise fairly simple task.

In the first chapter we constructed a functor of smooth pseudoisotopies by first passing to a category of choices and onwards to spectra. The first part is explained in Section 1.2.3, Section 1.2.4 and Section 1.2.5, all of which are of the following form: First, we use the tautological obstruction theory of Section 1.2.1 to reduce to certain extension problems. Then we use a well-known geometric fact to show that the desired choices can be made.

For the second part, we use a similar approach. However, the underlying geometry is less common and thus requires a careful argument. In Section 1.4 we again reduce to a geometric question while the geometry is taken care of in Section 1.3. From the author’s perspective, the main geometric result Lemma 1.3.0.10 and the definition of the map Φ in Definition 1.4.1.11 contain the two crucial arguments of the proof: The composition of transfers always results in “level sets”, which are in a coherent sense given by the lower half of the sphere, and we can ignore the differences in coordinates for the level sets by interpreting them as nicely trivialised tubular neighbourhoods which form a contractible subspace.

Essentially, the maps Φ provide us with explicit homotopies which correct the failure of strict functoriality. Given those, there are various ways to actually obtain a functor, but finding conceptually different homotopies does not seem easy.

In the second chapter we gave a zig-zag of natural weak equivalences between pseudoisotopies and the Whitehead spectrum. In the connective case we again reduce to a geometric question, this time by 2– and $(\infty, 1)$–functorial methods. In the author’s opinion the key steps are the existence of admissible retractions by Lemma 2.1.5.10 and the fact that they form a contractible space by Lemma 2.1.5.15. The contraction relies only on an Alexander trick and is thus far less elaborate than the construction of Φ in the first chapter.

In the case of spectra the argument becomes a little more complicated as it is not known whether a category of bundles of bounded polyhedra is a model for the one fold loops of the Whitehead spectrum. Our argument, then, has two steps. First we state that: "The construction given in the connected case admits an obvious analogue." which is, indeed, quite easy to check. We need some preparations to precisely state our second step, but we really only want to say that: "We can triangulate admissible retractions in a compatible fashion." The relevant tool to do so is Lemma 2.2.3.11. Since this is (almost) a standard result in piecewise linear topology, the actual construction of the natural transformation is quite straightforward.

As we have explained in the introduction, the whole purpose of the endeavour into pseudoisotopies is the eventual computation of the homotopy type of the automorphism spaces. Hence it seems feasible to turn one's attention to the comparison between the different $\mathbb{Z}/2$–actions.

A review of the literature reveals that a connection has already been established - however only on the level of homotopy groups. We summarize it here in some detail.

The $\mathbb{Z}/2$–action to be studied is introduced in [53] as an action on the spectrum with $\Omega(\mathrm{Top}^b(M \times \mathbb{R}^{n+1})/\mathrm{Top}^b(M \times \mathbb{R}^n))$ in level $n \in \mathbb{N}$. The level-wise action is compared to a $\mathbb{Z}/2$–action on the unstable pseudoisotopy space via a weak equivalence [53, Remark 1.9]. We expect that the action on pseudoisotopies can be extended as a homotopy coherent $\mathbb{Z}/2$–action to the stable space of pseudoisotopies. However, this is a non-trivial issue, as can be seen from the fact that Hatcher [23, p. 16] has shown that the stabilisation map anti-commutes with the involution.

To pass from the geometric action on pseudoisotopies to an action on A–theory, we note that this work has already been undertaken in [45], however only on the level of homotopy groups. A geometric action on h–cobordism spaces is introduced, which "turns the cylinder upside down". It is then shown that this action sends an h–cobordism, understood as a CW–complex, to its Whitehead dual - the latter being unique only up to homotopy.

Now, an involution on a category of suitable retractive CW–complexes is introduced which sends every CW–complex to its Whitehead dual and descends to an involution on A–theory. To the author's understanding, the main result of [45] is stated in terms of homotopy groups, because the geometric $\mathbb{Z}/2$–action on h–cobordism spaces is only shown to be compatible with the involution on A–theory up to homotopy.

However, studying the argument it seems plausible that a generalisation to a homotopy coherent framework is possible, i.e. the $\mathbb{Z}/2$–actions coincide up to coherent homotopy, instead of "just" homotopy.

Finally, it was shown in [28] and [29] that the $\mathbb{Z}/2$–action on A–theory defined in [45] is indeed compatible with the Bass-Heller-Swan decomposition in the best sense, i.e. it bijectively maps the positive and negative Nil-terms onto each other and restricts to an involution on each of the other two summands of the decomposition. This is enough for the computational application in [13], i.e. were these results shown, we would obtain the following computation:

Conjecture. *Let M be a smoothable aspherical closed manifold of dimension ≥ 10, whose fundamental group π is hyperbolic.*

Then we obtain for $1 \leq n \leq \min\{(\dim M - 7)/2, (\dim M - 4)/3\}$ isomorphisms

$$\pi_n(\mathrm{Top}(M)) \cong \pi_{n+2}\Big(\bigvee_C N_+ \operatorname{Wh}^{\mathrm{Top}}(BC)\Big)$$

and an exact sequence

$$1 \to \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \to \pi_0(\mathrm{Top}(M)) \to \mathrm{Out}(\pi) \to 1,$$

where C *ranges over the conjugacy classes of maximal infinite cyclic subgroups of* π *and* $\mathrm{Out}(\pi)$ *denotes the outer automorphisms.*

Finally, $N_+ \mathrm{Wh}^{\mathrm{Top}}(BC)$ *denotes one of the Nil-terms of the Bass-Heller-Swan decomposition of* $\mathrm{Wh}^{\mathrm{Top}}(BC)$*.*

Conditional proof. If the $\mathbb{Z}/2$–actions are compatible, this follows from [13, Theorem 1.3], stated as Theorem 0.0.0.6 in the introduction, the computation of homotopy groups of the Whitehead spectrum in low degrees by Hesselholt [25] and the fact that all homotopy groups of $\mathrm{Wh}^{\mathrm{Top}}(*)$ vanish. □

To a geometrically minded reader, this should seem unsatisfying: Although we have described the group, it is not at all clear what the actual homeomorphisms look like. Since we understand the pseudoisotopy functor quite well, we can easily reduce the question to the case of a single circle (note that for this discussion we operate on the level of homotopy groups - hence the results of this thesis are not really required).

In the case of the circle, Hatcher [23] provided an example, due to Farrell, of a geometric representative of a non-trivial element. A slightly more detailed account of the argument can be found on MathOverflow [21]. Also, the work by Hatcher and Wagoner, starting with [24], might provide insights into explicit geometric constructions.

Closely related to this question is the rich structure of topological cyclic homology. The calculations of Hesselholt [25] rely heavily on the various operations available in topological cyclic homology, and while some admit a fairly straightforward counterpart in pseudoisotopies, not all of them are easily translated. In the case of a circle, all non-trivial classes are generated by one "fundamental class" via application of restriction, Frobenius, Verschiebung and Connes' operator.

With the program initiated by Weiss and Williams ongoing, the new avenues established by Galatius and Randall-Williams, and the progress on the algebraic side of the story, the prospects for further research on automorphism spaces look promising. Moreover, the results established on the Farrell-Jones conjecture for pseudoisotopies in conjunction with the explicit calculations on the circle make it seem as if a complete understanding of isotopy classes of homeomorphisms of hyperbolic high-dimensional manifolds might be attainable in the not too distant future.

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