Small embeddings, forcing with side conditions, and large cardinal characterizations

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Summary

In this thesis, we provide new characterizations for several well-studied large cardinal notions. These characterizations will be of two types. Motivated by seminal work of Magidor, the first type characterizes large cardinals through the existence of so-called *small embeddings*, elementary embeddings between set-sized structures that map their critical point to the large cardinal in question. Building up on these characterizations, we also provide characterizations of many large cardinal notions through the validity of certain combinatorial principles at ω_2 in generic extensions. The combinatorial principles used in these characterizations are generalizations of large cardinal properties defined through small embeddings that can also hold at accessible cardinals and, for inaccessible cardinals, these principles are equivalent to the original large cardinal property. In this thesis, we focus on generic extensions obtained via the *pure side condition forcing* introduces by Neeman in his studies of forcing axioms and their generalizations. Our results will provide these two types of characterizations for some of the most prominent large cardinal notions, including inaccessible, Mahlo, Π^m_n-indescribable cardinals, subtle, λ-ineffable, and supercompact cardinals. In addition, we will derive small embedding characterizations of measurable, λ-supercompact and huge cardinals, as well as forcing characterizations of almost huge and super almost huge cardinals. As an application of techniques developed in this work, we provide new proofs of Weiß's results on the consistency strength of generalized tree properties, eliminating problematic arguments contained in his original proofs.

The work presented in this thesis is joint work with Peter Holy and Philipp Lücke. It will be published in the following papers.

- Peter Holy, Philipp Lücke and Ana Njegomir, *Small Embedding Characterizations for Large Cardinals*. Submitted to the *Annals of Pure and Applied Logic*, 23 pages, 2017.
- Peter Holy, Philipp Lücke and Ana Njegomir, *Characterizing large cardinals through Neeman's pure side condition forcing*. In preparation, 27 pages, 2018.

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Contents

Chapter 1

Introduction

Paul Cohen solved Hilbert's first problem by showing that the *Continuum Hypothesis*, stating that every uncountable set of subsets of the natural numbers has the cardinality of the set of all subsets of the natural numbers, is not decided by the standard axiomatization of set theory provided by the axioms of *Zermelo-Fraenkel Set Theory with the Axiom of Choice* (ZFC). In his proof, he introduced the technique of *forcing* that allows the construction of new models of set theory by extending existing ones to include so-called *generic object*. Today, this technique belongs to the most important tools of set theoretic research and its interaction with large cardinal axioms is a central topic in this area. Large cardinals have grown out of the work of Felix Hausdorff on *cardinal arithmetics* and Stanisław Ulam on the *measure problem*. These axioms postulate the existence of cardinal numbers having certain properties that make them very large, and whose existence cannot be proved in ZFC, because it implies the consistency of ZFC itself. In these interactions, a large cardinal is generically collapsed to become a successor cardinal, while certain combinatorial properties of the original cardinal are preserved. This approach provides a strong connection between large cardinals and combinatorial principles for small cardinals. The connection is further strengthened by fundamental results of inner model theory showing that combinatorial principles imply the existence of large cardinals in inner models. In many important cases, these results provide equiconsistencies between large cardinal axioms and combinatorial principles for small cardinals by recovering the type of large cardinals used to establish the consistency of the combinatorial principle in the first place. For example, results of Robert Solovay show that, if θ is a Mahlo cardinal above an uncountable regular cardinal κ, then forcing with the Lévy collapse $\mathrm{Col}(\kappa, {<}\theta)$ causes *Jensen's principle* $\Box_\kappa$(see [14]) to fail in the generic extension. In contrast, if κ is an uncountable cardinal such that $\Box_\kappa$ fails, then seminal results of Ronald Jensen in [14] show that κ^+ is a Mahlo cardinal in Gödel's constructible universe L.

The main goal of this thesis is to study strengthenings of the above connection, by investigating situations in which large cardinals are actually characterized through the validity of combinatorial principles in forcing extensions. More precisely, in these situations the forcing will cause the combinatorial principle to hold at a given cardinal θ in the generic extension if and only if θ has the corresponding large cardinal property in the ground model. Moreover, these characterizations will be *strong*, in the sense that the combinatorial principle used, if conjuncted with inaccessibility, will be equivalent to the corresponding large cardinal property. Therefore, one could say that the principles, that characterize large cardinals in the above way are their combinatorial remainder after their inaccessibility has been destroyed. It is easy to see that not all equiconsistency results necessarily lead to such characterizations. For example, in the case of the equiconsistency result for Mahlo cardinals described above, we can combine a result of Todorčević showing that the *Proper Forcing Axiom* PFA implies that $\square_\kappa$ fails for all uncountable cardinals κ (see [31]) with a result of Larson showing that PFA is preserved by $<\omega_2$-closed forcing (see [21]) to see that, if PFA holds and $\kappa < \theta$ are regular cardinals greater than ω_1, then $\square_\kappa$ fails in every $\mathrm{Col}(\kappa, {<}\theta)$-generic extension.

The following definition aims to formulate this approach in a more precise way. We use Card to denote the class of all infinite cardinals.

Definition 1.1. Let $\vec{\mathbb{P}} = \langle \mathbb{P}(\theta) \mid \theta \in \mathrm{Card}\rangle$ be a class-sequence of partial orders, and let $\Phi(v)$ and $\varphi(v)$ be formulas in the language of set theory.

(i) We say that $\vec{\mathbb{P}}$ *characterizes* Φ *through* φ if

$$\mathsf{ZFC} \vdash \forall \theta \in \mathrm{Card}\ [\Phi(\theta) \longleftrightarrow \mathbb{1}_{\mathbb{P}(\theta)} \Vdash \varphi(\check{\theta})].$$

(ii) If $\vec{\mathbb{P}}$ characterizes Φ through φ, then we say that this characterization is *strong* in case that

$$\mathsf{ZFC} \vdash \forall \theta \textit{ inaccessible } [\Phi(\theta) \longleftrightarrow \varphi(\theta)].$$

It can easily be seen that not every forcing that turns a large cardinal into a successor cardinal is applicable for characterizations as defined in the previous definition. For example, the *Lévy collapse* $\mathrm{Col}(\kappa, {<}\theta)$ is not suitable for such characterizations, as we will see in the next proposition.

Proposition 1.2. *Assume that the existence of an inaccessible cardinal is consistent with the axioms of* ZFC*. If $n < \omega$, then no formula in the language of set theory characterizes the class of inaccessible cardinals through the sequence $\langle \mathrm{Col}(\omega_n, {<}\theta) \mid \theta \in \mathrm{Card}\rangle$.*

Proof. Suppose for a contradiction that $\varphi(v)$ is a formula with this property, and that there is a model V of ZFC that contains an inaccessible cardinal θ. Let G be $\mathrm{Col}(\omega_n, {<}\theta)$-generic over V. Then, using that $\mathrm{Col}(\omega_n, {<}\theta) \times \mathrm{Col}(\omega_n, {<}\theta)$ and $\mathrm{Col}(\omega_n, {<}\theta)$ are forcing equivalent, we may find $H_0, H_1 \in \mathrm{V}[G]$ with the property that $H_0 \times H_1$ is $(\mathrm{Col}(\omega_n, {<}\theta) \times \mathrm{Col}(\omega_n, {<}\theta))$-generic over V and $\mathrm{V}[G] = \mathrm{V}[H_0 \times H_1]$. Since θ is inaccessible in V, our assumption implies that $\varphi(\theta)$ holds in $\mathrm{V}[G]$ and, since the partial order $\mathrm{Col}(\omega_n, {<}\theta)^{\mathrm{V}} = \mathrm{Col}(\omega_n, {<}\omega_{n+1})^{\mathrm{V}[H_0]}$ is weakly homogeneous in $\mathrm{V}[H_0]$, we can conclude that $\mathbb{1}_{\mathrm{Col}(\omega_n, <\theta)} \Vdash \varphi(\check{\theta})$ holds in $\mathrm{V}[H_0]$ by [18, Proposition 10.19]. But, again by our assumption, this shows that $\theta = \omega_{n+1}^{\mathrm{V}[H_0]}$ is inaccessible in $\mathrm{V}[H_0]$, a contradiction. □

In contrast, the main results of this thesis will show that the *pure side condition forcing* introduced by Itay Neeman in [28] can be used to characterize many important large cardinal notions through canonical combinatorial principles. Neeman introduced this forcing to provide a new proof of the consistency of PFA. Using finite sequences of elementary submodels, this forcing turns a large cardinal into ω_2, while preserving many combinatorial properties of the cardinal. The forcing has many strong structural properties. For example, it is strongly proper for a rich class of models and its quotient forcings have the σ-approximation property. These properties will be essential for many of our proofs. As an example of such a characterization, we will show (see Theorems 6.7 and 7.6) that a cardinal θ is a Mahlo cardinal if and only if Neeman's forcing causes $\varphi(\theta)$ to hold in the generic extension, where $\varphi(v)$ is the canonical formula stating that v is equal to $\aleph_2$, there are no special $\aleph_2$-Aronszajn trees, and there are no weak Kurepa trees.

In order to obtain such characterizations of large cardinals via Neeman's pure side condition forcing, we use characterizations of these cardinals through so-called *small embeddings*. Many large cardinal notions are characterized by the existence of non-trivial elementary embeddings with certain properties. There are two kinds of such characterizations. In the first kind of characterization, a large cardinal θ is characterized by the existence of elementary embeddings whose critical point is θ. Standard examples of cardinals that are defined in this way are measurable and supercompact cardinals. Results of Kai Hauser (see [10]) show that Π^m_n-indescribable cardinals can also be characterized in this way. In the second case, a large cardinal θ is characterized by the existence of elementary embeddings that map their critical point to θ. The following classical result of Menachem Magidor is the first example of a characterization of the second kind.

Theorem 1.3 ([24, Theorem 1]). *A cardinal θ is supercompact if and only if for every $\eta > \theta$, there is a non-trivial elementary embedding $j : \mathrm{V}_\alpha \longrightarrow \mathrm{V}_\eta$ with $\alpha < \theta$ and $j(\mathrm{crit}\,(j)) = \theta$.*

Other examples of large cardinal properties that are characterized by the existence of such embeddings are *subcompactness* (introduced by Ronald Jensen) and its generalizations (see [3]), and also Ralf Schindler's *remarkable cardinals* (see [29]).

We refer to characterizations of the latter kind as *small embedding characterizations*. More precisely, given cardinals $\theta < \vartheta$, we say that a non-trivial elementary embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ if $M \in \mathrm{H}(\vartheta)$ is transitive, and $j(\mathrm{crit}\,(j)) = \theta$ holds. The properties of cardinals θ studied in this thesis usually state that for sufficiently large[1] cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with certain elements of $\mathrm{H}(\vartheta)$ in its range, and with the property that the domain model M satisfies certain correctness properties with respect to the universe of sets V, sometimes in combination with some kind of smallness assumption about M. Note that the proof of Theorem 1.3 directly yields the following small embedding characterization of supercompactness.

Corollary 1.4. *The following statements are equivalent for every cardinal θ:*

(i) θ is supercompact.
(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $M = \mathrm{H}(\delta)$ for some cardinal $\delta < \theta$. □

The requirement that $M = \mathrm{H}(\delta)$ in the previous corollary can easily be interpreted as a correctness property of M (since $\mathrm{V} = \mathrm{H}(\mathrm{Ord})$), and that $\delta < \theta$ is a smallness assumption on M. In this thesis, we will see that many large cardinal properties can be characterized using small embeddings.

[1]Here, ϑ being a *sufficiently large cardinal* means that there is an $\alpha \geq \theta$ such that the corresponding statement holds for all cardinals $\vartheta > \alpha$.

In many situations, small embeddings characterizing certain large cardinals can be lifted to suitable forcing extensions and those lifted embeddings will still have most of the properties of the original small embeddings. The properties obtained by lifting the witnessing embeddings to a suitable collapse extension will be phrased as combinatorial principles that we call *internal large cardinals*. In many important cases, these principles capture strong fragments of the combinatorics of the collapsed cardinals. This approach also provides principles corresponding to large cardinal concepts for which no corresponding canonical combinatorial property exists, like higher degrees of indescribability. Most of our forcing characterizations of large cardinals will be based on these principles.

The observation that Neeman's results in [28, Chapter 5] can be used to characterize weak compactness was the starting point of a project with Philipp Lücke dealing with forcing characterizations of large cardinals and combinatorial principles that can be used for such characterizations. Meanwhile, Peter Holy and Philipp Lücke came up with the concepts of small embeddings and internal large cardinals. After we realized that this concept can be used to simplify several of our forcing characterizations, we joint these projects to produce the following two articles:

- Peter Holy, Philipp Lücke and Ana Njegomir, *Small Embedding Characterizations for Large Cardinals*. Submitted to the *Annals of Pure and Applied Logic*, 23 pages, 2017.
- Peter Holy, Philipp Lücke and Ana Njegomir, *Characterizing large cardinals through Neeman's pure side condition forcing*. In preparation, 27 pages, 2018.

Most of the results presented in this thesis are contained in the above articles. My main contributions to this work are the new results contained in the chapters 4, 6, 7, 9, 10, 11 and 12. The new results from the remaining chapters are due to Peter Holy and Philipp Lücke.

We will now summarize the content of this thesis. In Chapter 2, we will formulate the notion of strong properness, and show how it is related to the σ-approximation property. In the third chapter, we will precisely define Neeman's pure side condition forcing and provide proofs of some of its properties. In Chapter 4, we will present a side condition forcing in a special setting, that we will use later on to characterize certain large cardinals. In the fifth chapter, we will introduce the notion of small embeddings and use it to characterize large cardinal properties that are defined through the stationarity of certain sets of cardinals, like inaccessible and Mahlo cardinals. In Chapter 6, we will first characterize countably inaccessible cardinals via Neeman's pure side condition forcing. We will then provide a forcing characterization of inaccessible cardinals through the non-existence of certain trees in generic extensions. Although in this case it is not necessary, we will get the forcing characterization using the small embedding characterizations of inaccessible cardinals, in order to introduce some of the techniques that we will need in many of the later proofs. In the seventh chapter, we will introduce the notion of internal large cardinals and, using this principle, we will provide a forcing characterization of Mahlo cardinals. In Chapter 8, we will present small embedding characterizations for indescribable cardinals and use them to provide a forcing characterization of those cardinals. In Chapter 9, we will introduce small embedding characterizations of subtle and λ-ineffable cardinals. In Chapter 10, we will discuss theorems by Christoph Weiß and point out some problems occurring in his original proofs by showing that his proofs show something stronger that is provably wrong. In order to provide new proofs of them, in Chapter 11 we will introduce the notion of internally AP-ineffable cardinals and internally AP-subtle cardinals. In the twelfth chapter, we will produce a forcing characterization of λ-ineffable cardinals and subtle cardinals using the techniques developed earlier in this thesis. In Chapter 13, we will prove small embedding characterizations of some filter-based large cardinals. Chapter 14 will be dedicated to supercompact cardinals. Namely, we will introduce their internal version and we will provide a forcing characterization for them. In Chapter 15, we will present results about forcing characterizations of some larger large cardinals. Namely, generic elementary embeddings are used in order to characterize levels of supercompactness, almost huge cardinals and super almost huge cardinals through Neeman's pure side condition forcing. Finally, the last chapter contains some remarks and open questions regarding the work presented in this thesis.

Chapter 2

Strong properness

In the following, we are going to introduce the notion of *strong properness* that is central for much of the theory developed later on. The notion of strong properness and the results presented in this chapter are due to Wiliam Mitchell [27]. Our presentation follows the presentation by Neeman in [28, Section 3]. Before stating the definition of a strong master condition, we will recall the notion of a master condition, to which it is closely connected.

Definition 2.1. Let $\mathbb{P}$ be a partial order which is an element of a set M.

- A condition $p \in \mathbb{P}$ is a *master condition* for M if it forces the generic filter to intersect, inside M, every dense set $D \in M$ of $\mathbb{P}$.
- $\mathbb{P}$ is proper for M if every $p \in M \cap \mathbb{P}$ can be extended to some master condition for M.
- $\mathbb{P}$ is proper if for every sufficiently large θ and every countable $M \prec \mathrm{H}(\theta)$ with $\mathbb{P} \in M$ we have that $\mathbb{P}$ is proper for M.

Definition 2.2. Suppose that $\mathbb{P}$ is a partial order and that M is a set. We say that $p \in \mathbb{P}$ is a *strong master condition* for M if it forces the generic filter to meet every dense subset of $\mathbb{P} \cap M$.

Definition 2.3. Let $\mathbb{P}$ be a partial order and let M be a set.

- $\mathbb{P}$ is *strongly proper* for M if every condition of $\mathbb{P}$, which is an element of M, can be extended to a strong master condition for M.
- We say that $\mathbb{P}$ is strongly proper for a collection of sets $\mathcal{R}$ if it is strongly proper for every $M \in \mathcal{R}$.

The following observation clarifies the relationship between strong master conditions and master conditions.

Observation 2.4. *Let M be an elementary submodel of* $\mathrm{H}(\theta)$ *for some regular θ and let $\mathbb{P} \in M$ be a partial order. Suppose that $p \in \mathbb{P}$ is a strong master condition for M. Then p is also a master condition for M.*

Proof. Let D be a dense set of $\mathbb{P}$ which is an element of M. Then, by elementarity, $D \cap M$ is a dense subset of $\mathbb{P} \cap M$. Since p is a strong master condition for M, we know that p forces that $\dot{G} \cap D \cap M$ is nonempty, where $\dot{G}$ is the canonical name for the filter on $\mathbb{P}$. □

Together with the previous observation, the following one establishes the connection between properness and strong properness.

Observation 2.5. *Let K be a transitive set and let $\mathbb{P} \subseteq K$ be a forcing notion which is strongly proper for some $M \subseteq K$. Pick ϑ to be large enough that $K \subseteq \mathrm{H}(\vartheta)$ and let $M^* \prec \mathrm{H}(\vartheta)$ such that $M^* \cap K = M$. Then $\mathbb{P}$ is strongly proper for M^*.*

Proof. Let p be a condition in $M^* \cap \mathbb{P} \subseteq M$. Then there exists $q \leq_{\mathbb{P}} p$ a strong master condition for M. Observe that $M^* \cap \mathbb{P} = M \cap \mathbb{P}$, since $\mathbb{P} \subseteq K$ and $M^* \cap K = M$. Thus every dense set of $M^* \cap \mathbb{P}$ is also a dense set of $M \cap \mathbb{P}$ and hence q is a strong master condition for M^*. □

The following lemma is a standard consequence of properness, and due to the observations above, it is also a consequence of strong properness.

Lemma 2.6 ([28, Claim 3.3])**.** *Let ϑ be a sufficiently large regular cardinal, let M be an elementary submodel of $\mathrm{H}(\vartheta)$ and let $\mathbb{P}$ be a forcing partial order in M. Suppose that G is generic for $\mathbb{P}$ over V and that G contains a master condition for M. Then the following hold:*

(i) $M[G] \prec \mathrm{H}(\vartheta)[G]$ *and* $M[G] \cap V = M$.
(ii) Let $\dot{f}$ be a $\mathbb{P}$-name in M such that $\dot{f}^G$ is a function with ordinal domain. Let $\tau = \dot{f} \cap M$. Then we have that $\tau^G = \dot{f}^G \restriction M$.

In the following we are going to introduce a generalized notion of stationary sets, which we will need later.

Definition 2.7. Suppose that X is a non-empty set. A set S is stationary in $\mathcal{P}(X)$ if for every function $f : [X]^{<\omega} \to X$ there exists $y \in S$ such that $f[[y]^{<\omega}] \subseteq y$.

The following is a standard lemma (see for example [22, Lemma 2.1.3]) that we will later make use of.

Lemma 2.8. *Suppose that $\emptyset \neq X \subseteq Y$.*

(i) Let $S \subseteq \mathcal{P}(Y)$ be a stationary set in $\mathcal{P}(Y)$. Then $\{Z \cap X \mid Z \in S\}$ is stationary in $\mathcal{P}(X)$.
(ii) Let $S \subseteq \mathcal{P}(X)$ be a stationary set in $\mathcal{P}(X)$. Then $\{Z \subseteq Y \mid Z \cap X \in S\}$ is stationary in $\mathcal{P}(Y)$.

The following property of pairs of models of set theory was introduced by Joel Hamkins (see [9]). Later, we will see how it is related to strong properness.

Definition 2.9. For transitive classes $M \subseteq N$, we say that the pair (M, N) satisfies the *σ-approximation property* if $A \in M$ whenever $A \in N$ is such that $A \subseteq B$ for some $B \in M$, and $A \cap x \in M$ for every $x \in M$ which is countable in M.[1] We say that a partial order $\mathbb{P}$ satisfies the *σ-approximation property* in case the pair $(\mathrm{V}, \mathrm{V}[G])$ does so whenever G is $\mathbb{P}$-generic over V.

[1]In case M and N have the same ordinals and satisfy enough set theory, this definition is equivalent to the more common definition of the σ-approximation property where rather than requiring $A \subseteq B$ for some $B \in M$, one only requires that $A \subseteq M$.

The next lemma follows from a slight modification of the proof of [28, Lemma 3.6]. For the sake of completeness, we present a proof of this statement.

Lemma 2.10. *Suppose that a partial order $\mathbb{P}$ is strongly proper for unboundedly many countable elementary submodels of $\mathrm{H}(\vartheta)$, for some sufficiently large regular cardinal ϑ. Then $\mathbb{P}$ satisfies the σ-approximation property.*

Proof. Let G be $\mathbb{P}$-generic over V.

Claim 1. *Suppose that for every $f \in \mathrm{V}[G]$ that is a function from α to 2, for some $\alpha \in Ord$, such that $f \cap x \in \mathrm{V}$ whenever $x \in \mathrm{V}$ is countable in V, we have that $f \in \mathrm{V}$. Then $\mathbb{P}$ satisfies the σ-approximation property.*

Proof. Let $A \in \mathrm{V}[G]$ and $A \subseteq B$ for $B \in \mathrm{V}$. Pick a bijection $h : B \longrightarrow \alpha$ in V, for $\alpha \in \mathrm{Ord}$. Then the characteristic function of $h[A]$ is an element of ${}^{\alpha}2$. Assume $A \cap y \in \mathrm{V}$ whenever $y \in \mathrm{V}$ is countable in V. Hence $A \cap h^{-1}[y \cap \alpha] \in \mathrm{V}$. Thus $h[A] \cap y \cap \alpha \in \mathrm{V}$, since $h \in \mathrm{V}$. Hence $h[A] \in \mathrm{V}$ and thus $h^{-1}[h[A]] = A \in \mathrm{V}$. □

Let $f \in \mathrm{V}[G]$ be a function from α to 2, for some $\alpha \in \mathrm{Ord}$, and such that

$$f \cap x \in \mathrm{V} \text{ whenever } x \in \mathrm{V} \text{ is countable in V.} \tag{2.1}$$

By the previous claim we know that we are done if we show that $f \in \mathrm{V}$.

Claim 2. $f \in \mathrm{V}$.

Proof. Let $\dot{f}$ be a nice name for f, i.e. all elements of $\dot{f}$ are of the form $\langle\langle\beta,\gamma\rangle\check{}, r\rangle$ such that $\beta < \alpha$, $\gamma \in 2$ and $r \Vdash \dot{f}(\check{\beta}) = \check{\gamma}$, letting $\langle\beta,\gamma\rangle\check{}$ denote the canonical $\mathbb{P}$-name for the ordered pair $\langle\beta,\gamma\rangle$. Suppose for a contradiction that $f \notin \mathrm{V}$. Let $p \in G$ be a condition forcing this, and also forcing that Property 2.1 holds. By our assumption, we may pick a countable $M \prec \mathrm{H}(\vartheta)$ such that $\mathbb{P}, \dot{f}, p \in M$, and such that $\mathbb{P}$ is strongly proper for M. Then there exists a strong master condition $q \leq_{\mathbb{P}} p$ for M.

Let H be a $\mathbb{P}$-generic filter over V such that $q \in H$, and let $g = \dot{f} \cap M$. Thus, by Lemma 2.6, it holds that $g^H = \dot{f}^H \cap M$. Since H contains a strong master condition for M, $H \cap M$ is generic for $\mathbb{P} \cap M$. Note that g is a $\mathbb{P} \cap M$-name, hence $g^H = g^{H \cap M}$. By our assumption, it holds that $g^H = g^{H \cap M} \in \mathrm{V}$. Then there exists a condition $s \in H \cap M$ which decides all values of g, and such that $s \leq_{\mathbb{P} \cap M} p$. That means, for every $\beta \in \alpha \cap M$, there is $\gamma \in 2$ and $u \in \mathbb{P} \cap M$ such that $\forall t \leq_{\mathbb{P} \cap M} u \; t \parallel s$ and such that $\langle\langle\beta,\gamma\rangle\check{}, u\rangle \in \dot{f}$. By elementarity of M, this implies that for every $\beta \in \alpha$, there is $\gamma \in 2$ and $u \in \mathbb{P}$ such that $\forall t \leq_{\mathbb{P}} u \; t \parallel s$ and $\langle\langle\beta,\gamma\rangle\check{}, u\rangle \in \dot{f}$. Hence $s \Vdash_{\mathbb{P}} \dot{f} \in \mathrm{V}$, but since $s \leq_{\mathbb{P}} p$, this contradicts our assumption on p. □

This completes the proof of the lemma. □

We will end this chapter with the helpful preservation result. Although its proof is contained in [28], we will include it here for the sake of completeness.

Lemma 2.11 ([28, Claim 3.5])**.** *Let K be a set, let $\mathbb{P} \subseteq K$ be a partial order and let $\mathcal{S}$ be a collection of subsets of K. Suppose that $\mathbb{P}$ is strongly proper for $\mathcal{S}$. Let κ be a cardinal and let for each $\alpha < \kappa$ the set $\{M \in \mathcal{S} \mid \alpha \subseteq M \text{ and } |M| < \kappa\}$ be stationary in $\mathcal{P}(K)$. Then forcing with $\mathbb{P}$ preserves κ.*

Proof. Suppose for a contradiction that forcing with $\mathbb{P}$ collapses κ. Let $\dot{f}$ be a name for a surjective function from α to κ for some $\alpha < \kappa$. Pick $p \in \mathbb{P}$ that forces this. Let ϑ be large enough that $K, \mathbb{P} \in \mathrm{H}(\vartheta)$. Let $\mathcal{S}^* = \{M^* \subseteq \mathrm{H}(\vartheta) \mid M^* \cap K \in \mathcal{S},\ |M^*| < \kappa \text{ and } \alpha \subseteq M^*\}$. Since $\{M \in \mathcal{S} \mid \alpha \subseteq M \text{ and } |M| < \kappa\}$ is stationary in $\mathcal{P}(K)$, by Lemma 2.8 we know that $\mathcal{S}^*$ is stationary in $\mathcal{P}(\mathrm{H}(\vartheta))$. Thus, we may find $M^* \in \mathcal{S}^*$ such that $\dot{f}, p, \mathbb{P} \in M^* \prec \mathrm{H}(\vartheta)$. By Observation 2.5. we may take $q \leq_{\mathbb{P}} p$ a strong master condition for M^*. Then q forces the range of $\dot{f}$ to be contained in M^* by Lemma 2.6. But κ cannot be a subset of M^*, since M^* has size less than κ. Hence, this is a contradiction, since q is stronger than p and q forces the range of $\dot{f}$ not to be κ. □

Chapter 3

Neeman's pure side conditions forcing

In this chapter, we present the definition of Neeman's pure side condition forcing, whose conditions are finite sequences of models of two types, namely countable and transitive ones. The idea of including models into conditions originates from Stevo Todorčević in his [33]. Afterwards, we are going to discuss some strong structural properties of this forcing that were already presented by Neeman in [28]. We are including proofs for more restrictable results, as we are looking at the specific version of Neeman's pure side condition forcing.

Definition 3.1. Given a transitive set K and $\mathcal{S}, \mathcal{T} \subseteq K$, we define a condition in the partial order $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ to be a finite sequence $s = \langle M_i \mid i < n \rangle$ such that

(i) for each $i < n$, $M_i \in \mathcal{S} \cup \mathcal{T}$,
(ii) for all $i + 1 < n$, $M_i \in M_{i+1}$,
(iii) for every $i, j < n$, $M_i \cap M_j = M_k$ for some $k < n$.

Let $p = \langle M_i \mid i < n \rangle$ and $q = \langle N_i \mid i < m \rangle$ be conditions in $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$. Then we define $p \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} q$ if and only if $\operatorname{ran}(p) \supseteq \operatorname{ran}(q)$.

Remark 3.2. *Given a transitive set K and $\mathcal{S}, \mathcal{T} \subseteq K$, let $s = \langle M_i \mid i < n \rangle$ be a finite sequence satisfying Condition (i) and Condition (ii) of Definition 3.1. Note that for $j < k < n$, M_j has a smaller rank than M_k. Hence the order of s is determined uniquely from the elements of s. Thus we can identify conditions in $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ with their range. We will use this identification throughout the whole thesis without further mentioning it.*

Definition 3.3.

- We say that a set K is suitable if it is a transitive set such that $\omega_1 \in K$ and the model $(K, \in)$ satisfies a sufficient fragment of ZFC, in the sense that K is closed under the operations of pairing, union, intersection, set difference, cartesian product, and transitive closure, closed under the range and restriction operations on functions, and such that for each $x \in K$, the closure of x under intersections belongs to K, and there is an ordinal length sequence in K consisting of the members of x arranged in non-decreasing von Neumann rank.
- If K is suitable, then we say that $\mathcal{S}$ and $\mathcal{T}$ are *appropriate* for K if the following statements hold:

(i) Elements of $\mathcal{T} \subseteq K$ are transitive and countably closed elementary submodels of K, and elements of $\mathcal{S} \subseteq K$ are countable elementary submodels of K.
(ii) If $W \in \mathcal{T}$, $M \in \mathcal{S}$, and $W \in M$, then $M \cap W \in W$ and $M \cap W \in \mathcal{S}$.

If $\mathcal{S}$ and $\mathcal{T}$ are appropriate for a suitable set K, then we will refer to elements of $\mathcal{S}$ as *small nodes*, and to elements of $\mathcal{T}$ as *transitive nodes*.

The following remark is about some notation used later on in this chapter.

Remark 3.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be appropriate for a suitable set K. Let $s = \langle M_i \mid i < n \rangle$ for some $n \in \omega$ be a finite sequence of nodes such that Condition (i) and (ii) of Definition 3.1 are satisfied. Then we say that M_j occurs before M_k in s, whenever $j < k < n$. Also, we say that M_l is between M_j and M_k, if $j < l < k < n$. We call M_0 the left endpoint of s, and M_{n-1} we call the right endpoint of s. For given $i, j < n$, we denote $\langle M_k \mid i \leq k < j \rangle$ by $[M_i, M_j)$.*

The following observation is about certain relations between nodes. Its proof is trivial in our special setting, so we will omit it.

Observation 3.5 ([28, Claim 2.10 and Claim 2.11])**.** *Let $\mathcal{S}$ and $\mathcal{T}$ be appropriate for a suitable set K. Let p be a finite sequence of nodes such that Condition (i) and (ii) of Definition 3.1 are satisfied. Then the following holds:*

(i) If M and N are nodes of small type such that $M \in N$, then $M \subseteq N$.
(ii) Let W be a node of transitive type in p, and let M be a node in p that occurs before W. Then $M \in W$.
(iii) Let Q be a small node in p, and let M be a node in p that occurs before Q and there are no nodes of transitive type between M and Q, then $M \in Q$.
(iv) Suppose that M and N are nodes of p such that M occurs before N. If there are nodes of transitive type between M and N, then there are transitive nodes between them that belong to N.

Later on, we will use the following observation, in order to prove that some finite sequence of nodes is closed under intersections.

Observation 3.6 ([28, Claim 2.12])**.** *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let p be a finite sequence of nodes such that Condition (i) and (ii) of Definition 3.1 are satisfied. Suppose that the following holds.*

(iii)′ If W is a transitive node of p and M is a small node of p such that $W \in M$, then $M \cap W \in p$.

Then p is closed under intersection, hence $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$.

Proof. Suppose for a contradiction that this is not the case, and let $P, Q \in p$ witness that p is not closed under intersections, and such that P occurs before Q and Q is the minimal such node, meaning that for every node $R \in p$ that occurs between P and Q in p, we have $P \cap R \in p$.

First, if Q is a node of transitive type, then by Observation 3.5, we have that $P \subseteq Q$ which implies $P \cap Q = P \in p$.

Otherwise, let Q be a node of small type. If there are no transitive nodes between P and Q, then by Observation 3.5 we get that $P \in Q$. In case P is of small type, we again have that $P \subseteq Q$ and thus $P \cap Q = P \in p$. In the other case, when P is of transitive type, we may apply $(iii)'$ and get $P \cap Q \in p$. Thus, we may assume that there are transitive nodes between P and Q. By Observation 3.5, we get $W \in p$, that is of transitive type, occurs between P and Q, and $W \in Q$. By $(iii)'$, we have that $W \cap Q \in p$. It has to occur before W, hence also before Q. Since we have picked Q to be the minimal such node, we know that $(W \cap Q) \cap P$ is a node of p. Observe that since W is a transitive node above P, we have that $P \subseteq W$ by Observation 3.5. Thus $P \cap Q = (P \cap W) \cap Q \in p$ and we are done. □

A very important notion for Neeman's pure side condition forcing is the notion of *residues* of a condition.

Definition 3.7 (Neeman)**.** Let $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let $Q \in p$. Then we define the *residue* of p in Q to be $\text{res}_Q(p) = p \cap Q$.

In the following, we will see how residues look like. In order to do so, we will first introduce the notion of residue gaps.

Definition 3.8 (Neeman)**.** Let K be a suitable set and let $\mathcal{S}$ and $\mathcal{T}$ be appropriate for K. Suppose that $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let $M \in p$ be a node of small type. Let W be a transitive node of $\text{res}_M(p)$. Then the interval $[M \cap W, W)$ of p is called a residue gap of p in M.

Lemma 3.9 ([28, Lemma 2.17])**.** *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let Q be a node of p. If $Q \in \mathcal{T}$, then $res_Q(p)$ consists of all nodes that occur before M. Otherwise, if Q is of small type, then $res_Q(p)$ contains of all nodes of p that occur before Q and do not belong to residue gaps of p in Q.*

Proof. In case Q is a transitive node, by transitivity we have that all the nodes of p that occur before Q belong to $\text{res}_Q(p)$. The remaining nodes of p have higher rank than Q, so they cannot belong to Q.

Let us consider the other case, when Q is a node of small type. We will first show that nodes in residue gaps are not elements of $\text{res}_Q(p)$. Let $W \in \text{res}_Q(p)$ be a node of transitive type. Let N be any node of $\text{res}_Q(p)$ that occurs before W. By Observation 3.5,(ii) we know that $N \in W$, so $N \in Q \cap W$. Hence, N cannot be an element of the gap. Thus, the only thing we need to show is that all the nodes of p that occur before Q and are not elements of the residue gaps, are elements of $\text{res}_Q(p)$. Let M be such a node of p. Then, if there are no transitive nodes of p between M and Q, by Observation 3.5,(iii), we have that $M \in Q$, hence also $M \in \text{res}_Q(p)$. Otherwise, let $R \in p$ be the first transitive node that is an element of Q, meaning that there are no such nodes between M and R. We know that such a node exists by Observation 3.5,(iv). Hence, M cannot be an element of $[Q \cap R, R)$, because we have chosen M not to be an element of any residue gap. Hence M occurs before $Q \cap R$. By minimality, there are no transitive nodes between M and $Q \cap R$ that belong to $Q \cap R$, and hence there are no transitive nodes at all between them by Observation 3.5,(iv). Again, by Observation 3.5,(iii) we get that $M \in Q$, hence also $M \in \text{res}_Q(p)$ and we are done. □

Lemma 3.10 ([28, Lemma 2.18]). *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. If $p \in \mathbb{P}_{\mathcal{S},\mathcal{T}}$ and $Q \in p$, then $res_Q(p) \in \mathbb{P}_{\mathcal{S},\mathcal{T}}$.*

Proof. In case Q is a transitive node, we know that $\text{res}_Q(p)$ is an initial segment of p. Then trivially $\text{res}_Q(p) \in \mathbb{P}_{\mathcal{S},\mathcal{T}}$.

Assume now that Q is a small node. Condition (i) and (iii) of Definition 3.1 for the residue are satisfied immediately since $p \in \mathbb{P}_{\mathcal{S},\mathcal{T}}$ and since $Q \prec K$. We only need to show that Condition (ii) of the definition is satisfied.

Let $M \in \text{res}_Q(p)$ be of small type. Then, by Observation 3.5,(i) we have that $M \subseteq Q$ and hence the predecessor of M in the condition p is also in $\text{res}_Q(p)$. Otherwise, if M is of transitive type, then the predecessor of M in $\text{res}_Q(p)$ is also an element of M, by the transitivity of M. □

Let us now define a certain type of compatibility between conditions, that we will use in the rest of this chapter.

Definition 3.11. Let p and q be conditions in $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$. We say that p and q are *directly compatible* if they are compatible and the condition witnessing their compatibility can be taken to be exactly the closure of $p \cup q$ under intersections.

The following couple of lemmas will play a central role in proving that Neeman's pure side condition forcing is strongly proper for a reach class of models.

Lemma 3.12 ([28, Lemma 2.20]). *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let Q be a transitive node in p. Suppose that $q \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap Q$ and that it extends $res_Q(p)$. Then p and q are directly compatible.*

Proof. It suffices to show that $r := p \cup q$ is an element of $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$. Note that $r \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ trivially implies that p and q are directly compatible. Observe that r is the same as p above Q and the same as q below Q. Hence Condition (i) and (ii) of Definition 3.1 are satisfied trivially for r. In order to show that r is closed under the intersections, it is enough to show that for every $M \in \mathcal{S} \cap r$ and $W \in \mathcal{T} \cap r$ such that $W \in M$, we have that $M \cap W \in r$, by Observation 3.6. So, pick such W and M in r. If W and M occur in r below Q, then they both belong to q. Hence $W \cap M \in q$ and then also $W \cap M \in r$. Similarly, if W and M occur above Q, we get that $M \cap W \in p$, hence also $M \cap W \in r$. Let us consider the last case, when W occurs below Q and M occurs above Q. Then since Q is a transitive node, we have that $W \subseteq Q$. Hence $M \cap W = M \cap W \cap Q$. Observe that $M \cap Q$ is a node of p below Q, hence $M \cap Q \in \text{res}_Q(p)$. Since $q \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} \text{res}_Q(p)$, we know that $M \cap Q \in q$. Now we are done, since both $M \cap Q$ and W are nodes of q, and hence $M \cap W \cap Q \in q$, thus also $M \cap W = M \cap W \cap Q \in r$ by the remark above. □

Lemma 3.13 ([28, Lemma 2.21]). *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let Q be a small node in p. Suppose that $q \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap Q$ and that it extends $res_Q(p)$. Then p and q are directly compatible.*

Proof. First we will show that $p \cup q$ is an $\in$-increasing sequence. Let r denote the nodes of p that occur after Q. Then, $q \cup \{Q\} \cup r$ is an $\in$-increasing sequence, where nodes in q are ordered in the way they are ordered in q, followed by Q, followed by the nodes in r

ordered in the way they are ordered in p. This is an $\in$-increasing sequence since the last node in q is an element of Q, since $q \subseteq Q$ by elementarity, and other parts of the sequence are parts of the corresponding condition in the forcing.

Only nodes of p that do not belong to $q \cup \{Q\} \cup r$ are nodes in residue gaps of p in Q, because $q \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} \mathrm{res}_Q(p)$. We need to show that, if we add those nodes to $q \cup \{Q\} \cup r$ in a sufficiently nice way, then we get an $\in$-increasing sequence.

Claim 3. *Let $W \in res_Q(p)$ be a node of transitive type. If we add the nodes of $[Q \cap W, W)$ of p to $q \cup \{Q\} \cup r$ immediately before W and order it in the same way they are ordered in p, we will get an $\in$-increasing sequence.*

Proof. Since W is a node of condition p, and since the residue gap contains the node that occurs before W in p, we know that the last node of $[Q \cap W, W)$ of p is an element of W. The predecessor of W in q is contained in W, since q is a condition, and also in Q, since $q \in Q$. Thus the predecessor of W in q is an element of $Q \cap W$. Note that $[Q \cap W, W)$ is a segment of p, hence it is an $\in$-increasing sequence. Thus, $q \cup [Q \cap W, W) \cup \{Q\} \cup r$ is an $\in$-increasing sequence. □

By the previous claim, we may conclude that $p \cup q$ is an $\in$-increasing sequence. Our next goal is to add the nodes that are missing in $p \cup q$ in order to get a condition in our forcing. In order to do so, we will define E_W for every transitive node $W \in q \setminus p$. We consider two cases.

First case: if W is a transitive node of q which is not a node of p, and there are transitive nodes of p in the interval (W, Q) of $p \cup q$ and R is the first such node, then define E_W to be a sequence of small nodes of p, starting from $Q \cap R$, and ending with the predecessor of the first transitive node of p above $Q \cap R$. Observe that all the nodes in E_W contain W as an element. The first node of E_W is $Q \cap R$ and it contains W as an element since $W \in q \subseteq Q$ and since R is a transitive node of $p \cup q$, which is an $\in$-increasing sequence, and W occurs in $p \cup q$ before R. Observe that $Q \cap R$ is a subset of all the other nodes of E_W, since they form an $\in$-increasing sequence and they are countable elementary substructures of K. Thus W is an element of all nodes in E_W.

Second case: if W is a transitive node of q which is not an element of p and there are no transitive nodes of p in the interval (W, Q) in $p \cup q$, then define E_W to be the sequence of small nodes of p, starting from Q, and ending with the predecessor of the first transitive node of p above Q if such exists, otherwise it ends with the last small node of p. Note that W is an element of each node of E_W, since it is an element of the first node.

For every transitive $W \in q \setminus p$, let $F_W = \{M \cap W \mid M \in E_W\}$, with the ordering induced by the ordering of the nodes of E_W. Observe that all the nodes of F_W are small by Definition 3.3. Note that F_W is an $\in$-increasing sequence, since E_W is an $\in$-increasing sequence and $W \in M$ for all $M \in E_W$. Since all the nodes of F_W occur before W in an $\in$-increasing sequence $p \cup q$, then by Observation 3.5,(ii) we know that F_W is subset of W.

Let s be obtained from $p \cup q$ by adding all nodes in F_W for every transitive $W \in q \setminus p$ immediately before W, in the way they were ordered in F_W.

Claim 4. *s is an $\in$-increasing sequence.*

Proof. Observe that $p \cup q$ and F_W for $W \in q \setminus p$ are $\in$-increasing sequences. Hence, we only need to show that Condition (ii) of Definition 3.1. is satisfied at the endpoints of each F_W. Since $F_W \subseteq W$, we know that Condition (ii) of Definition 3.1 is satisfied at the right endpoint of each F_W. Let $W \in q \setminus p$. Then the first node of F_W is $Q \cap W$, by the way F_W was defined. Let N be the predecessor of W in s. We should show that N is an element of $Q \cap W$. If N is a node of q, then we are done, since $q \subseteq Q$ and $N \in W$ by Observation 3.5. Otherwise, assume that N is not an element of q. Then either N is an element of some residue gap of p in Q, or of added intervals $F_{\bar{W}}$ for some $\bar{W} \in q \setminus p$. If N would have been an element of some residue gap of p is Q, then this would imply that W is an element of p, which is not the case. Also, N cannot be an element of some $F_{\bar{W}}$ for some $\bar{W} \in q \setminus p$, since then W would have been an element of it, and this is a contradiction since W is a transitive node. This completes the proof of the claim. □

Now, we will end this proof by showing that s is closed under intersections. By Observation 3.6, it is enough to show that $(iii)'$ holds. So, let $W \in s$ be a node of transitive type and $M \in s$ be a node of small type, such that $W \in M$. We will show that $M \cap W$ is a node of s. Observe that for $\bar{W} \in q \setminus p$, $F_{\bar{W}}$ consists only of small nodes. Hence $W \in p \cup q$.

First, assume that $W \in p$. If W occurs above Q, we have that M occurs also above Q, hence both M and W are nodes in p, which implies that $M \cap W \in p$. Thus, assume that W occurs before Q. If $M \in p$, then it is clear that $M \cap W \in p$, hence $M \cap W \in s$. If M is an element of $F_{\bar{W}}$ for $\bar{W} \in q \setminus p$, then $M = \bar{M} \cap \bar{W}$, for some $\bar{M} \in p$. Note that $W \subseteq \bar{W}$, since M occurs before $\bar{W}$ in s and W occurs before M in s, and since $\bar{W}$ is a transitive node. Hence $M \cap W = \bar{M} \cap \bar{W} \cap W = \bar{M} \cap W$. But now, both $\bar{M}$ and W belong to p, hence $M \cap W = \bar{M} \cap W \in p$. Finally if $M \in q$, then $M \in Q$. Since M is a small node we have that $M \subseteq Q$ and hence $W \in Q$. This implies that $W \in \mathrm{res}_Q(p)$, hence also $W \in q$. Thus $M \cap W$ is a node of q, hence also it is a node of s.

Next, assume that W is not an element of p. Hence it is a node of $q \setminus p$. If $M \in q$, then $M \cap W \in q$ and thus also $M \cap W \in s$. If M is an element of $F_{\bar{W}}$ for $\bar{W} \in q \setminus p$, then $M = \bar{M} \cap \bar{W}$, for some $\bar{M} \in p$. Thus $M \cap W = \bar{M} \cap W$, since $W \subseteq \bar{W}$, by similar reasons as earlier in this proof. It is enough to show that $\bar{M} \cap W$ is an element of s. Hence we will just consider the case when $M \in p$. In case that there are transitive nodes $\bar{W}$ of p, that occur between W and M, we may replace M with $M \cap \bar{W}$, because $M \cap W = M \cap W \cap \bar{W}$ and $M \cap W$ is a node of p. Hence, we may assume that W is a node of $q \setminus p$, M is a node of $p \setminus q$, and there are no transitive nodes between W and M. Thus $M \in E_W$, which implies that $M \cap W \in F_W$. Hence $M \in s$. □

Corollary 3.14 ([28, Corollary 2.31])**.** *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let $s \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ and let Q be a node in s. Suppose that $t \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap Q$ and that it extends $\mathrm{res}_Q(s)$. Then s and t are directly compatible.* □

The following lemma is an easy consequence of the previous corollary.

Lemma 3.15 ([28, Corollary 2.32])**.** *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K. Let $M \in \mathcal{S} \cup \mathcal{T}$, and let $t \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ be an element of M. Then there is a condition $r \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} t$ which contains M, and r can be taken as the closure of $t \cup \{M\}$ under intersections.*

Proof. Let $s = \{M\} \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$. Observe that $\mathrm{res}_M(s) = \emptyset$. Since $t \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} \emptyset$, using Corollary 3.14, we know that s and t are directly compatible. Let r witness this. Then $r \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} t$, $M \in r$ and r is the closure of $t \cup \{M\}$ under intersections. □

Finally, the next lemma will show us for which models is Neeman's pure side condition strongly proper.

Lemma 3.16 ([28, Claim 4.1]). *Suppose that K is suitable and that $\mathcal{S}$ and $\mathcal{T}$ are appropriate for K.*

(i) Let $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$, and let M be a node in p. Then p is a strong master condition for M.
(ii) $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ is strongly proper for $\mathcal{S} \cup \mathcal{T}$.
(iii) Let $W \in \mathcal{S} \cup \mathcal{T}$. For any condition $p \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W$ and any node $M \in p$, we have that p is a strong master condition for M with respect to the forcing notion $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W$. It holds that $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W$ is strongly proper for $(\mathcal{S} \cup \mathcal{T}) \cap W$.

Proof. In order to prove (i), suppose for a contradiction that p is not a strong master condition for M. Thus, there is a condition $q \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ with $q \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} p$, that forces the generic filter for $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ not to intersect a dense set D of $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M$. Then by Lemma 3.10 we know that $\mathrm{res}_M(p) \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M$. Since D is a dense set of $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M$, there is $r \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M} \mathrm{res}_M(p)$ which is an element of D. Note that $r \in M$, since $D \subseteq M$. Observe that q and r are directly compatible by Corollary 3.14. Thus there is a condition $s \in \mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ that witnesses this direct compatibility. Note that $s \leq_{\mathbb{P}_{K,\mathcal{S},\mathcal{T}}} q$ and s forces that r is an element of the intersection of D with the generic object for $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$, which brings us to a contradiction, since q forces the generic filter for $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ not to intersect a dense set D.

By Lemma 3.15 and by (i), it follows that (ii) holds.

Let us now prove (iii). Since W is closed under intersections, one can see that if p and r in the proof of (i) both belong to W, then s also belongs to W. Hence the same proof can be used in order to prove the first part of (iii). By this, using Lemma 3.15 and the fact that W is closed under intersections, we may conclude that $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W$ is strongly proper for $(\mathcal{S} \cup \mathcal{T}) \cap W$. □

We will conclude this chapter with results showing that Neeman's pure side condition forcing and its quotient forcings satisfy the σ-approximation property. But before that, let us introduce a necessary definition and lemmas.

Definition 3.17. Let K be a transitive set, let $\mathcal{S}, \mathcal{T} \subseteq K$, and let $M \in \mathcal{S} \cup \mathcal{T}$. We let $\dot{\mathbb{Q}}^M_{K,\mathcal{S},\mathcal{T}}$ denote the canonical $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M)$-nice name for a suborder of $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ with the property that whenever G is $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M)$-generic over V, then $(\dot{\mathbb{Q}}^M_{K,\mathcal{S},\mathcal{T}})^G$ consists of all conditions p in $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ with $M \in p$ and $p \cap M \in G$.

Given a partial order $\mathbb{P}$ and a condition p, we will denote the suborder of $\mathbb{P}$ consisting of all conditions below p with $\mathbb{P}\lfloor p$.

Lemma 3.18. *Let K be a suitable set, let $\mathcal{S}$ and $\mathcal{T}$ be appropriate for K, and let $M \in \mathcal{S} \cup \mathcal{T}$. Then the map*

$$D^M_{K,\mathcal{S},\mathcal{T}} : \mathbb{P}_{K,\mathcal{S},\mathcal{T}}\lfloor\langle M\rangle \longrightarrow (\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M) * \dot{\mathbb{Q}}^M_{K,\mathcal{S},\mathcal{T}};\ p \longmapsto (p \cap M, \check{p})$$

is a dense embedding. Moreover, if G is $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$-generic over V *with $M \in \bigcup G$, then* $\mathrm{V}[G \cap M]$ *is a $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M)$-generic extension of* V *and* $\mathrm{V}[G]$ *is a $(\dot{\mathbb{Q}}^M_{K,\mathcal{S},\mathcal{T}})^{G\cap M}$-generic extension of* $\mathrm{V}[G \cap M]$.

Proof. The map $D^M_{K,\mathcal{S},\mathcal{T}}$ is a dense embedding by Corollary 3.14. Let G be $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$-generic over V with $M \in \bigcup G$. Then $\mathrm{V}[G \cap M]$ is a $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap M)$-generic extension of V and $\mathrm{V}[G]$ is a $(\dot{\mathbb{Q}}^M_{K,\mathcal{S},\mathcal{T}})^{G\cap M}$-generic extension of $\mathrm{V}[G \cap M]$, by Lemma 3.16,(i) and since $D^M_{K,\mathcal{S},\mathcal{T}}$ is a dense embedding. □

The following lemma summarizes two results of Neeman. The first statement follows from a standard application of strong properness, as is the proof of Lemma 2.11. The second statement is a consequence of the structural properties of $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ derived in this chapter, similar to the proof of Lemma 3.16.

Lemma 3.19 ([28, Claim 4.3 and Claim 4.4])**.** *Let K be a suitable set, let $\mathcal{S}, \mathcal{T}$ be appropriate for K, let $W \in \mathcal{T}$ and let G be $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W)$-generic over* V*. Define*

$$\hat{\mathcal{S}} \;=\; \{M \in \mathcal{S} \mid W \in M,\; M \cap W \in \bigcup G\}.$$

If $\mathcal{S}$ is a stationary subset of $\mathcal{P}(K)$ in V*, then $\hat{\mathcal{S}}$ is a stationary subset of $\mathcal{P}(K)$ in* $\mathrm{V}[G]$ *and the partial order $(\dot{\mathbb{Q}}^W_{K,\mathcal{S},\mathcal{T}})^G$ is strongly proper in* $\mathrm{V}[G]$ *for every element of $\hat{\mathcal{S}}$.*

Corollary 3.20. *Let K be a suitable set, let $\mathcal{S}, \mathcal{T}$ be appropriate for K, let $W \in \mathcal{T}$ and let G be $(\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \cap W)$-generic over* V*. If $\mathcal{S}$ is a stationary subset of $\mathcal{P}(K)$ in* V*, then the partial order $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ has the σ-approximation property, and the partial order $(\dot{\mathbb{Q}}^W_{K,\mathcal{S},\mathcal{T}})^G$ satisfies the σ-approximation property in* $\mathrm{V}[G]$.

Proof. This is an immediate consequence of Lemma 2.10, Lemma 3.16 and Lemma 3.19. □

Chapter 4

Some properties of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$

In this chapter, we are going to introduce Neeman's pure side condition forcing in a special setting. Namely, elements of conditions will be elementary submodels of $\mathrm{H}(\theta)$ for some infinite cardinal θ. Then we will present some results from [28], and prove some additional properties that we will need afterwards. From now on, we are going to use this specific forcing, which will be defined precisely in the following, in order to characterize some large cardinals.

Definition 4.1. Let θ be an infinite cardinal. Define $\mathcal{T}_\theta$ to be the set of all transitive and countably closed elementary submodels of $\mathrm{H}(\theta)$ that are elements of $\mathrm{H}(\theta)$, and $\mathcal{S}_\theta$ to be the set of all countable elementary submodels of $\mathrm{H}(\theta)$ that are elements of $\mathrm{H}(\theta)$.

In the following we will introduce countably inaccessible cardinals and show that θ being countably inaccessible implies that the forcing $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ fits into the framework introduced in Chapter 3.

Definition 4.2. A cardinal θ is *countably inaccessible* if and only if it is regular and for all $\delta < \theta$, we have that $\delta^\omega < \theta$ holds.[1]

The following lemma shows that countable inaccessibility suffices for the arguments present in [28, Section 5.1].

Lemma 4.3. *If θ is countably inaccessible, then $\mathrm{H}(\theta)$ is suitable and $\mathcal{S}_\theta$ and $\mathcal{T}_\theta$ are appropriate for $\mathrm{H}(\theta)$.*

Proof. It is easy to see that $\mathrm{H}(\theta)$ is suitable. Let $W \in \mathcal{T}_\theta$ and $M \in \mathcal{S}_\theta$ such that $W \in M$. Since W is countably closed and M is countable, we know that $W \cap M \in W$. Since there is a well-ordering of W in M we may define Skolem functions for W in M. By this and the fact that $M \prec \mathrm{H}(\theta)$ we may conclude that $M \cap W$ is an elementary submodel of W. Hence $M \cap W \prec \mathrm{H}(\theta)$. Thus $M \cap W \in \mathcal{S}_\theta$, since M is countable. $\square$

The next observations were made in [28, Section 5.1], and show that $\mathcal{T}_\theta$ and $\mathcal{S}_\theta$ are a large collection of models as well. We will lay out the easy arguments for the benefit of readers.

[1] Note that a countably inaccessible cardinal is always greater than ω_1.

Observation 4.4. *Assume that* θ *is countably inaccessible and let* $\alpha < \theta$. *Then* $\{M \in \mathcal{T}_\theta \mid \alpha \subseteq M\}$ *is stationary in* $\mathcal{P}(\mathrm{H}(\theta))$.

Proof. Pick $f : [\mathrm{H}(\theta)]^{<\omega} \to \mathrm{H}(\theta)$. Let $\vartheta > \theta$ be sufficiently large and regular. Construct an elementary continuous chain of models $\langle N_\gamma \prec \mathrm{H}(\vartheta) \mid \gamma < \theta\rangle$ such that $\alpha+1\cup\{f,\theta\} \subseteq N_0$, $|N_\gamma| < \theta$, $N_\gamma \cap \theta \in \theta$ and ${}^{\omega}N_\gamma \subseteq N_{\gamma+1}$ (we can do this since θ is countably inaccessible). For all γ with uncountable cofinality, we have that $N_\gamma \cap \mathrm{H}(\theta) \in \mathcal{T}_\theta$ and $f[[N_\gamma \cap \mathrm{H}(\theta)]^{<\omega}] \subset N_\gamma \cap \mathrm{H}(\theta)$. □

Observation 4.5. $\mathcal{S}_\theta$ *is stationary in* $\mathcal{P}(\mathrm{H}(\theta))$.

Proof. Let $f : [\mathrm{H}(\theta)]^{<\omega} \to \mathrm{H}(\theta)$. Pick $\vartheta > \theta$ to be sufficiently large and regular, and countable $M^* \prec \mathrm{H}(\vartheta)$ such that $\theta, f \in M^*$. Let $M = M^* \cap \mathrm{H}(\theta)$. Since $\theta \in M^*$, we know that M is a countable elementary submodel of $\mathrm{H}(\theta)$. Observe that M^* is closed under f. Thus M is also closed under f and we are done. □

Given an infinite cardinal θ and $M \in \mathcal{S}_\theta \cup \mathcal{T}_\theta$, we write $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ instead of $\mathbb{P}_{\mathrm{H}(\theta),\mathcal{S}_\theta,\mathcal{T}_\theta}$, $\dot{\mathbb{Q}}^M_\theta$ instead of $\dot{\mathbb{Q}}^M_{\mathrm{H}(\theta),\mathcal{S}_\theta,\mathcal{T}_\theta}$ and D^M_θ instead of $D^M_{\mathrm{H}(\theta),\mathcal{S}_\theta,\mathcal{T}_\theta}$.

Corollary 4.6. *Let* θ *be a countably inaccessible cardinal. Then the partial order* $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ *has the* σ*-approximation property and if* $W \in \mathcal{T}_\theta$ *and* G *is* $(\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap W)$*-generic over* V, *then the partial order* $(\dot{\mathbb{Q}}^W_\theta)^G$ *has the* σ*-approximation property in* $\mathrm{V}[G]$.

Proof. This is an immediate consequence of Corollary 3.20, Lemma 4.3 and Observation 4.5. □

The following lemma is proven by Itay Neeman [28, Claim 5.7] in a slightly different setting. We will include the proof of the lemma in our setting for the sake of completeness.

Lemma 4.7. *Suppose that* θ *is a countably inaccessible cardinal. Let* p *be a condition in* $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ *and let* $\alpha < \theta$ *be a cardinal with* $\mathrm{H}(\alpha) \in \mathcal{T}_\theta$. *Then there is* $r \leq_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} p$ *with* $\mathrm{H}(\alpha) \in r$. *Hence for every* $\mathrm{H}(\alpha) \in \mathcal{T}_\theta$ *and for every* G $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$*-generic over* V, *we have that* $\mathrm{H}(\alpha) \in \bigcup G$.

Proof. Suppose that α is a cardinal such that $\mathrm{H}(\alpha) \in \mathcal{T}_\theta$ and let p be a condition in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$. In case $p \subseteq \mathrm{H}(\alpha)$ then $p \cup \{\mathrm{H}(\alpha)\} \in \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$. Thus assume that $p \nsubseteq \mathrm{H}(\alpha)$. Let $M \in p$ be the first node which is not an element of $\mathrm{H}(\alpha)$. We will do induction on the rank of the least such M. If $M = \mathrm{H}(\alpha)$, then p satisfies the desired property. If α is an element of M, then $\mathrm{H}(\alpha)$ is an element of M. Hence $\mathrm{res}_M(p) \cup \{\mathrm{H}(\alpha)\} \in \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is an element of M and it is stronger than $\mathrm{res}_M(p)$. Thus by Corollary 3.14 there is some r which is stronger than both p and $\mathrm{res}_M(p) \cup \{\mathrm{H}(\alpha)\}$. Thus $\mathrm{H}(\alpha) \in r$ and $r \leq_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} p$ as desired. Hence, the only case that is left to consider is the case when $\alpha \notin M$ and $M \notin \mathrm{H}(\alpha)$. There exists some $\alpha < \beta \in M$, since $M \nsubseteq \mathrm{H}(\alpha)$. Thus M has to be an element of $\mathcal{S}_\theta$. Pick β^* to be the smallest ordinal greater than α in M. Then β^* is a cardinal by elementarity[2] and $M \cap \mathrm{H}(\beta^*) = M \cap \mathrm{H}(\alpha)$.

Claim 5. $\mathrm{H}(\beta^*) \prec \mathrm{H}(\theta)$.

[2]Otherwise the cardinality of β^* would be in M and it would be smaller or equal than α, which contradicts the fact that β^* is the smallest ordinal greater than α in M.

Proof. Suppose for a contradiction that there exist $a_1,...,a_n \in \mathrm{H}(\beta^*)$ and a formula ϕ such that $\mathrm{H}(\theta) \vDash \phi(a_1,...,a_n,y)$, but there is no such y in $\mathrm{H}(\beta^*)$[3]. Since $M \prec \mathrm{H}(\theta)$ and $\beta^* \in M$, we may by elementarity without loss of generality assume that $a_1,...,a_n \in M \cap \mathrm{H}(\beta^*)$. Note that $a_1,...,a_n \in \mathrm{H}(\alpha)$, since $M \cap \mathrm{H}(\beta^*) = M \cap \mathrm{H}(\alpha)$. By elementarity of $\mathrm{H}(\alpha)$, y can be found in $\mathrm{H}(\alpha)$ and hence also in $\mathrm{H}(\beta^*)$. □

Notice that β^* has uncountable cofinality, by elementarity of M and by the way β^* was picked.

Then $p \nsubseteq \mathrm{H}(\beta^*)$ and M is the first node of p that is not an element of $\mathrm{H}(\beta^*)$. Since $\beta^* \in M$, we know that $\mathrm{H}(\beta^*) \in M$. Thus $\mathrm{res}_M(p) \cup \{\mathrm{H}(\beta^*)\} \in \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is an element of M and it is stronger than $\mathrm{res}_M(p)$. Thus by Corollary 3.14 there is some p^* which is stronger than both p and $\mathrm{res}_M(p) \cup \{\mathrm{H}(\beta^*)\}$. Since $\mathrm{H}(\beta^*)$ occurs before M in p^* and it is not an element of $\mathrm{H}(\alpha)$, we may pick X_{p^*} to be the first node of p^* which is not an element of $\mathrm{H}(\alpha)$ and that occurs before M in p^*. Then $\mathrm{rank}(X_{p^*}) < \mathrm{rank}(M)$. By induction there is $q \leq_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} p^*$ such that $\mathrm{H}(\alpha) \in q$. □

Remark 4.8. *Suppose that θ is a countably inaccessible cardinal. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over* V. *Note that $\bigcup\bigcup G \subseteq \mathrm{H}(\theta)^{\mathrm{V}}$. By Lemma 3.15, by genericity, and since $\mathcal{S}_\theta$ is an unbounded set in $\mathcal{P}(\mathrm{H}(\theta))$, we know that $\mathrm{H}(\theta)^{\mathrm{V}} \subseteq \bigcup\bigcup G$. Hence, $\bigcup\bigcup G = \mathrm{H}(\theta)^{\mathrm{V}}$.*

The remaining results in this chapter are very useful regarding the forcing notions $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$.

Corollary 4.9. *Suppose that θ is a countably inaccessible cardinal. Then $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is proper, hence it preserves all uncountable cofinalities of ordinals and in particular it preserves ω_1.*

Proof. Let $\vartheta > \theta$ be a sufficiently large cardinal, and let M^* be a countable elementary submodel of $\mathrm{H}(\vartheta)$ such that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \in M^*$. Then $M := M^* \cap \mathrm{H}(\theta) \in \mathcal{S}_\theta$. By Lemma 4.3 and Lemma 3.16 we know that the partial order $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is strongly proper for $\mathcal{S}_\theta$. Using Observation 2.4 and since $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \subseteq \mathrm{H}(\theta)$, we know that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is proper for M^*. Hence $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is a proper forcing. □

Lemma 4.10. *Suppose that θ is an inaccessible cardinal. Let $\nu < \theta$ be a cardinal of uncountable cofinality. Then $\bar{\mathbb{P}}_{\mathcal{S}_\theta,\mathcal{T}_\theta} = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\nu)$ is a complete subforcing of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$.*

Proof. Let A be a maximal antichain of $\bar{\mathbb{P}}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$. By Lemma 4.7, we know that $\mathrm{H}(\nu) \in \bigcup G$ for any $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic G over V. Since Lemma 4.3 and Lemma 3.16,(i) show that $\langle \mathrm{H}(\nu) \rangle$ is a strong master condition for $\mathrm{H}(\nu)$, we know that $G \cap \mathrm{H}(\nu)$ is $(\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\nu))$-generic over V, and hence it intersects A, showing that A is a maximal antichain in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, as desired. □

We will close this chapter by showing that for inaccessible cardinals θ, partial orders of the form $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfy the θ-cc.

Lemma 4.11. *Let θ be an inaccessible cardinal. Then $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is θ-cc.*

[3]Note that here we use Tarski-Vaught test.

Proof. Let A be an antichain of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, pick a regular cardinal $\vartheta > \theta$, and pick $M^* \prec \mathrm{H}(\vartheta)$ of cardinality less than θ such that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ and A are both elements of M^*, and such that $M^* \cap \mathrm{H}(\theta) = \mathrm{H}(\nu) \in \mathcal{T}_\theta$. By elementarity, M^* thinks that A is a maximal antichain of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, which means that $A \cap \mathrm{H}(\nu)$ is a maximal antichain of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\nu)$. Lemma 4.10 implies that $A \cap \mathrm{H}(\nu)$ is a maximal antichain of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, and hence that $A = A \cap \mathrm{H}(\nu)$, and therefore, since θ is inaccessible, that A has size less than θ. $\square$

Chapter 5

Small embedding characterization of large cardinals

In this chapter, we are going to introduce small embedding characterizations of large cardinals. These kinds of characterizations will be used throughout this thesis. We will present small embedding characterizations of certain smaller large cardinals in this chapter, and, in the remaining part of this thesis, characterizations of this type of many other important large cardinals.

Throughout this thesis, we will show that whenever we have a direct implication between two large cardinals properties that we provide small embedding characterizations for, then amongst the embeddings witnessing the stronger property, we may also find such witnessing the weaker one. For the convenience of the reader, we remind the reader of the following definition, which already appeared in the introduction of the thesis.

Definition 5.1. Let $\theta < \vartheta$ be cardinals. A *small embedding for* θ is a non-trivial[1], elementary embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ with $j(\mathrm{crit}\,(j)) = \theta$ and $M \in \mathrm{H}(\vartheta)$ transitive.

The following lemma will directly yield small embedding characterizations of all the notions of large cardinals that can be characterized as being stationary limits of certain kinds of cardinals. The proof of the next lemma and the corollary following it, are very basic and shall perhaps be considered part of the set-theoretic folklore.

Lemma 5.2. *Given an $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$, the following statements are equivalent for every cardinal θ and every set x:*

(i) θ is a regular uncountable cardinal and the set of all ordinals $\lambda < \theta$ such that $\varphi(\lambda, x)$ holds is stationary in θ.

(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\varphi(\mathrm{crit}\,(j), x)$ and $x \in \mathrm{ran}(j)$.

Proof. First, assume that (i) holds, and pick a cardinal $\vartheta > \theta$ with $x \in \mathrm{H}(\vartheta)$. Let $\langle X_\alpha \mid \alpha < \theta\rangle$ be a continuous and increasing sequence of elementary substructures of $\mathrm{H}(\vartheta)$ of cardinality less than θ with $x \in X_0$ and $\alpha \subseteq X_\alpha \cap \theta \in \theta$ for all $\alpha < \theta$. By (i),

[1] An embedding j is non-trivial if there exists $x \in \mathrm{dom}(j)$ such that $j(x) \neq x$. The critical point of j is the smallest ordinal moved by j.

there is an $\alpha < \theta$ such that $\alpha = X_\alpha \cap \theta$ and $\varphi(\alpha, x)$ holds. Let $\pi : X_\alpha \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi^{-1} : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\varphi(\mathrm{crit}\,(\pi^{-1}), x)$ and $x \in \mathrm{ran}(\pi^{-1})$.

Now, assume that (ii) holds. Then there is a cardinal $\vartheta > \theta$ such that the formula φ is absolute between $\mathrm{H}(\vartheta)$ and V, and there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\varphi(\mathrm{crit}\,(j), x)$ holds and there is a $y \in M$ with $x = j(y)$. Then θ is uncountable, because elementarity implies that $j \restriction (\omega + 1) = \mathrm{id}_{\omega+1}$. Next, assume that θ is singular. Then $\mathrm{crit}\,(j)$ is singular in M and there is a cofinal function $c : \mathrm{cof}(\mathrm{crit}\,(j))^M \longrightarrow \mathrm{crit}\,(j)$ in M. In this situation, elementarity implies that $j(c) = c$ is cofinal in θ, a contradiction. Finally, assume that there is a club C in θ such that $\neg\varphi(\lambda, x)$ holds for all $\lambda \in C$. Then elementarity and our choice of ϑ imply that, in M, there is a club D in $\mathrm{crit}\,(j)$ such that $\neg\varphi(\lambda, y)$ holds for all $\lambda \in D$. Again, by elementarity and our choice of ϑ, we know that $j(D)$ is a club in θ with the property that $\neg\varphi(\lambda, x)$ holds for all $\lambda \in j(D)$. But elementarity also implies that $\mathrm{crit}\,(j)$ is a limit point of $j(D)$ and therefore $\mathrm{crit}\,(j)$ is an element of $j(D)$ with $\varphi(\mathrm{crit}\,(j), x)$, a contradiction. □

By changing the formula φ and only using the empty set as a parameter, we can use the above lemma to obtain small embedding characterizations of some of the smallest notions of large cardinals. Moreover, one can also characterize regular uncountable cardinals in such a way. Using the above lemma, the statements listed in the next corollary can be easily derived from the fact that weakly inaccessible cardinals are exactly regular stationary limits of cardinals, inaccessible cardinals are exactly regular stationary limits of strong limit cardinals, weakly Mahlo cardinals are exactly regular stationary limits of regular cardinals, and that Mahlo cardinals are exactly regular stationary limits of inaccessible cardinals.

Corollary 5.3. *Let θ be a cardinal.*

(i) θ is uncountable and regular if and only if for all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ.

(ii) θ is weakly inaccessible if and only if for all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\mathrm{crit}\,(j)$ *is a cardinal.*

(iii) θ is inaccessible if and only if for all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\mathrm{crit}\,(j)$ *is a strong limit cardinal.*

(iv) θ is weakly Mahlo if and only if for all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\mathrm{crit}\,(j)$ *is a regular cardinal.*

(v) θ is Mahlo if and only if for all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\mathrm{crit}\,(j)$ *is an inaccessible cardinal.* □

Trivially, for all types of large cardinals characterized in the previous corollary, we have that every small embedding witnessing the stronger large cardinal property also witnesses the weaker large cardinal property.

Remark 5.4. *In many important cases, and, in particular, in case of the characterizations provided by Corollary 5.3, the large cardinal properties in question can also be characterized by the existence of a single elementary embedding. For each of those, it suffices to require the existence of a single appropriate small embedding $j : M \longrightarrow \mathrm{H}(\theta^+)$, as* *can easily be seen from the proof of Lemma 5.2. For example, a cardinal θ is inaccessible if and only if there is a small embedding $j : M \longrightarrow \mathrm{H}(\theta^+)$ for θ with the property that $\mathrm{crit}\,(j)$ is a strong limit cardinal. This will in fact be the case for several of the small embedding characterizations that will follow, however we will not make any further mention of this.*

Proposition 5.5. *Any notion of large cardinal that implies weakly compact cardinals cannot be characterized as in Statement (ii) of Lemma 5.2, i.e. it cannot be characterized through the existence of small embeddings $j : M \longrightarrow \mathrm{H}(\vartheta)$ with $\varphi(\mathrm{crit}\,(j), x)$ and $x \in \mathrm{ran}(j)$ for a given formula $\varphi(v_0, v_1)$ and a set x.*

Proof. First, note that stationary subsets of weakly compact cardinals reflect to smaller inaccessible cardinals. Suppose for a contradiction that weakly compact cardinals can be characterized in such a way. Let θ be the smallest weakly compact cardinal. Then by Lemma 5.2, it satisfies (i) of the same lemma. By stationary reflection a smaller cardinal than θ satisfies (i) of Lemma 5.2, and again by the same lemma it satisfies (ii) of that lemma and this is a contradiction since we chose θ to be the smallest weakly compact cardinal. □

In the reminder of this chapter we will prove the following two technical lemmas that we will need later on.

Lemma 5.6. *The following statements are equivalent for every small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for a cardinal θ:*

(i) *θ is a strong limit cardinal.*
(ii) $\mathrm{crit}\,(j)$ *is a strong limit cardinal.*
(iii) $\mathrm{crit}\,(j)$ *is a cardinal and* $\mathrm{H}(\mathrm{crit}\,(j)) \subseteq M$.

Proof. Assume that (i) holds and pick a cardinal $\nu < \mathrm{crit}\,(j)$. Since $\mathrm{crit}\,(j)$ is a strong limit cardinal in M, we have $(2^\nu)^M < \mathrm{crit}\,(j)$. But then

$$2^\nu \;=\; j((2^\nu)^M) \;=\; (2^\nu)^M \;<\; \mathrm{crit}\,(j)$$

and this shows that (ii) holds. In the other direction, assume (i) fails. By elementarity, there is a cardinal $\nu < \mathrm{crit}\,(j)$ and an injection of $\mathrm{crit}\,(j)$ into $\mathcal{P}(\nu)$ in M. Then this injection witnesses that (ii) fails.

Now, again assume that (i) holds. Then elementarity implies that, in M, there is a bijection $s : \mathrm{crit}\,(j) \longrightarrow \mathrm{H}(\mathrm{crit}\,(j))$ with the property that $\mathrm{H}(\delta) = s[\delta]$ holds for every strong limit cardinal $\delta < \mathrm{crit}\,(j)$. Since we already know that (i) implies (ii), we have $\mathrm{H}(\mathrm{crit}\,(j)) = j(s)[\mathrm{crit}\,(j)]$. Fix $x \in \mathrm{H}(\mathrm{crit}\,(j))$ and $\alpha < \mathrm{crit}\,(j)$ with $j(s)(\alpha) = x$. Since $\mathrm{crit}\,(j)$ is a strong limit cardinal in M, we have $j \upharpoonright \mathrm{H}(\mathrm{crit}\,(j))^M = \mathrm{id}_{\mathrm{H}(\mathrm{crit}(j))^M}$ and this allows us to conclude that $x = j(s)(\alpha) = j(s(\alpha)) = s(\alpha) \in M$, and hence that (iii) holds.

Finally, assume for a contradiction that (iii) holds and (i) fails. Then, by elementarity, there is a minimal cardinal $\nu < \mathrm{crit}\,(j)$ such that either $(2^\nu)^M \geq \mathrm{crit}\,(j)$ or such that $\mathcal{P}(\nu)$ does not exist in M. By (iii), $\mathcal{P}(\nu) \subseteq M$. By elementarity, we may pick an injection $\iota : \mathrm{crit}\,(j) \longrightarrow \mathcal{P}(\nu)$ in M. Define $x = j(\iota)(\mathrm{crit}\,(j)) \in \mathcal{P}(\nu) \subseteq M$. Then $j(x) = x$, and elementarity yields an ordinal $\gamma < \mathrm{crit}\,(j)$ with $\iota(\gamma) = x$. But then $j(\iota)(\gamma) = x = j(\iota)(\mathrm{crit}\,(j))$, contradicting the injectivity of ι. □

Next, we isolate a property of small embedding characterizations, that will be important throughout this thesis.

Definition 5.7. Let $\Phi(v_0, v_1)$ be an $\mathcal{L}_\in$-formula and let x be a set. We call the pair (Φ, x) *restrictable* if for every cardinal θ, there is an ordinal α such that if

- $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\Phi(j, x)$ and $x \in \mathrm{ran}(j)$, and
- ν is a cardinal in M with $\nu > \mathrm{crit}\,(j)$ and $j(\nu) > \alpha$,

then $\Phi(j \restriction \mathrm{H}(\nu)^M, x)$ holds.

Note that the small embedding characterizations (i) – (v) provided by Corollary 5.3 are given by pairs (Φ, x) such that $x = \emptyset$ and $\Phi(j, \emptyset)$ states that $\varphi(\mathrm{crit}\,(j))$ holds for some formula $\varphi(v)$. In particular, these pairs (Φ, x) are trivially restrictable. Next, note that the small embedding formulation of Magidor's characterization of supercompactness in Corollary 1.4 is given by the pair (Φ, x) with $x = \emptyset$ and

$$\Phi(v_0, v_1) \;\equiv\; \text{“}\exists\delta \; \mathrm{dom}(v_0) = \mathrm{H}(\delta)\text{”},$$

and this pair is obviously restrictable as well. Finally, we remark that the pairs (Φ, x) used in the small embedding characterizations provided in the remainder of this thesis will all be restrictable. The verification of restrictability will be trivial in each case, and is thus left for the interested reader to check throughout. The following lemma will be the key consequence of restrictability.

Lemma 5.8. *Let (Φ, x) be restrictable and assume that θ is a cardinal with the property that for sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\Phi(j, x)$ and $x \in \mathrm{ran}(j)$. Then for all sets z and sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\Phi(j, x)$ and $z \in \mathrm{ran}(j)$.*

Proof. By our assumptions, there is an ordinal $\alpha > \theta$ such that the following statements hold:

(i) For all cardinals $\vartheta > \alpha$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\Phi(j, x)$ and $x \in \mathrm{ran}(j)$.
(ii) If $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ such that $\Phi(j, x)$ holds, $\nu > \mathrm{crit}\,(j)$ is a cardinal in M, $x \in \mathrm{ran}(j)$ and $j(\nu) > \alpha$, then $\Phi(j \restriction \mathrm{H}(\nu)^M, x)$ holds.

Assume for a contradiction that the conclusion of the lemma does not hold. Pick a strong limit cardinal $\vartheta > \alpha$ with the property that $\mathrm{H}(\vartheta)$ is sufficiently absolute in V and fix a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that $\Phi(j, x)$ holds, and fix $y \in M$ with $j(y) = x$. In this situation, our assumptions, the absoluteness of $\mathrm{H}(\vartheta)$ in V and the elementarity of j imply that there are $\beta, \mu, z \in M$ such that the following statements hold in M:

(a) If $k : N \longrightarrow \mathrm{H}(\eta)$ is a small embedding for $\mathrm{crit}\,(j)$ such that $\Phi(k, y)$ holds, $\nu > \mathrm{crit}\,(k)$ is a cardinal in N, $y \in \mathrm{ran}(k)$ and $k(\nu) > \beta$, then $\Phi(k \restriction \mathrm{H}(\nu)^N, y)$ holds.
(b) $\mu > \beta$ is a cardinal with $y, z \in \mathrm{H}(\mu)$ and there is no small embedding $k : N \longrightarrow \mathrm{H}(\mu)$ for $\mathrm{crit}\,(j)$ with $\Phi(k, y)$ and $z \in \mathrm{ran}(k)$.

By elementarity and our absoluteness assumptions on $\mathrm{H}(\vartheta)$, the above implies that the following statements hold in V:

(a)′ If $k : N \longrightarrow \mathrm{H}(\eta)$ is a small embedding for θ and $\mathrm{crit}\,(k) < \nu \in N$ is a cardinal in N such that $\Phi(k, x)$ holds, $x \in \mathrm{ran}(k)$ and $k(\nu) > j(\beta)$, then $\Phi(k \restriction \mathrm{H}(\nu)^N, x)$ holds.

(b)′ $j(\mu) > j(\beta)$ is a cardinal with $x, j(z) \in \mathrm{H}(j(\mu))$ and there is no small embedding $k : N \longrightarrow \mathrm{H}(j(\mu))$ for θ with $\Phi(k, x)$ and $j(z) \in \mathrm{ran}(k)$.

Since $j(\mu) > j(\beta)$, we can apply the statement (a)′ to $j : M \longrightarrow \mathrm{H}(\vartheta)$ and μ to conclude that $\Phi(j \restriction \mathrm{H}(\mu)^M, x)$ holds in V. But we also have $j(z) \in \mathrm{ran}(j \restriction \mathrm{H}(\mu)^M)$ and together these statements contradict (b)′. □

In the end of this chapter, let us show that the above lemma implies a somewhat stronger statement, essentially allowing us to switch the quantifiers on z and on ϑ.

Lemma 5.9. *Let (Φ, x) be restrictable and assume that θ is a cardinal with the property that for sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\Phi(j, x)$ and $x \in \mathrm{ran}(j)$. Then for all sufficiently large cardinals ϑ and for all $z \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\Phi(j, x)$ and $z \in \mathrm{ran}(j)$.*

Proof. Fix a sufficiently large cardinal ϑ and some $z \in \mathrm{H}(\vartheta)$. By Lemma 5.8, there is a cardinal ϑ' and a small embedding $j' : M' \longrightarrow \mathrm{H}(\vartheta')$ for θ with $\Phi(j', x)$ and $z, \vartheta \in \mathrm{ran}(j')$. Let j be the restriction of j' to $M = \mathrm{H}((j')^{-1}(\vartheta))^{M'}$. Then $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\Phi(j, x)$ and $z \in \mathrm{ran}(j)$. □

Chapter 6

Inaccessible cardinals

In this chapter, we will first characterize countably inaccessible cardinals via Neeman's pure side condition forcing. Then, we will produce a strong forcing characterization of those cardinals.

The forward direction of the next theorem was already shown for inaccessible cardinals in [28, Chapter 5.1]. Although the proof is the same for countably inaccessible cardinals, proofs for both directions will be included here for the sake of completeness.

Theorem 6.1. *Let θ be an infinite cardinal. Then the following statements are equivalent:*

(i) θ is countably inaccessible,

(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "$\check{\theta} = \omega_2$"

(iii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "$\check{\theta}$ is a regular cardinal greater than ω_1".

Proof. Assume that (i) holds. We will show that (ii) holds. By Corollary 4.9 we know that ω_1 is preserved. Note that $\{M \in \mathcal{T}_\theta \mid \alpha \subseteq M\}$ is stationary in $\mathcal{P}(\mathrm{H}(\theta))$ by Observation 4.4. Thus using Lemma 2.11, Lemma 3.16 and Lemma 4.3 we may conclude that θ is preserved. Hence it suffices to show the following claim.

Claim 6. *Forcing with $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ collapses all cardinals between ω_1 and θ to ω_1.*

Proof. Let G be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic filter over V. Observe that $\bigcup G$ is a set of nodes. Since any two different transitive nodes $W_1, W_2 \in \bigcup G$ have to be elements of some condition in G, and thus $W_1 \in W_2$ or $W_2 \in W_1$, we know that $T = \{W \in \bigcup G \mid W \in \mathcal{T}_\theta\}$ is a well-ordered set with respect to $\in$, and thus also with respect to $\subset$.
By Lemma 4.7, we know that $\bigcup T = \mathrm{H}(\theta)$. Let W and W^* be two successive nodes in T. Define C_{W,W^*} to be the set of all small nodes $M \in \bigcup G$ such that M is between W and W^*, i.e.

$$C_{W,W^*} = \{M \in \mathcal{S}_\theta \mid \exists p \in G \text{ with } W, M, W^* \in p \text{ such that } W \in M \in W^*\}.$$

Again, it is clear that C_{W,W^*} is a well-ordered set with respect to $\in$, and hence also with respect to $\subset$.
Now, we will show that $\bigcup C_{W,W^*} = W^*$. It holds trivially that $\bigcup C_{W,W^*} \subset W^*$. For the other direction, let $x \in W^*$. By genericity and by Lemma 3.15, there is $M \in \bigcup G \cap \mathcal{S}_\theta$ such that $\{W, x\} \subset M$. We know that all conditions in G are closed under intersections,

so $M \cap W^*$ is an element of $\bigcup G$, and hence $M \cap W^* \in C_{W,W^*}$. Thus $x \in \bigcup C_{W,W^*} = W^*$. Let α be some ordinal smaller than θ. Then by genericity and Lemma 3.15, there is some $W^* \in T$ with $\alpha \in W^*$. Let W be the predecessor of W^* in T. The size of C_{W,W^*} is at most $\aleph_1$, since C_{W,W^*} is a sequence of $\subset$-increasing countable models. Thus we have that $\bigcup C_{W,W^*} = W^*$ has size at most $\aleph_1$ in $\mathrm{V}[G]$. Hence the cofinality of α is at most ω_1 in $\mathrm{V}[G]$. Thus every ordinal between ω_1 and θ is collapsed to ω_1 in $\mathrm{V}[G]$. □

Observe that (ii) implies (iii) holds trivially. Assume that there is an infinite cardinal θ with the property that (i) fails and (iii) holds. Then we know that θ is a regular cardinal greater than ω_1 and there is a $\delta < \theta$ with $\delta^\omega \geq \theta$. Let δ_0 be minimal with this property.

Claim. $\mathcal{T}_\theta = \emptyset$.

Proof of the Claim. Assume, towards a contradiction, that there exists a W in $\mathcal{T}_\theta$. By elementarity, there is an ordinal $\gamma \in W$ with the property that for all $x \in W$, there is a function $f : \omega \longrightarrow \gamma$ in W with $f \notin x$. Let γ_0 be minimal with this property. Then $\gamma_0 \geq \delta_0$, because otherwise the minimality of δ_0 would imply that ${}^\omega\gamma_0 \in \mathrm{H}(\theta)$ and elementarity would then allow us to show that ${}^\omega\gamma_0 \in \mathrm{H}(\theta)$ is contained in W, contradicting our assumptions on γ_0. But then $\delta_0 \in W$ and the countable closure of W implies that ${}^\omega\delta_0 \in \mathrm{H}(\theta) \subseteq W$. By our assumption, we can conclude that $|W| \geq \delta_0^\omega \geq \theta$ and hence $W \notin \mathrm{H}(\theta)$, a contradiction. □

Claim. *Given $x \in \mathrm{H}(\theta)$, the set $\{p \in \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \mid x \in \bigcup p\}$ is dense in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$.*

Proof of the Claim. Fix a condition p in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ and let N denote the Skolem hull of $\{p, x\}$ in $\mathrm{H}(\theta)$. Since θ is uncountable and regular, we have $N \in \mathcal{S}_\theta$. Moreover, the above claim shows that $p \subseteq \mathcal{S}_\theta$ and this implies that $M \subseteq N$ holds for all $M \in p$. In particular, the set $p \cup \{N\}$ is a condition in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ below p. □

Now, let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. Then the above claims show that $\bigcup G \subseteq S_\theta^{\mathrm{V}}$ and $\mathrm{H}(\theta)^{\mathrm{V}} = \bigcup\bigcup G$. The first statement directly implies that $\bigcup G$ is well-ordered by $\subsetneq$ in $\mathrm{V}[G]$ and every proper initial segment of this well-order is a subset of an element of $\mathcal{S}_\theta^{\mathrm{V}}$. In combination with the second statement, this shows that θ is a union of ω_1-many countable sets in $\mathrm{V}[G]$, contradicting (iii). □

As we mentioned before, the main result of this section is that the sequence $\langle \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \mid \theta \in \mathrm{Card}\rangle$ strongly characterizes the class of inaccessible cardinals. In order to prove this, we will use a slight variation of the small embedding characterization of inaccessible cardinals introduced in Chapter 5. Although we do not really need to make use of such a characterization in this proof, we will do so, since we would like to introduce some techniques that will be necessary later on.

The proof of the following lemma easily follows from the proof of Lemma 5.2 using the fact that an uncountable regular cardinal θ is inaccessible if and only if the set of all strong limit cardinals of uncountable cofinality below θ forms a stationary set of θ.

Lemma 6.2. *The following statements are equivalent for every cardinal θ:*

(i) θ is inaccessible.

(ii) For all sufficiently large cardinals ϑ and all $x \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $x \in \mathrm{ran}(j)$ and $\mathrm{crit}\,(j)$ is a strong limit cardinal of uncountable cofinality.

The following lemma will be very useful later on. It will help us to establish some connections between small embeddings and Neeman's pure side condition forcing.

Lemma 6.3. *Let θ be an inaccessible cardinal and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ with the property that ϑ is regular and $\mathrm{crit}\,(j)$ is a strong limit cardinal of uncountable cofinality. If G is $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V, then the following statements hold.*

(i) $\mathrm{H}(\mathrm{crit}\,(j)) \in M \cap \mathcal{T}_\theta$ and $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}} = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\mathrm{crit}\,(j)) \in M$ is a complete suborder of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ with $j(\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}) = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$.

(ii) $G_j := G \cap \mathrm{H}(\mathrm{crit}\,(j))$ is $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-generic over V and $\mathrm{V}[G]$ is a $(\dot{\mathbb{Q}}_\theta^{\mathrm{H}(\mathrm{crit}(j))})^{G_j}$-generic extension of $\mathrm{V}[G_j]$.

(iii) $(\mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ has the σ-approximation property.

(iv) There is an elementary embedding $j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ with the property that $j_G(\dot{x}^{G_j}) = j(\dot{x})^G$ holds for every $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-name $\dot{x}$ in M.

Proof. Using the inaccessibility of θ and elementarity, we can find a surjection

$$e : \mathrm{crit}\,(j) \longrightarrow \mathrm{H}(\mathrm{crit}\,(j))^M$$

with the property that $j(e)[\kappa] = \mathrm{H}(\kappa)$ holds for every strong limit cardinal $\kappa \leq \theta$. But this implies that

$$\mathrm{H}(\mathrm{crit}\,(j)) \;=\; j(e)[\mathrm{crit}\,(j)] \;=\; e[\mathrm{crit}\,(j)] \;=\; \mathrm{H}(\mathrm{crit}\,(j))^M.$$

Since $\mathrm{crit}\,(j)$ is a strong limit cardinal with uncountable cardinality, we know that the set $\mathrm{H}(\mathrm{crit}\,(j))$ is a transitive, countably closed elementary substructure of $\mathrm{H}(\theta)$, $\mathcal{S}_{\mathrm{crit}(j)} = \mathcal{S}_\theta \cap \mathrm{H}(\mathrm{crit}\,(j))$, $\mathcal{T}_{\mathrm{crit}(j)} = \mathcal{T}_\theta \cap \mathrm{H}(\mathrm{crit}\,(j))$,

$$\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)}\mathcal{T}_{\mathrm{crit}(j)}} = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\mathrm{crit}\,(j)) = \mathbb{P}^M_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}},$$

and Lemma 4.10 implies that $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$ is a complete suborder of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$. Since the partial order $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is uniformly definable from the parameter θ, we also obtain that

$$j(\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}) = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}.$$

This proves (i). By Lemma 4.7 and by (i), we know that $\langle \mathrm{H}(\mathrm{crit}\,(j))\rangle \in G$. Then using Lemma 3.18 and Lemma 4.3, we get that G_j is $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-generic over V and $\mathrm{V}[G]$ is a $(\dot{\mathbb{Q}}_\theta^{\mathrm{H}(\mathrm{crit}(j))})^{G_j}$-generic extension of $\mathrm{V}[G_j]$. This shows (ii). Using the first two statements, Corollary 4.6 shows that the pair $(\mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ has the σ-approximation property. Finally, since $j(\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}) = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ and $j[G_j] = G_j \subseteq G$, we can apply [5, Proposition 9.1] and get an embedding

$$j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)[G]$$

with $j_G(\dot{x}^{G_j}) = j(\dot{x})^G$ for every $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-name $\dot{x}$ in M. Using that $\mathrm{H}(\vartheta)^{\mathrm{V}[G]} = \mathrm{H}(\vartheta)[G]$, this shows that (iv) holds. □

Let us now recall the definition of a set theoretic tree and some relevant notions which we will use later on.

Definition 6.4. A tree is a partial order $(\mathbb{T}, <_{\mathbb{T}})$ such that the following hold.

(i) $(\mathbb{T}, <_{\mathbb{T}})$ has a unique minimal element $\mathrm{root}(\mathbb{T})$.
(ii) For every $t \in \mathbb{T}$ set of all predecessors of t, $pred_{\mathbb{T}}(t) = \{s \in \mathbb{T} \mid s <_{\mathbb{T}} t\}$, is well-ordered by $<_{\mathbb{T}}$.

Definition 6.5. Let $(\mathbb{T}, <_{\mathbb{T}})$ be a tree.

(i) Given $t \in \mathbb{T}$, we define the length of t, $\mathrm{lh}_{\mathbb{T}}(t)$, to be the order-type of $pred_{\mathbb{T}}(t)$.
(ii) Define the height of $(\mathbb{T}, <_{\mathbb{T}})$ to be $\mathrm{ht}(\mathbb{T}) = \sup_{t\in\mathbb{T}} \mathrm{lh}_{\mathbb{T}}(t)$.
(iii) For every $\alpha < \mathrm{ht}(\mathbb{T})$ we define the α-th level of $\mathbb{T}$ by $\mathbb{T}(\alpha) = \{t \in \mathbb{T} \mid \mathrm{lh}_{\mathbb{T}}(t) = \alpha\}$.
(iv) A chain in $(\mathbb{T}, <_{\mathbb{T}})$ is a subset $c \subset \mathbb{T}$ that is linearly ordered by $<_{\mathbb{T}}$. A branch through $\mathbb{T}$ is a chain that is $\leq_{\mathbb{T}}$-downward closed in $\mathbb{T}$. The length of a branch is the order-type of it.

In the next definition we will introduce a combinatorial concept that will be used in our characterization of inaccessible cardinals via Neeman's pure side condition forcing.

Definition 6.6. A *weak Kurepa tree* is a tree of height ω_1 and cardinality $\aleph_1$ and with at least $\aleph_2$-many cofinal branches.

Theorem 6.7 ([12])**.** *Let θ be a countably inaccessible cardinal. Then the following are equivalent.*

(i) θ is an inaccessible cardinal.
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "There are no weak Kurepa trees".

Proof. Let us first consider the forward direction. Assume that θ is inaccessible. Let $\dot{\mathbb{T}}$ be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for a tree of height ω_1 and cardinality $\aleph_1$ and let $\dot{x}$ be a nice $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for a subset of ω_1 coding $\dot{\mathbb{T}}$. Suppose further that the inaccessibility of θ is witnessed by the small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ with $\dot{x} \in \mathrm{ran}(j)$, and such that $\mathrm{crit}\,(j)$ is a strong limit cardinal of uncountable cofinality, using Lemma 6.2. Since $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfies the θ-cc by Lemma 4.11, we know that $\dot{x} \in \mathrm{H}(\theta)$. Observe that Lemma 6.3 shows that $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}} = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\mathrm{crit}\,(j))$. By the above, elementarity shows that $\dot{x}$ is a $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-name which is an element of M. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. By Lemma 6.3, we know that

$$G_j := G \cap \mathrm{H}(\mathrm{crit}\,(j))$$

is $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-generic over V and $\mathrm{V}[G]$ is a $(\dot{\mathbb{Q}}_\theta^{\mathrm{H}(\mathrm{crit}(j))})^{G_j}$-generic extension of $\mathrm{V}[G_j]$. By the same lemma, the pair $(\mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ has the σ-approximation property and there is an embedding

$$j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$$

with $j_G(\dot{x}^{G_j}) = j(\dot{x})^G$ for every $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-name $\dot{x}$ in M. By Corollary 4.9 and since the pair $(\mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ has the σ-approximation property, we know that every cofinal branch through $\dot{\mathbb{T}}^G$ in $\mathrm{V}[G]$ is an element of $\mathrm{V}[G_j]$. Since $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$ has size less than θ, we know that θ is still inaccessible in $\mathrm{V}[G_j]$, and therefore that $\dot{\mathbb{T}}^G$ has less than θ-many cofinal branches in $V[G]$. But Theorem 6.1 shows that $\theta = \omega_2^{V[G]}$, which allows us to conclude that $\dot{\mathbb{T}}^G$ is not a weak Kurepa tree in $V[G]$.

For the reverse implication, suppose that θ is countably inaccessible, however not inaccessible. Pick the least $\lambda < \theta$ such that $2^\lambda \geq \theta$. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. Since θ is countably inaccessible, the cofinality of λ is uncountable in V, and hence ω_1 in $V[G]$, by Corollary 4.9. In $V[G]$, pick a strictly increasing cofinal sequence s of order type ω_1 of ordinals in λ, and let $\mathbb{T}$ be the tree that consists exactly of the levels of the tree ${}^{<\lambda}2$ that are indexed by the ordinals in the range of s. By the minimality of λ, the levels of $\mathbb{T}$ have size at most $\aleph_1$ in $V[G]$, however $\mathbb{T}$ has $2^\lambda \geq \aleph_2^{V[G]}$ many branches. Hence $\mathbb{T}$ is a weak Kurepa tree in $V[G]$. □

A combination of the above result with Theorem 6.1 now shows that the sequence $\langle \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \mid \theta \in \mathrm{Card} \rangle$ characterizes the class of inaccessible cardinals through the statement

> "*θ is a regular cardinal greater than ω_1 with the property that for every uncountable cardinal $\kappa < \theta$, every tree of cardinality and height κ has less than θ-many cofinal branches*".

Observe that this characterization is strong for trivial reasons.

Chapter 7

Mahlo cardinals

In this chapter, we will show that the sequence $\langle \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \mid \theta \in \mathrm{Card}\rangle$ strongly characterizes the class of all Mahlo cardinals through the non-existence of special $\aleph_2$-Aronszajn trees in generic extensions. We start by introducing the relevant notion of specialness.

Definition 7.1. (i) A tree $\mathbb{T}$ of height κ^+ is called *κ-special* if there is a function $f : \mathbb{T} \longrightarrow \kappa$ such that for all $x, y \in \mathbb{T}$ we have that $x <_{\mathbb{T}} y$ implies $f(x) \neq f(y)$. In other words a tree of height κ^+ is κ-special if and only if it is a union of κ-many antichains.

(ii) Let κ be a regular and uncountable cardinal. A tree $\mathbb{T}$ of height κ is called *special*, if there is a regressive map[1] $r : \mathbb{T} \to \mathbb{T}$ with the property that $r^{-1}[\{t\}]$ is the union of less than κ-many antichains in $\mathbb{T}$, for every $t \in \mathbb{T}$.

A result of Todorčević (see [32, Theorem 14]) shows that, given an uncountable cardinal κ, a tree of height κ^+ is κ-special if and only if it is special. Thus Definition 7.1,(ii) generalizes Definition 7.1,(i). In addition, Todorčević showed that an inaccessible cardinal θ is Mahlo if and only if there are no special θ-Aronszajn trees (see [35, Theorem 6.1.4])

It is easy to see that ZFC^- proves that special trees do not have cofinal branches. Moreover, note that the statement "$\mathbb{T}$ *is special*" is upwards-absolute between transitive models of ZFC^- in which the height of $\mathbb{T}$ remains regular.

In the case of small embedding characterizations of large cardinal properties that imply the Mahloness of the given cardinal, the combinatorics obtained by lifting the witnessing embeddings to a suitable collapse extension can be phrased as meaningful combinatorial principles, entitled *internal large cardinals*. The concept of internal large cardinals was introduced by Peter Holy and Philipp Lücke, and we will make use of this concept in many places. While the general setup will be postponed to the forthcoming [11], we will only introduce and make use of internal large cardinals with respect to the σ-approximation property. These principles describe strong fragments of large cardinal properties that were characterized by small embeddings, which however can also hold at potentially small cardinals θ, by postulating the existence of small embeddings $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ together with the existence of a transitive ZFC^- model N such that $M \in N \subseteq \mathrm{H}(\vartheta)$, for which the correctness property that held between M and V in the original small embedding characterization now holds between M and N, and some correctness property

[1] A map $r : \mathbb{T} \to \mathbb{T}$ is regressive if $r(t) <_{\mathbb{T}} t$ for every $t \in \mathbb{T} \setminus \{\mathrm{root}(\mathbb{T})\}$.

induced by the properties of the tails of the collapse forcing used holds between N and $\mathrm{H}(\vartheta)$. These definitions of internal large cardinals are strongly related to their small embedding characterizations ([13]). For that reason let us first introduce the following small embedding characterization for Mahlo cardinals that is a combination of Statement (v) in Corollary 5.3 and Lemma 5.9.

Lemma 7.2. *The following statements are equivalent for every cardinal θ:*

(i) θ is a Mahlo cardinal.

(ii) For every sufficiently large cardinal ϑ and all $x \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $x \in \mathrm{ran}(j)$ and $\mathrm{crit}\,(j)$ is inaccessible.

Now, we are ready to introduce the following internal version of Mahlo cardinals.

Definition 7.3. We say that a cardinal θ is *internally AP Mahlo* if for all sufficiently large regular cardinals ϑ and all $x \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ, and a transitive model N of ZFC^- such that $x \in \mathrm{ran}(j)$, and the following statements hold:

(i) $N \subseteq \mathrm{H}(\vartheta)$.

(ii) The pair $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.

(iii) $M \in N$, and $\mathcal{P}_{\omega_1}(\mathrm{crit}\,(j))^N \subseteq M$.

(iv) $\mathrm{crit}\,(j)$ is regular in N.

Lemma 7.4. *Suppose that θ is an internally AP Mahlo cardinal. Then θ is an uncountable and regular cardinal, and there are no special θ-Aronszajn trees.*

Proof. First, let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be any small embedding for θ. Then by elementarity we know that $\mathrm{crit}\,(j)$ is an uncountable and regular cardinal in M. Hence, again by elementarity, we have that θ is uncountable and regular in $\mathrm{H}(\vartheta)$.
Now, assume for a contradiction that there is a special θ-Aronszajn tree $\mathbb{T}$. Fix a level-by-level enumeration e of $\mathbb{T}$, and note that $\mathrm{dom}(e) = \theta$. Let c be the club subset of ordinals $\alpha < \theta$ such that $e[\alpha] = \mathbb{T}_{<\alpha}$. Let $\vartheta = \theta^+$, and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ with c and e in the range of j and N witness that θ is internally AP Mahlo with respect to ϑ. Note that $\mathbb{T} \in \mathrm{ran}(j)$. By elementarity and because $c \in \mathrm{ran}(j)$, it follows that $\mathrm{crit}\,(j)$ is a limit point of c and hence $\mathrm{crit}\,(j) \in c$, and hence because $e \in \mathrm{ran}(j)$, $\bar{\mathbb{T}} := j^{-1}(\mathbb{T}) = \mathbb{T}_{<\mathrm{crit}(j)}$ is a special $\mathrm{crit}\,(j)$-Aronszajn tree in M, and hence also in N, for $\mathrm{crit}\,(j)$ remains regular in N. Let t be a node of $\mathbb{T}$ on level $\mathrm{crit}\,(j)$, and note that t gives rise to a branch through $\bar{\mathbb{T}}$. Every proper initial segment of this branch is already in N, for it is definable as the set of $\bar{\mathbb{T}}$-predecessors of a single element of $\bar{\mathbb{T}}$. Since $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property, and $\mathrm{crit}\,(j)$ is regular and uncountable in N, we thus obtain $t \in N$. But this contradicts the fact that $\bar{\mathbb{T}}$ is an Aronszajn tree in N. □

Lemma 7.5. *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a Mahlo cardinal.

(ii) θ is internally AP Mahlo.

Proof. First, assume that (i) holds. Let ϑ be a sufficiently large regular cardinal in the sense of Lemma 7.2. Pick $x \in \mathrm{H}(\vartheta)$ and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ

such that $x \in \operatorname{ran}(j)$ and $\operatorname{crit}(j)$ inaccessible given by Lemma 7.2. Let $N = \mathrm{H}(\vartheta)$. Then j and N witness that (ii) holds. Next, assume that (ii) holds. Then Lemma 7.4 implies that there are no special θ-Aronszajn trees and [35, Theorem 6.1.4] then shows that θ is a Mahlo cardinal. □

Theorem 7.6 ([12]). *Let θ be an inaccessible cardinal. Then the following statements are equivalent.*

(i) θ is a Mahlo cardinal.
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "ω_2 is internally AP Mahlo".
(iii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "There are no special ω_2-Aronszajn trees".

Proof. To show that (i) implies (ii), assume that θ is a Mahlo cardinal. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V, let ϑ be sufficiently large in the sense of Lemma 7.2, and let $x \in \mathrm{H}(\vartheta)$. Since it follows by Lemma 4.11 that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfies the θ-cc, we can find a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name $\dot{x}$ in $\mathrm{H}(\vartheta)^{\mathrm{V}}$ with $\dot{x}^G = x$. Using Lemma 7.2, let Mahloness of θ be witnessed by the small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ such that $\dot{x}$ is an element of the range of j and $\operatorname{crit}(j)$ is inaccessible. Next, let $j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ be the embedding given by Proposition 6.3, and set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}$. Then $M[G_j] \in N \subseteq \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$, and the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ satisfies the σ-approximation property. Moreover, since $\operatorname{crit}(j)$ is inaccessible in V and $\mathbb{P}_{\mathcal{S}_{\operatorname{crit}(j)},\mathcal{T}_{\operatorname{crit}(j)}} = \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \cap \mathrm{H}(\operatorname{crit}(j))$, Theorem 6.1 implies that $\operatorname{crit}(j)$ is regular in N. Finally, a combination of Lemma 4.11 with Lemma 6.3 shows that $\mathcal{P}_{\omega_1}(\operatorname{crit}(j))^N \subseteq \mathrm{H}(\operatorname{crit}(j))^N \subseteq M[G_j]$. Therefore j_G and N witness that θ is internally AP Mahlo with respect to x in $\mathrm{V}[G]$.

Lemma 7.4 shows that (ii) implies (iii).

To show that (iii) implies (i), assume for a contradiction that θ is not a Mahlo cardinal. By [35, Theorem 6.1.4], there is a special θ-Aronszajn tree $\mathbb{T}$ in V. Let G be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. Then by Theorem 6.1, we know that $\theta = {\omega_2}^{\mathrm{V}[G]}$. Since being special is upwards absolute between transitive models of of ZFC^- in which the height of $\mathbb{T}$ remains regular, we have that $\mathbb{T}$ is a special ω_2-Aronszajn tree in $\mathrm{V}[G]$. This shows that (iii) fails. □

By combining the results of the previous chapter with Lemma 7.5, [35, Theorem 6.1.4] and the above theorem, we may conclude that the sequence $\langle \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} \mid \theta \in \mathrm{Card}\rangle$ provides a strong characterization of the class of Mahlo cardinals.

Chapter 8

Indescribable cardinals

In this chapter, we will first present small embedding characterizations for indescribable cardinals and then strong characterizations of indescribable cardinals through Neeman's pure side condition forcing. The results of [28] already essentially provide such a characterization for Π^1_1-indescribable cardinals (i.e. weakly compact cardinals) with the help of the *tree property*. If either m or n are greater than 1, Π^m_n-indescribable cardinals seem to be lacking such a canonical combinatorial characterization.

As we will have to work a lot with higher order objects in this section, let us indicate the order of free variables by a superscript attached to them, letting v^0 denote a standard first order free variable, letting v^1 denote a free variable that is to be interpreted by an element of the powerset of the domain, and so on. In the same way, we will also label higher order quantifiers. Remember that, given $0 < m, n < \omega$, an uncountable cardinal θ is Π^m_n-indescribable if for every Π^m_n-formula $\Phi(v^1)$ and every $A \subseteq \mathrm{V}_\theta$ such that $\mathrm{V}_\theta \models \Phi(A)$, there is a $\delta < \theta$ with $\mathrm{V}_\delta \models \Phi(A \cap \mathrm{V}_\delta)$. Moreover, remember that, given an uncountable cardinal θ, a transitive set M of cardinality θ is a *θ-model* if $\theta \in M$, ${}^{<\theta}M \subseteq M$ and M is a model of ZFC^-. The following small embedding characterizations of indescribable cardinals are build on the following embedding characterizations of these cardinals by Kai Hauser (see [10, Theorem 1.3]).

Theorem 8.1 (Hauser)**.** *The following statements are equivalent for every inaccessible cardinal θ and all $0 < m, n < \omega$:*

(i) θ is Π^m_n-indescribable.

(ii) For every θ-model M, there is a transitive set N and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \theta$ *such that the following statements hold:*

(a) N has cardinality $\beth_{m-1}(\theta)$, ${}^{<\theta}N \subseteq N$ and $j, M \in N$.

(b) If $m > 1$, then ${}^{\beth_{m-2}(\theta)}N \subseteq N$.

(c) We have

$$\mathrm{V}_\theta \models \varphi \iff (\mathrm{V}_\theta \models \varphi)^N$$

for all Π^m_{n-1}-formulas φ whose parameters are contained in $N \cap \mathrm{V}_{\theta+m}$.[1]

[1]Note that we write $(\mathrm{V}_\theta \models \varphi)^N$ to denote satisfaction for the higher order formula φ in the model V_θ *in N*, i.e. k-th order variables are interpreted as elements of $\mathrm{V}^N_{\theta+k}$.

Lemma 8.2. *Given $0 < m, n < \omega$, the following statements are equivalent for every cardinal θ:*

(i) *θ is Π^m_n-indescribable.*

(ii) *For every sufficiently large cardinal ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such ${}^{<\mathrm{crit}(j)}M \subseteq M$ and*

$$(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(A))^M \implies \mathrm{V}_{\mathrm{crit}(j)} \models \varphi(A)$$

for every Π^m_n-formula $\varphi(v^1)$ with parameter $A \in M \cap \mathrm{V}_{\mathrm{crit}(j)+1}$.

(iii) *For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with the property that*

$$(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi)^M \implies \mathrm{V}_{\mathrm{crit}(j)} \models \varphi$$

for every Π^m_n-formula φ whose parameters are contained in $M \cap \mathrm{V}_{\mathrm{crit}(j)+1}$.

(iv) *For all sufficiently large cardinals ϑ and all $x \in \mathrm{V}_{\theta+1}$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $x \in \mathrm{ran}(j)$ and with the property that*

$$(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi)^M \implies \mathrm{V}_{\mathrm{crit}(j)} \models \varphi$$

for every Π^m_n-formula φ using only $j^{-1}(x)$ as a parameter.

Proof. First, assume that (i) holds. Pick a cardinal $\vartheta > \beth_m(\theta)$ and a regular cardinal $\vartheta^* > \vartheta$ with $\mathrm{H}(\vartheta) \in \mathrm{H}(\vartheta^*)$. Since θ is inaccessible, there is an elementary submodel X of $\mathrm{H}(\vartheta^*)$ of cardinality θ with $\theta + 1 \cup \{\vartheta\} \subseteq X$ and ${}^{<\theta}X \subseteq X$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then M is a θ-model and Theorem 8.1 yields an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \theta$ that satisfies the properties (a)–(c) listed in the Statement (ii) of Theorem 8.1. Note that the assumption ${}^{<\theta}N \subseteq N$ implies that θ is inaccessible in N.

Claim. *We have*

$$(\mathrm{V}_\theta \models \varphi)^M \implies (\mathrm{V}_\theta \models \varphi)^N$$

for all Π^m_n-formulas φ whose parameters are contained in $M \cap \mathrm{V}_{\theta+1}$.

Proof of the Claim. Assume that $(\mathrm{V}_\theta \models \varphi)^M$ holds. This assumption implies that $\mathrm{V}_\theta \models \varphi$ holds, because $\pi^{-1} \restriction \mathrm{V}_{\theta+1} = \mathrm{id}_{\mathrm{V}_{\theta+1}}$ and $\mathrm{V}_{\theta+m} \in \mathrm{H}(\vartheta^*)$. By Statement (c) of Theorem 8.1, we can conclude that $(\mathrm{V}_\theta \models \varphi)^N$ holds. □

Set $\vartheta_* = \pi(\vartheta)$, $M_* = \mathrm{H}(\vartheta_*)^M$ and $j_* = j \restriction M_*$. Since $j, M \in N$, we also have $j_*, M_* \in N$. Moreover, in N, the map $j_* : M_* \longrightarrow \mathrm{H}(j(\vartheta_*))^N$ is a small embedding for $j(\theta)$. If φ is a Π^m_n-formula with parameters in $M_* \cap \mathrm{V}_{\theta+1}$ such that $(\mathrm{V}_\theta \models \varphi)^{M_*}$ holds, then $\vartheta > \beth_m(\theta)$ implies that $(\mathrm{V}_\theta \models \varphi)^M$ holds, and we can use the above claim to conclude that $(\mathrm{V}_\theta \models \varphi)^N$ holds. By elementarity, this shows that, in M, there is a small embedding $j' : M' \longrightarrow \mathrm{H}(\vartheta_*)$ for θ such that $\mathrm{crit}\,(j')$ is inaccessible and $\mathrm{V}_{\mathrm{crit}(j')} \models \varphi$ holds for every Π^m_n-formula φ with parameters in $M' \cap \mathrm{V}_{\mathrm{crit}(j)+1}$ with the property that $(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi)^{M'}$ holds. Since $\mathrm{V}_{\theta+m} \in \mathrm{H}(\vartheta^*)$, we can conclude that $\pi^{-1}(j')$ is a small embedding for θ witnessing that (ii) holds for ϑ. It holds trivially that (ii) implies (iii). Next, Lemma 5.9

shows that (iv) is a consequence of (iii). Hence, assume, towards a contradiction, that (iv) holds and that there is a Π^m_n-formula $\varphi(x)$ with $x \in \mathrm{V}_{\theta+1}$, $\mathrm{V}_\theta \models \varphi(x)$ and $\mathrm{V}_\delta \models \neg\varphi(x \cap \mathrm{V}_\delta)$ for all $\delta < \theta$. Pick a regular cardinal $\vartheta > \beth_m(\theta)$ such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ that satisfies the statements listed in (iii) with respect to x. Since $\mathrm{V}_{\theta+m} \in \mathrm{H}(\vartheta)$, elementarity yields that $(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x)))^M$. Thus our assumptions on j allow us to conclude that $\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x))$, contradicting the above assumption. □

In the case $m = 1$, the equivalence between Statements (i) and (iii) in Lemma 8.2 can be rewritten in the following way, using the fact that we can canonically identify Σ_n-formulas using parameters in $\mathrm{H}(\mathrm{crit}\,(j)^+)$ with Σ^1_n-formulas using parameters in $V_{\mathrm{crit}(j)+1}$, such that the given Σ_n-formula holds true in $\mathrm{H}(\mathrm{crit}\,(j)^+)$ if and only if the corresponding Σ^1_n-formula holds in $V_{\mathrm{crit}(j)}$.

Corollary 8.3. *Given $0 < n < \omega$, the following statements are equivalent for every cardinal θ:*

(i) θ is Π^1_n-indescribable.

(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $\mathrm{H}(\mathrm{crit}\,(j)^+)^M \prec_{\Sigma_n} \mathrm{H}(\mathrm{crit}\,(j)^+)$. □

Lemma 8.2 directly shows that small embeddings witnessing Π^m_n-indescribability also witnesses all smaller degrees of indescribability. Next, it is also easy to see that these embeddings possess the properties mentioned in the small embedding characterization of Mahlo cardinals provided by Corollary 5.3.

Corollary 8.4. *Given $0 < m, n < \omega$, let θ be Π^m_n-indescribable and let ϑ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing Π^m_n-indescribability of θ, as in Statement (iii) of Lemma 8.2. Then $\mathrm{crit}\,(j)$ is inaccessible, hence j witnesses the Mahloness of θ, as in Statement (v) of Corollary 5.3.* □

We now consider an internal version of indescribable cardinals.

Definition 8.5. Given $0 < m, n < \omega$, we say that a cardinal θ is *internally AP Π^m_n-indescribable* if for all sufficiently large regular cardinals ϑ and all $x \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ, and a transitive model N of ZFC^- such that $x \in \mathrm{ran}(j)$, and the following statements hold:

(i) $N \subseteq \mathrm{H}(\vartheta)$, and the pair $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.

(ii) $M \in N$, and $\mathcal{P}_{\omega_1}(\mathrm{crit}\,(j))^N \subseteq M$.

(iii) $\mathrm{crit}\,(j)$ is regular in N and

$$(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(A))^M \implies (\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(A))^N$$

for every Π^m_n-formula $\Phi(v^1)$ with parameter $A \in \mathcal{P}(\mathrm{H}(\mathrm{crit}\,(j)))^M$.

Note that the above definition directly implies that internally AP indescribable cardinals are internally AP Mahlo. As mentioned earlier, this principle may be viewed as a strong substitute for the tree property with respect to higher levels of indescribability. For the basic case of Π^1_1-indescribability, we easily obtain the following result.

Proposition 8.6. *Let θ be an internally AP Π^1_1-indescribable cardinal. Then θ is an uncountable regular cardinal with the tree property.*

Proof. By Lemma 7.4, we know that θ is uncountable and regular. Assume for a contradiction that there exists a θ-Aronszajn tree $\mathbb{T}$ with domain θ and pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that the properties listed in Definition 8.5 hold for $m = n = 1$ and $j(\mathbb{S}) = \mathbb{T}$ holds for some tree $\mathbb{S} \in M$ of height $\mathrm{crit}\,(j)$. Then $\mathbb{S}$ is a $\mathrm{crit}\,(j)$-Aronszajn tree in M and, since this statement can be formulated over $\mathrm{H}(\mathrm{crit}\,(j))^M$ by a Π^1_1-formula with parameter $\mathbb{S} \in \mathcal{P}(\mathrm{H}(\mathrm{crit}\,(j)))^M$, we can conclude that $\mathbb{S}$ is a $\mathrm{crit}\,(j)$-Aronszajn tree in N. As in the proof of Lemma 7.4, elementarity implies that $\mathbb{S}$ is an initial segment of $\mathbb{T}$, and the σ-approximation property implies that N contains a cofinal branch through $\mathbb{S}$, a contradiction. □

Before we may commence with the main results of this section, we need to make a few technical observations. Namely, we will want to identify countable subsets of $\mathrm{V}_{\theta+k}$ with certain elements of $\mathrm{V}_{\theta+k}$, and also view forcing statements about Π^m_n-formulas themselves as Π^m_n-formulas. The basic problem about this is that the forming of (standard) ordered pairs is rank-increasing. For example, names for elements of $\mathrm{V}_{\theta+k}$ are usually not elements of $\mathrm{V}_{\theta+k}$ when $k > 0$, even if θ is regular and the forcing satisfies the θ-chain condition. However, there are well-known alternative definitions of ordered pairs (see, for example, [2]) that possess all the nice properties of the usual ordered pairs that we will need, and which are, in addition, not rank-increasing. While it would be tedious to do so, it is completely straightforward to verify that one can base all set theory (like the definition of finite tuples) and forcing theory (starting with the definition of forcing names) on these modified ordered pairs, and preserve all of their standard properties, while additionally obtaining our desired properties. We will assume that we work with the modified ordered pairs for the remainder of this section. The following lemma shows how this approach allows us to formulate Π^m_n-statements in the forcing language in a Π^m_n-way.

Lemma 8.7. *Work in ZFC^-. Let $m \in \omega$, and let θ be a cardinal such that $\mathcal{P}^m(\mathrm{H}(\theta))$ exists. Assume that $\mathbb{P} \subseteq \mathrm{H}(\theta)$ is a partial order such that forcing with $\mathbb{P}$ preserves θ and such that for every $\mathbb{P}$-name τ for an element of $\mathrm{H}(\theta)$, there is a $\mathbb{P}$-name σ in $\mathrm{H}(\theta)$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\sigma = \tau$".*

(i) If τ is a $\mathbb{P}$-name for an element of $\mathcal{P}^{m+1}(\mathrm{H}(\theta))$, then there is a $\mathbb{P}$-name σ in $\mathcal{P}^{m+1}(\mathrm{H}(\theta))$ with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\sigma = \tau$".

(ii) If $\sigma_0, \ldots, \sigma_k$ is a finite sequence of $\mathbb{P}$-names in $\mathcal{P}^{m+1}(\mathrm{H}(\theta))$, and Φ is a Π^{m+1}_n-formula for some $n \in \omega$, then the statement $p \Vdash \Phi(\sigma_0, \ldots, \sigma_k)$ is equivalent to a Π^{m+1}_n-formula.

We are now ready to prove the main results of this chapter. The following result will show that our characterization of indescribability through internal AP indescribability presented below is strong.

Lemma 8.8. *Given $0 < m, n < \omega$, the following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a Π^m_n-indescribable cardinal.

(ii) θ is an internally AP Π^m_n-indescribable cardinal.

Proof. First, assume that (i) holds, let ϑ be sufficiently large in the sense of Lemma 8.2,(ii). Fix $x \in \mathrm{H}(\vartheta)$. By Lemma 5.9 and Lemma 8.2,(ii) there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ satisfying $x \in \mathrm{ran}(j)$, and satisfying the properties listed in Lemma 8.2,(ii). By the above remarks, $\mathrm{crit}\,(j)$ is an inaccessible cardinal and hence $\mathrm{H}(\mathrm{crit}\,(j)) = \mathrm{V}_{\mathrm{crit}(j)}$. Therefore, if we set $N = \mathrm{H}(\vartheta)$, then j and N witness that θ is internally AP Π^m_n-indescribable with respect to x.

In the other direction, assume that the inaccessible cardinal θ is internally AP Π^m_n-indescribable. Fix $A \subseteq \mathrm{V}_\theta = \mathrm{H}(\theta)$ and a Π^m_n-formula $\Phi(v^1)$ with $\mathrm{V}_\theta \models \Phi(A)$. Let $j : M \longrightarrow \mathrm{H}(\vartheta)$ and N witness that θ is internally AP Π^m_n-indescribable with respect to $\{A, \mathrm{V}_{\theta+\omega}\}$. Since θ is inaccessible, elementarity implies that $\mathrm{crit}\,(j)$ is a strong limit cardinal in V.

Claim. $\mathrm{V}_{\mathrm{crit}(j)} \subseteq M$ *and* $\mathrm{V}_{\mathrm{crit}(j)+\omega} \subseteq N$*.*

Proof of the Claim. The proof of Lemma 6.3 contains an argument that proves the first statement. Next, assume that the second statement fails. Then there is some $k < \omega$ and $x \subseteq \mathrm{V}_{\mathrm{crit}(j)+k}$ with $\mathrm{V}_{\mathrm{crit}(j)+k} \subseteq N$ and $x \notin N$. By the above remarks, we can identify countable subsets of $\mathrm{V}_{\mathrm{crit}(j)+k}$ with elements of $\mathrm{V}_{\mathrm{crit}(j)+k}$ in a canonical way. This shows that $\mathcal{P}_{\omega_1}(x) \subseteq N$, and hence the σ-approximation property implies that $x \in N$, a contradiction. □

Since our assumptions imply that $\mathrm{crit}\,(j)$ is regular in N, the above claim directly shows that $\mathrm{crit}\,(j)$ is an inaccessible cardinal in V. Moreover, by the above claim and our assumptions, we have $A \cap \mathrm{V}_{\mathrm{crit}(j)} \in M$ and $j(A \cap \mathrm{V}_{\mathrm{crit}(j)}) = A$. In addition, the above choices ensure that $(\mathrm{V}_\kappa \models \Phi(A))^{\mathrm{H}(\vartheta)}$ holds and hence elementarity implies that $(\mathrm{V}_{\mathrm{crit}(j)} \models \Phi(A \cap \mathrm{V}_{\mathrm{crit}(j)}))^M$. By Clause (iii) of Definition 8.5, we therefore know that $(\mathrm{V}_{\mathrm{crit}(j)} \models \Phi(A \cap \mathrm{V}_{\mathrm{crit}(j)}))^N$ holds, and, since the above claim shows that Π^m_n-formulas over $\mathrm{V}_{\mathrm{crit}(j)}$ are absolute between N and V, this allows us to conclude that $\mathrm{V}_{\mathrm{crit}(j)} \models \Phi(A \cap \mathrm{V}_{\mathrm{crit}(j)})$. □

We may now show that indescribability can be characterized by internal AP indescribability via Neeman's pure side condition forcing.

Theorem 8.9. *Given* $0 < m, n < \omega$*, the following statements are equivalent for every inaccessible cardinal* θ*:*

(i) θ *is a* Π^m_n*-indescribable cardinal.*
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ *"*ω_2 *is internally AP* Π^m_n*-indescribable".*

Proof. Assume first that (i) holds. Pick a regular cardinal $\vartheta > \theta$ that is sufficiently large in the sense of Lemma 8.2,(ii), let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V, and pick $x \in \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$. By Lemma 4.11, there is a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name $\dot{x} \in \mathrm{H}(\vartheta)^{\mathrm{V}}$ with $x = \dot{x}^G$. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ witnessing that θ is Π^m_n-indescribable with respect to $\dot{x}$ given by the combination of Lemma 8.2,(ii) and Lemma 5.9, and let $j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ be the embedding given by Lemma 6.3. Define $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}$ and pick a Π^m_n-formula $\Phi(v^1)$ and $A \in \mathcal{P}(\mathrm{H}(\mathrm{crit}\,(j)))^{M[G_j]}$ such that $(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(A))^{M[G_j]}$. Our assumption $({}^{<\mathrm{crit}(j)}M)^{\mathrm{V}} \subseteq M$ implies that $\mathrm{crit}\,(j)$ is an inaccessible cardinal in V, Theorem 6.1 shows that $\mathrm{crit}\,(j) = \omega_2^{\mathrm{V}[G_j]}$, and therefore Lemma 4.11 shows that

$\mathcal{P}_{\omega_1}(\mathrm{crit}\,(j))^N \subseteq \mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}[G_j]} \subseteq M[G_j]$. Using Lemma 8.7, there is a $\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}$-name $\tau \in \mathcal{P}(\mathrm{H}(\mathrm{crit}\,(j)))^M$ for an element of $\mathcal{P}(\mathrm{H}(\mathrm{crit}\,(j)))$, and a condition $r \in G_j$ such that $A = \tau^{G_j}$, and such that

$$r \Vdash^M_{\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}} \text{“}\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(\tau)\text{”}.$$

Again, by Lemma 8.7, the above forcing statement is equivalent to a Π^m_n-statement over $\mathrm{H}(\mathrm{crit}\,(j))$ and this statement holds true in M. By our assumptions, this statement holds in V, and therefore

$$r \Vdash^{\mathrm{V}}_{\mathbb{P}_{\mathcal{S}_{\mathrm{crit}(j)},\mathcal{T}_{\mathrm{crit}(j)}}} \text{“}\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(\tau)\text{”}.$$

This allows us to conclude that $(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(A))^N$ holds. Hence j_G and N witness that ω_2 is internally AP Π^m_n-indescribable with respect to x in $\mathrm{V}[G]$.

Now, assume that (ii) holds. Fix a subset A of $\mathrm{H}(\theta)$, and assume that $\Phi(v^1)$ is a Π^m_n-formula with $\mathrm{V}_\theta \models \Phi(A)$. Let C denote the club of strong limit cardinals below θ and fix a bijection $b : \theta \longrightarrow \mathrm{H}(\theta)$ with $b[\kappa] = \mathrm{H}(\kappa)$ for all $\kappa \in C$.

Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V, and work in $\mathrm{V}[G]$. By our assumption, we can find a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ and a transitive ZFC^--model N witnessing the internal AP Π^m_n-indescribability of θ with respect to $\{A, b, C, \mathrm{V}_{\kappa+\omega}\}$. By elementarity, we have $\mathrm{crit}\,(j) \in C$, $\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}} \in M \subseteq N$, $j(\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}}) = \mathrm{H}(\theta)^{\mathrm{V}}$, $\bar{A} = A \cap \mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}} \in M$ and $A = j(\bar{A})$. Moreover, since $\mathrm{crit}\,(j)$ is regular in N, the σ-approximation property between N and $\mathrm{H}(\vartheta)$ implies that $\mathrm{cof}(\mathrm{crit}\,(j)) > \omega$. Another application of the σ-approximation property then yields $\mathcal{P}(\mathrm{crit}\,(j))^{\mathrm{V}} \subseteq N$, and therefore $\mathrm{crit}\,(j)$ is a regular cardinal in V.

Claim. *Given $k < \omega$, we have $\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}} \subseteq N$, and there is a Π^k_0-formula $\Phi_k(v^1, w^k)$ satisfying*

$$\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}} = \{B \in \mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^N \mid (\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi_k(\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}}, B))^N\}$$

and

$$\mathcal{P}^k(\mathrm{H}(\theta))^{\mathrm{V}} = \{B \in \mathcal{P}^k(\mathrm{H}(\theta)) \mid \mathrm{H}(\theta) \models \Phi_k(\mathrm{H}(\theta)^{\mathrm{V}}, B)\}.$$

Proof of the Claim. Using induction, we will simultaneously define the formulas $\Phi_k(v^1, w^k)$, show that they satisfy the above statements, and also verify that $\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}} \subseteq N$. In order to start, set $\Phi_0(v^1, w^0) \equiv w^0 \in v^1$. Then Φ_0 is clearly as desired, and we already argued above that $\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}} \subseteq N$.

Now, assume that we arrived at stage $k+1$ of our induction. Assume, for a contradiction, that there is a subset B of $\mathcal{P}^{k+1}(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}}$ with $B \notin N$. Since the pair $(N, \mathrm{V}[G])$ satisfies the σ-approximation property, we can find $b \in \mathcal{P}_{\omega_1}(\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}})^N$ with $B \cap b \notin N$. Then Corollary 4.9 shows that the partial order $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ is proper in V, and we therefore find $c \in \mathcal{P}_{\omega_1}(\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j))))^{\mathrm{V}}$ with $b \subseteq c$. By identifying elements of $\mathcal{P}_{\omega_1}(\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j))))^{\mathrm{V}}$ with elements of $\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))$, we can conclude that $B \cap c \in N$ and hence $B \cap b \in N$, a contradiction. This shows that $\mathcal{P}^{k+1}(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}} \subseteq N$. Moreover, since Corollary 3.20 implies that the pair $(\mathrm{V}, \mathrm{V}[G])$ also satisfies the σ-approximation property, it follows that $\mathcal{P}^{k+1}(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}}$ exactly consists of all $B \in \mathcal{P}^{k+1}(\mathrm{H}(\mathrm{crit}\,(j)))^N$ with the property that for all $D \in \mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}}$ that code a countable subset d

of $\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}}$, there is an element of $\mathcal{P}^k(\mathrm{H}(\mathrm{crit}\,(j)))^{\mathrm{V}}$ coding the subset $B \cap d$. Furthermore, it also follows from the σ-approximation property for the pair $(\mathrm{V}, \mathrm{V}[G])$ that $\mathcal{P}^{k+1}(\mathrm{H}(\theta))^{\mathrm{V}}$ exactly consists of all $B \in \mathcal{P}^{k+1}(\mathrm{H}(\theta))$ with the property that for all $D \in \mathcal{P}^k(\mathrm{H}(\theta))$ that code a countable subset d of $\mathcal{P}^k(\mathrm{H}(\theta))^{\mathrm{V}}$, there is an element of $\mathcal{P}^k(\mathrm{H}(\theta))^{\mathrm{V}}$ coding $B \cap d$. Now, let $\Phi_{k+1}(v^1, w^{k+1})$ denote the canonical Σ_0^{k+1}-formula stating that for every $D \in \mathcal{P}^k(v^1)$ such that $\Phi_k(v^1, D)$ holds and D codes a countable subset d of $\mathcal{P}^k(v^1)$, there is $E \in \mathcal{P}^k(v^1)$ such that $\Phi_k(v^1, E)$ holds and E codes $d \cap w^{k+1}$. Then, the above remarks show that the two equalities stated in the above claim also hold at stage $k+1$. □

Let $\Phi_*(u^1, v^1)$ denote the relativisation of $\Phi(v^1)$ using the formulas $\Phi_k(v^1, w^k)$, i.e. we obtain Φ_* from Φ by replacing each subformula of the form $\exists^k x\ \psi$ by $\exists^k x\ [\psi\ \wedge\ \Phi_k(u^1, x)]$. Then Φ_* is again a Π_n^m-formula and, by the above claim and our assumptions, we know that $\mathrm{H}(\theta) \models \Phi_*(\mathrm{H}(\theta)^{\mathrm{V}}, A)$. Therefore we can use elementarity to conclude that

$$(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi_*(\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}}, \bar{A}))^M$$

and, since j and N witness the internal AP Π_n^m-indescribability of θ, we know that

$$(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi_*(\mathrm{H}(\mathrm{crit}\,(j))^{\mathrm{V}}, \bar{A}))^N.$$

But then the above claim shows that $(\mathrm{H}(\mathrm{crit}\,(j)) \models \Phi(\bar{A}))^{\mathrm{V}}$. These computations show that θ is Π_n^m-indescribable in V. □

Chapter 9

Small embedding characterization of subtle and λ-ineffable cardinals

In this chapter, we are going to derive small embedding characterizations for subtle and λ-ineffable cardinals. In order to define these large cardinals, we have to first introduce the notion of a list.

Definition 9.1. Given a set A, a sequence $\langle d_a \mid a \in A\rangle$ is an *A-list* if $d_a \subseteq a$ holds for all $a \in A$.

Now, we are ready to define subtle cardinals, which were introduced by Ronald Jensen and Kenneth Kunen in [15] in their studies of $\Diamond$-principles and higher Kurepa trees.

Definition 9.2 (Jensen and Kunen)**.** A regular uncountable cardinal θ is *subtle* if for every θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ and every club subset C of θ there are $\alpha < \beta$ both in C such that $d_\alpha = d_\beta \cap \alpha$.

Lemma 9.3. *The following statements are equivalent for every cardinal θ:*

(i) θ is subtle.

(ii) For all sufficiently large cardinals ϑ, for every θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ and for every club C in θ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $\vec{d}, C \in \mathrm{ran}(j)$ and $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$ for some $\alpha \in C \cap \mathrm{crit}\,(j)$.

Proof. First, assume first that θ is subtle. Pick a cardinal $\vartheta > \theta$, a club C in θ and a θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$. Let $\langle X_\alpha \mid \alpha < \theta\rangle$ be a continuous and increasing sequence of elementary substructures of $\mathrm{H}(\vartheta)$ of cardinality less than θ with $\vec{d}, C \in X_0$ and $\alpha \subseteq X_\alpha \cap \theta \in \theta$ for all $\alpha < \theta$. Set $D = \{\alpha \in C \mid \alpha = M_\alpha \cap \theta\}$. Then D is a club in θ and the subtlety of θ yields $\alpha, \beta \in D \subseteq C$ with $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$. Let $\pi : X_\beta \longrightarrow M$ denote the transitive collapse of X_β. Then $\pi^{-1} : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\mathrm{crit}\,(\pi^{-1}) = \beta$, $\vec{d}, C \in \mathrm{ran}(\pi^{-1})$ and $d_\alpha = d_{\mathrm{crit}(\pi^{-1})} \cap \alpha$.

Now, assume that (ii) holds. Then Corollary 5.3 implies that θ is uncountable and regular. Fix a θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ and a club C in θ. Let ϑ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $\vec{d}, C \in \mathrm{ran}(j)$ and $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$ for some $\alpha \in C \cap \mathrm{crit}\,(j)$. Since $C \in \mathrm{ran}(j)$, elementarity implies that $\mathrm{crit}\,(j)$ is a limit point of C and hence $\mathrm{crit}\,(j) \in C$. □

Remark 9.4. *Note that, unlike all other small embedding characterizations that we provide in this thesis, the above characterization of subtle cardinals is not based on a correctness property between the domain model M and V.*

Adapting the proof of [1, Theorem 3.6.3], it is easy to verify the following lemma, that will be needed later on. We will provide a proof for this result for the sake of completeness.

Lemma 9.5. *Let θ be a subtle cardinal. Then there is a θ-list $\vec{d}$ and a club subset C of θ with the property that whenever ϑ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the subtlety of θ with respect to $\vec{d}$ and C, as in Statement (ii) of Lemma 9.3, then $\mathrm{crit}\,(j)$ is a totally indescribable cardinal.*

In particular, any family of small embeddings witnessing the subtlety of θ as in Statement (ii) of Lemma 9.3 witnesses that θ is a stationary limit of totally indescribable cardinals, as in Statement (ii) of Lemma 5.2.

Proof. By inaccessibility, we know that $C := \{\alpha < \theta \mid |\mathrm{V}_\alpha| = \alpha\}$ is a club in θ. Let $h : \mathrm{V}_\theta \longrightarrow \theta$ be a bijection with $h[\mathrm{V}_\alpha] = \alpha$ for all $\alpha \in C$. Let $\prec\cdot,\cdot\succ$ denote the Gödel pairing function and let $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ be a θ-list with the following properties:

(i) If $\alpha \in C$ is not totally indescribable, then there is a Π^m_n-formula φ and a subset A of V_α such that these objects provide a counterexample to the Π^m_n-indescribability of α. Then $d_\alpha = \{\prec 0, \ulcorner\varphi\urcorner\succ\} \cup \{\prec 1, h(a)\succ \mid a \in A\}$, where $\ulcorner\varphi\urcorner \in \omega$ is the Gödel number of φ in some fixed Gödelization of m^{th} order set theory.
(ii) Otherwise, d_α is the empty set.

Let ϑ be a sufficiently large cardinal and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ that witnesses the subtlety of θ with respect to $\vec{d}$ and C, as in Lemma 9.3. Then $\mathrm{crit}\,(j) \in C$. Assume for a contradiction that $\mathrm{crit}\,(j)$ is not totally indescribable. Then there is a Π^m_n-formula φ and a subset A of V_α such that $d_\alpha = \{\prec 0, \ulcorner\varphi\urcorner\succ\} \cup \{\prec 1, h(a)\succ \mid a \in A\}$, $\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(A)$ and $\mathrm{V}_\alpha \models \neg\varphi(A \cap \mathrm{V}_\alpha)$ for all $\alpha < \mathrm{crit}\,(j)$. By our assumptions, there is an $\alpha \in C \cap \mathrm{crit}\,(j)$ with $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$. In this situation, our definition of d_α ensures that the formula φ and the subset $A \cap \mathrm{V}_\alpha$ of V_α provide a counterexample to the Π^m_n-indescribability of α. In particular, we know that $\mathrm{V}_\alpha \models \varphi(A \cap \mathrm{V}_\alpha)$ holds, a contradiction. □

Now we will define ineffable cardinals. They were introduced by Jensen and Kunen in [15] and arose out of their studies of $\Diamond$-principles.

Definition 9.6 (Jensen and Kunen)**.** A regular uncountable cardinal θ is *ineffable* if for every θ-list $\langle d_\alpha \mid \alpha < \theta\rangle$, there exists a subset D of θ such that the set $\{\alpha < \theta \mid d_\alpha = D \cap \alpha\}$ is stationary in θ.

Lemma 9.7. *The following statements are equivalent for every cardinal θ:*

(i) θ is ineffable.
(ii) For all sufficiently large cardinals ϑ and for every θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\vec{d} \in \mathrm{ran}(j)$ and $d_{\mathrm{crit}(j)} \in M$.

Proof. Assume first that θ is ineffable. Pick a θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ and a cardinal $\vartheta > \theta$. Using the ineffability of θ, we find a subset D of θ such that the set $S =$

$\{\alpha < \theta \mid d_\alpha = D \cap \alpha\}$ is stationary in θ. With the help of a continuous chain of elementary submodels of $\mathrm{H}(\vartheta)$, we then find $X \prec \mathrm{H}(\vartheta)$ of size less than θ such that $\vec{d}, D \in X$ and $X \cap \theta \in S$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi^{-1} : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\mathrm{crit}\,(j) \in S$, $\vec{d} \in \mathrm{ran}(\pi^{-1})$ and $d_{\mathrm{crit}(j)} = D \cap \mathrm{crit}\,(j) = \pi(D) \in M$.

Assume now that (ii) holds. Let $\vec{d} = \langle d_\alpha \mid \alpha < \theta \rangle$ be a θ-list and let ϑ be a sufficiently large cardinal such that there exists a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with $\vec{d} \in \mathrm{ran}(j)$ and $d_{\mathrm{crit}(j)} \in M$. Assume that there is a club C in θ with $d_\alpha \neq j(d_{\mathrm{crit}(j)}) \cap \alpha$ for all $\alpha \in C$. Since $\vec{d} \in \mathrm{ran}(j)$, elementarity implies that there is a club subset C_0 of $\mathrm{crit}\,(j)$ in M with $d_\alpha \neq j(d_{\mathrm{crit}(j)}) \cap \alpha$ for all $\alpha \in j(C_0)$. But $j(C_0)$ is a club in θ and elementarity implies that $\mathrm{crit}\,(j)$ is a limit point of $j(C_0)$ with $d_{\mathrm{crit}(j)} = j(d_{\mathrm{crit}(j)}) \cap \mathrm{crit}\,(j)$, a contradiction. This argument shows that the set $\{\alpha < \theta \mid d_\alpha = j(d_{\mathrm{crit}(j)}) \cap \alpha\}$ is stationary in θ. □

It is easy to see that every ineffable cardinal is subtle. The next lemma will show that every small embedding that is witnessing the ineffability of some cardinal, witnesses as well the subtlety of that cardinal.

Lemma 9.8. *Let θ be ineffable, let $\vec{d}$ be a θ-list and let C be a club in θ. If ϑ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the ineffability of θ with respect to $\vec{d}$, as in Statement (ii) of Lemma 9.7, then $C \in \mathrm{ran}(j)$*[1] *implies that j also witnesses the subtlety of θ with respect to $\vec{d}$ and C, as in Statement (ii) of Lemma 9.3.*

Proof. Pick a club subset C_0 of $\mathrm{crit}\,(j)$ in M with $j(C_0) = C$. Since $\mathrm{crit}\,(j)$ is an element of $j(C_0)$ with $d_{\mathrm{crit}(j)} = j(d_{\mathrm{crit}(j)}) \cap \mathrm{crit}\,(j)$, elementarity implies that there is an $\alpha \in C_0 \cap \mathrm{crit}\,(j)$ with $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$. Then α is an element of $C \cap \mathrm{crit}\,(j)$ with

$$d_\alpha = j(d_{\mathrm{crit}(j)}) \cap \alpha = d_{\mathrm{crit}(j)} \cap \alpha. \quad \square$$

We now show that small embeddings for θ witnessing that θ is ineffable also witness that θ is Π^1_2-indescribable. Note that the least ineffable cardinal is not Π^1_3-indescribable.

Lemma 9.9. *Let θ be ineffable and let $x \in \mathrm{V}_{\theta+1}$. Then there is a θ-list $\vec{d}$ and a subset h of V_θ with the property that whenever ϑ is a sufficiently large cardinal and $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ witnessing the ineffability of θ with respect to $\vec{d}$, as in Statement (ii) of Lemma 9.7, then $h, x \in \mathrm{ran}(j)$ implies that j witnesses the Π^1_2-indescribability of θ with respect to x, as in Statement (iv) of Lemma 8.2.*

Proof. Fix a bijection $h : \mathrm{V}_\theta \longrightarrow \theta$ with $h[\mathrm{V}_\alpha] = \alpha$ for every strong limit cardinal $\alpha < \theta$ and a θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta \rangle$ such that the following statements hold for all $\alpha < \theta$:

(i) If α is inaccessible, then $d_\alpha \neq \emptyset$ if and only if there is a Σ^1_1-formula $\psi_\alpha(v_0, v_1)$ and $\emptyset \neq y_\alpha \in \mathrm{V}_{\alpha+1}$ with the property that $\mathrm{V}_\theta \models \forall Z\ \psi_\alpha(x, Z)$ and $\mathrm{V}_\alpha \models \neg\psi_\alpha(x \cap \mathrm{V}_\alpha, y_\alpha)$. We let $d_\alpha = h[y_\alpha]$ in this case.

[1]The assumption that C is contained in the range of j is harmless by Lemma 5.9.

(ii) If α is a singular cardinal, then d_α is a cofinal subset of α of order-type $\mathrm{cof}(\alpha)$.
(iii) Otherwise, $d_\alpha = \emptyset$.

Let ϑ be a sufficiently large cardinal and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ with $\vec{d}, h, x \in \mathrm{ran}(j)$ and $d_{\mathrm{crit}(j)} \in M$. Then Lemma 5.6 implies that $\mathrm{crit}\,(j)$ is a strong limit cardinal. Since $\mathrm{crit}\,(j)$ is regular in M, our definition of $\vec{d}$ ensures that $\mathrm{crit}\,(j)$ is inaccessible. Then our assumptions imply that $j^{-1}(x) = x \cap \mathrm{V}_{\mathrm{crit}(j)} \in M$.

Assume that there is a Π^1_2-formula $\varphi(v)$ with

$$(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(x \cap \mathrm{V}_{\mathrm{crit}(j)}))^M \text{ and } \mathrm{V}_{\mathrm{crit}(j)} \models \neg\varphi(x \cap \mathrm{V}_{\mathrm{crit}(j)}).$$

Then elementarity implies that $\mathrm{V}_\theta \models \varphi(x)$ holds, and this allows us to conclude that the set $d_{\mathrm{crit}(j)}$ is not empty, $\mathrm{V}_\theta \models \forall Z\ \psi_{\mathrm{crit}(j)}(x, Z)$ and $\mathrm{V}_{\mathrm{crit}(j)} \models \neg\psi_{\mathrm{crit}(j)}(x \cap \mathrm{V}_{\mathrm{crit}(j)}, y_{\mathrm{crit}(j)})$. Since $d_{\mathrm{crit}(j)} \in M$ and $h \in \mathrm{ran}(j)$, we obtain that $y_{\mathrm{crit}(j)} \in M$, and elementarity implies that $(\mathrm{V}_{\mathrm{crit}(j)} \models \psi_{\mathrm{crit}(j)}(x \cap \mathrm{V}_{\mathrm{crit}(j)}, y_{\mathrm{crit}(j)}))^M$. Then Lemma 5.6 shows that $\mathrm{V}_{\mathrm{crit}(j)} \subseteq M$ and we can apply Σ^1_1-upwards absoluteness to conclude that

$$\mathrm{V}_{\mathrm{crit}(j)} \models \psi_{\mathrm{crit}(j)}(x \cap \mathrm{V}_{\mathrm{crit}(j)}, y_{\mathrm{crit}(j)}),$$

a contradiction. □

Let us now consider a generalized version of ineffable cardinals that were introduced by Magidor in [24], where he showed that a cardinal θ is supercompact if and only if it is λ-ineffable for all $\lambda \geq \theta$.

Definition 9.10 (Magidor)**.** A regular uncountable cardinal θ is λ-*ineffable*, for a cardinal $\lambda \geq \theta$, if for every $\mathcal{P}_\theta(\lambda)$-list $\langle A_x \mid x \in \mathcal{P}_\theta(\lambda)\rangle$, there exists $A \subseteq \lambda$ such that the set $S = \{x \in \mathcal{P}_\theta(\lambda) \mid A \cap x = A_x\}$ is stationary.

Next, we will provide a small embedding characterization of λ-ineffable cardinals.

Lemma 9.11. *The following statements are equivalent for all cardinals $\theta \leq \lambda$:*

(i) θ is λ-ineffable.
(ii) For all sufficiently large cardinals ϑ and every $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ such that $j(\delta) = \lambda$, $\vec{d} \in \mathrm{ran}(j)$ and $j^{-1}[d_{j[\delta]}] \in M$.

Proof. Assume first that θ is λ-ineffable. Fix a $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ and a cardinal ϑ with $\mathcal{P}_\theta(\lambda) \in \mathrm{H}(\vartheta)$. Then the λ-ineffability of θ yields a subset D of λ such that $S = \{a \in \mathcal{P}_\theta(\lambda) \mid d_a = D \cap a\}$ is stationary in $\mathcal{P}_\theta(\lambda)$. In this situation, we can find $X \prec \mathrm{H}(\vartheta)$ of cardinality less than θ such that $\vec{d}, D \in X$, $X \cap \theta \in \theta$ and $X \cap \lambda \in S$. Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse. Then $\pi(\lambda) < \theta$ and $\pi^{-1} : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ with $\vec{d} \in \mathrm{ran}(\pi^{-1})$. Moreover, we have

$$\pi[d_{\pi^{-1}[\pi(\lambda)]}] \;=\; \pi[d_{X \cap \lambda}] \;=\; \pi[D \cap X] \;=\; \pi(D) \;\in M.$$

Now, assume that (ii) holds, and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ be a $\mathcal{P}_\theta(\lambda)$-list. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ with $j(\delta) = \lambda$, $\vec{d} \in \mathrm{ran}(j)$ and

$d = j^{-1}[d_{j[\delta]}] \in \mathcal{P}(\delta)^M$. We define $S = \{a \in \mathcal{P}_\theta(\lambda) \mid d_a = j(d) \cap a\} \in \operatorname{ran}(j)$. Assume for a contradiction that the set S is not stationary in $\mathcal{P}_\theta(\lambda)$. Then there is a function $f : \mathcal{P}_\omega(\lambda) \longrightarrow \mathcal{P}_\theta(\lambda)$ with $\mathrm{Cl}_f \cap S = \emptyset$, where Cl_f denotes the set of all $a \in \mathcal{P}_\theta(\lambda)$ with $f(b) \subseteq a$ for all $b \in \mathcal{P}_\omega(a)$. Since $S \in \operatorname{ran}(j)$, elementarity yields a function $f_0 : \mathcal{P}_\omega(\delta) \longrightarrow \mathcal{P}_{\operatorname{crit}(j)}(\delta)$ in M with $\mathrm{Cl}_{j(f_0)} \cap S = \emptyset$. Pick $b \in \mathcal{P}_\omega(j[\delta])$. Then $b \in \operatorname{ran}(j)$, and hence $j^{-1}(b) = j^{-1}[b] \in M$, and there is $a \in \mathrm{Cl}^{\mathrm{M}}_{\mathrm{f_0}}$ with $j^{-1}[b] \subseteq a \in \mathcal{P}_{\operatorname{crit}(j)}(\delta)^M$. In this situation, we have $j(f_0)(b) = j(f_0(j^{-1}[b])) \subseteq j(a) = j[a] \subseteq j[\delta]$. These computations show that $j[\delta] \in \mathrm{Cl}_{j(f_0)}$. But we also have $j(d) \cap j[\delta] = j[d] = d_{j[\delta]}$, and this shows that $j[\delta] \in \mathrm{Cl}_{j(f_0)} \cap S$, a contradiction. □

It is easy to see that small embeddings witnessing certain degrees of ineffability also witness all smaller degrees.

Proposition 9.12. *Let θ be a λ-ineffable cardinal, let $\theta \leq \lambda_0 < \lambda$ be a cardinal and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda_0)\rangle$ be a $\mathcal{P}_\theta(\lambda_0)$-list. If ϑ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the λ-ineffability of θ with respect to $\langle d_{a\cap\lambda_0} \mid a \in \mathcal{P}_\theta(\lambda)\rangle$, as in Statement (ii) of Lemma 9.11, then $\vec{d} \in \operatorname{ran}(j)$ implies that j also witnesses the λ_0-ineffability of θ with respect to $\vec{d}$ in this way.* □

The next result reformulates the proof of [37, Proposition 3.2] to derive a strengthening of Lemma 5.6 for many small embeddings witnessing λ-ineffability. We will make use of this in Chapter 11.

Lemma 9.13. *Let θ be a λ-ineffable cardinal. If $\lambda = \lambda^{<\theta}$, then there is a $\mathcal{P}_\theta(\lambda)$-list $\vec{d}$ and a set x with the property that whenever ϑ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ witnessing the λ-ineffability of θ with respect to $\vec{d}$, as in Statement (ii) of Lemma 9.11, then $x \in \operatorname{ran}(j)$ implies that* $\operatorname{crit}(j)$ *is an inaccessible cardinal and $\mathcal{P}_{\operatorname{crit}(j)}(\delta) \subseteq M$.*

Proof. Fix a bijection $f : \mathcal{P}_\theta(\lambda) \longrightarrow \lambda$. Then Lemma 9.5 yields a club C in θ and a θ-list $\vec{e} = \langle e_\alpha \mid \alpha < \theta\rangle$ with the property that whenever ϑ is a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the subtlety of θ with respect to $\vec{e}$ and C as in Statement (ii) of Lemma 9.3, then $\operatorname{crit}(j)$ is an inaccessible cardinal.

Let A denote the set of all $a \in \mathcal{P}_\theta(\lambda)$ with the property that there is a cardinal $\nu_a > \lambda$ and an elementary submodel X_a of $\mathrm{H}(\nu_a)$ such that $f \in X_a$, $\alpha_a = X_a \cap \theta \in C$ is inaccessible and $\mathcal{P}_{\alpha_a}(X_a \cap \lambda) \nsubseteq X_a$. Given $a \in A$, pick $x_a \in \mathcal{P}_{\alpha_a}(X_a \cap \lambda) \setminus X_a$. Next, let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ denote the unique $\mathcal{P}_\theta(\lambda)$-list such that $d_a = x_a$ for all $a \in A$, $d_a = e_{a\cap\theta}$ for all $a \in \mathcal{P}_\theta(\lambda) \setminus A$ with $a \cap \theta \in C$, and $d_a = \emptyset$ otherwise.

Now, let ϑ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ and $\delta \in M \cap \theta$ witnessing the λ-ineffability of θ with respect to $\vec{d}$, as in Statement (ii) of Lemma 9.11, such that f, $\vec{e}$ and C are contained in $\operatorname{ran}(j)$. Assume for a contradiction that either $\operatorname{crit}(j)$ is not inaccessible or $\mathcal{P}_{\operatorname{crit}(j)}(\delta) \nsubseteq M$.

Next, assume also that $j[\delta] \notin A$. Since $j[\delta] \cap \theta = \operatorname{crit}(j) \in C$, $j^{-1}[d_{j[\delta]}] \in M$ implies that $e_{\operatorname{crit}(j)} \in M$. In this situation, the combination of Lemma 9.5 and Lemma 9.8 yields that $\operatorname{crit}(j) = j[M] \cap \theta$ is inaccessible. Since our assumptions imply that

$$\mathcal{P}_{\operatorname{crit}(j)}(j[M] \cap \lambda) \nsubseteq j[M],$$

we can conclude that $j[M]$ witnesses that $j[\delta] \in A$, a contradiction.

Hence $j[\delta] \in A$. Since we know that $\operatorname{crit}(j) = \alpha_{j[\delta]}$ and $j^{-1}[d_{j[\delta]}] \in M$, we know that $x_{j[\delta]} \in j[M]$. But this allows us to conclude that $f(x_{j[\delta]}) \in \lambda \cap j[M] \subseteq X_{j[\delta]}$ and hence $x_{j[\delta]} \in X_{j[\delta]}$, a contradiction. $\square$

Chapter 10

On a theorem by Christoph Weiss

In this chapter, we will point out problems that occur in proofs for theorems claimed by Christoph Weiß in his [38] and [39]. Before we can state the actual theorems, we will need to introduce a series of definitions. The key concept of *slenderness* originates from work of Saharon Shelah, and was isolated by Weiß in his [38]. It is a smallness property of lists, that allowed Weiß to define the *ineffable slender tree property* $\mathrm{ISP}(\theta, \lambda)$, which may be seen as the *combinatorial essence of a λ-ineffable cardinal θ*, which however can also hold accessible cardinals.

Remember that given an uncountable regular cardinal θ and a cardinal $\lambda \geq \theta$, a set $C \subseteq \mathcal{P}_\theta(\lambda)$ is a club if for every $x \in \mathcal{P}_\theta(\lambda)$ there is a $y \in C$ with $x \subseteq y$ (unboundedness), and if for every chain $x_0 \subseteq x_1 \subseteq ... \subseteq x_\xi \subseteq ...$, $\xi < \alpha$, of sets in C, with $\alpha < \theta$, the union $\bigcup_{\xi<\alpha} x_\xi$ is in C (closure).

Definition 10.1. Let θ be an uncountable regular cardinal and let $\lambda \geq \theta$ be a cardinal.

(i) A θ-list $\langle d_\alpha \mid \alpha < \theta\rangle$ is *slender* if there is a club C in θ with the property that for every $\gamma \in C$ and every $\alpha < \gamma$, there is a $\beta < \gamma$ with $d_\gamma \cap \alpha = d_\beta \cap \alpha$.

(ii) $\mathrm{SSP}(\theta)$ is the statement that for every slender θ-list $\langle d_\alpha \mid \alpha < \theta\rangle$ and every club C in θ, there are $\alpha, \beta \in C$ such that $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$.

(iii) A $\mathcal{P}_\theta(\lambda)$-list $\langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ is *slender* if for every sufficiently large cardinal ϑ, there is a club C in $\mathcal{P}_\theta(\mathrm{H}(\vartheta))$ with $b \cap d_{X\cap\lambda} \in X$ for all $X \in C$ and all $b \in X \cap \mathcal{P}_{\omega_1}(\lambda)$.

(iv) $\mathrm{ISP}(\theta, \lambda)$ is the statement that for every slender $\mathcal{P}_\theta(\lambda)$-list $\langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$, there exists $D \subseteq \lambda$ such that the set $\{a \in \mathcal{P}_\theta(\lambda) \mid d_a = D \cap a\}$ is stationary in $\mathcal{P}_\theta(\lambda)$.

The following theorem is presented in [38, Theorem 2.3.1].

Theorem 10.2. *Let $\tau < \theta$ be cardinals with τ uncountable and regular, and let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration such that the following statements hold for all inaccessible cardinals $\eta \leq \theta$:*

(i) $\vec{\mathbb{P}}_{<\eta} \subseteq \mathrm{H}(\eta)$[1] is the direct limit of $\langle\langle\mathbb{P}_{<\alpha} \mid \alpha < \eta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \eta\rangle\rangle$ and satisfies the η-chain condition.

(ii) If G is $\vec{\mathbb{P}}_{<\theta}$-generic over V and G_η is the filter on $\vec{\mathbb{P}}_{<\eta}$ induced by G, then the pair $(\mathrm{V}[G_\eta], \mathrm{V}[G])$ satisfies the σ-approximation property.

[1]Following [39], we make use of the convention that conditions in forcing iterations are only defined on their support.

(iii) If $\alpha < \eta$, then $\mathbb{P}_{<\alpha}$ is definable in $\mathrm{H}(\eta)$ from the parameters τ and α.

Then the following statements hold:

(1) If θ is a subtle cardinal, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash \mathrm{SSP}(\check{\theta})$.

(2) If θ is an ineffable cardinal, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash \mathrm{ISP}(\check{\theta}, \check{\theta})$.

Let us remind the reader of the following definition used later on.

Definition 10.3. Let $M \subseteq N$ be classes. We say that (M, N) satisfies the *ω_1-covering property* if whenever $A \in N$ is countable in N and $A \subseteq M$, then there is $B \in M$ which is countable in M and such that $A \subseteq B$. We say that a partial order $\mathbb{P}$ satisfies the *ω_1-covering property* in case the pair $(\mathrm{V}, \mathrm{V}[G])$ does so whenever G is $\mathbb{P}$-generic over V.

The next theorem is claimed in [38, Theorem 2.3.3], and it is a generalization of the second part of Theorem 10.2.

Theorem 10.4. *Let $\tau < \theta \leq \lambda$ be cardinals with τ uncountable and regular, and let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration such that the following statements hold for all inaccessible cardinals $\eta \leq \theta$:*

(i) $\vec{\mathbb{P}}_{<\eta} \subseteq \mathrm{H}(\eta)$ is the direct limit of $\langle\langle\mathbb{P}_{<\alpha} \mid \alpha < \eta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \eta\rangle\rangle$ and satisfies the η-chain condition.

(ii) If G is $\vec{\mathbb{P}}_{<\theta}$-generic over V and G_η is the filter on $\vec{\mathbb{P}}_{<\eta}$ induced by G, then the pair $(\mathrm{V}[G_\eta], \mathrm{V}[G])$ satisfies the σ-approximation property.

(iii) If $\alpha < \eta$, then $\mathbb{P}_{<\alpha}$ is definable in $\mathrm{H}(\eta)$ from the parameters τ and α.

(iv) If G_η is $\vec{\mathbb{P}}_{<\eta}$-generic over V, then the pair $(\mathrm{V}, \mathrm{V}[G_\eta])$ satisfies the ω_1-covering property.

Then, if θ is $\lambda^{<\theta}$-ineffable for some cardinal $\lambda \geq \theta$, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash \mathrm{ISP}(\check{\theta}, \check{\lambda})$.

As pointed out in [39, Section 5], William Mitchell's classical proof of the consistency of the tree property at successors of regular cardinals in [26] shows that for every uncountable regular cardinal τ and every inaccessible cardinal $\theta > \tau$, there is a forcing iteration $\vec{\mathbb{P}}$ satisfying the statements (i)-(iv) listed in Theorem 10.4 such that $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash$ "$\check{\theta} = \check{\tau}^+$" and forcing with $\vec{\mathbb{P}}_{<\theta}$ preserves all cardinals less than or equal to τ.

Now, we will discuss what appears to be a serious problem in the arguments used to derive the above statements in [38] and [39]. Latter in this chapter, we will use the small embedding characterizations of subtlety and of λ-ineffability from the previous chapter to provide a corrected versions of the proofs of Theorem 10.2 and of Theorem 10.4. The results will also be slightly improved, since we will weaken their assumptions.

We would first like to point out where the problematic step in Weiß's proof of Statements (2) of Theorem 10.2 and of Theorem 10.4 seems to be, and argue that it is indeed a problem, for Weiß's proof would in fact show a stronger result, one that is provably wrong. Let θ be a λ-ineffable cardinal with $\lambda = \lambda^{<\theta}$, let $\vec{\mathbb{P}} = \langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration satisfying Statements (i)-(iv) listed in Theorem 10.4, let G be $\vec{\mathbb{P}}_{<\theta}$-generic over V, and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)^{\mathrm{V}[G]}\rangle$ be a slender $\mathcal{P}_\theta(\lambda)$-list in $\mathrm{V}[G]$. The proofs of [38, Theorem 2.3.1] and [39, Theorem 5.4] then claim that there is a stationary subset T of $\mathcal{P}_\theta(\lambda)$ in V and $d \in \mathcal{P}(\lambda)^{\mathrm{V}[G]}$ such that $d_a = d \cap a$ holds for all $a \in T$. Since $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition in V and therefore preserves the stationarity of T, this argument would actually yield a strengthening of $\mathrm{ISP}(\theta,\lambda)$ stating that every instance of the principle is witnessed by a stationary subset of $\mathcal{P}_\theta(\lambda)$ contained in the ground model V. In particular, this conclusion would imply that if G is $\vec{\mathbb{P}}_{<\theta}$-generic over V and $\langle d_\alpha \mid \alpha < \theta\rangle$ is a θ-list in V[G], then there is a stationary subset S of θ in V such that $d_\alpha = d_\beta \cap \alpha$ holds for all $\alpha, \beta \in S$ with $\alpha < \beta$. The following proposition shows that this statement provably fails if forcing with $\vec{\mathbb{P}}_{<\theta}$ destroys the ineffability of θ.

Proposition 10.5. *Let $\langle\langle\vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle\dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration with the property that θ is an uncountable regular cardinal, $\vec{\mathbb{P}}_{<\theta}$ is a direct limit and $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition. Let G be $\vec{\mathbb{P}}_{<\theta}$-generic over* V *and, given $\alpha < \theta$, let G_α denote the filter on $\vec{\mathbb{P}}_{<\alpha}$ induced by G. Then one of the following statements holds:*

(i) There is an $\alpha < \theta$ such that for all $\alpha \leq \beta < \theta$, the partial order $\dot{\mathbb{P}}_\beta^{G_\beta}$ is trivial.
(ii) There is a slender θ-list $\langle d_\alpha \mid \alpha < \theta\rangle$ in V[G] *with the property that for every stationary subset S of θ in* V*, there are $\alpha, \beta \in S$ with $\alpha < \beta$ and $d_\alpha \neq d_\beta \cap \alpha$.*

Proof. Pick a sequence $\langle\langle \dot{q}_\alpha^0, \dot{q}_\alpha^1\rangle \mid \alpha < \theta\rangle$ in V such that the following statements hold for all $\alpha < \theta$:

(a) $\dot{q}_{\alpha,0}$ and $\dot{q}_{\alpha,1}$ are both $\vec{\mathbb{P}}_{<\alpha}$-names for a condition in $\dot{\mathbb{P}}_\alpha$.
(b) If H is $\vec{\mathbb{P}}_{<\alpha}$-generic over V, then the conditions $\dot{q}_{\alpha,0}^H$ and $\dot{q}_{\alpha,1}^H$ are compatible in $\dot{\mathbb{P}}_\alpha^H$ if and only if the partial order $\dot{\mathbb{P}}_\alpha^H$ is trivial.

Now, assume that (i) fails, and work in V[G]. Let $g : \theta \longrightarrow \theta$ denote the unique function with the property that for all $\beta < \theta$, $g(\beta)$ is the minimal ordinal greater than or equal to $\sup_{\alpha<\beta} g(\alpha)$ such that $\dot{\mathbb{P}}_{g(\beta)}^{G_{g(\beta)}}$ is a non-trivial partial order. Since $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition, there is a club subset C of θ in V with $g(\alpha) < \beta$ for all $\alpha < \beta$ whenever $\beta \in C$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ denote the unique θ-list with the property that

$$d_\alpha = 0 \iff d_\alpha \neq 1 \iff \dot{q}_{g(\alpha),0}^{G_{g(\alpha)}} \in G^{g(\alpha)}$$

holds for every $\alpha < \theta$, where G^β denotes the filter on $\dot{\mathbb{P}}_\beta^{G_\beta}$ induced by G for all $\beta < \theta$. Then $\vec{d}$ is a slender θ-list.

Assume for a contradiction that there is a stationary subset S of θ in V such that $d_\alpha = d_\beta \cap \alpha$ holds for all $\alpha, \beta \in S$ with $\alpha < \beta$. Then there is an $i < 2$ with $d_\alpha = i$ for all $\alpha \in S$. Let $\dot{g}$ be a $\vec{\mathbb{P}}_{<\theta}$-name for a function from θ to θ with $g = \dot{g}^G$ and let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\theta}$-name for a θ-list with $\vec{d} = \dot{d}^G$. Let p be a condition in G forcing all of the above statements. Pick a condition q in $\vec{\mathbb{P}}_{<\theta}$ below p. Then there is $\alpha \in C \cap S$ with $q \in \vec{\mathbb{P}}_{<\alpha}$. By density, we can find a condition $s \in G$ below q, and $\alpha \leq \beta < \theta$ with $g(\alpha) = \beta$, $s \in \vec{\mathbb{P}}_{<\beta+1}$ and $s(\beta) = \dot{q}_{\beta,1-i}$. But then $\dot{\mathbb{P}}_\beta^{G_\beta}$ is non-trivial, $\dot{q}_{\beta,1-i}^{G_\beta} \in G^\beta$ and $d_\alpha = 1 - i$, a contradiction. □

In the argument that is supposed to prove Theorem 10.4, Weiß constructs a club C in $\mathcal{P}_\theta(\lambda)$ in V such that $d_a \in \mathrm{V}[G_{a\cap\theta}]$ holds for every $a \in C$ with the property that $a \cap \theta$ is an inaccessible cardinal in V. The problematic step then seems to be his conclusion that there exists a sequence $\langle \dot{d}_a \mid a \in C\rangle$ in V with the property that for all $a \in C$ with

$a \cap \theta$ inaccessible in V, $\dot{d}_a$ is a $\vec{\mathbb{P}}_{<(a\cap\theta)}$-name with $d_a = \dot{d}_a^G$. Assuming the existence of such a sequence of names in V, it is easy to code the name $\dot{d}_a$ as a subset of a and then use the λ-ineffability of θ in V to obtain a stationary subset of $\mathcal{P}_\theta(\lambda)$ in V that witnesses the strengthening of $\mathrm{ISP}(\theta, \lambda)$ formulated above. Therefore, the above observation shows that such a sequence cannot exist in the ground model V. Since a similar argument is used in the proof of Statement (1) of Theorem 10.2 presented in [38], it is also not clear if these arguments can be modified to produce a correct proof of that statement.

Based on our small embedding characterizations of subtlety and of λ-ineffability, in the next chapter we will provide new proofs of Theorem 10.4 (which of course has Theorem 10.2,(2) as a special case) and of Theorem 10.2. Roughly the first halves of both those proofs closely resemble Weiß's original proofs.

Chapter 11

Internally AP-ineffable and subtle cardinals

In this chapter, we provide applications of our small embedding characterizations of subtlety, ineffability and λ-ineffability. We will use them to provide new proofs for theorems by Christoph Weiß that were discussed in the previous chapter. We will first introduce the concept of internally AP subtle, of internally AP ineffable, and of internally AP λ-ineffable cardinals for a proper class of cardinals λ and then using those concepts we will prove a slight strengthening of Theorem 10.2 and Theorem 10.4. In our improved versions of these theorems, we will not rely on any kind of definability, and we will not have to assume any kind of covering property of our iteration in the case of λ-ineffable cardinals.

Let us first define the notion of internally AP-subtle cardinals. This principle is based on the small embedding characterization of subtlety in Lemma 9.3 and on the σ-approximation property.

Definition 11.1. A cardinal θ is *internally AP subtle* if for all sufficiently large regular cardinals ϑ, all $x \in \mathrm{H}(\vartheta)$, every club C in θ, and every θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta \rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and a transitive model N of ZFC^- such that $\vec{d}, x, C \in \mathrm{ran}(j)$ and the following statements hold:

(i) $N \subseteq \mathrm{H}(\vartheta)$ and the pair $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.
(ii) $M \in N$ and $\mathcal{P}_{\omega_1}(\mathrm{crit}\,(j))^N \subseteq M$.
(iii) If $d_{\mathrm{crit}(j)} \in N$, then there is $\alpha \in C \cap \mathrm{crit}\,(j)$ with $d_\alpha = d_{\mathrm{crit}(j)} \cap \alpha$.

Now, we will see some consequences of internal subtlety that we will use later on.

Lemma 11.2. *If θ is an internally AP subtle cardinal, then* $\mathrm{SSP}(\theta)$ *holds.*

Proof. Fix a slender θ-list $\vec{d} = \langle d_\alpha \mid \alpha < \theta \rangle$ and a club C_0 in θ. Let $C \subseteq C_0$ be a club witnessing the slenderness of $\vec{d}$ and let ϑ be a sufficiently large regular cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ and a transitive ZFC^--model N witnessing the internal AP subtlety with respect to $\vec{d}$ and C. Then elementarity implies that $\mathrm{crit}\,(j) \in C \subseteq C_0$.

Assume for a contradiction that $d_{\mathrm{crit}(j)} \notin N$. Then the σ-approximation property yields an $x \in \mathcal{P}_{\omega_1}(\mathrm{crit}\,(j))^N$ with $d_{\mathrm{crit}(j)} \cap x \notin N$. Then $x \in M$ and, since $\mathrm{crit}\,(j)$ is a regular cardinal in M, there is an $\alpha < \operatorname{crit}(j) \in C$ with $x \subseteq \alpha$. In this situation, the slenderness of $\vec{d}$ yields a $\beta < \operatorname{crit}(j)$ with $d_{\operatorname{crit}(j)} \cap \alpha = d_\beta \cap \alpha$. But then we have

$$d_{\operatorname{crit}(j)} \cap x \;=\; d_{\operatorname{crit}(j)} \cap x \cap \alpha \;=\; d_\beta \cap x \cap \alpha.$$

Since $\vec{d} \in \operatorname{ran}(j)$, we have $d_\beta \in M \subseteq N$ and hence $d_{\operatorname{crit}(j)} \cap x \in N$, a contradiction.

The above computations show that $d_{\operatorname{crit}(j)} \in N$ and therefore our assumptions yield an $\alpha < \operatorname{crit}(j)$ with $\alpha \in C \subseteq C_0$ and $d_\alpha = d_{\operatorname{crit}(j)} \cap \alpha$. □

Corollary 11.3. *If θ is an internally AP subtle cardinal, then θ is a subtle cardinal in* L.

Proof. This statement follows directly from [38, Theorem 2.4.1] and the above lemma. □

Corollary 11.4. *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a subtle cardinal.
(ii) θ is an internally AP subtle cardinal.

Proof. The forward direction is a direct consequence of Lemma 9.3. In the other direction, the results of [38, Section 1.2] show that the inaccessibility of θ implies that every θ-list is slender and we can apply Lemma 11.2 to conclude that (i) is a consequence of (ii). □

In combination with Lemma 11.2, the following theorem directly yields a proof of Statement (1) of Theorem 10.2. As already mentioned above, the results of [26] show that there are forcing iterations with these properties that turn inaccessible cardinals into the successor of an uncountable regular cardinal. In particular, it is possible to establish the consistency of $\mathrm{SSP}(\omega_2)$ from a subtle cardinal.

Theorem 11.5. *[13] Let $\vec{\mathbb{P}} = \langle\langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration with θ an uncountable and regular cardinal, such that the following statements hold for all inaccessible $\nu \leq \theta$:*

(i) $\vec{\mathbb{P}}_{<\nu} \subseteq \mathrm{H}(\nu)$ is the direct limit of $\langle\langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha < \nu\rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \nu\rangle\rangle$ and satisfies the ν-chain condition.
(ii) If G is $\vec{\mathbb{P}}_{<\theta}$-generic over V *and G_ν is the filter on $\vec{\mathbb{P}}_{<\nu}$ induced by G, then the pair* $(\mathrm{V}[G_\nu], \mathrm{V}[G])$ *satisfies the σ-approximation property.*

Then, if θ is a subtle cardinal, then $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash$ "θ is internally AP subtle".

Proof. Let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\theta}$-name for a θ-list, let $\dot{C}$ be a $\vec{\mathbb{P}}_{<\theta}$-name for a club in θ and let $\dot{x}$ be any $\vec{\mathbb{P}}_{<\theta}$-name. Since $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition, there is a club $C \subseteq \mathrm{Lim}$ in θ such that $\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash$ "$\check{C} \subseteq \dot{C}$" and all elements of C are closed under the Gödel pairing function $\prec\cdot,\cdot\succ$. Given $\alpha < \theta$, let $\dot{d}_\alpha$ be a $\vec{\mathbb{P}}_{<\theta}$-nice name for the α-th component of $\dot{d}$. Pick a regular cardinal $\vartheta > 2^\theta$ with $\dot{d}, \dot{x}, C, \dot{C}, \vec{\mathbb{P}} \in \mathrm{H}(\vartheta)$, which is sufficiently large with respect to Statement (ii) in Lemma 9.3. Let G be $\vec{\mathbb{P}}_{<\theta}$-generic over V.

First, assume that there is an inaccessible cardinal $\nu < \theta$ in V and a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for θ in V such that $\dot{d}, \dot{x}, C, \dot{C}, \vec{\mathbb{P}} \in \operatorname{ran}(j)$, $\nu = \operatorname{crit}(j)$ and $\dot{d}_\nu^G \notin \mathrm{V}[G_\nu]$. Then our assumptions on $\vec{\mathbb{P}}$ imply that $\vec{\mathbb{P}}_{<\nu} \in M$, $j(\vec{\mathbb{P}}_{<\nu}) = \vec{\mathbb{P}}_{<\theta}$ and $j \restriction \vec{\mathbb{P}}_{<\nu} = \mathrm{id}_{\vec{\mathbb{P}}_{<\nu}}$.

Hence it is possible to lift j in order to obtain a small embedding $j_* : M[G_\nu] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ for θ in $\mathrm{V}[G]$ with $\dot{d}^G, \dot{x}^G, \dot{C}^G \in \mathrm{ran}(j_*)$. Set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_\nu]}$. Then our assumptions imply that $M[G_\nu] \in N \subseteq \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ and the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ satisfies the σ-approximation property. Moreover, we also have $\mathrm{H}(\nu)^N \subseteq M[G_\nu]$, because Lemma 5.6 shows that $\mathrm{H}(\nu)^{\mathrm{V}} \subseteq M$ and $\vec{\mathbb{P}}_{<\nu}$ satisfies the ν-chain condition in V. Since $\dot{d}^G_\nu \notin \mathrm{V}[G_\nu]$, this shows that the embedding j_* and the model N witness that θ is internally *AP* subtle with respect to $\dot{d}^G$, $\dot{x}^G$ and $\dot{C}^G$ in $\mathrm{V}[G]$.

Next, assume that $\dot{d}^G_\nu \in \mathrm{V}[G_\nu]$ holds for every ν contained in the set A of all inaccessible cardinals $\nu < \theta$ in V with the property that there is small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for θ in V with $\mathrm{crit}\,(j) = \nu$ and $\dot{d}, \dot{x}, C, \dot{C}, \vec{\mathbb{P}} \in \mathrm{ran}(j)$. Let $p \in G$ be a condition forcing this statement. Work in V and pick a condition q below p in $\vec{\mathbb{P}}_{<\theta}$. We let A_* denote the set of all $\nu \in A$ with $q \in \vec{\mathbb{P}}_{<\nu}$. With the help of our assumption and the fact that $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition, we find a function $g : A_* \longrightarrow \theta$ and sequences $\langle q_\nu \mid \nu \in A_* \rangle$, $\langle \dot{r}_\nu \mid \nu \in A_* \rangle$ and $\langle \dot{e}_\nu \mid \nu \in A_* \rangle$ such that the following statements hold for all $\nu \in A_*$:

(1) $g(\nu) > \nu$ and $\dot{d}_\nu$ is a $\vec{\mathbb{P}}_{<g(\nu)}$-name.
(2) q_ν is a condition in $\vec{\mathbb{P}}_{<\nu}$ below q.
(3) $\dot{r}_\nu$ is a $\vec{\mathbb{P}}_{<\nu}$-name for a condition in the corresponding tail forcing $\dot{\mathbb{P}}_{[\nu, g(\nu))}$. [1]
(4) $\dot{e}_\nu$ is a $\vec{\mathbb{P}}_{<\nu}$-name for a subset of ν with $\langle q_\nu, \dot{r}_\nu \rangle \Vdash_{\vec{\mathbb{P}}_{<\nu} * \dot{\mathbb{P}}_{[\nu, g(\nu))}}$ "$\dot{d}_\nu = \dot{e}_\nu$".

Given $\nu \in A_*$, let E_ν denote the set of all triples $\langle s, \beta, i \rangle \in \vec{\mathbb{P}}_{<\nu} \times \nu \times 2 \subseteq \mathrm{H}(\nu)$ with

$$s \Vdash_{\vec{\mathbb{P}}_{<\nu}} \text{“}\check{\beta} \in \dot{e}_\nu \;\longleftrightarrow\; i = 1\text{”}.$$

Let $\vec{c} = \langle c_\alpha \mid \alpha < \theta \rangle$ be the θ-list, and let C_* be the club in θ, obtained from an application of Lemma 9.5. Fix a bijection $f : \theta \longrightarrow \mathrm{H}(\theta)$ with $f[\nu] = \mathrm{H}(\nu)$ for every inaccessible cardinal $\nu < \theta$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \theta \rangle$ be the unique θ-list such that the following statements hold for all $\alpha < \theta$:

(a) If $\alpha \in A_*$, then

$$d_\alpha \;=\; \{\prec 0, 0 \succ\} \cup \{\prec f^{-1}(q_\alpha), 1 \succ\} \cup \{\prec f^{-1}(e), 2 \succ \mid e \in E_\alpha\} \;\subseteq\; \alpha.$$

(b) If $\omega \subseteq \alpha \notin A_*$ and α is closed under $\prec \cdot, \cdot \succ$, then

$$d_\alpha \;=\; \{\prec 1, 0 \succ\} \cup \{\prec \beta, 1 \succ \mid \beta \in c_\alpha\} \;\subseteq\; \alpha.$$

(c) Otherwise, d_α is the empty set.

Let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ witnessing the subtlety of θ with respect to $\vec{d}$ and $C \cap C_*$, as in Statement (ii) of Lemma 9.3, such that

$$\vec{c}, \dot{d}, f, g, q, \dot{x}, C, C_*, \dot{C}, \vec{\mathbb{P}} \in \mathrm{ran}(j).$$

Set $\nu = \mathrm{crit}\,(j)$ and pick $\alpha \in C \cap C_* \cap \nu$ with $d_\alpha = d_\nu \cap \alpha$. Then $\omega \leq \alpha < \nu$ and both α and ν are closed under $\prec \cdot, \cdot \succ$.

[1] Let us point out that the problematic argument in Weiß's original proof can be seen as him assuming that the name $\dot{r}_\nu$ is just the name for the trivial condition in the corresponding tail forcing.

Assume for a contradiction that $\nu \notin A_*$. This implies that $\prec 1, 0 \succ \in d_\alpha$ and therefore $\alpha \notin A_*$. But then $c_\alpha = c_\nu \cap \alpha$, and j witnesses the subtlety of θ with respect to $\vec{c}$ and C_*, as in Statement (ii) of Lemma 9.3. By Lemma 9.5, this implies that ν is inaccessible, and hence j witnesses that ν is an element of A_*, a contradiction.

Hence $\nu \in A_*$, and this implies that $\prec 0, 0 \succ \in d_\alpha$, $\alpha \in A_*$, $g(\alpha) < \nu$, $q_\alpha = q_\nu \in \vec{\mathbb{P}}_{<\alpha}$ and $E_\alpha \subseteq E_\nu$. Pick a condition u in $\vec{\mathbb{P}}_{<\theta}$ such that the canonical condition in $\vec{\mathbb{P}}_{<\alpha} * \dot{\mathbb{P}}_{[\alpha,\nu)}$ corresponding to $u \restriction \nu$ is stronger than $\langle q_\alpha, \dot{r}_\alpha \rangle$ and the canonical condition in $\vec{\mathbb{P}}_{<\nu} * \dot{\mathbb{P}}_{[\nu,\theta)}$ corresponding to u is stronger than $\langle u \restriction \nu, \dot{r}_\nu \rangle$. Let H be $\vec{\mathbb{P}}_{<\theta}$-generic over V with $u \in H$, let H_α denote the filter on $\vec{\mathbb{P}}_{<\alpha}$ induced by H and let H_ν denote the filter on $\vec{\mathbb{P}}_{<\nu}$ induced by H. Then $\dot{d}^H_\alpha = \dot{e}^{H_\alpha}_\alpha \in \mathrm{V}[H_\alpha]$, and $\dot{d}^H_\nu = \dot{e}^{H_\alpha}_\nu \in \mathrm{V}[H_\nu]$. If $\beta \in \dot{d}^H_\alpha$, then there is $s \in H_\alpha \subseteq H_\nu$ with $\langle s, \beta, 1 \rangle \in E_\alpha \subseteq E_\nu$, and this implies that $\beta \in \dot{d}^H_\nu$. In the other direction, if $\beta \in \alpha \setminus \dot{d}^H_\alpha$, then there is $s \in H_\alpha$ with $\langle s, \beta, 0 \rangle \in E_\alpha$, and hence $\beta \notin \dot{d}^H_\nu$. This shows that $\dot{d}^H_\alpha = \dot{d}^H_{\mathrm{crit}(j_*)} \cap \alpha$. Set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[H_\nu]}$ and let $j_* : M[H_\nu] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$ denote the lift of j in $\mathrm{V}[H]$. Then j_* is a small embedding for θ in $\mathrm{V}[H]$ with $\dot{d}^H, \dot{x}^H, \dot{C}^H \in \mathrm{ran}(j)$, $M[H_\nu] \in N \subseteq \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$, and the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[H]})$ satisfies the σ-approximation property. Moreover, since Lemma 5.6 shows that $\mathrm{H}(\nu)^{\mathrm{V}} \subseteq M$ and $\vec{\mathbb{P}}_{<\nu}$ satisfies the ν-chain condition in V, we also have $\mathrm{H}(\nu)^N \subseteq M[H_\nu]$. Finally, we have $\alpha \in \dot{C}^H \cap \mathrm{crit}\,(j_*)$ with $\dot{d}^H_\alpha = \dot{d}^H_{\mathrm{crit}(j_*)} \cap \alpha$.

Since $u \leq_{\vec{\mathbb{P}}_{<\theta}} q$ holds in the above computations, a density argument shows that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ for θ in $\mathrm{V}[G]$ witnessing that θ is internally AP subtle with respect to $\dot{d}^G$, $\dot{x}^G$ and $\dot{C}^G$ in $\mathrm{V}[G]$. □

Next, we turn our attention towards the hierarchy of ineffable cardinals. The small embedding characterization for λ-ineffable cardinals from Lemma 9.11 also gives rise to an internal large cardinal principle.

Definition 11.6. Given cardinals $\theta \leq \lambda$, the cardinal θ is *internally AP λ-ineffable* if for all sufficiently large regular cardinals ϑ, all $x \in \mathrm{H}(\vartheta)$, and every $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda) \rangle$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ, an ordinal $\delta \in M \cap \theta$ and a transitive model N of ZFC^- such that $j(\delta) = \lambda$, $\vec{d}, x \in \mathrm{ran}(j)$, and the following statements hold:

(i) $N \subseteq \mathrm{H}(\vartheta)$ and the pair $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.
(ii) $M \in N$ and $\mathcal{P}_{\omega_1}(\delta)^N \subseteq M$.
(iii) If $j^{-1}[d_{j[\delta]}] \in N$, then $j^{-1}[d_{j[\delta]}] \in M$.

Analogous to the above study of internal subtlety, we will present some consequences of this principle.

Lemma 11.7. *If θ is internally AP λ-ineffable, then* $\mathrm{ISP}(\theta, \lambda)$ *holds.*

Proof. Fix a slender $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda) \rangle$ and a sufficiently large cardinal ν such that there is a function $f : \mathcal{P}_\omega(\mathrm{H}(\nu)) \longrightarrow \mathcal{P}_\theta(\mathrm{H}(\nu))$ with the property that Cl_f is a club in $\mathcal{P}_\theta(\mathrm{H}(\nu))$ witnessing the slenderness of $\vec{d}$. Let ϑ be a sufficiently large regular cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ with $f \in \mathrm{ran}(j)$, $\delta \in M$ and a transitive ZFC^--model N witnessing the internal AP λ-ineffability of θ with respect to $\vec{d}$. Pick $\varepsilon \in M$ with $j(\varepsilon) = \nu$. As in the proof of Lemma 9.11, we then have $X = j[\mathrm{H}(\varepsilon)^M] \in \mathrm{Cl}_f$.

Assume for a contradiction that $j^{-1}[d_{j[\delta]}] \notin N$. Then the σ-approximation property yields an $x \in \mathcal{P}_{\omega_1}(\delta)^N$ with $x \cap j^{-1}[d_{j[\delta]}] \notin N$. Then our assumptions imply that x is an element of $\mathcal{P}_{\omega_1}(\delta)^M$. But then $j(x) \in X \cap \mathcal{P}_{\omega_1}(\lambda)$, and the slenderness of $\vec{d}$ implies that $j(x) \cap d_{j[\delta]} \in X$. Then, we can conclude that

$$x \cap j^{-1}[d_{j[\delta]}] \ = \ j^{-1}[j(x) \cap d_{j[\delta]}] \ = \ j^{-1}(j(x) \cap d_{j[\delta]}) \ \in \ M \ \subseteq \ N,$$

a contradiction.

The above computations show that $j^{-1}[d_{j[\delta]}] \in N$, and hence our assumptions imply that this set is also an element of M. Let $D = j(j^{-1}[d_{j[\delta]}])$, and assume for a contradiction that the set $S = \{a \in \mathcal{P}_\theta(\lambda) \mid d_a = D \cap a\}$ is not stationary in $\mathcal{P}_\theta(\lambda)$. By elementarity, there is a function $f_0 : \mathcal{P}_\omega(\delta) \longrightarrow \mathcal{P}_{\mathrm{crit}(j)}(\delta)$ in M such that $\mathrm{Cl}_{j(f_0)} \cap S = \emptyset$. But then $j[\delta] \in \mathrm{Cl}_{j(f_0)} \cap S$, a contradiction. □

Corollary 11.8. *If θ is an internally AP θ-ineffable cardinal, then θ is an ineffable cardinal in* L.

Proof. This statement follows directly from [38, Theorem 2.4.3] and the above lemma. □

Corollary 11.9. *The following statements are equivalent for every inaccessible cardinal θ and every cardinal $\lambda \geq \theta$ satisfying $\lambda = \lambda^{<\theta}$:*

(i) θ is a λ-ineffable cardinal.
(ii) θ is an internally AP λ-ineffable cardinal.

Proof. Lemma 9.11 directly shows that (i) implies (ii) with $N = \mathrm{H}(\theta)$. In the other direction, the results of [39, Section 2] show that an inaccessible cardinal θ is λ-ineffable if and only if $\mathrm{ISP}(\theta, \lambda)$ holds. Therefore, Lemma 11.7 shows that (ii) implies (i). □

A variation of the proof of Theorem 11.5, using Lemma 9.13, allows us to establish the consistency of the principle $\mathrm{ISP}(\kappa, \lambda)$ for accessible cardinals κ with the help of small embeddings. Note that since $\lambda^{<\kappa} = (\lambda^{<\kappa})^{<\kappa}$ and $\mathrm{ISP}(\kappa, \lambda^{<\kappa})$ implies $\mathrm{ISP}(\kappa, \lambda)$ (see [39, Proposition 3.4]), a combination of Lemma 11.7 and of the following result implies Statement (2) of Theorem 10.2 and Theorem 10.4. Moreover, note that results of Chris Johnson in [16] show that if κ is λ-ineffable and $\mathrm{cof}(\lambda) \geq \kappa$, then $\lambda = \lambda^{<\kappa}$ (see also [38, Proposition 1.5.4]).

Theorem 11.10 ([13])**.** *Let $\vec{\mathbb{P}} = \langle\langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha \leq \theta\rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \theta\rangle\rangle$ be a forcing iteration with θ an uncountable and regular cardinal, such that the following statements hold for all inaccessible $\nu \leq \theta$:*

(i) $\vec{\mathbb{P}}_{<\nu} \subseteq \mathrm{H}(\nu)$ is the direct limit of $\langle\langle \vec{\mathbb{P}}_{<\alpha} \mid \alpha < \nu\rangle, \langle \dot{\mathbb{P}}_\alpha \mid \alpha < \nu\rangle\rangle$ and satisfies the ν-chain condition.
(ii) If G is $\vec{\mathbb{P}}_{<\theta}$-generic over V and G_ν is the filter on $\vec{\mathbb{P}}_{<\nu}$ induced by G, then the pair $(\mathrm{V}[G_\nu], \mathrm{V}[G])$ satisfies the σ-approximation property.

Then, if θ is a λ-ineffable cardinal with $\lambda = \lambda^{<\theta}$, then

$$\mathbb{1}_{\vec{\mathbb{P}}_{<\theta}} \Vdash \text{“}\theta \text{ is internally AP } \lambda\text{-ineffable”}.$$

Proof. Let $\dot{d}$ be a $\vec{\mathbb{P}}_{<\theta}$-name for a $\mathcal{P}_\theta(\lambda)$-list and let $\dot{x}$ be any $\vec{\mathbb{P}}_{<\theta}$-name. Given $a \in \mathcal{P}_\theta(\lambda)$, let $\dot{d}_a$ be a $\vec{\mathbb{P}}_{<\theta}$-nice name for the a-th component of $\dot{d}$. Fix a bijection $f : \theta \longrightarrow \mathrm{H}(\theta)$ with $f[\nu] = \mathrm{H}(\nu)$ for every inaccessible cardinal $\nu < \theta$. Pick a regular cardinal $\vartheta > 2^\lambda$ with $\dot{d}, \dot{x}, \vec{\mathbb{P}} \in \mathrm{H}(\vartheta)$, which is sufficiently large with respect to Statement (ii) in Lemma 9.11. Let G be $\vec{\mathbb{P}}_{<\theta}$-generic over V.

First, assume that there is an inaccessible cardinal $\nu < \theta$ in V, a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for θ in V and $\delta \in M \cap \theta$ such that $\dot{d}, f, \dot{x}, \vec{\mathbb{P}} \in \mathrm{ran}(j)$, $\nu = \mathrm{crit}\,(j)$, $j(\delta) = \lambda$, $\mathcal{P}_\nu(\delta)^{\mathrm{V}} \subseteq M$ and $\dot{d}^G_{j[\delta]} \notin \mathrm{V}[G_\nu]$. Let $j_* : M[G_\nu] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ denote the corresponding lift of j. Then j_* is a small embedding for θ in $\mathrm{V}[G]$ with $\dot{d}^G, \dot{x}^G \in \mathrm{ran}(j_*)$. If we define $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_\nu]}$, then our assumptions imply that $M[G_\nu] \in N \subseteq \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ and the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ satisfies the σ-approximation property. Moreover, since $\mathcal{P}_\nu(\delta)^{\mathrm{V}} \subseteq M$ and $\vec{\mathbb{P}}_{<\nu}$ satisfies the ν-chain condition in V, we know that $\mathcal{P}_\nu(\delta)^N \subseteq M[G_\nu]$. Since $\dot{d}^G_{j[\delta]} \notin \mathrm{V}[G_\nu]$ implies that $j_*^{-1}[\dot{d}^G_{j_*[\delta]}] \notin N$, we can conclude that j_*, δ and N witness that θ is internally AP λ-ineffable with respect to $\dot{d}^G$ and $\dot{x}^G$ in $\mathrm{V}[G]$.

Next, assume that $\dot{d}^G_a \in \mathrm{V}[G_\nu]$ holds for all elements of the set A of all $a \in \mathcal{P}_\theta(\lambda)^{\mathrm{V}}$ with the property that there is small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for θ in V and $\delta \in M \cap \theta$ with $j(\delta) = \lambda$, $a = j[\delta]$, $\nu_a = \mathrm{crit}\,(j) = a \cap \theta$ is an inaccessible cardinal in V, $\mathcal{P}_{\nu_a}(\delta)^{\mathrm{V}} \subseteq M$ and $\dot{d}, f, \dot{x}, \vec{\mathbb{P}} \in \mathrm{ran}(j)$. Let $p \in G$ be a condition forcing this statement. Work in V and pick a condition q below p in $\vec{\mathbb{P}}_{<\theta}$. We let A_* denote the set of all $a \in A$ with $q \in \vec{\mathbb{P}}_{<\nu_a}$. Then all elements of A_* are closed under $\prec\cdot,\cdot\succ$ and, with the help of our assumption and the fact that $\vec{\mathbb{P}}_{<\theta}$ satisfies the θ-chain condition, we find sequences $\langle q_a \mid a \in A_* \rangle$, $\langle \dot{r}_a \mid a \in A_* \rangle$ and $\langle \dot{e}_a \mid a \in A_* \rangle$ such that the following statements hold for all $a \in A_*$:

(1) q_a is a condition in $\vec{\mathbb{P}}_{<\nu_a}$ below q.
(2) $\dot{r}_a$ is a $\vec{\mathbb{P}}_{<\nu_a}$-name for a condition in the corresponding tail forcing $\dot{\mathbb{P}}_{[\nu_a,\theta)}$.[2]
(3) $\dot{e}_a$ is a $\vec{\mathbb{P}}_{<\nu_a}$-nice name for a subset of a with $\langle q_a, \dot{r}_a \rangle \Vdash_{\vec{\mathbb{P}}_{<\nu_a} * \dot{\mathbb{P}}_{[\nu_a,\theta)}}$ "$\dot{d}_a = \dot{e}_a$".

Let $\vec{c} = \langle c_a \mid a \in \mathcal{P}_\theta(\lambda) \rangle$ be the $\mathcal{P}_\theta(\lambda)$-list given by Lemma 9.13 and let $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda) \rangle$ be the unique $\mathcal{P}_\theta(\lambda)$-list with

$$d_a \;=\; \{\prec f^{-1}(s), \beta \succ \mid \langle \check{\beta}, s \rangle \in \dot{e}_a\} \;\subseteq\; a$$

for all $a \in A_*$ and $d_a = c_a$ for all $a \in \mathcal{P}_\theta(\lambda) \setminus A_*$. Pick a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ that witness the λ-ineffability of θ with respect to $\vec{d}$, as in Statement (ii) of Lemma 9.11, such that $\vec{c}, \dot{d}, f, q, \dot{x}, \vec{\mathbb{P}} \in \mathrm{ran}(j)$.

Assume for a contradiction that $j[\delta] \notin A_*$. Then $d_{j[\delta]} = c_{j[\delta]}$ and hence $j^{-1}[c_{j[\delta]}] \in M$. This shows that j and δ witness the λ-ineffability of θ with respect to $\vec{c}$, and Lemma 9.13 implies that $\mathrm{crit}\,(j)$ is an inaccessible cardinal and $\mathcal{P}_{\mathrm{crit}(j)}(\delta) \subseteq M$. But then j and δ also witness that $j[\delta]$ is an element of A_*, a contradiction.

Hence $j[\delta] \in A_*$. Pick a condition u in $\vec{\mathbb{P}}_{<\theta}$ such that the canonical condition in $\vec{\mathbb{P}}_{<\nu_{j[\delta]}} * \dot{\mathbb{P}}_{[\nu_{j[\delta]},\theta)}$ corresponding to u is stronger than $\langle q_{j[\delta]}, \dot{r}_{j[\delta]} \rangle$. Let H be $\vec{\mathbb{P}}_{<\theta}$-generic over V with $u \in H$ and let H_j denote the filter on $\vec{\mathbb{P}}_{<\nu_{j[\delta]}}$ induced by H. Then $\dot{d}^H_{j[\delta]} = \dot{e}^{H_j}_{j[\delta]} \in \mathrm{V}[H_j]$.

[2]Let us point out that the problematic argument in Weiß's original proof can be seen as him assuming that the name $\dot{r}_a$ is just the name for the trivial condition in the corresponding tail forcing.

Given $\gamma < \delta$, we have $j(\gamma) \in \dot{d}^H_{j[\delta]}$ if and only if there is an $s \in H_j$ with $\prec f^{-1}(s), j(\beta) \succ \in d_{j[\delta]}$. Since $f \upharpoonright \nu_{j[\delta]} \in M$ with $j(f \upharpoonright \nu_{j[\delta]}) = f$, this shows that $j^{-1}[\dot{d}^H_{j[\delta]}]$ is equal to the set of all $\gamma < \delta$ such that there is an $s \in H_j$ with $\prec (f \upharpoonright \nu_{j[\delta]})^{-1}(s), \gamma \succ \in j^{-1}[d_{j[\delta]}]$. This shows that $j^{-1}[\dot{d}^H_{j[\delta]}]$ is an element of $M[H_j]$.

Set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[H_j]}$ and let $j_* : M[H_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$ denote the induced lift of j. Then j_* is a small embedding for θ in $\mathrm{V}[H]$ such that $\dot{d}^H, \dot{x}^H, \in \mathrm{ran}(j)$, $\dot{d}^H_{j_*[\delta]} \in M[H_j]$, $M[H_j] \in N \subseteq \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$ and the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[H]})$ satisfies the σ-approximation property. Since $\mathcal{P}_{\nu_{j[\delta]}}(\delta)^{\mathrm{V}} \subseteq M$ and $\vec{\mathbb{P}}_{<\nu_{j[\delta]}}$ satisfies the $\nu_{j[\delta]}$-chain condition in V, we also know that $\mathcal{P}_{\mathrm{crit}(j_*)}(\delta)^N \subseteq M[H_j]$.

As above, a density argument shows that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ for θ in $\mathrm{V}[G]$ witnessing that θ is internally AP λ-ineffable with respect to $\dot{d}^G$ and $\dot{x}^G$ in $\mathrm{V}[G]$. □

Chapter 12

Characterization of subtle and λ-ineffable cardinals via $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$

In this chapter, we are going to provide strong characterizations of subtle and λ-ineffable cardinals for a proper class of cardinals λ. These results will also produce a strong characterization of supercompactness, and this will be shown in latter chapter.

Since Neeman's pure side condition forcing has similar structural properties as the forcing iterations used in the previous chapter, the proofs of the following two theorems are very similar to the proofs of Theorem 11.5 and Theorem 11.10 respectively. We start with subtlety.

Theorem 12.1 ([12]). *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a subtle cardinal.
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "ω_2 is internally AP subtle".
(iii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash \mathrm{SSP}(\omega_2)$.

Proof. First, assume that (iii) holds. Let $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ be a θ-list, and let C be a club subset of θ. Since θ is inaccessible, we know that $\vec{d}$ is slender. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. Since slenderness of θ-lists is clearly upwards absolute to models that preserve the regularity of θ, our assumption implies that there are $\alpha, \beta \in C$ with $\alpha < \beta$ and $d_\alpha = d_\beta \cap \alpha$. These computations show that (i) holds.

Now, assume that (i) holds. Let $\dot{d}$ be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for a θ-list, let $\dot{C}$ be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for a club in θ, and let $\dot{x}$ be any $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name. By Lemma 4.11, we can find a club C in θ consisting only of limit ordinals that are closed under the Gödel pairing function $\prec\cdot,\cdot\succ$, with $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$"$\check{C} \subseteq \dot{C}$". For every $\alpha < \theta$, let $\dot{d}_\alpha$ be a nice $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for the α-th element of $\dot{d}$. Let $\vartheta > 2^\theta$ be a regular cardinal that is sufficiently large with respect to Lemma 9.3, and which satisfies $\dot{d}, \dot{x} \in \mathrm{H}(\vartheta)$. Define A to be the set of all inaccessible cardinals κ less than θ for which there exists a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with critical point κ and $\dot{d}, \dot{x}, C, \dot{C} \in \mathrm{ran}(j)$. Finally, let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V.

Assume first that there is $\kappa < \theta$ and a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ in V so that j witnesses that κ is an element of A, and $\dot{d}_\kappa^G \notin \mathrm{V}[G_j]$, with $G_j = G \cap \mathrm{H}(\kappa)$ defined as in Lemma 6.3. Let $j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ be the lifting of j provided by Lemma 6.3, and set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}$. Then $\dot{d}^G, \dot{x}^G, \dot{C}^G \in \mathrm{ran}(j_G)$, the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ satisfies the σ-approximation property, Theorem 6.1 implies that $\kappa = \omega_2^N$, and another application of Lemma 4.11 yields $\mathcal{P}_{\omega_1}(\kappa)^N \subseteq \mathrm{H}(\kappa)^N \subseteq M[G_j]$. Since $\dot{d}_\kappa^G \notin N$, we can conclude that j_G and N witness that θ is internally AP subtle in $\mathrm{V}[G]$ with respect to $\dot{d}^G, \dot{x}^G$ and $\dot{C}^G$.

Otherwise, assume that whenever $j : M \longrightarrow \mathrm{H}(\vartheta)$ is a small embedding for θ in V that witnesses that some $\kappa < \theta$ is an element of A, then $\dot{d}_\kappa^G \in \mathrm{V}[G_j]$. Fix a condition $p \in G$ that forces this statement, pick some $q \leq_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} p$, and work in V. Let B denote the set of all $\kappa \in A$ such that q is a condition in $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$. Since $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfies the θ-chain condition, we can find a function $g : B \longrightarrow \theta$ and sequences $\langle q_\kappa \mid \kappa \in B\rangle, \langle \dot{r}_\kappa \mid \kappa \in B\rangle$ and $\vec{e} = \langle \dot{e}_\kappa \mid \kappa \in B\rangle$ such that the following statements hold for all $\kappa \in B$:

(a) $g(\kappa) > \kappa$ is inaccessible and $\dot{d}_\kappa$ is a $\mathbb{P}_{\mathcal{S}_{g(\kappa)},\mathcal{T}_{g(\kappa)}}$-name.
(b) q_κ is a condition in $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$ below q.
(c) $\dot{r}_\kappa$ is a $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$-name for a condition in $\dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa)}$ that is an element of $\mathrm{H}(g(\kappa))$.
(d) $\dot{e}_\kappa$ is a $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$-name for a subset of κ with $\langle q_\kappa, \dot{r}_\kappa\rangle \Vdash_{\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa} * \dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa)}}$ "$\dot{d}_\kappa = \dot{e}_\kappa$".

Given $\kappa \in B$, let E_κ denote the set of all triples $\langle s, \beta, i\rangle \in \mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa} \times \kappa \times 2 \subseteq \mathrm{H}(\kappa)$ with

$$s \Vdash_{\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}} \text{"}\check{\beta} \in \dot{e}_\kappa \longleftrightarrow i = 1\text{"}.$$

Fix a bijection $b : \theta \longrightarrow \mathrm{H}(\theta)$ with $b[\kappa] = \mathrm{H}(\kappa)$ for every inaccessible cardinal $\kappa \leq \theta$, and define $\vec{d} = \langle d_\alpha \mid \alpha < \theta\rangle$ to be the unique θ-list with

$$d_\alpha = \{\prec 0, 0\succ\} \cup \{\prec b^{-1}(q_\alpha), 1\succ\} \cup \{\prec b^{-1}(e), 2\succ \mid e \in E_\alpha\} \subseteq \alpha$$

for all $\alpha \in B$, and with $d_\alpha = \emptyset$ for all $\alpha \in \theta \setminus B$. Next, let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ which witnesses the subtlety of θ with respect to C, $\vec{d}$ and $\{b, \dot{d}, g, q, \dot{C}\}$, and let κ denote the critical point of j. Then there is an $\alpha \in C \cap \kappa$ with $d_\alpha = d_\kappa \cap \alpha$. In this situation, the embedding j witnesses that κ is an element of A and, by elementarity, $q \in \mathrm{ran}(j)$ implies that $q \in \mathrm{H}(\kappa)$ and $\kappa \in B$. But then, $\prec 0, 0\succ \in d_\kappa \cap \alpha$, and therefore $\alpha \in B$. By Lemma 6.3, this shows that $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$ is a complete suborder of $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, $\mathbb{P}_{\mathcal{S}_\alpha,\mathcal{T}_\alpha}$ is a complete suborder of $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$ and $\mathrm{H}(\alpha) \in \mathcal{T}_\kappa \subseteq \mathcal{T}_\theta$. Moreover, the above coherence implies that $q_\alpha = q_\kappa \in \mathbb{P}_{\mathcal{S}_\alpha,\mathcal{T}_\alpha}$ and $E_\alpha \subseteq E_\kappa$. By elementarity, we have $g(\alpha) < \kappa$, and therefore the above remarks show that $\dot{r}_\alpha$ is also a $\mathbb{P}_{\mathcal{S}_\alpha,\mathcal{T}_\alpha}$-name for a condition in $\dot{\mathbb{Q}}_\kappa^{\mathrm{H}(\alpha)}$. Hence, there is a condition u in $\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}$ satisfying

$$D_\kappa^{\mathrm{H}(\alpha)}(u) \leq_{\mathbb{P}_{\mathcal{S}_\alpha,\mathcal{T}_\alpha} * \dot{\mathbb{Q}}_\kappa^{\mathrm{H}(\alpha)}} (q_\alpha, \dot{r}_\alpha).$$

Then $u \leq_{\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}} u \cap \mathrm{H}(\alpha) \leq_{\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa}} q_\alpha = q_\kappa$, and we may find a condition v in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ with

$$D_\theta^{\mathrm{H}(\kappa)}(v) \leq_{\mathbb{P}_{\mathcal{S}_\kappa,\mathcal{T}_\kappa} * \dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa)}} (u, \dot{r}_\kappa).$$

Let H be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V with $v \in H$. Then $q \in H$, $u \in H_j = H \cap \mathrm{H}(\kappa)$ and the above choices ensure that $\dot{d}_\kappa^H = \dot{e}_\kappa^{H_j}$ and $\dot{d}_\alpha^H = \dot{e}_\alpha^{H \cap \mathrm{H}(\alpha)}$.

Claim. $\dot{d}_\alpha^H = \dot{d}_\kappa^H \cap \alpha$.

Proof of the Claim. Pick $\beta \in \dot{d}^H_\alpha$. By the above computations, we have $\beta \in \dot{e}_\alpha^{H\cap \mathrm{H}(\alpha)}$, and hence there is an $s \in H \cap \mathrm{H}(\alpha) \subseteq H_j$ with $\langle s, \beta, 1\rangle \in E_\alpha \subseteq E_\kappa$. But then, $\beta \in \dot{e}_\kappa^{H_j} = \dot{d}^H_\kappa$. In the other direction, pick $\beta \in \alpha \setminus \dot{d}^H_\alpha$. Then $\beta \in \alpha \setminus \dot{e}_\alpha^{H\cap\mathrm{H}(\alpha)}$, and there is an $s \in H \cap \mathrm{H}(\alpha)$ with $\langle s, \beta, 0\rangle \in E_\alpha \subseteq E_\kappa$. Thus $\beta \notin \dot{e}_\kappa^{H_j} = \dot{e}^H_\kappa$. □

Let $j_H : M[H_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$ be the small embedding for θ provided by an application of Lemma 6.3, and set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[H_j]}$. As in the first case, we know that $\dot{d}^H, \dot{x}^H, \dot{C}^H \in \mathrm{ran}(j_H)$, the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[H]})$ satisfies the σ-approximation property, $\kappa = \omega_2^N$ and $\mathcal{P}_{\omega_1}(\kappa)^N \subseteq \mathrm{H}(\kappa)^N \subseteq M[H_j]$. By the above claim, this shows that j_H and N witness that θ is internally AP subtle in $\mathrm{V}[H]$ with respect to $\dot{d}^H, \dot{x}^H$ and $\dot{C}^H$. In particular, there is a condition in H below q that forces this statement.

This density argument shows that, in $\mathrm{V}[G]$, we can find a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ that witnesses the internal AP subtlety of θ with respect to $\dot{d}^G, \dot{x}^G, \dot{C}^G$. In particular, these computations show that (ii) holds. □

Note that the previous characterization of subtle cardinals is strong by Corollary 11.3.

The next theorem yields a characterization of the set of all λ-ineffable cardinals θ with $\lambda = \lambda^{<\theta}$. In particular, it shows that Neeman's pure side condition forcing can be used to characterize the class of all ineffable cardinals. Moreover, this result will also directly give rise to such a characterization of supercompactness, as we will see in Chapter 14. Corollary 11.9 already shows that these characterizations are strong.

Theorem 12.2 ([12])**.** *The following statements are equivalent for every inaccessible cardinal θ and every cardinal λ with $\lambda = \lambda^{<\theta}$:*

(i) θ is a λ-ineffable cardinal.

(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "ω_2 is interally AP $\check{\lambda}$-ineffable".

(iii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash \mathrm{ISP}(\omega_2, \check{\lambda})$.

Proof. First, assume that (iii) holds. Since Corollary 4.6 shows that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfies the σ-approximation property, and Lemma 4.11 implies that $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfies the θ-chain condition, we can combine our assumptions with [37, Theorem 6.3] and [39, Proposition 2.2] to conclude that θ is λ-ineffable.

Now, assume that (i) holds. Let $\dot{d}$ be a $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for a $\mathcal{P}_\theta(\lambda)$-list, and let $\dot{x}$ be an arbitrary $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name. For every $a \in \mathcal{P}_\theta(\lambda)$, let $\dot{d}_a$ be a nice $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-name for the component of $\dot{d}$ that is indexed by $\check{a}$. Fix a bijection $b : \theta \longrightarrow \mathrm{H}(\theta)$ such that $b[\kappa] = \mathrm{H}(\kappa)$ holds for every inaccessible cardinal $\kappa \leq \theta$. Pick a regular cardinal $\vartheta > 2^\lambda$ that is sufficiently large with respect to Lemma 9.11, and which satisfies $\dot{d}, \dot{x} \in \mathrm{H}(\vartheta)$. Define A to be the set of all $a \in \mathcal{P}_\theta(\lambda)$ for which there exists a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ with $j(\delta) = \lambda$, $a = j[\delta]$, $\kappa_a = \mathrm{crit}\,(j) = a \cap \theta$ inaccessible, $\mathcal{P}_{\kappa_a}(\delta) \subseteq M$ and $b, \dot{d}, \dot{x} \in \mathrm{ran}(j)$. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V.

Assume first that there exists $a \in \mathcal{P}_\theta(\lambda)^{\mathrm{V}}$, a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ for θ in V, and $\delta \in M \cap \theta$ such that j and δ witness that a is an element of A, and such that $\dot{d}^G_a \notin \mathrm{V}[G_j]$. Let $j_G : M[G_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$ be the lifting of j provided by Lemma 6.3, and set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[G_j]}$. Then $\dot{d}^G, \dot{x}^G \in \mathrm{ran}(j_G)$, Corollary 3.20 shows that the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[G]})$ satisfies the σ-approximation property and, by Lemma 4.11, $\mathcal{P}_{\kappa_a}(\delta)^{\mathrm{V}} \subseteq M$

implies that $\mathcal{P}_{\kappa_a}(\delta)^N \subseteq M[G_j]$. Since $\dot{d}_a^G \notin N$, we can conclude that j_G, δ and N witness the internal AP λ-ineffability of θ with respect to $\dot{d}^G$ and $\dot{x}^G$ in $\mathrm{V}[G]$.

Otherwise, assume that whenever $j : M \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}}$ is a small embedding for θ in V that witnesses that some $a \in \mathcal{P}_\theta(\lambda)^{\mathrm{V}}$ is an element of A, then $\dot{d}_a^G \in \mathrm{V}[G_j]$. Pick a condition p in G which forces this statement. Work in V, fix a condition q below p in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, and define B to be the set of all $a \in A$ such that $q \in \mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}}$. By our assumption and by Lemma 4.11, we can find sequences $\langle q_a \mid a \in B\rangle$, $\langle \dot{r}_a \mid a \in B\rangle$ and $\langle \dot{e}_a \mid a \in B\rangle$, such that the following statements hold for all $a \in B$:

(a) q_a is a condition in $\mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}}$ below q.
(b) $\dot{r}_a$ is a $\mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}}$-name for a condition in $\dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa_a)}$.
(c) $\dot{e}_a$ is a $\mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}}$-nice name for a subset of a with $\langle q_a, \dot{r}_a\rangle \Vdash_{\mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}} * \dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa_a)}}$ "$\dot{d}_a = \dot{e}_a$".

Given $a \in B$, we have $b^{-1}[\mathbb{P}_{\mathcal{S}_{\kappa_a},\mathcal{T}_{\kappa_a}}] \subseteq b^{-1}[\mathrm{H}(\kappa_a)] = \kappa_a \subseteq a$, and elementarity implies that the set a is closed under $\prec\cdot,\cdot\succ$. This shows that there is a unique $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ with

$$d_a \;=\; \{\prec b^{-1}(s), \beta\succ \mid \langle \check{\beta}, s\rangle \in \dot{e}_a\} \;\subseteq\; a$$

for all $a \in B$, and with $d_a = \emptyset$ for all $a \in \mathcal{P}_\theta(\lambda)\setminus B$. Fix a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ that witness the λ-ineffability of θ with respect to $\vec{d}$ and $\{b, \dot{d}, q, \dot{x}\}$, as in Lemma 9.11. Then j and δ witness that $j[\delta] \in B$. Pick u in $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ with

$$D_\theta^{\mathrm{H}(\kappa_{j[\delta]})}(u) \;\leq_{\mathbb{P}_{\mathcal{S}_{\kappa_{j[\delta]}},\mathcal{T}_{\kappa_{j[\delta]}}} * \dot{\mathbb{Q}}_\theta^{\mathrm{H}(\kappa_{j[\delta]})}}\; \langle q_{j[\delta]}, \dot{r}_{j[\delta]}\rangle,$$

and let H be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V with $u \in H$. Since $q_{j[\delta]} \in H_j$, we have $\dot{d}_{j[\delta]}^H = \dot{e}_{j[\delta]}^{H_j}$. Note that this implies that for all $\gamma < \delta$, we have $\gamma \in j^{-1}[\dot{d}_{j[\delta]}^H]$ if and only if there is an $s \in H_j$ with $\prec b^{-1}(s), j(\gamma)\succ \in d_{j[\delta]}$. Observe that $b \restriction \kappa_{j[\delta]} \in M$, $j(b \restriction \kappa_{j[\delta]}) = b$ and $j \restriction H_j = \mathrm{id}_{H_j}$. Hence, $j^{-1}[\dot{d}_{j[\delta]}^H]$ is equal to the set of all $\gamma < \delta$ with the property that there is an $s \in H_j$ with $\prec (b \restriction \kappa_{j[\delta]})^{-1}(s), \gamma\succ \in j^{-1}[d_{j[\delta]}]$. Since the above choices ensure that $j^{-1}[d_{j[\delta]}] \in M$, we can conclude that $j^{-1}[\dot{d}_{j[\delta]}^H]$ is an element of $M[H_j]$. Let $j_H : M[H_j] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[H]}$ denote the lifting of j provided by Lemma 6.3, and set $N = \mathrm{H}(\vartheta)^{\mathrm{V}[H_j]}$. As above, we have $\dot{d}^H, \dot{x}^H \in \mathrm{ran}(j_H)$, the pair $(N, \mathrm{H}(\vartheta)^{\mathrm{V}[H]})$ satisfies the σ-approximation property, and $\mathcal{P}_{\kappa_{j[\delta]}}(\delta)^N \subseteq M[H_j]$. Since $j^{-1}[\dot{d}_{j[\delta]}^H] \in M[H_j]$, we can conclude that j_H and δ witness that θ is internally AP λ-ineffable with respect to $\dot{d}^H$ and $\dot{x}^H$ in $\mathrm{V}[H]$.

Using a standard density argument, the above computations allow us to conclude that θ is internally AP λ-ineffable in $\mathrm{V}[G]$. In particular, these arguments show that (i) implies (ii). $\square$

Chapter 13

Filter-based large cardinals

In this chapter, we will show that large cardinal notions defined through the existence of certain normal filters can also be characterized through the existence of small embeddings. First, we will provide a small embedding characterization of measurable cardinals. Then we will present a small embedding characterization of λ-supercompact cardinals. Although the proof of the characterization of λ-supercompact cardinals is similar to the characterization of measurable cardinals, we will include both proofs, for the convenience of the reader. We will close this chapter by a characterization of huge cardinals.

Lemma 13.1. *The following statements are equivalent for every cardinal θ:*

(i) θ is measurable.

(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ with

$$\{A \in \mathcal{P}(\mathrm{crit}\,(j))^M \mid \mathrm{crit}\,(j) \in j(A)\} \ \in \ M.$$

Proof. Assume that U is a normal ultrafilter on θ witnessing the measurability of θ and let $j_{\mathcal{U}} : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ denote the corresponding ultrapower embedding. Pick a cardinal $\vartheta > 2^\theta$ and an elementary submodel X of $\mathrm{H}(\vartheta)$ of cardinality θ containing $\{U\} \cup (\theta + 1)$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then the map

$$k = j_U \circ \pi^{-1} : N \longrightarrow \mathrm{H}(j_U(\vartheta))^{\mathrm{Ult}(\mathrm{V},U)}$$

is a non-trivial elementary embedding with $\mathrm{crit}\,(k) = \theta$ and $k(\mathrm{crit}\,(k)) = j_U(\theta)$. Since $\mathrm{Ult}(\mathrm{V}, U)$ is closed under θ-sequences in V and $N \in \mathrm{H}(\theta^+) \subseteq \mathrm{Ult}(\mathrm{V}, U)$, we have $k, N \in \mathrm{Ult}(\mathrm{V}, U)$ and the map $k : N \longrightarrow \mathrm{H}(j_U(\vartheta))^{\mathrm{Ult}(\mathrm{V},U)}$ is a small embedding for $j_U(\theta)$ in $\mathrm{Ult}(\mathrm{V}, U)$. Given $A \in \mathcal{P}(\theta)^N$, we have $\pi^{-1}(A) = A = \pi(A)$ and hence

$$\theta \in k(A) \iff \theta \in j_U(A) \iff A \in U \iff A \in \pi(U).$$

This allows us to conclude that

$$\pi(U) \ = \ \{A \in \mathcal{P}(\theta)^N \mid \theta \in k(A)\} \ \in \ N.$$

Using elementarity, we can find a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ in V with the property stated in (ii).

Now, assume that (ii) holds. Pick a sufficiently large cardinal ϑ and a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $\vartheta > 2^\theta$ and the set U of all $A \in \mathcal{P}(\mathrm{crit}\,(j))^M$ with $\mathrm{crit}\,(j) \in j(A)$ is contained in M. Then U is a normal ultrafilter on $\mathrm{crit}\,(j)$ in M. Since $\vartheta > 2^\theta$, this shows that $j(U)$ is a normal ultrafilter on θ that witnesses the measurability of θ. □

The following observation connects the above result with the small embedding characterizations of smaller large cardinal notions, by showing that witnessing small embeddings for measurability are also witnessing embeddings for all large cardinal notions considered so far that are direct consequences of measurability.

Lemma 13.2. *Let θ be a measurable cardinal, let $\vartheta > 2^\theta$ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the measurability of θ, as in Statement (ii) of Lemma 13.1.*

(i) The embedding j witnesses that θ is a stationary limit of Ramsey cardinals, as in Statement (ii) of Lemma 5.2.
(ii) If $x \in \mathrm{V}_{\theta+1} \cap \mathrm{ran}(j)$, then j witnesses the Π^2_1-indescribability of θ with respect to x, as in Statement (iv) of Lemma 8.2.

Proof. Let U denote the set of all $A \in \mathcal{P}(\mathrm{crit}\,(j))^M$ with $\mathrm{crit}\,(j) \in j(A)$. Then U is an element of M and $j(U)$ is a normal ultrafilter on θ.

(i) Since θ is a Ramsey cardinal in the ultrapower $\mathrm{Ult}(\mathrm{V}, j(U))$, it follows that the set of all Ramsey cardinals less than θ is an element of $j(U)$ and this implies that $\mathrm{crit}\,(j)$ is a Ramsey cardinal.

(ii) Let $\varphi(v)$ be a Π^2_1-formula with $(\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x)))^M$. Then elementarity implies $\mathrm{V}_\theta \models \varphi(x)$ and we can apply [17, Proposition 6.5] to conclude that the set

$$\{\alpha < \theta \mid \mathrm{V}_\alpha \models \varphi(x \cap \mathrm{V}_\alpha)\}$$

is an element of $j(U)$. Since Lemma 5.6 implies that $j^{-1}(x) = x \cap \mathrm{V}_{\mathrm{crit}(j)}$, this shows that $\mathrm{V}_{\mathrm{crit}(j)} \models \varphi(j^{-1}(x))$. □

Lemma 13.1 directly generalizes to a small embedding characterization of certain degrees of supercompactness.

Lemma 13.3. *The following statements are equivalent for all cardinals $\theta \leq \lambda$:*

(i) θ is λ-supercompact.
(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ such that $j(\delta) = \lambda$ and

$$\{A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M \mid j[\delta] \in j(A)\} \in M.$$

Proof. Assume that there is a normal ultrafilter U on $\mathcal{P}_\theta(\lambda)$ witnessing the λ-supercompactness of θ. Let $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ denote the corresponding ultrapower embedding. Then $\lambda < j_U(\theta)$. Fix a cardinal ϑ with $U \in \mathrm{H}(\vartheta)$ and an elementary submodel X of $\mathrm{H}(\vartheta)$ of cardinality λ with $\{U\} \cup (\lambda + 1) \subseteq X$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then the closure of $\mathrm{Ult}(\mathrm{V}, U)$ under λ-sequences in V implies that the map

$$k = j_U \circ \pi^{-1} : N \longrightarrow \mathrm{H}(j_U(\vartheta))^{\mathrm{Ult}(\mathrm{V},U)}$$

is an element of $\mathrm{Ult}(\mathrm{V},\mathcal{U})$, and this map is a small embedding for $j_U(\theta)$ with $\mathrm{crit}\,(k) = \theta$ and $k(\lambda) = j_U(\lambda)$ in $\mathrm{Ult}(\mathrm{V}, U)$. Then $k[\lambda] = j_U[\lambda]$ and therefore we have

$$k[\lambda] \in k(A) \iff j_U[\lambda] \in j_U(\pi^{-1}(A)) \iff \pi^{-1}(A) \in U \iff A \in \pi(U)$$

for all $A \in \mathcal{P}(\mathcal{P}_\theta(\lambda))^N$. These computations show that

$$\pi(U) \;=\; \{A \in \mathcal{P}(\mathcal{P}_\theta(\lambda))^N \mid k[\lambda] \in k(A)\} \;\in\; N.$$

In this situation, we can use elementarity between V and $\mathrm{Ult}(\mathrm{V}, U)$ to find a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M$ such that $\delta < \theta$, $j(\delta) = \lambda$ and

$$\{A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M \mid j[\delta] \in j(A)\} \in M.$$

Now, assume that (ii) holds. Fix a cardinal ϑ such that $\mathcal{P}(\mathcal{P}_\theta(\lambda)) \in \mathrm{H}(\vartheta)$ and such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and $\delta \in M \cap \theta$ as in (ii). Then the set U of all $A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M$ with $j[\delta] \in j(A)$ is an element of M and the assumption $\delta < \theta$ implies that this set is a normal ultrafilter on $\mathcal{P}_{\mathrm{crit}(j)}(\delta)$ in M. Since $\mathcal{P}(\mathcal{P}_\theta(\lambda)) \in \mathrm{H}(\vartheta)$, we can conclude that $j(U)$ is a normal filter on $\mathcal{P}_\theta(\lambda)$ that witnesses the λ-supercompactness of θ. □

Lemma 13.4. *Let θ be a λ-supercompact cardinal, let ϑ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the λ-supercompactness of θ, as in Statement (ii) of Lemma 13.3.*

(i) The embedding j witnesses the measurability of θ, as in Lemma 13.1.
(ii) If $\theta \leq \lambda_0 < \lambda$ is a cardinal in $\mathrm{ran}(j)$, then the embedding j witnesses the λ_0-supercompactness of θ, as in Statement (ii) of Lemma 13.3.
(iii) If $\vec{d} \in \mathrm{ran}(j)$ is a $\mathcal{P}_\theta(\lambda)$-list then j witnesses the λ-ineffabilty of θ with respect to $\vec{d}$ like in Lemma 9.11.

Proof. Pick $\delta \in M \cap \theta$ with $j(\delta) = \lambda$ and let $U \in M$ denote the set of all $A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta))^M$ with $j[\delta] \in j(A)$.

(i) Define F to be the set of all $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$ with the property that the set $F_x = \{a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M \mid \mathrm{otp}\,(a \cap \mathrm{crit}\,(j)) \in x\}$ is an element of U. Then our assumptions imply that F is an element of M and we have

$$x \in F \iff F_x \in U \iff j[\delta] \in j(F_x) \iff \mathrm{crit}\,(j) = \mathrm{otp}\,(j[\delta] \cap \theta) \in j(x)$$

for all $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$.

(ii) Pick $\delta_0 \in M$ with $j(\delta_0) = \lambda_0$. Let F denote the set of all $A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta_0))^M$ with the property that the set $\{a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M \mid a \cap \delta_0 \in A\}$ is an element of U. Then F is an element of M and it is equal to the set of all $A \in \mathcal{P}(\mathcal{P}_{\mathrm{crit}(j)}(\delta_0))^M$ with $j[\delta_0] \in j(A)$.

(iii) Pick a $\mathcal{P}_\theta(\lambda)$-list $\vec{d} = \langle d_a \mid a \in \mathcal{P}_\theta(\lambda)\rangle$ in $\mathrm{ran}(j)$ and a $\mathcal{P}_{\mathrm{crit}(j)}(\delta)$-list $\vec{e} = \langle e_a \mid a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)\rangle$ in M with $j(\vec{e}) = \vec{d}$. Let D denote the set of all $\gamma < \delta$ with the property that the set $D_\gamma = \{a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M \mid \gamma \in e_a\}$ is contained in U. Then D is an element of M and we have

$$\gamma \in D \iff D_\gamma \in U \iff j[\delta] \in j(D_\gamma) \;=\; j(\gamma) \in d_{j[\delta]}$$

for all $\gamma < \delta$. This shows that $D = j^{-1}[d_{j[\delta]}] \in M$. □

The next proposition shows that the domain models of small embeddings witnessing λ-supercompactness possess certain closure properties. These closure properties will allow us to connect the characterization of supercompactness provided by Lemma 13.3 with Magidor's characterization in Corollary 1.4.

Proposition 13.5. *Let θ be a λ-supercompact cardinal and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ witnessing the λ-supercompactness of θ, as in Statement (ii) of Lemma 13.3. If $\delta \in M \cap \theta$ with $j(\delta) = \lambda$ and $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$, then $j(x) \cap \delta \in M$. Moreover, if λ is a strong limit cardinal, then δ is a strong limit cardinal and $\mathrm{H}(\delta) \in M$.*

Proof. Fix some $x \in \mathcal{P}(\mathrm{crit}\,(j))^M$. Given $\gamma < \delta$, set

$$A_\gamma = \{a \in \mathcal{P}_{\mathrm{crit}(j)}(\delta)^M \mid \gamma \in a,\ \mathrm{otp}\,(a \cap \gamma) \in x\}.$$

Then

$$j[\delta] \in j(A_\gamma) \iff \mathrm{otp}\,(j[\delta] \cap j(\gamma)) \in j(x) \iff \gamma \in j(x)$$

for all $\gamma < \delta$. By our assumptions, these equivalences imply that the subset $j(x) \cap \delta$ is definable in M.

Now, assume that λ is a strong limit cardinal. Fix a sequence $s = \langle s_\alpha \mid \alpha < \mathrm{crit}\,(j)\rangle$ in M such that $s_\alpha : (2^{|\alpha|})^M \longrightarrow \mathcal{P}(\alpha)^M$ is a bijection for every $\alpha < \mathrm{crit}\,(j)$. Define

$$x \;=\; \{\prec\alpha, \prec\beta, \gamma\succ\succ \mid \alpha < \mathrm{crit}\,(j),\ \beta < 2^{|\alpha|}, \gamma \in s_\alpha(\beta)\} \;\in\; \mathcal{P}(\mathrm{crit}\,(j))^M.$$

Elementarity implies that δ is a strong limit cardinal in M, and the above computations show that $j(x) \cap \delta$ is an element of M. Assume for a contradiction that δ is not a strong limit cardinal. Pick a cardinal $\nu < \delta$ with $2^\nu \geq \delta$. Then the injection $j(s)_\nu \restriction \delta : \delta \longrightarrow \mathcal{P}(\nu)$ can be defined from $j(x) \cap \delta$, and therefore this function is contained in M, a contradiction. Since the above computations show that the sequence $\langle j(s)_\alpha \mid \alpha < \delta\rangle$ can be defined from the subset $j(x) \cap \delta$ of δ and this subset is contained in M, it follows that $\mathrm{H}(\delta)$ is an element of M. □

Corollary 13.6. *Let θ be a supercompact cardinal.*

(i) Let $\lambda \geq \theta$ be a cardinal and let $\vartheta > 2^{(\lambda^{<\theta})}$ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the supercompactness of θ, as in Statement (ii) of Corollary 1.4. If $\lambda \in \mathrm{ran}(j)$, then j witnesses the λ-supercompactness of θ, as in Statement (ii) of Lemma 13.3.

(ii) Let $\vartheta > \theta$ be a cardinal, let $\lambda \geq \vartheta$ be a strong limit cardinal and let μ be sufficiently large such that there exists a small embedding $j : M \longrightarrow \mathrm{H}(\mu)$ for θ witnessing the λ-supercompactness of θ, as in Statement (ii) of Lemma 13.3. If there is a $\delta \in M$ with $j(\delta) = \vartheta$, then $j \restriction \mathrm{H}(\delta)^M : \mathrm{H}(\delta) \longrightarrow \mathrm{H}(\vartheta)$ witnesses the supercompactness of θ, as in Statement (ii) of Corollary 1.4. □

In the remainder of this chapter, we turn our attention to *huge* cardinals. Remember that, given $0 < n < \omega$, an uncountable cardinal θ is *n-huge* if there is a sequence $\theta = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ of cardinals and a θ-complete normal ultrafilter U on $\mathcal{P}(\lambda_n)$ with $\{a \in \mathcal{P}(\lambda_n) \mid \mathrm{otp}\,(a \cap \lambda_{i+1}) = \lambda_i\} \in U$ for all $i < n$. A cardinal is *huge* if it is 1-huge. Note that, if $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ and U witness the n-hugeness of θ and $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ is the induced ultrapower embedding, then $\mathrm{crit}\,(j_U) = \theta$, $j_U(\lambda_i) = \lambda_{i+1}$ for all $i < n$, $U = \{A \in \mathcal{P}(\mathcal{P}(\lambda_n)) \mid j_U[\lambda_n] \in j_U(A)\}$ and $\mathrm{Ult}(\mathrm{V}, U)$ is closed under λ_n-sequences. In particular, each λ_i is measurable. Moreover, since U concentrates on the subset $[\lambda_n]^{\lambda_{n-1}}$ of all subsets of λ_n of order-type λ_{n-1}, we may as well identify U with an ultrafilter on this set of size λ_n.

Lemma 13.7. *Given $0 < n < \omega$, the following statements are equivalent for all cardinals θ:*

(i) θ is n-huge.

(ii) For all sufficiently large cardinals ϑ, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ such that $j^i(\mathrm{crit}\,(j)) \in M$ for all $i \leq n$ and

$$\{A \in \mathcal{P}(\mathcal{P}(j^n(\mathrm{crit}\,(j))))^M \mid j[j^n(\mathrm{crit}\,(j))] \in j(A)\} \ \in \ M.$$

Proof. First, assume that $\lambda_0 < \lambda_1 < \ldots < \lambda_n$ and U witness the n-hugeness of θ and let $j_U : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, U)$ denote the corresponding ultrapower embedding. Pick a cardinal ϑ with $U \in \mathrm{H}(\vartheta)$ and an elementary submodel X of $\mathrm{H}(\vartheta)$ of cardinality λ_n with $\mathrm{H}(\lambda_n) \cup \{U\} \subseteq X$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Then $k = j_U \circ \pi : N \longrightarrow \mathrm{H}(j_U(\vartheta))^{\mathrm{Ult}(\mathrm{V},U)}$ is a non-trivial elementary embedding and $\pi^{-1} \restriction \mathrm{H}(\lambda_n) = \mathrm{id}_{\mathrm{H}(\lambda_n)}$ implies that $\mathrm{crit}\,(k) = \theta$, $k[\lambda_n] = j_U[\lambda_n]$ and $k(\lambda_i) = \lambda_{i+1}$ for all $i < n$. Since $N \in \mathrm{H}(\lambda_n^+) \subseteq \mathrm{Ult}(\mathrm{V}, U)$, the closure of $\mathrm{Ult}(\mathrm{V}, U)$ under λ_n-sequences implies that k is an element of $\mathrm{Ult}(\mathrm{V}, U)$, and the above computations show that k is a small embedding for $j_U(\theta)$ in $\mathrm{Ult}(\mathrm{V}, U)$. Then

$$k[\lambda_n] \in k(A) \iff j_U[\lambda_n] \in j_U(\pi^{-1}(A)) \iff \pi^{-1}(A) \in U \iff A \in \pi(U).$$

for all $A \in \mathcal{P}(\mathcal{P}(\lambda_n))^N$. This shows that

$$\pi(U) \ = \ \{A \in \mathcal{P}(\mathcal{P}(k^n(\mathrm{crit}\,(k))))^N \mid k[k^n(\mathrm{crit}\,(k))] \in k(A)\} \ \in \ N$$

and, by elementarity, we find a small embedding for θ as in (ii).

Now, assume that (ii) holds. Fix a sufficiently large cardinal ϑ and a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ as in (ii). Let U denote the set of all A in $\mathcal{P}(\mathcal{P}(j^n(\mathrm{crit}\,(j))))^M$ with $j[j^n(\mathrm{crit}\,(j))] \in j(A)$ and set $\lambda_i = j^i(\mathrm{crit}\,(j))$ for all $i \leq n$. Then U is an element of M and our assumptions imply that U is a $\mathrm{crit}\,(j)$-complete, normal ultrafilter on $\mathcal{P}(\lambda_n)$ with $\{a \in \mathcal{P}(\lambda_n)^M \mid \mathrm{otp}\,(a \cap \lambda_{i+1}) = \lambda_i\} \in U$ for all $i < n$. Hence $\lambda_0 < \lambda_1 < \ldots < \lambda_n$

and U witness that $\mathrm{crit}\,(j)$ is an n-huge cardinal in M. This allows us to conclude that $\lambda_1 < \ldots < \lambda_n < j(\lambda_n)$ and $j(U)$ witness that $\mathrm{crit}\,(j)$ is an n-huge cardinal in $\mathrm{H}(\vartheta)$. Then $j(U)$ concentrates on the subset $[j(\lambda_n)]^{\lambda_n}$ of $\mathcal{P}(j(\lambda_n))$. Since $j(\lambda_n)$ is inaccessible and therefore $\mathcal{P}([j(\lambda_n)]^{\lambda_n})$ is contained in $\mathrm{H}(\vartheta)$, we can conclude that θ is n-huge. □

The next lemma shows that the domain models of small embeddings witnessing n-hugeness also possess certain closure properties. These closure properties will directly imply that these embeddings also witness weaker large cardinal properties.

Lemma 13.8. *Let $0 < n < \omega$, let θ be an n-huge cardinal and let $j : M \longrightarrow \mathrm{H}(\vartheta)$ be a small embedding for θ witnessing the n-hugeness of θ, as in Statement (ii) of Lemma 13.7. Then $\mathcal{P}(j^n(\mathrm{crit}\,(j))) \cap \mathrm{ran}(j)$ is contained in M. In particular, $\mathrm{H}(j^n(\mathrm{crit}\,(j)))$ is an element of M.*

Proof. Fix $A \in \mathcal{P}(j^{n-1}(\mathrm{crit}\,(j)))^M$. Given $\gamma < j^n(\mathrm{crit}\,(j))$, define

$$A_\gamma \;=\; \{a \in \mathcal{P}(j^n(\mathrm{crit}\,(j)))^M \mid \gamma \in a,\ \mathrm{otp}\,(a \cap \gamma) \in A\}.$$

For each $\gamma < j^n(\mathrm{crit}\,(j))$, we then have

$$\begin{aligned} A_\gamma \in U \iff & j[j^n(\mathrm{crit}\,(j))] \in j(A_\gamma) \iff \mathrm{otp}\,(j(\gamma) \cap j[j^n(\mathrm{crit}\,(j))]) \in j(A) \\ \iff & \mathrm{otp}\,(j[\gamma]) \in j(A) \iff \gamma \in j(A). \end{aligned}$$

This shows that $j(A)$ is equal to the set $\{\gamma < j^n(\mathrm{crit}\,(j)) \mid A_\gamma \in U\}$. Since the sequence $\langle A_\gamma \mid \gamma < j^n(\mathrm{crit}\,(j))\rangle$ is an element of M, this shows that $j(A) \in M$.

The final statement of the lemma follows from the fact that elementarity implies that there is a subset of $j^n(\mathrm{crit}\,(j))$ in $\mathrm{ran}(j)$ that codes all elements of $\mathrm{H}(j^n(\mathrm{crit}\,(j)))$. □

Corollary 13.9. *Let $0 < n < \omega$, let θ be an n-huge cardinal and let ϑ be a sufficiently large cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ witnessing the n-hugeness of θ, as in Statement (ii) of Lemma 13.7.*

(i) If $0 < m < n$, then j also witnesses the m-hugeness of θ, as in Statement (ii) of Lemma 13.7.
(ii) If $\theta \leq \lambda < j(\theta)$ with $\lambda \in \mathrm{ran}(j)$, then j also witnesses the λ-supercompactness of θ, as in Statement (ii) of Lemma 13.3. □

Chapter 14

Characterization of supercompact cardinals via $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$

In this chapter, we will introduce an internal version of supercompact cardinals and then we will use results of Matteo Viale and Christoph Weiß from [37] to show that internal AP supercompactness is equivalent to the generalized tree property $\mathrm{ISP}(\theta,\lambda)$. Since the results of [37] show that the *Proper Forcing Axiom* PFA implies this tree property for ω_2, these arguments will also yield a consistency proof for the internal AP supercompactness of ω_2. We will end this chapter by presenting a strong characterization of supercompact cardinals via $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$.

Definition 14.1. A cardinal θ is *internally AP supercompact* if for all sufficiently large regular cardinals ϑ and all $x \in \mathrm{H}(\vartheta)$, there is a small embedding $j : M \longrightarrow \mathrm{H}(\vartheta)$ for θ and a transitive model N of ZFC^- such that $x \in \mathrm{ran}(j)$ and the following statements hold:

(i) $N \subseteq \mathrm{H}(\vartheta)$ and the pair $(N, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.
(ii) $M \in N$ and $M = \mathrm{H}(\delta)^N$ for some N-cardinal $\delta < \theta$.

The following lemma is the main result of this chapter.

Lemma 14.2. *A regular cardinal $\theta > \omega_1$ is internally AP supercompact if and only if $\mathrm{ISP}(\theta,\lambda)$ holds for all cardinals $\lambda \geq \theta$.*

In order to prove this statement, we need to introduce more concepts from [37].

Definition 14.3. Let ϑ be an uncountable cardinal and let $X \prec \mathrm{H}(\vartheta)$.

(i) A set d is *X-approximated* if $b \cap d \in X$ for all $b \in X \cap \mathcal{P}_{\omega_1}(X)$.
(ii) A set d is *X-guessed* if $d \cap X = e \cap X$ for some $e \in X$.
(iii) Given $\rho \in \mathrm{Ord}$, X is a *ρ-guessing model* if every X-approximated $d \subseteq \rho$ is X-guessed.
(iv) X is a *guessing model* if it is ρ-guessing for every $\rho \in X \cap \mathrm{Ord}$.

Proposition 14.4. *Let $\vartheta > \omega_1$ be a cardinal, let $X \prec \mathrm{H}(\vartheta)$ and let $\pi : X \longrightarrow M$ be the corresponding transitive collapse. Then X is a guessing model if and only if the pair $(M, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property.*

Proof. First, assume that X is a guessing model. Pick $B \in M$ and $A \in \mathrm{H}(\vartheta)$ with the property that $A \subseteq B$ and $A \cap x \in M$ for every $x \in M$ that is countable in M. Pick a bijection $f : B \longrightarrow \rho$ in M with $\rho \in \mathrm{Ord}$. We define

$$d \;=\; (\pi^{-1} \circ f)[A] \;\subseteq\; \pi^{-1}(\rho) \;\in\; X \cap \mathrm{Ord}.$$

Fix $b \in X \cap \mathcal{P}_{\omega_1}(X)$. Then $f^{-1}[\pi(b) \cap \rho] \in M$ is countable in M and this implies that $A \cap f^{-1}[\pi(b) \cap \rho] \in M$. Since elementarity implies that $\pi(b) = \pi[b]$, this yields

$$b \cap d \;=\; (\pi^{-1} \circ f)[A \cap f^{-1}[\pi(b) \cap \rho]] \;\in\; X.$$

These computations show that $d \subseteq \pi^{-1}(\rho)$ is X-approximated. Since X is a guessing model, d is X-guessed and there is an $e \in X$ with $d = d \cap X = e \cap X$. But then $\pi(e) = \pi[d] = f[A] \in M$ and hence A is an element of M.

For the other direction, assume that the pair $(M, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property. Pick $\rho \in X \cap \mathrm{Ord}$ and $d \subseteq \rho$ that is X-approximated. Set

$$A = \pi[d \cap X] \subseteq \pi(\rho) \in M.$$

Fix an $x \in M$ that is countable in M. Then $\pi^{-1}(x) \in X \cap \mathcal{P}_{\omega_1}(X)$, $d \cap \pi^{-1}(x) \in X$ and therefore

$$A \cap x \;=\; \pi[d \cap X] \cap \pi[\pi^{-1}(x)] \;=\; \pi[d \cap \pi^{-1}(x)] \;=\; \pi(d \cap \pi^{-1}(x)) \in M,$$

because $\pi^{-1}(x) = \pi^{-1}[x]$ and $\pi(d \cap \pi^{-1}(x)) = \pi[d \cap \pi^{-1}(x)]$. By our assumption, it follows that $A \in M$. Since $\pi^{-1}(A) \cap X = d \cap X$, we can conclude that d is X-guessed. □

We are now ready to show that the internal AP supercompactness of a cardinal κ is equivalent to the statement that $\mathrm{ISP}(\kappa, \lambda)$ holds for all cardinals $\lambda \geq \kappa$.

Proof of Lemma 14.2. By [37, Proposition 3.2 & 3.3], the following statements are equivalent for every uncountable regular cardinal θ and all cardinals $\lambda \geq \theta$:

(i) $\mathrm{ISP}(\theta, \lambda)$.
(ii) If ν is a cardinal with $|\mathrm{H}(\nu)| = \lambda$, then the set of all guessing models $X \prec \mathrm{H}(\nu)$ with $|X| < \theta$ and $X \cap \theta \in \theta$ is stationary in $\mathcal{P}_\theta(\mathrm{H}(\nu))$.
(iii) For sufficiently large cardinals ν, there exists a λ-guessing model $X \prec \mathrm{H}(\nu)$ with $|X| < \theta$, $X \cap \theta \in \theta$ and $\lambda^+ \in X$.

First, assume that $\theta > \omega_1$ is a regular cardinal with the property that $\mathrm{ISP}(\theta, \lambda)$ holds for all cardinals $\lambda \geq \theta$. Fix some regular cardinal $\vartheta > \theta$ and $x \in \mathrm{H}(\vartheta)$. Pick some $\nu > |\mathrm{H}(\vartheta)|$ and use (ii) to find a guessing model $X \prec \mathrm{H}(\nu)$ of cardinality less than θ with $\theta, \vartheta, x \in X$ and $X \cap \theta \in \theta$. Let $\pi : X \longrightarrow N$ denote the corresponding transitive collapse. Define $M = \mathrm{H}(\pi(\vartheta))^M$ and $j = \pi^{-1} \restriction M : M \longrightarrow \mathrm{H}(\vartheta)$. Then j is a small embedding for θ with $x \in \mathrm{ran}(j)$ and N is a transitive model of ZFC^- with $N \subseteq \mathrm{H}(\vartheta)$ and $M = \mathrm{H}(\delta)^N$ for some N-cardinal δ. Since Proposition 14.4 shows that the pair $(N, \mathrm{H}(\vartheta))$ has the σ-approximation property, we can conclude that θ is internally AP supercompact.

In the other direction, assume that θ is internally AP supercompact. Fix cardinals $\lambda \geq \theta$ and $\vartheta > \lambda^+$. Let $\nu > \vartheta$ be a sufficiently large strong limit cardinal such that there is a small embedding $j : M \longrightarrow \mathrm{H}(\nu^+)$ for θ and a transitive ZFC^--model N witnessing the internal AP supercompactness of θ with respect to the pair $\langle\lambda^+,\vartheta\rangle$. Then there is an N-cardinal $\varepsilon < \theta$ with $M = \mathrm{H}(\varepsilon)^N$. Pick $\delta \in M$ with $j(\delta) = \vartheta$. In this situation, elementarity implies that $\delta < \varepsilon$, $|\mathrm{H}(\delta)^M|^N < \varepsilon < \theta$ and $\mathrm{H}(\delta)^M = \mathrm{H}(\delta)^N$. Since the pair $(N, \mathrm{H}(\nu))$ satisfies the σ-approximation property, this implies that the pair $(\mathrm{H}(\delta)^M, \mathrm{H}(\vartheta))$ satisfies the σ-approximation property. If we define $X = j[\mathrm{H}(\delta)^M]$, then $X \prec \mathrm{H}(\vartheta)$, $j^{-1} \restriction X$ is the transitive collapse of X and Proposition 14.4 shows that X is a guessing model satisfying $|X| \leq |\mathrm{H}(\delta)^M|^N < \theta$, $X \cap \theta = \mathrm{crit}\,(j) \in \theta$ and $\lambda^+ \in X$. By the above equivalences, this shows that $\mathrm{ISP}(\theta,\lambda)$ holds. □

Corollary 14.5. *The following statements are equivalent for every inaccessible cardinal* θ*:*

(i) θ *is supercompact.*
(ii) θ *is internally AP supercompact.*

Proof. The implication from (i) to (ii) directly follows from a combination of Corollary 1.4 and Lemma 5.9. In the other direction, assume that θ is internally AP supercompact. Then Lemma 14.2 shows that $\mathrm{ISP}(\theta,\lambda)$ holds for all $\lambda \geq \theta$. Since θ is inaccessible, every $\mathcal{P}_\theta(\lambda)$-list is slender (see [39, Proposition 2.2]) and therefore θ is λ-ineffable for all $\lambda \geq \theta$. By the results of [24] that we mentioned earlier, this implies that θ is supercompact. □

Corollary 14.6. PFA *implies that* ω_2 *is internally AP supercompact.*

Proof. By [37, Theorem 4.8], PFA implies that $\mathrm{ISP}(\omega_2,\lambda)$ holds for all cardinals $\lambda \geq \omega_2$. In combination with Lemma 14.2, this yields the statement of the corollary. □

Corollary 14.7. *The following statements are equivalent for every inaccessible cardinal* θ*:*

(i) θ *is a supercompact cardinal.*
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ *"*ω_2 *is internally AP supercompact".*
(iii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ *"ISP*(ω_2,λ) *holds for all cardinals* $\lambda \geq \omega_2$*".*

Proof. Using the results of [25] and Theorem 12.2, the equivalence between (i) and (iii) follows directly from the fact that there is a proper class of cardinals λ satisfying $\lambda = \lambda^{<\theta}$ and the fact that $\mathrm{ISP}(\theta,\lambda_1)$ implies $\mathrm{ISP}(\theta,\lambda_0)$ for all cardinals $\theta \leq \lambda_0 \leq \lambda_1$. The equivalence between (ii) and (iii) is given by Lemma 14.2. □

Chapter 15

Characterization of some larger large cardinals via Neeman's pure side condition forcing

In this chapter, we will present some results based on ideas of Peter Holy and Philipp Lücke that allow the characterization of some larger large cardinals. They showed that it is possible to characterize levels of supercompactness, almost huge cardinals and super almost huge cardinals through Neeman's pure side condition forcing. Since either no small embedding characterizations for these properties are known, or the ones presented on Chapter 13 are not suitable for these purposes, these characterizations instead make use of the classical concept of *generic elementary embeddings*, a variation of large cardinals given by elementary embeddings $j : V \longrightarrow M$ that exist in generic extension $\mathrm{V}[G]$ of V.

The following lemma lies at the heart of these results. Its proof heavily relies on the concepts and results presented in [37, Section 6].

Lemma 15.1. *Let* $\mathrm{V}[G]$ *be a generic extension of the ground model* V, *let* $\mathrm{V}[G, H]$ *be a generic extension of* $\mathrm{V}[G]$ *and let* $j : \mathrm{V}[G] \longrightarrow M$ *be an elementary embedding definable in* $\mathrm{V}[G, H]$ *with critical point* θ. *Assume that the following statements hold:*

(i) θ *is an inaccessible cardinal in* V.
(ii) The pair $(\mathrm{V}, \mathrm{V}[G])$ *satisfies the* σ*-approximation and the* θ*-cover property.*
(iii) The pair $(\mathrm{V}[G], \mathrm{V}[G, H])$ *satisfies the* σ*-approximation property.*

In this situation, if $\theta \leq \gamma < j(\theta)$ *is an ordinal with* $j[\gamma] \in M$, *then* $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$, *and the set*

$$\mathcal{U} \;=\; \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}} \mid j[\gamma] \in j(A)\}$$

is an element of V.

Proof. The above assumptions imply that $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G]} = \omega_1^{\mathrm{V}[G,H]}$, and hence θ is an uncountable regular cardinal greater than ω_1 in $\mathrm{V}[G]$.

Claim. $\mathcal{U} \in \mathrm{V}[G]$.

Proof of the Claim. Assume, towards a contradiction, that the set $\mathcal{U}$ is not an element of $\mathrm{V}[G]$. Then there is $u \in \mathcal{P}_{\omega_1}(\mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}})^{\mathrm{V}[G]}$ with $\mathcal{U} \cap u \notin \mathrm{V}[G]$. Define

$$d : \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \longrightarrow \mathcal{P}(u)^{\mathrm{V}[G]};\ x \longmapsto \{A \in u \mid x \in A\}.$$

By our assumptions, there is $c \in \mathcal{P}_\theta(\mathcal{P}(\mathcal{P}_\theta(\gamma)))^{\mathrm{V}}$ with $u \subseteq c$. In the following, let $a : \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \longrightarrow \mathcal{P}(c)^{\mathrm{V}}$ denote the unique function with $a(x) = \{A \in c \mid x \in A\}$ for all $x \in \mathrm{dom}(a) \cap \mathrm{V}$ and $a(x) = \emptyset$ for all $x \in \mathrm{dom}(a) \setminus \mathrm{V}$. Since $d(x) = \emptyset$ for all $x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \setminus \mathrm{V}$, we then have $d(x) = a(x) \cap u$ for all $x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]}$ and

$$\mathrm{ran}(d) \ = \ \{a(x) \cap u \mid x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]}\} \ \subseteq \ \{u \cap y \mid y \in \mathcal{P}(c)^{\mathrm{V}}\}.$$

Since θ is inaccessible in V, this implies that $\mathrm{ran}(d)$ has cardinality less than θ in $\mathrm{V}[G]$ and there is a bijection $b : \mu \longrightarrow \mathrm{ran}(d)$ in $\mathrm{V}[G]$ for some $\mu < \theta$. In this situation, we have $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]})$, and elementarity yields an $\alpha < \mu$ with $j(b)(\alpha) = j(d)(j[\gamma])$. But then

$$j[(b(\alpha))] \ = \ j(b)(\alpha) \ = \ j(d)(j[\gamma]) \ = \ \{j(A) \mid A \in u,\ j[\gamma] \in j(A)\} \ = \ j[(\mathcal{U} \cap u)],$$

and this implies that $\mathcal{U} \cap u = b(\alpha) \in \mathrm{V}[G]$, a contradiction. □

Claim. $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$.

Proof of the Claim. Assume, towards a contradiction, that the set $j[\gamma]$ is not an element of $j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$. By our assumptions on V and $\mathrm{V}[G]$, there is a function $a : \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \longrightarrow \mathcal{P}_{\omega_1}(\gamma)^{\mathrm{V}}$ in $\mathrm{V}[G]$ with $a(x) = \emptyset$ for all $x \in \mathrm{dom}(a) \cap \mathrm{V}$ and $a(x) \cap x \notin \mathrm{V}$ for all $x \in \mathrm{dom}(a) \setminus \mathrm{V}$. Define

$$d : \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \longrightarrow \mathcal{P}_{\omega_1}(\gamma)^{\mathrm{V}[G]};\ x \longmapsto a(x) \cap x,$$

and set $D = \{\alpha < \gamma \mid j(\alpha) \in j(d)(j[\gamma])\}$. Then our assumption and elementarity imply that $D \neq \emptyset$.

Subclaim. $D \in \mathrm{V}[G]$.

Proof of the Subclaim. Assume, towards a contradiction, that D is not an element of $\mathrm{V}[G]$. Then there is $u \in \mathcal{P}_{\omega_1}(\gamma)^{\mathrm{V}[G]}$ with $D \cap u \notin \mathrm{V}[G]$. Define

$$R \ = \ \{d(x) \cap u \mid u \subseteq x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]}\},$$

and fix $c \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}}$ with $u \subseteq c$. Then

$$R \ = \ \{a(x) \cap c \cap u \mid u \subseteq x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]}\} \ \subseteq \ \{u \cap y \mid y \in \mathcal{P}_{\omega_1}(c)^{\mathrm{V}}\},$$

and, since θ is inaccessible in V, there is a bijection $b : \mu \longrightarrow R$ in $\mathrm{V}[G]$ with $\mu < \theta$.

We now have $j(d)(j[\gamma]) \cap j(u) \in j(R)$, because $j(u) = j[u] \subseteq j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]})$. Hence there is an $\alpha < \mu$ with

$$j[(b(\alpha))] \ = \ j(b)(\alpha) \ = \ j(d)(j[\gamma]) \cap j(u) \ = \ j[(D \cap u)],$$

and this implies that $D \cap u = b(\alpha) \in \mathrm{V}[G]$, a contradiction. □

Define $U = \{x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]} \mid d(x) = D \cap x\} \in \mathrm{V}[G]$.

Subclaim. *In* V[G]*, the set* U *is unbounded in* $\mathcal{P}_\theta(\gamma)$.

Proof of the Subclaim. We have $j(d)(j[\gamma]) = j[D] = j(D) \cap j[\gamma]$, and this shows that $j[\gamma] \in j(U)$. Now, if $x \in \mathcal{P}_\theta(\gamma)^{\mathrm{V}[G]}$, then $j(x) = j[x] \subseteq j[\gamma] \in j(U)$, and hence elementarity yields a $y \in U$ with $x \subseteq y$. □

Now, work in V[G] and use our assumptions together with the last claim to construct a sequence $\langle x_\alpha \mid \alpha \leq \omega_1 \rangle$ of elements of U and a sequence $\langle y_\alpha \mid \alpha \leq \omega_1 \rangle$ of elements of $\mathcal{P}_\theta(\gamma)^{\mathrm{V}}$, such that $d(x_0) \neq \emptyset$, and such that

$$\bigcup \{y_{\bar{\alpha}} \mid \bar{\alpha} < \alpha\} \ \subseteq \ x_\alpha \ \subseteq \ y_\alpha$$

for all $\alpha \leq \omega_1$. Then we have

$$d(x_{\bar{\alpha}}) \ = \ D \cap x_{\bar{\alpha}} \ \subseteq \ D \cap x_\alpha \ = \ d(x_\alpha)$$

for all $\bar{\alpha} \leq \alpha \leq \omega_1$. Since $d(x_{\omega_1})$ is a countable set, this implies that there is an $\alpha_* < \omega_1$ with $d(x_{\alpha_*}) = d(x_\alpha)$ for all $\alpha_* \leq \alpha \leq \omega_1$. Then

$$\begin{aligned} d(x_{\alpha_*}) \ &= \ d(x_{\alpha_*+1}) \cap x_{\alpha_*} \ \subseteq \ a(x_{\alpha_*+1}) \cap y_{\alpha_*} \\ &\subseteq \ a(x_{\alpha_*+1}) \cap x_{\alpha_*+1} \ = \ d(x_{\alpha_*+1}) \ = \ d(x_{\alpha_*}) \end{aligned}$$

and therefore $\emptyset \neq d(x_{\alpha_*}) = a(x_{\alpha_*+1}) \cap y_{\alpha_*} \in \mathrm{V}$, a contradiction. □

Assume, towards a contradiction, that $\mathcal{U}$ is not an element of V. Since $\mathcal{U} \in \mathrm{V}[G]$, this implies that there is a $u \in \mathcal{P}_{\omega_1}(\mathcal{P}(\mathcal{P}_\theta(\gamma)))^{\mathrm{V}}$ with $\mathcal{U} \cap u \notin \mathrm{V}$. Define

$$d : \mathcal{P}_\theta(\gamma)^{\mathrm{V}} \longrightarrow \mathcal{P}(u)^{\mathrm{V}}; \ x \longmapsto \{A \in u \mid x \in A\}.$$

Since θ is inaccessible in V, we can find a bijection $b : \mu \longrightarrow \mathrm{ran}(d)$ in V with $\mu < \theta$. By the above claim, we have $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$ and hence there is an $\alpha < \mu$ with $j(d)(j[\gamma]) = j(b)(\alpha)$. But then

$$j[(b(\alpha))] \ = \ j(b)(\alpha) \ = \ j(d)(j[\gamma]) \ = \ \{j(A) \mid A \in u, \ j[\gamma] \in j(A)\} \ = \ j[(\mathcal{U} \cap u)],$$

and this implies that $\mathcal{U} \cap u = b(\alpha) \in \mathrm{V}$, a contradiction. □

We now study typical situations in which the assumptions of Lemma 15.1 are satisfied.

Definition 15.2. Given an uncountable regular cardinal θ and an ordinal $\gamma \geq \theta$, we say that a partial order $\mathbb{P}$ *witnesses that θ is generically γ-supercompact* if there is a $\mathbb{P}$-name $\dot{\mathcal{U}}$ such that $\dot{\mathcal{U}}^G$ is a fine, V-normal, V-$<\theta$-complete ultrafilter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}}$ in V[G], with the property that the corresponding ultrapower $\mathrm{Ult}(\mathrm{V}, \dot{\mathcal{U}}^G)$ is well-founded whenever G is $\mathbb{P}$-generic over V.

The proof of the following proposition uses standard arguments about generic ultrapowers (see [8, Chapter 2]).

Proposition 15.3. *Let $\mathbb{P}$ be a partial order witnessing that an uncountable regular cardinal θ is generically γ-supercompact, and let $\dot{\mathcal{U}}$ be the corresponding $\mathbb{P}$-name. If G is $\mathbb{P}$-generic over V, and $j : \mathrm{V} \longrightarrow \mathrm{Ult}(\mathrm{V}, \dot{\mathcal{U}}^G)$ is the corresponding ultrapower embedding defined in $\mathrm{V}[G]$, then j has critical point θ, $j(\theta) > \gamma$, and $j[\gamma] \in \mathrm{Ult}(\mathrm{V}, \dot{\mathcal{U}}^G)$.*

Proof. The V-$<\theta$-completeness of $\dot{\mathcal{U}}^G$ yields $j \restriction \theta = \mathrm{id}_\theta$. The fineness and V-normality of $\dot{\mathcal{U}}^G$ imply that $j[\gamma] = [\mathrm{id}_{\mathcal{P}_\theta(\gamma)^{\mathrm{V}}}]_{\dot{\mathcal{U}}^G} \in \mathrm{Ult}(\mathrm{V}, \dot{\mathcal{U}}^G)$, and moreover

$$\theta \leq \gamma \;=\; [a \mapsto \mathrm{otp}\,(a)]_{\dot{\mathcal{U}}^G} \;<\; [a \mapsto \theta]_{\dot{\mathcal{U}}^G} = j(\theta). \qquad \square$$

The following results yield strong characterizations of measurable and of supercompact cardinals through Neeman's pure side condition forcing.

Lemma 15.4. *The following statements are equivalent for every inaccessible cardinal θ and every ordinal $\gamma \geq \theta$:*

(i) θ is a γ-supercompact cardinal.
(ii) There is a partial order with the σ-approximation property that witnesses that θ is generically γ-supercompact.

Proof. If (i) holds, then the trivial partial order clearly witnesses that θ is generically γ-supercompact. In order to verify the reverse direction, let $\mathbb{P}$ be a partial order with the σ-approximation property that witnesses θ to be generically γ-supercompact, let H be $\mathbb{P}$-generic over V, and let $j : \mathrm{V} \longrightarrow M$ be the elementary embedding definable in $\mathrm{V}[H]$ that is provided by an application of Proposition 15.3. In this situation, an application of Lemma 15.1 with $\mathrm{V} = \mathrm{V}[G]$ shows that the set $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}} \mid j[\gamma] \in j(A)\}$ is an element of V and it is easy to see that $\mathcal{U}$ is a fine, $<\theta$-complete, normal ultrafilter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))$ in V. Hence, $\mathcal{U}$ witnesses that θ is γ-supercompact in V. $\square$

The following result shows how γ-supercompactness can be characterized through Neeman's pure side condition forcing. Note that in particular, this theorem yields a strong characterization of measurability and yet another strong characterization of supercompactness.

Theorem 15.5. *The following statements are equivalent for every inaccessible cardinal θ and every ordinal $\gamma \geq \theta$:*

(i) θ is a γ-supercompact cardinal.
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta, \mathcal{T}_\theta}} \Vdash$ "There is a partial order with the σ-approximation property that witnesses ω_2 to be generically $\check{\gamma}$-supercompact".

Proof. First, assume that (i) holds, and let $j : \mathrm{V} \longrightarrow M$ be an elementary embedding witnessing the γ-supercompactness of θ. Set $K = \mathrm{H}(j(\theta))^M$, $\mathcal{S} = \mathcal{S}^M_{j(\theta)}$ and $\mathcal{T} = \mathcal{T}^M_{j(\theta)}$. Then $\mathbb{P}_{K, \mathcal{S}, \mathcal{T}} = \mathbb{P}^M_{j(\theta)} = j(\mathbb{P}_{\mathcal{S}_\theta, \mathcal{T}_\theta})$.

Claim. *The set K is suitable and the pair $(\mathcal{S}, \mathcal{T})$ is appropriate for K.*

Proof of the Claim. Since $\omega_1^M = \omega_1 < \theta < j(\theta)$, elementarity directly yields the above statements. $\square$

Note that the closure properties of M imply that $\mathrm{H}(\theta) \in M$, $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} = \mathbb{P}^M_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ and $\mathrm{H}(\theta) \in \mathcal{T}$. Moreover, Lemma 4.7 and elementarity imply that $\mathbb{P}_{K,\mathcal{S},\mathcal{T}} \downharpoonright \langle \mathrm{H}(\theta) \rangle$ is dense in $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$. Define $\dot{\mathbb{Q}} = \dot{\mathbb{Q}}^{\mathrm{H}(\theta)}_{K,\mathcal{S},\mathcal{T}}$. Then Lemma 4.11 and the closure properties of M imply that $\dot{\mathbb{Q}} = (\dot{\mathbb{Q}}^{\mathrm{H}(\theta)}_{j(\theta)})^M$. Let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V.

Claim. *The partial order $\dot{\mathbb{Q}}^G$ has the σ-approximation property in* $\mathrm{V}[G]$.

Proof of the Claim. By Corollary 3.20, it suffices to show that $\mathcal{S}$ is a stationary subset of $\mathcal{P}(K)$ in V. Work in V and fix a function $f : [K]^{<\omega} \longrightarrow K$. Then the closure properties of M imply that M contains a sequence $\langle X_n \mid n < \omega \rangle$ of countable elementary substructures of K with the property that $f[[X_n]^{<\omega}] \subseteq X_{n+1}$ for all $n < \omega$. But then $\bigcup\{X_n \mid n < \omega\} \in C_f \cap \mathcal{S} \neq \emptyset$. □

If H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$ and F is the filter on $\mathbb{P}_{K,\mathcal{S},\mathcal{T}}$ induced by the embedding $D^{\mathrm{H}(\theta)}_{K,\mathcal{S},\mathcal{T}}$ and the filter $G * H$, then $j \restriction \mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} = \mathrm{id}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}}$ implies that $j[G] \subseteq F$ and hence there is an embedding $j_{G,H} : \mathrm{V}[G] \longrightarrow M[F]$ that extends j and is definable in $\mathrm{V}[G,H]$. Let $\dot{\mathcal{U}}$ denote the canonical $\dot{\mathbb{Q}}^G$-name in $\mathrm{V}[G]$ with the property that whenever H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$, then

$$\dot{\mathcal{U}}^H \ = \ \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}[G]} \mid j[\gamma] \in j_{G,H}(A)\}$$

and therefore standard arguments show that $\dot{\mathcal{U}}^H$ is a fine, $\mathrm{V}[G]$-normal, $\mathrm{V}[G]$-$<\theta$-complete ultrafilter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}[G]}$ with the property that $\mathrm{Ult}(\mathrm{V}[G], \dot{U}^H)$ is well-founded. This allows us to conclude that (ii) holds.

Now, assume that (ii) holds and let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. In $\mathrm{V}[G]$, there is a partial order $\mathbb{Q}$ with the σ-approximation property that witnesses that θ is generically γ-supercompact. Let H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$. Then Proposition 15.3 yields an elementary embedding $j : \mathrm{V}[G] \longrightarrow M$ definable in $\mathrm{V}[G,H]$ with critical point θ, $j(\theta) > \gamma$, and $j[\gamma] \in M$. In this situation, Corollary 3.20 and Lemma 4.11 show that the assumptions of Lemma 15.1 are satisfied, and therefore $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$ and $\mathcal{U} = \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}} \mid j[\gamma] \in j(A)\} \in \mathrm{V}$. Since it is easy to see that $\mathcal{U}$ is a fine, $<\theta$-complete, normal ultrafilter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))$ in V, it follows that θ is γ-supercompact in V, as desired. □

Now, we will characterize almost huge cardinals. Remember that a cardinal θ is *almost huge* if there is an elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{crit}\,(j) = \theta$ and ${}^{<j(\theta)}M \subseteq M$. If such an embedding j exists, then we say that θ *is almost huge with target* $j(\theta)$. Our characterization of almost hugeness will rely on a generic large cardinal concept for almost hugeness. The following lemma provides us with an adaption of Lemma 15.1 to the setting of almost huge cardinals.

The proof of the following lemma uses standard characterization of almost hugeness as in [18, Theorem 24.11].

Lemma 15.6. *Let* $\mathrm{V}[G]$ *be a generic extension of the ground model* V, *let* $\mathrm{V}[G,H]$ *be a generic extension of* $\mathrm{V}[G]$, *let θ be an uncountable regular cardinal in* $\mathrm{V}[G]$ *and let $\lambda > \theta$ be an uncountable regular cardinal in* $\mathrm{V}[G,H]$. *Assume that the following statements hold:*

(i) θ and λ are inaccessible cardinals in V.

(ii) The pair $(\mathrm{V}, \mathrm{V}[G])$ *satisfies the* σ*-approximation and the* θ*-cover property.*
(iii) The pair $(\mathrm{V}[G], \mathrm{V}[G,H])$ *satisfies the* σ*-approximation property.*
(iv) There is an elementary embedding $j : \mathrm{V}[G] \longrightarrow M$ *definable in* $\mathrm{V}[G,H]$ *with the property that* $\mathrm{crit}\,(j) = \theta$*,* $j(\theta) = \lambda$ *and* $j[\gamma] \in M$ *for all* $\gamma < \lambda$*.*

Then θ *is almost huge with target* λ *in* V*.*

Proof. Given $\theta \leq \gamma < \lambda$, define

$$\mathcal{U}_\gamma \;=\; \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}} \mid j[\gamma] \in j(A)\} \;\in\; \mathrm{V}[G,H].$$

Then $\mathcal{U}_\gamma = \{\{a \cap \gamma \mid a \in A\} \mid A \in \mathcal{U}_\delta\}$ for all $\theta \leq \gamma \leq \delta < \lambda$. Moreover, we can apply Lemma 15.1 to conclude that for every $\theta \leq \gamma < \lambda$, we have $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$, and $\mathcal{U}_\gamma$ is an element of V. Define

$$\mathbb{U} \;=\; \{\mathcal{U}_\gamma \mid \theta \leq \gamma < \lambda\} \;\in\; \mathrm{V}[G,H].$$

Claim. $\mathbb{U} \in \mathrm{V}$.

Proof of the Claim. First, assume, towards a contradiction, that $\mathbb{U} \notin \mathrm{V}[G]$. Then our assumptions imply that there is $u \in \mathrm{V}[G]$ that is countable in $\mathrm{V}[G]$ with the property that $\mathbb{U} \cap u \notin \mathrm{V}[G]$. Since λ is regular and uncountable in $\mathrm{V}[G,H]$, we can find $\theta \leq \delta < \lambda$ with

$$\mathbb{U} \cap u \;=\; \{\mathcal{U}_\gamma \mid \gamma < \delta\} \cap u \;=\; \{\{\{a \cap \gamma \mid a \in A\} \mid A \in \mathcal{U}_\delta\} \mid \gamma < \delta\} \cap u.$$

But then $\mathcal{U}_\delta \in \mathrm{V} \subseteq \mathrm{V}[G]$ implies that $\mathbb{U} \cap u \in \mathrm{V}[G]$, a contradiction.

Since we already know that $\mathbb{U} \subseteq \mathrm{V}$, we can use the same argument to show that the set $\mathbb{U}$ is an element of V. □

Claim. *If* $\theta \leq \gamma < \lambda$ *and* $f \in \left({}^{\mathcal{P}_\theta(\gamma)}\theta\right)^{\mathrm{V}}$ *with* $\{a \in \mathcal{P}_\theta(\gamma) \mid \mathrm{otp}\,(a) \leq f(a)\} \in \mathcal{U}_\gamma$*, then there is* $\gamma \leq \delta < \lambda$ *with* $\{a \in \mathcal{P}_\theta(\delta) \mid f(a \cap \gamma) = \mathrm{otp}\,(a)\} \in \mathcal{U}_\delta$*.*

Proof of the Claim. Since $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}}) = \mathrm{dom}(j(f))$, there is a $\delta < \lambda = j(\theta)$ with $\delta = j(f)(j[\gamma])$. Then, we have $\gamma = \mathrm{otp}\,(j[\gamma]) \leq \delta < \lambda$, and

$$j(f)(j(\gamma) \cap j[\delta]) \;=\; j(f)(j[\gamma]) \;=\; \delta \;=\; \mathrm{otp}\,(j[\delta]).$$

This shows that $\{a \in \mathcal{P}_\theta(\delta) \mid f(a \cap \gamma) = \mathrm{otp}\,(a)\} \in \mathcal{U}_\delta$. □

For every $\theta \leq \gamma < \lambda$, $j[\gamma] \in j(\mathcal{P}_\theta(\gamma)^{\mathrm{V}})$ implies that $\mathcal{U}_\gamma$ is a fine, normal, θ-complete filter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))$ in V. Let $M_\gamma = \mathrm{Ult}(\mathrm{V}, \mathcal{U}_\gamma)$ denote the corresponding ultrapower and let $j_\gamma : \mathrm{V} \longrightarrow M_\gamma$ denote the induced ultrapower embedding. Given $\theta \leq \gamma \leq \delta < \lambda$, we have $\mathcal{U}_\gamma = \{\{a \cap \gamma \mid a \in A\} \mid A \in \mathcal{U}_\delta\}$, and the map

$$k_{\gamma,\delta} : M_\gamma \longrightarrow M_\delta;\; [f]_{\mathcal{U}_\gamma} \longmapsto [a \mapsto f(a \cap \gamma)]_{\mathcal{U}_\delta}$$

is an elementary embedding with $j_\delta = k_{\gamma,\delta} \circ j_\gamma$.

Now, work in V, and fix $\theta \leq \gamma < \lambda$ and $\theta \leq \xi < j_\gamma(\theta)$. Then $\xi = [f]_{\mathcal{U}_\gamma}$ for some function $f : \mathcal{P}_\theta(\gamma) \longrightarrow \theta$, and therefore $\{a \in \mathcal{P}_\theta(\gamma) \mid \mathrm{otp}\,(a) \leq f(a)\} \in \mathcal{U}_\gamma$. In this situation, the last claim yields an ordinal $\gamma \leq \delta < \lambda$ with the property that $\{a \in \mathcal{P}_\theta(\delta) \mid f(a \cap \gamma) = \mathrm{otp}\,(a)\} \in \mathcal{U}_\delta$, and this implies that

$$k_{\gamma,\delta}(\xi) \;=\; k_{\gamma,\delta}([f]_{\mathcal{U}_\gamma}) \;=\; [a \mapsto f(a \cap \gamma)]_{\mathcal{U}_\delta} \;=\; [a \mapsto \mathrm{otp}\,(a)]_{\mathcal{U}_\delta} \;=\; \delta.$$

Since λ is inaccessible in V, the above computation allow us to apply [17, Theorem 24.11] to conclude that θ is almost huge with target λ in V. □

We will now discuss the typical situation in which the assumptions of the previous lemma are satisfied.

Definition 15.7. Given an uncountable regular cardinal θ and an inaccessible cardinal $\lambda > \theta$, we say that a partial order $\mathbb{P}$ *witnesses that θ is generically almost huge with target λ* if the following statements hold:

(i) Forcing with $\mathbb{P}$ preserves the regularity of λ.
(ii) There is a sequence $\langle \dot{\mathcal{U}}_\gamma \mid \theta \leq \gamma < \lambda \rangle$ of $\mathbb{P}$-names such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V:
   (a) If $\theta \leq \gamma < \lambda$, then $\dot{\mathcal{U}}^G$ is a fine, V-normal, V-$<\theta$-complete filter on $\mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}}$ with the property that the corresponding ultrapower $\mathrm{Ult}(\mathrm{V},\dot{\mathcal{U}}^G)$ is well-founded.
   (b) If $\theta \leq \gamma \leq \delta < \lambda$, then $\dot{\mathcal{U}}^G_\gamma = \{\{a \cap \gamma \mid a \in A\} \mid A \in \dot{\mathcal{U}}^G_\delta\}$.
   (c) If $\theta \leq \gamma < \lambda$ and $f \in ({}^{\mathcal{P}_\theta(\gamma)}\theta)^{\mathrm{V}}$, then there is $\gamma \leq \delta < \lambda$ with
$$\{a \in \mathcal{P}_\theta(\delta)^{\mathrm{V}} \mid f(a \cap \gamma) \leq \mathrm{otp}\,(a)\} \;\in\; \dot{\mathcal{U}}^G_\delta.$$

The name of the property defined above is justified by the following proposition and by [7, Lemma 3] stating that, in the setting of that proposition, $j[\gamma] \in M$ implies $\mathcal{P}(\gamma)^{\mathrm{V}} \in M$ for all $\gamma < \theta$.

Proposition 15.8. *Given an uncountable regular cardinal θ and an inaccessible cardinal $\lambda > \theta$, if a partial order $\mathbb{P}$ witnesses that θ is generically almost huge with target λ and G is $\mathbb{P}$-generic over* V*, then there is an elementary embedding $j : \mathrm{V} \longrightarrow M$ definable in* $\mathrm{V}[G]$ *with* $\mathrm{crit}\,(j) = \theta$*, $j(\theta) = \lambda$ and $j[\gamma] \in M$ for all $\gamma < \lambda$.*

Proof. Let $\langle \dot{\mathcal{U}}_\gamma \mid \gamma \leq \gamma < \lambda \rangle$ be the corresponding sequence of $\mathbb{P}$-names and let G be $\mathbb{P}$-generic over V. Given $\theta \leq \gamma < \lambda$, let $M_\gamma = \mathrm{Ult}(\mathrm{V},\dot{\mathcal{U}}^G_\gamma)$ denote the corresponding generic ultrapower and let $j_\gamma : \mathrm{V} \longrightarrow M_\gamma$ denote the corresponding elementary embedding. Then Proposition 15.3 shows that j_γ has critical point θ and $j_\gamma[\gamma] \in M_\gamma$ for all $\theta \leq \gamma < \lambda$. Moreover, if $\theta \leq \gamma \leq \delta < \lambda$, then the function

$$k_{\gamma,\delta} : M_\gamma \longrightarrow M_\delta;\; [f]_{\mathcal{U}_\gamma} \longmapsto [a \mapsto f(a \cap \gamma)]_{\mathcal{U}_\delta}$$

is an elementary embedding with $j_\delta = k_{\gamma,\delta} \circ j_\gamma$. In addition, it is easy to see that $k_{\gamma,\epsilon} = k_{\delta,\epsilon} \circ k_{\gamma,\delta}$ holds for all $\theta \leq \gamma \leq \delta \leq \epsilon < \lambda$. Since λ has uncountable cofinality in $\mathrm{V}[G]$, the corresponding limit

$$\langle M,\; \langle k_\gamma : M_\gamma \longrightarrow M \mid \theta \leq \gamma < \lambda \rangle \rangle$$

of the resulting directed system

$$\langle\langle M_\gamma \mid \theta \leq \gamma < \lambda\rangle, \ \langle k_{\gamma,\delta} : M_\gamma \longrightarrow M_\delta \mid \theta \leq \gamma \leq \delta < \lambda\rangle\rangle$$

is well-founded, and we can identify M with its transitive collapse. If $j : \mathrm{V} \longrightarrow M$ is the unique map with $j = k_\gamma \circ j_\gamma$ for all $\theta \leq \gamma < \lambda$, then the above remarks directly imply that j is an elementary embedding with critical point θ.

Now, fix $\theta \leq \gamma < \lambda$. If $\alpha < \gamma$, then $j_\gamma(\alpha) \in j_\gamma[\gamma]$, and therefore $j(\alpha) \in k_\gamma(j_\gamma[\gamma])$. In the other direction, pick $\beta \in k_\gamma(j_\gamma[\gamma])$. Then we can find $\gamma \leq \delta < \lambda$ and $\beta_0 \in k_{\gamma,\delta}(j_\gamma[\gamma]) = [a \mapsto a \cap \gamma]_{\dot{\mathcal{U}}_\delta^G}$ with $\beta = k_\delta(\beta_0)$. In this situation, V-normality implies that there is an $\alpha < \gamma$ with $\beta_0 = j_\delta(\alpha)$ and hence $\beta = j(\alpha)$. In combination, these arguments show that $j[\gamma] = k_\gamma(j_\gamma[\gamma]) \in M$ for all $\gamma < \lambda$. But this also implies that $\gamma = \mathrm{otp}\,(j[\gamma]) = k_\gamma(\mathrm{otp}\,(j_\gamma[\gamma])) = k_\gamma(\gamma)$ holds for all $\theta \leq \gamma < \lambda$.

Finally, fix $\beta < j(\theta)$. Then there is a $\theta \leq \gamma < \lambda$ and a function $f \in ({}^{\mathcal{P}_\theta(\gamma)}\theta)^{\mathrm{V}}$ such that $\beta = k_\gamma([f]_{\mathcal{U}_\gamma})$. By Definition 15.7, we can find an ordinal $\gamma \leq \delta < \lambda$ with $\{a \in \mathcal{P}_\theta(\delta)^{\mathrm{V}} \mid f(a \cap \gamma) \leq \mathrm{otp}\,(a)\} \in \dot{\mathcal{U}}_\delta^G$. This implies that $k_{\gamma,\delta}([f]_{\dot{\mathcal{U}}_\gamma^G}) \leq \delta$ and hence $\beta \leq k_\delta(\delta) = \delta$. This shows that $j(\theta) \leq \lambda$. Since we obviously also have $j(\theta) \geq \lambda$, we can conclude that $j(\theta) = \lambda$. □

The following theorem contains our characterization of almost hugeness through Neeman's pure side condition forcing.

Theorem 15.9. *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is an almost huge cardinal.

(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "There is an inaccessible cardinal λ and a partial order $\mathbb{P}$ with the σ-approximation property that witnesses that ω_2 is generically almost huge with target λ".

Proof. First, assume that (i) holds, and let the almost hugeness of θ be witnessed by the embedding $j : \mathrm{V} \longrightarrow M$. Then $\lambda = j(\kappa)$ is an inaccessible cardinal, $\mathrm{H}(\lambda) \subseteq M$, $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta} = \mathbb{P}^M_{\mathcal{S}_\theta,\mathcal{T}_\theta}$, $j(\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}) = \mathbb{P}_\lambda$ and $\mathrm{H}(\theta) \in \mathcal{T}_\lambda$. Set $\dot{\mathbb{Q}} = \dot{\mathbb{Q}}_\lambda^{\mathrm{H}(\theta)}$ and let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. Then λ is inaccessible in $\mathrm{V}[G]$ and Corollary 3.20 implies that $\dot{\mathbb{Q}}^G$ has the σ-approximation property in $\mathrm{V}[G]$. Now, if H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$ and F is the filter on $\mathbb{P}_\lambda$ induced by the embedding $D_\lambda^{\mathrm{H}(\theta)}$ and the filter $G * H$, then $j[G] = G \subseteq F$ and there is an embedding $j_{G,H} : \mathrm{V}[G] \longrightarrow M[F]$ that extends j and is definable in $\mathrm{V}[G,H]$. Given $\theta \leq \gamma < \lambda$, let $\dot{\mathcal{U}}_\gamma$ be the canonical $\dot{\mathbb{Q}}^G$-name in $\mathrm{V}[G]$ such that

$$\dot{\mathcal{U}}_\gamma^H \ = \ \{A \in \mathcal{P}(\mathcal{P}_\theta(\gamma))^{\mathrm{V}[G]} \mid j[\gamma] \in j_{G,H}(A)\}$$

holds whenever H is $\dot{\mathbb{Q}}^G$-generic over $\mathrm{V}[G]$. Then forcing with $\dot{\mathbb{Q}}^G$ over $\mathrm{V}[G]$ preserves the regularity of λ and, as in the proof of Lemma 15.6, we can also show the sequence $\langle \dot{\mathcal{U}}_\gamma \mid \theta \leq \gamma < \lambda\rangle$ of $\dot{\mathbb{Q}}^G$-names satisfies the statements listed in Item (ii) of Definition 15.7 in $\mathrm{V}[G]$. In particular, $\dot{\mathbb{Q}}^G$ witnesses θ to be generically almost huge with target λ in $\mathrm{V}[G]$.

In the other direction, assume that (ii) holds and let G be $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$-generic over V. In V[$G$], there is an inaccessible cardinal $\lambda > \theta$ and a partial order $\mathbb{Q}$ with the σ-approximation property that witnesses that θ is generically almost huge with target λ. Let H be $\mathbb{Q}$-generic over V[G]. An application of Proposition 15.8 shows that there is an elementary embedding $j : \mathrm{V}[G] \longrightarrow M$ definable in $\mathrm{V}[G, H]$ with $\mathrm{crit}\,(j) = \theta$, $j(\theta) = \lambda$ and $j[\gamma] \in M$ for all $\gamma < \lambda$. Since Corollary 3.20 and Lemma 4.11 show that the assumptions of Lemma 15.6 are satisfied, it follows by Lemma 15.6 that θ is almost huge with target λ in V. □

The following lemma shows that the above characterization of almost hugeness is strong.

Lemma 15.10. *The following statements are equivalent for all inaccessible cardinals $\theta < \lambda$:*

(i) θ is almost huge with target λ.
(ii) There is a partial order with the σ-approximation property that witnesses that θ is generically almost huge with target λ.

Proof. If θ is almost huge with target λ, then the trivial partial order witnesses that θ is generically almost huge with target λ by [17, Theorem 24.11]. In order to verify the reverse direction, let $\mathbb{P}$ be a partial order with the σ-approximation property that witnesses that θ is generically almost huge with target λ, let H be $\mathbb{P}$-generic over V and let $j : \mathrm{V} \longrightarrow M$ be the elementary embedding definable in V[H] that is provided by an application of Proposition 15.8. Then, an application of Lemma 15.6 with $\mathrm{V} = \mathrm{V}[G]$ shows that θ is almost huge with target λ in V. □

The arguments contained in the above proofs also allow us to prove the analogous results for even larger large cardinals, namely *super almost huge cardinals* (see, for example, [4] and [36]), i.e. cardinals θ with the property that for every $\gamma > \theta$, there is an inaccessible cardinal $\lambda > \gamma$ such that θ is almost huge with target λ.

Lemma 15.11. *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a super almost huge cardinal.
(ii) For every $\gamma > \theta$, there is an inaccessible cardinal $\lambda > \gamma$ and a partial order $\mathbb{P}$ with the σ-approximation property that witnesses that θ is generically almost huge with target λ. □

Theorem 15.12. *The following statements are equivalent for every inaccessible cardinal θ:*

(i) θ is a super almost huge cardinal.
(ii) $\mathbb{1}_{\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}} \Vdash$ "For every ordinal γ, there is an inaccessible cardinal $\lambda > \gamma$ and a partial order $\mathbb{P}$ with the σ-approximation property that witnesses that ω_2 is generically almost huge with target λ". □

Chapter 16

Concluding remarks and open questions

In order to strongly characterize inaccessible, Mahlo, Π^m_n-indescribable, subtle and λ-ineffable cardinals, we were using various structural properties of Neeman's forcing. The most important ones are given by Corollary 3.19, Lemma 4.11 and Corollary 4.9. As we have mentioned in the introduction of this thesis, Mitchell showed in [26] that the consistency of weakly compact cardinal implies consistency of the tree property at ω_2. The partial order that he used in order to show this is yet another example of forcing that satisfies all the relevant properties. Hence, it is also possible to use partial order of this form to characterize inaccessibility, Mahloness, Π^m_n-indescribability, subtlety, λ-ineffability and supercompactness.

The fact that quotients of forcing notions of the form $\mathbb{P}_{\mathcal{S}_\theta,\mathcal{T}_\theta}$ satisfy the σ-approximation property is very important for almost all large cardinal characterizations presented in this thesis. This property implies that these quotients add new real numbers, and this causes the Continuum Hypothesis to fail in their final forcing extension. In addition, if we want to use some sequence of collapse forcing notions to characterize inaccessibility as in Theorem 6.7, then these collapses have to force failures of the GCH below the relevant cardinals.[1] This shows that, in order to obtain large cardinal characterizations based on forcing notions whose quotients do not add new reals, one has to work with different combinatorial principles. Since Proposition 1.2 shows that the canonical collapse forcing with this quotient behavior, the *Lévy Collapse* $\mathrm{Col}(\kappa,{<}\theta)$, is not suitable for the type of large cardinal characterization as in Definition 1.1, it is then natural to consider the two-step iteration $\mathbb{C}_\theta = \mathrm{Add}(\omega,1) * \mathrm{Col}(\kappa,{<}\theta)$ that first adds a Cohen real and then collapses some cardinal θ to become the successor of a regular uncountable cardinal κ. Using results of [9], showing that forcings of this form satisfy the σ-approximation and cover property, it is possible to modify the characterizations obtained in the early chapters of this thesis in order to characterize inaccessibility, Mahloness and weak compactness with the help of the sequence $\langle \mathbb{C}_\theta \mid \theta \in \mathrm{Card}\rangle$. In these modifications, we replace statements about the non-existence of certain trees by statements that claim that these trees contain *Cantor subtrees*, i.e. that there is an embedding $\iota : {}^{\leq\omega}2 \longrightarrow \mathbb{T}$ of the full binary tree ${}^{\leq\omega}2$ of height $\omega+1$ into the given tree $\mathbb{T}$, that satisfies $\mathrm{lh}_{\mathbb{T}}(\iota(s)) = \sup_{n<\omega} \mathrm{lh}_{\mathbb{T}}(\iota(s \restriction n))$ and

[1] This is because if CH holds, the full binary tree of height ω_1 is a weak Kurepa tree.

$\mathrm{lh}_{\mathbb{T}}(\iota(s \restriction n)) = \mathrm{lh}_{\mathbb{T}}(\iota(t \restriction n))$ for all $s, t \in {}^{\omega}2$ and $n < \omega$. Using results from [30] and ideas contained in the proof of [23, Theorem 7.2], it is then possible to obtain the following characterizations:

- An infinite cardinal θ is inaccessible if and only if $\mathbb{C}_\theta$ forces θ to become ω_2 and every tree of height ω_1 with $\aleph_2$-many cofinal branches to contain a Cantor subtree.
- An inaccessible cardinal θ is a Mahlo cardinal if and only if $\mathbb{C}_\theta$ forces all special ω_2-Aronszajn trees to contain a Cantor subtree.
- An inaccessible cardinal θ is weakly compact if and only if $\mathbb{C}_\theta$ forces all ω_2-Aronszajn trees to contain a Cantor subtree.

In addition, it is also possible to use [9, Theorem 10] and arguments from the proof of Lemma 15.1 to prove analogues of the results of the previous chapter for the sequence $\langle \mathbb{C}_\theta \mid \theta \in \mathrm{Card} \rangle$:

- An inaccessible cardinal θ is λ-supercompact for some cardinal $\lambda \geq \theta$ if and only if in every $\mathbb{C}_\theta$-generic extension, there is a σ-closed partial order witnessing that ω_2 is generically λ-supercompact.
- An inaccessible cardinal θ is almost huge with target $\lambda > \theta$ if and only if in every $\mathbb{C}_\theta$-generic extension, there is a σ-closed partial order witnessing that ω_2 is generically almost huge with target λ.

It follows directly that the large cardinal characterizations obtained in this way are all strong. The details of these results will be presented in the forthcoming [11]. Note that the above arguments provide no analogues for the results of chapters 8 and 12. We do not know which combinatorial principles could replace the ones used in these chapters in order to allow characterizations of the corresponding large cardinal properties using the sequence $\langle \mathbb{C}_\theta \mid \theta \in \mathrm{Card} \rangle$. These observations motivate the following question:

Question 1. *Does the sequence* $\langle \mathbb{C}_\theta \mid \theta \in \mathrm{Card} \rangle$ *characterize* Π^m_n*-indescribability, subtlety or* λ*-ineffability?*

Proposition 1.2 shows that the Levy Collapse is not suitable for large cardinal characterizations in the sense of Definition 1.1, by showing that it cannot characterize inaccessibility in this way. However, we do not know whether it could be used to characterize stronger large cardinal properties if we restrict the desired provable equivalences to inaccessible cardinals. In particular, we cannot answer the following sample question:

Question 2. *Is there a formula* $\varphi(v)$ *in the language of set theory with the property that*

$$\mathsf{ZFC} \vdash \forall \theta \text{ inaccessible } [\theta \text{ is weakly compact } \longleftrightarrow \mathbb{1}_{\mathrm{Col}(\omega_1, <\theta)} \Vdash \varphi(\theta)] \text{ ?}$$

In the remainder of this chapter, we present some arguments suggesting that if it is possible to characterize stronger large cardinal properties of inaccessible cardinals using forcings of the form $\mathrm{Col}(\omega_1, <\theta)$, then the combinatorial principles to be used in these equivalences are not as canonical as the ones that appear in the above characterization through Neeman's pure side condition forcing. The proof of the following result is based upon a classical construction of Kunen from [20].

Theorem 16.1. *If* θ *is a weakly compact cardinal, then the following statements hold in a regularity preserving forcing extension* $\mathrm{V}[G]$ *of the ground model* V*:*

(i) *θ is an inaccessible cardinal that is not weakly compact.*
(ii) $\mathbb{1}_{\mathrm{Col}(\omega_1,<\theta)} \Vdash$ *"Every θ-Aronszajn tree contains a Cantor subtree".*

Proof. By classical results of Silver, we may assume that

$$\mathbb{1}_{\mathrm{Add}(\theta,1)} \Vdash \text{“}\check{\theta} \textit{ is weakly compact”}.$$

Given $D \subseteq \theta$, let π_D denote the unique automorphism of the tree ${}^{<\theta}2$ with the property that

$$\pi_D(t)(\alpha) \ = \ t(\alpha) \iff \alpha \notin D$$

holds for all $t \in {}^{<\theta}2$ and $\alpha \in \mathrm{dom}(t)$. Moreover, given $s, t \in {}^{<\theta}2$, we set

$$\Delta(s,t) \ = \ \{\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\alpha) \neq t(\alpha)\}.$$

Note that $\pi_{\Delta(s,t)}(s) = t$ holds for all $s, t \in {}^{<\theta}2$ with $\mathrm{dom}(s) = \mathrm{dom}(t)$.

Define $\mathbb{P}$ to be the partial order whose conditions are either $\emptyset$, or normal, σ-closed subtrees S of ${}^{<\theta}2$ of cardinality less than θ and height $\alpha_S + 1 < \theta$,[2] with the additional property that for all $s, t \in S$ with $\mathrm{dom}(s) = \mathrm{dom}(t)$, the map $\pi_{\Delta(s,t)} \upharpoonright S$ is an automorphism of S. Let $\mathbb{P}$ be ordered by reverse end-extension. If G is $\mathbb{P}$-generic over V, then $\bigcup\bigcup G$ is a subtree of ${}^{<\theta}2$. Let $\dot{\mathbb{S}}$ be the canonical $\mathbb{P}$-name for the forcing notion corresponding to the tree $\bigcup\bigcup G$, and let

$$D \ = \ \{\langle S, \check{s}\rangle \in \mathbb{P} * \dot{\mathbb{S}} \mid S \in \mathbb{P}, \ s \in S(\alpha_S)\}.$$

Then it is easy to see that D is dense in $\mathbb{P} * \dot{\mathbb{S}}$.

Claim. *Let $\lambda < \theta$, and let $\langle S_\gamma \mid \gamma < \lambda\rangle$ be a descending sequence in $\mathbb{P}$. Define $\alpha = \sup_{\gamma<\lambda} \alpha_{S_\gamma}$, $S = \bigcup\{S_\gamma \mid \gamma < \lambda\}$ and $[S] = \{t \in {}^{\alpha}2 \mid \forall \gamma < \lambda \ t \upharpoonright \alpha_{S_\gamma} \in S_\gamma\}$.*

(a) *If $\mathrm{cof}(\lambda) = \omega$, then $[S] \neq \emptyset$ and $S \cup [S]$ is the unique condition T in $\mathbb{P}$ with $\alpha_T = \alpha$ and $T \leq_{\mathbb{P}} S_\gamma$ for all $\gamma < \lambda$.*
(b) *If $\mathrm{cof}(\lambda) > \omega$ and $[S] \neq \emptyset$, then $S \cup [S]$ is a condition in $\mathbb{P}$ below S_γ for all $\gamma < \lambda$.*
(c) *If $\mathrm{cof}(\lambda) > \omega$, G is the subgroup of the group of all automorphisms of ${}^{<\theta}2$ that is generated by the set $\{\pi_{\Delta(s,t)} \mid s, t \in S, \ \mathrm{dom}(s) = \mathrm{dom}(t)\}$, $u \in [S]$ and $B = \{\pi(u) \mid \pi \in G\}$, then $S \cup B$ is a condition in $\mathbb{P}$ below S_γ for all $\gamma < \lambda$.*

In particular, the dense suborder D of $\mathbb{P} * \dot{\mathbb{S}}$ is $<\theta$-closed, $\mathbb{P} * \dot{\mathbb{S}}$ is forcing equivalent to $\mathrm{Add}(\theta, 1)$*, and forcing with $\mathbb{P}$ preserves the inaccessibility of θ.*

By the above claim, there is a winning strategy Σ for player Even in the game $G_\theta(\mathbb{P})$ of length θ associated to the partial order $\mathbb{P}$ (see [5, Definition 5.14]), with the property that whenever $\langle S_\gamma \mid \gamma < \theta\rangle$ is a run of $G_\theta(\mathbb{P})$ in which player Even played according to Σ, then the following statements hold:

(1) There is a sequence $\langle t_\gamma \mid \gamma < \theta\rangle$ of elements of ${}^{<\theta}2$ with the property that $\langle\langle S_{2\cdot\gamma}, \check{t}_\gamma\rangle \mid \gamma < \theta\rangle$ is a strictly descending sequence of conditions in D.
(2) The set $\{\alpha_{S_{2\cdot\gamma}} \mid \gamma < \theta\}$ is a club in θ.

[2] In this situation, normality means that if $s \in S$ with $\mathrm{dom}(s) \in \alpha_S$, then $s^\frown\langle i\rangle \in S$ for all $i < 2$, and there is a $t \in S(\alpha_S)$ with $s \subseteq t$.

(3) If $\lambda \in \mathrm{Lim} \cap \theta$ and $S = \bigcup\{S_\gamma \mid \gamma < \lambda\}$, then $S_\lambda = S \cup [S]$.

In particular, Σ witnesses that $\mathbb{P}$ is θ-strategically closed.

Claim. $\mathbb{1}_{\mathbb{P}} \Vdash$ "*$\dot{\mathbb{S}}$ is a σ-closed $\check{\theta}$-Souslin tree*".

Proof of the Claim. It is immediate that $\dot{\mathbb{S}}$ is forced to be a tree of height θ whose levels all have cardinality less than θ, and that the tree $\dot{\mathbb{S}}$ is forced to be σ-closed. It remains to show that its antichains have size less than θ.

Therefore, let S_* be a condition in $\mathbb{P}$, let $\dot{A} \in \mathrm{V}$ be a $\mathbb{P}$-name for a maximal antichain in $\dot{\mathbb{S}}$, and let $\dot{C} \in \mathrm{V}$ be the induced $\mathbb{P}$-name for the club of all ordinals less than θ with the property that the intersection of $\dot{A}$ with the corresponding initial segment of $\dot{\mathbb{S}}$ is a maximal antichain in this initial segment. Then there is a run $\langle S_\gamma \mid \gamma < \theta\rangle$ of $G_\theta(\mathbb{P})$ in which player Even played according to Σ, $S_1 \leq_{\mathbb{P}} S_*$, and there exist sequences $\langle \beta_\gamma \mid \gamma < \theta\rangle$ and $\langle A_\gamma \mid \gamma < \theta\rangle$ with the properties that $\alpha_{S_{2\cdot\gamma+1}} > \beta_\gamma$ and

$$S_{2\cdot\gamma+1} \Vdash_{\mathbb{P}} \text{"}\check{\beta}_\gamma = \min(\dot{C} \setminus \check{\alpha}_{S_{2\cdot\gamma}}) \ \wedge \ \check{A}_\gamma = \dot{A} \cap {}^{<\check{\beta}_\gamma}2\text{"}$$

for all $\gamma < \theta$. Since $C = \{\alpha_{S_{2\cdot\gamma}} \mid \gamma < \theta\}$ is a club in θ, we can find an inaccessible cardinal $\eta < \theta$ with $\eta = \alpha_{S_\eta}$ and $|S_\gamma| < \eta$ for all $\gamma < \eta$. Set $A = \bigcup\{A_\gamma \mid \gamma < \eta\}$ and $S = \bigcup\{S_\gamma \mid \gamma < \eta\}$. Then we have

$$S_\eta \Vdash \text{"}\check{\eta} \in \dot{C} \ \wedge \ \check{A} = \dot{A} \cap {}^{<\check{\eta}}2 \ \wedge \ \check{S} = \dot{\mathbb{S}} \cap {}^{<\check{\eta}}2\text{"}.$$

Hence S is a normal tree of cardinality and height η, and A is a maximal antichain in S. Fix an enumeration $\langle \pi_\gamma \mid \gamma < \eta\rangle$ of the subgroup of the group of all automorphisms of ${}^{<\theta}2$ generated by all automorphisms of the form $\pi_{\Delta(s,t)}$ with $s, t \in S$ and $\mathrm{dom}(s) = \mathrm{dom}(t)$. Since $S \cap {}^{\gamma}2 = [S \cap {}^{<\gamma}2]$ holds for all $\gamma \in C \cap \eta$, we can now inductively construct a continuous increasing sequence $\langle s_\gamma \mid \gamma < \eta\rangle$ of elements of S with the property that for every $\gamma < \eta$, we have $\mathrm{dom}(s_\gamma) \in C$, and there is a $t_\gamma \in A$ with $\pi_\gamma^{-1}(t_\gamma) \subseteq s_{\gamma+1}$. Set $s = \bigcup\{s_\gamma \mid \gamma < \eta\} \in [S]$, $B = \{\pi_\gamma(s) \mid \gamma < \eta\}$ and $T = S \cup [B]$. By the above claim, T is a condition in $\mathbb{P}$ below S_*. By the construction of s, for every $u \in B$, there is a $t \in A$ with $t \subseteq u$. Hence $T \Vdash_{\mathbb{P}}$ "$\dot{A} = \check{A}$". □

Let G be $\mathbb{P}$-generic over V, set $\mathbb{S} = \dot{\mathbb{S}}^G$, and let H be $\mathbb{S}$-generic over V$[G]$. Then the above computations ensure that θ is weakly compact in V$[G, H]$. Set

$$\mathbb{C} = \mathrm{Col}(\omega_1, {<}\theta)^{\mathrm{V}[G,H]},$$

and let K be $\mathbb{C}$-generic over V$[G, H]$. Since the partial order $\mathbb{S}$ is ${<}\theta$-distributive in V$[G]$, we have $\mathbb{C} = \mathrm{Col}(\omega_1, {<}\theta)^{\mathrm{V}[G]}$, and V$[G, H, K]$ is a $(\mathbb{C} \times \mathbb{S})$-generic extension of V$[G]$. Moreover, $\mathbb{C}$ is a σ-closed, θ-Knaster partial order in V$[G]$, and therefore $\mathbb{S}$ remains a σ-closed θ-Souslin tree in V$[G, K]$. But this shows that the partial order $\mathbb{C} \times \mathbb{S}$ is σ-distributive in V$[G]$.

Let $\mathbb{T}$ be a θ-Aronszajn tree in V$[G, K]$. First, assume that $\mathbb{T}$ has a cofinal branch in V$[G, H, K]$. Then, in V$[G, K]$, there is a σ-closed forcing that adds a cofinal branch through $\mathbb{T}$, and therefore standard arguments show that $\mathbb{T}$ contains a Cantor subtree in V$[G, H, K]$. In the other case, assume that $\mathbb{T}$ is a θ-Aronszajn tree in V$[G, H, K]$. Since θ

is weakly compact in $\mathrm{V}[G,H]$, results from [30] show that $\mathbb{T}$ contains a Cantor subtree in $\mathrm{V}[G,H,K]$. Let $\iota : {}^{\leq\omega}2 \longrightarrow \mathbb{T}$ be an embedding in $\mathrm{V}[G,H,K]$ witnessing this. Since the above remarks show that $({}^{\omega}\mathrm{V}[G,K])^{\mathrm{V}[G,H,K]} \subseteq \mathrm{V}[G,K]$, the map $\iota \upharpoonright ({}^{<\omega}2)$ is an element of $\mathrm{V}[G,K]$. Pick $\alpha < \theta$ with $\iota[{}^{\omega}2] \subseteq \mathbb{T}(\alpha)$. Given $x \in ({}^{\omega}2)^{\mathrm{V}[G,K]}$, we then know that there is an element t of $\mathbb{T}(\alpha)$ with $\iota(x \upharpoonright n) \leq_{\mathbb{T}} t$ for all $n < \omega$. This allows us to conclude that, in $\mathrm{V}[G,K]$, there is an embedding from ${}^{\leq\omega}2$ into $\mathbb{T}$ that extends $\iota \upharpoonright ({}^{<\omega}2)$ and witnesses that $\mathbb{T}$ contains a Cantor subtree. □

Note that, in combination with [19, Theorem 3.9], the above proof shows that the existence of a weakly compact cardinal is equiconsistent with the existence of a non-weakly compact inaccessible cardinal θ with the property that every θ-Aronszajn tree contains a Cantor subtree. In contrast, the proof of the following result shows that the corresponding statement for special Aronszajn trees has much larger consistency strength. In particular, it shows that the inconsistency of certain large cardinal properties strengthening measurability would imply that the Mahloness of inaccessible cardinals can be characterized by partial orders of the form $\mathrm{Col}(\omega_1, {<}\theta)$ in a canonical way.

Theorem 16.2. *Let θ be an inaccessible cardinal with property that one of the following statements holds:*

(i) Every special θ-Aronszajn tree contains a Cantor subtree.
(ii) $\mathbb{1}_{\mathrm{Col}(\omega_1,<\theta)} \Vdash$ "Every special ω_2-Aronszajn tree contains a Cantor subtree".

If θ is not a Mahlo cardinal, then there is an inner model that contains a stationary limit of measurable cardinals of uncountable Mitchell order.

Proof. Fix a closed and unbounded subset D of θ that consists of singular strong limit cardinals and assume that the above conclusion fails. Then, the proof of [6, Theorem 1] shows that *Jensen's □-principle* holds up to θ, i.e. there is a sequence $\langle B_\alpha \mid \alpha \in \mathrm{Lim} \cap \theta \textit{ singular}\rangle$ such that for all singular limit ordinals $\alpha < \theta$, the set B_α is a closed and unbounded subset of α of order-type less than α, and, if $\beta \in \mathrm{Lim}(B_\alpha)$, then $\mathrm{cof}(\beta) < \beta$ and $C_\beta = C_\alpha \cap \beta$. Then, we may pick a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \mathrm{Lim} \cap \theta\rangle$ satisfying the following statements for all $\alpha \in \mathrm{Lim} \cap \theta$:

(i) If $\alpha \in \mathrm{Lim}(D)$ and $B_\alpha \cap D$ is unbounded in α, then $C_\alpha = B_\alpha \cap D$.
(ii) If $\alpha \in \mathrm{Lim}(D)$ and $\max(B_\alpha \cap D) < \alpha$, then C_α is an unbounded subset of α of order-type ω with $\min(C_\alpha) > \max(B_\alpha \cap D)$.
(iii) If $\max(D \cap \alpha) < \alpha$, then $C_\alpha = (\max(D \cap \alpha), \alpha)$.

It is easy to check that $\vec{C}$ is a $\square(\theta)$-sequence (see [35, Definition 7.1.1]).

Claim. *$\vec{C}$ is a special $\square(\theta)$-sequence (see [35, Definition 7.2.11]).*

Proof of the Claim. Given $\alpha \leq \beta < \theta$, let $\rho_0^{\vec{C}}(\alpha,\beta) : \beta \longrightarrow {}^{<\omega}\theta$ denote the *full code of the walk from β to α through $\vec{C}$*, as defined in [35, Section 7.1]. Let $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$ be the tree of all functions of the form $\rho_0^{\vec{C}}(\,\cdot\,,\beta) \upharpoonright \alpha$ with $\alpha \leq \beta < \theta$. Then, the results of [35, Section 7.1] show that $\mathbb{T}$ is a θ-Aronszajn tree. Fix a bijection $b : \theta \longrightarrow {}^{<\omega}\theta$ with $b[\kappa] = {}^{<\omega}\kappa$ for every cardinal $\kappa \leq \theta$. Now, fix $\alpha \leq \beta < \theta$ with $\alpha \in D$, and let $\langle \gamma_0, \ldots, \gamma_n\rangle$ denote the walk from β to α through $\vec{C}$. If $C_{\gamma_{n-1}} \cap \alpha$ is unbounded in α, then the above definitions ensure that $\gamma_{n-1} \in \mathrm{Lim}$, $\mathrm{cof}(\gamma_{n-1}) < \gamma_{n-1}$, $\alpha \in \mathrm{Lim}(B_{\gamma_{n-1}})$, and therefore $\mathrm{otp}\left(C_{\gamma_{n-1}} \cap \alpha\right) \leq \mathrm{otp}\left(B_{\gamma_{n-1}} \cap \alpha\right) = \mathrm{otp}\left(B_\alpha\right) < \alpha$. This shows that we always have $\mathrm{otp}\left(C_{\gamma_{n-1}} \cap \alpha\right) < \alpha$, and hence there is an $\varepsilon < \alpha$ with $b(\varepsilon) = \rho_0^{\vec{C}}(\alpha, \beta)$. Define $r(\rho_0^{\vec{C}}(\,\cdot\,, \beta) \restriction \alpha) = \rho_0^{\vec{C}}(\,\cdot\,, \beta) \restriction \varepsilon$. Then, the proof of [35, Theorem 6.1.4] shows that the resulting regressive function $r : \mathbb{T} \restriction D \longrightarrow \mathbb{T}$ witnesses that the set D is non-stationary with respect to $\mathbb{T}$. Since D is a club in θ, this implies that the tree $\mathbb{T}$ is special and, by the results of [34], this conclusion is equivalent to the statement of the claim. □

The above claim now allows us to use [19, Theorem 3.14] to conclude that there is a special θ-Aronszajn tree $\mathbb{T}$ without Cantor subtrees and therefore (i) fails. Since the partial order $\mathrm{Col}(\omega_1, {<}\theta)$ is σ-closed, we may argue as in the last part of the proof of Theorem 16.1 to show that (ii) implies (i) and therefore the above assumption also implies a failure of (ii). □

The next proposition shows that examples of inaccessible non-Mahlo cardinals satisfying statement (i) in Theorem 16.2 can be obtained using supercompactness.

Proposition 16.3. *Let $\kappa < \theta$ be uncountable regular cardinals. If κ is θ-supercompact, then the following statements hold:*

(i) Every θ-Aronszajn tree contains a Cantor subtree.
(ii) $\mathbb{1}_{\mathrm{Col}(\omega_1, <\kappa)} \Vdash$ "Every $\check{\theta}$-Aronszajn tree contains a Cantor subtree".

Proof. Fix an elementary embedding $j : \mathrm{V} \longrightarrow M$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \theta$ and ${}^\theta M \subseteq M$. Set $\nu = \sup(j[\theta]) < j(\theta)$. Let G be $\mathrm{Col}(\omega_1, {<}\kappa)$-generic over V, let H be $\mathrm{Col}(\omega_1, [\kappa, j(\kappa)))$-generic over $\mathrm{V}[G]$ and let $j_* : \mathrm{V}[G] \longrightarrow M[G, H]$ denote the canonical lifting of j.

Fix a θ-Aronszajn tree $\mathbb{T}$ in $\mathrm{V}[G]$. By standard arguments, we may, without loss of generality, assume that every node in $\mathbb{T}$ has at most two direct successors, and that all elements of the limit levels of $\mathbb{T}$ are uniqueley determined by their sets of predecessors. Pick a node $t \in j_*(\mathbb{T})(\nu)$, and define $b = \{s \in \mathbb{T} \mid j_*(s) \leq_{j(\mathbb{T})} t\} \in \mathrm{V}[G, H]$. Then b is a branch through $\mathbb{T}$, and the above assumptions on $\mathbb{T}$ imply that b does not have a maximal element. Set $\lambda = \mathrm{otp}\,(b, \leq_{\mathbb{T}}) \leq \theta$.

Claim. *$b \notin \mathrm{V}[G]$.*

Proof of the Claim. Assume, towards a contradiction, that $b \in \mathrm{V}[G]$. Since $\mathbb{T}$ is a θ-Aronszajn tree, we know that $\lambda \in \mathrm{Lim} \cap \theta$. This implies that t extends every element of the branch $j_*(b)$ through the tree $j_*(\mathbb{T})$, and therefore $j_*(b)$ is equal to the set of all predecessors of some node in the level $j_*(\mathbb{T})(j(\lambda))$. By elementarity, there is a node u in $\mathbb{T}(\lambda)$ with the property that b consists of all predecessors of u in $\mathbb{T}$. But then, the above assumptions on $\mathbb{T}$ imply that $j_*(u) \leq_{j(\mathbb{T})} t$, and hence that $u \in b$, a contradiction. □

Since $b \notin \mathrm{V}[G]$ and $\mathrm{Col}(\omega_1, [\kappa, j(\kappa)))$ is σ-closed in $\mathrm{V}[G]$, we thus know that $\mathrm{cof}(\lambda)^{\mathrm{V}[G]} > \omega$. This shows that, in $\mathrm{V}[G]$, there is a σ-closed notion of forcing that adds a new branch of uncountable cofinality through $\mathbb{T}$. In this situation, standard arguments show that $\mathbb{T}$ contains a Cantor subtree in $\mathrm{V}[G]$. These computations show that (ii) holds and, by applying the arguments used in the last part of the proof of Theorem 16.1, we know that this also yields (i). □

The above arguments leave open the possibility that Statement (ii) in Theorem 16.2 provably fails for inaccessible non-Mahlo cardinals, and therefore motivate the following question, asking whether the Mahloness of inaccessible cardinals can be characterized by the existence of Cantor subtrees of special Aronszajn trees in collapse extensions.

Question 3. *Is the existence of an inaccessible non-Mahlo cardinal θ with*

$$\mathbb{1}_{\mathrm{Col}(\omega_1,<\theta)} \Vdash \text{"Every special } \omega_2\text{-Aronszajn tree contains a Cantor subtree"}$$

consistent with the axioms of ZFC*?*

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