String Compactifications from the Worldsheet and Target Space Point of View

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von
Andreas Gerhardus

aus
Kirchen (Sieg)

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1. Gutachter: Prof. Dr. Albrecht Klemm
2. Gutachter: PD Dr. Stefan Förste

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Abstract

The research presented in this thesis revolves around compactifications of type II superstring theories from both the worldsheet and target space point of view. We employ techniques from the fields of supersymmetric gauge theory and geometry to analyze the moduli structure of such compactifications, with particular emphasis on their close interconnectedness.

The thesis begins with a non-technical introduction to the wider research field and explains the context in which the considered research questions arise. We then give a more detailed review of the physical and mathematical concepts that form the basis for the subsequents parts of the thesis, including type II superstring theories, the gauged linear sigma model and Picard–Fuchs operators.

Thereafter, we turn to a study of certain correlation functions in the gauged linear sigma model and their geometric significance. We demonstrate that these correlation functions are subject to non-trivial and universal linear dependencies, which in a Hilbert space interpretation correspond to differential operators that annihilate the moduli dependent gauge theory ground state. For conformal theories these are identified as the Picard–Fuchs operators on the quantum Kähler moduli space and we present an algorithm to determine them from the defining gauge theory spectrum directly. For several classes of Calabi–Yau geometries we moreover derive universal formulas that express their Picard–Fuchs operators in terms of the gauge theory correlators.

While these findings are also applicable to gauged linear sigma models with non-Abelian gauge groups, the involved calculations quickly get out of hand. In order to circumvent these computational difficulties, we in the next chapter build on the Givental I-function to propose explicit formulas for the holomorphic solutions to the Picard–Fuchs operators of models with general non-Abelian gauge groups and a large class of matter spectra. These formulas are ready-to-use and thereby provide a computationally efficient way of determining the operators. We also briefly comment on the idea of reconstructing gauged linear sigma models from given Picard–Fuchs operators.

As an application of the various concepts and techniques introduced at this point, we then consider Calabi–Yau fourfolds that arise as target spaces of non-Abelian gauged linear sigma models. The quantum cohomology ring of such geometries is not necessarily generated by products of the marginal Kähler deformations alone, rather certain irrelevant operators need to be additionally included. This translates into the existence of non-zero quantum periods that vanish in the classical large volume limit. We explain why and under which conditions this phenomenon arises and in an example discuss the construction of integral quantum periods. These are used to obtain new types of flux superpotentials and to determine geometric invariants such as genus zero worldsheet instanton numbers.

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List of Publications

Significant parts of this thesis are based on the following publications of the author:

- A. Gerhardus, H. Jockers, and U. Ninad, *The Geometry of Gauged Linear Sigma Model Correlation Functions*, Nucl. Phys. **B933** (2018) 65, `arXiv:1803.10253 [hep-th]`
- A. Gerhardus and H. Jockers, *Quantum periods of Calabi–Yau fourfolds*, Nucl. Phys. **B913** (2016) 425, `arXiv:1604.05325 [hep-th]`

Contents

CHAPTER 1

Introduction

This thesis presents research in the context of superstring theory whose shared objective is to improve our understanding of the so-called process of compactification. As we will explain below, the latter is an essential step in the endeavor of building phenomenologically realistic models within superstring theory and requires the use of a particular type of conformal field theories. However, their construction is difficult and only few examples are known explicitly. A certain gauge theory that allows to circumvent this difficulty will therefore figure prominently. This gauge theory provides tools with which various examples of the desired conformal field theories can at least be constructed indirectly and thereby opens the possibility of systematically analyzing their properties. As usual, the gauge theory depends on a set of coupling constants. These can be interpreted as parameters that continuously deform the theory and the physical implication of varying them is of particular interest. We will thus study the space of deformations and to this end employ several techniques from the mathematical discipline of geometry. Special emphasis is laid on the close interconnection between the methods derived from gauge theory on the one hand and those from geometry on the other hand.

In the next parts of this chapter we will give a largely non-technical introduction to the main physical concepts that underly and motivate the research presented here. If not cited otherwise, the below exposition of particle physics, quantum field theory and string theory is based on references [1–4]. We then briefly comment on the approach and key findings of our research and conclude with a detailed outline of the following chapters.

Motivation

It is the aim of physics is to mathematically describe and understand the various phenomena that occur in nature. In doing so it takes a reductive point of view and tries to unify seemingly different effects by attributing them to the same underlying principle. In this sense theories fall into an order, where theories referred to as *less fundamental* are required to be consistent with those that are believed to be *more fundamental*. A natural scale for such an order is provided by the length or, equivalently, the energy at which a theory is applicable. Namely, phenomena that predominantly occur and are well described at a given energy scale should nevertheless obey the principles that govern nature at higher energies. We stress, however, that more fundamental theories do not have an intrinsically higher scientific value. While less fundamental theories are required to be consistent with the more fundamental principles, it is often infeasible to explain a certain phenomenon by only employing the underlying laws of more fundamental theories. A good example for this is statistical mechanics, which introduces new technologies for describing large ensembles of particles that go beyond the microscopic dynamics.

The standard model of particle physics is a shining example of a theory that has successfully unified a variety of phenomena. It accurately describes the behavior of subatomic particles in the framework of relativistic quantum field theory, i.e., the synthesis of quantum mechanics and special relativity. Its predictions have been verified in various experiments, a prominent example of which is the anomalous magnetic momentum of the electron. Matter particles such as the electron and quarks are described as fermions with spin $1/2$ that interact via their coupling to several particles of spin 1. Since these force mediating particles fall into three classes, it is natural to classify physical phenomena by which class of force mediator is involved. This leads to the notion of the strong, weak and electromagnetic interaction. Finally, there is a spin 0 particle known as the Higgs boson. It plays a pivotal role in the standard model by giving mass to the matter particles and mediators of the weak force.

In addition to several problems such as the instability of the electro-weak scale against quantum corrections and the fact that it does not account for the experimentally verified non-zero neutrino masses, the standard model is certainly an incomplete description of nature: it does not describe the effects of gravity. At this point it is natural to wonder whether the standard model can be augmented to include the gravitational interaction within the framework of relativistic quantum field theory. The answer to this question is positive: it can be achieved by adding a massless field of spin 2, the so-called graviton, which corresponds to fluctuations in the metric and upon quantization requires the theory to exhibit general covariance. Since Newton's constant G_N is of negative mass dimension, the interactions of a massless spin 2 field are not perturbatively renormalizable. While this means that the theory cannot be extrapolated to higher energy scales, it is perfectly consistent from an effective point of view. However, gravity is so weak that the quantum predictions of this effective theory are not measurable by current experiments.

These considerations clearly demonstrate the need for a more fundamental theory of both particle physics and gravity. This theory should remain valid at the Planck scale $M_P = \sqrt{1/G_N} = 10^{19}$ GeV, where the true quantum nature of gravity is extrapolated to be relevant.

String Theory

A promising candidate for such a theory of quantum gravity is string theory. Its key postulate is the existence of one-dimensional objects, so-called strings. These can be open or closed (forming a loop) and they propagate in a predefined space-time M, which for now we take to be d-dimensional Minkowski space $\mathbb{M}^d$. In complete analogy to (zero-dimensional) particles tracing out a one-dimensional worldline in space-time, these one-dimensional strings trace out a two-dimensional surface that is referred to as the worldsheet. Unlike particles, however, strings have vibrations as internal degrees of freedom. Consider this from a space-time point of view: when an observer measures at energies much lower than the string length scale l_s, they will not be able to resolve the spatial extension of the string and thus perceive it as point-like, i.e., as a particle. In this way string theory yields a theory of particles.

This intuition is formalized in the worldsheet formulation of string theory, where the string is described by maps X from the worldsheet into space-time that are treated as two-dimensional quantum fields whose excitations correspond to the vibrations mentioned above. For $d = 26$ the theory exhibits space-time Poincaré invariance and the excitations arrange themselves in representations of the 26-dimensional Poincaré group, such that they can indeed be interpreted as particles. This dimensional dependence results from requiring the worldsheet quantum field theory to be conformally invariant, with the line of reasoning being as follows: the string dynamics are postulated to obey the so-called Polyakov action, which couples the quantum field X to the metric h on the worldsheet. For future reference, we note that the coupling constant of this theory is denoted as α' and is related to the string length scale by $l_s = 2\pi\sqrt{\alpha'}$. A key property of the Polyakov action is its invariance under two-dimensional diffeomorphisms as well as Weyl transformations, the latter of which are local rescalings of the metric h. These symmetries can be used to gauge-fix the action into a form that is invariant under two-dimensional conformal transformations. This gauge-fixed action describes the theory of d free bosons X^N, where N is a space-time index, plus a set of Faddeev–Popov ghost fields introduced by gauge-fixing. The latter constitute an independent conformal field theory with central charge $c = -26$. Since conformal symmetry arose as a part of the gauge symmetry, it is necessary to cancel the conformal anomaly, i.e., the total central charge c_{tot} needs to vanish. This can be achieved by noting that each X^N adds $c = 1$, such that we get the desired result $c_{\text{tot}} = -26 + d \cdot 1 = 0$ in $d = 26$ space-time dimensions.

We now specialize to $d = 26$ — referred to as the critical dimension of string theory — and analyze the spectrum of closed string excitations. Most importantly, it exhibits a collection of excitations that transform as a symmetric massless rank-two tensor in space-time. These are precisely the properties of a graviton, which hints at string theory's capacity to incorporate gravity. It further gives a massless antisymmetric tensor as well as a massless scalar, the latter of which is referred to as dilaton. In addition, there is a whole tower of massive excitations with various tensor structures and masses proportional to the string mass scale $M_s = \sqrt{1/\alpha'}$. These states are only relevant at energies comparable to the inverse string length scale l_s and therefore typically not considered in a low energy approximation. The entirely unexcited string corresponds to a state of negative squared mass, i.e., it is a so-called tachyon and signals an instability of the theory. Moreover, all excitations are space-time bosons. The here outlined *bosonic* string theory can thus not be a complete description of nature.

String Perturbation Theory

Before further addressing these problems, let us recapitulate in what string theory differs from the more familiar framework of four-dimensional relativistic quantum field theory. Certainly, the notion of particles arises in a different way. In string theory they do not appear as excitations of a four-dimensional quantum field on space-time, but rather as excitations of the string. The latter is described as a *two-dimensional* quantum field *on the worldsheet*, and we have so far introduced string theory as the quantum theory of this field and its excitations. The string itself is an object that moves within space-time and as such is treated on similar footing as particles in first quantization.

From a more formal point of view, one might argue that string theory appears to be 'just' a particular two-dimensional quantum field theory. This is not true, a conceptual difference arises in the prescription with which scattering amplitudes are calculated. In quantum field theory in- and out-going particles are represented by asymptotic in- and out-states, and transition amplitudes between these are typically calculated perturbatively by summing over all allowed Feynman diagrams. These depict processes of interaction between particles, including their creation and annihilation. Perturbation theory corresponds to an expansion in a set of coupling constants and its order is given by the number of interaction vertices in the diagram or, equivalently, the number of loops. This prescription can be derived for example from path integral quantization and relies on the framework's ability to describe the creation and destruction of particles. The path integral specifies transition amplitudes even non-perturbatively.

The construction of a similar non-perturbative formalism for the interaction of strings that includes their creation and annihilation — referred to as string field theory, see for instance the review [5] — remains an open research question. While for some particular backgrounds the AdS/CFT correspondence [6] might be argued to provide a non-perturbative definition of string theory, in the general case the rules for calculating scattering amplitudes are imposed by hand. In- and out-going strings are, similar to the quantum field theory setup, represented by asymptotic in- and out-states. These states additionally carry the information about the vibrational degrees of freedom, thereby specifying which particles are scattered from a space-time point of view. Transition amplitudes are calculated perturbatively by summing over all worldsheet topologies in the sense depicted in Figure 1.1 for the scattering of four closed strings. These diagrams can be interpreted to represent

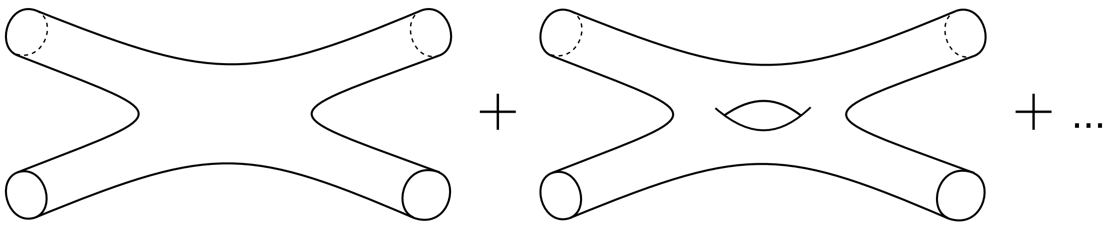

Figure 1.1: This picture illustrates the perturbation series for calculating the scattering amplitude between four closed strings.

the joining and splitting of strings and differ in the number of their 'holes', mathematically known as the genus g of the surface. On each such surface 'lives' a worldsheet quantum field theory and the corresponding diagram is calculated by a path integral, which involves integration over all string maps X and worldsheet metrics h that are compatible with the topology of the respective surface. This perturbation series corresponds to an expansion in the string coupling constant g_s and the order of perturbation theory is counted by the Euler characteristic $\chi = 2g - 2$ of the surface. However, g_s can be shown to be the space-time background value of the dilaton and is thus not a free dimensionless parameter. It is precisely this prescription of summing over worldsheet genera that elevates string theory to be more than the two-dimensional quantum field theory of the string excitations. Note that this definition of string theory is intrinsically perturbative in the coupling constant g_s, as opposed to the non-perturbative definition of quantum field theories.

Superstring Theory

As we saw above, bosonic string theory exhibits a state of negative squared mass and does not describe space-time fermions. Both problems are solved in *superstring theory*. It is obtained by extending the bosonic theory to exhibit supersymmetry on the worldsheet, thereby introducing additional fermionic fields with excitations that behave as space-time fermions. The possible structure of these theories is severely constrained and in flat space-time there are exactly five consistent superstring theories: type I, type IIA and IIB, heterotic SO(32) and heterotic $E_8 \times E_8$.

In this thesis the type II theories figure prominently and will in the next chapter be explained in greater detail. We here content ourselves with stating a few key facts: After gauge-fixing local symmetries, their closed string sector is described by a $N = (2, 2)$ superconformal field theory on the worldsheet. In the present setup this is the theory of d free bosons and their d fermionic superpartners, plus a ghost field sector which contributes $c = -15$ to the central charge. Since a free fermion adds $c = 1/2$, the total central is $c_{\text{tot}} = -15 + d \cdot (1 + 1/2)$ and vanishes in $d = 10$ space-time dimensions. This also implies space-time Poincaré invariance. For later reference we note that the type II spectra contain several massless antisymmetric tensor fields that are known as p-form gauge fields. If not made explicit otherwise, when speaking of 'string theory' without attribute we refer to the superstring theories of type II.

D-Branes

In the introduction of open strings we have so far ignored an important detail: to fully determine their behavior, one needs to specify boundary conditions at their endpoints. The simplest possibilities are of two types. First, there are Dirichlet conditions that fix the space-time position of the endpoints. Second, Neumann conditions set the momentum flowing of the string's ends to zero. While the time direction is (typically) chosen to obey Neumann conditions, the conditions for the spatial directions can be chosen independently. Let us consider an open string with $(p + 1)$ von Neumann and $(9 - p)$ Dirichlet directions, where $0 \leq p \leq 9$. This choice confines the string's endpoints to a $(p + 1)$-dimensional subspace of space-time. Further, translational symmetry is broken along the $(9-p)$ Dirichlet directions and hence the corresponding momenta are not conserved. In order to ensure this symmetry breaking to be spontaneous, the $(p + 1)$-dimensional subspace itself needs to be interpreted as a dynamical object that absorbs the outflowing momentum. These entities are referred to as Dp-branes, they are extended along p space directions and in space-time sweep out a $(p + 1)$-dimensional so-called world-volume. In the type II theories they are essential for obtaining vector fields with non-Abelian gauge symmetry. The tension of Dp-branes can be calculated to scale with $1/g_s$, such that they are non-perturbative from the string perturbation theory point of view. It is also possible to impose Dirichlet conditions in the time directions, which leads to the notion of D-instantons that only exist at a given point in time.

Effective Space-Time Description

String perturbation theory allows the calculation of scattering amplitudes between the various string excitations and, in particular, can be used to derive the low energy interactions among all massless particles in the string spectrum. In order to obtain a better space-time interpretation one determines a ten-dimensional quantum field theory on $\mathbb{M}^{10}$ that reproduces these low energy results. The latter is referred to as the (low energy) effective space-time theory, which for type IIA and IIB string theory are the supergravity theories of type IIA and IIB. Note that the full string theory corrects this leading order result in *two* ways. First, there are corrections in the string coupling g_s due to summing over worldsheets with genera $g \geq 1$. Higher order terms correspond to processes with virtual string loops. The leading order, $g = 0$, is similar to a tree level approximation in quantum field theory and in this sense sometimes referred to as *classical limit*. Additionally, there are non-perturbative effects that scale with inverse powers of g_s, for example the Dp-branes discussed above. Second, there are corrections in the worldsheet coupling constant α'. Since α' is related to the string length scale by $l_s = 2\pi\sqrt{\alpha'}$, terms of higher order in α' are due to the string having a finite spatial extension. The leading order thus corresponds to sending the string length to zero and is referred to as the *field theory limit*. Corrections can be both perturbative and non-perturbative.

Compactification

Having learned about the closed string excitations, we can reinterpret the choice of space-time as choosing a background value for the graviton field. In the above discussion we chose d-dimensional Minkowski space-time $M = \mathbb{M}^d$, which for superstring theories led to the remarkable conclusion that $d = 10$. This is, however, in conflict with our experience of living in four and not in ten macroscopic space-time dimensions. Clearly, a different choice is necessary.

The so-called ansatz of *compactification* amounts to writing space-time as the product of four-dimensional Minkowski space and a six-dimensional compact manifold M_c referred to as the internal space, i.e., to the choice $M = \mathbb{M}^4 \times M_c$. Provided the internal space is small enough, current experiments will not be able to resolve its spatial extension. This is analogous to a two-dimensional sheet of paper appearing as one-dimensional when being curled up tightly and observed from a distant point of view. We can thus very well think to live in four dimensions, while the strings still propagate in the full ten-dimensional space-time. Although the internal space might not be directly detectable, its shape nevertheless has significant observable consequences: the four-dimensional low energy effective theory that governs physics in $\mathbb{M}^4$ is obtained by the dimensional reduction of the ten-dimensional effective space-time theory on the manifold M_c. Different choices of M_c result in different four-dimensional effective theories and it is a well posed physical problem to search for an internal space that yields a phenomenologically viable scenario.

To gain some intuition, let us recall the perhaps historically first example of compactification. This is the so-called Kaluza–Klein theory, which considers five-dimensional general relativity on the product space $\mathbb{M}^4 \times S^1$. By dimensional reduction along the circle, the internal space of this example, the four-dimensional physics can be demonstrated to be gravity coupled to electrodynamics (a massless vector field) plus an additional scalar. The appearance of massless scalars is also a generic feature of string compactifications, these fields are referred to as *moduli* and are in conflict with the non-observation of any massless scalars in nature. There are approaches that try to avoid the existence of moduli, for example so-called flux-compactifications in which one turns on non-zero background fluxes for the p-form gauge fields [7], see also the reviews [8–10].

We do not consider these approaches and turn off such background fluxes. String theory at leading order in α' then predicts the vacuum Einstein equations in space-time. This is tantamount to conformal symmetry on the worldsheet and requires the internal space to be Ricci-flat. On top of that, not much can be said about M_c in general. A generic choice will break space-time supersymmetry entirely, which is actually the phenomenologically favored situation. However, calculational control is largely lost and the effective four-dimensional theory on Minkowski space $\mathbb{M}^4$ cannot be derived. It is therefore common to consider cases that retain one quarter, i.e., eight supercharges. This requires the internal space M_c to have (exactly) one covariantly constant spinor, which identifies M_c as a so-called Calabi–Yau manifold. The effective four-dimensional theory then exhibits $\mathcal{N} = 2$ space-time supersymmetry that is assumed to be dynamically broken at some lower energy scale independent of the compactification.

Let us now focus on the case of eight unbroken supercharges and employ a worldsheet point of view. Recall that, when ignoring the fixed ghost sector, the closed string sector of type II superstring theory is described by a two-dimensional $N = (2, 2)$ superconformal field theory with central charge $c = 15$. Corresponding to the initial choice of ten-dimensional Minkowski space-time, this above was the theory of ten free bosons and their superpartners. The choice a different space-time translates into the choice a different $N = (2, 2)$ superconformal field theory with appropriate central charge. In particular, compactifications to four-dimensional theories with eight supercharges can be realized by decomposing the worldsheet theory into two separate factors. First, corresponding to the well-known four-dimensional Minkowski space $\mathbb{M}^4$, the theory of *four* free bosons plus their superpartners. Second, some *internal* two-dimensional $N = (2, 2)$ superconformal theory with $c = 9$. Contact with the space-time picture is made by the observation that these superconformal theories may (but also may not) admit a geometric interpretation as a non-linear sigma model on some manifold. This manifold is referred to as the target space of the superconformal theory and is identified with the internal space $\mathcal{M}_c$. Such a geometric interpretation is, however, not necessary for the formalism. Moduli are also generically present in this worldsheet approach to compactification and here correspond to so-called marginal couplings. These are terms that continuously deform the theory, such that we actually deal with an entire *family* of superconformal theories. We note that a potential target space interpretation is in any case strictly valid at certain boundary components of the deformation space only, where quantum corrections in α' are strongly suppressed. After all, the condition of Ricci-flatness is a leading order result that is corrected at the four-loop level and by worldsheet instantons [11–13]. Superconformal field theories can vary significantly with their moduli and the study of this dependence is a central topic of this thesis.

Gauged Linear Sigma Model

This discussion demonstrates the central role that two-dimensional $N = (2, 2)$ superconformal field theories play in compactifications of type II superstring theories. Explicitly known examples that are suitable for type II string theories are free theories, orbifolds and Gepner models [14, 15].

In the important work [16] Witten introduced a powerful method to indirectly construct a variety of further examples of the desired type. He proposes to employ a certain gauge theory, known as the gauged linear sigma model, that is chosen to closely resemble the properties of the superconformal theories: it is two-dimensional, exhibits the correct amount of supersymmetry, features two classical U(1) symmetries and depends on a set of coupling constants. His key insight was that, as long as the gauge theory spectrum is chosen appropriately, these coupling constants are renormalized by a finite amount only and thus can be interpreted as free parameters that continuously deform theory. The same choice guarantees the U(1) symmetries to be non-anomalous and as a result the renormalization group drives the theory to a family of non-trivial $N = (2, 2)$ superconformal field theories in the infrared (IR). While the renormalization is in general not traceable, certain quantities are largely protected by supersymmetry. The strategy therefore is to calculate such protected quantities within the ultraviolet (UV) gauge theory and to determine what they correspond to in the IR superconformal field theory. This opens the possibility for an indirect study of the latter. The free parameters of the UV gauge theory correspond to the moduli of the superconformal theory, such that the gauged linear sigma model can be used to extrapolate between different points in the moduli space of the superconformal theory. In addition, the IR central charge can be immediately read off from the defining UV gauge spectrum. This allows us to deliberately engineer exactly those superconformal theories that correspond to compactifications with four-dimensional $N = (2, 2)$ effective theories. A potential target space interpretation arises naturally within the formalism.

We stress that the gauged linear sigma model can be employed in a wider context. First, the IR superconformal theory can have central charges $c \neq 9$. This corresponds to compactifications with other than four macroscopic space-time dimensions. Second, it can also be used to study target space geometries that do not give rise to families of superconformal field theories.

Picard–Fuchs Operators

As a second powerful tool for analyzing the moduli structure of superconformal field theories we will frequently employ so-called Picard–Fuchs operators. These differential operators capture the dependence of the manifolds of our interest on their moduli, in particular, on the choice of complex structure and Kähler class. To understand the connection to the previous discussions, consider a superconformal field theory with geometric target space interpretation. The complex structure moduli of this target space exactly agree with several moduli of the superconformal theory. In case of the Kähler moduli such an identification is — due to the above mentioned worldsheet corrections — valid at leading order in α' only, which can be interpreted as a deformation of the geometric Kähler moduli space by quantum effects. While Picard–Fuchs operators arise naturally from a target space picture, their validity extends beyond regions with a geometric interpretation and they capture global properties of the moduli space. As we will demonstrate, structures similar to Picard–Fuchs operators arise within the gauged linear sigma model even without any reference to a target space.

On the Research Presented in This Thesis

This thesis combines the work of different research projects, all of which are related to compactifications of type II superstring theories and have the common objective to improve our understanding of the superconformal worldsheet theories arising in this context. Our aim is not an immediate construction of a phenomenologically viable model, but rather to enhance our conceptual understanding and to draw general lessons through a careful study of toy models. As main tools we will frequently employ gauged linear sigma model techniques as well as Picard–Fuchs operators.

The thesis presents new ways in which these tools encode the moduli structure of the superconformal field theories. Moreover, we establish direct connections between the two approaches that seem to diminish the role of knowing about a potential target space geometry. A more detailed overview now follows.

Outline of the Thesis

The following list briefly summarizes the remaining parts of the thesis. For benefit of an expert reader we here refer to concepts beyond those introduced above.

- **Chapter two** elaborates on the above concepts and reviews the techniques that underly the original research presented in the later chapters. We first discuss superstring theories of type IIA and IIB in more detail, with a focus on their compactification to four macroscopic space-time dimensions. This motivates the introduction of the gauged linear sigma model, where we put particular emphasis on demonstrating that its properties are chosen to closely resemble the $N = (2, 2)$ superconformal field theories that appear in type II compactifications. We also discuss the low energy dynamics of the model and explain how the notion of a target space arises. Lastly, we introduce Picard–Fuchs operators as a central tool for studying the moduli dependence of the superconformal field theories of our interest.

- The **third chapter** is based on the author's publication [17]. It is devoted to a detailed study of certain correlation functions in the gauged linear sigma model. Our results demonstrate that this set of field theory observables encodes the moduli structure of the IR superconformal field theory to a large extent. Remarkably enough, this does not require an explicit geometric construction of a potential target space and, for the most part, not even knowledge thereof. We begin by showing that correlation functions with different powers of field insertions are not mutually independent but rather subject to strong linear relations. Our proof is constructive and yields an algorithm for deriving these dependencies in concrete examples. This is a pure gauge theory result and does not require the existence of an IR family of superconformal field theories. In that special case, however, there is a deep connection to the moduli structure of the superconformal theory. In particular, we show that the set of linear dependencies between correlation functions corresponds to the ideal of Picard–Fuchs operators governing the target space quantum geometry. By combining these two results we obtain an elementary algorithm to determine the Picard–Fuchs operator of a given model, which as input only requires the defining gauge theory data, i.e., the choice of gauge group and matter spectrum. Neither does it require the potentially involved calculation of field theory observables nor any knowledge about an associated geometry. For fixed classes of target space geometries, specified by their dimension and number of Kähler moduli, we lastly derive formulas that express the Picard–Fuchs operators in terms of correlation functions.

- In **chapter four** we continue to study the connection between gauged linear sigma models and Picard–Fuchs operators, with the aim of finding a practical method to determine the operators associated to non-Abelian models in a computationally efficient and straightforward way. As central result we propose a formula that gives a closed form expression for the fundamental period — i.e., the holomorphic solution to the differential equation defined by the Picard–Fuchs operator — of gauged linear sigma models with arbitrary non-Abelian gauge groups and large classes of chiral matter spectra. Given a concrete model, the Picard–Fuchs operator is easily found by requiring it to annihilate the holomorphic solution as determined by this formula. The derivation heavily relies on the so-called Givental I-function [18], which is a concept that we will introduce in chapter two and also briefly employ in the third chapter. We further comment on the idea of reconstructing gauged linear sigma models from given Picard–Fuchs operators.

- In the **fifth chapter** we apply the concepts introduced in the earlier parts of the thesis to study quantum periods of Calabi–Yau fourfolds that arise as target spaces of non-Abelian gauged linear sigma models. Contrary to a widely spread belief, their quantum cohomology need not be fully generated by marginal deformations in the chiral–anti-chiral ring of the two-dimensional superconformal worldsheet theory. Our focus is on models with a single Kähler parameter, in case of which this phenomenon results in a non-factorizable Picard–Fuchs operator of order six or higher. We explain why and when this effect arises, and demonstrate how so-called integral quantum periods — a special choice of solution to the Picard–Fuchs differential equation — can be determined for such operators of non-minimal order. This enables us to construct interesting new types of flux-induced superpotentials that can, for example, be purely instanton generated. Lastly, we explain how the integral periods determine geometrical invariants such as instanton numbers. The chapter is based on the author's publication [19].

- **Chapter six** provides a short summary of the thesis. We also comment on open questions and propose directions for future research.

CHAPTER 2

Basics of Type II Superstrings and Their Compactification

In this chapter we review several physical and mathematical concepts that form the basis for the research presented in the later chapters of the thesis. The first part begins with a short introduction to superstring theories of type IIA and IIB, including their ten-dimensional massless spectrum. We then introduce the concepts of compactification and dimensional reduction in a geometric setting and explain how the choice of internal space influences the effective low energy theory. Our focus is on $\mathcal{N} = 2$ supersymmetric effective theories in four space-time dimensions, which leads us to consider complex three-dimensional Calabi–Yau manifolds. The second part of this chapter defines the gauged linear sigma model, with an emphasize on how it properties closely resembles those of the string worldsheet theories. We discuss its low energy behavior and demonstrate how a potential target space interpretation arises. The third part is devoted to an introduction of Picard–Fuchs operators.

The intention of this chapter is pedagogical and to not require prior research knowledge. That being said, we may use some concepts that not every reader is familiar with. For an accessible introduction to various notions from differential geometry and topology we recommend ref. [20] and for detailed expositions of string and conformal field theory refer to the textbooks [4, 21–25] and [26, 27]. References [28, 29] are further valuable resources.

2.1 Type II Superstring Theories

We here give a short introduction to superstring theories of type II, which form the physical foundation for the research presented in the later parts of this thesis. Our review follows the textbooks cited in the previous paragraph.

2.1.1 Worldsheet Description

Let us begin with recalling some basic notions from the previous chapter. The fundamental string is a one-dimensional object that moves in space-time M, which for now we assume to be $M = \mathbb{M}^d$, i.e., d-dimensional Minkowski space. It sweeps out the two-dimensional worldsheet Σ and is described by maps $X : \Sigma \to M$ whose individual space-time components are X^N. Their dynamics is governed by the Polyakov action that for $d = 26$ exhibits Poincaré invariance, in case of which the excitations of

X^N can be interpreted as particles. However, this only yields bosonic particles and there is a state with negative squared mass. Both problems are solved in superstring theory, of which there are five consistent types in flat space-time. These are known as type I, type IIA and type IIB, as well as heterotic $E_8 \times E_8$ and heterotic SO(32).

In this thesis we consider the type II theories only, both of which do not contain open strings. Their worldsheet description is obtained by supersymmetrizing the Polyakov action to exhibit $N = (2, 2)$ worldsheet supersymmetry, which means there are two left as well as two right moving supercharges — for a more precise discussion of this notion see subsection 2.2.1, although this is not necessary to follow the below discussion. This action is invariant under various local worldsheet symmetries, including diffeomorphisms and Weyl transformations. These can be used to bring the action into the form

$$S = -\frac{1}{8\pi} \int_\Sigma d^2x\, \eta_{MN} \left[\frac{2}{\alpha'} \left(\partial_\alpha X^M\right) \left(\partial^\alpha X^N\right) + 2i\overline{\psi}^M \rho^\alpha \partial_\alpha \psi^N \right], \tag{2.1}$$

referred to as superconformal gauge. Here α is a worldsheet index, ρ^α are the Gamma matrices in two dimensions, α' is the worldsheet coupling constant and η_{MN} is the space-time Minkowski metric. The ψ^N are superpartners of X^N, they are Weyl fermions on the worldsheet and space-time vectors. Equation (2.1) constitutes a conformal field theory with central charge $c = d \cdot (1 + 1/2)$, where each X^N and ψ^N respectively contributes $c = 1$ and $c = 1/2$. The Faddeev–Popov ghosts that are introduced by gauge fixing add $c = -15$, such that the total conformal anomaly vanishes for $d = 10$. This is necessary since conformal symmetry arose as part of gauge symmetries, and moreover implies space-time Poincaré invariance.

2.1.2 Massless Spectrum in Ten Dimensions

We now specialize to type II theories on ten-dimensional Minkowski space and briefly explain their massless space-time spectra. The equations of motions for X^N and ψ^N deduced from eq. (2.1) are solved by a Fourier expansion whose coefficients are turned into operators by canonical quantization. These act on an oscillator ground state and thereby create a tower of excited states. The local symmetries translate into a set of constraints that define a subspace of physical states with positive norm and the boundary conditions of the closed string require an equal number of excitations in the left and right moving sector. Since ψ^N are worldsheet fermions, they may obey periodic or anti-periodic boundary conditions along the spatial extension of the closed string. Whereas space-time Poincaré invariance requires that the boundary condition are the same for all space-time directions N, the choice can be made independently for the two spinor components ψ_+^N and ψ_-^N. The states thus fall into four sectors, abbreviate as (R,R), (NS,NS), (R,NS) and (NS,R). Here, ‘R’ is for ‘Ramond’ and amounts to periodic conditions, whereas ‘NS‘ stands for ‘Neveu-Schwarz’ and denotes anti-periodic conditions. States in the first two sectors behave as space-time bosons, the third and forth sector are fermionic.

The space-time spectrum obtained in this way still features a tachyon and is non-supersymmetric. These problems are cured by the GSO projection, which uses the fact that the states carry a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quantum number and amounts to only retaining states of particular charges with respect to this symmetry. There are two inequivalent choices of such a projection and these respectively define type IIA and type IIB superstring theory. They agree in the (NS,NS) sector and differ in the others. The truncation is consistent: no states that were projected out are created through scattering of those states that were retained.

Type IIA

The massless excitations of type IIA superstring theory are listed in Table 2.1, including their number of on-shell degrees of freedom (d.o.f.). This spectrum agrees with that of ten-dimensional type IIA supergravity, which is the low energy effective space-time description. As signaled by the presence of two gravitini and an equal number of bosonic and fermionic degrees of freedom, the spectrum exhibits $\mathcal{N} = 2$ space-time supersymmetry corresponding to a total of 32 supercharges. Since the gravitini are of opposite chirality, the theory is non-chiral.

Sector	Type	On-shell d.o.f	Name
(NS, NS)	graviton	35	g
	2-form field	28	B
	dilaton	1	ϕ
(R, R)	3-form field	56	$C^{(3)}$
	1-form field	8	$C^{(1)}$
(NS, R)	gravitino of chirality +	56	λ_+
	dilatino of chirality −	8	η_-
(R, NS)	gravitino of chirality −	56	λ_-
	dilatino of chirality +	8	η_+

Table 2.1: The massless spectrum of type IIA superstring theory and type IIA supergravity in ten-dimensional Minkowski space.

A p-form gauge field ω is a higher-dimensional generalization of the familiar vector field. Its p indices are fully antisymmetric and its field strength $d\omega$ is a $(p+1)$-form. In this language vector fields are 1-form fields and scalars can be regarded as 0-form fields. We note that the Hodge dual field strength $*d\omega$ in ten dimensions is a $(9-p)$-form, which can be reinterpreted as the field strength of a $(8-p)$-form. A p-form field is thus dual to a $(8-d)$-form field and they respectively represent electric and magnetic degrees of freedom.

Type IIB

The massless spectrum of the type IIB theory is summarized in Table 2.2. It exhibits $\mathcal{N} = 2$ supersymmetry in space-time and, due to both gravitini having the same chirality, is chiral. The 4-form gauge field $C^{(4)}$ obeys the self duality constraint $*dC^{(4)} = dC^{(4)}$ and the low-energy effective space-time theory is ten-dimensional type IIB supergravity.

D-Branes

Recall from the introduction that in addition to the fundamental string there are the Dp-branes, which are non-perturbative in the string coupling g_s. While type IIA string theory features Dp-branes with p even, the branes in type IIB have p odd. Since the world-volume of a Dp-brane is $(p+1)$-dimensional, the brane naturally couples to $(p+1)$-form gauge fields from the (R,R) sector (as well as their duals). This is analogous to point-particle electrodynamics, where the 1-form vector field couples to the one-dimensional worldlines of particles. We also note that the 2-form field B from the (NS,NS) sector always couples to the fundamental string.

Sector	Type	On-shell d.o.f	Name
(NS, NS)	graviton	35	g
	2-form field	28	B
	dilaton	1	ϕ
(R, R)	self dual 4-form field	35	$C^{(4)}$
	2-form field	28	$C^{(2)}$
	0-form field	1	$C^{(0)}$
(NS, R)	gravitino of chirality +	56	λ_1
	dilatino of chirality −	8	η_1
(R, NS)	gravitino of chirality +	56	λ_2
	dilatino of chirality −	8	η_2

Table 2.2: The massless spectrum of type IIB superstring theory and type IIB supergravity in ten-dimensional Minkowski space.

2.1.3 Compactification

In subsection 2.1.1 we saw that superstring theory on flat d-dimensional Minkowski space requires $d = 10$. This is in conflict with our experience of living in four and not ten space-time dimensions, such that a more general space-time ansatz needs to be made. The ansatz of compactification amounts to choosing

$$M = \mathbb{M}^4 \times M_c \,, \tag{2.2}$$

which decomposes space-time into the product of four-dimensional Minkowski space with a real six-dimensional manifold M_c. The latter is referred to as the internal space and can, intuitively, not be observed if its spatial extension is below the currently accessible length scales. While there are approaches to obtain large extra dimensions from string theory [30–32], we follow the intuition and assume M_c to be small and compact. From the space-time point of view the ansatz (2.2) amounts to the non-trivial vacuum expectation value $\langle g \rangle = g_M$ for the graviton, where g_M is the metric on M. From the worldsheet point of view it requires a generalization of the action (2.1) to the so-called non-linear sigma model, in which the space-time Minkowski metric η is replaced by g_M. We do not consider compactifications in which other fields than the metric have non-trivial background fluxes.

This is a good place to introduce some notation. Indices of the full space-time M are denoted with capital latin letters, for the internal space M_c we use lower case latin letters and Greek letters for four-dimensional Minkowski space.

Conformal Invariance at Leading Order in α'

String theory requires conformal invariance. In the non-linear sigma model with a general metric g this symmetry is broken by a non-zero beta function $\beta(g)$ for the metric. It is therefore necessary to impose the constraint $\beta(g_M) = 0$ on g_M and in extension on the internal space M_c, which is sometimes referred to as the string or supergravity equation of motion. A perturbative calculation [33] shows that the one-loop beta function is proportional to the Ricci tensor $\mathrm{Ric}(g_M)$, such that at leading order in α' string theory requires that M_c is a Ricci flat manifold.

Worldsheet Supersymmetry

An essential ingredient of ten-dimensional type II theories is $N = (2, 2)$ supersymmetry on the worldsheet. In absence of any other background fluxes this symmetry requires M and hence M_c to be a complex Kähler manifold [34]. This means that the metric on M_c can be put into a Hermitian form and the associated Kähler form ω is closed.

Space-time Supersymmetry

A general internal space will break all of the 32 space-time supercharges, in case of which there is little computational control. Demanding some amount of unbroken space-time supersymmetry requires M_c to have a covariantly constant spinor, which in turn implies M_c to be a complex Kähler manifold and restricts its holonomy group from SO(6) to SU(3). If the holonomy is exactly SU(3), i.e., no proper subgroup thereof, there is exactly one covariantly constant spinor. This keeps one quarter of the space-time supersymmetry, i.e., eight supercharges and thus results in $\mathcal{N} = 2$ supersymmetry of the effective four-dimensional theory. If the holonomy drops further, more space-time supersymmetry is retained: exact SU(2) holonomy keeps 16 supercharges and trivial holonomy keeps all 32 of them, which respectively correspond to $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories in four dimensions. By turning on background values for the field strengths of the p-form gauge fields it is also possible to obtain type II compactification with $\mathcal{N} = 1$ supersymmetry in four dimensions [7].

Complex n-dimensional Kähler manifolds with exact SU(n) holonomy are known as n-dimensional Calabi–Yau manifolds, in short Calabi–Yau n-folds. In case the holonomy may be SU(n) *or* a proper subgroup thereof, we speak of a *generalized* Calabi–Yau manifold. As an important fact, generalized Calabi–Yau manifolds are Ricci flat. Provided the compactification has a geometric interpretation as non-linear sigma model on a manifold — which we assume — the requirement of some unbroken space-time supersymmetry thus implies worldsheet supersymmetry and leading order conformal invariance. The reverse is also true: a Ricci flat complex Kähler manifold is a Calabi–Yau manifold, such that leading order conformal invariance and worldsheet supersymmetry imply some amount of unbroken space-time supersymmetry. If the compactification has no geometric interpretation, the situation is more subtle. Worldsheet supersymmetry and leading order conformal invariance are then not enough to ensure space-time supersymmetry, but need to be supplemented by a condition on the charges in the theory such that the spectral flow operator is well defined [35–37], see also the review [28].

Corrections of Higher Order in α'

We now specialize to the case in which M_c is a Calabi–Yau threefold and explain the effects of higher-loop corrections to the beta function $\beta(g_M)$ as established by the works [11–13, 38–41]. Since the two- and three-loop contributions can be written as covariant derivatives of the Ricci tensor, they automatically vanish on Ricci flat manifolds and in this sense do not modify the leading order conclusion. However, at four-loop order there is a new contribution to the beta function that does not vanish on Ricci flat manifolds. This modifies the condition for conformal invariance, $\beta(g_M) = 0$, and string theory requires M_c to deviate from the Calabi–Yau geometry. There are also non-perturbative corrections by worldsheet instantons [13, 42–45]. All these corrections are strongly suppressed when the size of M_c tends to infinity, such that M_c in this limit remains to be a Calabi–Yau manifold. We note that the preceding perturbative discussion and a geometric interpretation altogether are valid only in the 'vicinity' of such large volume boundary limits. The precise meaning of 'vicinity' will be clarified later.

Despite all these corrections to g_M, its Kähler class $[\omega]$ is largely protected by supersymmetry. It is effected at one-loop order only with a correction proportional to the first Chern class $c_1(M_c)$ of M_c. The latter is the trace of the curvature class and vanishes on Ricci flat manifolds, in case of which $[\omega]$ is not corrected at all. This is particularly useful due to Yau's theorem [46], which states that a complex Kähler manifold with Kähler form ω and vanishing first Chern class can always be equipped with a unique Ricci flat metric whose Kähler class is $[\omega]$. It is believed that there similarly exists a unique metric g_* which fulfills the string equation of motion $\beta(g_*) = 0$ and whose Kähler class is $[\omega]$, at least in the vicinity of a large volume boundary limit.

Dimensional Reduction

The purpose of the compactification ansatz (2.2) is to conform to our experience of living in four space-time dimensions. Although the internal space is chosen small enough in order to not be directly observable, it strongly influences the four-dimensional effective low energy theory. Namely, the latter is obtained by dimensionally reducing the ten-dimensional theory on M_c and we now demonstrate that even the spectrum of the four-dimensional theory depends on the choice of internal space. Since we are interested in an effective low energy theory (and do not consider cases in which the string mass scale is significantly smaller than the Planck scale), we can restrict our attention to fields that are massless in ten dimensions.

Let us first consider a scalar field $\phi_{10}(x, y)$. It depends on coordinates x on $\mathbb{M}^4$ as well as coordinates y on the compact manifold M_c and obeys the ten-dimensional equation of motion $\Delta_{10}\,\phi_{10} = 0$. With the product ansatz in eq. (2.2) this decomposes as

$$0 = \Delta_{10}\,\phi_{10}(x, y) = \Delta_4\,\phi_{10}(x, y) + \Delta_6\,\phi_{10}(x, y)\,, \tag{2.3}$$

where the Laplace operators Δ_4 of $\mathbb{M}^4$ and Δ_6 of M_c respectively only act on x and y. This equation is solved by expanding ϕ_{10} in a basis of eigenfunctions α_n of the six-dimensional Laplace operator,

$$0 = \Delta_{10}\left[\sum_{n=0}^{\infty} \phi_{4,n}(x)\,\alpha_n(y)\right] = \sum_{n=0}^{\infty} \alpha_n(y)\,\left[\Delta_4 + \lambda_n\right]\,\phi_{4,n}(x)\,, \tag{2.4}$$

where λ_n is the eigenvalue of α_n. These are non-negative, discrete and have finite multiplicity — see for instance the textbook [47] for mathematical background. Independence of the eigenfunctions requires every term on the right hand side to vanish separately, such that the single ten-dimensional field ϕ_{10} gives rise to an entire tower of four-dimensional fields $\phi_{4,n}$ with masses proportional to $\sqrt{\lambda_n}$. Since the positive eigenvalues grow with the inverse of the typical length scale of M_c, the massive fields of this so-called Kalazu–Klein tower decouple and only the massless $\phi_{4,n}$ remain in the low energy theory. There is one such field per linearly independent solution to the equation $\Delta_6\,\alpha_n = 0$, which states that α_n is a harmonic function on M_c. Since on compact manifolds only constant functions are harmonic and since M_c is connected, ϕ_{10} gives rise to exactly one massless scalar in the four-dimensional theory.

Second, let us generalize to a ten-dimensional massless p-form gauge field $A^{(p)}(x, y)$. This field still obeys the equation of motion $\Delta_{10}\,A^{(p)}(x, y) = 0$ but now additionally carries p totally antisymmetric space-time indices. We employ the decomposition

$$A^{(p)}(x,y) = \sum_{n=0}^{\infty} \sum_{q=\max(0,\,p-6)}^{\min(p,\,2)} A_{4,n}^{(q)}(x) \wedge \alpha_n^{(p-q)}(y)\,, \tag{2.5}$$

where the $\alpha_n^{(k)}$ are k-eigenforms of Δ_6 and the $A_{4,n}^{(q)}$ are q-form gauge fields in four dimensions. The summation over q is constrained because real n-dimensional manifolds do not support forms of degree $(n+1)$ or higher. Moreover, 3- and 4-form fields are non-dynamical in four dimensions and therefore excluded from the decomposition. The single ten-dimensional $A^{(p)}(x,y)$ thus gives rises to b^{p-q} four-dimensional q-form fields, where b^{p-q} is the number of linearly independent harmonic $(p-q)$-forms on M_c. Since harmonic forms are defined through the Laplace operator, they depend on the metric and are difficult to determine. At this point it beneficial to employ the Hodge theorem, which states that the space of harmonic k-forms on M_c is isomorphic to the k-th de Rahm cohomology group $H^k_{\mathrm{dR}}(M_c)$ of M_c. This identifies b^k as the k-th Betti number, i.e., the dimension of $H^k_{\mathrm{dR}}(M_c)$. The cohomology groups are topological and do not depend on the details of the metric. In light of the higher order α' corrections it is anyway more natural to work with such quantities.

These considerations explain the dimensional reduction of all bosonic degrees of freedom except for the graviton. In order to understand this and to determine the Betti numbers, we now discuss some topological and geometric properties of Calabi–Yau threefolds.

2.1.4 Basic Properties of Calabi–Yau Threefolds

The previous subsection has demonstrated that, although they are not exact solutions to the string equation of motion, Calabi–Yau threefolds are a good starting point for studying type II string compactifications with $\mathcal{N}=2$ space-time supersymmetry in four dimensions. We here briefly summarize their relevant topological and geometric properties, which we base on the more exhaustive and still accessible review [28].

Cohomology

We saw that the Betti numbers b^p of M_c, i.e., the real dimensions of the de Rahm cohomology groups $H^p_{\mathrm{dR}}(M_c)$ of the internal space M_c determine the number and type of massless fields in four dimensions. Since M_c is a Kähler manifold, its de Rahm cohomology groups split into the direct sum

$$H^p_{\mathrm{dR}}(M_c) = \bigoplus_{r+s=p} H^{r,s}(M_c) \tag{2.6}$$

of the Dolbeault cohomology groups $H^{r,s}(M_c)$. These are the cohomology groups with respect to the anti-holomorphic part $\overline{\partial}$ of the differential and the integers r and s respectively count the number of holomorphic and anti-holomorphic form indices. The real dimensions $h^{r,s}$ of $H^{r,s}(M_c)$ are referred to as Hodge numbers, and due to the above equation b^p is the sum of all $h^{r,s}$ with $r+s=p$. It will turn out to be useful to study this finer structure.

To this end, we first recall that the Hodge star operator defines an isomorphism between $H^{r,s}(M_c)$ and $H^{n-r,n-s}(M_c)$. Here n is the complex dimension of M_c and we conclude $h^{r,s}=h^{n-r,n-s}$. Complex conjugation and the Kähler property further ensure the symmetry $h^{r,s}=h^{s,r}$, and $b^0=h^{0,0}=1$ is a result of M_c being connected. Finally, Calabi–Yau n-folds — which we defined to have exact SU(n) holonomy and no subgroup thereof — obey $h^{r,0} = 0$ for $1 \le r < n$ and $h^{n,0} = 1$. For Calabi–Yau threefolds ($n = 3$) these conditions determine all $h^{r,s}$ except for $h^{1,1}$ and $h^{2,1}$, which is conveniently visualized by the so-called Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.7}$$

The Betti numbers are easily obtained by summing across horizontal lines of the Hodge diamond and hence seen to be

$$b^0 = b^6 = 1\,, \quad b^1 = b^5 = 0\,, \quad b^2 = b^4 = h^{1,1}\,, \quad b^3 = 2h^{2,1} + 2\,. \tag{2.8}$$

While this completes our original motivation for studying the cohomology, let us introduce some further concepts for later use.

Since $h^{3,0} = 1$ holds for all Calabi–Yau threefolds, there is an up to scaling unique element in their cohomology group $H^{3,0}(M_c)$. This object is commonly referred to as the holomorphic (3, 0)-form Ω and vanishes at no point of M_c. The combinations

$$H_{\text{vert}}(M_c) = \bigoplus_{k=0}^{n} H^{k,k}(M_c)\,, \qquad H_{\text{hor}}(M_c) = \bigoplus_{k=0}^{n} H^{n-k,k}(M_c) \tag{2.9}$$

are respectively known as the vertical (or even) cohomology and the horizontal cohomology. The former carries a ring structure with the wedge product defining multiplication.

Moduli

We recall that a manifold is of the Calabi–Yau type if and only if it is Kähler and has a Ricci-flat metric. It is natural to wonder whether a given Kähler manifold can be equipped with different Ricci-flat metrics, thereby defining different Calabi–Yau manifolds. In other words, the question is whether a given Ricci-flat metric can be deformed by small perturbations such that the manifold remains Ricci flat and of the Kähler type. In the complex three-dimensional case this question is for example studied in ref. [48] and we here briefly summarize their results. A general perturbation reads

$$\delta g = \left(\delta g_{i\bar{j}}\, dz^i\, d\bar{z}^{\bar{j}} + \text{ c.c. }\right) + \left(\delta g_{\bar{i}\,\bar{j}}\, d\bar{z}^{\bar{i}}\, d\bar{z}^{\bar{j}} + \text{ c.c}\right) \tag{2.10}$$

with (anti-)holomorphic coordinates z^i ($\bar{z}^{\bar{j}}$) and where the terms in the two brackets are locally independent. The first term is a perturbation with mixed type indices that keeps the metric in a Hermitian form. It deforms the Kähler class of g by $\delta\omega = g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$, where the real (1, 1)-form $\delta\omega$ is required to be harmonic and, by the Hodge theorem, therefore uniquely corresponds to an element of $H^{1,1}(M_c)$. The second term is a pure type perturbation and leads to a deformed metric that is no longer hermitian. With the holomorphic (3, 0)-form Ω it defines the complex (2, 1)-form $\Omega_{ijk}g^{i\bar{i}}g_{\bar{i}\bar{j}}dz^j \wedge dz^k \wedge \bar{z}^{\bar{j}}$, which is also required to be harmonic and therefore uniquely corresponds to an element of $H^{2,1}$. There is a non-holomorphic coordinate transformation that puts the metric back into a hermitian form, such that pure type perturbations are deformations of the complex structure.

The parameter space $\mathcal{M}$ of these deformations is known as the moduli space of the Calabi–Yau threefold. Due to the local independence of the two perturbation types it locally takes the product form

$$\mathcal{M}(M_c) = \mathcal{M}_{\text{K}}(M_c) \times \mathcal{M}_{\text{CS}}(M_c)\,, \tag{2.11}$$

where points in the factors $\mathcal{M}_{\text{K}}$ and $\mathcal{M}_{\text{CS}}$ respectively correspond to choices of the Kähler class and complex structure. This shows that Calabi–Yau threefolds are typically part of continuously connected families of manifolds and we respectively identify $h^{1,1}$ and $2h^{2,1}$ as the real dimensions of $\mathcal{M}_{\text{K}}$ and $\mathcal{M}_{\text{CS}}$. To further understand the importance of moduli, we now study dimensional reduction on Calabi–Yau threefolds.

2.1.5 Dimensional Reduction on Calabi–Yau Threefolds

In subsection 2.1.3 we discussed the dimensional reduction of ten-dimensional scalars and p-form gauge fields. Through knowledge of the Betti numbers, see eq. (2.8), we now exactly know which and how many four-dimensional massless fields these give rise to. We here explain the full massless spectrum for type IIA and type IIB string compactifications on Calabi–Yau threefolds.

Reduction of the Graviton

The dimensional reduction of the graviton amounts to decomposing the ten-dimensional Ricci scalar into the sum of four-dimensional Ricci scalar plus additional terms. This calculation is quite involved, for an explicit discussion we for instance refer to ref. [49]. As the result, the ten-dimensional graviton gives rise to the four-dimensional graviton as well as $h^{1,1}$ real and $h^{2,1}$ complex scalars. We denote the former as t^a for $a = 1, \ldots, h^{1,1}$ and the latter as ξ^α for $\alpha = 1, \ldots, h^{2,1}$. These scalars intuitively correspond to the moduli of the internal space.

Four-Dimensional $\mathcal{N} = 2$ Multiplets

Since the effective four-dimensional theory is guaranteed to exhibit $\mathcal{N} = 2$ space-time supersymmetry, its massless fields need to assemble themselves into $\mathcal{N} = 2$ multiplets. It is therefore not necessary to explicitly discuss the reduction of fermions, the multiplet structure is already uniquely determined by the bosons. Let us briefly recall the three relevant multiplets and their bosonic degrees of freedom, see e.g. [50]. First, there is the gravity multiplet with one symmetric rank two tensor (the graviton) and one vector (the graviphoton). Second, the vector multiplet with one vector and one complex scalar. Third, the hyper multiplet with four real scalars.

Type IIA

The massless bosonic fields of type IIA string compactifications on Calabi–Yau threefolds are summarized in Table 2.3. We note that the scalar b_* is the dualized version of an anti-symmetric tensor

Type	Symbol	Multiplicity	10-dim. field
Graviton	$g_{\mu\nu}$	1	g
Vector	$C_\mu^{(1)}$	1	$C^{(1)}$
	$C_\mu^{(3),a}$	$h^{1,1}$	$C^{(3)}$
Complex scalar	ξ^α	$h^{2,1}$	g
Real scalar	t^a	$h^{1,1}$	g
	b^a	$h^{1,1}$	B
	b_*	1	B
	ϕ_4	1	ϕ
	$C^{(3),\beta}$	$2h^{2,1}+2$	$C^{(3)}$

Table 2.3: Massless bosonic fields in type IIA string compactifications with $\mathcal{N}=2$ space-time supersymmetry in four dimensions. Greek letters are four-dimensional space-time indices, $a=1,\ldots,h^{1,1}$, $\alpha=1,\ldots,h^{2,1}$ and $\beta=1,\ldots,2h^{2,1}+2$.

$b_{\mu\nu}$ arising from the the (NS,NS) 2-form field B, whereas type and number of the other fields follow from preceding discussions. The spectrum indeed assembles itself into $\mathcal{N}=2$ multiplets, namely

$$
\begin{array}{lll}
1 & \text{gravity multiplet:} & \left(g_{\mu\nu}, C_\mu^{(1)}, \ldots\right) \\
h^{1,1} & \text{vector multiplets:} & \left(C_\mu^{(3),a}, t^a + ib^a, \ldots\right) \\
1 & \text{hyper multiplet:} & \left(b_*, \phi_4, 2\times C^{(3),\beta}, \ldots\right) \\
h^{2,1} & \text{hyper multiplets:} & \left(\xi^\alpha, 2\times C^{(3),\beta}, \ldots\right)
\end{array}
$$

where ... stands for fermions. The real Kähler moduli t^a combine with the scalars b^a into complex degrees of freedom and belong to vector multiplets. Two of the scalars $C^{(3),\beta}$ — those corresponding to the harmonic $(3,0)$- and $(0,3)$-forms — combine with b_* and the four-dimensional dilaton ϕ_4 into the so-called *universal* hyper multiplet. The other $C^{(3),\beta}$ as well as the complex structure moduli ξ^α constitute additional hyper multiplets.

Type IIB

Table 2.4 lists the massless bosonic fields of type IIB string compactification on Calabi–Yau threefolds. The scalar $C_*^{(2)}$ is the dualized version of a 2-form field $C_{\mu\nu}^{(2)}$, and the self-duality constraint on $C^{(4)}$ is taken into account by retaining only $h^{2,1}+1$ (as opposed to twice as much) vectors $C_\mu^{(4),\beta}$ and by not including the $h^{1,1}$ scalar duals of 2-forms $C_{\mu\nu}^{(4),a}$. These fields assemble themselves into $\mathcal{N}=2$ multiplets according to

$$
\begin{array}{lll}
1 & \text{gravity multiplet:} & \left(g_{\mu\nu}, C_\mu^{(4),\beta}, \ldots\right) \\
h^{2,1} & \text{vector multiplets:} & \left(C_\mu^{(4),\beta}, \xi^\alpha, \ldots\right) \\
1 & \text{hyper multiplet:} & \left(b_*, \phi_4, C_4^{(0)}, C_*^{(2)}, \ldots\right) \\
h^{1,1} & \text{hyper multiplets:} & \left(t^a + ib^a, C^{(2),a}, C^{(4),a}, \ldots\right) .
\end{array}
$$

One of the $h^{2,1}+1$ vectors $C_\mu^{(4),\beta}$ falls into the gravity multiplet, the other belong to the $h^{2,1}$ vector multiplets. While the complexified Kähler moduli are part of hyper multiplets, the complex structure

Type	Symbol	Multiplicity	10-dim. field
Graviton	$g_{\mu\nu}$	1	g
Vector	$C_\mu^{(4),\beta}$	$h^{2,1}+1$	$C^{(4)}$
Complex scalar	ξ^α	$h^{2,1}$	g
Real scalar	t^a	$h^{1,1}$	g
	b^a	$h^{1,1}$	B
	b_*	1	B
	ϕ_4	1	ϕ
	$C^{(4),a}$	$h^{1,1}$	$C^{(4)}$
	$C^{(2),a}$	$h^{1,1}$	$C^{(2)}$
	$C_*^{(2)}$	1	$C^{(2)}$
	$C_4^{(0)}$	1	$C^{(0)}$

Table 2.4: Massless bosonic fields in type IIB string compactifications with $\mathcal{N}=2$ space-time supersymmetry in four dimensions. Greek letters are four-dimensional space-time indices, $a=1,\ldots,h^{1,1}$, $\alpha=1,\ldots,h^{2,1}$ and $\beta=1,\ldots,h^{2,1}+1$.

moduli reside in vector multiplets. The four-dimensional dilaton ϕ_4 is again part of the so-called universal hyper multiplet.

Summary

As the important point to take away from this discussion, we note that the moduli of the Calabi–Yau threefold M_c strongly influence the low energy dynamics of the compactification. They correspond to massless scalars t^a and ξ^α and their number determines the supersymmetric multiplet structure. For both type II theories the moduli either belong to vector or hyper multiplets, which is why we write the moduli space of the compactification as

$$\mathcal{M}^{\text{IIA / IIB}}(M_c) = \mathcal{M}^{\text{IIA / IIB}}_{\text{vector}}(M_c) \times \mathcal{M}^{\text{IIA / IIB}}_{\text{hyper}}(M_c). \tag{2.12}$$

In both cases the Kähler moduli t^a are complexified by the scalars b^a that arise from the (NS,NS) B-field. The connection to the geometric moduli space of the Calabi–Yau threefold M_c, see eq. (2.11), will be explained further below.

Mirror Symmetry

It should not go unnoticed that the two types of moduli exchange their roles between type IIA and IIB. This is a manifestation of mirror-symmetry — see the textbooks [29, 51] for an exhaustive discussion of this subject — which states that type IIA compactified on a Calabi–Yau threefold X is equivalent to type IIB compactified on the *mirror manifold* Y of X with the identification

$$\mathcal{M}^{\text{IIA}}_{\text{vector}}(X) = \mathcal{M}^{\text{IIB}}_{\text{vector}}(Y) \quad \text{and} \quad \mathcal{M}^{\text{IIA}}_{\text{hyper}}(Y) = \mathcal{M}^{\text{IIB}}_{\text{hyper}}(X)\,. \tag{2.13}$$

An immediate consequence of this are the identities $h^{1,1}(X) = h^{2,1}(Y)$ and $h^{1,1}(Y) = h^{2,1}(X)$. While mirror symmetry is not the focus of this thesis, we will come back to it at several points below.

2.1.6 Compactification from the Worldsheet Point of View

In order to explain corrections to the above observations and to obtain a more general formulation of type II string compactifications on Calabi–Yau threefolds, we now return to the worldsheet description. Recall that for $M = \mathbb{M}^{10}$ space-time the worldsheet theory (2.1) is a two-dimensional free $N = (2, 2)$ superconformal field theory with central charge (excluding ghosts) $c = 15$. In this language the compactification ansatz in eq. (2.2) amounts to decomposing the worldsheet theory into two independent factors. First, the four-dimensional version of the theory in eq. (2.1), whichcorresponds to the $\mathbb{M}^4$ factor and contributes $c = 6$ to the central charge. Second, the non-linear sigma model on the Calabi–Yau threefold M_c. As we saw in subsection 2.1.3, this second theory still exhibits $N = (2, 2)$ supersymmetry on the worldsheet, is conformal at leading order in α' and adds $c = 9$.

Superconformal Moduli

The theories of the second type typically possess truly marginal couplings. These are terms that can be added to the theory without breaking conformal invariance or supersymmetry, thereby defining a slightly perturbed version of the original theory. Put differently, $N = (2, 2)$ superconformal field theories typically have a moduli space $\mathcal{M}_{\text{scft}}$ that corresponds to different choices for the parameters multiplying the truly marginal couplings. In this thesis we are predominantly interested in the moduli of the (NS,NS) sector, whose moduli space for both type II theories locally takes the product form

$$\mathcal{M}^{(\text{NS,NS})}_{\text{scft}}(M_c) = \mathcal{M}_{(a,c)}(M_c) \times \mathcal{M}_{(c,c)}(M_c)\,, \tag{2.14}$$

where the two factors respectively corresponds to truly marginal couplings in the (a, c) (chiral–anti-chiral) and (c, c) (chiral–chiral) ring of the superconformal theory [44, 52], see also the review [28]. This is reminiscent of the factorization of the geometric and compactification moduli spaces in eqs. (2.11) and (2.12). We now relate these various notions of moduli spaces, for the purpose of which we also discuss corrections in g_s and α'.

Corrections in g_s

While we consider the superconformal theory at fixed worldsheet genus $g = 0$, the moduli spaces (2.12) of string compactifications are potentially subject to higher order g_s corrections through worldsheets with higher genera as well as non-perturbative effects such as D-instantons [53, 54]. However, we recall that the four-dimensional dilaton always belongs to the universal hyper multiplet and that g_s is the background value of the dilaton. Since in addition vector and hyper multiplets do not mix as a result of $\mathcal{N} = 2$ space-time supersymmetry, the vector multiplets can be described exactly with worldsheet techniques, at least up to the second derivative order.

Corrections in α'

We therefore restrict our attention to the vector multiplet sector. The worldsheet description as superconformal field theory is by definition exact in α' and we have

$$\begin{aligned}\mathcal{M}^{\text{IIA}}_{\text{vector}}(M_c) &= \mathcal{M}^{\text{IIA}, g_s\to 0}_{\text{vector}}(M_c) = \mathcal{M}^{\text{IIA}}_{(a,c)}(M_c)\,,\\ \mathcal{M}^{\text{IIB}}_{\text{vector}}(M_c) &= \mathcal{M}^{\text{IIB}, g_s\to 0}_{\text{vector}}(M_c) = \mathcal{M}^{\text{IIB}}_{(c,c)}(M_c)\,,\end{aligned} \tag{2.15}$$

where the respective identifications with the (a, c) and (c, c) ring follow from a more detailed analysis of the superconformal theories, see for instance the review [28]. Now recall from subsection 2.1.5 that in type IIA the complexified Kähler moduli of M_c belong to the vector multiplet sector, whereas in type IIB the complex structure moduli do. The correct identifications are

$$\mathcal{M}^{\text{IIA}, \alpha' \to 0}_{\text{vector}}(M_c) = \mathcal{M}_{\text{K}}(M_c)^* \quad \text{and} \quad \mathcal{M}^{\text{IIB}}_{\text{vector}}(M_c) = \mathcal{M}_{\text{CS}}(M_c)\,, \tag{2.16}$$

where $*$ denotes complexification. As a result of the higher-loop and non-perturbative α' corrections to the metric, see the discussion in subsection 2.1.3, for type IIA theory the identification is valid only at leading order in α'. The deviation for finite α' gives rise to the notion of the quantum Kähler moduli space, which we define as

$$\mathcal{M}^{\text{IIA}}_{\text{QK}}(M_c) = \mathcal{M}^{\text{IIA}}_{\text{vector}}(M_c) = \mathcal{M}^{\text{IIA}}_{(a,c)}(M_c)\,. \tag{2.17}$$

The interpretation of the superconformal theory as a non-linear sigma model on the Calabi–Yau threefold M_c is therefore strictly valid only at certain boundary components of $\mathcal{M}^{\text{IIA}}_{\text{QK}}(M_c)$, where the volume of M_c tends to infinity such that α' corrections are strongly suppressed. Away from these boundary components the internal space is of finite size and the string equation of motion requires its metric to be not Ricci flat. Moreover, this deformed geometric interpretation is possible only in the vicinity of such boundary components and entirely lost at a generic point in $\mathcal{M}^{\text{IIA}}_{\text{QK}}(M_c)$. There may even be other large volume limits with an interpretation as non-linear sigma model on a *different* Calabi–Yau threefold, a prominent example of which is discussed in ref.[55]. In a different example this effect has also been observed in the author's publication [56]. This shows that the description of $\mathcal{N} = 2$ compactifications as two-dimensional $N = (2, 2)$ superconformal field theories is more general than the geometric non-linear sigma model.

How to Proceed

In this thesis we are interested in the vector multiplet sector of type IIA string compactifications, which as stated in eq. (2.16) is subject to α' corrections. A classical way to deal with this difficulty is the mirror symmetry relation

$$\mathcal{M}^{\text{IIA}}_{\text{QK}}(X) = \mathcal{M}^{\text{IIA}}_{\text{vector}}(X) = \mathcal{M}^{\text{IIB}}_{\text{vector}}(Y) = \mathcal{M}_{\text{CS}}(Y)\,, \tag{2.18}$$

which reduces the problem to the study of the geometric complex structure moduli space of the mirror manifold Y of X. The approach of this thesis is different, we will analyze $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X)$ directly in type IIA theory and not rely on knowledge of the mirror manifold. Our study employs two central tools, the gauged linear sigma model and Picard–Fuchs operators. We will explain these in the next two sections of this chapter.

2.2 Gauged Linear Sigma Models

The gauged linear sigma model was introduced by Witten in ref. [16] and thereafter employed in various works, see e.g. ref. [57] for a careful application to the analysis of worldsheet instantons and refs. [16, 55, 56, 58–68] for the generalization to non-Abelian gauge groups. It is a gauge theory in two dimensions that — if its spectrum is chosen appropriately — realizes a family of non-trivial

$N = (2,2)$ superconformal field theories at its infrared renormalization group fixed point. Its use in the context of string compactifications is therefore twofold. First, it provides an easy way to indirectly construct the appropriate worldsheet theories and to smoothly interpolate across the entire quantum Kähler moduli space. Second, it allows to probe the complicated dynamics of the superconformal theory through certain largely unrenormalized observables. The research presented in chapter 3 is an example of such an analysis.

Unless cited otherwise, our review of the gauged linear sigma model follows refs. [16, 29, 57]. We put particular emphasis on how its properties are chosen to closely resemble those of type II worldsheet theories.

2.2.1 Two-Dimensional $N = (2,2)$ Gauge Theories

The worldsheet theories of supersymmetric type II compactifications exhibit $N = (2,2)$ supersymmetry. Since we cannot expect this symmetry to be installed by the renormalization group flow to the infrared, we will define the gauged linear sigma model as an $N = (2,2)$ gauge theory in two dimensions.

$N = (2,2)$ Superspace

A particularly elegant method for constructing supersymmetric gauge theories is the superspace formalism. For a general introduction to this formalism we e.g. refer to the textbook [69], our conventions follow ref. [29]. Let us denote the temporal and spatial worldsheet coordinates respectively as x^0 and x^1. We combine these into $x^\pm = x^0 \pm x^1$ and further define the four fermionic coordinates

$$\theta^\pm \quad \text{and} \quad \overline{\theta}^\pm = \left(\theta^\pm\right)^* , \tag{2.19}$$

which in combination with the bosonic $x^\pm$ constitute two-dimensional $N = (2,2)$ superspace. The fermionic coordinates mutually anti-commute and therefore square to zero. Supersymmetry is generated by the four supercharges

$$Q_\pm = +\frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm , \quad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm , \tag{2.20}$$

where $\partial_\pm$ are partial derivatives with respect to the coordinates $x^\pm$. Since the only non-trivial anti-commutators are

$$\left\{Q_\pm, \overline{Q}_\pm\right\} = -2i\partial_\pm , \tag{2.21}$$

their algebra splits into two independent sectors with two supercharges each. The first involves '+' indices and is referred to as left-moving or anti-holomorphic, the '−' sector is said to be right-moving or holomorphic. These indices moreover display the opposite chirality under two-dimensional Lorentz transformations, with the convention that downstairs '±' indices transform as upstairs indices '∓'.

Superfields

Superfields are functions defined on superspace. Their expansion in the fermionic coordinates contains at most $2^4 = 16$ terms, and hence they are expressed in terms of 16 coefficient functions that depend on $x^\pm$ only. Since a general superfields does not correspond to an irreducible representations of the supersymmetry algebra, we define special types of superfields with fewer degrees of freedom. This is done with the differential operators

$$D_\pm = +\frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm\,, \quad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm\,, \tag{2.22}$$

which anti-commute with the supercharges (2.20) and can therefore be used to impose constraints that are invariant under supersymmetry transformations. This leads to the following definitions:

- A chiral superfield Φ is subject to $\overline{D}_\pm\Phi = 0$. Its bosonic degrees of freedom are complex scalars ϕ and F, the latter of which is a non-dynamical auxiliary field.
- A twisted chiral superfield Σ is defined by $\overline{D}_+\Sigma = D_-\Sigma = 0$. Its bosonic coefficient functions are the complex scalars σ and E, where E is non-dynamical.
- A vector superfield is subject to the reality condition $V = V^\dagger$. We will elaborate on this type of superfield when discussing gauge symmetries below.

In addition, the hermitian conjugate of a (twisted) chiral superfield is a (twisted) anti-chiral superfield and vice versa. All of these superfields transform as Lorentz scalars.

Supersymmetric Actions

Let $\mathcal{F}$ denote the collection of all superfields in the theory, and similarly $\mathcal{F}_c$ and $\mathcal{F}_{tc}$ be the collection of all chiral and twisted chiral superfields. A supersymmetric action then takes the general form

$$\begin{aligned} S = \int d^2x\, d^4\theta\, K(\mathcal{F}) + \int d^2x \left[d^2\theta\, W(\mathcal{F}_c)\big|_{\overline{\theta}^\pm=0} + \text{h.c.}\right] \\ + \int d^2x \left[d^2\tilde{\theta}\, \widetilde{W}(\mathcal{F}_{tc})\big|_{\theta^-=\overline{\theta}^+=0} + \text{h.c.}\right], \end{aligned} \tag{2.23}$$

where $d^4\theta = d\theta^+ d\theta^- d\overline{\theta}^- d\overline{\theta}^+$, $d^2\theta = d\theta^+ d\theta^-$, $d^2\tilde{\theta} = d\theta^+ d\overline{\theta}^-$ and 'h.c' abbreviates 'hermitian conjugate'. The function K is referred to as the Kähler potential, whereas W and $\widetilde{W}$ are respectively known as superpotential and twisted superpotential. Reality of the action requires $K(\mathcal{F}) = K(\mathcal{F})^\dagger$ and supersymmetry demands that W and $\widetilde{W}$ are holomorphic. Lorentz invariance is automatic.

Pure Supersymmetric Gauge Theory

To incorporate gauge symmetry into the superspace formalism, let G denote a in general non-Abelian compact Lie-group and $\mathfrak{g}$ be its Lie algebra. Gauging of G requires the introduction of a vector superfield V with values in $\mathfrak{g}$, which is the superspace generalization of an ordinary gauge field. Gauge transformations act as

$$\exp(V) \longmapsto \exp\left(-i\Lambda^\dagger\right)\, \exp(V)\, \exp(i\Lambda)\,, \tag{2.24}$$

where Λ is a chiral superfield with values in $\mathfrak{g}$ that generalizes the gauge parameter. This transformation is consistent with the reality condition $V = V^\dagger$ and reduces the number of independent component fields in V. Its bosonic degrees of freedom are a complex scalar σ, the ordinary two-component gauge field v and a non-dynamical auxiliary field D. The field strength is generalized by the twisted chiral superfield

$$\Sigma = \frac{1}{2}\left\{e^V \overline{D}_+ e^{-V},\, e^{-V} D_- e^V\right\}, \tag{2.25}$$

to which we refer as the super field strength. Pure supersymmetric gauge theory is obtained by choosing

$$K_{\text{gauge}} = \text{tr}_{\text{adj}}\left(\Sigma^\dagger \Sigma\right) \tag{2.26}$$

as the Kähler potential in eq. (2.23), where the trace is taken in the adjoint representation. Both the superpotential and twisted superpotential are required to be a gauge invariant.

Charged Chiral Matter

The gauged linear sigma model is not a pure gauge theory but additionally comprises chiral superfields. These are charged under the gauge group and transforms as

$$\Phi \mapsto \exp\left[-i\Lambda(\rho_\Phi)\right]\Phi\,, \tag{2.27}$$

where Λ here is in the representation of ρ_Φ of Φ. The transformation is consistent with the chirality constraint on Φ and the basic bilinear gauge invariant combination reads

$$K_{\text{matter}} = \Phi^\dagger \exp\left[V(\rho_\Phi)\right]\Phi = \widetilde{\Phi}^\dagger\,\widetilde{\Phi}\,. \tag{2.28}$$

This term is added to the Kähler potential in eq. (2.26) and includes both matter kinetic terms as well as matter-gauge interactions. The last equality defines $\widetilde{\Phi}$, a covariant chiral superfield [16].

2.2.2 The $N = 2$ Superconformal Algebra and R-Symmetries

As our next guideline in constructing the gauged linear sigma model we recall that it is supposed to flow to a non-trivial $N = (2, 2)$ superconformal field theory in the infrared. Therefore, we now discuss the algebra of conserved currents in such superconformal theories.

$N = 2$ Superconformal Algebra

The full algebra splits into a holomorphic and an anti-holomorphic sector, and it is sufficient to only discuss the holomorphic one. Conformal symmetry gives rise to the energy momentum tensor $T(z)$, which in the non-supersymmetric case constitutes the Virasoro algebra with central charge c. The two right-moving supersymmetries extend this by the two supercurrents $G^{(\pm)}(z)$, whose operator product further requires the inclusion of an additional U(1) current $J(z)$. These four currents constitute the $N = 2$ superconformal algebra [70–72] and their scaling dimensions as well as U(1) charges are summarized in the following diagram:

U(1) charge	1	0	−1	scaling dim.
		$T(z)$		2
	$G^{(+)}(z)$		$G^{(-)}(z)$	3/2
		$J(z)$		1

(2.29)

It is important to note that the superscripts '(±)' in $G^{(\pm)}(z)$ refer to the U(1) charge with respect to $J(z)$ and not to the chirality under two-dimensional Lorentz transformations. The supercurrents $G^{(+)}(z)$ and $G^{(-)}(z)$ respectively correspond to the supercharges Q_- and $\overline{Q}_-$.

R-Symmetries

This shows that together with the anti-holomorphic sector there are two U(1) currents in the desired type of infrared fixed point theory. Therefore, we choose to equip the gauged linear sigma model with two associated U(1) R-symmetries that are referred to as $\mathrm{U}(1)_L$ (left moving, anti-holomorphic) and $U(1)_R$ (right moving, holomorphic). They assign non-zero charges to the fermionic coordinates in the corresponding sector, namely

$$\begin{aligned} \mathrm{U}(1)_L: \quad & \mathfrak{q}_R(\theta^+) = -\mathfrak{q}_R(\overline{\theta}^+) = 1\,, \\ \mathrm{U}(1)_R: \quad & \mathfrak{q}_L(\theta^-) = -\mathfrak{q}_L(\overline{\theta}^-) = 1\,. \end{aligned} \tag{2.30}$$

Invariance under these symmetries puts additional constraints on the supersymmetric action functional (2.23) and allowed charge assignments:

1. Since the gauge symmetry must commute with the R-symmetries, the vector superfield V is required to have charge zero under both $\mathrm{U}(1)_L$ and $U(1)_R$. The definition (2.25) of the super field strength then implies $\mathfrak{q}_L(\Sigma) = -\mathfrak{q}_R(\Sigma) = 1$.

2. The Kähler potential K needs to be uncharged under both $\mathrm{U}(1)_L$ and $U(1)_R$. For K as in eqs. (2.26) and (2.28) this is fulfilled automatically.

3. The superpotential W is required to have $\mathfrak{q}_L(W) = \mathfrak{q}_R(W) = 1$.

4. The twisted superpotential $\widetilde{W}$ needs to have $\mathfrak{q}_L(\widetilde{W}) = -\mathfrak{q}_R(\widetilde{W}) = 1$.

It is sometimes more convenient to instead work with the vector and axial R-symmetries $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_A$, which are defined by their charges $\mathfrak{q}_V = \mathfrak{q}_L + \mathfrak{q}_R$ and $\mathfrak{q}_A = \mathfrak{q}_L - \mathfrak{q}_R$. We will move back and forth between these two bases.

2.2.3 Definition of the Gauged Linear Sigma Model

Having discussed several important symmetries, we are now ready to complete our definition of the gauged linear sigma model.

Specification of a Particular Model

Contrary to what the name might suggest, the gauged linear sigma model is not one specific model but rather a framework that subsumes a variety of models. The definition of a *particular* gauged linear sigma model requires the specification of its gauge theory spectrum. This is a triple $(G, \mathrm{Irrep}(G), Q_V)$ that consists of a compact Lie group G together with a set $\mathrm{Irrep}(G)$ of irreducible G-representations and a set Q_V of integers, both of which have finite cardinality N. In physics terminology G is the gauge group of the model and it takes the general form

$$G = \frac{\mathrm{U}(1)^\ell \times G_1 \times \ldots \times G_m}{\Gamma}\,. \tag{2.31}$$

Here the G_k with $k = 1, \ldots, m$ are compact simple Lie groups and Γ is a discrete normal subgroup of the product $\mathrm{U}(1)^\ell \times G_1 \times \ldots \times G_m$. The U(1) factors will be indexed with $l = 1, \ldots, \ell$. Each element ρ_i of Irrep(G), where $i = 1, \ldots, N$, corresponds to a chiral multiplet Φ_i and defines its gauge representation $\rho_i = \rho(\Phi_i)$. The set Q_V specifies the vector R-charges $\mathfrak{q}_i = \mathfrak{q}_V(\Phi_i)$, whereas we choose zero axial R-charge for all Φ_i. The R-charges may always be redefined by adding integer multiplets of the gauge charges.

Twisted Superpotential

Let us now understand the implications of these definitions for the twisted superpotential. Associated to the gauge symmetry there are the vector field V and its twisted chiral super field strength Σ, both of which take values in $\mathfrak{g}$. These decompose as $V = \sum V_l + \sum V_k$ and $\Sigma = \sum \Sigma_l + \sum \Sigma_k$, where the individual terms correspond to the ℓ Abelian and the m non-Abelian factors in G. Since there are no additional twisted chiral superfields, the twisted superpotential is a function of Σ alone. Gauge symmetry together with the first and fourth constraint by R-symmetry — see the list in the previous subsection — thus imply

$$\widetilde{W}(\Sigma) = \frac{1}{2}\sum_{q=1}^{\ell+m} \tau_q \, \mathrm{tr}_{\mathrm{adj}}\, \Sigma_q = \frac{1}{2}\sum_{l=1}^{\ell} \tau_l \, \Sigma_l \, . \tag{2.32}$$

Here τ_l for $l = 1, \ldots, \ell$ are complex numbers known as complexified Fayet–Iliopoulos (FI) parameters and the overall constant of proportionality was chosen for convenience. The second equality follows since the generators of simple Lie groups are traceless, which is why there is precisely one τ_l for each U(1) factor in G. After fermionic integration as specified by eq. (2.23) this twisted linear superpotential gives rise to the real action,

$$S_{\widetilde{W}} = \sum_{l=1}^{\ell}\left[-\int d^2x\, r_l\, D_l + \frac{\theta_l}{2\pi}\int dv_l\right] \quad \text{with} \quad \tau_l = r_l - i\frac{\theta_l}{2\pi} \, . \tag{2.33}$$

The first term inside the sum is the standard FI term that involves the auxiliary D-field of V_l, and in the second term θ_l is the two-dimensional theta angle that multiplies the field strength dv_l of the ordinary U(1) gauge field v_l. Both terms do not exist for simple gauge group factors. We denote the collection of τ_l as vector $\vec{\tau}$, and similarly for other parameter types.

The Action

In order to complete our definition of the gauged linear sigma model, we explicitly state its classical action

$$\begin{aligned} S_{\mathrm{glsm}} = &\int d^2x\, d^4\theta \left[\sum_{i=1}^{N} \widetilde{\Phi}_i^\dagger \widetilde{\Phi}_i - \sum_{l=1}^{\ell} \frac{1}{e_l^2} \Sigma_l^\dagger \Sigma_l - \sum_{k=1}^{m} \frac{c_k}{g_k^2}\, \mathrm{tr}\left(\Sigma_k^\dagger \Sigma_k\right)\right] \\ &+ \int d^2x \left[d^2\theta\, W(\{\Phi_i\})\big|_{\bar{\theta}^\pm=0} + \mathrm{h.c.}\right] + \int d^2x \left[d^2\tilde{\theta}\, \widetilde{W}(\Sigma)\big|_{\theta^-=\bar{\theta}^+=0} + \mathrm{h.c.}\right] . \end{aligned} \tag{2.34}$$

Here e_l and g_k are gauge coupling constants, the number c_k depends on the non-Abelian gauge group factor G_k and the twisted superpotential $\widetilde{W}$ is as in eq. (2.32). The superpotential W is defined as the most general holomorphic polynomial of the chiral fields that is consistent with the symmetries — in other words: the most general holomorphic polynomial that is gauge invariant and has vector R-charge $\mathfrak{q}_V(W) = 2$. Its precise form depends on the gauge theory spectrum and typically is the sum of several terms. We schematically write

$$W = \sum_{\alpha=1}^{n_\alpha} y_\alpha \, M_\alpha(\{\Phi_i\}) \,, \tag{2.35}$$

where the M_α are linearly independent monomials and y_α are complex constants whose interpretation we clarify below. The individual parameters y_α are summarized into the vector $\vec{y}$.

Twisted Masses

There is an additional ingredient to the gauged linear sigma model that will play an important role in the next chapter. To understand this, let us consider a flavor symmetry group F that acts in some representation R_F on the chiral multiplets. It is then possible to turn on an a background vector superfield V_F with values in the corresponding representation $\mathfrak{r}_\mathfrak{f}$ of the Lie algebra $\mathfrak{f}$ of F. As a common supersymmetry preserving choice, we take its bosonic component σ_F to be constant and set all other component fields of V_F to zero [73].

We distinguish two cases. First, in absence of a superpotential there is always the Abelian subgroup $F' = \mathrm{U}(1)^N \subset F$ — where N is the number of chiral multiplets — that acts by phase rotations on the individual multiplets. While the full flavor symmetry F may be bigger and in particular non-Abelian, we choose to only turn on a vector superfield for F'. Second, in presence of a superpotential with generic parameters $\vec{y}$ the flavor symmetry is typically smaller than $\mathrm{U}(1)^N$. In the next chapter we will then nevertheless turn on $V_{F'}$ for $F' = \mathrm{U}(1)^N$, which physically requires a non-generic choice of superpotential parameters. While this does change the theory, the observables we will be concerned with are independent of the $\vec{y}$ such that this method is admissible.

To summarize, in both cases we turn on a background vector superfield for the flavor symmetry group $F' = \mathrm{U}(1)^N$. Its Lie algebra $\mathfrak{f}'$ is $\mathbb{R}^N$ and the components $\sigma_{F'}(\rho_i) = \mathfrak{m}_i$ are real valued constants that we refer to as twisted masses. A full specification of a particular model requires us to additionally specify these constants. We may certainly choose all twisted masses $\mathfrak{m}_i$ to be zero, which unless specified otherwise is understood to be the case.

2.2.4 Anomaly of the R-Symmetries

In the previous subsection we have deliberately constructed the gauged linear sigma model to be classically invariant under the R-symmetries $\mathrm{U}(1)_L$ and $\mathrm{U}(1)_R$. On the quantum level these symmetries potentially suffer from anomalies Δ_L and Δ_R of the form [16, 57]

$$\Delta_L = -\Delta_R = \frac{i}{2\pi} \sum_{l=1}^{\ell} S_l \int dv_l \quad \text{with} \quad S_l = \sum_{\rho_i \in \mathrm{Irrep}(G)} \dim \rho_i \, Q_l(\rho_i) \,. \tag{2.36}$$

Here $Q_l(\rho_i)$ is the charge of the chiral multiplet Φ_i under l-th $\mathrm{U}(1)$ gauge group factor, $\dim \rho_i$ is the dimension of its representation ρ_i and dv_l is the ordinary field strength two-form. Since the sum of the two anomalies $\Delta_V = \Delta_L + \Delta_R = 0$ vanishes, the vector R-symmetry $\mathrm{U}(1)_V$ is automatically non-anomalous

Conformal Models

The axial R-symmetry $U(1)_A$ can and in general will be anomalous. Its anomaly cancels if and only if the condition

$$S_l \;=\; \sum_{\rho_i \in \operatorname{Irrep}(G)} \dim \rho_i \, Q_l(\rho_i) = 0 \tag{2.37}$$

is fulfilled for all l, in case of which the gauged linear sigma model is said to be *conformal*. The low energy limit then is a non-trivial $N = (2,2)$ superconformal field theory with central charge

$$c \;=\; -3 \dim \mathfrak{g} + 3 \sum_{\rho_i \in \operatorname{Irrep}(G)} \left(1 - \mathfrak{q}_i\right) \cdot \dim \rho_i \;. \tag{2.38}$$

By an appropriate choice of the gauge theory spectrum we can thus deliberately cancel the anomaly and choose the desired central charge. A potential spontaneous breakdown of the R-symmetries does not pose a problem since the $N = 2$ superconformal algebra is a statement about operators and not about states. Conformal gauged linear sigma models figure prominently in this thesis.

2.2.5 Renormalization

We are eventually interested in the low energy behavior of the gauged linear sigma model. To obtain a better understanding thereof, we here shortly discuss effects of renormalization.

Superpotential

The superpotential is strongly protected by $N = (2,2)$ supersymmetry. Arguments of holomorphy constrain quantum corrections to one-loop order in perturbation theory and to non-perturbative effects [74], see also the review [75]. In the gauged linear sigma model these are absent [16, 57], such that the superpotential is entirely unrenormalized. The coupling constants $\vec{y}$ can therefore be chosen freely and are true moduli of the infrared fixed point theory.

Twisted Superpotential

The arguments of holomorphy remain applicable to the twisted superpotential, i.e., quantum corrections can only appear at one-loop order or through non-perturbative effects. There indeed is a one-loop divergence that results in the real FI parameter $\vec{r}$ running according to [16, 29, 57]

$$\vec{r}(\mu) = \vec{S} \, \log\left(\frac{\mu}{\Lambda}\right) + \vec{r}(\Lambda) \;. \tag{2.39}$$

Here Λ is the dynamically generated renormalization group invariant scale, μ is the floating energy scale and $\vec{S}$ is the vector of S_l defined as in eq. (2.36). The effect of taking the infrared limit $\mu \to 0^+$ heavily depends on the spectrum dependent numbers $\vec{S}$, namely

$$\lim_{\mu \to 0^+} r_l(\mu) \;=\; \begin{cases} -\infty & \text{if } S_l > 0 \\ r_l(\Lambda) = r_l(\Lambda_{\mathrm{UV}}) & \text{if } S_l = 0 \\ +\infty & \text{if } S_l < 0 \end{cases} \;, \tag{2.40}$$

where Λ_{UV} is the ultraviolet cutoff energy. In the first and third case the infrared value $r_l(0^+)$ is unambiguously fixed by the renormalization group flow, and hence r_l does not correspond to a modulus of the infrared theory. This is unlike the second case, where we can freely choose $r_l(0^+) = r_l(\Lambda_{\mathrm{UV}})$ in terms of the bare value $r_l(\Lambda_{\mathrm{UV}})$. A similar discussion applies to the theta angles [16, 57], such that for conformal gauged linear sigma models the complexified FI parameters $\vec{\tau}$ are moduli of the infrared fixed point theory.

Kähler Potential

The Kähler potential K is not holomorphic and therefore significantly less protected by supersymmetry. It will not remain in the canonical diagonal form as chosen in eq. (2.34) but rather be more general at energy scales $\mu < \Lambda_{\mathrm{UV}}$. To obtain some qualitative insight into its renormalization, we now proceed with a discussion of vacuum states.

2.2.6 Low Energy Limit

In this section we more concretely study the low energy behavior of gauged linear sigma models. Our aim is to find the vacuum states of the theory, i.e., field configurations that minimize the scalar potential. These are typically not unique but rather define an entire space of vacuum configurations known as the low energy target space. Until specified otherwise, we consider conformal gauged linear sigma models.

Classical Scalar Potential

Since we cannot follow the renormalization of the Kähler potential explicitly, we cannot determine the fully quantum corrected scalar potential. As a starting point we therefore analyze the classical scalar potential U, which directly follows from expanding the action (2.34) and takes the form [16, 55]

$$
\begin{aligned}
U = \sum_{l=1}^{\ell} \frac{1}{2e_l^2} D_l^2 + \sum_{k=1}^{m} \sum_{a=1}^{\dim \mathfrak{g}_k} \frac{1}{2g_k^2} \left(D_k^a\right)^2 + \sum_{i=1}^{N} F_i^\dagger F_i \\
+ \frac{1}{2} \sum_{i=1}^{N} \phi_i^\dagger \left\{ \sigma(\rho_i), \sigma(\rho_i)^\dagger \right\} \phi_i + \sum_{k=1}^{m} \frac{c_k}{2g_k^2} \operatorname{tr}_{\mathrm{adj}} \left[\sigma_k, \sigma_{k^\dagger} \right] .
\end{aligned}
\tag{2.41}
$$

Here ϕ_i is the complex scalar of Φ_i, $\sigma(\rho_i)$ the complex scalar of the vector multiplet V in the representation of Φ_i and σ_k the component of σ that is associated to the non-Abelian gauge group factor G_k. The auxiliary F- and D-fields are integrated out and read

$$
D_l = -e_l^2 \sum_{i=1}^{N} \left[Q_l(\rho_i)\, \phi_i^\dagger \phi_i - r_l \right] , \quad D_k^a = -g_k^2 \sum_{i=1}^{N} \phi_i^\dagger T_k^a(\rho_i)\, \phi_i , \quad F_i = \left(\frac{\partial W}{\partial \phi_i} \right)^\dagger \tag{2.42}
$$

where T_k^a are the generators of the non-Abelian group G_k. As a consequence of supersymmetry U cannot be negative, which is confirmed by the fact that all five terms in eq. (2.41) are positive definite. The second and fifth term vanish in case of an Abelian gauge group.

Classical Ground States

The space of classical ground states is the set of gauge inequivalent field configurations that minimize the classical scalar potential U. It strongly depends on the choice of real FI parameters $\vec{r}$, which enter through the Abelian D-fields in eq. (2.42). While there might be choices of $\vec{r}$ for which the minimal value of U is strictly positive such that supersymmetry seems to be spontaneously broken, this do not occur upon including quantum corrections [16, 57]. We therefore define

$$\mathcal{X}_{\vec{r}} = {U^{-1}(0)}\big/{G} \tag{2.43}$$

and consider choices of $\vec{r}$ for which this space of classical ground states is non-empty. Since the individual terms of U are non-negative, all of them separately vanish at a minimum.

Branches and Phases

Consider setting the Abelian D-terms to zero. This will introduce vacuum expectation values for the chiral scalars ϕ_i whose precise form depend on the choice of $\vec{r}$. That region in $\vec{r}$-space for which G is broken to a discrete subgroup is referred to as the Higgs branch [57]. Through the second line in eq. (2.41) the components of σ then get massive and thus have vanishing vacuum expectation value. As a result, $\mathcal{X}_{\vec{r}}$ is parameterized by the ϕ_i only and is given by the common zeros of all D- and F-terms. The Higgs branch further decomposes into several regions with different vacuum spaces $\mathcal{X}_{\vec{r}}$, referred to as the different phases of the gauged linear sigma model.

Those values of $\vec{r}$ that leave at least one generator of G unbroken constitute the so called mixed Coulomb-Higgs branch, and those for which G is completely unbroken the Coulomb branch [16, 55, 57]. At the classical level $\mathcal{X}_{\vec{r}}$ is then non-compact since the components of σ associated to the unbroken generators are unconstrained. This results in a singular low energy theory. The Coulomb-Higgs branch is typically of real codimension one in $\vec{r}$-space and separates the different Higgs branch phases from each other. In the space of complexified FI parameters $\vec{\tau}$ it typically is of *complex* codimension one, such that different phases actually are smoothly connected to each other by variations of $\vec{\tau}$.

Higgs Branch

We now consider the Higgs branch in more detail. It turns out to be useful to reinterpret the D-terms in a more mathematical way, for which we define the vector space $V = \mathrm{span}(\phi_i) = \mathbb{C}^N$ spanned by the chiral scalars ϕ_i. This space is equipped with the canonical Kähler form $\omega_V = \sum d\phi^i \wedge d\overline{\phi}^{\bar{i}}$ that corresponds to the diagonal Kähler potential chosen in eq. (2.34). In this formulation the D-terms in eq. (2.42) are components of the moment map $\mu : V \longrightarrow \mathfrak{g}^*$ of the G-action on V with respect to the Kähler form ω_V and the real FI parameters $\vec{r}$ appear as constants of integration for the Abelian factors of G [16, 57]. The superpotential W is a gauge invariant function on V and the F-terms (2.42) are components of its differential dW. The target space (2.43) therefore is

$$\mathcal{X}_{\vec{r}} = Y_{\vec{r}} \cap dW^{-1}(0) \quad \text{with} \quad Y_{\vec{r}} = {\mu^{-1}(\vec{r})}\big/{G}\,. \tag{2.44}$$

While $\mathcal{X}_{\vec{r}}$ and $Y_{\vec{r}}$ in general do not have a geometric interpretation as manifolds, in the following discussion we assume they do. For conformal models the first Chern class of $\mathcal{X}_{\vec{r}}$ vanishes [16].

Gauge Theory Parameters and Geometric Moduli

This formulation is advantageous because it clarifies the role of the FI and superpotential parameters $\vec{\tau}$ and $\vec{y}$. Assuming a geometric interpretation, the Marsden-Weinstein-Meyer theorem guarantees $Y_{\vec{r}}$ to be a Kähler manifold and the cohomology class of its Kähler form ω_Y depends linearly on $\vec{r}$ [76, 77], see also the review [78] for mathematical background. The Kähler form ω_X on $X_{\vec{r}}$ descents from ω_Y, such that also the Kähler class of ω_X depends on $\vec{r}$. Consequently, the FI parameters correspond to Kähler moduli of $X_{\vec{r}}$ and these are complexified by the theta angles $\vec{\theta}$. We also conclude that the number ℓ of U(1) factors in G equals the number of Kähler moduli. The complex structure moduli of $X_{\vec{r}}$ are induced by the intersection of the ambient space $Y_{\vec{r}}$ with $dW^{-1}(0)$ and correspond to the superpotential parameters $\vec{y}$. Since the choice of $\vec{y}$ determines the form of the defining equations $dW^{-1}(0)$, this is intuitively plausible.

Quantum Corrections and Superconformal Moduli

A careful analysis [16, 57] of the vacuum states demonstrates that the above classical discussion becomes a good approximation at certain boundary components in $\vec{\tau}$-space. In these limits the volume of the target space $X_{\vec{r}}$ tends to infinity and the infrared $N = (2, 2)$ superconformal field theory reduces to a non-linear sigma model on $X_{\vec{r}}$. The Kähler potential will then be renormalized precisely such that the Kähler manifold $X_{\vec{r}}$ is equipped with a Ricci-flat metric, i.e., $X_{\vec{r}}$ becomes a generalized Calabi–Yau manifold. Moreover, both the complexified FI parameters $\vec{\tau}$ and superpotential parameters $\vec{y}$ are guaranteed to be true moduli of the superconformal theory. It is thus natural to employ a type IIA interpretation in which $\vec{\tau}$ and $\vec{y}$ are identified with the moduli in the (a, c) and (c, c) ring of the superconformal theory. The FI parameters $\vec{\tau}$ are quantum Kähler moduli and allow us to extrapolate around the entire quantum Kähler moduli space $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X_{\vec{r}})$. In particular, when $\vec{\tau}$ is not at but only in vicinity of large volume the low energy theory has an interpretation as non-linear sigma model on a deformed (not Ricci-flat) version of $X_{\vec{r}}$. For a generic choice of $\vec{\tau}$ the geometric picture of the low energy theory is lost entirely and there may be other boundary components where a different geometric target space arises. While we cannot follow the renormalization group flow exactly, we know that the Kähler potential will be renormalized precisely such that it agrees with this structure [16].

We note that this line of reasoning will typically fail if a non-Abelian gauge symmetry remains unbroken. The gauge theory may then be strongly coupled, such that even at large volume points the quantum physics significantly deviates from the classical discussion. One approach to such cases is the use of strong-weak coupling dualities [55, 61, 79], with which the theory may be rewritten in a dual weakly coupled form.

Coulomb Branch Singularities

As we recall, the classical analysis predicts special choices of the FI parameters $\vec{r}$ for which some components of σ become unconstrained. These values constitute the Coulomb and Coulomb-Higgs branches, where the space of classical ground states becomes non-compact and the low energy theory is singular. For Abelian gauge groups these singularities are on the quantum level carefully analyzed in refs. [16, 29, 57]. The result of this analysis is that — due to a finite quantum correction to the twisted superpotential — the singular choices cannot be read off from the classical potential. Rather, they are computed as follows: let $H \subset G$ be any continuous subgroup of the gauge group $G = \mathrm{U}(1)^\ell$

and I be an index set that enumerates all chiral multiplet Φ_i with non-zero charge under H. Those values of $\vec{\tau}$ for which the equations

$$\prod_{i\in I}\left[\sum_{a=1}^{\ell} Q_a(\rho_i)\,x_a\right]^{Q_l(\rho_i)} = e^{-2\pi\,\tau_l} \quad \text{with} \quad l = 1,\ldots,\ell \tag{2.45}$$

can be simultaneously solved by some value of the auxiliary variables x_a constitute the Coulomb-Higgs branch (if $H \neq G$) and the Coulomb branch (if $H = G$). When speaking of the Coulomb branch, we henceforth mean this to also include the Coulomb-Higgs branch. As we will explain further below, for a different reason the low energy theory is also singular at large volume boundary components of the moduli space.

Discussions of singularities in several examples of non-Abelian gauged linear sigma models can for instance be found in refs. [55, 61]. It is also possible to employ the following heuristic, yet general approach: consider the non-Abelian gauge group being broken to its maximal Abelian subgroup $\mathrm{U}(1)^{\dim\mathfrak{g}}$ and turn on auxiliary FI parameters for the additional U(1) factors. In this modified theory we can employ eq. (2.45) in order to determine the singular locus. By then turning off the auxiliary parameters we go back to the original non-Abelian theory and correspondingly restrict the singular locus. While this restriction subsumes the singularities of the non-Abelian theory, it may still be too big. For a definite check of whether a given choice of $\vec{\tau}$ is singular we may employ the Picard–Fuchs operators that will be introduced in the next section or the correlation functions that feature in the next chapter.

Example

In order to illustrate these concepts, we now discuss a concrete example that goes back to Witten's original work [16]. Consider a gauged linear sigma model with gauge group $G = \mathrm{U}(1)$ and chiral matter spectrum as listed in Table 2.5. The condition for anomaly cancellation (2.37) is fulfilled and

Chiral multiplets	$G = \mathrm{U}(1)$ charge	Vector R-charge $\mathfrak{q}_i$
$\Phi_i,\ i = 1,\ldots,5$	$+1$	0
P	-5	2

Table 2.5: Matter spectrum of the gauged linear sigma model of the quintic Calabi–Yau threefold $\mathbb{P}^4[5]$.

eq. (2.38) shows that the infrared central charge is $c = 9$. As a result, the model flows to an $N = (2,2)$ superconformal field theory that is a valid internal theory for a supersymmetric compactification to four dimensions. Gauge and $\mathrm{U}(1)_V$ invariance constrain the superpotential to take the form

$$W = P\cdot H(\Phi_1,\ldots,\Phi_5)\,, \tag{2.46}$$

where H is a polynomial of homogeneous degree five in the five variables Φ_i. It generically contains 126 (= 9 choose 5) complex parameters y_α, only 101 = 126 − 25 of which are independent due to a $GL(5,\mathbb{C})$ equivalence of the S_i. The classical scalar potential (2.41) reads

$$U = \frac{e^2}{2}\left(|\phi|^2 - 5|p|^2 - r\right)^2 + |H|^2 + |p|^2\left|\nabla_\phi H\right|^2 + |\sigma|^2\left(25|p|^2 + |\phi|^2\right), \tag{2.47}$$

where p is the scalar in P and ϕ the vector of scalars in the Φ_i. We now determine the classical vacua for different choices of r.

We begin with $r \gg 0$. Setting the first term in eq. (2.47) to zero then requires $\phi \neq 0$. This breaks the gauge group completely and the fourth term can be zero for $\sigma = 0$ only. For generic choices of superpotential parameters $\vec{y}$ the differential dH will be of full rank, such that the third term requires $p = 0$. The D-term then shows the ambient space to be

$$Y_{r\gg 0} = \left\{\phi \in \mathbb{C}^5 \,\middle|\, |\phi|^2 = r \neq 0\right\} \big/ \mathrm{U}(1) = \mathbb{P}^4 \,. \tag{2.48}$$

Since H is homogeneous polynomial (a result of gauge symmetry), it is well defined on $\mathbb{P}^4$ and we arrive at

$$\mathcal{X}_{r\gg 0} = \left\{\phi \in \mathbb{P}^4 \mid H(\phi) = 0\right\} = \mathbb{P}^4[5]. \tag{2.49}$$

This space has vanishing first Chern class and therefore can be equipped with a Ricci-flat metric. The limit $r \to \infty$ is a large volume point, at which the low energy fixed point theory is the non-linear sigma model on $\mathbb{P}^4[5]$ and the Kähler potential is renormalized such that its metric becomes Ricci-flat. We refer to $\mathbb{P}^4[5]$ with Ricci-flat metric as the quintic Calabi–Yau threefold.

Now consider the phase $r \ll 0$. Vanishing of the D-field here implies that p cannot be zero and with dH having full rank the third term in eq. (2.47) then requires $\phi = 0$. Given this, H is zero automatically and $\sigma = 0$ follows from the fourth term. The modulus of p is fixed to be $\sqrt{-r/5}$ by the first term and division by the gauge group U(1) further fixes its phase. Consequently, the space of classical vacua $\mathcal{X}_{r\ll 0}$ is a single point. The expectation value for p breaks G to a discrete $\mathbb{Z}_5$ subgroup, which acts by multiplication with a 5-th root of unity on the ϕ_i and trivially on p. For $r \to -\infty$ the gauged linear sigma model thus flows to a Landau-Ginzburg orbifold [16], which is smoothly connected to the $r \to \infty$ region with its completely different nature. While this fact is naturally understood in the gauged linear sigma model, it is rather miraculous from a pure geometric non-linear sigma model point of view.

Lastly, eq. (2.45) with $H = G$ shows that a Coulomb branch arises for $2\pi\tau = 5\ln(5) + i\pi$. This point in moduli space is known as the conifold point.

Non-Conformal Models

In this subsection we have so far specialized to conformal gauged linear sigma models. The low energy limit $\mu \to 0^+$ is indeed very different in the non-conformal case. As an example, in case of $G = \mathrm{U}(1)$ the true vacuum states of the theory are the isolated points [16, 29]

$$\sigma = \Lambda \cdot \exp\left(\frac{2\pi i\, n}{S}\right) \quad \text{with} \quad n = 0, \ldots |S| - 1\,, \tag{2.50}$$

where $S = S_1 \neq 0$ is the sum of chiral charges as defined in eq. (2.36). All other fields have vanishing vacuum expectation value.

For energies μ that are close to the ultraviolet cutoff Λ_{UV} and thereby far above the dynamical scale Λ, the classical target space (2.44) is — unless a continuous non-Abelian subgroup of G remains unbroken — nevertheless a good approximation to the low energy states of the theory [29]. The real FI parameters r_l can no longer be chosen freely but are required to be close to their bare values

$r_l(\Lambda_{\text{UV}}) = \text{sign}(S_l) \cdot \infty$ as determined by eq. (2.39). We will see examples of non-conformal gauged linear sigma models in the next chapter.

Summary

This concludes our introduction of the gauged linear sigma model. There are two key messages to take away. First, with an appropriate choice of gauge theory spectrum the low energy fixed point theory is guaranteed to be a family of $N = (2, 2)$ superconformal field theory with the desired central charge. Second, the FI parameters allow us to extrapolate around the entire quantum Kähler moduli space of this superconformal theory.

2.3 Picard–Fuchs Operators

In this section we continue our discussion of quantum Kähler moduli spaces in $N = (2, 2)$ superconformal field theories. This naturally leads us to the introduction of Picard–Fuchs operators, which figure prominently in the later chapters of this thesis. We also briefly explain the mirror symmetry interpretation. Our review is short and practical, with the focus on introducing the techniques that will be employed in the subsequent chapters.

2.3.1 Quantum Kähler Moduli Space

Let X be a Calabi–Yau threefold. We here explain how the quantum Kähler moduli space $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X)$ can in certain regions be interpreted as a deformation of the complexified geometric Kähler moduli space.

Correlation Functions and Quantum Product

Let H_i with $i = 1, \ldots, h^{1,1}(X)$ be a basis of $H^2_{\text{dR}}(X, \mathbb{Z})$ and $\widetilde{H}^i$ be the dual basis of $H^4_{\text{dR}}(X, \mathbb{Z})$, such that $\int_X \widetilde{H}^i \wedge H_j = \delta^i_j$. These H_i correspond to operators ϕ_i in the (a, c) ring of the associated superconformal field theory, whose triple correlation function reads [80, 81]

$$\langle \phi_i\, \phi_j\, \phi_k \rangle = \kappa_{ijk} + \sum_{\vec{d} \in H_2(X,\mathbb{Z})} N_{\vec{d}}\; d_i\, d_j\, d_k\, \frac{\vec{q}^{\;\vec{d}}}{1 - \vec{q}^{\;\vec{d}}} \quad \text{with} \quad \vec{q}^{\;\vec{d}} = \textstyle\prod_{i=1}^{h^{1,1}} q_i^{d_i}\,. \tag{2.51}$$

Here κ_{ijk} is the classical contribution, whereas the sum arises to due instanton interactions that are characterized by the vector $\vec{d}$ and counted by the integers $N_{\vec{d}}$. The instanton action is $\prod q_i^{d_i}$ and a multi-covering formula gives the presented form. These various quantities also a have a geometric interpretation. Namely, the classical contribution κ_{ijk} is the intersection number $\int_X H_i \wedge H_j \wedge H_k$ and the sum runs over holomorphic curves in X. Further, d_i is the integral of H_i over these curves and the q_i are a distinguished set of coordinates on $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X)$ that we will further explain below. Finally, the integers $N_{\vec{d}}$ are known as the genus zero integral Gromov–Witten invariants and count genus zero holomorphic curves of a fixed degree, see for instance the textbook [82].

The quantum product $*$ is a deformation of the wedge product $\wedge$ between forms in the vertical cohomology $H_{\text{vert}}(X)$ introduced in eq. (2.9), see the review [81]. It is defined by the two requirements that $A * B = A \wedge B$ if $A \wedge B$ is a form of top degree and that the correlation function (2.51) agrees with integral over the triple quantum product $H_i * H_j * H_k$. This implies

$$H_i * H_j = H_i \wedge H_j + \sum_{\vec{d} \in H_2(X,\mathbb{Z})} N_{\vec{d}} \, d_i \, d_j \, d_k \, \frac{\vec{q}^{\,\vec{d}}}{1 - \vec{q}^{\,\vec{d}}} \, \widetilde{H}^k \,, \tag{2.52}$$

where summation over k is implicit. The vertical cohomology $H_{\mathrm{vert}}(X)$ as vector space equipped with the quantum product defines the quantum cohomology ring $QH_{\mathrm{vert}}(X)$ [81].

Givental I-Function

The Givental I-function, introduced in [18] for complete intersections X in compact weak Fano toric varieties, is a mathematical entity that encodes the structure of the quantum cohomology ring. While it can be formulated entirely in terms of geometric quantities, we here choose to describe it with gauged linear sigma model language so as to facilitate our discussions in the subsequent chapters.

Since the geometries covered in [18] arise as target spaces of Abelian models, we specialize to the gauge group $G = \mathrm{U}(1)^\ell$ where ℓ equals the number of Kähler moduli. We assume the FI parameters to be in the vicinity of a geometric large volume boundary component at $\vec{\tau} \to \infty$, such that in this limit the low energy dynamics reduces to the non-linear sigma model on the target space (2.44). It is useful to employ the coordinates

$$Q_l = e^{-2\pi\tau_l} \quad \text{for} \quad l = 1, \ldots, \ell \,, \tag{2.53}$$

which are in vicinity of the large volume boundary component at $\vec{Q} \to 0$. We collect the gauge charges of the chiral multiplets Φ_i into the vectors $\vec{\rho}_i$ and their vector R-charges $\mathfrak{q}_i$ are constrained to be either zero or two. The I-function of the target space (2.44) then reads

$$I(\vec{Q}, \mathfrak{m}_i, \epsilon) = \sum_{\vec{k} \in \gamma_m^+} \prod_{i=i}^{N} \frac{\prod_{s=1+\vec{k}\cdot\vec{\rho}_i}^{\infty} \left[\vec{H} \cdot \vec{\rho}_i + \mathfrak{m}_i + \epsilon\left(s - \frac{\mathfrak{q}_i}{2}\right)\right]}{\prod_{s=1}^{\infty} \left[\vec{H} \cdot \vec{\rho}_i + \mathfrak{m}_i + \epsilon\left(s - \frac{\mathfrak{q}_i}{2}\right)\right]} \vec{Q}^{\,\vec{k} + \frac{\vec{H}}{\epsilon}} \,, \tag{2.54}$$

where $\vec{H} = (H_1, \ldots, H_\ell)$ is the same basis of $H^2_{\mathrm{dR}}(X,\mathbb{Z})$ as above and $\gamma_m^+ = \mathbb{Z}^\ell_{\geq 0}$ a subset of the magnetic charge lattice $\gamma_m = \mathbb{Z}^\ell$. To obtain a compact target space, we often assume that those Φ_i with $\mathfrak{q}_i = 0$ are such that $\vec{k} \cdot \vec{\rho}_i \geq 0$ for all $\vec{k} \in \gamma_m^+$ whereas $\vec{k} \cdot \vec{\rho}_i \geq 0$ for those Φ_i with $\mathfrak{q}_i = 2$. Through the parameter ϵ the I-function also captures Gromov–Witten invariants with insertions of ψ^k at their marked points [18, 83], where ψ is the first Chern class of the universal cotangent line bundle over the moduli space of stable maps and k is a positive integer. The twisted masses $\mathfrak{m}_i$ correspond to equivariant parameters of $\mathbb{C}^*$ symmetries of X and the I-function is understood to take values in the equivariant vertical cohomology ring of X. Note that the infinite products are employed for notational convenience only, almost all terms cancel between numerator and denominator. For the remainder of this section we set $\mathfrak{m}_i = 0$ for all i and write $I(\vec{Q}, \epsilon) = I(\vec{Q}, 0, \epsilon)$.

A mathematical generalization of the I-function to geometries beyond toric varieties, which typically arise as target spaces of gauged linear sigma models with non-Abelian gauge groups, can be found in ref. [84]. In the later chapters, most prominently in chapter 5, we will deal with such cases by means of the associated Abelian Cartan theories.

Quantum Periods and Flat Coordinates

Setting $\mathfrak{m}_i = 0$ in eq. (2.54), the cohomology elements $\vec{H}$ follow the classical ring structure of $H_{\text{vert}}(X)$ given by the wedge product. An expansion of the I-function in these variables gives rise to finitely many terms, whose coefficients are known as the quantum periods — in short ‘periods’ — of X [83]. The coefficients of at most linear terms,

$$
\begin{aligned}
\Pi_0(\vec{Q}) &= I(\vec{Q}, \epsilon)\big|_{\vec{H}=0}\,, \\
\Pi_l(\vec{Q}) &= \epsilon \frac{\partial}{\partial H_l} I(\vec{Q}, \epsilon)\big|_{\vec{H}=0} = \Pi_0(\vec{Q}) \log Q_l + \pi_l(\vec{Q})\,,
\end{aligned}
\tag{2.55}
$$

are of particular interest. Here we have suppressed a potential ϵ dependence, which for conformal gauged linear sigma models with $\mathfrak{m}_i = 0$ indeed cancels out. We respectively refer to $\Pi_0(\vec{Q})$ and $\pi_l(\vec{Q})$ as the *fundamental* and *singly logarithmic* periods. The other quantum periods are found as appropriate linear combinations of higher derivatives and they involve higher powers of logarithms. The functions $\Pi_0(\vec{Q})$ and $\pi_l(\vec{Q})$ are power series with a finite radius of convergence in $\vec{Q}$-space. This more precisely defines the notion of being in the vicinity of large volume. The strict large volume limit $\vec{Q} \to 0$ is due to the presence of logarithms not well defined, which signals a singularity of the superconformal theory at this point. In a string theory context this is a result of having neglected higher Kalazu–Klein modes, although in a large volume case these too become massless.

Note from eq. (2.54) that $\Pi_0(\vec{Q})$ does not vanish in the large volume limit $\vec{Q} \to 0$. The coordinates $\vec{q}$ employed in eq. (2.52) are determined by

$$
\log q_l(\vec{Q}) = b_l + \frac{\Pi_l(\vec{Q})}{\Pi_0(\vec{Q})} = b_l + \log Q_l + \frac{\pi_l(\vec{Q})}{\Pi_0(\vec{Q})}\,,
\tag{2.56}
$$

where the constants b_l are fixed by requiring the Gromov–Witten invariants to be non-negative and the logarithms of $\vec{q}$ are known as *flat* coordinates [48, 85–89]. This equation connects $\vec{\tau}$, which are parameters of the ultraviolet gauge theory and in the conformal case coordinates on $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X)$, to a different set $\vec{q}$ of coordinates on $\mathcal{M}^{\text{IIA}}_{\text{QK}}(X)$ that are more natural from the low energy target space point of view. We refer to eq. (2.56) as the UV-IR map, and note that for a reason explained below it is alternatively known as the mirror map. According to this definition $\vec{q}$ vanishes in the large volume limit, such that in its vicinity the quantum product (2.52) really is a deformation of the wedge product.

Introduction of Picard–Fuchs Operators

Observe that a derivative of the I-function with respect to Q_l brings down a power of H_l and that since $H_{\text{vert}}(X)$ is finite-dimensional there are only finitely many distinct monomials in the H_l. Consequently, there will be differential operators

$$
\mathcal{L}(\vec{Q}) = \sum_{m=1}^{M} c_m(\vec{Q}) \mathcal{D}_m[\partial_Q]
\tag{2.57}
$$

that annihilate the I-function in cohomology. Here M is a finite integer and the $\mathcal{D}_m[\partial_Q]$ are monomials of the partial derivatives with respect to the Q_l. Such operators $\mathcal{L}(\vec{Q})$ are known as Picard–Fuchs operators and their set forms an ideal. Namely, any linear combination of Picard–Fuchs operators as well as the multiplication of an arbitrary operator to the left of a Picard–Fuchs operator still annihilates the I-function. This ideal is finitely generated [90] and we denote a choice of generators as $\mathcal{L}_k(\vec{Q})$ with $k = 1, \ldots, K$. We sometimes refer to these $\mathcal{L}_k(\vec{Q})$ as *the* Picard–Fuchs operators.

Every Picard–Fuchs operator annihilates all quantum periods of X, which motivates us to define the set of equations

$$\mathcal{L}_k(\vec{Q})\,\Pi(\vec{Q}) = 0 \quad \text{for} \quad k = 1, \ldots, K\,. \tag{2.58}$$

A function $\Pi(\vec{Q})$ that obeys these so-called Picard–Fuchs differential equations is annihilated by all operators in the ideal and necessarily is a linear combination of the quantum periods defined by the expansion of the I-function [83]. Conversely, the quantum periods can be written as linear combinations of a basis of solutions to the Picard–Fuchs differential equations (2.58). Such functions $\Pi(\vec{Q})$ are therefore also referred to as quantum periods.

The Picard–Fuchs operators are for instance practically determined by making an ansatz and requiring it to annihilate the quantum periods. Finding a set of generators $\mathcal{L}_k(\vec{Q})$ is, however, in general a difficult question. For cases with only few moduli other arguments determine the order of the generating differential operators, which significantly simplifies the problem.

Yukawa Coupling

We now explain how these methodologies allow us to determine the Gromov–Witten invariants. For this we specialize to a single Kähler modulus, $h^{1,1}(X) = \ell = 1$, which is the case of our main interest. We define

$$W^{0,3}(Q) = \left(\frac{Q}{q}\frac{\partial q}{\partial Q}\right)^3 \int_X H * H * H\,, \tag{2.59}$$

where H is the single generator of $H^2_{\mathrm{dR}}(X, \mathbb{Z})$. This quantity has two interpretations. The first, which we will explain below, involves mirror symmetry and identifies $W^{0,3}$ as the B-model Yukawa coupling [13]. It can be shown to obey a differential equation governed by the Picard–Fuchs operator, by which it is determined exactly. While the derivation of this method involves mirror symmetry, the calculation can proceed without knowledge of the mirror manifold Y as long as the Picard–Fuchs operator is known. In the second interpretation $W^{0,3}$ is a certain correlation function of the gauged linear sigma model, which by modern localization techniques [91, 92] can also be calculated exactly [73, 93]. This approach does not make reference to mirror symmetry at all and will be covered in the next chapter. Regardless of the interpretation, the Gromov–Witten invariants N_d are determined by expanding both sides of eq. (2.59) in terms of q and comparing coefficients.

Example

To clarify the above definitions and discussions, let us come back to the gauged linear sigma model discussed at the end of subsection 2.2.6. For $r \gg 0$ the low energy target space was found to be the quintic Calabi–Yau threefold threefold $\mathbb{P}^4[5]$, whose Givental I-function according eq. (2.54) is

$$I_{\mathbb{P}^4[5]}(Q, \epsilon) = \sum_{k=0}^{\infty} (-1)^k \frac{\prod_{s=1}^{5k} (5H + \epsilon \cdot s)}{\prod_{s=1}^{k} (H + \epsilon \cdot s)^5}\, Q^{k+\frac{H}{\epsilon}}\,. \tag{2.60}$$

The quantum periods Π_0 and Π_1 defined in eq. (2.55) follow from an expansion of $I_{\mathbb{P}^4[5]}$ up to linear order in H, they read

$$\Pi_0(Q) = \sum_{k=0}^{\infty} (-Q)^k \frac{(5k)!}{k!^5}\,, \quad \Pi_1(Q) = \sum_{k=0}^{\infty} (-Q)^k \frac{(5k)!}{k!^5} \cdot \left[\log Q + 5\left(h_{5k} - h_k\right)\right] \tag{2.61}$$

where h_k denotes the k-th harmonic number. By inverting the exponential of eq. (2.56) we find Q as a function of q,

$$Q = \frac{q}{b'} + 770\left(\frac{q}{b'}\right)^2 + 171\,525\left(\frac{q}{b'}\right)^3 + O\left(q^4\right)\,, \tag{2.62}$$

with b' the exponential of b_1. The quantum periods are annihilated by the fourth-order Picard–Fuchs operator

$$\mathcal{L}_{\mathbb{P}^4[5]}(Q) = \sum_{k=0}^{4} c_k(Q)\,\Theta^k = \Theta^4 + 5Q\prod_{l=1}^{4}(5\Theta + l) \quad \text{with} \quad \Theta = Q\,\partial_Q\,. \tag{2.63}$$

The fact that the coefficients $c_k(Q)$ are polynomial in Q is not obvious from how we introduced Picard–Fuchs operators in general and is a consequence of $\mathbb{P}^4[5]$ being a compact geometry. As we will show below, $W^{0,3}(Q)$ obeys the differential equation

$$\left[2c_4(Q)\,\Theta + c_3(Q)\right] W^{0,3}(Q) = 0\,, \tag{2.64}$$

where $c_k(Q)$ is the coefficient of Θ^k in $\mathcal{L}_{\mathbb{P}^4[5]}(Q)$. This differential equations is readily solved and gives

$$W^{0,3}(Q) = \frac{a}{1 + 5^5 Q} \tag{2.65}$$

with some constant a. Finally, the triple intersection number is $\int_X H \wedge H \wedge H = 5$. Equations (2.59) and (2.52) then give the famous numbers [13, 94, 95]

$$N_1 = 2\,875\,, \quad N_2 = 609\,250\,, \quad \ldots \tag{2.66}$$

as well as $a = 5$ and $b' = -1$. It is important to note that $W^{0,3}(Q)$ is singular for $Q = -5^5$, at which we also observe $c_4(Q)$ to vanish. This is not a coincidence, it precisely is the location of the Coulomb branch as predicted by eq. (2.45). The periods in eq. (2.61) converge for $|Q| < 5^5$.

Why Picard–Fuchs Operators are Useful

Let us now summarize why we are interested in Picard–Fuchs operators and not just content ourselves with the I-function. As the first and most important reason, they allow us to continue the quantum periods across the entire quantum Kähler moduli space. To do this, we rewrite the operators in terms of coordinates centered around a different point and then solve the new Picard–Fuchs differential equations. The solutions obtained this way are analytic continuations of the periods defined by the I-function in vicinity of large volume. We will use this technique in chapter 5. While this method works for all points and boundary limits of moduli space, in special cases the analytic continuation can without reference to a Picard–Fuchs operator also be done by means of a Mellin–Barnes integral representation, see for instance ref. [13].

For discussion further reasons we specialize to compact Calabi–Yau threefolds with a single Kähler modulus. In this case the generating operator immediately determines the B-model Yukawa coupling $W^{0,3}$ through eq. (2.64) and moreover encodes the locations at which the superconformal theory becomes singular. For this we rescale the operator such that the coefficients $c_k(Q)$ in eq. (2.63) are polynomials with no common factor. The superconformal theory can then only be singular at the origin, at infinity, or at points where $c_4(Q) = 0$. On the contrary, not all points at which $c_4(Q) = 0$ are necessarily singular.

Calabi–Yau Manifolds of Different Dimension

The above discussions on the I-function, periods and Picard–Fuchs operators equally apply to Calabi–Yau n-folds of different complex dimensions than $n = 3$. While its precise form changes, the quantum product can also be generalized, see ref. [81].

Non-Conformal Models

The notion of the Givental I-function extends to manifolds that are not of the Calabi–Yau type. These arise as target spaces of non-conformal gauged linear sigma models and eq. (2.54) is physically understood to be at an energy scale μ close to the ultraviolet cutoff Λ_{UV}. While the I-function is still annihilated by differential operators, these are not referred to as Picard–Fuchs operators. Similarly, the solutions to the differential equations defined by the operators are still of interest but not referred to as periods.

2.3.2 Complex Structure Moduli Space

There are also Picard–Fuchs operators on the complex structure moduli spaces $\mathcal{M}_{\text{CS}}$. To be precise, we consider $\mathcal{M}^{\text{IIB}}_{\text{vector}}(Y) = \mathcal{M}_{\text{CS}}(Y)$ where Y is the mirror manifold of X. The operators are then literally the same as those studied in the previous subsection and we are able to explain some of the assertions made there.

Introduction of Picard–Fuchs Operators

The holomorphic (3, 0)-form Ω on Y is the (up to scaling) unique element in $H^{3,0}(Y)$ and depends on the choice of complex structure on Y. We hence write $\Omega = \Omega(\vec{\xi}\,)$, where the components ξ_α of $\vec{\xi}$ with $\alpha = 1, \ldots, h^{2,1}(Y) = h^{1,1}(X)$ are coordinates on $\mathcal{M}_{\text{CS}}(Y)$. Derivatives of $\Omega(\vec{\xi})$ with respect to ξ_α are still closed differential forms of total form degree three, i.e., are elements of the horizontal cohomology $H_{\text{hor}}(Y)$ defined in eq. (2.9). Since this space is finite-dimensional, there necessarily will be linear dependencies between $\Omega(\vec{\xi}\,)$ and its derivatives. These translate into differential operators $\mathcal{L}(\vec{\xi}\,)$ that annihilate $\Omega(\vec{\xi}\,)$ in cohomology,

$$\mathcal{L}(\vec{\xi}\,)\,\Omega(\vec{\xi}\,) = [0] \quad \text{with} \quad \mathcal{L}(\vec{\xi}\,) = \sum_{m=1}^{M} c_m(\vec{\xi}\,)\,\mathcal{D}_m[\partial_\xi]\,. \tag{2.67}$$

Here M is a finite integer and the $\mathcal{D}_m[\partial_\xi]$ are monomials in the partial derivatives with respect to the complex structure coordinates. These $\mathcal{L}(\vec{\xi}\,)$ are the Picard–Fuchs operators.

Griffith Transversality and B-Model Correlation Functions

A tighter connection to the structure of $H_{\text{hor}}(Y) = QH_{\text{vert}}(X)$ is drawn by so-called Griffith transversality [96]. To explain this, let us restrict to cases with $h^{2,1}(Y) = h^{1,1}(X) = 1$ and generalize to arbitrary complex dimensions n. The statement is in terms of the holomorphic bundles

$$\mathcal{F}^{n-m} = \bigoplus_{k=0}^{m} H^{n-k,k}(Y) \quad \text{for} \quad 0 \leq m \leq n\,, \tag{2.68}$$

which define a filtration $F^n \subset \ldots \subset F^0 = H^n_{\text{dR}}(Y, \mathbb{C})$ of the n-th de Rahm cohomology group. Derivatives of the holomorphic $(n,0)$-form obey

$$\partial_\xi^m\, \Omega(\xi) \begin{cases} \in \mathcal{F}^{n-m} \quad \text{and} \quad \notin \mathcal{F}^{n-(m-1)} & \text{for} \quad n \geq m \geq 1 \\ \in \mathcal{F}^0 & \text{for} \quad m \geq n+1 \end{cases} . \tag{2.69}$$

Let us now restrict to Calabi–Yau threefolds, i.e., we take $n = 3$. The above equation then implies that there cannot be a Picard–Fuchs operator of order three or lower, but there necessarily is one of order four as confirmed by eq. (2.63) — for a more detailed exposition of this argument see subsection 5.2.1. Lastly, we consider the quantities

$$W^{a,b}(\xi) = \int_Y \Theta_\xi^a\, \Omega(\xi) \wedge \Theta_\xi^b\, \Omega(\xi) \quad \text{with} \quad \Theta_\xi = \xi\, \partial_\xi\,. \tag{2.70}$$

This generalizes the B-model Yukawa coupling in eq. (2.59), which agrees with $W^{0,3}(\xi)$ when Q and ξ identified. From Griffith transversality we find $W^{0,b} = 0$ for $b \leq 2$ and an elementary calculation gives

$$0 = \Theta_\xi^2\, W^{0,2}(\xi) - 2\Theta_\xi\, W^{0,3}(\xi) + W^{0,4}(\xi) = -2\Theta_\xi\, W^{0,3}(\xi) + W^{0,4}(\xi)\,. \tag{2.71}$$

In order to derive the differential equation (2.64) that allowed us to solve for $W^{0,3}$, we make the additional observation

$$0 = \int_X \Omega(\xi) \wedge \mathcal{L}(\xi)\, \Omega(\xi) = c_4(\xi)\, W^{0,4}(\xi) + c_3(\xi)\, W^{0,3}(\xi)\,. \tag{2.72}$$

In combination with eq. (2.71) this gives the desired result. We note that this derivation does not require detailed knowledge about the mirror geometry Y and can be used as long as Y exists.

Periods

For completeness, we briefly introduce quantum periods in this context. These are integrals of the holomorphic $(3,0)$-form over $\omega \in H_3(X, \mathbb{C})$,

$$\Pi_\omega(\vec{\xi}\,) = \int_\omega \Omega(\vec{\xi}\,)\,. \tag{2.73}$$

They are annihilated by the entire ideal of Picard–Fuchs operators and for appropriate choices of ω agree with the quantum periods defined by the I-function of X. If ω is an element of $H_3(X, \mathbb{Z})$, the above integral is a so-called *integral* quantum period. This concept will play a role in chapter 5.

CHAPTER 3

The Geometry of Gauged Linear Sigma Model Correlation Functions

The previous chapter has introduced the gauged linear sigma model and Picard–Fuchs operators as two complementary tools for studying the moduli structure of superconformal worldsheet theories that arise in supersymmetric type II string compactifications. This chapter demonstrates that these two tools are, in fact, closely connected to each other. In particular, we show that the Picard–Fuchs operators arise — and can be determined — from a certain set of observables in the gauged linear sigma model. A similar result applies to non-conformal models.

This chapter is based on the author's publication [17].

3.1 Introduction and Results

The central player of this chapter are correlation functions of the complex scalar σ in the vector superfield of gauged linear sigma models, to which we refer as 'correlators'. We systematically analyze these observables from a gauge theory point of view and study their target space interpretation. Our analysis is based on a localization computation [73] by Closset, Cremonesi and Park which opens the possibility to calculate the correlators quantum exactly — including all perturbative and non-perturbative effects — for any value of the complexified Fayet–Iliopoulos (FI) parameters. This computation generalizes the methods of ref. [57] by means of modern localization techniques [91, 92], in which curved space supersymmetry is realized by a suitable off-shell supergravity background. Since the particular background chosen in ref. [73] relates to A-twisted gauged linear sigma models in the context of mirror symmetry [97], the correlators are expected to contain information about the quantum cohomology of the target space. This manifests itself in the results of refs. [98, 99], whose authors conjecture and for a certain class of target spaces prove the correlators to arise from a bilinear pairing of the Givental I-function [18]. Correlators of gauged linear sigma models are also calculated in ref. [93]. Our central results are the following:

- Starting from the localization formula of ref. [73], we demonstrate that there are universal and non-trivial linear dependencies among the set of correlators. Our prove is constructive and yields a combinatorial algorithm that determines these relations from the defining gauge theory spectrum directly, without the need to explicitly calculate any correlator.

- By employing a Hilbert space interpretation we map these universal correlator relations to differential operators that annihilate the gauge theory ground state. In case of a geometric target space we use the connection of correlators and Givental I-function to argue that these differential operators generate the GKZ system of differential equations governing the target space quantum cohomology.

- We find that for conformal gauged linear sigma models these differential operators are the Picard–Fuchs operators on the quantum Kähler moduli space of the low energy $N = (2, 2)$ superconformal field theory.

- For several classes of Calabi–Yau manifolds — specified by a fixed complex dimension and number of Kähler moduli — we derive formulas that universally express the generating Picard–Fuchs operators in terms of the gauge theory correlator. These formulas automatically obey certain non-trivial constraints, for example $\mathcal{N} = 2$ special geometry [88] in the case of Calabi–Yau threefolds.

Focusing on the physically important case of conformal models, these findings provide us with two methods to determine the Picard–Fuchs operators on the quantum Kähler moduli space of $N = (2, 2)$ superconformal field theories from the correlators of the corresponding gauged linear sigma model. The correlators thereby encode the quantum cohomology the target space — including for example the genus zero integral Gromov–Witten invariants encountered in eq. (2.51) — and more generally speaking the moduli structure of the superconformal theories. Traditionally, the Picard–Fuchs operators are often indirectly determined via mirror symmetry [13]. This is particularly powerful for complete intersection Calabi–Yau manifolds in toric varieties, since these admit a systematic mirror construction [100, 101]. Our methods are complementary and determine the Picard–Fuchs operators without the need to construct a mirror geometry, which for Calabi–Yau compactifications beyond complete intersection in toric varieties is not always known. Moreover, there is no need to factor a higher order differential operator as is common in other approaches, see e.g. ref. [102].

We note that in addition to the correlator based approach presented here, there are other gauged linear sigma model observables that encode infrared gauge theory quantities without employing mirror symmetry. In particular, the sphere partition function computes the quantum-exact Kähler metric on the quantum Kähler moduli space [103–105] and the hemisphere partition functions directly yields exact expressions for the quantum periods [106–108].

3.2 Abelian A-Twisted Correlators

This section introduces A-twisted correlators of $N = (2, 2)$ two-dimensional gauge theories. These are the central object of study in this chapter. To simplify the discussion and to not obscure the main ideas with technical details, we here specialize to Abelian gauged linear sigma models and refer to section 3.5 for the generalization to non-Abelian gauge groups.

3.2.1 General Properties

Our notation follows that of previous chapter. The Abelian gauge group reads $G = \mathrm{U}(1)^\ell$ and there are N chiral multiplets Φ_i whose gauge and vector R-charges respectively are $\vec{\rho}_i \in \mathbb{Z}^\ell$ and $\mathfrak{q}_i$. We turn on generic twisted masses $\mathfrak{m}_i$, which are an important technical ingredient in our calculations as will be discussed further below.

The complex scalar σ in the vector superfield V decomposes as $\sigma = \sum \sigma_l$, where the components σ_l with $l = 1, \ldots, \ell$ correspond to the individual U(1) factors in G. We collect them into the vector $\vec{\sigma} = (\sigma_1, \ldots, \sigma_\ell)$. The authors of ref. [73] calculate A-twisted correlators of monomials of these σ_l, for the purpose of which they put the gauge theory on a two-sphere with a suitable off-shell supergravity background that realizes a topological A-twist. Field insertions are then only BRST invariant at the sphere's north pole (subscript 'N') and south pole (subscript 'S'), such that the correlators take the general form

$$\left\langle \vec{\sigma}_N^{\vec{n}} \vec{\sigma}_S^{\vec{m}} \right\rangle = \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) \quad \text{with} \quad \vec{n}, \vec{m} \in \mathbb{Z}^\ell_{\geq 0} \,. \tag{3.1}$$

We here use the short-hand notation $\sigma^{\vec{n}} = \sigma_1^{n_1} \cdots \sigma_\ell^{n_\ell}$, the variables $\vec{Q} = (Q_1, \ldots, Q_\ell)$ are defined in terms of the complexified FI parameters $\vec{\tau}$ as in eq. (2.53), the entries of $\vec{n}$ and $\vec{m}$ are non-negative integers and ϵ is a parameter of the supergravity background. We view the correlators as functions of $\vec{Q}$, the twisted masses $\mathfrak{m}_i$ and ϵ — and not as functions of the locations of field insertions, on which they depend only through $\vec{n}$ and $\vec{m}$. The A-twist moreover guarantees that the correlators are independent of the parameters $\vec{y}$ in the superpotential.

Since the metric dependence of the correlators is encoded in ϵ, their value at $\epsilon = 0$ is topological and insensitive to the location of the field insertion [73]. This implies the symmetry

$$\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, 0) = \kappa_{\vec{n}',\vec{m}'}(\vec{Q}, \mathfrak{m}_i, 0) \quad \text{for all} \quad \vec{n} + \vec{m} = \vec{n}' + \vec{m}' \in \mathbb{Z}^\ell_{\geq 0} \,. \tag{3.2}$$

Let us now think of $\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)$ being expanded as a power series of its argument. The axial R-symmetry (even in the anomalous case) gives a selection rule due to which only those terms that are in accord with the equation

$$d + \#(\epsilon) + \#(\mathfrak{m}_i) + \sum_{l=1}^{\ell} S_l \cdot \#(Q_l) = |\vec{n}|_1 + |\vec{m}|_1 \tag{3.3}$$

can be non-zero [57, 73]. Here $\#(\,\cdot\,)$ denotes the exponent of its argument, we write $|\vec{n}|_1 \equiv \sum_k |n_k|$ and S_l is the sum of gauge charges as defined in eq. (2.36). The integer d is $1/3$ times the right hand side of eq. (2.38), which in case of a geometric target space agrees with its complex dimension and for conformal models equals $1/3$ times the central charge of the low energy superconformal theory. We now distinguish three special cases.

1. In case of *conformal* gauged linear sigma models we have $S_l = 0$ for all l, such that the selection rule (3.3) does not constrain the $\vec{Q}$ dependence of $\kappa_{\vec{n},\vec{m}}$. Assuming a model with compact target space, the correlators are guaranteed to be finite except at those values of $\vec{Q}$ where a Coulomb branch arises, see eq. (2.45). Their correlators are rational functions of $\vec{Q}$ with poles at these singularities only and remain finite in the limits $\epsilon \to 0$ and $\mathfrak{m}_i \to 0$. The selection rule (3.3) then implies $\kappa_{\vec{n},\vec{m}} = 0$ for $|\vec{n}|_1 + |\vec{m}|_1 < d$. In case of a non-compact target space some $\kappa_{\vec{n},\vec{m}}$ are expected to diverge in the limit $\mathfrak{m} \to 0$ of vanishing twisted masses.

2. Gauged linear sigma models with $S_l > 0$ for all l are said to possess the *Fano* property. The real FI parameters $\vec{r}$ are then physically required to be close to their bare values $r_l(\Lambda_{\text{UV}}) \to +\infty$ for all l, in case of which the classical target space becomes a good approximation to the low

energy states of the theory. The correlators are finite for $\vec{Q} \to 0$ and according to the selection rule (3.3) are polynomial in $\vec{Q}$ as long as the limits $\epsilon \to 0$ and $\mathfrak{m}_i \to 0$ exist.

3. Gauged linear sigma models with $S_l < 0$ for all l are referred to have the *ample canonical bundle* property. Since the real FI parameters are required to be close to their bare values $r_l(\Lambda_{\mathrm{UV}}) \to -\infty$ for all l, we should then rather work with $1/Q_l$ instead of Q_l. In terms of these inverted variables the discussion parallels the previous case.

We stress that a general gauged linear sigma model does not belong to any of these three classes. In such cases we can a priori say less about the structure of the correlators.

3.2.2 Localization Formula

As starting point for our analysis in the next section, we will employ the localization formula of ref. [73] according to which

$$\kappa_{\vec{n},\vec{m}}(\vec{Q},\mathfrak{m}_i,\epsilon) = \sum_{\vec{k}\in\gamma_m} \vec{Q}^{\vec{k}}\, \mathrm{Res}^{\vec{r}}_{\vec{\sigma},\vec{k}} \left[\left(\vec{\sigma} - \frac{\epsilon}{2}\vec{k}\right)^{\vec{n}} \left(\vec{\sigma} + \frac{\epsilon}{2}\vec{k}\right)^{\vec{m}} Z_{\vec{k}}(\vec{\sigma},\mathfrak{m}_i,\epsilon) \right] . \tag{3.4}$$

Here $\vec{Q}^{\vec{k}} = Q_1^{k_1} \cdots Q_\ell^{k_\ell}$ is the classical action in the topological sector labeled by $\vec{k}$ and the sum is over the co-character lattice $\gamma_m \simeq \mathbb{Z}^\ell$ of the gauge group $G = \mathrm{U}(1)^\ell$, which physically is the magnetic charge lattice. The one loop determinant $Z_{\vec{k}}(\vec{\sigma},\mathfrak{m}_i,\epsilon)$ of the chiral multiplets reads

$$Z_{\vec{k}}(\vec{\sigma},\mathfrak{m}_i,\epsilon) = \prod_{i=1}^{N} Z^{(i)}_{\vec{k}}(\vec{\sigma},\mathfrak{m}_i,\epsilon)\,, \quad Z^{(i)}_{\vec{k}}(\vec{\sigma},\mathfrak{m}_i,\epsilon) = \epsilon^{\mathfrak{q}_i-\vec{\rho}_i\cdot\vec{k}-1} \frac{\Gamma\left(\frac{\vec{\rho}_i\cdot\vec{\sigma}+\mathfrak{m}_i}{\epsilon} + \frac{\mathfrak{q}_i-\vec{\rho}_i\cdot\vec{k}}{2}\right)}{\Gamma\left(\frac{\vec{\rho}_i\cdot\vec{\sigma}+\mathfrak{m}_i}{\epsilon} - \frac{\mathfrak{q}_i-\vec{\rho}_i\cdot\vec{k}}{2}+1\right)}\,, \tag{3.5}$$

where each individual factor corresponds to a single multiplet. Note that the ratio of gamma functions always reduces to a rational function.

This localization formula is quantum exact, it includes all perturbative and non-perturbative effects. For conformal gauged linear sigma models the FI parameters $\vec{\tau}$ can be freely chosen and eq. (3.4) is applicable irrespective of this choice. In particular, it is not restricted to regions of the moduli space in which the low energy superconformal field theory enjoys a geometric interpretation.

Residue Symbol

Let us now explain the precise meaning of the residue operation in eq. (3.4) To this end we first consider a set of ℓ chiral multiplets $\Phi_{i_1},\ldots,\Phi_{i_\ell}$ with linearly independent charge vectors $\vec{\rho}_{i_1},\ldots,\vec{\rho}_{i_\ell}$. These vectors span an ℓ-dimensional cone $\sigma(i_1,\ldots,i_\ell)$ in the electric charge lattice $\gamma_e \simeq \mathbb{Z}^\ell$ of G. We define Σ as the set of all such cones and $\Sigma(\vec{r})$ as the subset of cones that contain the vector $\vec{r}$ of real FI parameters. Second, we let $\Pi(\vec{k}\,|\,i_1,\ldots,i_\ell)$ be the countable set of poles in the variables of integration $\vec{\sigma} \in \mathbb{C}^\ell$ that arise from the fields $\Phi_{i_1},\ldots,\Phi_{i_\ell}$ in the topological sector $\vec{k}$. In other words, this set is given by the intersection of the equations $Z^{(i_a)}_{\vec{k}}(\vec{\sigma},\mathfrak{m}_{i_a},\epsilon)^{-1} = 0$ for $a = 1,\ldots,\ell$. In terms of these objects the residue symbol used in eq. (3.4) is defined by

$$\mathrm{Res}^{\vec{r}}_{\vec{\sigma},\vec{k}}\left(\ldots\right) = \sum_{\sigma(i_1,\ldots,i_\ell)\in\Sigma(\vec{r})}\ \sum_{\vec{x}\in\Pi(\vec{k}\,|\,i_1,\ldots,i_\ell)} \mathrm{Res}_{\vec{\sigma}=\vec{x}}\left(\ldots\right)\,. \tag{3.6}$$

Here $\mathrm{Res}_{\vec{\sigma}=\vec{x}}$ denotes the conventional higher-dimensional residue, a rigorous definition of which can for instance be found in the textbook [109].

Phase Independence

The residue operation defined by eq. (3.6) clearly depends on the gauge theory phase as specified by the real FI parameters $\vec{r}$. As we recall, for gauged linear sigma models with the Fano or ample canonical bundle property the r_l are required to be close to their bare value $r_l(\Lambda_{\mathrm{UV}}) = \pm\infty$ for all l. For conformal models, however, we can freely choose $\vec{r}$ and with this the phase to calculate in, the result will be the same rational function. This is particularly useful for non-Abelian models — see section 3.5 for an explanation of how the calculation of their correlators can be reduced to eq. (3.4) — as these may have both weakly and strongly coupled phases. We can then apply eq. (3.4) in the weakly coupled phase and thereby obtain a result that is equally valid in the strong coupling regime.

Generic vs. Non-Generic Twisted Masses

We stress that the localization formula (3.4) together with the residue operation (3.6) assumes generically chosen twisted masses $\mathfrak{m}_i$. To be precise, we require that the pole sets $\Pi(\vec{k}\,|\,i_1,\ldots,i_\ell)$ for different $i_1,\ldots,i_\ell$ but same $\vec{k}$ are mutually disjoint. Correlators for non-generic twisted masses $\mathfrak{m}_i^0$ are defined by taking the limit $\mathfrak{m}_i \to \mathfrak{m}_i^0$ after the residue operation, as long as this limit exists.

For non-generic twisted masses there are points at which the singular divisors arising from strictly more than ℓ chiral multiplets intersect. At these point the conventional residue $\mathrm{Res}_{\vec{\sigma}=\vec{x}}$ is not always well defined and, even if it is, will not necessarily give the correct contribution to $\kappa_{\vec{n},\vec{m}}$. One method of dealing with such degenerate points is to temporarily introduce auxiliary parameters that pull the intersecting poles apart, then employ eq. (3.6) and finally take the auxiliary parameters back to zero — see the Appendix of ref. [56] for a related discussion in the context of sphere partition functions. This is effectively equivalent to working with generic twisted masses and then taking the limit $\mathfrak{m}_i \to \mathfrak{m}_i^0$.

Practical Comments

In practice we cannot calculate all terms of the infinite sum in eq. (3.4). This is where the algebraic structure of the correlators as discussed in subsection 3.2.1 becomes important. For gauged linear sigma model with the Fano property — and equivalently for those of the ample canonical bundle type when Q_l is replaced by $1/Q_l$ — we expect that only finitely many terms are non-zero. Given fixed values for $\vec{n}$ and $\vec{m}$, the order of $\vec{Q}$ above which there will be no contributions can be inferred from the structure of the one loop determinant. For conformal gauged linear sigma models the correlators are rational functions, such that the sum will never truncate. We therefore calculate up to a given order of $\vec{Q}$, rewrite the results to this order as a rational function, for example by using a Padé approximant, and increase the cutoff order until the result stabilizes. The rational function obtained this way has to be finite except at the singularities predicted by eq. (2.45). As we will see below, there are symmetries and other identities between correlators for different $\vec{n}$ and $\vec{m}$ that furnish additional consistency checks.

When we are only interested in the correlators at certain non-generic twisted masses $\mathfrak{m}_i^0$, it is often not necessary to begin with a different generic value $\mathfrak{m}_i$ for every single field. For example, those subsets of fields that become equivalent in the non-generic limit — i.e., have the same gauge charges, vector R-charges and $\mathfrak{m}_i^0$ — can safely be assigned the same value $\mathfrak{m}_i$. This speeds up calculations significantly. In general, it is important to avoid an overlap of poles associated to cones in $\Sigma(\vec{r})$ with those associated to cones in $\Sigma \setminus \Sigma(\vec{r})$. For a related discussion see ref. [110].

3.2.3 Connection to the Givental I-Function

For gauged linear sigma models with a compact geometric target space $\mathcal{X}$ at $\vec{Q} \to 0$ the A-twisted correlators follow from the Givental I-function (2.54) of $\mathcal{X}$ with the formula [99]

$$\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) = \int_{\mathcal{X}} (-\epsilon\vec{\Theta})^{\vec{n}} I_{\mathcal{X}}(\vec{Q}, \mathfrak{m}_i, -\epsilon) \cup (\epsilon\vec{\Theta})^{\vec{m}} I_{\mathcal{X}}(\vec{Q}, \mathfrak{m}_i, \epsilon) . \tag{3.7}$$

Here Θ_l is the logarithmic derivative with respect to Q_l and $\vec{\Theta}^{\vec{n}} = \Theta_1^{n_1} \dots \Theta_\ell^{n_\ell}$. In addition to complete intersections in toric varieties — corresponding to Abelian gauged linear sigma models — the above equation is also proven for Grassmannian target spaces [99]. This builds on the generalization of the I-function to more general geometries [84] and corresponds to models with non-Abelian gauge groups. Equation (3.7) allows us to interpret the correlators geometrically in the context of Gromov–Witten theory [18, 83].

We also note the formal similarity to eq. (2.70). Upon setting the twisted masses to zero, the quantities $W^{a,b}$ of the mirror of $\mathcal{X}$ and the correlators of $\mathcal{X}$ agree up to minus signs and powers of ϵ. We will explicitly observe this in the example of subsection 3.7.2.

3.3 Correlator Relations in Abelian Models

In this section we define and derive one of the central findings of this chapter, namely the aforementioned correlator relations. Until specified otherwise, the twisted masses are generic.

3.3.1 Definition

Recall that the Givental I-function maps to the finite-dimensional equivariant vertical cohomology ring of the target space. Therefore, it seems plausible that the integrals on the right hand side of eq. (3.7) do not give rise to infinitely many independent quantities. We rather except linear dependencies between different $\kappa_{\vec{n},\vec{m}}$ and try to capture these in the below definition of correlator relations. We stress, however, that the discussion does not make reference to the I-function or a geometric target space at all and is valid independently of these concepts.

Our objective is to find linear dependencies between correlators $\kappa_{\vec{n},\vec{m}}$, where we keep $\vec{n}$ fixed but arbitrary and sum over different $\vec{m}$ with coefficients that are polynomial in the variables Q_l. In formulas, we set out determine relations of the form

$$0 = R_S = \sum_{\vec{m}=0}^{\vec{M}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)\, \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) = \sum_{\vec{m}=0}^{\vec{M}} \sum_{\vec{p}=0}^{\vec{s}} c_{\vec{m},\vec{p}}(\mathfrak{m}_i, \epsilon)\, \vec{Q}^{\vec{p}}\, \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) . \tag{3.8}$$

Here we abbreviate $\vec{Q}^{\vec{p}} = Q_1^{p_1} \cdots Q_\ell^{p_\ell}$, the vectors $\vec{M}$ and $\vec{s}$ are arbitrary elements of $\mathbb{Z}^\ell_{\geq 0}$ and we write $\vec{a} \leq \vec{b}$ if $a_i \leq b_i$ for all i. This definition is non-trivial because it is required to hold for all powers $\vec{n} \in \mathbb{Z}^\ell_{\geq 0}$ of the north pole insertion $\vec{\sigma}_N$. Consequently, R_S is referred to as a universal south pole correlator relation.

We may analogously define universal north pole correlator relations R_N, which are required to hold for any power $\vec{m}$ of the south pole insertion $\vec{\sigma}_S$. However, the localization formula (3.4) together with the one loop determinant (3.5) implies the symmetry property

$$\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) = (-1)^{d+|\vec{n}|_1+|\vec{m}|_1} \kappa_{\vec{m},\vec{n}}((-1)^{\vec{S}}\vec{Q}, -\mathfrak{m}_i, \epsilon) \tag{3.9}$$

where $(-1)^{\vec{S}}\vec{Q} = ((-1)^{S_1}Q_1, \ldots, (-1)^{S_\ell}Q_\ell)$. This shows that north and south pole relations are in a one-to-one correspondence through the involution $c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) \to (-1)^{|\vec{m}|_1} c_{\vec{m}}((-1)^{\vec{S}}\vec{Q}, -\mathfrak{m}_i, \epsilon)$, such that we would not obtain new information by additionally studying north pole relations. To abbreviate, we henceforth refer to universal south pole correlator relations simply as correlator relations.

3.3.2 Derivation

The factors $\vec{Q}^{\vec{p}}$ on the right hand side of eq. (3.8) will combine with factors $\vec{Q}^{\vec{k}}$ that come from summation over $\vec{k} \in \gamma_m$ within the localization formula (3.4). It would be convenient to define $\vec{k}' = \vec{p} + \vec{k}$ and to rewrite the sum over $\vec{k} \in \gamma_m$ as a sum over $\vec{k}' \in \gamma_m$. However, since the residue operation in (3.4) depends on $\vec{k}$, this would result in a $\vec{p}$ dependent residue operation and we would then not be able to pull the summation over $\vec{p}$ inside the residue.

Modified Residue Symbol

In order to remedy this technical difficulty we introduce a modified residue operation. For this purpose we let $P(i_1, \ldots, i_\ell) \subset \mathbb{C}^\ell$ be the smallest lattice that contains the pole sets $\Pi(\vec{k} \,|\, i_1, \ldots, i_\ell)$ associated to a given cone $\sigma(i_1, \ldots, i_\ell) \in \Sigma$ for all $\vec{k} \in \gamma_m$. Due to our assumption of generic twisted masses, these lattices $P(i_1, \ldots, i_\ell)$ are still mutually disjoint. Therefore, we can define

$$\widetilde{\operatorname{Res}}^{\vec{r}}_{\vec{\sigma}} \Big(\ldots \Big) = \sum_{\sigma(i_1,\ldots,i_\ell) \in \Sigma(\vec{r})} \; \sum_{\vec{x} \in P(i_1,\ldots,i_\ell)} \operatorname{Res}_{\vec{\sigma}=\vec{x}} \Big(\ldots \Big) \tag{3.10}$$

and without changing the correlators use this modified residue operation in the localization formula (3.4). As compared to the prescription given in eq. (3.6), we essentially add a lot of zeros.

Constraint Equation

The actual derivation starts with the assumption that eq. (3.8) holds for some choice of coefficients $c_{\vec{m}} = c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)$ that are yet to be determined. We then insert the localization formula (3.4) together with the modified residue prescription (3.10) and employ the replacement $\vec{k} = \vec{k}' - \vec{p}$. This results in summation over $\vec{k}' \in \gamma_m$, where the individual terms come with the powers $\vec{Q}^{\vec{k}'}$. Any two different values of $\vec{k}'$ thus have a different $\vec{Q}$ dependence and every term in the sum needs to vanish separately. This gives a $\vec{k}'$ dependent residue expression — the function inside the residue depends on $\vec{k}'$, not the residue symbol itself — that is required to vanish for all $\vec{k}'$. After some further rearrangements the function inside the residue factors into a $\vec{k}'$ and $\vec{n}$ dependent but $c_{\vec{m}}$ independent first part, as well as a $\vec{k}'$ and $\vec{n}$ independent but $c_{\vec{m}}$ dependent second part. Since the residue is required to vanish for all $\vec{k}'$ and $\vec{n}$, we conclude that the seccond factor needs to be zero by itself. This is a constraint on the coefficients $c_{\vec{m}}$, which is conveniently written as

$$0 = \sum_{\vec{p}=0}^{\vec{s}} \alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_i, \epsilon) \cdot g_{\vec{p}}(\vec{w}, \mathfrak{m}_i, \epsilon) \,. \tag{3.11}$$

Here the $\alpha_{\vec{p}}$ are polynomials in the variables $\vec{w}$ that are determined by the choice of $c_{\vec{m}}$ and the $g_{\vec{p}}$ are rational functions of $\vec{w}$ that only depend on the choice of gauge theory spectrum, namely

$$\begin{aligned} \alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_i, \epsilon) &= \sum_{\vec{m}=0}^{\vec{M}} c_{\vec{m},\vec{p}}(\mathfrak{m}_i, \epsilon)\,(\vec{w} - \epsilon\,\vec{p})^{\vec{m}} \,, \\ g_{\vec{p}}(\vec{w}, \mathfrak{m}_i, \epsilon) &= \prod_{i=1}^{N} \frac{\prod_{s=1}^{\infty} \left(\vec{w}\cdot\vec{\rho}_i + \mathfrak{m}_i + \epsilon\,(1 - \frac{\mathfrak{q}_i}{2} - s)\right)}{\prod_{s=1+\vec{\rho}_i\cdot\vec{p}}^{\infty} \left(\vec{w}\cdot\vec{\rho}_i + \mathfrak{m}_i + \epsilon\,(1 - \frac{\mathfrak{q}_i}{2} - s)\right)} \,. \end{aligned} \tag{3.12}$$

We stress that this derivation is valid for all Abelian gauged linear sigma models, independent of the spectrum and target space interpretation. For a few more intermediate steps in the derivation we refer to ref. [17].

Module of Relations

The rational functions $g_{\vec{p}}$ are fixed by the gauge theory spectrum alone and known as soon as a particular gauged linear sigma model has been specified. As an essential observation, note that the constraint equation (3.11) is entirely independent of the power $\vec{n}$ of north pole insertions $\vec{\sigma}_N$. Consequently, any set of polynomials $\alpha_{\vec{p}}$ that satisfies the constraint equation (3.11) defines a universal south pole correlator relation as

$$0 = R_S = \sum_{\vec{p}=0}^{\vec{s}} \vec{Q}^{\vec{p}} \left\langle \vec{\sigma}_N^{\vec{n}}\, \alpha_{\vec{p}}(\vec{\sigma}_S + \epsilon\vec{p}, \mathfrak{m}_i, \epsilon) \right\rangle \,, \tag{3.13}$$

where for convenience we have employed the notation of eq. (3.1). Finding correlator relations of a given gauge theory thus reduces to a well-studied problem in commutative algebra. Namely, the set M_S of polynomial solutions $\alpha_{\vec{p}}$ to eq. (3.11) forms the syzygy module over the polynomial ring $\mathbb{C}(\mathfrak{m}_i)[\vec{w}, \epsilon]$ of the rational function $g_{\vec{p}}$ — where $\mathbb{C}(\mathfrak{m}_i)$ denotes the field of complex rational functions in the twisted masses $\mathfrak{m}_i$. While the powers $\vec{m}$ of south pole insertions $\vec{\sigma}_S$ that appear in an expansion of eq. (3.13) are automatically determined by the choice of $\alpha_{\vec{p}}$, the maximal power $\vec{s}$ of $\vec{Q}$ remains as an input to the constraint equation. In order to find a set of generators for M_S, we need to iterate over increasing choices of $\vec{s}$ until the results stabilizes. In examples we typically observe that small choices are sufficient.

We note that every gauged linear sigma model is subject to infinitely many correlator relations. To see this, let us choose α_0 and $\alpha_{\vec{p}_0}$ respectively as the numerator and minus the denominator of

$g_{\vec{p}_0}$ for some $0 \neq \vec{p}_0 \in \mathbb{Z}^\ell$ and put all other $\alpha_{\vec{p}}$ to zero. This is in accord with eq. (3.11) and defines a correlator relation. Similarly, for any choice of three or more indices $\vec{p}_i$ we can find a solution where only the $\alpha_{\vec{p}_i}$ with these indices do not vanish. For this we first multiply eq. (3.11) with appropriate factors to make it a polynomial equation, then choose the $\alpha_{\vec{p}}$ to bring every term to the least common multiple of all terms and finally multiply the $\alpha_{\vec{p}}$ with factors $\beta_{\vec{p}_i}$ that sum to zero. The non-trivial question is, whether there are more clever choices for the $\alpha_{\vec{p}_i}$ that are of smaller degree in $\vec{w}$. There are specialized programming languages, for example 'Singular', which are well suited for such calculations.

Finally, let us observe that the constraint equation (3.11) does not depend on the FI parameters $\vec{\tau}$. For conformal gauged linear sigma models the correlator relations (3.13) are therefore valid in all the different gauge theory phases. Since the correlators are rational functions on moduli space, this is in fact expected. (Side remark: This phase independence formally even extends to non-conformal gauged linear sigma models, although the other phases may not be of physical significance.)

3.3.3 Non-Generic Twisted Masses

An important ingredient in the above derivation of correlator relations was the assumption of generic twisted masses $\mathfrak{m}_i$. We now consider non-generic choices $\mathfrak{m}_i^0$ at which the correlators remain finite.

Limiting Syzygy Module

Recall that $\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i^0, \epsilon)$ is defined by taking the limit $\mathfrak{m}_i \to \mathfrak{m}_i^0$ on $\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)$. We can certainly take the same limit on the level of the syzygy module and define $M_S^{\lim} = \lim_{\mathfrak{m}_i \to \mathfrak{m}_i^0} M_S$. Every element $R_S^{\lim}$ of $M_S^{\lim}$ is the limit of a relation $R_S \in M_S$ and, since the correlators are continuous in the twisted masses, a valid relation for the non-generic choice $\mathfrak{m}_i^0$.

Non-Generic Module

It is tempting to instead take the limit $\mathfrak{m}_i \to \mathfrak{m}_i^0$ on the level of rational functions $g_{\vec{p}}$ defined in eq. (3.12). These non-generic rational functions $g_{\vec{p}}^0(\vec{w}, \mathfrak{m}_i^0, \epsilon) = g_{\vec{p}}(\vec{w}, \mathfrak{m}_i^0, \epsilon)$ define the non-generic syzygy module M_S^0, whose elements we denote as R_S^0.

Since there may be cancellations between factors that originate from different chiral multiplets, the numerator and denominator of $g_{\vec{p}}^0(\vec{w}, \mathfrak{m}_i^0, \epsilon)$ may be of lesser degree in $\vec{w}$ than their counterparts in $g_{\vec{p}}(\vec{w}, \mathfrak{m}, \epsilon)$. In this case we expect additional elements $R_S^0 \in M_S^0$ that are not part of the limiting syzygy module $M_S^{\lim}$. Conversely, any limiting relation $R_S^{\lim}$ is an element of M_S^0 — just think of reinstalling the factors that cancelled by multiplying with one. In summary, we find the inclusion of modules $M_S^0 \supset M_S^{\lim}$.

Condition for Validity of Non-Generic Relations

This raises the question whether elements in $M_S^0 \setminus M_S^{\lim}$ are valid correlator relations. Since for $\mathfrak{m} = \mathfrak{m}_i^0$ the pole lattices $P(i_1, \ldots, i_\ell)$ that are used to define the modified residue operation (3.10) may no longer be disjoint, this is not guaranteed by our derivation in the previous subsection.

As a sufficient criterion for validity of all elements in M_S^0 we find the following: shift variables to $\vec{v} = \vec{\sigma} - \frac{\epsilon}{2}\vec{k}$ and consider the union of all pole sets (not pole lattices) associated to cones in Σ and similar the union of all pole sets associated to cones in $\Sigma(\vec{r}) \setminus \Sigma$, namely

$$\begin{aligned} \Theta(\vec{r}, \mathfrak{m}_i) &= \bigcup_{\sigma(i_1,\ldots,i_\ell)\in\Sigma(\vec{r})} \left\{ \vec{v} \in \mathbb{C}^r \,\middle|\, Z_{\vec{k}}^{(i_a)}\left(\vec{v} + \tfrac{\epsilon}{2}\vec{k}, \mathfrak{m}_{i_a}, \epsilon\right)^{-1} = 0 \;\; \forall \;\; 1 \le a \le \ell \right\}, \\ \Omega(\vec{r}, \mathfrak{m}_i) &= \bigcup_{\sigma(i_1,\ldots,i_\ell)\in\Sigma\setminus\Sigma(\vec{r})} \left\{ \vec{v} \in \mathbb{C}^r \,\middle|\, Z_{\vec{k}}^{(i_a)}\left(\vec{v} + \tfrac{\epsilon}{2}\vec{k}, \mathfrak{m}_{i_a}, \epsilon\right)^{-1} = 0 \;\; \forall \;\; 1 \le a \le \ell \right\}. \end{aligned} \tag{3.14}$$

Since the gamma functions in the numerator of the one loop determinant (3.5) are independent of $\vec{k}$ when written in terms of $\vec{v}$, these sets are independent of $\vec{k}$. For generic twisted masses $\mathfrak{m}_i$ the intersection $\Theta(\vec{r}, \mathfrak{m}_i) \cap \Omega(\vec{r}, \mathfrak{m}_i)$ is empty by construction. If the intersection is still empty for non-generic values $\mathfrak{m}_i^0$, the elements of M_S^0 are guaranteed to be valid relations for the choice $\mathfrak{m}_i^0$, symbolically:

$$\Theta(\vec{r}, \mathfrak{m}_i^0) \cap \Omega(\vec{r}, \mathfrak{m}_i^0) = \emptyset \quad \Longrightarrow \quad \text{all } R_S^0 \in M_S^0 \text{ are valid correlator relations.} \tag{3.15}$$

This condition intuitively ensures that there is no overlap between those poles that for given $\vec{r}$ contribute to the correlators and those poles that do not. We will come back to this subtle issue in the discussion of examples in section 3.7.

3.4 Differential Operators from Correlator Relations in Abelian Models

We now explain why and how the correlator relations turn into differential operators that annihilate the gauge theory ground state. For models with a compact geometric target we show these operators to also annihilate the Givental I-function, which for conformal gauged linear sigma model identifies them as Picard–Fuchs operators on the quantum Kähler moduli space.

3.4.1 Ideal of Differential Operators

As discussed in refs. [93, 111], the localization formula (3.4) decomposes into a quadratic form of suitable holomorphic blocks. Due to this factorization the correlators can be interpreted as matrix elements in a Hilbert space of states [112]. The field insertions $\vec{\sigma}_N$ and $\vec{\sigma}_S$ then correspond to operators $\boldsymbol{\vec{\sigma}}_N$ and $\boldsymbol{\vec{\sigma}}_S$, and the correlators read

$$\left\langle \vec{\sigma}_N^{\vec{n}}\, \vec{\sigma}_S^{\vec{m}} \right\rangle = \left\langle \Omega(\vec{\tau}) \middle| \boldsymbol{\vec{\sigma}}_N^{\vec{n}}\, \boldsymbol{\vec{\sigma}}_S^{\vec{m}} \middle| \Omega(\vec{\tau}) \right\rangle \tag{3.16}$$

in terms of the FI parameter dependent gauge theory ground state $|\Omega(\vec{\tau})\rangle$. As we will see below, it becomes necessary to promote $\vec{Q}$ to an operator $\boldsymbol{\vec{Q}}$ as well. By inserting eq. (3.16) into the definition (3.8) and by assuming that $\boldsymbol{\vec{Q}}$ and $\boldsymbol{\vec{\sigma}}_N$ commute we find

$$0 = \left\langle \Omega(\vec{\tau}) \middle| \boldsymbol{\vec{\sigma}}_N^{\vec{n}}\, \boldsymbol{R}_S \middle| \Omega(\vec{\tau}) \right\rangle = \left\langle \Omega(\vec{\tau}) \middle| \boldsymbol{\vec{\sigma}}_N^{\vec{n}} \sum_{\vec{p}=0}^{\vec{s}} \boldsymbol{\vec{Q}}^{\vec{p}}\, \alpha_{\vec{p}}(\boldsymbol{\vec{\sigma}}_S + \epsilon\vec{p}, \mathfrak{m}_i, \epsilon) \middle| \Omega(\vec{\tau}) \right\rangle, \tag{3.17}$$

where the second equality defines the operator $\boldsymbol{R}_S = \boldsymbol{R}_S(\vec{\boldsymbol{Q}}, \vec{\boldsymbol{\sigma}}_S, \mathfrak{m}_i, \epsilon)$ associated to the correlator relation R_S. Since the matrix element in the middle of this equation is zero for all powers $\vec{n}$ of $\vec{\boldsymbol{\sigma}}_N^{\vec{n}}$, we conclude

$$\boldsymbol{R}_S \, , \left|\Omega(\vec{\tau})\right\rangle = 0 \, . \tag{3.18}$$

Conversely, any operator of this type — meaning it is independent of $\vec{\boldsymbol{\sigma}}_N$ and all $\vec{\boldsymbol{\sigma}}_S$ are to the right of $\vec{\boldsymbol{Q}}$ — that annihilates the gauge theory ground state defines a corresponding correlator relation.

Let us now consider the operator $\boldsymbol{R}'_S = \vec{\boldsymbol{\sigma}}_S^{\vec{m}} \boldsymbol{R}_S$. It clearly annihilates the ground state as well and therefore defines an associated relation R'_S. In order for this to be consistent, we impose the non-trivial commutation relation

$$\left[\boldsymbol{\sigma}_{S,l}, \boldsymbol{Q}_k\right] = \delta_{lk} \, \epsilon \, \boldsymbol{Q}_k \, . \tag{3.19}$$

An explicit calculation then shows that $\boldsymbol{R}'_S$ corresponds to a relation R'_S whose defining polynomials $\alpha'_{\vec{p}}$ are related to the $\alpha_{\vec{p}}$ of R_S by multiplication with $\vec{w}^{\vec{m}}$. We similarly need to require that when $\boldsymbol{R}_S$ is multiplied with any polynomial of the $\boldsymbol{Q}_l$ from the left, the resulting operator $\boldsymbol{R}''_S$ still annihilates the ground state and defines yet another relation R''_S. This is fulfilled automatically, for details see ref. [17]. As the result of these considerations, the set of operators $\boldsymbol{R}_S$ is seen to define a left ideal $\boldsymbol{\mathcal{I}}_S$ in the non-commutative ring $\mathbb{C}(\mathfrak{m}_i)\langle \vec{\boldsymbol{Q}}, \vec{\boldsymbol{\sigma}}_S, \epsilon \rangle$ that is characterized by eq. (3.19). Finally, observe that this commutation relation can be represented by

$$\boldsymbol{Q}_l = Q_l \, , \qquad \boldsymbol{\sigma}_{S,l} = \epsilon Q_l \partial_{Q_l} = \epsilon \Theta_l \, . \tag{3.20}$$

This can be interpreted as representation with respect to the eigenstates of the monopole operators $\boldsymbol{Q}_\ell$ and turns $\boldsymbol{R}_S$ into the differential operator

$$\boldsymbol{R}_S(\vec{Q}, \epsilon\vec{\Theta}, \mathfrak{m}_i, \epsilon) = \sum_{\vec{p}=0}^{\vec{s}} \vec{Q}^{\vec{p}} \alpha_{\vec{p}}(\epsilon\vec{\Theta} + \epsilon\vec{p}, \mathfrak{m}_i, \epsilon) = \sum_{\vec{m}=0}^{\vec{M}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)(\epsilon\vec{\Theta})^{\vec{m}} \, . \tag{3.21}$$

For this reason we refer to $\boldsymbol{\mathcal{I}}_S$ as the differential ideal and from here on understand $\boldsymbol{R}_S$ to be in this representation.

3.4.2 Connection to the Givental I-Function

Let us now consider models with a compact geometric target space, such that the correlators are determined from the Givental I-function with eq. (3.7). By inserting this correspondence into the definition of correlator relations in eq. (3.8) we find

$$0 = R_S = (-1)^{|n|_1} \int_X (\epsilon\vec{\Theta})^{\vec{n}} \, I_X(\vec{Q}, \mathfrak{m}_i, -\epsilon) \cup \left[\sum_{\vec{m}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)(\epsilon\vec{\Theta})^{\vec{m}} \, I_X(\vec{Q}, \mathfrak{m}_i, \epsilon) \right] . \tag{3.22}$$

Since this is true for all $\vec{n} \in \mathbb{Z}^{\ell}_{\geq 0}$, the expression in square brackets needs to vanish by itself (in cohomology). This gives

$$0 = \boldsymbol{R}_S(\vec{Q}, \epsilon\vec{\Theta}, \mathfrak{m}_i, \epsilon) \, I_X(\vec{Q}, \mathfrak{m}_i, \epsilon) \, , \tag{3.23}$$

where $\boldsymbol{R}_S$ is the differential operator associated to R_S as defined by eq. (3.21). For conformal models this identifies $\boldsymbol{R}_S$ as a Picard–Fuchs operator on the quantum Kähler moduli space of X.

3.4.3 Non-Generic Twisted Masses

Lastly, we turn to non-generic choices $\mathfrak{m}_i^0$ of twisted masses for which all correlators are well defined and M_S^0 becomes strictly larger than M_S^{lim}. These modules similarly define the non-generic differential ideal $\boldsymbol{\mathcal{I}}_S^0$ as well as the limiting differential ideal $\boldsymbol{\mathcal{I}}_S^{\text{lim}}$. They are related by the proper inclusion $\boldsymbol{\mathcal{I}}_S^{\text{lim}} \subset \boldsymbol{\mathcal{I}}_S^0$, which is due to the fact that for the non-generic choice of twisted masses some generically irreducible operators $\boldsymbol{R}_S$ factor according to $\boldsymbol{R}_S^{\text{lim}} = \boldsymbol{C}\,\boldsymbol{R}_S^0$. The condition on the left hand side of eq. (3.15) then ensures that the factor $\boldsymbol{R}_S^0 \in \boldsymbol{\mathcal{I}}_S^0$ annihilates the ground stand already by itself, and — in case it applies — the I-function.

This is of particular importance for models with compact Calabi–Yau target spaces in the case of vanishing twisted masses. As we will explicitly observe in an example further below, the proper inclusion of $\boldsymbol{\mathcal{I}}_S^{\text{lim}}$ into $\boldsymbol{\mathcal{I}}_S^0$ then amounts to the proper inclusion of the system of GKZ operators into the full system of Picard–Fuchs operators.

3.5 Non-Abelian Gauge Groups

This section gives an outlook on how the previous findings are generalized to models with non-Abelian gauge groups. We keep the exposition short, with a focus on new features and conceptual differences as compared to the Abelian case. For details we refer to ref. [17].

3.5.1 A-Twisted Correlators

We consider gauged linear sigma models with a non-Abelian compact gauge group of the general form given in eq. (2.31). Its rank is written as $r = \operatorname{rank} G$ and we note that $r > \ell$ where ℓ is the number of U(1) factors in G. There are N chiral multiplets Φ_i that transform in the G-representations ρ_i, their vector R-charges are $\mathfrak{q}_i$ and they have generic twisted masses $\mathfrak{m}_i$.

The vector superfield V and its complex scalar $\underline{\sigma}$ take values in $\mathfrak{g}$ and transform in the adjoint representation — we use the underscore to clearly distinguish the adjoint valued field $\underline{\sigma}$ from its components. Since correlation functions correspond to physical measurements, they must be independent of gauge choices. This restricts our attention to correlators of the form

$$\left\langle f(\underline{\sigma}_N, \underline{\sigma}_S)\right\rangle = \kappa_f(\vec{Q}, \mathfrak{m}_i, \epsilon)\,, \tag{3.24}$$

where $f(\underline{\sigma}_N, \underline{\sigma}_S)$ is a gauge invariant polynomial of the non-Abelian field insertions $\underline{\sigma}_N$ and $\underline{\sigma}_S$ at the north and south pole. Since G is compact, any such polynomial can be expressed in terms of a finite generating set of gauge invariant expressions [113]. As an example, for $G = \mathrm{U}(2)$ all gauge invariant polynomials $f(\underline{\sigma}_N, \underline{\sigma}_S)$ can be expressed in terms of the five basic gauge invariants $\operatorname{tr}(\underline{\sigma}_{N/S})$, $\operatorname{tr}(\underline{\sigma}_{N/S}^2)$ and $\operatorname{tr}(\underline{\sigma}_N\underline{\sigma}_S)$. A special subclass of correlators are those based on factored polynomials $f(\underline{\sigma}_N, \underline{\sigma}_S) = f_N(\underline{\sigma}_N)\cdot f_S(\underline{\sigma}_S)$,

$$\left\langle f_N(\underline{\sigma}_N) f_S(\underline{\sigma}_S)\right\rangle = \kappa_{f_N,f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon)\,, \tag{3.25}$$

where $f_N(\underline{\sigma}_N)$ and $f_S(\underline{\sigma}_S)$ are gauge invariant independently. In the $G = \mathrm{U}(2)$ example this subclass excludes the mixed combination $\operatorname{tr}(\underline{\sigma}_N\underline{\sigma}_S)$.

non-Abelian theory		Cartan theory		$U(1)_V$	twisted
multiplet	G-rep.	multiplet	T-rep.	R-charge	mass
vector mult. V	adj(G)	chiral mult. W_1	$\vec{\alpha}_1$	2	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		chiral mult. $W_{\dim\mathfrak{g}-r}$	$\vec{\alpha}_{\dim\mathfrak{g}-r}$	2	0
chiral mult.	ρ_i	chiral mult. Λ_{i,β_1}	$\vec{\lambda}_{i,\beta_1}$	$\mathfrak{q}_i$	$\mathfrak{m}_i$
Φ_i		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i = 1, \ldots, N$		chiral mult. $\Lambda_{i,\beta_{\dim\rho_i}}$	$\vec{\lambda}_{i,\beta_{\dim\rho_i}}$	$\mathfrak{q}_i$	$\mathfrak{m}_i$

Table 3.1: Decomposition of the non-Abelian gauge theory spectrum into the spectrum of its associated Cartan theory with Abelian gauge group $T = \mathrm{U}(1)^r / \Gamma$.

The axial R-symmetry still imposes a selection rule and the non-Abelian correlators follow similar algebraic properties as those discussed in subsection 3.2.1.

3.5.2 Cartan Theory and Localization

The calculation of correlators in a non-Abelian model can be reduced to the Abelian case with its powerful localization formula (3.4) by means of the associated Cartan theory.

Cartan Theory

The latter — sometimes also referred to as Coulomb branch theory or Abelianization — is obtained by spontaneously breaking the non-Abelian gauge group G to its maximal Abelian subgroup $T = \mathrm{U}(1)^r / \Gamma$ with a generic expectation value for the adjoint-valued field $\underline{\sigma}$. A chiral multiplet Φ_i then decomposes into a set of chiral multiplets Λ_{i,β_i} with $\beta_i = 1, \ldots, \dim\rho_i$, whose charge vectors $\vec{\lambda}_{i,\beta_i}$ with respect to the unbroken group T are the weights of the representation ρ_i. Their vector R-charges and twisted masses remain unaltered, i.e., $\mathfrak{q}_{i,\beta_i} = \mathfrak{q}_i$ and $\mathfrak{m}_{i,\beta_i} = \mathfrak{m}_i$. In addition, $\underline{\sigma}$ decomposes into several neutral and charged components. The former are its U(1) components $\sigma_1, \ldots, \sigma_\ell$ as well as the Z-bosons $\sigma_{\ell+1}, \ldots, \sigma_r$ of the spontaneous symmetry breaking. We collect these into the r-dimensional vector $\vec{\sigma} = (\sigma_1, \ldots, \sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_r)$. Its charged components are the W-bosons, which we label by $\beta = 1, \ldots, \dim\mathfrak{g} - r$. These correspond [73] to chiral multiplets W_β with vector R-charge $\mathfrak{q}_\beta = 2$ and twisted mass $\mathfrak{m}_\beta = 0$, whose charge vectors $\vec{\alpha}_\beta$ with respect to the Cartan gauge group T are given by the non-zero roots of G. The spectrum of the Cartan theory and that of the original non-Abelian theory are summarized in Table 3.1.

A few remarks are in order. First, it is possible to rewrite the Cartan gauge group $T = \mathrm{U}(1)^r / \Gamma$ as $T \simeq T' = \mathrm{U}(1)^r$ by redefining the U(1) generators. We will *not* do this and keep working in the basis of T, because this allows for more universal formulas. Second, there are $r > \ell$ complexified FI parameters $\vec{\tau} = (\vec{\tau}_{\mathrm{na}}, 0, \ldots, 0)$ and equivalently $r > \ell$ parameters $\vec{Q} = (\vec{Q}_{\mathrm{na}}, 1, \ldots, 1)$ in the Cartan theory. Here $\vec{\tau}_{\mathrm{na}}$ and $\vec{Q}_{\mathrm{na}}$ correspond to the ℓ complexified FI parameters of the original U(1) factors in G, which respectively are supplemented by $r - \ell$ zeros and ones. Since the Cartan theory is Abelian, we can formally turn on additional variables $Q_{\ell+1}, \ldots, Q_r$ that replace the ones in $\vec{Q}$. While this may be helpful at intermediate steps in calculations, we must eventually set them back to one — otherwise we would not be dealing with a theory that arose from a spontaneous breakdown of a non-Abelian model. Unless stated otherwise, we work with $\vec{Q} = (\vec{Q}_{\text{na}}, 1, \ldots, 1)$.

Correlators of the Cartan Theory

Correlators of the Cartan theory are not conceptually different from those of any other Abelian model. They are defined as in eq. (3.1) and denoted

$$\left\langle \vec{\sigma}_N^{\vec{n}} \vec{\sigma}_S^{\vec{m}} \right\rangle = \kappa^{\text{Cartan}}_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) \quad \text{with} \quad \vec{n}, \vec{m} \in \mathbb{Z}^r_{\geq 0}\,, \tag{3.26}$$

where we keep in mind that $\vec{\sigma} = (\sigma_1, \ldots, \sigma_\ell, \sigma_{\ell+1}, \ldots, \sigma_r)$ in addition to the components of the original U(1) factors also subsumes the Z-bosons of the spontaneous symmetry breaking. The localization formula (3.4) is applicable and the chiral multiplets W_β corresponding to the W-bosons contribute to the one loop determinant (3.5) in the same was as any other chiral multiplet. Their combined contribution takes the form

$$Z^G_{\vec{k}}(\vec{\sigma}, \epsilon) = \prod_{\vec{\alpha}_\beta > 0} (-1)^{\vec{\alpha}_\beta \cdot \vec{k} + 1} \left(\vec{\alpha}_\beta \cdot \vec{\sigma} - \frac{\vec{\alpha}_\beta \cdot \vec{k}}{2} \epsilon \right) \left(\vec{\alpha}_\beta \cdot \vec{\sigma} + \frac{\vec{\alpha}_\beta \cdot \vec{k}}{2} \epsilon \right) . \tag{3.27}$$

Here the product taken over positive roots $\vec{\alpha}_\beta$, we used that non-zero roots come in pairs $(\vec{\alpha}_\beta, -\vec{\alpha}_\beta)$ and employed Euler's reflection identity for the gamma function. This shows that the W-bosons do not introduce new poles. By comparing with eq. (3.4), the above product can also be reinterpreted as a polynomial in the operator insertions $\vec{\sigma}_S$ and $\vec{\sigma}_N$.

Correlators of the Non-Abelian Theory

The central point of the entire construction is that the non-Abelian gauge invariants $f(\underline{\sigma}_N, \underline{\sigma}_S)$ can be expressed as polynomials in the components of $\vec{\sigma}_N$ and $\vec{\sigma}_S$. As a result, the non-Abelian correlators κ_f are found as certain linear combinations of the Cartan correlators. To clarify this, let us consider $G = \text{U}(2) = \text{U}(1) \times \text{SU}(2)/\mathbb{Z}_2$. We then have $\vec{\sigma} = (\sigma_1, \sigma_2)$, where σ_1 corresponds to the original U(1) factor in G and σ_2 to a Z-boson and for example find

$$\begin{aligned} \text{tr}(\underline{\sigma}) &= \sigma_1 && \Longrightarrow && \kappa_{\text{tr}(\underline{\sigma}_N)} = \kappa^{\text{Cartan}}_{(1,0),(0,0)}\,, \\ \text{tr}(\underline{\sigma}^2) &= \frac{\sigma_1^2 + \sigma_2^2}{2} && \Longrightarrow && \kappa_{\text{tr}(\underline{\sigma}_S^2)} = \frac{1}{2}\left[\kappa^{\text{Cartan}}_{(0,0),(2,0)} + \kappa^{\text{Cartan}}_{(0,0),(0,2)}\right], \end{aligned} \tag{3.28}$$

Here we are working in the basis of $T = \text{U}(1)^2/\mathbb{Z}_2$. We note, however, that not every Cartan correlator can conversely be expressed in terms of the non-Abelian correlators. The gauge group G still acts by its Weyl group $\mathcal{W}_G$ on $\vec{\sigma}$ and only correlators of polynomials that are invariant under this action can be expressed as correlators of the non-Abelian theory. For $G = \text{U}(2)$ the Weyl group is $\mathbb{Z}_2$ and acts by $\vec{\sigma} = (\sigma_1, \sigma_2) \mapsto (\sigma_1, -\sigma_2)$. We note that the combinations in eq. (3.28) are indeed invariant under this action.

3.5.3 Correlator Relations in Non-Abelian Models

The definition of universal south pole correlator relations R_S of non-Abelian models parallels the Abelian case in eq. (3.8). We require

$$0 = R_S^G = \sum_{f_S} c_{f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon)\, \kappa_{f_N, f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon)\,, \tag{3.29}$$

where the sum is over a finite collection of gauge invariant polynomials $f_S(\underline{\sigma}_S)$ and the c_{f_S} are polynomial in the variables Q_l. The equation is non-trivial because we require it to hold for all gauge invariant polynomials $f_N(\underline{\sigma}_N)$. We note that this approach is only based on the special subclass of correlator introduced in eq. (3.25) and does not involve mixed type gauge invariants.

We determine such relations in two steps. First, we use the methods of section 3.3 to determine a correlator relation R_S^{Cartan} of the Abelian Cartan theory. Second, we need to express the involved Cartan correlators in terms of the correlators of the original non-Abelian theory. Since some of the correlators may not be G-invariant, this is not always possible. For this reason we project out the G-variant parts by summing the relation R_S^{Cartan} over its Weyl group orbit. According to the Luna–Richardson theorem [114] this sum can be unambiguously expressed in terms of the non-Abelian correlators and we arrive at a non-Abelian relation R_S^G.

However, this method is more restrictive than the definition (3.29) requires. Namely, the Abelian correlator relation R_S^{Cartan} holds for all values of $\vec{Q}\,'$, even those involving general unphysical parameters $Q_{\ell+1}, \ldots, Q_r$. They are certainly valid for the physical choice $\vec{Q} = (\vec{Q}_{\text{na}}, 1, \ldots, 1)$ too, but we may have missed relations that exist for this physical choice only. Moreover, relations of the Cartan theory hold for all powers $\vec{n}$ of the north pole insertion $\vec{\sigma}_N$. This not only guarantees that R_S^G is valid for all gauge invariant polynomials $f_N(\underline{\sigma}_N)$, but actually requires it to hold for all — not necessarily gauge invariant — polynomials. Therefore, it is not clear whether all non-Abelian correlator relations of the type (3.29) can be found by this method.

3.5.4 Differential Operators from Non-Abelian Correlator Relations

Lastly, let us understand whether the non-Abelian correlator relations defined in eq. (3.29) also correspond to differential operators. To this end, consider the Abelian Cartan theory with all unphysical parameters $Q_{\ell+1}, \ldots, Q_r$ turned on. The discussion of section 3.4 then fully applies and every relation R_S^{Cartan} corresponds to a differential operator that is obtained by the identifications

$$\begin{aligned} \boldsymbol{Q}_l &= Q_l\,, & \boldsymbol{\sigma}_{S,l} &= \epsilon Q_l \partial_{Q_l} & &\text{for all} \quad l = 1, \ldots, \ell\,, \\ \text{and} \quad \boldsymbol{Q}_{l'} &= Q_{l'}\,, & \boldsymbol{\sigma}_{S,l'} &= \epsilon Q_{l'} \partial_{Q_{l'}} & &\text{for all} \quad l' = \ell+1, \ldots, r\,. \end{aligned} \tag{3.30}$$

Going back to the non-Abelian theory, the variables $Q_{l'}$ in the second line of this equation do not exist anymore. This is easily accounted for by simply setting them to one, which we in fact automatically do this when deriving non-Abelian relations as explained in the previous subsection. The problem is that we also must not use the derivatives $\partial_{Q_{l'}}$ anymore, although we cannot simply set them to zero or another fixed value. (Remark: This is an elementary statement about partial differential equations. Say the equation is in two real variables x and y, and we are only interested in $y = 1$. While we may set $y = 1$, we cannot simply ignore derivatives with respect to y. These involve a comparison of quantities for different y and prevent a simple restriction.)

This shows that we can only turn those non-Abelian relations R_S^G into operators, that do not implicitly depend on the insertions $\sigma_{S,l'}$ with $l' = \ell + 1, \ldots, r$. For example, in case of $G = \mathrm{U}(2)$ the relation R_S^G must not involve $\mathrm{tr}(\underline{\sigma}_S^2) = (\sigma_{S,1}^2 + \sigma_{S,2}^2)/2$ but only $\mathrm{tr}(\underline{\sigma}_S) = \sigma_{S,1}$. This is true in general: while the linear gauge invariants of $\underline{\sigma}_S$ are in one-to-one correspondence with the U(1) factors of G and can be represented as differential operators, this is not the case for non-linear gauge invariants. These depend on the insertions $\sigma_{\ell+1}, \ldots, \sigma_r$ associated to the Z-bosons of the simple gauge group factors, which cannot be represented as differential operators. In order to find special R_S^G that only involve the linear gauge invariants, we have to find polynomial solutions $\alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_i, \epsilon)$ to the constraint equation (3.11) of the Cartan theory that only depend on the first ℓ components of the vector $\vec{w} = (w_1, \ldots, w_\ell, w_{\ell+1}, \ldots, w_r)$. To be precise, it is enough for this to be true after summing the Abelian relation over its Weyl group orbit.

While being a well-posed mathematical problem, this is a computationally quite expensive task and it would be beneficial to have a different method of determining the differential operators that govern the quantum cohomology of non-Abelian models. This is the topic of the next section, as well as the entire next chapter.

3.6 Universal Formulas for Picard–Fuchs Operators

The above findings allow us to determine universal linear dependencies among A-twisted correlators directly from the gauge theory spectrum, without the need to calculate any correlator. In Abelian models these relations turn into differential operators, which in the case of a geometric target space annihilate the Givental I-funcion and for conformal models are Picard–Fuchs operators on the quantum Kähler moduli space of the low energy superconformal field theory. For non-Abelian gauge groups the correspondence between relations and operators is more complicated. This section presents an alternative way of deriving the operators from the A-twisted correlators, which does not make a distinction between Abelian and non-Abelian models. It comes with the price of having to calculate some, though only few correlators explicitly.

3.6.1 Methodology

We start with the assumption that some independent argument guarantees the existence of a correlator relation

$$0 = R_S = \sum_{\vec{m} \in I_{\vec{m}}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)\, \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)\,, \tag{3.31}$$

where $I_{\vec{m}} \subset \mathbb{Z}_{\geq 0}^{\ell}$ is some fixed and finite index set. There is no distinction between Abelian and non-Abelian gauge groups. In case of the latter the correlators are understood as

$$\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) = \langle \sigma_{N,1}^{n_1} \cdots \sigma_{N,\ell}^{n_\ell}\, \sigma_{S,1}^{m_1} \cdots \sigma_{S,\ell}^{m_\ell} \rangle\,, \tag{3.32}$$

where σ_l with $l = 1, \ldots, \ell$ are those components of the adjoint valued $\underline{\sigma}$ that are associated to the U(1) factors in G. (Side remark: The connection to the Cartan correlators in eq. (3.26) is $\kappa_{\vec{n},\vec{m}} = \kappa^{\mathrm{Cartan}}_{\vec{n}',\vec{m}'}$, where $\vec{n}' = (\vec{n}, 0, \ldots, 0)$ and $\vec{m}' = (\vec{m}, 0, \ldots, 0)$ with $r - \ell$ zeros.) Without any reference to an explicit gauge theory realization, we regard eq. (3.31) as an infinite family of homogeneous linear equations and by using these for sufficiently many values of $\vec{n} \in \mathbb{Z}^{\ell}$ try to solve for the coefficients $c_{\vec{m}}$

in terms of the correlators $\kappa_{\vec{n},\vec{m}}$. In case of geometric target space eqs. (3.20) and (3.21) immediately turn these coefficients $c_{\vec{m}}$ into a differential operator that annihilates its I-function. This equally applies to models with non-Abelian gauge groups, since the relation in eq. (3.31) by definition only involves powers of the linear gauge invariants. We now distinguish two cases.

First, we may pick a particular model and explicitly calculate the required correlators. Since the relation (3.31) exists by assumption, the equations it specifies will — once the correlators are inserted — for any number and choices of $\vec{n}$ always yield a solution for the coefficients $c_{\vec{m}}$. For two reasons the solution may, however, not be unique. First, there may be other valid correlator relations R'_S whose index set $I'_{\vec{m}}$ is a (not necessarily proper) subset of $I_{\vec{m}}$. Second, there may also be solutions that do not correspond to valid correlator relations. If present, these will eventually disappear upon including the equations corresponding to more choices of $\vec{n}$. Since we in practice have to restrict to finitely many $\vec{n}$, we may not always be able to distinguish between these two situations. In order for this approach to be useful, we therefore need to know how many independent correlator relations there are for a given index set $I_{\vec{m}}$. If we find exactly as many solutions for the $c_{\vec{m}}$ with the given index set $I_{\vec{m}}$, all of them are guaranteed to be valid relations.

Second, we may try to solve the equations for yet unspecified values of $\kappa_{\vec{n},\vec{m}}$. Even given the existence of a relation, the equations it imposes may not automatically have a kernel but only become solvable when using certain identities between different $\kappa_{\vec{n},\vec{m}}$. Assume we know how many independent relations exist for a given index set $I_{\vec{m}}$ and that we have solved the corresponding equations for a given set of vectors $\vec{n}$. If we find more solutions than there are relations, we will in general not be able to identify the valid relations among them. If we find exactly as many or less solutions, we still need to check whether we may impose identities between different $\kappa_{\vec{n},\vec{m}}$ that would result in an increased number of solutions. If the number of solutions agrees with the number of relations *and* if it is not possible to obtain even more solution by imposing additional identities on the correlators, all solutions are guaranteed to be valid relations. It is this approach that we follow below.

3.6.2 Calabi–Yau Target Spaces

We now turn off all twisted masses and specialize to conformal gauged linear sigma models whose target spaces $\mathcal{X}$ are compact Calabi–Yau manifolds. As discussed in subsection 3.4.3, the non-generic differential ideal $\mathcal{I}^0_S$ then captures the full ideal of Picard–Fuchs operators on the quantum Kähler moduli space of $\mathcal{X}$. Apart from being physically interesting, there are two technical reasons for this restriction. First, for some complex dimensions $d = \dim_{\mathbb{C}} \mathcal{X}$ and numbers ℓ of Kähler parameters of $\mathcal{X}$, the number and order of the generating Picard–Fuchs operators are uniquely fixed by Griffith transversality, see eq. (2.69). This tells us how $\mathcal{I}^0_S$ is generated, such that we know how many correlator relations there are for a given index set $I_{\vec{m}}$. Second, the correlators $\kappa_{\vec{n},\vec{m}} = \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon)$ are subject to the three restrictive identities

$$\kappa_{\vec{n},\vec{m}} = \quad \text{for} \quad |\vec{n}|_1 + |\vec{m}|_1 < d\,, \tag{3.33}$$

$$\kappa_{\vec{n},\vec{m}} = (-1)^{d+|\vec{n}|_1+|\vec{m}|_1} \kappa_{\vec{m},\vec{n}}\,, \tag{3.34}$$

$$\epsilon Q_l \partial_{Q_l} \kappa_{\vec{n},\vec{m}} = \epsilon \Theta_l \, \kappa_{\vec{n},\vec{m}} = \kappa_{\vec{n},\vec{m}+\vec{e}_l} - \kappa_{\vec{n}+\vec{e}_l,\vec{m}}\,. \tag{3.35}$$

The first of these is the selection rule (3.3) under the assumption of a finite limit $\epsilon \to 0$ and the second follows from the (anti-)symmetry (3.9) together with $\mathfrak{m}_i = 0$ and $\vec{S} = 0$. The third identity, where $\vec{e}_l$

denotes the l-th unit vector, directly follows from the localization formula (3.4) and is valid for all gauged linear sigma models — independent of the spectrum and target space interpretation.

For some choices of d and ℓ the combination of these properties allows us to solve for the coefficients $c_{\vec{m}}$ of the generating operators without using explicit values for the correlators, such that we obtain universally applicable formulas for the Picard–Fuchs operators of several classes of Calabi–Yau manifolds. Let us discuss a few examples.

3.6.3 Elliptic Curves

We start with elliptic curves, i.e., complex one-dimensional Calabi–Yau manifolds as our simplest example. These are always parameterized by a single Kähler modulus, such that we have $d = 1$ and $\ell = 1$. The selection rule (3.33) does here not carry additional information since $\kappa_{0,0} = 0$ is already guaranteed by eq. (3.34).

The Hodge numbers $h^{0,0} = h^{1,1} = 1$ together with Griffith transversality of the mirror manifold imply that $\mathcal{I}_S^0$ is generated by a single Picard–Fuchs operator

$$\mathcal{L}(Q,\epsilon) = \sum_{m \in I_m} c_m(Q,\epsilon)\,(\epsilon\Theta)^m \tag{3.36}$$

of order two. This specifies the index set, see eq. (3.31), as $I_m = \{0, 1, 2\}$ and we try to solve for the coefficients c_m of the Picard–Fuchs operator by using eq. (3.31) for three values of n. Choosing $0 \le n \le 2$ we get

$$0 \;=\; M \cdot \vec{c} \;=\; \begin{pmatrix} 0 & \kappa_{0,1} & \kappa_{0,2} \\ -\kappa_{0,1} & 0 & -\kappa_{1,2} \\ -\kappa_{0,2} & \kappa_{1,2} & 0 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}, \tag{3.37}$$

where the matrix M is defined by $M_{nm} = (-1)^n \kappa_{nm}$ with $0 \le n, m \le 2$. Its explicit form on the right hand side follows from the correlator properties eqs. (3.33) and (3.34), and the conventional factor of $(-1)^n$ was inserted to make M anti-symmetric. Since M is also odd-dimensional, it necessarily has a kernel. We find this to be

$$c_0 = -\kappa_{1,2}\,, \qquad c_1 = -\kappa_{0,2} = -\epsilon\Theta\,\kappa_{0,1}\,, \qquad c_2 = +\kappa_{0,1}\,, \tag{3.38}$$

which gives a candidate solution for the coefficients of the generating Picard–Fuchs operator. Here we used the elementary derivative property (3.35) in order to reduce the number of independent correlators that enter the formulas.

We still have to check whether the dimension of the kernel may be increased by imposing additional identities on the $\kappa_{n,m}$. From the 2×2 matrix in the upper left corner of M this is seen to require $\kappa_{0,1} = 0$. Since $\kappa_{0,1}$ is the one-dimensional analog of the Yukawa coupling in eq. (2.59), this can at most happen at special points in moduli space. The formula

$$\mathcal{L}(Q,\epsilon) = \kappa_{0,1}\,(\epsilon\Theta)^2 - \kappa_{0,2}\,(\epsilon\Theta) - \kappa_{1,2} = \kappa_{0,1}\,(\epsilon\Theta)^2 - \left(\epsilon\Theta\,\kappa_{0,1}\right)(\epsilon\Theta) - \kappa_{1,2} \tag{3.39}$$

corresponding to the solution (3.38) therefore universally expresses the Picard–Fuchs operator of elliptic curves in terms of two (or three) A-twisted gauged linear sigma model correlators. This is equally true for models with non-Abelian gauge groups.

3.6.4 One-Parameter Calabi–Yau Threefolds

As second example we turn to the physically important case of Calabi–Yau threefolds, which in the context of type II string compactifications (with no non-trivial background fluxes for the p-form gauge fields) give rise to $\mathcal{N} = 2$ supersymmetric four-dimensional theories. We have $d = 3$ and further choose $\ell = h^{1,1} = 1$. Large parts of the discussion parallel that of the previous subsection.

The Hodge numbers $h^{0,0} = h^{1,1} = h^{2,2} = h^{3,3} = 1$ and Griffith transversality imply that $\mathcal{I}_S^0$ is generated by a single Picard–Fuchs operator of order four. The index set is $I_m = \{0, 1, 2, 3, 4\}$ and we consider eq. (3.31) for $n = 0, \ldots, 4$. These equations are summarized into the matrix equation $0 = M \cdot \vec{c}$, where $\vec{c} = (c_0, \ldots, c_4)$ and the matrix M is defined by $M_{nm} = (-1)^n \kappa_{n,m}$ with $0 \leq n, m \leq 4$. Due to eq. (3.34) this odd-dimensional matrix is anti-symmetric and automatically has a one-dimensional kernel. With eq. (3.33) we deduce that its kernel can only increase if $\kappa_{0,3}$ vanishes, which due to the identification of $\kappa_{0,3}$ with the Yukawa coupling in eq. (2.59) can happen at non-generic points in moduli space only. The generic kernel therefore captures the generating operator, which we find to be

$$\begin{aligned}\mathcal{L}(Q, \epsilon) &= \kappa_{0,3}^2(\epsilon\Theta)^4 - \kappa_{0,3}\kappa_{0,4}(\epsilon\Theta)^3 + \left(\kappa_{0,4}\kappa_{1,3} - \kappa_{0,3}\kappa_{1,4}\right)(\epsilon\Theta)^2 \\ &\quad + \left(\kappa_{0,4}\kappa_{2,3} - \kappa_{0,3}\kappa_{2,4}\right)(\epsilon\Theta) + \left(\kappa_{1,4}\kappa_{2,3} - \kappa_{1,3}\kappa_{2,4} - \kappa_{0,3}\kappa_{3,4}\right) .\end{aligned} \tag{3.40}$$

With the derivative rule (3.35) we can reduce the number of correlators that enter this formula down to three and rewrite it as

$$\begin{aligned}\mathcal{L}(Q, \epsilon) \ = \ &+ \kappa_{0,3}^2(\epsilon\Theta)^4 - 2\kappa_{0,3}\left(\epsilon\Theta\, \kappa_{0,3}\right)(\epsilon\Theta)^3 + \left[2\left(\epsilon\Theta\, \kappa_{0,3}\right)^2 - \kappa_{0,3}\left(\epsilon^2\Theta^2\, \kappa_{0,3} + \kappa_{2,3}\right)\right](\epsilon\Theta^2) \\ &+ \left[2\kappa_{2,3}\left(\epsilon\Theta\, \kappa_{0,3}\right) - \kappa_{0,3}\left(\epsilon\Theta\, \kappa_{2,3}\right)\right](\epsilon\Theta) \\ &+ \left[\kappa_{2,3}^2 - \kappa_{0,3}\kappa_{3,4} - \left(\epsilon\Theta\, \kappa_{0,3}\right)\left(\epsilon\Theta\, \kappa_{2,3}\right) + \kappa_{2,3}\left(\epsilon^2\Theta^2\, \kappa_{0,3}\right)\right] .\end{aligned} \tag{3.41}$$

Both formulas are valid for all Calabi–Yau threefolds with a single Kähler modulus and express their generating Picard–Fuchs operator in terms A-twisted gauged linear sigma model correlators.

We now make an interesting observation. The generating Picard–Fuchs operator is constituted by its five coefficients c_m with $m = 1, \ldots, 4$. Since these may be freely rescaled by a common factor, the invariant information is encoded in *four* independent ratios of the c_m. However, eq. (3.41) expresses the operator entirely in terms of the *three* correlators $\kappa_{0,3}$, $\kappa_{2,3}$, $\kappa_{3,4}$. This means there needs to be one differential-algebraic relation between the five c_m. We find this to be

$$\begin{aligned}8c_1c_4^2 = &- 8c_3\left(\epsilon\Theta c_4\right)^2 + 8c_4\left(\epsilon\Theta c_3\right)\left(\epsilon\Theta c_4\right) + 4c_3c_4\left(\epsilon^2\Theta^2\, c_4\right) - 4c_4^2\left(\epsilon^2\Theta^2\, c_3\right) \\ &+ 6c_3^2\left(\epsilon\Theta\, c_4\right) - 6c_3c_4\left(\epsilon\Theta\, c_3\right) + 8c_4^2\left(\epsilon\Theta\, c_2\right) - 8c_2c_4\left(\epsilon\Theta\, c_4\right) - c_3^3 + 4c_2c_4c_3 ,\end{aligned} \tag{3.42}$$

which can be traced back to the single independent non-trivial selection rule $\kappa_{0,1} = 0$. The equation is invariant under a common rescaling of all c_m and unambiguously fixes c_1 in terms of the c_m with $2 \leq m \leq 5$. The coefficient c_0 does not appear because it is the only place where $\kappa_{3,4}$ enters. As proven in [115, 116], this equality is a consequence of the underlying $\mathcal{N} = 2$ special geometry of Calabi–Yau threefolds [88] and plays an important role in the classification of Picard–Fuchs operators for Calabi–Yau threefolds with a single Kähler modulus [117]. It is remarkable that this constraint is naturally recovered from our gauge theory considerations.

3.6.5 One-Parameter Polarized K3 Surfaces

As third and last example we consider polarized K3 surfaces with a single Kähler parameter. These are complex two-dimensional Calabi–Yau manifolds, such that $d = 2$ in eq. (3.34). By interpolating between the cases of $d = 1$ and $d = 3$ discussed above, we expect that the generating Picard–Fuchs operator is of order three. The index set reads $I_m = \{1, 2, 3, 4\}$ and we collect eq. (3.31) for $0 \leq n \leq 3$ into the matrix equation

$$0 = M \cdot \vec{c} = \begin{pmatrix} 0 & 0 & \kappa_{0,2} & \kappa_{0,3} \\ 0 & -\kappa_{1,1} & -\kappa_{1,2} & -\kappa_{1,3} \\ \kappa_{0,2} & -\kappa_{1,2} & \kappa_{2,2} & \kappa_{2,3} \\ \kappa_{0,3} & -\kappa_{1,3} & \kappa_{2,3} & -\kappa_{3,3} \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{3.43}$$

where the matrix M is defined as usual, i.e., $M_{nm} = (-1)^n \kappa_{n,m}$ with $0 \leq n, m \leq 3$. Since $d = 2$ is even, M is now symmetric rather than anti-symmetric and without imposing additional identities beyond those given by eqs. (3.33) through (3.35) does not have a kernel. Requiring its determinant to vanish is equivalent to the condition

$$\begin{aligned} \kappa_{3,3} = \left(16\,\kappa_{0,2}^3\right)^{-1} \Big\{ & 9\left(\epsilon\Theta\,\kappa_{0,2}\right)^4 - 12\kappa_{0,2}\left(\epsilon\Theta\,\kappa_{0,2}\right)^2\left(\epsilon^2\Theta^2\,\kappa_{0,2} - \kappa_{2,2}\right) \\ & 4\kappa_{0,2}^2\left[4\kappa_{2,2}^2 + \left(\epsilon^2\Theta^2\,\kappa_{0,2}\right)^2 + 4\kappa_{2,2}\left(\epsilon^2\Theta^2\,\kappa_{0,2}\right) - 6\left(\epsilon\Theta\,\kappa_{0,2}\right)\left(\epsilon\Theta\,\kappa_{2,2}\right)\right]\Big\} \end{aligned} \tag{3.44}$$

and results in a one-dimensional kernel. A higher dimensional kernel requires $\kappa_{0,2} = 0$, which again is only possible at special points in moduli space and therefore not of relevance. We arrive at the Picard–Fuchs operator

$$\begin{aligned} \mathcal{L}(Q, \epsilon) = & + 8\kappa_{0,2}^3\left(\epsilon\Theta\right)^3 - 12\kappa_{0,2}^2\left(\epsilon\Theta\kappa_{0,2}\right)\left(\epsilon\Theta\right)^2 \\ & + \left[6\kappa_{0,2}\left(\epsilon\Theta\kappa_{0,2}\right)^2 - 4\kappa_{0,2}^2\left(2\kappa_{2,2} + \Theta^2\epsilon^2\kappa_{0,2}\right)\right]\left(\epsilon\Theta\right) \\ & + 3\left(\epsilon\Theta\kappa_{0,2}\right)^3 - 4\kappa_{0,2}^2\left(\epsilon\Theta\kappa_{2,2}\right) - 2\kappa_{0,2}\left(\epsilon\Theta\kappa_{0,2}\right)\left(\Theta^2\epsilon^2\kappa_{0,2} - 4\kappa_{2,2}\right), \end{aligned} \tag{3.45}$$

which only involves the two correlator $\kappa_{0,2}$ and $\kappa_{2,2}$. Similar to the case of $d = 3$, there is one differential-algebraic relation between the four c_k, see ref. [17] for details.

If condition (3.44) were not fulfilled, the generating Picard–Fuchs operator would need to be of order four or higher. This would require additional elements of $H^{1,1}$ that participate in the quantum product, which is not possible for gauged linear sigma models with a single U(1) factor in their gauge group. For Calabi–Yau fourfolds, however, a similar but slightly different complication can indeed happen. This is the topic of chapter 5.

3.6.6 Generalizations

Let us now generalize to one-parameter Calabi–Yau manifolds of general odd complex dimension $d \geq 5$. As before, we define a $(d + 2) \times (d + 2)$-dimensional matrix M by $M_{nm} = (-1)^n \kappa_{n,m}$ with $0 \leq n, m \leq d + 1$. This matrix is antisymmetric by eq. (3.34) and, since $\kappa_{0,d}$ can at most vanish at special points in moduli spaces, generically has a one-dimensional kernel. This corresponds to a unique differential operator of order $d + 1$, which is a plausible candidate for the Picard–Fuchs operator that generates the non-generic ideal $\mathcal{I}_S^0$. While Griffith transversality implies the order of the generating operator to be at least $d + 1$, the author is not aware of a reason why it cannot be of order $d + 2$ or higher. In such cases the candidate solution would need to run into a contradiction with eq. (3.31) for some $n \geq d + 2$, which without a detailed analysis can indeed not be excluded. It would be interesting to consider this question further.

We can also generalizes to one-parameter Calabi–Yau manifolds of even complex dimensions $d \geq 4$. Defining the matrix M as usual, it is symmetric and without imposing further identities has full rank. By a condition similar to that in eq. (3.44) can impose its determinant to vanish, whereas a kernel of dimension two or higher is only possible at non-generic points in moduli space. Griffith transversality guarantees that the order of the generating Picard–Fuchs operator is at least $d + 1$, but it may be $d + 2$ or higher. For the case of Calabi–Yau fourfolds, which is $d = 4$, there are examples with a generating operator of order six, see refs. [19, 56, 118]. In fact, the entire fifth chapter of this thesis is devoted to a study of this phenomenon. The correlator formulas for the generating Picard–Fuchs operators of one-parameter Calabi–Yau fourfolds can, both for the order five and six case, be found in ref. [17].

Formulas for the two generating Picard–Fuchs operators of Calabi–Yau threefolds with two Kähler moduli are presented in ref. [17]. For higher complex dimensions or yet more Kähler parameters we do not except to find universal formulas, at least they would become rather unwieldy.

3.7 Examples

In this section we apply the above findings to several concrete gauged linear sigma models. We specialize to the gauge group $G = \mathrm{U}(1)$, for non-Abelian and multi-parameter examples see ref. [17]. In order to keep formulas compact, we suppress writing any functional dependencies except those on twisted masses.

3.7.1 Projective Space $\mathbb{P}^{N-1}$

We start with a gauged linear sigma model with gauge group $G = \mathrm{U}(1)$ and charged matter spectrum as listed in Table 3.2. Due to the positive sum of gauge charges, $S = N$, the bare value of the real FI

Chiral multiplets	$G = \mathrm{U}(1)$ charge	Vector R-charge $\mathfrak{q}_i$	Twisted masses
$\Phi_i,\ i = 1, \ldots, N$	$+1$	0	$\mathfrak{m}_i$

Table 3.2: Matter spectrum of the gauged linear sigma model of the projective space $\mathbb{P}^{N-1}$.

parameter is $r(\Lambda_{\mathrm{UV}}) = \infty$ and we need to consider $r \gg 0$. Since all chiral multiplets have zero vector R-charge, there is no superpotential. The classical target space, see eq. (2.44), is the complex projective space $\mathbb{P}^{N-1}$. In order to determine correlator relations, we first spell out the rational functions g_p defined in eq. (3.12),

$$g_p(\mathfrak{m}_i) = \prod_{s=1}^{p} \prod_{i=1}^{N} \left[w + \mathfrak{m}_i + \epsilon\,(1 - s) \right] . \tag{3.46}$$

The simplest solution to the constraint eq. (3.11) is $\alpha_0 = g_1$ and $\alpha_1 = -1$ together with $\alpha_p = 0$ for $p \geq 2$, which determines the relation

$$0 = R_S(\mathfrak{m}_i) = \left\langle \sigma_N^n (\sigma_S + \mathfrak{m}_i) \cdots (\sigma_S + \mathfrak{m}_N) \right\rangle - Q \left\langle \sigma_N^n \right\rangle \tag{3.47}$$

that is guaranteed to hold for all non-negative integers n. For $N = 2$ an explicit calculation of the localization formula (3.4) gives the correlators

$$\begin{aligned} &\kappa_{0,0} = 0\,, && \kappa_{0,1} = 1\,, && \kappa_{0,2} = -(\mathfrak{m}_1 + \mathfrak{m}_2)\,, \\ &\kappa_{1,0} = 1\,, && \kappa_{1,1} = -(\mathfrak{m}_1 + \mathfrak{m}_2)\,, && \kappa_{1,2} = (\mathfrak{m}_1 + \mathfrak{m}_2)^2 - \mathfrak{m}_1\mathfrak{m}_2 + Q\,. \end{aligned} \tag{3.48}$$

These are indeed polynomial in Q as discussed in subsection 3.2.1 and demonstrate validity of the relation (3.47) for $n = 0$ and $n = 1$. (Side remark: For this model *all* poles of the one loop determinant contribute to the residue. Their combined contribution can be rewritten as the residue at infinity, which speeds up calculations significantly.) According to eq. (3.21) the relation defines the differential operator

$$\boldsymbol{R}_S(\mathfrak{m}_i) = (\epsilon\Theta + \mathfrak{m}_1) \cdots (\epsilon\Theta + \mathfrak{m}_N) - Q\,, \tag{3.49}$$

where $\Theta = Q\partial_Q$ is the logarithmic derivative with respect to Q. The equivariant Givental I-function (2.54), of the target space $\mathbb{P}^{N-1}$ reads

$$I_{\mathbb{P}^{N-1}}(\mathfrak{m}_i) = \sum_{k=0}^{\infty} \frac{1}{\prod_{s=1}^{k} (H + \mathfrak{m}_1 + s\epsilon) \cdots (H + \mathfrak{m}_N + s\epsilon)} Q^{\frac{H}{\epsilon} + k}\,, \tag{3.50}$$

with H the generator of $H^{1,1}(\mathbb{P}^N, \mathbb{Z})$. This series is indeed annihilated by $\boldsymbol{R}_S$, where the $k = 0$ term vanishes due to the identification $(H + \mathfrak{m}_1) \cdots (H + \mathfrak{m}_N) \sim 0$ in the equivariant cohomology ring $\mathbb{C}[H, \mathfrak{m}_i] / (H + \mathfrak{m}_1) \cdots (H + \mathfrak{m}_N)$. Since there cannot be any cancellations in the g_p, see eq. (3.46), there is no special choice of twisted masses for which an enhanced non-generic module M_S^0 and differential ideal $\mathcal{I}_S^0$ arise. Correspondingly, $\boldsymbol{R}_S$ generates the entire ideal of operators and never factorizes.

3.7.2 Quintic Calabi–Yau Threefold $\mathbb{P}^4[5]$

As second example we reconsider the model discussed at the end of section 2.2, with the difference of now turning on generic twisted masses as specified in Table 3.3. This model is conformal and its $r \gg 0$ target space (2.49) is the quintic Calabi–Yau threefold $\mathbb{P}^4[5]$. The first two rational functions read

$$g_0(\mathfrak{m}_i, \mathfrak{m}_P) = 1\,, \quad g_1(\mathfrak{m}_i, \mathfrak{m}_P) = -\frac{(w + \mathfrak{m}_1) \cdots (w + \mathfrak{m}_5)}{(5w - \mathfrak{m}_P) \cdots (5w - \mathfrak{m}_P - 4\epsilon)}\,, \tag{3.51}$$

which are now — since P has negative charge — except for g_0 not polynomial anymore. We readily determine a relation $R_S = R_S(\mathfrak{m}_i, \mathfrak{m}_P)$,

$$\begin{aligned} 0 = R_S &= \left\langle \sigma_N^n (\sigma_S + \mathfrak{m}_1) \cdots (\sigma_S + \mathfrak{m}_5) \right\rangle + Q \left\langle \sigma_N^n (5\sigma_S - \mathfrak{m}_P + \epsilon) \cdots (5\sigma_S - \mathfrak{m}_P + 5\epsilon) \right\rangle\,, \\ \boldsymbol{R}_S &= (\epsilon\Theta + \mathfrak{m}_1) \cdots (\epsilon\Theta + \mathfrak{m}_5) + Q(5\epsilon\Theta - \mathfrak{m}_p + \epsilon) \cdots (5\epsilon\Theta - \mathfrak{m}_p + 5\epsilon)\,, \end{aligned} \tag{3.52}$$

Chiral multiplets	$G = \mathrm{U}(1)$ charge	Vector R-charge $\mathfrak{q}_i$	Twisted masses
Φ_i, $i = 1, \ldots, 5$	$+1$	0	$\mathfrak{m}_i$
P	-5	2	$\mathfrak{m}_P$

Table 3.3: Matter spectrum of the gauged linear sigma model of the quintic Calabi–Yau threefold $\mathbb{P}^4[5]$ with generic twisted masses.

together with its associated differential operator $\boldsymbol{R}_S = \boldsymbol{R}_S(\mathfrak{m}_i, \mathfrak{m}_P)$. This operator indeed annihilates the equivariant Givental I-function

$$I_{\mathbb{P}^4[5]}(\mathfrak{m}_i, \mathfrak{m}_P) = \sum_{k=0}^{\infty} (-1)^k \frac{\prod_{s=1}^{5k} (5H - \mathfrak{m}_P + s\epsilon)}{\prod_{s=1}^{k} (H + \mathfrak{m}_1 + s\epsilon) \cdots (H + \mathfrak{m}_5 + s\epsilon)} Q^{\frac{H}{\epsilon}+k} , \tag{3.53}$$

where the $k = 0$ term vanishes due to the identification $(H + \mathfrak{m}_{i_1}) \cdots (H + \mathfrak{m}_{i_5}) \sim 0$ in the equivariant cohomology ring of the target space $\mathbb{P}^4[5]$.

We observe that the non-generic choice of twisted masses given by the equality $\mathfrak{m}_P = -5\mathfrak{m}_i$ results in a cancellation of the i-th factor in the numerator of g_1 with the first factor in its denominator. For definiteness, let us consider $\mathfrak{m}_P = -5\mathfrak{m}_5$. This gives a non-generic correlator relation $R^0_S(\mathfrak{m}_i)$, whose associated differential operator reads

$$\boldsymbol{R}^0_S(\mathfrak{m}_i) = (\epsilon\Theta + \mathfrak{m}_1) \cdots (\epsilon\Theta + \mathfrak{m}_4) + 5\,Q\,(5\epsilon\Theta + 5\mathfrak{m}_5 + \epsilon) \cdots (5\epsilon\Theta + 5\mathfrak{m}_5 + 4\epsilon) . \tag{3.54}$$

We now have to check condition (3.15) and for this purpose determine the sets defined in eq. (3.14). These read

$$\Theta(r > 0) = \bigcup_{i=1}^{5} \left\{\epsilon\mathbb{Z}_{\leq 0} - \mathfrak{m}_i\right\} , \quad \Omega(r > 0) = \left\{\tfrac{1}{5}\epsilon\mathbb{Z}_{>0} + \tfrac{1}{5}\mathfrak{m}_P\right\} , \tag{3.55}$$

and for $\mathfrak{m}_P = -5\mathfrak{m}_5$ their intersection is still empty. Consequently, $\boldsymbol{R}^0_S$ is a valid operator and we can check that it indeed annihilates the non-generic I-function $I_{\mathbb{P}^4[5]}(\mathfrak{m}_i, \mathfrak{m}_P = -5\mathfrak{m}_i)$. With the commutator $[\Theta, Q] = Q$ we also find the factorization

$$\boldsymbol{R}^{\lim}_S(\mathfrak{m}_i) = \boldsymbol{R}_S(\mathfrak{m}_i, \mathfrak{m}_P = -5\mathfrak{m}_5) = \left(\epsilon\Theta + \mathfrak{m}_5\right) \boldsymbol{R}^0_S(\mathfrak{m}_i) . \tag{3.56}$$

While $\boldsymbol{R}^0_S$ generates the entire ideal of operators for the non-generic choice $\mathfrak{m}_P = -5\mathfrak{m}_5$, the limiting operator $\boldsymbol{R}^{\lim}_S$ only generates a proper subideal. Let us now additionally put $\mathfrak{m}_i = 0$, which we are free to do without checks since there are no additional cancellations in g_1. The operator then reduces to

$$\boldsymbol{R}^0_S(\mathfrak{m}_i = 0) = \epsilon^4 \left[\Theta^4 + 5Q\,(5\Theta + 1) \cdots (5\Theta + 4)\right] . \tag{3.57}$$

Up to an inconsequential overall factor of ϵ^4 this precisely is the generating Picard–Fuchs operator of the quintic Calabi–Yau threefold, see eq. (2.63).

The methods of section 3.6 provide an alternative way of deriving this operator. To this end, we use the localization formula (3.4) to calculate the three correlators

$$\kappa_{0,3} = \frac{5}{1 + 5^5 Q} , \quad \kappa_{2,3} = \frac{-6\,250 Q}{(1 + 5^5 Q)^2}\epsilon^2 , \quad \kappa_{3,4} = \frac{100Q(-6 + 59\,375Q)}{(1 + 5^5 Q)^3}\epsilon^4 \tag{3.58}$$

for vanishing twisted masses. From the universal formula (3.41) for Picard–Fuchs operators of one-parameter Calabi–Yau threefold we then get

$$\mathcal{L} = \frac{5^2\epsilon^4}{(1+5^5Q)^3}\left[\Theta^4 + 5Q\,(5\Theta+1)\cdots(5\Theta+4)\right], \tag{3.59}$$

which up to an overall factor agrees with $\boldsymbol{R}^0_S(\mathfrak{m}_i = 0)$. As long as it is feasible to calculate correlators quickly, this method is rather simple.

We see that the correlators in eq. (3.58) are, in accord with the discussion of subsection 3.2.1, rational functions of Q with a pole at the point $Q = -5^{-5}$ of the Coulomb branch as predicted by eq. (2.45). Moreover, the correlator $\kappa_{0,3}$ agrees with the B-model Yukawa coupling (2.65) that we derived from the differential equation (2.64). We can even re-derive this differential equation from our gauge theory approach. For this we take the non-generic relation $R^0_S(\mathfrak{m}_i = 0)$ and employ properties (3.33) and (3.35) to get

$$0 = R^0_S(\mathfrak{m}_i = 0) = c_4\,\kappa_{0,4} + c_3\,\kappa_{0,3} = \left[2c_4\,\epsilon\Theta + c_3\right]\,\kappa_{0,3}\,. \tag{3.60}$$

Here $c_k = c_k(Q,\epsilon)$ are the coefficients of σ^k_S in $R^0_S(\mathfrak{m}_i = 0)$ and, equivalently, the coefficients of $(\epsilon\Theta)^k$ in the Picard–Fuchs operator $\boldsymbol{R}^0_S(\mathfrak{m}_i = 0)$. This argument equally applies to all one-parameter Calabi–Yau threefolds.

As we have seen, the Picard–Fuchs operator follows straightforwardly from the gauged linear sigma model. This means that we can easily find the genus zero integral Gromov–Witten invariants — which appear in the quantum product (2.52) and thereby determine the triple correlation function (2.51) — without using mirror symmetry.

3.7.3 Local Calabi–Yau Threefold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$

As third and last example we consider the $G = \mathrm{U}(1)$ gauged linear sigma model with matter spectrum as listed in Table 3.4. While the chosen twisted masses are not entirely generic, they are generic

Chiral multiplets	$G = U(1)$ charge	Vector R-charge $\mathfrak{q}_i$	Twisted masses
$\Phi_i,\ i = 1,2$	$+1$	0	0
$\Psi_i,\ i = 1,2$	-1	0	$\mathfrak{m}^i_\psi$

Table 3.4: Matter spectrum of the gauged linear sigma model of the non-compact conifold Calabi–Yau threefold $\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1$.

enough to ensure applicability of the modified residue symbol (3.10) and guarantee that all correlator relations are valid. The model is conformal and in the phase $r \gg 0$ has the Calabi–Yau target space

$$\mathcal{X} = \left\{(\phi,\psi)\in\mathbb{C}^4 \,\middle|\, |\phi|^2 = r + |\psi|^2 > 0\right\}\Big/\mathrm{U}(1) = \mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1\,, \tag{3.61}$$

which is non-compact due to the unconstrained directions ψ_i. In the phase $r \ll 0$ the target space is actually the same.the multiplets ϕ_i and ψ_i simply swap their roles. We work with $r \gg 0$ and analyze a basic correlator relation both for $\mathfrak{m}^i_\psi$ generic, as well as for the non-generic choice given by the condition $\mathfrak{m}_\psi^2 = -\epsilon$. From eq. (3.12) we find the rational functions

$$g_0(\mathfrak{m}_\psi^i) = 1\,, \quad g_1(\mathfrak{m}_\psi^i) = \frac{w^2}{(w-\mathfrak{m}_\psi^1-\epsilon)(w-\mathfrak{m}_\psi^2-\epsilon)}\,, \tag{3.62}$$

and the cancellation in $g_1(\mathfrak{m}_\psi^1, \mathfrak{m}_\psi^2 = -\epsilon)$ signals an enhanced syzygy module M_S^0. With eqs. (3.11) and (3.13) we get

$$\begin{aligned} 0 &= R_S(\mathfrak{m}_\psi^i) = \left\langle \sigma_N^n\,\sigma_S^2 \right\rangle - Q\left\langle \sigma_N^n\,(\sigma_S-\mathfrak{m}_\psi^1)(\sigma_S-\mathfrak{m}_\psi^2)\right\rangle\,,\\ 0 &\stackrel{?}{=} R_S^0(\mathfrak{m}_\psi^1) = \left\langle \sigma_N^n\,\sigma_S \right\rangle - Q\left\langle \sigma_N^n\,(\sigma_S-\mathfrak{m}_\psi^1)\right\rangle\,, \end{aligned} \tag{3.63}$$

where R_S^0 is unlike R_S not guaranteed to be a valid correlator relations. For this we need to check the condition (3.15) and determine the pole sets defined in (3.14), which here read

$$\Theta(r>0) = \epsilon\mathbb{Z}_{\leq 0}\,, \quad \Omega(r>0,\mathfrak{m}_\psi^i) = \cup_{i=1}^2\left\{\epsilon\mathbb{Z}_{\geq 0}+\mathfrak{m}_\psi^i\right\}\,. \tag{3.64}$$

Since the intersection $\Theta(r>0)\cap\Omega(r>0,\mathfrak{m}_\psi^1,\mathfrak{m}_\psi^2=-\epsilon) = \{-\epsilon, 0\}$ is non-empty for the non-generic choice of twisted masses, such that we do not expect R_S^0 to be a valid relation. The correlators

$$\begin{aligned} \kappa_{0,0} &= \frac{\mathfrak{m}_\psi^1+\mathfrak{m}_\psi^2}{\left(\mathfrak{m}_\psi^1\mathfrak{m}_\psi^2\right)^2}\,, & \kappa_{0,1} &= \frac{1}{\mathfrak{m}_\psi^1\mathfrak{m}_\psi^2}\,, & \kappa_{0,2} &= 0\,,\\ \kappa_{1,0} &= \frac{1}{\mathfrak{m}_\psi^1\mathfrak{m}_\psi^2}\,, & \kappa_{1,1} &= 0\,, & \kappa_{1,2} &= \frac{Q}{1-Q} \end{aligned} \tag{3.65}$$

confirm these considerations explicitly for $n=0$ and $n=1$. A similar discussion applies to the Givental I-function, which hear reads

$$I_{\mathcal{X}}(\mathfrak{m}_\psi^i) = \sum_{k=0}^{\infty}\frac{\prod_{s=0}^{k-1}(H-\mathfrak{m}_\psi^1+s\epsilon)(H-\mathfrak{m}_\psi^2+s\epsilon)}{\prod_{s=1}^{k}(H+s\epsilon)^2}\,Q^{\frac{H}{\epsilon}+k}\,. \tag{3.66}$$

It is annihilated by the operator $\boldsymbol{R}_S(\mathfrak{m}_\psi^i)$ associated to $R_S(\mathfrak{m}_\psi^i)$, where the $k=0$ term vanishes thanks to the identification $H^2\sim 0$ in the equivariant cohomology ring of $\mathcal{X}$ for $\mathfrak{m}_\phi^i = 0$. This suggests that eq. (3.23) is also valid for non-compact target spaces. The limiting operator factors,

$$\boldsymbol{R}_S^{\lim}(\mathfrak{m}_\psi^1) = \boldsymbol{R}_S(\mathfrak{m}_\psi^1,\mathfrak{m}_\psi^2-\epsilon) = \epsilon\Theta\left[\epsilon\Theta - Q(\epsilon\Theta-\mathfrak{m}_\psi^1)\right] = \epsilon\Theta\,\boldsymbol{R}_S^0(\mathfrak{m}_\psi^1)\,, \tag{3.67}$$

where $\boldsymbol{R}_S^0$ is the operator obtained from R_S^0. This operator does, however, not annihilate the non-generic I-function $I_{\mathcal{X}}(\mathfrak{m}_\psi^1,\mathfrak{m}_\psi^2=-\epsilon)$. Interestingly, this only fails at the classical order $k=0$.

Since the target space $\mathcal{X}$ is non-compact, we cannot take the limit of vanishing twisted masses in all of the correlators presented in eq. (3.65). However, those correlators that remain finite in this limit behave similar to correlators of models with a compact Calabi–Yau target space of the same dimension. In particular, $\kappa_{1,2}$ is rational in Q with a pole at the position of the Coulomb branch.

CHAPTER **4**

Fundamental Periods of Non-Abelian Gauged Linear Sigma Models

In this chapter we continue on the study of the connection between gauged linear sigma models and Picard–Fuchs operators. We present yet another method to determine the latter from the gauge theory spectrum, which is particularly useful for non-Abelian models and clearly outperforms the methods of the previous chapter in terms of computational efficiency. The discussion also applies to non-conformal models and the operators describing their target space quantum cohomology.

This chapter presents new, unpublished work by the author. Useful discussions with Dr. Hans Jockers and Christoph Nega are acknowledged.

4.1 Introduction

The previous chapter demonstrated tight connections between gauged linear sigma models on the one hand and differential operators governing the target space quantum cohomology on the other hand. In particular, we presented two methods that determine the operators from the gauge theory without the use of mirror symmetry. Let us here briefly recall them. First, in section 3.3 we derived universal linear dependencies between gauged linear sigma model correlation functions that, as in explained in section 3.4, turn into differential operators annihilating the moduli dependent gauge theory ground state. For Abelian models with a geometric target space we have further shown these to annihilate the Givental I-function [18, 83], which for conformal models identifies them as the Picard–Fuchs operators on the quantum Kähler moduli space of the low energy superconformal field theory. Second, for several classes of Calabi–Yau manifolds — specified by their complex dimension and number of Kähler parameters — we in section 3.6 presented formulas that universally express their Picard–Fuchs operators in terms of the gauge theory correlators.

As an alternative practical approach, we may also make use of quantities that are known to be annihilated by the differential operators. The latter can then be found by making an ansatz and requiring it to annihilate these expressions. Examples of such quantities are the Givental I-function and, for conformal models, the quantum periods it defines. We here want to apply this simple idea to non-Abelian gauged linear sigma models. For this purpose we aim to find explicit formulas for their fundamental periods — or, more generally speaking and also applicable to non-conformal models, the holomorphic solutions to the differential operators — in terms of the defining gauge theory spectrum. Our approach starts with the I-function of the associated Abelian Cartan theory and we make a proposal for how the fundamental period (holomorphic solution) of the original non-Abelian model is recovered in a certain non-trivial limit. We derive a simple sufficient condition that in combination with some restrictions on the chiral matter spectrum guarantees the limit to exist and in the same time determines its finite value. For a few low-rank non-Abelian gauge groups we check this condition explicitly and spell out general formulas for the corresponding fundamental periods (holomorphic solutions). We apply these to several examples of non-Abelian gauged linear sigma models and find agreement with the literature. Moreover, we briefly comment on how the method may be generalized to also yield the other quantum periods, i.e, the other solutions of the differential operator. The formulas for the fundamental periods provide a particularly efficient way to determine the Picard–Fuchs operator of a given non-Abelian model and — since they are applicable to a large class of matter spectra — an automated scan over models with the aim of finding, for example, phases with Calabi–Yau threefold target spaces presents itself as a promising application. Lastly, we discuss the idea of reconstructing gauged linear sigma models from given differential operators. As we will explain, the formulas for the fundamental period may play a central role for such a program.

The hemisphere partition function of gauged linear sigma models is also known to be annihilated by the Picard–Fuchs operators and, in fact, computes the quantum periods [106–108]. While this quantity is general in the sense that it applies without restriction on the matter spectrum, it still needs to be evaluated for a given model. The formulas presented here have the advantage that they are ready to use and can essentially be typed into the computer directly, without the need to do a calculation before. It would be interesting to re-derive the formulas from the hemisphere partition function. A related quantity is the two-sphere partition function [65, 66], which computes the quantum-exact Kähler metric on the quantum Kähler moduli space and too is annihilate by the Picard–Fuchs operators [103–105]. Moreover, it would be interesting to understand how the findings of this chapter connect to the related mathematical literature, such as for example ref. [84].

The below discussions equally apply to conformal and non-conformal gauged linear sigma models, as well as for generic and non-generic choices of twisted masses. In order to simplify notation and reading, we will from here on not distinguish between these cases unless explicitly stated. When speaking of (fundamental) periods, we implicitly also refer to the (holomorphic) solutions of the differential operators that govern the quantum cohomology of non-conformal models, as well as cases with generically chosen twisted masses.

4.2 Formulas for Fundamental Periods of Non-Abelian Models

In this section we propose general formulas for the fundamental periods of gauged linear sigma models with non-Abelian gauge groups and large classes of chiral matter spectra. As valuable resource for the structure of simple Lie algebras we employ section 13 of the textbook [26].

4.2.1 Problem Specification and Cartan Theory I-Function

We consider gauged linear sigma models with non-Abelian gauge groups of the general form specified by eq. (2.31). Without loss of generality, we assume that the ℓ complexified Fayet–Iliopoulos (FI) parameters $\vec{\tau}$ are in the phase $\vec{r} = \mathrm{Re}(\vec{\tau}) \gg 0$. This can always be achieved by reversing signs of the U(1) generators if needed. For non-conformal models the physical consistency condition $r_l(\Lambda_{\mathrm{UV}}) = +\infty$ is understood to hold for all l.

Our aim is to derive general formulas for the fundamental periods of such non-Abelian models. We propose to approach this by means of the Givental I-function [18, 83] of the associated Cartan theories introduced in subsection 3.5.2, in which the non-Abelian gauge groups are spontaneously broken to their maximal Abelian subgroups T according to

$$G = \frac{\mathrm{U}(1)^\ell \times G_1 \times \ldots \times G_m}{\Gamma} \longrightarrow T = \frac{\mathrm{U}(1)^\ell \times \mathrm{U}(1)^{r-\ell}}{\Gamma} . \tag{4.1}$$

Here we write $r = \operatorname{rank} G \geq \ell + 1$ and each factor of the discrete quotient group Γ acts non-trivially on at least one of the non-Abelian groups $G_1, \ldots, G_m$ — else we redefine the generators of $\mathrm{U}(1)^\ell$ to get rid of this factor. The chiral spectrum of the Cartan theory is summarized in Table 3.1. We now want to employ eq. (2.54) for the Givental I-function, which in this form is only valid for genuine Abelian gauge groups $\mathrm{U}(1)^r$ and here requires some modifications. Let us explain the individual ingredients of the formula one after another:

- The set γ_m^+ is a subset of the magnetic charge lattice γ_m of T, which for a non-trivial quotient group Γ is *not* equal to $\mathbb{Z}^r$. Even for a trivial quotient, γ_m^+ is in the present setup *not* necessarily equal to $\mathbb{Z}^r_{\geq 0}$. We will specify γ_m^+ below.

- The vector $\vec{Q}$ reads $\vec{Q} = (\vec{Q}_{\mathrm{na}}, 1, \ldots, 1)$, where the variables $\vec{Q}_{\mathrm{na}} = (Q_1, \ldots, Q_\ell)$ are specified by the FI parameters $\vec{\tau}$ of the non-Abelian theory as in eq. (2.53). There are $r - \ell$ additional ones, which means we *do not turn on* auxiliary FI parameters.

- We take $\vec{H} = (\vec{H}_{\mathrm{na}}, \vec{H}_{\mathrm{aux}})$, where $\vec{H}_{\mathrm{na}} = (H_1, \ldots, H_\ell)$ generate the cohomology ring of the non-Abelian model's target space. Unlike for the FI parameters, we do turn on auxiliary variables $\vec{H}_{\mathrm{aux}} = (H_{\ell+1}, \ldots, H_r)$.

- The chiral spectrum of the Cartan theory includes the W-bosons of the spontaneous symmetry breaking, whose combined contribution for $\vec{k} \in \gamma_m^+$ fixed reads

$$I^W(\vec{H}_{\mathrm{aux}}, \vec{k}) = (-1)^{\vec{k}\cdot\sum_{\vec{\alpha}_\beta > 0}\vec{\alpha}_\beta} \prod_{\vec{\alpha}_\beta > 0} \left[1 + \epsilon \frac{\vec{\alpha}_\beta \cdot \vec{k}}{\vec{\alpha}_\beta \cdot \vec{H}} \right] . \tag{4.2}$$

  Here the product is over positive roots $\vec{\alpha}_\beta > 0$ and we used that non-zero roots come in pairs $(\vec{\alpha}_\beta, -\vec{\alpha}_\beta)$. The first ℓ entries of $\vec{\alpha}_\beta$ correspond to the ℓ Abelian U(1) factors and are zero, such that $\vec{\alpha}_\beta \cdot \vec{H}$ does not involve $\vec{H}_{\mathrm{na}}$ but only depends on the auxiliary variables $\vec{H}_{\mathrm{aux}}$. These are thus required to define the I-function of the Cartan theory in the first place and setting them back to zero is a non-trivial problem. The overall power of (-1) can be simplified by using that the sum of positive roots equals $(0, \ldots, 2, \ldots, 2)$. Since roots are (independent of the quotient Γ) always elements of the electric charge lattice, this in particular implies $2k_{\ell+1} + \ldots + 2k_r \in \mathbb{Z}$.

- Further, there are the chiral multiplets Λ_{i,β_i} that result from the decomposition of the original non-Abelian matter spectrum. As for the genuine Abelian case in eq. (2.54), we restrict their vector R-charges to be either zero or two. All fields of the former type are collected into a first class whose charge vectors and R-charges we relabel as $\vec{\lambda}_a^+$ and $\mathfrak{q}_a^+ = 0$ with $a = 1, \ldots, N_+$. The others form a second class and we similarly write $\vec{\lambda}_b^-$ and $\mathfrak{q}_b^- = 2$ with $b = 1, \ldots, N_-$. Their combined contribution for $\vec{k} \in \gamma_m^+$ fixed reads

$$I^M(\vec{H}, \vec{k}) = \frac{1}{\epsilon^{\vec{k}\cdot\vec{S}^+}(-\epsilon)^{\vec{k}\cdot\vec{S}^-}} \prod_{a=1}^{N_+} \frac{\Gamma\left[1 + \frac{\mathfrak{m}_a^+ + \vec{\lambda}_a^+ \cdot \vec{H}}{\epsilon}\right]}{\Gamma\left[1 + \frac{\mathfrak{m}_a^+ + \vec{\lambda}_a^+ \cdot (\vec{H} + \epsilon \vec{k})}{\epsilon}\right]} \prod_{b=1}^{N_-} \frac{\Gamma\left[1 - \frac{\mathfrak{m}_b^- + \vec{\lambda}_b^- \cdot (\vec{H} + \epsilon \vec{k})}{\epsilon}\right]}{\Gamma\left[1 - \frac{\mathfrak{m}_b^- + \vec{\lambda}_b^- \cdot \vec{H}}{\epsilon}\right]}, \tag{4.3}$$

where $\mathfrak{m}_a^+$ and $\mathfrak{m}_b^-$ are twisted masses. We further use the quantities $\vec{S}^+ = \sum_a \vec{\lambda}_a^+$ and $\vec{S}^- = \sum_b \vec{\lambda}_b^-$, which for conformal models are subject to $\vec{S}^+ + \vec{S}^- = 0$.

- The set γ_m^+ is defined as the collection of all $\vec{k} \in \gamma_m$ for which $\vec{k} \cdot \vec{\lambda}_a^+ \geq 0$ holds for all a.
- Finally, we impose further restrictions on the non-Abelian matter fields. First, with the unit vectors $\vec{e}_i$ we require $\vec{e}_i \cdot \vec{\lambda}_a^+ > 0$ for all a and $i = 1, \ldots, \ell$. In other words, all non-Abelian chiral matter multiplets with zero vector R-charge are positively charged under all U(1) factors in G. Second, we require $\vec{k} \cdot \vec{\lambda}_b^- \leq 0$ for all b and $\vec{k} \in \gamma_m^+$.

With these definitions and restrictions the Givental I-function (2.54) of the Cartan theory assumes the form

$$I^{\text{Cartan}}(\vec{H}, \vec{Q}_{\text{na}}) = \sum_{\vec{k} \in \gamma_m^+} Q_1^{k_1 + \frac{H_1}{\epsilon}} \cdots Q_\ell^{k_\ell + \frac{H_\ell}{\epsilon}} \cdot I^W(\vec{H}_{\text{aux}}, \vec{k}) \cdot I^M(\vec{H}, \vec{k}) \,. \tag{4.4}$$

For brevity we suppress writing functional dependencies on ϵ and the twisted masses, whereas we make explicit that no auxiliary FI parameters are turned on.

4.2.2 Proposal for Fundamental Periods of Non-Abelian Models

As we recall from eq. (2.55), the fundamental periods of Abelian theories are recovered from the Givental I-function by setting $\vec{H} = 0$. In direct generalization of this we propose the limit

$$\Pi_0(\vec{Q}_{\text{na}}) = \lim_{\vec{H} \to 0} I^{\text{Cartan}}(\vec{H}, \vec{Q}_{\text{na}}) = \lim_{\vec{H}_{\text{aux}} \to 0} I^{\text{Cartan}}(\vec{H}_{\text{na}} = 0, \vec{H}_{\text{aux}}, \vec{Q}_{\text{na}}) \tag{4.5}$$

as the fundamental period of the original non-Abelian theory, where the second equalities used that the Cartan theory I-function I^{Cartan} is regular at $\vec{H}_{\text{na}} = 0$. This proposal comes in two parts.

First, the W-boson contribution I^W specified in eq. (4.2) is clearly singular in the limit $\vec{H}_{\text{aux}} \to 0$ and it is not immediately clear whether the above expression is well-defined in the first place. We conjecture that — since the Cartan theory is invariant under the Weyl group $\mathcal{W}_G$ of G — all singular terms cancel in the sum over magnetic charges γ_m^+ and the limit exists. We will prove this for several low-rank non-Abelian groups and derive explicit formulas for the finite values of the corresponding limits. In short, we find that I^W is replaced by a differential operator involving derivatives with respect to the auxiliary variables $\vec{H}_{\text{aux}}$ that act on the matter contribution I^M. The methods of this proof generalize and we state a simple condition with which the existence of the limit as well as its finite value can be established for fixed but arbitrary non-Abelian gauge groups.

Second, we propose that Π_0 is the actual fundamental period that correctly captures the non-Abelian model's quantum cohomology. In order for this notion to make sense, we in addition to the technical restrictions of the previous subsection assume the model to have a geometric target space in the phase

$\vec{r} \gg 0$ under consideration. For several applicable examples of non-Abelian gauged linear sigma models we compare Π_0 as given by eq. (4.5) to the known expressions from the literature and find agreement in all cases. The associated differential operators are easily found by making an ansatz and requiring it to annihilate the expansion of Π_0. For conformal models with vanishing twisted masses these are the Picard–Fuchs operators on the quantum Kähler moduli space and the present approach provides a computationally efficient way of determining them for non-Abelian gauged linear sigma models. As a caveat, in all examples that we consider the non-Abelian gauge group is spontaneously broken to an Abelian subgroup. This might be an additional requirement for identifying the above limit of the Cartan theory I-function with the fundamental period of the non-Abelian model.

4.2.3 Comments on the Definition of γ_m^+ and the Restrictions on the Non-Abelian Matter Spectrum

Our definition of γ_m^+ and the restrictions on the non-Abelian matter spectrum imply several desirable properties. First, the components $k_1, \ldots, k_\ell$ — associated to the original U(1) factors in G — of all magnetic charges $\vec{k} \in \gamma_m^+$ are non-negative. This implies that I^{Cartan} is holomorphic in the variables $Q_1, \ldots, Q_\ell$ that correspond to the FI parameters $\vec{\tau}$ of the non-Abelian theory. Second, even for $\mathfrak{m}_a^+ = \mathfrak{m}_b^- = 0$ the matter contribution (4.3) remains well behaved in the sense that none of the gamma functions become zero or singular at $\vec{H}_{\text{aux}} = 0$. This will help us to prove that the limit of sending $\vec{H}_{\text{aux}} \to 0$ involved in eq. (4.5) is indeed well defined.

To the author's knowledge, the assumptions are fulfilled for all gauged linear sigma model phases that realize a compact geometric target space which spontaneously breaks non-Abelian gauge group to an Abelian subgroup. In these cases we find agreement with the literature. The assumptions do not hold for several strongly coupled gauged linear sigma model phases in which a non-Abelian subgroup is left unbroken and that nevertheless realize a compact geometric target space. Examples of this type are the $r \ll 0$ phase of the model in section five of ref. [55] and those in ref. [56] — recall: while we here consider $r \gg 0$ only, the sign of r can be reversed by redefining the U(1) generator. We do not claim that eq. (4.5) gives the correct I-function for these cases.

Finally, the above definition of γ_m^+ implicitly depends on the non-Abelian matter spectrum. While this does give the correct answer in all examples we consider, it also conceivable that γ_m^+ is universally fixed by the choice of gauge group. To follow this idea we define $\tilde{\gamma}_m^+$ as the set of all $\vec{k} \in \gamma_m$ whose first ℓ components are non-negative. At least for the examples discussed below, this bigger set is automatically truncated back to γ_m^+ by poles of gamma functions in the denominator of eq. (4.3), which outweigh potential poles in its numerator. As candidate for an alternative definition, we might replace γ_m^+ by $\tilde{\gamma}_m^+$ and restrict the non-Abelian matter spectrum such that I^M with $\mathfrak{m}_a^+ = \mathfrak{m}_b^- = 0$ remains non-singular in the limit $\vec{H}_{\text{aux}} \to 0$ for all $\vec{k} \in \tilde{\gamma}_m^+$. It would be interesting to obtain a better understanding of these aspects.

4.2.4 Single Non-Abelian Factor SU(2)

For models with a single non-Abelian gauge group factor $G_1 = \text{SU}(2)$ we now prove that the limit in eq. (4.5) exists. To be precise, we consider gauge groups of the form

$$G = \frac{\text{U}(1)^\ell \times \text{SU}(2)}{\Gamma} . \tag{4.6}$$

While the discrete quotient Γ does affect the magnetic charge lattice γ_m and in extension the summation set γ_m^+, it does not influence the argument for the existence of the limit. Intuitively, the W-boson contribution I^W only depends on the Lie algebra $\mathfrak{g}$ of G and on the group's global structure only implicitly via the allowed magnetic charges. Therefore, we are able to derive a universal result.

To begin, we spell out the Cartan theory I-function (4.4). Together with eq. (4.2) and using that there is a single positive root $\vec{\alpha}_\beta = (0, \ldots, 0, 2)$ we get

$$I^{\text{Cartan}}(\vec{H}, \vec{Q}_{\text{na}}) = \sum_{\vec{k} \in \gamma_m^+} Q_1^{k_1 + \frac{H_1}{\epsilon}} \cdots Q_\ell^{k_\ell + \frac{H_\ell}{\epsilon}} (-1)^{2k_2} \left(1 + \epsilon \frac{k_r}{H_r}\right) I^M(\vec{H}, \vec{k}) \,. \tag{4.7}$$

Our claim is that the singular term with the power H_r^{-1} cancels in the sum over k_r, which we recall to be the last component of $\vec{k} = (k_1, \ldots, k_\ell, k_r) \in \gamma_m^+$. To show this, let us consider an irreducible representation ρ of G. It is labelled by an ℓ-dimensional integer vector $\vec{\rho}_q$ that specifies the various U(1) charges and the Dynkin label ρ_{hw} of the highest SU(2) weight in the representation. For non-trivial Γ there is an additional constraint on the integers $\vec{\rho}_q$ and ρ_{hw}, whose precise form depends on how the quotient acts. Independent of the quotient, a non-Abelian multiplet in the representation ρ gives rise to $\rho_{hw} + 1$ chiral matter fields in the Cartan theory. Their charge vectors are $\vec{\lambda}_n = (\vec{\rho}_q, \rho_{hw} - 2n)$ with $n = 0, 1, \ldots, \rho_w$ and they all share the same vector R-charge and twisted mass. This means that, including the W-bosons, for each chiral field of the Cartan theory with charge vector $\vec{\lambda} = (\rho_q, \rho_w \neq 0)$ there is another field with $\vec{\lambda}\,' = (\rho_q, -\rho_w)$ that is equivalent in all other respects. Put differently, the weights of a SU(2)-representation — and in extension those of a G-representation — are permuted by the action of the Weyl group $\mathcal{W}_{\text{SU}(2)} = \mathbb{Z}_2$ on the weight lattice of $\mathfrak{su}(2)$, whose non-trivial element acts by $\omega_1 \to -\omega_1$ on the fundamental weight and hence by sign reversal of the Dynkin labels ρ_{hw}. This has two important implications.

- First, the set γ_m^+ is mapped to itself under the transformation $k_r \to -k_r$ that implements the Weyl group action on the magnetic charges $\vec{k}$. To see this, let us denote the action of $w \in \mathcal{W}_{\text{SU}(2)}$ on weights $\vec{\lambda}$ as $w_e[\vec{\lambda}]$. We note that $\vec{\lambda}$ can be a root, which is why the argument equally applies to the W-bosons. Since all fields descending from the same G-representation have the same vector R-charge, we observe that $\{\vec{k} \cdot \vec{\lambda}_a^+ \geq 0\} = \{\vec{k} \cdot w_e[\vec{\lambda}_a^+] \geq 0\}$ and $\{\vec{k} \cdot \vec{\lambda}_b^- \leq 0\} = \{\vec{k} \cdot w_e[\vec{\lambda}_b^-] \leq 0\}$ — where the respective sets are obtained by joining over all a or b. Lastly, the action of w on magnetic charges $\vec{k}$ is dual to its action on weights, i.e., it is defined by the requirement $w_m[\vec{k}] \cdot \vec{\lambda} = \vec{k} \cdot w_e[\vec{\lambda}]$ for all $\vec{k}$ and $\vec{\lambda}$. Given the above equalities of sets, this proves the claim that γ_m^+ is invariant under application of $w_m[\cdot]$.

- Second, $I^W \cdot I^M$ is invariant under application of $w_m[\cdot]$ when it acts on both $\vec{k}$ and $\vec{H}$. We write this as $I^W \cdot I^M = w_m(\vec{k}) \circ w_m(\vec{H})[I^W \cdot I^M] = w_m(\vec{k}) \circ w_m(\vec{H})[I^W \cdot I^M]$, where $\circ$ denotes the composition of maps. The two individual transformations only act on the arguments given in round brackets and therefore clearly commute. We stress that $I^W \cdot I^M$ is *not* invariant under the individual transformations. Moreover, all of this also applies independently to factors I^W and I^M. The powers of $\vec{Q}_{\text{na}}$ are anyway invariant under all these transformations, which is why we do not consider them here.

These two observation are central to the following discussion and will be used frequently. The first implies that the Cartan theory I-function given in eq. (4.7) remains unchanged when $I^W \cdot I^M$ is replaced with its average over the Weyl group action $w_m(\vec{k})$ defined by

$$\mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right] = \frac{1}{|\mathcal{W}_{\mathrm{SU}(2)}|} \sum_{w \in \mathcal{W}_{\mathrm{SU}(2)}} w_m(\vec{k}) \left[I^W \cdot I^M\right] = \frac{I^W \cdot I^M + I^W \cdot I^M{}_{|k_r \to -k_r}}{2} \,. \tag{4.8}$$

Here the Weyl group acts only on $\vec{k}$, and neither on $\vec{H}$ nor on the charge vectors. The validity of this replacement amounts to an appropriate relabeling of the summation set γ_m^+ for the second term of the sum (the one that is non-trivially acted upon). We claim that this averaged expression is manifestly finite in the limit $\vec{H}_{\mathrm{aux}} = (H_r) \to 0$, without the need to first sum over k_r.

In order to understand this, we recall that — due to the restrictions put on the non-Abelian matter spectrum — the term I^M is regular at $H_r = 0$. We can therefore expand it in the variable H_r and get

$$I^M(\vec{H}_{\mathrm{na}}, H_r, \vec{k}) = I^M(\vec{H}_{\mathrm{na}}, 0, \vec{k}) + H_r \cdot \partial_{H_r} I^M(\vec{H}_{\mathrm{na}}, 0, \vec{k}) + O(H_r^2) \,. \tag{4.9}$$

Since I^M is invariant under the combined application of $w_m(\vec{k}) \circ w_m(\vec{H})$ and since terms with different H_r powers do not mix, we find that $I^M(\vec{H}_{\mathrm{na}}, 0, \vec{k})$ and $\partial_{H_r} I^M(\vec{H}_{\mathrm{na}}, 0, \vec{k})$ respectively are even and odd under application of $w_m^{(1)}(\vec{k})$, where $w^{(1)}$ is the non-identity element of $\mathcal{W}_{\mathrm{SU}(2)}$. By using the explicit form of I^W and that $2k_r$ is an integer — which is guaranteed independent of the quotient Γ, see the fourth bullet point in subsection 4.2.1 — eq. (4.8) simplifies to

$$\mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right] = (-1)^{2k_r} \left(1 + k_r\, \partial_{H_r}\right) I^M(\vec{H}, \vec{k})\big|_{H_r = 0} + O(H_r) \,. \tag{4.10}$$

Here the singular term with power H_r^{-1} has cancelled since it is odd under $w_m^{(1)}(\vec{k})$ and the $O(H_r)$ terms will automatically vanish in the limit of interest. In effect, I^W is replaced by the differential operator $\mathcal{D}_{\mathfrak{su}(2)} = (-1)^{2k_r}\left(1 + k_r\, \partial_{H_r}\right)$ involving a derivative with respect to the auxiliary variable H_r that acts on the matter contribution in I^M. We arrive at

$$\Pi_0(\vec{Q}_{\mathrm{na}}) = \sum_{\vec{k} \in \gamma_m^+} Q_1^{k_1} \cdots Q_\ell^{k_\ell}\, \mathcal{D}_{\mathfrak{su}(2)} \left[I^M(\vec{H}, \vec{k})\right]\Big|_{\vec{H} = 0} \tag{4.11}$$

as the finite result of the limit and propose this to be fundamental period of models with gauge group as given by eq. (4.6) and matter spectra in accord with the assumptions of subsection 4.2.1. An explicit evaluation of the derivative yields

$$\begin{aligned} \mathcal{D}_{\mathfrak{su}(2)} \left[I^M(\vec{H}, \vec{k})\right]\Big|_{\vec{H}=0} = I^M(\vec{H}, \vec{k})\big|_{\vec{H}=0} \cdot \Bigg[& 1 - k_r \sum_{a=1}^{N_+} \left(\vec{e}_r \cdot \vec{\lambda}_a^+\right) \psi^{(0)}\big(1 + \tfrac{\mathfrak{m}_a^+}{\epsilon} + \vec{\lambda}_a^+ \cdot \vec{k}\,\big) \\ & - k_r \sum_{b=1}^{N_-} \left(\vec{e}_r \cdot \vec{\lambda}_b^-\right) \psi^{(0)}\big(1 - \tfrac{\mathfrak{m}_b^-}{\epsilon} - \vec{\lambda}_b^- \cdot \vec{k}\,\big) \Bigg] \cdot (-1)^{2k_r} \,, \end{aligned} \tag{4.12}$$

where $\psi^{(0)}$ is the digamma function and $\vec{e}_r$ the unit vector in the r-th direction. The operator $\mathcal{D}_{\mathfrak{su}(2)}$ commutes with putting $\vec{H}_{\mathrm{na}} = 0$ and with all those gamma functions that correspond to Abelian fields. Upon setting $\vec{H}_{\mathrm{aux}} = (H_r) = 0$ after application of the derivatives, it moreover commutes with the collection of all $\vec{k}$-independent gamma functions — this is even independently true for the collection of all $\vec{k}$-independent gamma functions that correspond to a single non-Abelian multiplet.

An important special case are conformal models with vanishing twisted masses. For later use we here spell out that their fundamental periods take the form

$$\Pi_0(\vec{Q}_{\text{na}}) = \sum_{\vec{k}\in\gamma_m^+} (-1)^{2k_r+\vec{k}\cdot\vec{S}^-} Q_1^{k_1}\cdots Q_\ell^{k_\ell} \frac{\prod_{b=1}^{N_-}\Gamma\left(1-\vec{\lambda}_b^-\cdot\vec{k}\right)}{\prod_{a=1}^{N_+}\Gamma\left(1+\vec{\lambda}_a^+\cdot\vec{k}\right)}\cdot\Bigg[1-k_r\sum_{a=1}^{N_+}\left(\vec{e}_r\cdot\vec{\lambda}_a^+\right)h_{\vec{\lambda}_a^+\cdot\vec{k}} - k_r\sum_{b=1}^{N_-}\left(\vec{e}_r\cdot\vec{\lambda}_b^-\right)h_{-\vec{\lambda}_b^-\cdot\vec{k}}\Bigg], \tag{4.13}$$

where h_n denotes the n-th harmonic number — we deviate from the standard notation H_n to avoid confusion with the variables $\vec{H}$ — and the gamma functions may also be written as factorials. For a given model, we can straightforwardly determine the Picard–Fuchs operators by requiring them to annihilate the expansion of this expression.

4.2.5 Single Non-Abelian Factor SU(3)

Let us now generalize these arguments to gauge groups with a single non-Abelian factor $G_1 = \text{SU}(3)$. These are of the general form

$$G = \frac{\text{U}(1)^\ell\times\text{SU}(3)}{\Gamma}, \tag{4.14}$$

where the precise form of the quotient Γ will again not effect the existence of the limit in eq. (4.5). There are three positive roots — $\vec{\alpha}_{\beta_1} = (\ldots, 2, -1)$, $\vec{\alpha}_{\beta_2} = (\ldots, 1, 1)$ and $\vec{\alpha}_{\beta_3} = (\ldots, -1, 2)$ where the dots stand for ℓ zeros — and the W-boson contribution (4.2) reads

$$I^W(\vec{H}_{\text{aux}},\vec{k}) = (-1)^{2k_{r-1}+2k_r}\left(1+\frac{2k_{r-1}-k_r}{2H_{r-1}-H_r}\right)\left(1+\frac{k_{r-1}+k_r}{H_{r-1}+H_r}\right)\left(1+\frac{-k_{r-1}+2k_r}{-H_{r-1}+2H_r}\right). \tag{4.15}$$

The weights of a SU(3)-representation — and in extension those of a G-representation — are permuted by the action of the Weyl group $\mathcal{W}_{\text{SU}(3)}$ on the weight lattice of $\mathfrak{su}(3)$, which is constituted by the six elements $\{1, w^{(1)}, w^{(2)}, w^{(1)}w^{(2)}, w^{(2)}w^{(1)}, w^{(1)}w^{(2)}w^{(1)}\}$. Its generators w_1 and w_2 act by

$$w_m^{(1)}[\vec{k}\,] = (\ldots, k_r-k_{r-1}, k_r), \quad w_m^{(2)}[\vec{k}\,] = (\ldots, k_{r-1}, k_{r-1}-k_r) \tag{4.16}$$

on magnetic charges $\vec{k}\in\gamma_m^+$, where … stands for the unaffected Abelian components $k_1, \ldots, k_\ell$. For the same reason as in the SU(2) case, these transformations map γ_m^+ to itself. Therefore, the Cartan theory I-function (4.4) remains unchanged upon replacing the product $I^W\cdot I^M$ with the average over its Weyl group orbit,

$$I^W\cdot I^M \mapsto \text{avg}_{w_m(\vec{k})}\left[I^W\cdot I^M\right] = \frac{1}{|\mathcal{W}_{\text{SU}(3)}|}\sum_{w\in\mathcal{W}_{\text{SU}(3)}} w_m(\vec{k})\left[I^W\cdot I^M\right], \tag{4.17}$$

where as before the Weyl group acts only on $\vec{k}$. This amounts to an appropriate relabelling of the summation set γ_m^+ for all those (five) terms that are non-trivially acted upon. For the same reason as in the SU(2) case, the product $I^W \cdot I^M$ is for every $v \in \mathcal{W}_{\mathrm{SU}(3)}$ invariant under application of $v_m(\vec{k}) \circ v_m(\vec{H})$. Since the Weyl group is Abelian, the same is true for $\mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right]$ and we find the first step in the sequence of equalities

$$\begin{aligned} \mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right] &= \mathrm{avg}_{v_m(\vec{H}) \circ v_m(\vec{k})} \circ \mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right] \\ &= \mathrm{avg}_{v_m(\vec{H})} \circ \mathrm{avg}_{w_m(\vec{k})}\left[I^W \cdot I^M\right] \\ &= \mathrm{avg}_{w_m(\vec{k})} \circ \mathrm{avg}_{v_m(\vec{H})}\left[I^W \cdot I^M\right] \\ &= \mathrm{avg}_{w_m(\vec{k}) \circ w_m(\vec{H})} \circ \mathrm{avg}_{v_m(\vec{H})}\left[I^W \cdot I^M\right] \\ &= \mathrm{avg}_{v_m(\vec{H})}\left[I^W \cdot I^M\right] . \end{aligned} \tag{4.18}$$

The second step uses that expressions which have already been averaged over the Weyl group action on $\vec{k}$ are invariant under $v_m(\vec{k})$ and the third that transformations on $\vec{k}$ and $\vec{H}$ commute. Steps four and five then go backwards with the roles of $\vec{k}$ and $\vec{H}$ exchanged. In summary, we have so far shown that the replacement

$$I^W \cdot I^M \mapsto \mathrm{avg}_{v_m(\vec{H})}\left[I^W \cdot I^M\right] \tag{4.19}$$

does not affect the Cartan theory I-function in eq. (4.4). This is an important intermediate result which we will now use to prove that I^{Cartan} remains finite in the limit $\vec{H}_{\mathrm{aux}} \to 0$.

To this end, let us multiply out the products in eq. (4.15) and for brevity reintroduce the symbols $\vec{\alpha}_{\beta_i}$ for the three positive roots. We get

$$I^W(\vec{H}_{\mathrm{aux}}, \vec{k}) = \eta + \eta \sum_{a=1}^{3} \sum_{1 \le \beta_1 < \ldots < \beta_a \le 3} \frac{(\vec{\alpha}_{\beta_1} \cdot \vec{k}) \cdots (\vec{\alpha}_{\beta_a} \cdot \vec{k})}{(\vec{\alpha}_{\beta_1} \cdot \vec{H}) \cdots (\vec{\alpha}_{\beta_a} \cdot \vec{H})} , \tag{4.20}$$

where we abbreviate $\eta = (-1)^{2k_{r-1}+2k_r}$. Expanding the matter contribution I^M in its variables H_r and H_{r-1}, the right hand side of eq. (4.20) will be multiplied by monomials of the form $H_{r-1}^p H_r^q$ times a coefficient that is H_{r-1}- and H_r-independent. In the limit of interest, $\vec{H}_{\mathrm{aux}} \to 0$, we only need to consider monomials with $p + q \le 3$. An explicit evaluation of the average over the Weyl group action on $\vec{H}$ — as on the right hand side of eq. (4.19) — for a fixed term in the above sum gives

$$\frac{1}{|\mathcal{W}_{\mathrm{SU}(3)}|} \sum_{v \in \mathcal{W}_{\mathrm{SU}(3)}} v_m(\vec{H}) \left[\frac{\sum_{p=0}^{a} \sum_{q=0}^{a-p} b_{p,q} H_{r-1}^p H_r^q}{(\vec{\alpha}_{\beta_1} \cdot \vec{H}) \cdots (\vec{\alpha}_{\beta_a} \cdot \vec{H})} \right] = \sum_{p=0}^{a} c_{\beta_1, \ldots, \beta_a}^{(p,a-p)} b_{p,a-p} . \tag{4.21}$$

This equally applies for all $1 \le \beta_1 < \ldots < \beta_a \le 3$ and $a = 1, 2, 3$. The $c_{\beta_1, \ldots, \beta_a}^{(p,a-p)}$ are numerical constants that are determined by the equation and $b_{p,q}$ abbreviates

$$b_{p,q} = \frac{1}{p!q!} \partial_{H_{r-1}}^p \partial_{H_r}^q I^M \Big|_{\vec{H}_{\mathrm{aux}} = (H_r, H_{r-1}) = 0} \tag{4.22}$$

Since on the right hand side of eq. (4.21) all dependence on the two auxiliary variables H_{r-1} and H_r has canceled, the limit $\vec{H}_{\mathrm{aux}} \to 0$ can now safely be taken. The equation intuitively states that there are no Weyl group invariant functions of $\vec{H}_{\mathrm{aux}}$ — the original variables $\vec{H}_{\mathrm{na}}$ are invariant anyway —

that are singular for $\vec{H}_{\text{aux}} = 0$ and at the same time consistent with the form of I^W. In the present example of $G_1 = \text{SU}(3)$ we find

$$\begin{aligned}
&c_1^{(1,0)} = \tfrac{1}{2}\,, && c_{1,3}^{(2,0)} = \tfrac{1}{2} c_{1,3}^{(1,1)} = c_{1,3}^{(0,2)} = \tfrac{1}{6}\,,\\
&c_2^{(1,0)} = c_2^{(0,1)} = \tfrac{1}{2}\,, && c_{2,3}^{(2,0)} = -c_{2,3}^{(1,1)} = -\tfrac{1}{2} c_{2,3}^{(0,2)} = -\tfrac{1}{6}\,,\\
&c_3^{(0,1)} = \tfrac{1}{2}\,, && c_{1,2,3}^{(1,2)} = c_{1,2,3}^{(2,1)} = \tfrac{1}{6}\,,\\
&c_{1,2}^{(2,0)} = +2c_{1,2}^{(1,1)} = -2c_{1,2}^{(0,2)} = \tfrac{1}{3}\,,
\end{aligned} \tag{4.23}$$

whereas all other $c_{\beta_1,\ldots,\beta_a}^{(p,a-p)}$ are zero. The purpose of these numbers is to define the differential operator $\mathcal{D}_{\mathfrak{su}(3)}$ by

$$\begin{aligned}
\eta^{-1}\mathcal{D}_{\mathfrak{su}(3)} &= +1 + \sum_{a=1}^{3} \sum_{1\le\beta_1<\ldots<\beta_a\le 3} \sum_{p=0}^{a} (\vec{\alpha}_{\beta_1}\cdot\vec{k})\cdots(\vec{\alpha}_{\beta_a}\cdot\vec{k})\, c_{\beta_1,\ldots,\beta_a}^{(p,a-p)} \frac{\partial_{H_{r-1}}^p \partial_{H_r}^q}{p!q!}\\
&= +1 + \tfrac{3}{2} k_{r-1}\,\partial_{H_{r-1}} + \tfrac{3}{2} k_r\,\partial_{H_r} + \tfrac{1}{4}\left(k_{r-1}^2 + 2k_{r-1}k_r - 2k_r^2\right)\partial_{H_{r-1}}^2\\
&\quad + \tfrac{1}{2}\left(-k_{r-1}^2 + 4k_{r-1}k_r - k_r^2\right)\partial_{H_{r-1}}\partial_{H_r} + \tfrac{1}{4}\left(-2k_{r-1}^2 + 2k_{r-1}k_r + k_r^2\right)\partial_{H_r}^2\\
&\quad - \tfrac{1}{12}\left(k_{r-1} - 2k_r\right)\left(2k_{r-1} - k_r\right)\left(k_{r-1} + k_r\right)\partial_{H_{r-1}}\partial_{H_r}\left(\partial_{H_{r-1}} + \partial_{H_r}\right)\,,
\end{aligned} \tag{4.24}$$

which involves derivatives with respect to the auxiliary variables H_{r-1} and H_r. The finite result of the limit in eq. (4.5) is obtained by replacing I^W with $\mathcal{D}_{\text{SU}(3)}$ that acts on the matter contribution I^M,

$$\Pi_0(\vec{Q}_{\text{na}}) = \sum_{\vec{k}\in\gamma_m^+} Q_1^{k_1}\cdots Q_\ell^{k_\ell}\,\mathcal{D}_{\mathfrak{su}(3)}\left[I^M(\vec{H},\vec{k})\right]\Big|_{\vec{H}=0}\,, \tag{4.25}$$

and we propose this as the fundamental periods for models with non-Abelian gauge groups of the type given in eq. (4.14). This result is independent of the quotient group Γ and valid as long as the restrictions on the non-Abelian matter spectrum explained in subsection 4.2.1 are fulfilled. The operator $\mathcal{D}_{\mathfrak{su}(3)}$ commutes with setting $\vec{H}_{\text{na}} = 0$, with all gamma functions in I^M associated to Abelian fields and with the collection of all $\vec{k}$-independent gamma functions when evaluated at $H_{r-1} = H_r = 0$. Since the formula would become rather lengthy, we refrain from explicitly executing the derivatives.

4.2.6 General Non-Abelian Gauge Groups

The methods of this proof and derivation immediately carry over to general gauge groups with simple non-Abelian factors $G_1, \ldots, G_m$ as in eq. (2.31).

Let $\mathcal{W}_G$ and n_α^G respectively denote the Weyl group of G and the number of positive roots in the Lie algebra $\mathfrak{g}$ of G. A sufficient condition for the existence of the limit $\vec{H}_{\text{aux}} \to 0$ in eq. (4.5) is that for all $1 \le \beta_1 < \ldots < \beta_a \le n_\alpha^G$ with $1 \le a \le n_\alpha^G$ there are numbers $c_{\beta_1,\ldots,\beta_a}^{(\vec{p})}$ for which the equation

$$\frac{1}{|\mathcal{W}_G|}\sum_{v\in\mathcal{W}_G} v_m(\vec{H})\left[\frac{\sum_{\vec{p}\in I_a} b_{\vec{p}}\, H_{\ell+1}^{p_1}\cdots H_r^{p_{r-\ell}}}{(\vec{\alpha}_{\beta_1}\cdot\vec{H})\cdots(\vec{\alpha}_{\beta_a}\cdot\vec{H})}\right] = \sum_{\vec{p}\in J_a} c_{\beta_1,\ldots,\beta_a}^{(\vec{p})} b_{\vec{p}} \tag{4.26}$$

holds. Here J_a is the set of all $(r-\ell)$-dimensional integer vectors $\vec{p} = (p_1, \ldots, p_{r-\ell})$ with non-negative components whose sum equals a, the set I_a is the union of all J_b with $0 \leq b \leq a$ and $b_{\vec{p}}$ are arbitrary placeholders that later are identified with

$$b_{\vec{p}} = \frac{1}{p_1! \cdots p_{r-\ell}!} \partial_{H_{\ell+1}}^{p_1} \cdots \partial_{H_r}^{p_{r-\ell}} I^M \big|_{\vec{H}_{\text{aux}} = (H_{\ell+1}, \ldots, H_r) = 0} \tag{4.27}$$

The Weyl group acts on $\vec{H}$ as if they were magnetic charges, which is the action dual to its action on weights and leaves $b_{\vec{p}}$ unaffected. We note that eq. (4.26) is a statement about the Lie algebra $\mathfrak{g}$ and therefore independent of a potential discrete quotient Γ. For a given gauge group G — or rather its Lie algebra $\mathfrak{g}$ — it can be checked by an explicit calculation, which in the same time also determines the $\mathfrak{g}$-dependent numbers $c^{(\vec{p})}_{\beta_1, \ldots, \beta_a}$. We conjecture that the equation holds for all gauge groups of the type given in eq. (2.31), independent of the precise non-Abelian factors $G_1, \ldots, G_m$ and their number m. It would be interesting to find a formal proof and to see whether the numbers $c^{(\vec{p})}_{\beta_1, \ldots, \beta_a}$ enjoy a more direct interpretation in terms of the structure of $\mathfrak{g}$ than the one given by their definition in eq. (4.26). In cases with multiple non-Abelian factors, the equation decomposes into the m corresponding equations for the individual factors G_1 to G_m. This can also be understood from the observation that — provided they exist — the limits of sending to zero the various subparts of $\vec{H}_{\text{aux}}$ associated to the individual factors commute. Therefore, it is enough to prove eq. (4.26) for a single but general non-Abelian factor.

Assuming the equation is found to hold, the numbers $c^{(\vec{p})}_{\beta_1, \ldots, \beta_a}$ determine a differential operator $\mathcal{D}_{\mathfrak{g}}$ that involves derivatives with respect to the auxiliary variables $H_{\ell+1}, \ldots, H_r$ by

$$\eta^{-1} \mathcal{D}_{\mathfrak{g}} = 1 + \sum_{a=1}^{n_\alpha^G} \sum_{1 \leq \beta_1 < \ldots < \beta_a \leq n_\alpha^G} \sum_{\vec{p} \in J_a} (\vec{\alpha}_{\beta_1} \cdot \vec{k}) \cdots (\vec{\alpha}_{\beta_a} \cdot \vec{k})\, c^{(\vec{p})}_{\beta_1, \ldots, \beta_a} \frac{\partial_{H_{\ell+1}}^{p_1} \cdots \partial_{H_r}^{p_{r-\ell}}}{p_1! \cdots p_{r-\ell}!}\,, \tag{4.28}$$

where we abbreviate $\eta = (-1)^{2k_{\ell+1} + \ldots + 2k_r}$. In case of multiple non-Abelian factors, the operator correspondingly factorizes into $\mathcal{D}_{\mathfrak{g}} = \mathcal{D}_{\mathfrak{g}_g} \cdots \mathcal{D}_{\mathfrak{g}_m}$ where $\mathfrak{g}_i$ is the Lie algebra of the factor G_i and $\mathcal{D}_{\mathfrak{g}_i}$ its associated operator. The finite result of the limit in eq. (4.5) reads

$$\Pi_0(\vec{Q}_{\text{na}}) = \sum_{\vec{k} \in \gamma_m^+} Q_1^{k_1} \cdots Q_\ell^{k_\ell}\, \mathcal{D}_{\mathfrak{g}} \left[I^M(\vec{H}, \vec{k}) \right] \Big|_{\vec{H} = 0}\,, \tag{4.29}$$

and we propose this as the fundamental period of general non-Abelian models that are consistent with the assumptions made in subsections 4.2.1. The above equation does not change if we let $\mathcal{D}_{\mathfrak{g}}$ only act on those gamma functions that are $\vec{k}$-dependent and associated to a non-Abelian field. Moreover, we can put $\vec{H}_{\text{na}} = 0$ before application of the operator.

4.2.7 Other Rank Two Non-Abelian Factors

To provide further evidence for our conjecture that eq. (4.26) is always fulfilled — which itself is a sufficient condition for the existence of the limit in eq. (4.5) — we briefly consider the remaining semi-simple Lie algebras of rank two, i.e., $\mathfrak{su}(2)^2 = \mathfrak{su}(2) \times \mathfrak{su}(2)$, $\mathfrak{sp}(4)$ and G_2. An explicit calculation confirms eq. (4.26) and the associated differential operators read

$$
\begin{aligned}
\eta^{-1}\,\mathcal{D}_{\mathfrak{su}(2)^2} =& + (1 + k_{r-1}\partial_{H_{r-1}})(1 + k_r\partial_{H_r}) \\
\eta^{-1}\,\mathcal{D}_{\mathfrak{sp}(4)} =& + 1 + 2k_{r-1}\partial_{H_{r-1}} - 3k_r\left(k_r - k_{r-1}\right)\partial^2_{H_{r-1}} - \tfrac{1}{6}k_{r-1}\left(k^2_{r-1} - 6k_r k_{r-1} + 6k_r^2\right)\partial^3_{H_{r-1}} \\
& + 2k_r\partial_{H_r} - \tfrac{3}{2}\left(k^2_{r-1} - 4k_r k_{r-1} + 2k_r^2\right)\partial_{H_{r-1}}\partial_{H_r} - \tfrac{1}{2}\left(k^3_{r-1} - 3k_r k^2_{r-1} + 2k_r^3\right)\partial^2_{H_{r-1}}\partial_{H_r} \\
& - \tfrac{1}{6}k_{r-1}k_r\left(k^2_{r-1} - 3k_r k_{r-1} + 2k_r^2\right)\partial^3_{H_{r-1}}\partial_{H_r} + \tfrac{3}{4}k_{r-1}\left(2k_r - k_{r-1}\right)\partial^2_{H_r} \\
& - \tfrac{1}{4}\left(2k_r - k_{r-1}\right)\left(-k^2_{r-1} - 2k_r k_{r-1} + 2k_r^2\right)\partial_{H_{r-1}}\partial^2_{H_r} \\
& - \tfrac{1}{4}k_{r-1}k_r\left(k_r - k_{r-1}\right)\left(2k_r - k_{r-1}\right)\partial^2_{H_{r-1}}\partial^2_{H_r} - \tfrac{1}{12}k_r\left(3k^2_{r-1} - 6k_r k_{r-1} + 2k_r^2\right)\partial^3_{H_r} \\
& - \tfrac{1}{12}k_{r-1}k_r\left(k^2_{r-1} - 3k_r k_{r-1} + 2k_r^2\right)\partial_{H_{r-1}}\partial^3_{H_r} \\
\eta^{-1}\,\mathcal{D}_{G_2} =& + 1 + 3k_{r-1}\partial_{H_{r-1}} - \tfrac{5}{4}\left(3k^2_{r-1} - 6k_r k_{r-1} + 2k_r^2\right)\partial^2_{H_{r-1}} \\
& - \tfrac{5}{3}k_{r-1}\left(2k^2_{r-1} - 3k_r k_{r-1} + k_r^2\right)\partial^3_{H_{r-1}} + \tfrac{5}{144}\left(-9k^4_{r-1} + 18k_r^2k^2_{r-1} - 12k_r^3k_{r-1} + 2k_r^4\right)\partial^4_{H_{r-1}} \\
& + \tfrac{1}{360}k_{r-1}\left(9k^4_{r-1} - 45k_r k^3_{r-1} + 60k_r^2k^2_{r-1} - 30k_r^3k_{r-1} + 5k_r^4\right)\partial^5_{H_{r-1}} + 3k_r\partial_{H_r} \\
& - \tfrac{15}{2}\left(3k^2_{r-1} - 4k_r k_{r-1} + k_r^2\right)\partial_{H_{r-1}}\partial_{H_r} + \tfrac{5}{3}\left(9k^3_{r-1} - 9k_r k^2_{r-1} + k_r^3\right)\partial^2_{H_{r-1}}\partial_{H_r} \\
& + \tfrac{5}{12}k_r\left(-12k^3_{r-1} + 18k_r k^2_{r-1} - 8k_r^2k_{r-1} + k_r^3\right)\partial^3_{H_{r-1}}\partial_{H_r} \\
& + \tfrac{1}{72}\left(27k^5_{r-1} - 90k_r k^4_{r-1} + 90k_r^2k^3_{r-1} - 30k_r^3k^2_{r-1} + k_r^5\right)\partial^4_{H_{r-1}}\partial_{H_r} \\
& + \tfrac{1}{720}k_{r-1}k_r\left(18k^4_{r-1} - 45k_r k^3_{r-1} + 40k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + 2k_r^4\right)\partial^5_{H_{r-1}}\partial_{H_r} \\
& - \tfrac{15}{4}\left(6k^2_{r-1} - 6k_r k_{r-1} + k_r^2\right)\partial^2_{H_r} - 5\left(3k^3_{r-1} - 3k_r^2k_{r-1} + k_r^3\right)\partial_{H_{r-1}}\partial^2_{H_r} \\
& + \tfrac{5}{8}\left(9k^4_{r-1} - 36k_r k^3_{r-1} + 36k_r^2k^2_{r-1} - 12k_r^3k_{r-1} + k_r^4\right)\partial^2_{H_{r-1}}\partial^2_{H_r} \\
& + \tfrac{1}{12}\left(18k^5_{r-1} - 45k_r k^4_{r-1} + 30k_r^2k^3_{r-1} - 5k_r^4k_{r-1} + k_r^5\right)\partial^3_{H_{r-1}}\partial^2_{H_r} \\
& + \tfrac{1}{96}k_{r-1}k_r\left(18k^4_{r-1} - 45k_r k^3_{r-1} + 40k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + 2k_r^4\right)\partial^4_{H_{r-1}}\partial^2_{H_r} \\
& - \tfrac{5}{3}k_r\left(9k^2_{r-1} - 9k_r k_{r-1} + 2k_r^2\right)\partial^3_{H_r} \\
& + \tfrac{5}{4}k_{r-1}\left(9k^3_{r-1} - 24k_r k^2_{r-1} + 18k_r^2k_{r-1} - 4k_r^3\right)\partial_{H_{r-1}}\partial^3_{H_r} \\
& + \tfrac{1}{12}\left(27k^5_{r-1} - 45k_r k^4_{r-1} + 30k_r^3k^2_{r-1} - 15k_r^4k_{r-1} + 2k_r^5\right)\partial^2_{H_{r-1}}\partial^3_{H_r} \\
& + \tfrac{1}{36}k_{r-1}k_r\left(18k^4_{r-1} - 45k_r k^3_{r-1} + 40k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + 2k_r^4\right)\partial^3_{H_{r-1}}\partial^3_{H_r} \\
& - \tfrac{5}{16}\left(-18k^4_{r-1} + 36k_r k^3_{r-1} - 18k_r^2k^2_{r-1} + k_r^4\right)\partial^4_{H_r} \\
& + \tfrac{1}{8}\left(9k^5_{r-1} - 30k_r^2k^3_{r-1} + 30k_r^3k^2_{r-1} - 10k_r^4k_{r-1} + k_r^5\right)\partial_{H_{r-1}}\partial^4_{H_r} \\
& + \tfrac{1}{32}k_{r-1}k_r\left(18k^4_{r-1} - 45k_r k^3_{r-1} + 40k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + 2k_r^4\right)\partial^2_{H_{r-1}}\partial^4_{H_r} \\
& + \tfrac{1}{40}k_r\left(45k^4_{r-1} - 90k_r k^3_{r-1} + 60k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + k_r^4\right)\partial^5_{H_r} \\
& + \tfrac{1}{80}k_{r-1}k_r\left(18k^4_{r-1} - 45k_r k^3_{r-1} + 40k_r^2k^2_{r-1} - 15k_r^3k_{r-1} + 2k_r^4\right)\partial_{H_{r-1}}\partial^5_{H_r}
\end{aligned} \tag{4.30}
$$

where $\eta = (-1)^{2k_{r-1}+2k_r}$. The complexity of $\mathcal{D}_{\mathfrak{g}}$ clearly scales more with the number of positive roots than with the rank of $\mathfrak{g}$.

4.3 Generalization to Other Quantum Periods

At this point it is natural to wonder whether the other quantum periods — i.e., the other solutions of the Picard–Fuchs operator that annihilates the fundamental period — can also be determined from the Cartan theory I-function. The answer to this question seems to be positive, although the result is model dependent and not what one might intuitively expect.

As we recall from subsection 2.3.1, for Abelian models quantum periods other than the fundamental period are obtained as appropriate derivatives of the Givental I-function with respect to the variables $\vec{H}$. Since in our present setup $\vec{H}_{\rm aux}$ are auxiliary variables of the Cartan theory that do not exist in the original non-Abelian model, an intuitive guess is that the quantity

$$I^{\rm lim}(\vec{H}_{\rm na}, \vec{Q}_{\rm na}) = \lim_{\vec{H}_{\rm aux}\to 0} I^{\rm Cartan}(\vec{H}, \vec{Q}_{\rm na}) \tag{4.31}$$

generalizes the notion of the Givental I-function to non-Abelian gauged linear sigma models. This limit exists for the very same reason as why the limit in eq. (4.5) exists — of course subject to the same assumptions — and is calculated from the right hand side of eq. (4.29) with the difference of not putting $\vec{H}_{\rm na} = 0$. The fundamental period is then clearly recovered as $\Pi_0(\vec{Q}_{\rm na}) = I^{\rm lim}(\vec{H}_{\rm na} = 0, \vec{Q}_{\rm na})$, which leads to the expectation that appropriate derivatives of $I^{\rm lim}$ with respect to the variables $\vec{H}_{\rm na}$ yield the other quantum periods of the Picard–Fuchs operator. As we observe in examples, this expectation is true for first derivatives — which give the single logarithmic quantum periods, see eq. (2.55), and in extension the flat coordinates (2.56) — but in general wrong for second or higher derivatives. These are not necessarily annihilated by the operator.

As the examples demonstrate, we also need to consider derivatives of the Cartan theory I-function with respect to the auxiliary variables $\vec{H}_{\rm aux}$ and only thereafter take the limit $\vec{H}_{\rm aux} \to 0$. We can find linear combinations of derivatives with respect to elements of both $\vec{H}_{\rm na}$ and $\vec{H}_{\rm aux}$ that, when evaluated at $\vec{H} = 0$, are indeed annihilated by the operator and therefore are quantum periods. In order to calculate derivatives with respect to the auxiliary variables $\vec{H}_{\rm aux}$, we need to generalize the right hand side of eq. (4.29) to include higher powers of $\vec{H}_{\rm aux}$. With the technologies developed by now, this is not difficult. The restriction to powers constant in $\vec{H}_{\rm aux}$ — or rather at most constant in $\vec{H}_{\rm aux}$, where the singular terms were found to cancel for several examples of gauge groups and conjectured to be absent in general — amounts to on the left hand side of eq. (4.26) only summing over monomials $H_{\ell+1}^{p_1} \cdots H_r^{p_{r-\ell}}$ of combined degree at most a. Assume we are interested in calculating $I^{\rm Cartan}$ to combined order n in the auxiliary variables $\vec{H}_{\rm aux}$. The appropriate generalization of eq. (4.26) then reads

$$\frac{1}{|\mathcal{W}_G|} \sum_{v\in \mathcal{W}_G} v_m(\vec{H}) \left[\frac{\sum_{\vec{p}\in I_{a+n}} b_{\vec{p}}\, H_{\ell+1}^{p_1} \cdots H_r^{p_{r-\ell}}}{(\vec{\alpha}_{\beta_1}\cdot \vec{H}) \cdots (\vec{\alpha}_{\beta_a}\cdot \vec{H})} \right] = \sum_{i=0}^{n} \sum_{j=1}^{m_i^{\rm inv}} \sum_{\vec{p}\in J_{a+i}} c^{(\vec{p})\,(i,j)}_{\beta_1,\dots,\beta_a} b_{\vec{p}}\, f^{\rm inv}_{i,j}(\vec{H}_{\rm aux}) \tag{4.32}$$

where on the left hand side $\vec{p}$ is summed over all $(r-\ell)$-dimensional integer vectors $\vec{p} = (p_1, \dots, p_{r-\ell})$ with non-negative components whose sum is a most $a+n$. The $f^{\rm inv}_{i,j}(\vec{H}_{\rm aux})$ are Weyl invariant polynomials — i.e., they invariant under application of $v_m(\vec{H})$ for all $v \in \mathcal{W}_G$ — that are of homogeneous degree i in the variables $\vec{H}_{\rm aux}$, and $m_i^{\rm inv}$ counts their number up to linear dependence. Lastly, the numbers $c^{(\vec{p})\,(i,j)}_{\beta_1,\dots,\beta_a}$ are defined by the equation and dependent on the Lie algebra $\mathfrak{g}$ (as well as the normalization of invariants). By setting $n = 0$ we recover eq. (4.26), in case of which there is the single invariant $f^{\text{inv}}_{0,1}(\vec{H}_{\text{aux}}) = 1$ and we have the identification $c^{(\vec{p})(0,1)}_{\beta_1,\dots,\beta_a} = c^{(\vec{p})}_{\beta_1,\dots,\beta_a}$. The generalization of the operator $\mathcal{D}_{\mathfrak{g}}$ defined in eq. (4.28) reads

$$\begin{aligned}\mathcal{D}^{(n)}_{\mathfrak{g}}(\vec{H}_{\text{aux}}) = \eta + \eta \sum_{a=1}^{n^G_\alpha} \sum_{1\le\beta_1<\dots<\beta_a\le n^G_\alpha} \sum_{\vec{p}\in J_{a+i}} (\vec{\alpha}_{\beta_1}\cdot\vec{k})\cdots(\vec{\alpha}_{\beta_a}\cdot\vec{k}) \\ \cdot \sum_{i=0}^{n}\sum_{j=1}^{m^{\text{inv}}_i} c^{(\vec{p})(i,j)}_{\beta_1,\dots,\beta_a}\, f^{\text{inv}}_{i,j}(\vec{H}_{\text{aux}})\, \frac{\partial^{p_1}_{H_{\ell+1}}\cdots\partial^{p_{r-\ell}}_{H_r}}{p_1!\cdots p_{r-\ell}!}\,,\end{aligned} \tag{4.33}$$

where $\eta = (-1)^{2k_{\ell+1}+\dots+2k_r}$. For $n = 0$ we recover the original operator $\mathcal{D}_{\mathfrak{g}}$, namely $\mathcal{D}^{(0)}_{\mathfrak{g}}(\vec{H}_{\text{aux}}) = \mathcal{D}_{\mathfrak{g}}$. The expansion of the Cartan theory I-function up to combined order n in its variables $\vec{H}_{\text{aux}}$ is then given by

$$I^{\text{Cartan}}(\vec{H},\vec{Q}_{\text{na}}) = \sum_{\vec{k}\in\gamma^+_m} Q_1^{k_1+\frac{H_1}{\epsilon}}\cdots Q_\ell^{k_\ell+\frac{H_\ell}{\epsilon}}\, \mathcal{D}^{(n)}_{\mathfrak{g}}(\vec{H}_{\text{aux}}) \left[I^M(\vec{H},\vec{k})\right]\Big|_{\vec{H}_{\text{aux}}=0} + O(\vec{H}^{n+1}_{\text{aux}})\,, \tag{4.34}$$

where after evaluating the derivatives specified by $\mathcal{D}^{(n)}_{\mathfrak{g}}(\vec{H}_{\text{aux}})$ we set $\vec{H}_{\text{aux}} = 0$ inside the gamma functions and its derivatives — but not inside the factors $f^{\text{inv}}_{i,j}(\vec{H}_{\text{aux}})$ within $\mathcal{D}^{(n)}_{\mathfrak{g}}(\vec{H}_{\text{aux}})$.

Since there are no Weyl group invariants $f^{\text{inv}}_{i,j}(\vec{H}_{\text{aux}})$ of homogenous order $i = 1$, eq. (4.34) does not involve terms linear in $H_{\ell+1},\dots,H_r$. This explains our earlier statement that the singly-logarithmic periods are given as partial derivatives of I^{Cartan} with respect to the variables $H_1,\dots,H_\ell$. In contrast, the precise linear combinations of second or higher derivatives that yield the other quantum periods are model dependent and not universal. For a given example they can be established by means of the Picard–Fuchs operator found from the fundamental period or — as we suspect — by classical intersection theory on the model's target space. If one is just interested in finding *the expansion of* quantum periods other than the fundamental and singly-logarithmic ones, it therefore makes more sense to simply determine them by solving the Picard–Fuchs differential equation. However, the combination of the Picard–Fuchs operator with the methods presented here allows us to find *closed form expressions* also for these periods.

To conclude this section, let us consider the non-Abelian factor $\mathfrak{g} = \mathfrak{su}(2)$ as an example. The associated operator reads

$$\mathcal{D}^{(n)}_{\mathfrak{su}(2)}(H_r) = (-1)^{2k_r} \sum_{m=0}^{\lfloor n/2\rfloor} H_r^{2m} \frac{\partial^{2m}_{H_r}}{(2m)!}\left(1 + \frac{k_r}{2m+1}\partial_{H_r}\right)\,, \tag{4.35}$$

where only even powers of H_r appear since the non-identity element of the Weyl group acts on $\vec{H}$ by sending $H_r \to -H_r$. In particular, there is no term linear in H_r.

4.4 Application to Concrete Non-Abelian Models

As the central result of section 4.2, we proposed that formula (4.29) gives the fundamental periods of gauged linear sigma models with non-Abelian gauge groups and large classes of chiral matter spectra.

We here test this proposal by applying the formula to several models for which the fundamental period and the annihilating differential operator are known from different considerations. In all cases we find agreement with the literature. For most of the examples we moreover concretize the considerations of the previous section by writing the other quantum periods as appropriate derivatives of the Cartan theory I-function. Let us further recall that formula (4.29) is equally applicable to non-conformal gauged linear sigma models, in case of which it gives the holomorphic solution to the differential operator that governs the target space quantum cohomology. We also consider one example of this type. For simplicity we set all twisted masses to zero and consider one-parameter cases only.

4.4.1 Gauge Group U(2)

We begin by considering three gauged linear sigma models with gauge group $G = \mathrm{U}(2)$. Since $\mathrm{U}(2) = \mathrm{U}(1) \times \mathrm{SU}(2)/\mathbb{Z}_2$ involves the single non-Abelian factor $G_1 = \mathrm{SU}(2)$, the relevant discussion is that of subsection 4.2.4.

Rødland Calabi–Yau Threefold

Our first example is one of the models studied in [55], which has gauge group $G = \mathrm{U}(2)$ and is specified by the chiral matter spectrum given in Table 4.1. This conformal model is conceptually

Matter multiplet	$G = \mathrm{U}(2)$ representation	Vector R-charge	T charges
$\Phi_i,\ i = 1, \ldots, 7$	$(1, 1)$	0	$(1, +1)$ $(1, -1)$
$P_j,\ j = 1, \ldots, 7$	$(-2, 0)$	2	$(-2, 0)$

Table 4.1: This table shows the chiral matter spectrum of a gauged linear sigma model studied in ref. [55]. Representations of $G = \mathrm{U}(2)$ are specified by a pair of integers as explained in the main text. The table moreover lists vector R-charges and the decomposition of the fields in the Cartan theory with gauge group $T = \mathrm{U}(1)^2/\mathbb{Z}_2$.

very interesting, since in both phases the target space is a compact Calabi–Yau threefold [55]. These manifolds were first constructed in [119] and the fact that they appear in the same quantum Kähler moduli space means they are derived-equivalent [120, 121]. We focus on the $r \gg 0$ phase, where as shown in [55] the target space is the intersection of seven generic hyperplanes X_{1^7} in the Grassmannian ambient space $\mathrm{Gr}(2, 7)$.

Representations of U(2) are specified by a pair of integers (ρ_q, ρ_{hw}), where ρ_q is the U(1) charge and ρ_{hw} the Dynkin label of the highest SU(2) weight. The $\Gamma = \mathbb{Z}_2$ quotient is such that $\rho_q + \rho_{hw} \in 2\mathbb{Z}$ is required, which is in accord with the spectrum and specifies the electric charge lattice. The magnetic charge lattice γ_m — the dual lattice — hence reads

$$\gamma_m = \left\{ (k_1, k_2) \in (\mathbb{Z}/2) \times (\mathbb{Z}/2) \,\middle|\, k_1 + k_2 \in \mathbb{Z} \right\} . \tag{4.36}$$

From its definition in subsection 4.2.1 and the matter spectrum in Table 4.1 the set of summation γ_m^+ is found to be

$$\gamma_m^+ = \left\{ (k_1, k_2) \in \gamma_m \,\middle|\, 0 \le k_1 , -k_1 \le k_2 \le k_1 \right\} , \tag{4.37}$$

which is indeed mapped to itself under the Weyl group action $k_2 \to -k_2$. All assumptions of subsection 4.2.1 are fulfilled, in particular we check that $\vec{\lambda}_{P_j} \cdot \vec{k} = -2k_1 \leq 0$ for all $\vec{k} \in \gamma_m^+$. This shows that formula (4.13) for the fundamental period is applicable and we find

$$\begin{aligned}\Pi_0(Q) &= \sum_{0 \leq k_1 \in \mathbb{Z}/2} \sum_{k_2=-k_1}^{k_1} Q^{k_1} \binom{2k_1}{k_1+k_2}^7 \left[1 + 7k_2\left(h_{k_1-k_2} - h_{k_1+k_2}\right)\right] \\ &= 1 - 5\sqrt{Q} + 109Q - 3\,317\sqrt{Q}^3 + 121\,501Q^2 - 4\,954\,505\sqrt{Q}^5 + \ldots ,\end{aligned} \tag{4.38}$$

where we have used that $(-1)^{2k_2-14k_1} = 1$ for all $(k_1, k_2) \in \gamma_m^+$. The appearance of half integral powers of Q is not a mistake and can be understood as follows: the periodicity of the theta angle θ results from the fact that if too much energy is stored in the Abelian background electric field, the vacuum energy can be reduced by electron-positron pair creation [16]. Due to the $\mathbb{Z}_2$ quotient in U(2), the electron and positron of a U(2) gauge theory respectively have Abelian charge ±2. This results in a 4π-periodic rather than 2π-periodic theta angle. The definitions in eqs. (2.33) and (2.53) then demonstrate that the variable $\tilde{Q} = \sqrt{Q}$ does not suffer from a branch cut.

The fundamental period is easily expanded to higher orders, from which we determine the annihilating Picard–Fuchs operator to be

$$\begin{aligned}\mathcal{L}(\tilde{Q}) &= (3+\tilde{Q})^2(1+57\tilde{Q}-289\tilde{Q}^2-\tilde{Q}^3)\tilde{\Theta}^4 + 4\tilde{Q}(3+\tilde{Q})(85-867\tilde{Q}-149\tilde{Q}^2-\tilde{Q}^3)\tilde{\Theta}^3 \\ &\quad + 2\tilde{Q}(408-7\,597\tilde{Q}-2\,353\tilde{Q}^2-239\tilde{Q}^3-3\tilde{Q}^4)\tilde{\Theta}^2 \\ &\quad + 2\tilde{Q}(153-4\,773\tilde{Q}-675\tilde{Q}^2-87\tilde{Q}^3-2\tilde{Q}^4)\tilde{\Theta} + \tilde{Q}(45-2\,166\tilde{Q}-12\tilde{Q}^2-26\tilde{Q}^3-\tilde{Q}^4)\end{aligned} \tag{4.39}$$

in terms of the new variable $\tilde{Q}$ and with the logarithmic derivative $\tilde{\Theta} = \tilde{Q}\partial_{\tilde{Q}}$. This operator as well as the expansion of the fundamental period are in agreement with the literature [119], where the analytic expression of the period interestingly takes a completely different form. This demonstrated that, at least for this example, eq. (4.13) is valid. We find that this method of determining the Picard–Fuchs operator is computationally far more efficient than the methods presented in chapter 3, see ref. [17] for a use of formula (3.41) in terms of gauge theory correlators.

Having determined the Picard–Fuchs operator from the fundamental period, the other quantum periods are straightforwardly found as its three other, logarithmic solutions. With the methods of section 4.3 these can alternatively be expressed as certain derivatives of the Cartan theory I-function. Equations (4.34) and (4.35) determine the latter as

$$\begin{aligned}I^{\mathrm{Cartan}}(\vec{H}, Q) &= \sum_{0 \leq k_1 \in \mathbb{Z}/2} \sum_{k_2=-k_1}^{k_1} \tilde{Q}^{2k_1+2\frac{H_1}{\epsilon}}(-1)^{2k_2}\left[1 + k_2\partial_{H_2} + \frac{H_2^2}{2}\partial_{H_2}^2\left(1+\frac{k_2}{3}\partial_{H_2}\right)\right] \\ &\quad \frac{\Gamma\left(1+2k_1+2\frac{H_1}{\epsilon}\right)^7 \Gamma\left(1+\frac{H_1+H_2}{\epsilon}\right)^7 \Gamma\left(1+\frac{H_1-H_2}{\epsilon}\right)^7}{\Gamma\left(1+2\frac{H_1}{\epsilon}\right)^7 \Gamma\left(1+k_1+k_2+\frac{H_1+H_2}{\epsilon}\right)^7 \Gamma\left(1+k_1-k_2+\frac{H_1-H_2}{\epsilon}\right)^7}\Big|_{H_2=0} + O(H_2^4)\end{aligned} \tag{4.40}$$

where after executing the derivatives specified by the term in square brackets we set $H_2 = 0$ inside the gamma functions and its derivatives — but we keep the H_2^2 factor inside the square bracket itself. The expressions is exact up to and including third powers of the auxiliary variable $\vec{H}_{\mathrm{aux}} = (H_2)$, such that we can safely differentiate three times with respect to H_2. With the Picard–Fuchs operator in eq. (4.39) we find that

$$\Pi_k(Q) = \left(\frac{\epsilon}{2}\,\partial_{H_1} \pm i\frac{\epsilon}{2\sqrt{3}}\,\partial_{H_2}\right)^k I^{\text{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0} \quad \text{with} \quad k = 0,\ldots,3 \tag{4.41}$$

are four linearly independent solutions. Since I^{Cartan} only involves even powers of H_2, terms with odd powers of ∂_{H_2} are zero automatically. This explains the sign ambiguity, shows that no square roots remain and that Π_1 does not receive contributions from derivatives with respect to H_2. For the higher logarithmic periods Π_2 and Π_3 those terms cannot be neglected. We suspect that the precise linear combination of ∂_{H_1} and ∂_{H_2} is linked to classical intersection theory on the target space.

The fundamental period of the second Calabi–Yau target space in the phase $r \ll 0$ is not captured by eq. (4.13). To see this, we swap the phases by reversing the sign of the U(1) generator. The field P_j has then positive gauge charge and non-zero vector R-charge, which is in conflict with the assumptions ofsubsection 4.2.1. We note that this phase leaves the non-Abelian gauge group factor SU(2) unbroken.

Skew Symplectic Sigma Model $\text{SSSM}_{1,12,6}$

Our second example is the skew symplectic sigma model $\text{SSSM}_{1,12,6}$ introduced in ref. [56], which has gauge group $G = \text{U}(2)$ and the chiral matter spectrum given in Table 4.2. As demonstrated

Matter multiplet	$G = \text{U}(2)$ representation	Vector R-charge	T charges
$\Phi_a, i = 1,\ldots,12$	$(+2,0)$	0	$(+2,0)$
$P_{[ij]}, 1 \le i < j \le 6$	$(-2,0)$	2	$(-2,0)$
$X_i, i = 1,\ldots,6$	$(+1,1)$	0	$(+1,+1)$ $(+1,-1)$
R	$(-3,1)$	2	$(-3,+1)$ $(-3,-1)$

Table 4.2: The non-Abelian chiral matter spectrum of the $G = \text{U}(2)$ gauged linear sigma model $\text{SSSM}_{1,12,6}$ studied in ref. [56] and its decomposition under the Cartan gauge group.

in ref. [56], this model also exhibits two geometric phases with distinct Calabi–Yau target spaces. Focusing on $r \gg 0$, the set of γ_m^+ is found to be the same as in eq. (4.37) — a consequence of the fact that both models have a field in the representation $(\rho_q, \rho_{hw}) = (1,1)$ — and all assumptions made in subsection 4.2.1 are fulfilled. Application of eq. (4.13) gives the fundamental period

$$\begin{aligned}\Pi_0(Q) = &\sum_{0\le k_1\in\mathbb{Z}/2}\ \sum_{k_2=-k_1}^{k_1} Q^{k_1}(-1)^{2k_2}\frac{(2k_1)!^3(3k_1+k_2)!(3k_1-k_2)!}{(k_1+k_2)!^6(k_1-k_2)!^6}\\ &\cdot\left[1 + k_2\left(h_{3k_1+k_2} - h_{3k_1-k_2} - 6h_{k_1+k_2} + 6h_{k_1-k_2}\right)\right]\\ = &\ 1 + 7\sqrt{Q} + 199Q + 8\,359\sqrt{Q}^3 + 423\,751Q^2 + 23\,973\,757\sqrt{Q}^5 + \ldots\,.\end{aligned} \tag{4.42}$$

This is in agreement with ref. [122], where the manifold realized as the target space of this gauged linear sigma model was first constructed, as well as ref. [56], where the same expression was found from a calculation of the two-sphere partition function. The expansion straightforwardly determines the fourth-order Picard–Fuchs operator, whereas the methods of chapter 3 are computationally quite involved for this example.

Given the operator, we check that four linearly independent solutions to the Picard–Fuchs differential equation are determined from the Cartan theory I-function as

$$\Pi_k(Q) = \left(\frac{\epsilon}{2}\partial_{H_1} \pm i\frac{\epsilon}{2}\sqrt{\frac{5}{11}}\,\partial_{H_2}\right)^k I^{\text{Cartan}}(\vec{H}, Q)\big|_{\vec{H}=0} \quad \text{with} \quad k = 0, \ldots, 3\,. \tag{4.43}$$

Not surprisingly, the precise linear combination of ∂_{H_1} and ∂_{H_2} whose powers generate the four solutions has changed in comparison to the previous example.

The $r \ll 0$ phase, which we can map to $r \gg 0$ by reversing the U(1) generator, leaves the non-Abelian factor SU(2) unbroken and is strongly coupled [56]. We find that the assumptions of subsection 4.2.1 are not fulfilled.

Grassmannian Gr(2,4)

Let us now demonstrate that eq. (4.13) is also valid for non-conformal models, in case of which it gives the holomorphic solution to the differential operator that governs the target space quantum cohomology. For this purpose, our third example is the $G = \text{U}(2)$ gauged linear sigma model with matter spectrum as specified by Table 4.3. Since the sum of U(1) charges is positive, the bare coupling

Matter multiplet	$G = \text{U}(2)$ representation	Vector R-charge	T charges
Φ_i, $i = 1, \ldots, 4$	$(\mathbf{1}, 1)$	0	$(1, +1)$ $(1, -1)$

Table 4.3: The chiral matter spectrum of a $G = \text{U}(2)$ gauged linear sigma model with $r \gg 0$ target space Gr(2, 4) and its decomposition under the Cartan gauge group.

is $r(\Lambda_{\text{UV}}) = \infty$ and we can consistently consider the $r \gg 0$ phase. Its target space is the complex Grassmannian Gr(2, 4) and all assumptions of subsection 4.2.1 are fulfilled. The set γ_m^+ is as in eq. (4.37) and the holomorphic solution (4.13) here reads

$$\begin{aligned}\Pi_0(Q) &= \sum_{0\leq k_1\in\mathbb{Z}/2}\ \sum_{k_2=-k_1}^{k_1} Q^{k_1}\,\epsilon^{-8k_1}\,(-1)^{2k_2}\,\frac{1+4k_2\left(h_{k_1-k_2}-h_{k_1+k_2}\right)}{(k_1+k_2)!^4(k_1-k_2)!^4} \\ &= 1 + 2\sqrt{Q}\epsilon^{-4} + \tfrac{3}{8}Q\epsilon^{-8} + \tfrac{5}{324}\sqrt{Q}^{\,3}\epsilon^{-12} + \ldots\,.\end{aligned} \tag{4.44}$$

In terms of the variable $\tilde{Q}$ and with the logarithmic derivative $\tilde{\Theta} = \tilde{Q}\partial_{\tilde{Q}}$ this expansion is annihilated by the differential operator

$$\mathcal{L} = (\epsilon\tilde{\Theta})^5 - 2\tilde{Q}\,(\epsilon\tilde{\Theta} + \epsilon)\,. \tag{4.45}$$

If we were given this operator without reference to the non-Abelian model and asked to determine its holomorphic solution, we would probably arrive at the simpler representation

$$\Pi_0(\tilde{Q}) = \sum_{n=0}^{\infty} \tilde{Q}^n \, \epsilon^{-4n} \frac{(2n)!}{n!^6} = 1 + 2\tilde{Q}\epsilon^{-4} + \tfrac{3}{8}\tilde{Q}^2\epsilon^{-8} + \tfrac{5}{324}\tilde{Q}^3\epsilon^{-12} + \ldots \,. \tag{4.46}$$

By inspection of formula (2.54) for the Givental I-function of Abelian theories and comparison with the model discussed in section 2.3, this is seen to be the expansion obtained for the model with target space $\mathbb{P}^5[2]$ — see subsection 4.5.1 for a more detailed explanation of such type of reasoning. We thus correctly recover the isomorphism $\mathrm{Gr}(2,4) \simeq \mathbb{P}^5[2]$ given by the Plücker embedding and confirm eq. (4.13) for the present example.

Moreover, also the discussion of section 4.3 is applicable to this non-conformal example. We here find that

$$\begin{aligned} \Pi_k(Q) &= \left(\frac{\epsilon}{2}\partial_{H_1} \pm i\frac{\epsilon}{2}\partial_{H_2}\right)^k I^{\mathrm{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0} \quad \text{with} \quad k = 0, \ldots, 3\,, \\ \Pi_4(Q) &= \left[\left(\frac{\epsilon}{2}\partial_{H_1} \pm i\frac{\epsilon}{2}\partial_{H_2}\right)^4 + \frac{\epsilon^4}{4}\partial_{H_2}^4\right] I^{\mathrm{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0} \end{aligned} \tag{4.47}$$

are five linearly independent solutions to differential equation defined by the operator (4.45). As opposed to the previous two examples, the full set of solutions is not generated by powers of a single linear combination of ∂_{H_1} and ∂_{H_2} applied to the Cartan theory I-function. This fails at the fourth derivative level. We note that, although this term is in principle allowed by the Weyl symmetry, the second derivative of I^{Cartan} with respect to H_2 is zero in the sum over k_2. The derivative ∂_{H_2} is thus only relevant for $k = 3$ and $k = 4$.

4.4.2 Gauge Group U(3)

Motivated by the of above examples, let us now increase the rank of the non-Abelian factor and move on to the gauge group $G = \mathrm{U}(3) = \mathrm{U}(1) \times \mathrm{SU}(3)/\mathbb{Z}_3$. The relevant discussion is that of subsection 4.2.5. As concrete example we consider the model specified by Table 4.4, which is similar

Matter multiplet	$G = \mathrm{U}(3)$ representation	Vector R-charge	T charges
Φ_i, $i = 1, \ldots, 6$	$(1,1,0)$	0	$(1,1,0)$ $(1,-1,1)$ $(1,0,-1)$
P_j, $j = 1, \ldots, 6$	$(-3,0,0)$	2	$(-3,0,0)$

Table 4.4: The chiral matter spectrum of a $G = \mathrm{U}(3)$ gauged linear sigma model with $r \gg 0$ target space $X_{1^6} \subset \mathrm{Gr}(3,6)$ together with its decomposition under the Cartan gauge group $T = \mathrm{U}(1)^3/\mathbb{Z}_3$. Representations of U(3) are specified as explained in the main text.

to the first example of the previous subsection and in analogy realizes the intersection X_{1^6} of six hyperplanes in the ambient Grassmannian $\mathrm{Gr}(3,6)$ as its $r \gg 0$ target space [55].

Representations of U(3) are specified by the triple $(\rho_q, \rho_{hw}^{(1)}, \rho_{hw}^{(2)})$, where ρ_q is the integer U(1) charge and $(\rho_{hw}^{(1)}, \rho_{hw}^{(2)})$ the pair of Dynkin labels (both integers) of the highest SU(3) weight of the representations. The $\Gamma = \mathbb{Z}_3$ gives the condition $\rho_q + 2\rho_{hw}^{(1)} + \rho_{hw}^{(2)} \in 3\mathbb{Z}$, which is in accord with the spectrum. As a result, the magnetic charge lattice γ_m and the set of summation γ_m^+ are

$$\begin{aligned} \gamma_m &= \left\{(k_1, k_2, k_3) \in (\mathbb{Z}/3)^3 \,\middle|\, k_1 + k_2 \in \mathbb{Z}, k_1 - k_3 \in \mathbb{Z}\right\} , \\ \gamma_m^+ &= \left\{(k_1, k_2,, k_3) \in \gamma_m \,\middle|\, 0 \leq k_1, -2k_1 \leq k_3 \leq k_1, -k_1 \leq k_2 \leq k_1 + k_3\right\} . \end{aligned} \tag{4.48}$$

This form of γ_m^+ applies as soon as there is a non-Abelian multiplet in the representation (1, 1, 0), which is the fundamental of SU(3) with U(1) charge one. The Weyl group action (4.16) is confirmed to map both γ_m and γ_m^+ to itself and we further check that all assumptions of subsection 4.2.1 are fulfilled. Formula (4.25) for the fundamental period here reads

$$\begin{aligned} \Pi_0(Q) = \sum_{\vec{k} \in \gamma_m^+} & Q^{k_1 + \frac{H_1}{\epsilon}} (3k_1)!^6 \, \mathcal{D}_{\mathfrak{su}(3)} \left[\Gamma\left(1 + k_1 + k_2 + \tfrac{H_2}{\epsilon}\right)\right. \\ & \left.\cdot \Gamma\left(1 + k_1 - k_2 + k_3 + \tfrac{-H_2+H_3}{\epsilon}\right) \Gamma\left(1 + k_1 - k_3 - \tfrac{H_3}{\epsilon}\right)\right]^{-6}_{\big| H_2 = H_3 = 0} , \end{aligned} \tag{4.49}$$

where the differential operator $\mathcal{D}_{\mathfrak{su}(3)}$ is given by eq. (4.24) and we have already put $H_1 = 0$. The expansion of this expression reads

$$\Pi_0(\tilde{Q}) = 1 + 6\,\tilde{Q} + 126\,\tilde{Q}^2 + 3\,948\,\tilde{Q}^3 + 149\,310\,\tilde{Q}^4 + 6\,300\,756\,\tilde{Q}^5 + \ldots \tag{4.50}$$

in terms of the variable $\tilde{Q} = \sqrt[3]{Q}$ that does not suffer from branch cuts. These fractional powers of Q appear due to the $\mathbb{Z}_3$ quotient in G, which results in a 6π-periodic theta angle. The fundamental period is easily expanded to higher orders, from which we find the annihilating Picard–Fuchs operator

$$\mathcal{L} = \tilde{\Theta}^4 - \tilde{Q}\,(6 + 40\tilde{\Theta} + 105\tilde{\Theta}^2 + 130\tilde{\Theta}^3 + \tilde{\Theta}^4) + 4\tilde{Q}^2\,(4\tilde{\Theta} + 5)(4\tilde{\Theta} + 3)(\tilde{\Theta} + 1)^2 . \tag{4.51}$$

Here $\tilde{\Theta} = \tilde{Q}\partial_{\tilde{Q}}$ is the logarithmic derivative with respect $\tilde{Q}$. This is in agreement with the result of ref. [123] and validates eq. (4.25) for this non-Abelian model.

With the methods of section 4.3 we can also determine the three other, logarithmic solutions of this Picard–Fuchs operator from the Cartan theory I-function. For this we use eqs. (4.34) and eq. (4.33) to expand I^{Cartan} up to combined order $n = 3$ in the auxiliary variables H_2 and H_3. An explicit calculation then demonstrates that

$$\Pi_k(Q) = \left[\frac{\epsilon}{3}\partial_{H_1} \pm i\frac{2\sqrt{2}\epsilon}{3\sqrt{7}}\left(\partial_{H_2} + \partial_{H_3}\right)\right]^k I^{\text{Cartan}}(\vec{H}, Q)_{\big|\vec{H}=0} \quad \text{with} \quad k = 0, \ldots, 3 \tag{4.52}$$

are four linearly independent solutions to the Picard–Fuchs differential equation. While there is the order $i = 2$ invariant $f_{2,1}^{\text{inv}}(\vec{H}_{\text{aux}}) = H_2^2 - H_2H_3 + H_3^2$, the corresponding second derivatives $\partial_{H_2}^2$, $\partial_{H_3}^2$ and $\partial_{H_2}\partial_{H_3}$ of the Cartan theory I-function vanish in the sum over k_2 and k_3. At the third derivative level the order two invariant contributes in combination with one derivative with respect to H_1, whereas the order $i = 3$ invariant $f_{3,1}^{\text{inv}}(\vec{H}_{\text{aux}}) = H_2H_3(H_2 - H_3)$ is annihilated by $(\partial_{H_2} + \partial_{H_3})^3$. This shows that no terms with square roots survive. There are other linear combinations of the three partial derivatives that yield the same result, we here chose the (up to scaling) unique one among those that is invariant under the exchange of ∂_{H_2} and ∂_{H_3}.

4.4.3 Gauge Group $\mathrm{U}(1) \times \mathrm{USp}(4)/\mathbb{Z}_2$

Our last example is the skew symplectic sigma model $\mathrm{SSSM}_{2,9,6}$ that was introduced and studied in ref. [56]. Its gauge group is $G = \mathrm{U}(1)\times\mathrm{USp}(4)/\mathbb{Z}_2$ and the chiral matter spectrum is listed in Table 4.5. The single non-Abelian factor $G_1 = \mathrm{USp}(4)$ is the compact Lie group $\mathrm{USp}(4) = \mathrm{U}(4) \cap \mathrm{Sp}(4,\mathbb{C})$ with Lie algebra $\mathfrak{sp}(4)$. Representations of G are specified by the triple $(\rho_q, \rho_{hw}^{(1)}, \rho_{hw}^{(2)})$, where ρ_q is the

Matter multiplet	$G = \mathrm{U}(2)$ representation	Vector R-charge	T charges
$\Phi_a, i = 1, \ldots, 9$	$(+2, 0, 0)$	0	$(+2, 0, 0)$
$P_{[ij]}, 1 \le i < j \le 6$	$(-2, 0, 0)$	2	$(-2, 0, 0)$
$X_i, i = 1, \ldots, 6$	$(1, 1, 0)$	0	$(1, +1, 0)$
			$(1, -1, +1)$
			$(1, +1, -1)$
			$(1, -1, 0)$
R	$(-3, 1, 0)$	2	$(-3, +1, 0)$
			$(-3, -1, +1)$
			$(-3, +1, -1)$
			$(-3, -1, 0)$

Table 4.5: The chiral matter spectrum of the $G = \mathrm{U}(1) \times \mathrm{USp}(4)/\mathbb{Z}_2$ gauged linear sigma model $\mathrm{SSSM}_{2,9,6}$ studied in ref. [56] and its decomposition under the Cartan gauge group $T = \mathrm{U}(1)^2/\mathbb{Z}_2$. Representations of G are specified as explained in the main text.

integer U(1) charge and $(\rho_{hw}^{(1)}, \rho_{hw}^{(2)})$ the pair of Dynkin labels (both integers) of the highest USp(4) weight of the representations. Due to the $\Gamma = \mathbb{Z}_2$ quotient there is the constraint $\rho_q + \rho_{hw}^{(1)} \in 2\mathbb{Z}$, which is in accord with the spectrum. As a result, the magnetic charge lattice γ_m and the set of summation γ_m^+ are found to be

$$\begin{aligned} \gamma_m &= \left\{ (k_1, k_2, k_3) \in (\mathbb{Z}/2)^2 \times \mathbb{Z} \,\middle|\, k_1 + k_2 \in \mathbb{Z} \right\} , \\ \gamma_m^+ &= \left\{ (k_1, k_2,, k_3) \in \gamma_m \,\middle|\, 0 \le k_1, -k_1 \le k_2 \le k_1, k_2 - k_1 \le k_3 \le k_2 + k_1 \right\} . \end{aligned} \tag{4.53}$$

All assumptions of subsection 4.2.1 are fulfilled, in particular we see that $\vec{k} \cdot \vec{\lambda}_b^- \le 0$ for all $\vec{k} \in \gamma_m^+$ and T-charge vectors $\vec{\lambda}_b^-$ of fields whose vector R-charge equals 2. The two generators of the Weyl group act on magnetic charges $\vec{k}$ as

$$w_1[\vec{k}\,] = (k_1, k_3 - k_2, k_3) , \quad w_2[\vec{k}\,] = (k_1, k_2, 2k_2 - k_3) , \tag{4.54}$$

which can be checked to map both γ_m and γ_m^+ to itself. The fundamental period is determined from eq. (4.29) with the operator $\mathcal{D}_{\mathrm{sp}(4)}$ given in eq. (4.30). Its expansion is found to be

$$\Pi_0(Q) = 1 - 11\sqrt{Q} + 559Q - 42\,923\sqrt{Q}^3 + 3\,996\,751Q^2 - 416\,148\,761\sqrt{Q}^5 + \ldots, \qquad (4.55)$$

which is in agreement with refs. [56, 122] and confirms eq. (4.29) for this non-Abelian model. We may also use the above expansion to efficiently determine the Picard–Fuchs operator of this model.

4.5 Reconstructing Gauged Linear Sigma Models from Differential Operators

At this point of the thesis we have discussed various ways in which we can associate differential operators to a given gauged linear sigma model. Having seen these tight connections, it is natural to wonder whether we can turn the arguments around and reconstruct a gauged linear sigma model from a given differential operator. From a physics point of view this would be particularly exciting for conformal models, since it would allow us to specify $N = (2, 2)$ two-dimensional superconformal field theories — relevant for example as internal worldsheet theories of type II string compactifications — in terms of appropriate differential operators. Moreover, there are efforts to classify [117] so-called Calabi–Yau operators [115, 116, 124] that are defined through some key properties shared by the Picard–Fuchs operators of compact Calabi–Yau threefolds with a single Kähler modulus. The currently known operators of this type also includes examples for which one is not aware of geometries that realize them as their Picard–Fuchs operators. If we were able to write down a gauged linear sigma model for these cases, we could likely close this gap by determining its target space. In addition, this would allow us to approach the question whether there necessarily needs to be an associated Calabi–Yau threefold geometry — or whether Calabi–Yau operators and hence the superconformal field theories of our interest are also consistent without a geometric large volume limit.

The below discussion is meant to advertise this idea of reconstructing a gauged linear sigma model from a given differential operator. We begin with a clarifying Abelian example and then make a high-level proposal of how such a program might be implemented in general, for which formula (4.29) for the fundamental periods of non-Abelian models plays a central role. Lastly, we consider a concrete non-Abelian model in order to highlight the difficulty arising. While we here focus on one-parameter conformal models with central charge $c = 9$ — having in mind the classification program mentioned above — the concepts are also applicable for different central charges and non-conformal models.

4.5.1 An Abelian Example

Let us further motivate and clarify these considerations with an example. We start with the Picard–Fuchs operator

$$\mathcal{L}(\tilde{Q}) = \tilde{\Theta}^4 - 12\tilde{Q}\,(3\tilde{\Theta}+1)(2\tilde{\Theta}+1)^2(3\tilde{\Theta}+2) \quad \text{with} \quad \tilde{\Theta} = \tilde{Q}\,\partial_{\tilde{Q}}\,, \qquad (4.56)$$

which is listed as number 5 in appendix A of ref. [117]. Our goal is to reconstruct a conformal gauged linear sigma model that realizes this as the Picard–Fuchs operator on the quantum Kähler moduli space of its low energy superconformal field theory. We approach this step by step.

First, we determine the fundamental period $\Pi_0(\tilde{Q})$ as the holomorphic solution to the Picard–Fuchs differential equation. With the power series ansatz $\Pi_0(\tilde{Q}) = \sum_{k=0}^{\infty} a_k \tilde{Q}^k$ the condition $\mathcal{L}(\tilde{Q})\,\Pi_0(\tilde{Q}) = 0$

Chiral multiplets	$G = \mathrm{U}(1)$ charge	Vector R-charge
Φ_i, $i = 1, \dots, 7$	$+1$	0
$P_i^{(1)}$, $i = 1, 2$	-2	2
$P^{(2)}$	-3	2

Table 4.6: Matter spectrum of the gauged linear sigma model corresponding to the Picard–Fuchs operator in eq. (4.56). Its $r \gg 0$ target space is the Calabi–Yau threefold $\mathbb{P}^6[2, 2, 3]$.

becomes equivalent to the recurrence relation

$$0 = (k+1)^4 a_{k+1} - 12(3k+1)(2k+1)^2(3k+2)a_k \quad \text{for} \quad k = 0, 1, \dots, \tag{4.57}$$

which we complement with the boundary condition $a_0 = 1$. The same ansatz works for all operators that are listed in ref. [117].

Second, we use eq. (2.55) according to which the fundamental period is obtained by setting $\vec{H} = 0$ in the Givental I-function. Under the assumption — or rather with the ansatz — of a $G = \mathrm{U}(1)$ gauge group, the latter is given by eq. (2.54) and we obtain the condition

$$\Pi_0(\tilde{Q}) \stackrel{!}{=} I(Q = \alpha\tilde{Q}, \epsilon) = \sum_{k=0}^{\infty} b_k \tilde{Q}^k = \sum_{k=0}^{\infty} (-1)^{k \cdot \sum_{i=1}^{N_1} \cdot \lambda_i} \frac{\prod_{i=1}^{N_1} \Gamma(1 - k \cdot \lambda_i)}{\prod_{i=1}^{N_2} \Gamma(1 + k \cdot \rho_i)} \alpha^k \tilde{Q}^k \, . \tag{4.58}$$

Here we assumed a chiral matter spectrum of N_1 multiplets with vector R-charge $\mathfrak{q}_i = 2$ and negative gauge charges $\lambda_i < 0$, as well as N_2 multiplets with $\mathfrak{q}_i = 0$ and positive gauge charges $\rho_i > 0$. Moreover, we assume that $0 = \sum_{i=1}^{N_1} \lambda_i + \sum_{i=1}^{N_2} \rho_i$ in order to cancel the axial anomaly and set all twisted masses to zero. Lastly, we have included a constant multiple α to match the variable Q defined by the gauge theory with the variable $\tilde{Q}$ in terms of which the Picard–Fuchs operator (4.56) is expressed.

Third, we set out to determine a recurrence relation for the coefficients b_k in the above equation and then choose the matter spectrum to obtain agreement with the recurrence (4.57) specified by the operator. Given that $a_0 = b_0 = 1$ holds per construction, the equality $a_k = b_k$ is then guaranteed for all k and we will have found a gauged linear sigma model realization. From the right hand side of eq. (4.58) we find

$$b_{k+1} \cdot \prod_{i=1}^{N_2} \prod_{s=0}^{\rho_i - 1} (1 + k \cdot \rho_i + s) - b_k \cdot \alpha \cdot (-1)^{\sum_{i=1}^{N_1} \lambda_i} \prod_{i=1}^{N_1} \prod_{s=0}^{-\lambda_i - 1} (1 - k \cdot \lambda_i + s) = 0 \tag{4.59}$$

with $k \in \mathbb{Z}_{\geq 0}$. In comparing this with eq. (4.57) we begin with those irreducible factors $(c \cdot k + d)$ that have maximal c. These are the terms $(3k+2)$ and $(3k+1)$, and since they multiply a_k we include a single field with $\lambda_i = -3$ and $\mathfrak{q}_i = 2$. We then proceed with those factors of next to maximal c and so on. The next term is $(2k+1)^2$, and since it multiplies a_k we add two fields with $\lambda_i = -2$ and $\mathfrak{q}_i = 2$. When inserting this in the products multiplying b_k in eq. (4.59), these three fields additionally give rise to the factor $(3k+3)(2k+2)^2 = 12(k+1)^3$ such that we need to multiply eq. (4.57) with $(k+1)^3$ to match this. Hence, there effectively is a factor $(k+1)^7$ in front of a_{k+1} and we include seven fields with $\rho_i = 1$ and $\mathfrak{q}_i = 0$. The recurrence relations fully match for $\alpha = -1$ and we summarize the spectrum of the deduced gauged linear sigma model in Table 4.6.

The condition for anomaly free axial R-invariance in eq. (2.37) is fulfilled and eq. (2.38) gives $c = 9$ as the central charge of the low energy superconformal field theory. From eq. (2.44) the $r \gg 0$ target space is seen to be the Calabi–Yau threefold $\mathbb{P}^6[2, 2, 3]$, i.e., the intersection of one generic cubic and two generic quadratics in projective space $\mathbb{P}^6$. As an ultimate check on our calculation, we can write down the rational functions g_p that are defined in eq. (3.12) and solve for polynomial solutions α_p of eq. (3.11). From this we indeed recover the operator we started with up to an inconsequential overall power of ϵ^4.

4.5.2 Abstraction to a General Approach

Having discussed this example, let us package the central steps into a general procedure. As the main complication, the gauged linear sigma model description of a given operator is — assuming it exists in the first place — not guaranteed to have gauge group U(1). We rather need to iterate over different choices of gauge groups, for which we propose the following high-level algorithm:

1. Take a Calabi–Yau operator from ref. [117] and derive the recurrence relation for the coefficients of its fundamental period.
2. Pick a gauge group G and use the I-function to write down the fundamental period for a general chiral matter spectrum.
3. Derive a recurrence relation for this general expression and match it with the result of item 1 by an appropriate choice of spectrum.
4. If no match can be found, go back to item 2 and pick a different gauge group. If there is a match, we have successfully found a gauged linear sigma model realization. We may still decide to go back to item 2 in order to search for a different realization.

It is perfectly conceivable that by this procedure we obtain different gauged linear sigma models for the same Calabi–Yau operator. This is not a flaw but rather a feature, since such a situation would be strong evidence for a duality between the different models. The first of these steps follows directly from making a power series ansatz, and for the second we can rely on formula (2.54) for the Givental I-function of Abelian gauged linear sigma models as well as our proposal (4.29) for the fundamental period of models with non-Abelian gauge groups. As we will see in the next subsection, the third step appears to be highly non-trivial.

We also note that, in a slight modification of the above algorithm, we may attempt to directly solve the recurrence relation of the given operator to obtain a closed form expression for the coefficients of its holomorphic solution. This is essentially equivalent and it is not obvious whether one of the approaches has a clear advantage over the other.

4.5.3 A Non-Abelian Example

To demonstrate the difficulty of the third step in the above algorithm, we here reconsider the first example discussed in subsection 4.4.1. For this model we know both the Picard–Fuchs operator *and* the analytic expression

$$\Pi_0(Q) = \sum_{n=0}^{\infty} b_n \tilde{Q}^n \quad \text{with} \quad b_n = \sum_{k_2=-n/2}^{n/2} \binom{n}{\frac{n}{2}+k_2}^7 \left[1 + 7k_2 \left(h_{n/2-k_2} - h_{n/2+k_2}\right)\right] \tag{4.60}$$

of its fundamental period. This is obtained from eq. (4.38) by using $\tilde{Q} = \sqrt{Q}$ and $2n = k_1$. The concrete form of the operator, given eq. (4.39), implies the recurrence relation

$$\begin{aligned} 0 = &+ 9n^4\, b_n + 3(18 - 114n + 290n^2 - 352n^3 + 173n^4)\, b_{n-1} \\ &- 2(-267 - 1\,359n + 4\,501n^2 - 4\,000n^3 + 1\,129n^4)\, b_{n-2} \\ &- 2(16\,485 - 33\,531n + 24\,223n^2 - 7\,488n^3 + 843n^4)\, b_{n-3} \\ &+ (-43\,586 + 49\,986n - 21\,502n^2 + 4\,112n^3 - 295n^4)\, b_{n-4} - (-4 + n)^4\, b_{n-5} \end{aligned} \tag{4.61}$$

where $n \in \mathbb{Z}_{\geq 0}$ and with the understanding that $b_q = 0$ for $q < 0$. For arbitrary but fixed values of n this equation can be checked to be in agreement with values of b_n obtained from the above closed form expression. The author is, however, not aware of an analytic proof of this, which would require to keep n arbitrary.

We stress that this problem is still a lot easier than what we actually need to do. Namely, to derive a recurrence for the fundamental period (4.13) with a yet undetermined matter spectrum. Even this would only cover models with gauge groups $G = \mathrm{U}(1) \times \mathrm{SU}(2)/\,\Gamma$ and the generalization to higher rank non-Abelian groups with their fundamental periods given by eq. (4.29) would only be more complicated. The alternative approach to obtain the closed form expression for the periods by directly solving recurrence relations such as eq. (4.61) does not appear to be a trivial problem either. A practical implementation of the proposed reconstruction program therefore remains for future work. Since eq. (4.29) allows us to efficiently determine the Picard–Fuchs operators associated to concrete models, it seems worthwhile to alternatively use it for an automated scan over various gauge groups and matter spectra with the aim of finding models with geometric Calabi–Yau target spaces.

CHAPTER 5

Non-Minimal Period Geometry of Calabi–Yau Fourfolds

In this chapter we apply the concepts introduced in the earlier parts of the thesis to study an interesting property of one-parameter Calabi–Yau fourfolds that arise as target spaces of non-Abelian gauged linear sigma models. As we will explain, the order of their generating Picard–Fuchs operator is not guaranteed to be five but may be higher in general.

This chapter is based on the author's publication [19].

5.1 Introduction

From the physics point of view we have so far mostly focused on Calabi–Yau threefolds, which arise in type II string compactifications to $\mathcal{N} = 2$ space-time supersymmetric four-dimensional theories. Both the gauged linear sigma model and Picard–Fuchs operators have in chapter 2 been motivated as tools for studying the internal worldsheet theories of such compactifications. In the later chapters we then demonstrated that both concepts are applicable in a wider context. Namely, the gauged linear sigma model can be equally used to study compactifications to — and Calabi–Yau manifolds of — different dimensions and it even allows us to analyze the quantum cohomology of non Ricci-flat target spaces. Picard–Fuchs operators are also not limited to complex three-dimensional Calabi–Yau spaces, and there is a similar notion in the non Ricci-flat case. In this chapter we apply the concepts introduced in the earlier parts of this thesis to the study of Calabi–Yau fourfolds. The compactification of type II superstring theories on these geometries results in two-dimensional effective theories with $\mathcal{N} = (2, 2)$ space-time supersymmetry [125–127], which is the same number of supercharges as in minimal supersymmetry in four dimensions and therefore of particular interest. In case they exhibit an elliptic fibration, Calabi–Yau fourfolds can moreover be used for the compactification of F-theory to $\mathcal{N} = 1$ space-time supersymmetric theories in four space-time dimensions [128–131].

As we will explain, Calabi–Yau fourfolds that arise as target spaces of non-Abelian gauged linear sigma models may exhibit the interesting feature that their quantum cohomology ring is not fully generated by products of the marginal Kähler, i.e., chiral–anti-chiral deformations of the two-dimensional worldsheet theory alone. Rather, certain irrelevant chiral–anti-chiral operators need to be additionally included to obtain a set of generators. This is unlike the case of Calabi–Yau threefolds, where due to $\mathcal{N} = 2$ special geometry [88, 132, 133] the number of generators of the quantum cohomology ring is essentially determined by the dimension of the Kähler moduli space. The necessity to include these irrelevant operators results in the existence of additional quantum periods, which describe even-dimensional cycles of the Calabi–Yau fourfold whose quantum volume is non-zero although their classical Kähler volume vanishes. Our focus is on geometries with a single Kähler modulus, where this phenomenon leads to a non-factorizable generating Picard–Fuchs operator of order six or higher. The large volume boundary component of the Kähler moduli space does then not have maximally unipotent monodromy with respect to the Picard–Fuchs operator — see for instance ref. [81] for a mathematical review of this notion — and there are additional solutions to the Picard–Fuchs differential equation that vanish in this limit. These complicate the determination of integral quantum periods, because the integration constants are not entirely determined by the perturbative asymptotic behavior as for instance computed by the Gamma class of the Calabi–Yau fourfold [108, 134–138]. We will demonstrate that the knowledge of the monodromy at an additional regular singular point in combination with numerical analytic continuation techniques is sufficient to fix the integral quantum periods unambiguously. In addition, there is more than one tower of genus zero worldsheet instanton corrections to the operator product of marginal Kähler deformations. Finding their numbers (after employing a suitable multi-covering formula) to be integral, provides a strong consistency check on the integral periods. Lastly, the additional quantum periods also have an interesting phenomenological implication: in Calabi–Yau fourfold compactifications of the type IIA superstring to two dimensions they allow for flux-induced superpotentials that are entirely instanton generated — plus, if one chooses to put them, a constant or a term linear in the flat coordinate. If the mirror Calabi–Yau fourfold has a suitable elliptic fibration, these superpotentials can in the context of F-theory be reinterpreted in four space-time dimensions.

In the next section we give an intuitive explanation for why and when the phenomenon of additional quantum cohomology elements can appear. We then demonstrate this effect and its implications explicitly in a concrete example.

5.2 On the Order of the Picard–Fuchs Operator

We here explain the central observation that the generating Picard–Fuchs operator of Calabi–Yau fourfolds with a single Kähler modulus can be of order six or higher.

5.2.1 Calabi–Yau Threefolds

To build up some intuition, let us first discuss the simpler case Calabi–Yau threefolds X with a single Kähler modulus. As we will see, and in fact have already stated at several points above, their generating Picard–Fuchs operator is always of order four. This is most easily explained on the mirror side, where we consider the holomorphic $(3,0)$-form $\Omega = \Omega(\xi)$ on the mirror manifold Y of X in dependence on the single complex structure modulus ξ of Y. Equations (2.69) and (2.68) then state that

$$
\begin{aligned}
\Omega \in \mathcal{F}^3 &= H^{3,0}\,, & & \\
\partial\,\Omega \in \mathcal{F}^2 &= H^{3,0} \oplus H^{2,1} & \text{and}\quad \partial\,\Omega &\notin \mathcal{F}^3\,,\\
\partial^2\,\Omega \in \mathcal{F}^1 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} & \text{and}\quad \partial^2\,\Omega &\notin \mathcal{F}^2\,,\\
\partial^3\,\Omega \in \mathcal{F}^0 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} & \text{and}\quad \partial^3\,\Omega &\notin \mathcal{F}^1\,,
\end{aligned}
\tag{5.1}
$$

where ∂ denotes the derivative with respect to the modulus ξ and all cohomology groups are those of Y. This shows that Ω and its first three derivatives live in different vector spaces, and hence they are linearly independent. Recalling that $h^{3,0} = h^{0,3} = 1$ holds for all Calabi–Yau threefolds and that $h^{2,1} = h^{1,2} = 1$ by the assumption of a single modulus, these four elements are seen to form a basis of the horizontal cohomology $H_{\text{hor}}(Y) = \mathcal{F}^0$. The forth derivative $\partial^4\,\Omega$ — which as all derivatives of Ω is an element of $\mathcal{F}^0$ — therefore necessarily is a linear combination of Ω and its lower derivatives. This immediately translates into a fourth order differential operator that annihilates Ω. By mirror symmetry this is the generating Picard–Fuchs operator on the quantum Kähler moduli space of X and we have demonstrated our claim.

5.2.2 Calabi–Yau Fourfolds

We now turn to Calabi–Yau fourfolds X with a single Kähler modulus as the case of our actual interest. Again employing the mirror interpretation, eqs. (2.69) and (2.68) now give

$$
\begin{aligned}
\Omega \in \mathcal{F}^4 &= H^{4,0}\,, && \\
\partial\,\Omega \in \mathcal{F}^3 &= H^{4,0} \oplus H^{3,1} && \text{and} \quad \partial\,\Omega \notin \mathcal{F}^4 \\
\partial^2\,\Omega \in \mathcal{F}^2 &= H^{4,0} \oplus H^{3,1} \oplus H^{2,2} && \text{and} \quad \partial^2\,\Omega \notin \mathcal{F}^3 \\
\partial^3\,\Omega \in \mathcal{F}^1 &= H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} && \text{and} \quad \partial^3\,\Omega \notin \mathcal{F}^2 \\
\partial^4\,\Omega \in \mathcal{F}^0 &= H^{4,0} \oplus H^{3,1} \oplus H^{2,2} \oplus H^{1,3} \oplus H^{0,4} && \text{and} \quad \partial^4\,\Omega \notin \mathcal{F}^1\,,
\end{aligned}
\tag{5.2}
$$

where the cohomology groups are those of Y (the mirror of X). Some of the observations made in the previous subsection generalize. First, since they live in different vector spaces, Ω and its four lowest derivatives are linearly independent. Hence, there cannot be a Picard–Fuchs operator of order four or lower. Second, Calabi–Yau geometry gives $h^{4,0} = h^{0,4} = 1$ and we have $h^{3,1} = h^{1,3}$ by assumption of a single modulus. As the central complication, the dimension $h^{2,2}$ of the middle cohomology group can be bigger than one. The above five elements do in general not form a basis of the horizontal cohomology, which means they do not necessarily express the fifth derivative $\partial^5\,\Omega$ as a linear combination. We can thus *not* conclude that the generating Picard–Fuchs operator is always of order five. As an example, it may happen that the third derivative generates an element of $H^{2,2}$ that has not previously been generated by the second derivative. Intuitively, this additional element requires two more derivatives to reach $H^{0,4}$ and yet another one to become linearly dependent. The Picard–Fuchs operator would then be of order six. In general, the order of the operator equals four plus the number of linearly independent elements of $H^{2,2}$ that are generated by the derivatives.

To concretize this, let us return to the original Calabi–Yau fourfold X. A standard technique for studying its quantum cohomology ring $QH_{\text{vert}}(X)$ uses a quantum version of the Lefschetz hyperplane theorem [83, 139, 140], which infers information about $QH_{\text{vert}}(X)$ from the quantum cohomology of some ambient space. For Calabi–Yau fourfolds embedded in toric ambient spaces X_Σ of complete fans Σ, one therefore typically studies the quantum cohomology ring of those elements that are induced via pullback from the cohomology ring $H_{\text{vert}}(X_\Sigma)$ of X_Σ. The Jurkiewicz–Danilov theorem for complete compact toric varieties X_Σ guarantees the entire cohomology ring of X_Σ to be generated by the elements in $H^{1,1}(X_\Sigma)$. As a result, the part of the quantum cohomology ring $QH_{\text{vert}}(X)$ of X that is induced from the embedding of X into X_Σ is also generated by the elements in $H^{1,1}(X)$. In other words, for compact smooth Calabi–Yau fourfolds embedded as complete intersections in toric varieties the part of the quantum cohomology that is induced from the toric ambient space is always generated by marginal operators in the chiral–anti-chiral ring of the associated superconformal field theory. Hence, their Picard–Fuchs operator is of order five. Such geometries arise as target spaces of Abelian gauged linear sigma models.

The situation is different for Calabi–Yau fourfolds that are embedded in ambient spaces Z whose vertical cohomology ring is not just generated by $H^{1,1}(Z)$. This for instance happens for non-toric GIT quotients, examples of which are complex Grassmannians. To be concrete, for $Z = \mathrm{Gr}(k, n)$ with $k > 2$ the middle cohomology group $H^{2,2}(Z)$ is two-dimensional and therefore not fully generated by products of the single marginal Kähler deformation. If both these elements participate in the quantum product, the Picard–Fuchs operator will be of order six. Similarly, if $H^{2,2}(Z)$ is generated by more than two elements and if these participate in the quantum product, the Picard–Fuchs operator of X will be of order even higher than six. We say that generating Picard–Fuchs operators with order higher than five have *non-minimal order*. From a physics point of view such geometries arise as target spaces of non-Abelian gauged linear sigma models [16, 55, 56, 58–64, 141] and refs. [56, 118] have indeed observed examples in which the generating operators are of non-minimal order six.

5.3 Discussion of an Example

Having obtained an understanding of why and for which geometries the Picard–Fuchs operator can be of non-minimal order, we here demonstrate this phenomenon and its implications in a concrete example [19, 118].

5.3.1 Gauged Linear Sigma Model Realization

The example arises from the gauged linear sigma model with non-Abelian gauge group $G = \mathrm{U}(2)$ and matter spectrum as listed in Table 5.1. This model is quite similar to the first example discussed in subsection 4.2.4. The condition for non-anomalous axial R-invariance in eq. (2.37) is fulfilled and

Matter multiplet	$G = \mathrm{U}(2)$ representation	Vector R-charge	T charges
Φ_i, $i = 1, \ldots, 5$	$(1, 1)$	0	$(1, +1)$ $(1, -1)$
$P^{(1)}$	$(-2, 0)$	2	$(-2, 0)$
$P^{(2)}$	$(-8, 0)$	2	$(-8, 0)$

Table 5.1: The chiral matter spectrum of $G = \mathrm{U}(2)$ gauged linear sigma model with $r \gg 0$ target space $X_{1,4} \subset \mathrm{Gr}(2, 5)$ together with its decomposition under the Cartan gauge group $T = \mathrm{U}(1)^2/\mathbb{Z}_2$. For an explanation of the notation we refer to subsection 4.2.4

with eq. (2.38) the central charge of the low energy superconformal field theory is seen to be $c = 12$. This is required for a Calabi–Yau fourfold target space. Concretely, for $r \gg 0$ the target space is $X_{1,4} \subset \mathrm{Gr}(2, 5)$, i.e., the intersection of one generic hyperplane with one generic quartic in the ambient Grassmannian $\mathrm{Gr}(2, 5)$ [55, 118].

5.3.2 Picard–Fuchs Operator

We begin by finding the generating Picard–Fuchs operator on the quantum Kähler moduli space of this model. We use two of the methods introduced in the earlier parts of this thesis.

Picard–Fuchs Operator from Fundamental Period

Our first choice is the method of chapter 4, which is based on explicit formulas for the Givental I-function of non-Abelian gauged linear sigma models. The present model is in accord with all assumptions of subsection 4.2.1 and from eq. (4.11) we find its fundamental period as

$$\begin{aligned}\Pi_0(Q) &= \sum_{0\leq k_1\in\mathbb{Z}/2}\sum_{k_2=-k_1}^{k_1} Q^{k_1}\frac{(2k_1)!(8k_1)!\left[1+5k_2\left(h_{k_1-k_2}-h_{k_1+k_2}\right)\right]}{(k_1+k_2)!^5(k_1-k_2)!^5}\\ &= 1-72\sqrt{Q}+47\,880Q-54\,331\,200\sqrt{Q}^3+78\,891\,813\,000Q^2+\ldots\,.\end{aligned} \tag{5.3}$$

The expansion can easily be extended to higher order, from which we in accord with ref. [118] determine the Picard–Fuchs operator

$$\begin{aligned}\mathcal{L}(\tilde{Q},\tilde{\Theta}) =& (\tilde{\Theta}-1)\tilde{\Theta}^5+8\tilde{Q}\,\tilde{\Theta}(2\tilde{\Theta}+1)(4\tilde{\Theta}+1)(4\tilde{\Theta}+3)(11\tilde{\Theta}^2+11\tilde{\Theta}+3)\\ &-64\tilde{Q}^2(2\tilde{\Theta}+1)(2\tilde{\Theta}+3)(4\tilde{\Theta}+1)(4\tilde{\Theta}+3)(4\tilde{\Theta}+5)(4\tilde{\Theta}+7)\,.\end{aligned} \tag{5.4}$$

Here we introduced the variable $\tilde{Q}=\sqrt{Q}$ — which does not suffer from a branch, see subsection 4.4.1 for an explanation — and $\tilde{\Theta}=\tilde{Q}\partial_{\tilde{Q}}$ is the logarithmic derivative. This operator is indeed of order six and therefore of non-minimal order.

We note that this approach requires us to make an ansatz for the operator and therein specify the highest power with which the variable $\tilde{Q}$ appears. One might thus object that we could simply have missed an order five operator with powers of $\tilde{Q}$ higher than allowed by the ansatz. If this was true, the above operator would factor into an order one and order five operator, the second of which would need to involve high powers of $\tilde{Q}$. Since we included powers up to fifty and since the above order six operator involves $\tilde{Q}$ at most quadratically, we are confident that this is not the case. Nevertheless, we see this as additional motivation for finding technologies to derive recurrence relations for the closed form expressions of periods that arise from non-Abelian models, see also the discussions in subsections 4.5.2 and 4.5.3.

Picard–Fuchs Operator from Gauged Linear Sigma Model Correlators

An alternative approach is the generalization of the methods of section 3.6 to one-parameter Calabi–Yau fourfolds. We here briefly summarize the key facts and refer to ref. [17] for details. Among the cases discussed in section 3.6, one-parameter Calabi–Yau fourfolds are in terms of gauge theory correlators most similar to one-parameter polarized K3 surfaces, see subsection 3.6.5. We there observed that the existence of an order three Picard–Fuchs operator imposed the non-trivial restriction (3.44) on the correlators. For Calabi–Yau fourfolds the existence of an order five operator imposes a similar non-trivial equality, which by an explicit calculation of the involved correlators is seen to be violated in the present model. Given this, there is a yet different restriction on the correlators that is necessary for an order six operator. This condition is fulfilled and application of the universal correlator formula for order six operators of one-parameter Calabi–Yau fourfolds derived in ref. [17] confirms the operator given in eq. (5.4).

Also this argument has potential pitfalls. First, the universal correlator formula does not guarantee but rather assume the existence of an order six operator. Second, as explained at the end of subsection 3.2.2, in the practical determination of correlators we have to truncate the calculation at some power of Q after which the rational functions appear to have stabilized. While it seems quite far fetched, it is conceptually possible that the rational functions are stable only for a finite range of powers and thereafter begin to change again.

5.3.3 Picard–Fuchs Differential Equation

The agreement of these two independent methods is by itself strong evidence for correctness of the Picard–Fuchs operator in eq. (5.4). Conceptually, the absence of an order five operator is consistent with the fact that there are two elements in the middle cohomology group $H^{2,2}$ of the ambient Grassmannian Gr(2,5) in which the target space $X_{1,4}$ is embedded. In order to provide further evidence, we now study the quantum periods of the operator. As reference for ordinary differential equations and their solution in terms of power series we refer for instance to the textbook [142]

Solutions Around Large Volume

We begin by solving the Picard–Fuchs differential equation in vicinity of the large volume limit at $\tilde{Q} \to 0$. To obtain an idea about the structure of the solutions, we first consider the indicial equation

$$\tilde{Q}^{-\alpha} \cdot \mathcal{L}(\tilde{Q}, \tilde{\Theta})\, \tilde{Q}^{\alpha} = \alpha^5(\alpha - 1) + O(\tilde{Q}) \stackrel{!}{=} O(\tilde{Q}) \tag{5.5}$$

for the characteristic exponents α. The two distinct solutions to this equation are $\alpha_1 = 0$ and $\alpha_2 = 1$, which is why we expect two holomorphic solutions of the type

$$\begin{aligned} \Pi_0^{(1)}(\tilde{Q}) &= \tilde{Q}^{\alpha_1} + O(\tilde{Q}^{\alpha_1+1}) = 1 + O(\tilde{Q})\,, \\ \Pi_0^{(2)}(\tilde{Q}) &= \tilde{Q}^{\alpha_2} + O(\tilde{Q}^{\alpha_2+1}) = \tilde{Q} + O(\tilde{Q}^2)\,. \end{aligned} \tag{5.6}$$

While the existence of $\Pi_0^{(1)}(\tilde{Q})$ is guaranteed, the second solution $\Pi_0^{(2)}(\tilde{Q})$ may run into a contradiction at higher orders in $\tilde{Q}$ since $\alpha_2 = 1$ differs by an integer from the smaller solution $\alpha_1 = 0$. In the present example this does not happen and we find that $\Pi_0^{(2)}(\tilde{Q})$ consistently extends to higher orders. Without further information $\Pi_0^{(1)}(\tilde{Q})$ is thus only defined up to adding multiples of $\Pi_0^{(2)}(\tilde{Q})$, which itself is unambiguously fixed by setting the coefficient of $\tilde{Q}$ to one. The form of the remaining four solutions is inferred from the fact that $\alpha_1 = 0$ solves eq. (5.5) with multiplicity five. Hence, there are four logarithmic solutions associated to $\Pi_0^{(1)}(\tilde{Q})$ and we make the ansatz

$$\begin{aligned} \Pi_1^{(1)}(\tilde{Q}) &= \Pi_0^{(1)}(\tilde{Q})\, \log \tilde{Q} + p_1(\tilde{Q})\,, \\ \Pi_1^{(2)}(\tilde{Q}) &= \Pi_0^{(1)}(\tilde{Q})\, \log^2 \tilde{Q} + 2p_1(\tilde{Q}) \log \tilde{Q} + p_2(\tilde{Q})\,, \\ \Pi_1^{(3)}(\tilde{Q}) &= \Pi_0^{(1)}(\tilde{Q})\, \log^3 \tilde{Q} + 3p_1(\tilde{Q}) \log^2 \tilde{Q} + 3p_2(\tilde{Q}) \log \tilde{Q} + p_3(\tilde{Q})\,, \\ \Pi_1^{(4)}(\tilde{Q}) &= \Pi_0^{(1)}(\tilde{Q})\, \log^4 \tilde{Q} + 4p_1(\tilde{Q}) \log^3 \tilde{Q} + 6p_2(\tilde{Q}) \log^2 \tilde{Q} + 4p_3\tilde{(}Q) \log \tilde{Q} + p_4(\tilde{Q})\,. \end{aligned} \tag{5.7}$$

The functions $p_k(\tilde{Q})$ are holomorphic in $\tilde{Q}$ and by using the freedom to add multiples of $\Pi_a^{(1)}$ to $\Pi_b^{(1)}$ with $b > a$ we set their constant terms to zero. For $p_4(\tilde{Q})$ we have the additional freedom to add multiples of $\Pi_0^{(2)}$, which we fix by choosing the leading power of $p_4(\tilde{Q})$ to be $\tilde{Q}^2$. The differential equation is then solved by

$$\begin{aligned} \Pi_0^{(1)}(\tilde{Q}) &= +1 - 72\tilde{Q} + 47\,880\tilde{Q}^2 + \dots\,, & \Pi_0^{(2)}(\tilde{Q}) &= \tilde{Q} - \tfrac{2\,625}{4}\tilde{Q}^2 + \dots \\ p_1(\tilde{Q}) &= -432\tilde{Q} + 327\,744\tilde{Q}^2 + \dots & p_2(\tilde{Q}) &= -944\tilde{Q} + 1\,101\,004\tilde{Q}^2 + \dots\,, \\ p_3(\tilde{Q}) &= +2\,832\tilde{Q} - 526\,770\tilde{Q}^2 + \dots\,, & p_4(\tilde{Q}) &= -7\,574\,016\tilde{Q}^2 + \dots\,. \end{aligned} \tag{5.8}$$

As important observation, the ansatz has uniquely fixed $\Pi_0^{(1)}(\tilde{Q}) = \Pi_0(\tilde{Q})$ and it agrees with the fundamental period given in eq. (5.3).

Due to the logarithms, the quantum periods are subject to a monodromy when circumventing the large volume boundary component. This amounts to sending $\tilde{Q}$ to $e^{2\pi i}\tilde{Q}$ and is captured by the monodromy matrix M_0 defined through

$$\vec{\Pi}(e^{2\pi i}\tilde{Q}) = M_0^T \cdot \vec{\Pi}(\tilde{Q}) \quad \text{with} \quad \vec{\Pi}^T = \left(\Pi_0^{(1)}, \tfrac{1}{(2\pi i)}\Pi_1^{(1)}, \dots, \tfrac{1}{(2\pi i)^4}\Pi_4^{(1)}, \Pi_0^{(2)}\right)^T, \tag{5.9}$$

where we have chosen the period vector $\vec{\Pi}$ for our convenience. From the structure of the logarithmic solutions, as specified by eq. (5.7), we immediately find

$$M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

The five periods $\Pi_k^{(1)}$ with $0 \leq k \leq 4$ transform into linear combinations of each other and behave in the same was as the solutions of an order five operator at a point of maximally unipotent monodromy would, whereas $\Pi_0^{(2)}$ appears to be entirely decoupled. This re-raises the earlier concern that the operator (5.4) might factor into an order one times an order five operator, where the order five operator would describe the logarithmic block associated to $\Pi_0^{(1)}$ and be the true Picard–Fuchs operator.

Global Solution Structure

This concern is disproven by the global structure of the quantum periods. In addition to the large volume point $\tilde{Q} = 0$ there are other points in the quantum Kähler moduli space around which the periods are subject to a monodromy transformation. These are $\tilde{Q} = \infty$ as well as the zero loci $\tilde{Q}_1 \simeq 0.043$ and $\tilde{Q}_2 \simeq -3.5 \cdot 10^{-4}$ of the discriminant factor $\Delta(\tilde{Q})$,

$$\Delta(\tilde{Q}) = 1 + 2\,816\tilde{Q} - 65\,536\tilde{Q}^2\,, \tag{5.11}$$

which is defined as the coefficient of the highest power of $\tilde{\Theta}$ in the operator. To find the solutions at these points, we rewrite the Picard–Fuchs operator in terms of local variables centered there and solve the corresponding differential equations in terms of the new variables. The solution structure is conveniently summarized by the Riemann P-symbol, which for the present example reads

$$\left\{ \begin{array}{cccc} 0 & \infty & Q_1 & Q_2 \\ \hline 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 1 \\ 0 & \frac{3}{4} & 2 & 2 \\ 0 & \frac{5}{4} & 3 & 3 \\ 0 & \frac{3}{2} & 4 & 4 \\ 1 & \frac{7}{4} & \frac{3}{2} & \frac{3}{2} \end{array} \right\} . \tag{5.12}$$

The first row lists the regular singular points of the differential equation, i.e, points around which there is a monodromy and the respective columns list the characteristic exponents defined by the indicial equations at these points. At other points in moduli space all four solutions are holomorphic.

The solutions around any of the regular singular points are only guaranteed to converge in a circular region whose radius equals the distance to the closest other regular singular point. In the intersection of two such discs both expansion are valid and the two sets of local solutions can be matched to each other by a general linear transformation. This opens the possibility for a numerical analytic continuation of the periods around the entire quantum Kähler moduli space, see for instance ref. [19] for a more detailed technical explanation thereof. This technique shows that the monodromy around the regular singular point $\tilde{Q}_2$ transforms the large volume period vector $\vec{\Pi}$ according to

$$M_{\tilde{Q}_1}^T \cdot \vec{\Pi}(\tilde{Q}) = \begin{pmatrix} \frac{137}{144} & \frac{55i\zeta(3)}{\pi^3} & -\frac{37}{12} & 0 & -\frac{5}{6} & -\frac{1}{\pi^2} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{259}{8\,640} & \frac{407i\zeta(3)}{12\pi^3} & -\frac{649}{720} & 0 & -\frac{37}{72} & -\frac{37}{60\pi^2} \\ -\frac{77i\zeta(3)}{96\pi^3} & -\frac{1815\zeta(3)^2}{2\pi^6} & -\frac{407i\zeta(3)}{8\pi^3} & 1 & -\frac{55i\zeta(3)}{4\pi^3} & -\frac{33i\zeta(3)}{2\pi^5} \\ -\frac{49}{17\,280} & \frac{77i\zeta(3)}{24\pi^3} & -\frac{259}{1\,440} & 0 & \frac{137}{144} & -\frac{7}{120\pi^2} \\ -\frac{7\pi^2}{103\,680} & \frac{11i\zeta(3)}{144\pi} & -\frac{37\pi^2}{8\,640} & 0 & -\frac{\pi^2}{864} & \frac{719}{720} \end{pmatrix} \cdot \vec{\Pi}(\tilde{Q}) . \tag{5.13}$$

The periods $\Pi_k^{(1)}$ with $k \neq 1$ transform into linear combinations that involve a non-zero contribution of the second holomorphic solution, which demonstrates that the latter does not globally decouple from the other five solutions. These can therefore not be consistently described by an order five operator. This excludes the possibility of a factorization and we have confirmed eq. (5.4) to give the correct Picard–Fuchs operator.

Since the derivation of eq. (5.13) involves numerical analytic continuation, we do not literally find the exact coefficients that are specified in the above matrix. However, the numerical precision is high enough in order to identify the numerical values with the numbers stated.

Integral Periods

A distinguished set of solutions to the Picard–Fuchs differential equation are the so-called *integral* quantum periods. These are associated to topological B-branes on the Calabi–Yau fourfold and enjoy the interpretation of moduli dependent central charges, whose magnitudes are the BPS masses of the branes — for a review of branes in string theory we for instance refer to refs. [143, 144]. As explained and employed in ref. [19], there are techniques that determine the asymptotic behavior of the integral quantum periods in the large volume limit. For the present example these methods yield

$$\vec{\Pi}^{\text{asy}}(t) = \Big(1, t,\ 10t^2 + 20t + \tfrac{107}{6},\ 4t^2 - 4t + \tfrac{7}{2},\\ -\tfrac{10}{3}t^3 - 5t^2 - \tfrac{19}{2}t - \tfrac{47}{12} + \tfrac{55i\zeta(3)}{\pi^3},\ \tfrac{5}{6}t^4 + \tfrac{37}{12}t^2 - \tfrac{55i\zeta(3)}{\pi^3}t + \tfrac{7}{144}\Big)^T , \tag{5.14}$$

where each entry specifies the value of an integral period in the limit $\tilde{Q} \to 0$ and t is the flat coordinate given by

$$t(\tilde{Q}) = \frac{1}{2\pi i} \cdot \frac{\Pi_1^{(1)}(\tilde{Q})}{\Pi_0^{(1)}(\tilde{Q})} = \frac{\log \tilde{Q}}{2\pi i} + O(\tilde{Q}) . \tag{5.15}$$

From this and eq. (5.7) we see that the k-fold logarithmic period $\Pi_k^{(1)}$ and t^k, where $0 \leq k \leq 4$, share the same non-zero asymptotic value. This motivates the introduction of the modified period vector

$$\vec{\Pi}_{\vec{\beta}} = C_{\vec{\beta}}^T \cdot \vec{\Pi} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{\beta_0}{\pi^2} \\ 0 & 1 & 0 & 0 & 0 & \frac{\beta_1}{\pi^2} \\ \frac{107}{6} & 20 & 10 & 0 & 0 & \frac{\beta_2}{\pi^2} \\ \frac{7}{2} & -4 & 4 & 0 & 0 & \frac{\beta_3}{\pi^2} \\ -\frac{47}{12} + \frac{55i\zeta(3)}{\pi^3} & -\frac{19}{2} & -5 & -\frac{10}{3} & 0 & \frac{\beta_4}{\pi^2} \\ \frac{7}{144} & -\frac{55i\zeta(3)}{\pi^3} & \frac{37}{12} & 0 & \frac{5}{6} & \frac{\beta_5}{\pi^2} \end{pmatrix} \cdot \vec{\Pi} , \tag{5.16}$$

which has the same asymptotics as $\vec{\Pi}^{\text{asy}}$. Since the second holomorphic solution $\Pi_0^{(2)}$ limits to zero for $\tilde{Q} \to 0$, the numbers β_k with $0 \leq k \leq 5$ that measure its respective contributions are not fixed by the large volume asymptotics and need to be determined by different means. This complication does not occur for large volume points with maximially unipotent monodromy, as they for example occur for Calabi–Yau threefolds and Calabi–Yau fourfolds with an order five operator.

Because the second holomorphic solution relates to the existence of B-branes on the two non-trivial algebraic cycles corresponding to the described cohomology classes in $H^4(X_{1,4}, \mathbb{Z})$, there are no such ambiguities for the quantum periods $\Pi_0^{(1)}$ and $\Pi_1^{(1)}$ that are associated to B-branes in higher codimension. From this we conclude $\beta_0 = \beta_1 = 0$. In order to determine the other coefficients, we employ the Strominger–Yau–Zaslow picture of mirror symmetry for Calabi–Yau fourfolds that conjectures the existence of a singular point $\tilde{Q}^*$ in quantum Kähler moduli space where the 8-brane — corresponding to the last entry of $\vec{\Pi}^{\text{asy}}$ and $\vec{\Pi}_{\vec{\beta}}$ — becomes massless [145]. As explained in ref. [19], this fixes the monodromy matrix $M_{\tilde{Q}^*}$ which for the present model takes the form

$$M_{\tilde{Q}^*} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -24 & -6 & 7 & -1 \end{pmatrix} \tag{5.17}$$

when expressed in the basis specified by $\vec{\Pi}_{\vec{\beta}}$. Using numerical analytical continuation to determine all four monodromy matrices, this form of the monodromy is seen to only be possible at the point $\tilde{Q}_2$. This gives

$$M_{\tilde{Q}^*} = C_{\vec{\beta}}^{-1} \cdot M_{\tilde{Q}_2} \cdot C_{\vec{\beta}} \tag{5.18}$$

as condition on the basis change from $\vec{\Pi}$ to $\vec{\Pi}_{\vec{\beta}}$, which is uniquely solved by choosing the coefficients β_k as

$$\beta_0 = \beta_1 = \beta_2 = \beta_4 = 0\,, \quad \beta_3 = 24\,, \quad \beta_5 = 1\,. \tag{5.19}$$

The vector $\vec{\Pi}^{\text{int}}$ of integral quantum periods is given by $\vec{\Pi}_{\vec{\beta}}$ with these values of β_k. Since β_3 and β_5 are non-zero, the second holomorphic solution is necessary to define a basis of integral quantum periods. This gives a physical explanation for why the operator (5.4) cannot factor. In terms of the basis specified by $\vec{\Pi}^{\text{int}}$ all monodromy matrices are integer valued, see ref. [19] for further details.

All Solutions from Cartan Theory I-Function

As a brief detour, we mention that also for this example all six solutions of the Picard–Fuchs differential equation are found as appropriate derivatives of the Cartan theory I-function. Equations (4.34) and (4.35) give

$$I^{\text{Cartan}}(\vec{H},Q) = \sum_{\vec{k}\in\gamma_m^+} \tilde{Q}^{2k_1+2\frac{H_1}{\epsilon}}(-1)^{2k_2}\left[1 + k_2\partial_{H_2} + \tfrac{H_2^2}{2}\partial_{H_2}^2\left(1+\tfrac{k_2}{3}\partial_{H_2}\right) + \tfrac{H_2^4}{24}\partial_{H_2}^4\left(1+\tfrac{k_2}{5}\partial_{H_2}\right)\right]\cdot$$
$$\frac{\Gamma\left(1+2k_1+2\frac{H_1}{\epsilon}\right)\Gamma\left(1+8k_1+8\frac{H_1}{\epsilon}\right)\Gamma\left(1+\frac{H_1+H_2}{\epsilon}\right)^5\Gamma\left(1+\frac{H_1-H_2}{\epsilon}\right)^5}{\Gamma\left(1+2\frac{H_1}{\epsilon}\right)\Gamma\left(1+8\frac{H_1}{\epsilon}\right)\Gamma\left(1+k_1+k_2+\frac{H_1+H_2}{\epsilon}\right)^5\Gamma\left(1+k_1-k_2+\frac{H_1-H_2}{\epsilon}\right)^5}\Big|_{H_2=0} + O(H_2^6) \tag{5.20}$$

where after executing the derivatives we set $H_2 = 0$ inside the gamma functions and its derivative, but we keep the factors H_2^2 and H_2^4 inside the square brackets in the first line. Summation over $\vec{k} = (k_1, k_2)$ is as exactly as in eq. (5.3), which arises from the set γ_m^+ given in eq. (4.37). The expression is exact up to and including third powers of the auxiliary variable $\vec{H}_{\text{aux}} = (H_2)$ and an explicit calculation demonstrates that

$$\begin{aligned}
\Pi_k^{(1)} &= \left(\tfrac{\epsilon}{2}\,\partial_{H_1} \pm i\tfrac{\epsilon}{2}\sqrt{\tfrac{3}{5}}\,\partial_{H_2}\right)^k I^{\text{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0} \quad \text{with} \quad k = 0,\ldots,3\,,\\
\Pi_4^{(1)} &= \left[\left(\tfrac{\epsilon}{2}\,\partial_{H_1} \pm i\tfrac{\epsilon}{2}\sqrt{\tfrac{3}{5}}\,\partial_{H_2}\right)^4 + \tfrac{\epsilon^4}{25}\partial_{H_2}^4\right] I^{\text{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0}\,,\\
\Pi_0^{(2)} &= -\tfrac{\epsilon}{240}\partial_{H_2}^2\, I^{\text{Cartan}}(\vec{H},Q)\big|_{\vec{H}=0}
\end{aligned} \tag{5.21}$$

are six linearly independent solutions to the Picard–Fuchs differential equation. Similar to the third example discussed in subsection 4.4.1, the five solutions $\Pi_k^{(1)}$ are not generated by powers of a fixed linear combination of ∂_{H_1} and ∂_{H_2} applied to the Cartan theory I-function. This again fails at the fourth derivative level. However, unlike that previous example, the second derivative with respect to ∂_{H_2} does here not sum to zero but rather give the second holomorphic solution. Consequently, all (both) second derivatives that are not automatically zero by Weyl symmetry are solutions to the Picard–Fuchs differential equation. This suggests that for models with gauge group $G = \mathrm{U}(1) \times \mathrm{SU}(2)/\Gamma$ and Calabi–Yau fourfold target space the order of the generating Picard–Fuchs operator is always either five or six, but not higher. For general non-Abelian gauge groups with a single U(1) factor we similarly expect that the order of the operator is at most five plus the number m_2^{inv} of Weyl group invariants of homogenous order $i = 2$.

5.3.4 Implications

Having fully established that the generating Picard–Fuchs operator has the non-minimal order six, we here comment on two interesting implications thereof.

New Types of Flux-Induced Superpotentials

As seen from eqs (5.16) and (5.19), the existence of the second holomorphic solution $\Pi_0^{(2)}$ to the order six Picard–Fuchs differential equation implies the existence of two doubly-logarithmic integral quantum periods, namely

$$\begin{aligned} \Pi_2 &= 10 \cdot \frac{\Pi_2^{(1)}}{(2\pi i)^2} + 20 \cdot \frac{\Pi_1^{(1)}}{2\pi i} + \frac{107}{6} \cdot \Pi_0^{(1)} \,, \\ \Pi_3 &= 4 \cdot \frac{\Pi_2^{(1)}}{(2\pi i)^2} - 4 \cdot \frac{\Pi_1^{(1)}}{2\pi i} + \frac{7}{2} \cdot \Pi_0^{(1)} + \frac{24}{\pi^2} \cdot \Pi_0^{(2)} \,. \end{aligned} \tag{5.22}$$

Here and below we denote the k-th entry of the integral period vector $\vec{\Pi}^{\mathrm{int}}$ as Π_k. In the context of Calabi–Yau fourfold compactifications of type IIA superstring theory to two dimensions, non-trivial background fluxes for the field strength of the (R,R) 3-form field can be used to generate a superpotential of the form

$$W_{\mathrm{flux}} = \frac{1}{\Pi_0} \sum_{k=0}^{5} n_k \cdot \Pi_k \quad \text{with} \quad n_k \in \mathbb{Z} \,, \tag{5.23}$$

where the n_k enjoy the interpretation of flux quantum numbers. As a side remark, for manifolds with odd second Chern class the n_k would rather be required to be half-integral [146]. Further, the quotient by the fundamental period amounts to a field redefinition in which the worldsheet instanton corrections become apparent when expressed in terms of the flat coordinate t given by eq. (5.15). In the present example there are choices for n_k that yield the three superpotentials

$$\begin{aligned} W_{\mathrm{flux}}^{(1)} &= \frac{1}{\Pi_0} \left(109\Pi_0 + 360\Pi_1 - 12\Pi_2 + 30\Pi_3\right) = \frac{2\,880}{4\pi^2} e^{2\pi i t} + O(e^{4\pi i t}) \,, \\ W_{\mathrm{flux}}^{(2)} &= \frac{1}{\Pi_0} \left(60\Pi_1 - 2\Pi_2 + 5\Pi_3\right) = -\frac{109}{6} + \frac{480}{4\pi^2} e^{2\pi i t} + O(e^{4\pi i t}) \,, \\ W_{\mathrm{flux}}^{(3)} &= \frac{1}{\Pi_0} \left(109\Pi_0 - 12\Pi_2 + 30\Pi_3\right) = -360\, t + \frac{2\,880}{4\pi^2} e^{2\pi i t} + O(e^{4\pi i t}) \,. \end{aligned} \tag{5.24}$$

As wee see, these are given by a non-zero tower of instanton corrections plus an at most linear term in t. Without two doubly-logarithmic integral quantum periods it is impossible to obtain such types of superpotentials, because the presence of non-zero instanton corrections requires a non-zero contribution from a doubly-logarithmic (or higher logarithmic) period and without a second one the t^2 term can then not be cancelled. All three cases indeed include both doubly-logarithmic periods Π_2 and Π_3, the latter of which involves the second holomorphic solution whose presence itself originates in the Picard–Fuchs operator of non-minimal order.

Two Towers of Gromov–Witten Invariants

The fact that two independent elements $\phi_2^{(1)}$ and $\phi_2^{(2)}$ of $H^{(2,2)}$ participate in the quantum product results in two independent towers of worldsheet instanton corrections, each counted by genus zero Gromov–Witten invariants. Denoting the single generator of $H^{1,1}$ as ϕ_1, the general structure of the quantum product yields

$$\phi_1 * \phi_1 \;=\; \phi_2^{(1)} \left[c^{(1)} + \sum_{d=1}^{\infty} n_{0,d}^{(1)} \frac{d^2 q^d}{1-q^d} \right] + \phi_2^{(2)} \left[c^{(2)} + \sum_{d=1}^{\infty} n_{0,d}^{(2)} \frac{d^2 q^d}{1-q^d} \right] . \tag{5.25}$$

Here the numbers $c^{(1)}$ and $c^{(2)}$ are defined by the classical cup product $\phi_1 \cup \phi_1 = \sum_a c^{(a)} \phi_2^{(a)}$, the variable $q = e^{2\pi i t}$ is the exponential of the flat coordinate (5.15), and $n_{0,d}^{(a)}$ are the integral genus zero Gromov–Witten invariants where the superscript refers to a single marked point that is constrained to lie on the algebraic cycle class $\phi_2^{(a)}$. This equation should be regarded as the adaption of eq. (2.52), which defines the quantum product for Calabi–Yau threefolds, to the present case of one-parameter Calabi–Yau fourfolds with a Picard–Fuchs operator of non-minimal order six.

As explained in detail in ref. [19], the two towers of Gromov–Witten invariants can be found from the integral quantum periods that we determined in the previous subsection. Intuitively speaking, eq. (5.25) needs to be multiplied by some element of $H^{(2,2)}$ in order to yield a differential form of top degree that can then be integrated over the target space $X_{1,4}$. Two such integrals are obtained from the doubly-logarithmic integral periods Π_2 and Π_3 together with the fundamental period Π_0 as

$$\frac{\partial^2}{\partial t^2} \frac{\Pi_k(\tilde{Q}(t))}{\Pi_0(\tilde{Q}(t))} \;=\; \int_{X_{1,4}} (\phi_1 * \phi_1) \cup \mathrm{ch}\left(\Pi_k\right) \quad \text{with} \quad k \;=\; 2,\,3\,. \tag{5.26}$$

Here the original variable $\tilde{Q}$ is expressed in terms of the flat coordinate — which we achieve by inverting the exponential of eq. (5.15), see subsection 2.3.1 for a similar calculation for the quintic Calabi–Yau threefold — and $\mathrm{ch}(\Pi_k)$ with $k = 2,\,3$ denote the Chern characters of the 4-branes associated to Π_2 and Π_3. These take values in $H^{(2,2)} \oplus H^{(3,3)} \oplus H^{(4,4)}$, such that the product of $\phi_1 * \phi_1$ with $\mathrm{ch}(\Pi_k)$ indeed yields a form of top degree. This gives the equality $(\phi_1 * \phi_1) * \mathrm{ch}(\Pi_k) = (\phi_1 * \phi_1) \cup \mathrm{ch}(\Pi_k)$, as already used in writing eq. (5.26), and qualifies the equations for a straightforward determination of the Gromov–Witten invariants. The right hand side of the equation is evaluated by a calculation in the classical cohomology ring, which for the present example of $X_{1,4} \subset \mathrm{Gr}(2,5)$ yields

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \frac{\Pi_2}{\Pi_0} &= 20 + \sum_{d=1}^{\infty} d^2 \frac{q^d}{1-q^d} \left(8 n_{0,d}^{(1)} + 12 n_{0,d}^{(2)}\right) , \\ \frac{\partial^2}{\partial t^2} \frac{\Pi_3}{\Pi_0} &= 8 + \sum_{d=1}^{\infty} d^2 \frac{q^d}{1-q^d} \left(4 n_{0,d}^{(1)} + 4 n_{0,d}^{(2)}\right) . \end{aligned} \tag{5.27}$$

d	$n^{(1)}_{0,d}$	$n^{(2)}_{0,d}$
1	400	520
2	208 240	226 480
3	175 466 480	191 464 760
4	196 084 534 160	213 155 450 240
5	255 402 582 828 400	277 092 686 601 400
6	367 048 595 782 193 680	397 700 706 634 553 680
7	564 810 585 071 858 496 880	611 416 342 763 726 567 800
8	913 929 133 261 543 393 001 760	988 670 017 271 687 389 572 480
9	1 536 929 129 164 031 410 293 358 720	1 661 748 145 541 449 358 296 013 440
10	2 664 576 223 763 330 924 317 069 072 400	2 879 777 881 450 393 936 532 565 976 400

Table 5.2: Genus zero integral Gromov–Witten invariants $n^{(1)}_{0,d}$ and $n^{(2)}_{0,d}$ of the Calabi–Yau fourfold $X_{1,4} \subset \mathrm{Gr}(2,5)$ associated to $\phi^{(1)}_2 = \sigma_{1,1}$ and $\phi^{(2)}_2 = \sigma_2$ up to degree $d = 10$.

Here we have chose $\phi^{(1)}_2 = \sigma_{1,1}$ and $\phi^{(2)}_2 = \sigma_2$ in terms of the two Schubert classes $\sigma_{1,1}$ and σ_2 that generate the cohomology group $H^4(\mathrm{Gr}(2,5),\mathbb{Z})$ of the ambient space in which $X_{1,4}$ is embedded — see for instanace the textbook [109] for mathematical background. By expanding these equations in the variables q, we obtain two independent equations for each degree d that determine the unknowns $n^{(1)}_{0,d}$ and $n^{(2)}_{0,d}$. In Table 5.2 they are listed up to degree $d = 10$ and their integrality confirms our choice of integral quantum periods. The integral periods moreover determine the topological limit F^{top}_1 of the generalized topological index of the $N = (2,2)$ superconformal worldsheet theory associated to the Calabi–Yau fourfold under consideration [133, 147–149]. Together with the symmetric Klemm–Pandharipande meeting invariants defined in [149], which themselves are found recursively from the genus zero worldsheet instanton numbers, this quantity encodes the integral genus one invariants of the Calabi–Yau fourfold. As demonstrated in [19], these numbers are also found to be integral and thereby additionally confirm our choice of integral quantum periods.

CHAPTER **6**

Conclusion and Outlook

In this thesis we have presented various results that connect gauged linear sigma models and Picard–Fuchs operators, both of which are powerful tools for studying the moduli structure of superconformal worldsheet theories that arise in compactifications of type II superstring theories. As application we moreover considered an interesting aspect of Calabi–Yau fourfold geometry that previously has mostly been overlooked.

The first chapter gave a non-technical introduction to — and motivation for — the bigger research field in which this work is set. In the second chapter we then reviewed several physical and mathematical concepts that are central to the presented research.

In the third chapter we used the result of a modern localization computation [73] for a detailed study of certain correlation functions in the gauged linear sigma model and connected these to Picard–Fuchs operators. To this end we first derived universal and non-trivial linear dependencies amongst the family of these correlators, which in a Hilbert space interpretation were shown to define differential operators that annihilate the moduli dependent ground state. Using the connection between the correlators and a quadratic pairing of the Givental I-function [99], we demonstrated that these operators in case of geometric target space also annihilate the I-function. For conformal models this identified them as Picard–Fuchs operators on the quantum Kähler moduli space. The combination of these findings provided an elementary combinatorial algorithm that allows to determine the Picard–Fuchs operators from the defining gauge theory spectrum directly, without the need to calculate the correlators. Since this algorithm does not use mirror symmetry, it is equally applicable to cases without a known mirror geometry. For several classes of Calabi–Yau manifolds, specified by a fixed complex dimension and number of Kähler parameters, we moreover derived universal formulas that express the generating Picard–Fuchs operator in terms of the gauge theory correlators. These formulas are automatically in accord with non-trivial constraints, such as $\mathcal{N} = 2$ special geometry for Calabi–Yau threefolds [88].

By means of the Abelian Cartan theories we also derived universal linear dependencies of correlators in non-Abelian gauged linear sigma models. However, as discussed in subsection 3.5.3, it is not clear whether this approach captures the full structure of the non-Abelian theory. It would be interesting to answer this question and to carry out the derivation directly in the non-Abelian model. This is expected to greatly aid the practical determination of Picard–Fuchs operators.

For non-Abelian gauge groups and complicated matter spectra the required calculations can become computationally challenging. In the fourth chapter we therefore presented an alternative method for deriving the operators associated to non-Abelian models, which in practice is computationally more efficient than those of chapter three. For this we employed the Givental I-functions of the Abelian Cartan theories and proposed that these encode the holomorphic solution — for conformal models known as the fundamental period — of the non-Abelian model's operator in a certain non-trivial limit. We demonstrated that the existence of this limit is, under certain assumptions on the matter spectrum, equivalent to a statement about the algebra of the semi-simple gauge group factors. For several low-rank non-Abelian groups we checked this condition explicitly and conjectured it to hold in general. We also presented a formula for the finite result of the limit, which is ready-to-use and allows to efficiently determine the Picard–Fuchs operator of a given model by requiring it to annihilate the expansion of this expression. In addition, we found that the other solutions of the operator — for conformal models these are the other quantum periods — are given by appropriate, model dependent linear combinations of derivatives applied to the Cartan theory I-function. We concluded the chapter by discussing the idea of reconstructing gauged linear sigma models from given differential operators.

There are various directions for future research in this context. First, the formulas for the holomorphic solutions are applicable to a large class of non-Abelian matter spectra and allow to efficiently determine the associated differential operators. They thus open the possibility to scan over various gauge theory spectra with the aim of finding models of a given desired type, for example models with a compact Calabi–Yau threefold target space. Second, the hemisphere partition function of the gauged linear sigma model is known to calculate the quantum periods [106–108] and it would be interesting to use it for a re-derivation of the formulas that we presented. Third, although it is less clear how feasible this is, a generalization of the formulas to strongly coupled phases in which a non-Abelian gauge group factor is left unbroken would be very useful for the study of strong-weak coupling dualities between different gauged linear sigma models. As a first step towards this, one might attempt an analytic continuation to other phases. Lastly, it would be very interesting to find a practical implementation of the proposed program for reconstructing gauged linear sigma models from differential operators. This would essentially put gauged linear sigma models and Picard–Fuchs operators in a one-to-one correspondence and thereby entirely bypass the need to understand or even know the target space geometry. Since there are efforts to classify Picard–Fuchs operators by their analytical and algebraic properties [115, 116, 124], it would moreover offer a roadmap towards a classification of gauged linear sigma models and, in extension, string vacua.

In the fourth chapter we employed the various techniques introduced at this point of the thesis to study Calabi–Yau fourfolds that arise as target spaces of non-Abelian gauged linear sigma models. As opposed to Calabi–Yau threefolds and target spaces of Abelian theories, the quantum cohomology of these geometries is not guaranteed to be generated by products of marginal Kähler deformations. Examples of this type have previously been observed in refs. [56, 118]. As we explained, this phenomenon is due to additional elements of the middle cohomology group $H^{2,2}$ that participate in the quantum product. For cases with a single Kähler modulus this was shown to result in a non-factorizable Picard–Fuchs operator of degree six or higher. We demonstrated the effect explicitly in an example and discussed its implication of additional quantum periods that vanish in the large volume limit but are non-zero in general. While the integral quantum periods of the model are thus not entirely determined by the large volume asymptotics, we have shown that they could still be found by using the global monodromy structure. Lastly, we demonstrated that the integral periods allow for new types of flux superpotentials that are entirely instanton generated and calculated the genus zero worldsheet instanton numbers.

It would be interesting to find and analyze multi-parameter examples of this type, as well as one-parameter examples in which the generating Picard–Fuchs operator is of yet higher order than six. For the latter purpose, we suspect models with gauge groups $G = \mathrm{U}(1) \times \mathrm{SU}(2)^2 / \Gamma$ to be the simplest choice. The effect of additional quantum cohomology elements is expected to also occur for Calabi–Yau manifolds of complex dimension higher than four. However, as briefly mentioned in subsection 3.6.6, from the correlator point of view there is some indication that it might not happen in the complex odd-dimensional case. It would be interesting answer this question.

As a recurring theme throughout the thesis, general non-Abelian theories are not yet particularly well understood. It would be a big step to obtain a better control over them, which goes beyond using the Abelian Cartan theories and then taking back an appropriate limit. This is, however, not likely to be an easy problem.

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