Functional Inequalities and Heat Kernel Asymptotics on Some Classes of Singular Riemannian Manifolds

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Abstract

This thesis consists of two parts.

In the first part we are focusing on stratified pseudomanifolds equipped with an iterated edge metric. More specifically, in Chapter 1 we give the basic definitions and review some basic constructions concerning stratified spaces and iterated edge metrics. Furthermore we introduce the notion of edge vector fields and weighted Sobolev spaces which naturally arise in these spaces, and prove some of their properties. In Chapter 2 we are focusing on stratified pseudomanifolds of depth 1, the so called simple edge spaces. We introduce Sobolev spaces and we compare them with the weighted Sobolev spaces we previously defined. Furthermore, by taking into account the special structure of simple edge spaces we prove the validity of the classical functional inequalities (Sobolev, Poincare, Sobolev-Poincare). Moreover, we examine the existence of appropriate cut-off functions and as an application we obtain an optimality result on the B-constant of the Sobolev inequality.

In the second part of the thesis we are focusing on the Dirichlet heat kernel and it's asymptotics as $t \to 0$. More precisely, in Chapter 3 we consider the case of compact manifolds with corners satisfying a specific assumption on the metric. Under this assumption we construct a heat calculus that contains information about the asymptotic behaviour and examine it's properties. After elaborating on this, we prove that the Dirichlet heat kernel belongs in this calculus and therefore we are able to obtain a complete asymptotic expansion as $t \to 0$.

Part I

Sobolev Spaces on Stratified Spaces

Sobolev Spaces on Stratified Spaces

Stratified spaces constitute an important part of singular spaces. Informally speaking, a stratified space is a topological space that can be partitioned into smooth manifolds (strata) of different dimension. Although this statement is far from complete, it is the fundamental guiding principle behind the idea of stratified spaces. The study of these spaces was initiated by Whitney ([Whi47]), Thom ([Tho69]) and Mather ([Mat73]) among others. Later, Goresky, MacPherson and Cheeger studied the intersection homology and L^2-cohomology of these spaces ([GM80] and [CGM82]). It was Cheeger with his seminal paper ([Che83]) that initiated the study of these spaces from an analytical point of view and more precisely the properties of the Laplace operator on manifolds with conical singularities. The program of laying the analytic foundations of these spaces was taken up since, and still is a very active area of research.

In this first part of the thesis, we restrict our attention to a class of stratified pseudomanifolds endowed with an iterated edge metric g. Roughly speaking, a stratified pseudomanifold X of depth k is a union of smooth manifolds of varying dimension (called the strata) with the property that the top dimensional stratum $\mathrm{reg}(X)$ is dense, and that for every singular stratum Y, there exists an open neighborhood U such that

$$U \simeq Y \times C(L),$$

where L is again a stratified space of depth $\leq k-1$, and $C(L)$ is the cone over L. Stratified spaces of depth 0 are closed manifolds. The Riemannian metric g, which is defined on $\mathrm{reg}(X)$, takes on $U \cap \mathrm{reg}(X)$ the form

$$g \simeq h + dr^2 + r^2 g_L,$$

where h is a metric on Y, r the radial variable of the cone, and g_L an iterated edge metric on $\mathrm{reg}(L)$. For precise definitions and statements we refer to Chapter 1.

An important role for the study of analytical questions is played by Sobolev spaces and their properties. For example, in the classical case of a compact manifold without boundary, the domain of elliptic differential operators is described in terms of Sobolev spaces. This fact, under some further assumptions and for geometric differential operators, still holds true in the case of a stratified pseudomanifold endowed with an iterated edge metric (see Theorem 10.4, Corollary 10.6 in [HLV18]). See also [ALMP12],[GKM13], [Les97]. However, in contrast to the classical case, Sobolev spaces on stratified pseudomanifolds are much more subtle and one can define them in different ways. The scope of Chapter 1 is to give precise definitions about stratified spaces and iterated edge metrics, to describe in more detail the (weighted) Sobolev spaces that appear in these references and also provide the proofs for some of their properties that are usually omitted in the literature.

Another direction concerning Sobolev spaces and Sobolev inequalities on stratified pseudomanifolds endowed with an iterated edge metric was taken in [ACM14]. There the authors dealt with the Yamabe problem on smooth metric measure spaces, which are a generalisation of stratified pseudomanifolds. In order to do so, among else, they proved the validity of the Sobolev inequality on stratified pseudomanifolds with exponent $p = 2$, namely, that there exist $A, B > 0$ such that for every $u \in W_0^{1,2}(X)$ we have

$$\|u\|^2_{\frac{2m}{m-2}} \leq A \int_X |\nabla u|^2 dv + B \int_X |u|^2 dv,$$

where $W_0^{1,2}(X)$ is defined as the closure of $C_c^\infty(\mathrm{reg}(X))$ with respect to the norm $\|u\|^2_{W^{1,2}} = \|\nabla u\|^2_{L^2} + \|u\|^2_{L^2}$. These Sobolev spaces are different, but related with the weighted Sobolev spaces mentioned before. Their exact relation is discussed in subsection 2.3.1.

However, Sobolev inequality, and in general functional inequalities (Poincare inequality, Sobolev-Poincare inequality) for general exponents p have an independent interest and it is valuable to know whether they are true in the case of compact simple edge spaces. These are the main results of Chapter 2. But before proving these inequalities we first need to build some other tools. More precisely, the first theorem we prove gives sufficient conditions for the existence of a sequence of cut-off functions, i.e. functions with compact support on the regular part, that converge to 1, and the L^p norm of their first and second order covariant derivative converges to 0. More precisely, we obtain the following

Theorem. Let X be a compact stratified pseudomanifold of dimension m, endowed with an iterated edge metric on $\mathrm{reg}(X)$ and let $k = 1$ or $k = 2$.

Suppose that for every singular stratum Y of X we have the condition

$$\operatorname{codim}(Y) = m - i > kp, \text{ where } i = \dim(Y).$$

Then

- If $k = 1$, then X admits a sequence of $(1, p)-$cut-offs.
- For $k = 2$, if $\operatorname{depth}(X) = 1$, or the strata Y with $\operatorname{depth}(Y) > 1$ satisfy Assumption 2.2.1, then X admits a sequence of $(2, p)-$cut-offs.

This result generalises the result obtained in [BG17] to second order cut-offs, and it is used as a step to prove a density theorem about Sobolev spaces, as well as it is used to provide estimates for the optimal B-constant in the Sobolev inequality.

After proving this result, we restrict our attention to compact simple edge spaces, which are stratified pseudomanifolds of depth 1. The reason for doing so, is that the neighborhood of a singular stratum Y is locally Euclidean. This, together with a Hardy inequality allow us to obtain the Sobolev inequality for $p \in [1, m)$, with $m = \dim(X)$, i.e. there exists $A, B > 0$, such that for every $u \in W_0^{1,p}(X)$ we have

$$\|u\|_{\frac{mp}{m-p}} \leq A\left(\int_X |\nabla u|^p dv\right)^{1/p} + B\left(\int_X |u|^p dv\right)^{1/p}, \qquad (I_p)$$

where $p^* = \frac{mp}{m-p}$.

Apart from the Sobolev inequality, we prove the validity of the Rellich embedding, i.e. that for X compact simple edge space of dimension $m > 1$, with p, q that satisfy $1 \leq p < m$, $p \neq m - \dim(Y)$ for every singular stratum Y of X and $q < p^*$, the embedding

$$W_0^{1,p}(X) \hookrightarrow L^q(X)$$

is compact. Rellich embedding implies Poincare inequality, i.e. for p satisfying the above condition and also p satisfying the condition $p < m - \dim(Y)$ for every singular stratum Y of X, we obtain a $C > 0$ such that for every $u \in W_0^{1,p}(X)$ we have

$$\|u - u_X\|_p \leq C\|\nabla u\|_p,$$

where $u_X = \frac{1}{\operatorname{vol}(X)} \int_X u(x)dv(x)$. Combining this with the Sobolev inequality, ones obtains a Sobolev-Poincare inequality, i.e. for $u \in W_0^{1,p}(X)$, we have

$$\|u - u_X\|_{p^*} \leq C\|\nabla u\|_p.$$

Finally, all these constructions and chain of inequalities lead to the main theorem of Chapter 2, namely:

Theorem. Let X be a connected, compact simple edge space of dimension $m > 1$. Then if $1 \leq p < m - \dim(Y)$ for every singular stratum Y of X, there exists $A > 0$ such that

$$\|u\|_{p^*} \leq A(\int_X |\nabla u|^p dv)^{1/p} + \mathrm{vol}(X)^{-\frac{1}{m}}(\int_X |u|^p dv)^{1/p}. \qquad (I_{p,B_{opt}})$$

Moreover, the constant $\mathrm{vol}(X)^{-\frac{1}{m}}$ is optimal, in the sense that if there exists a $B > 0$ such that (I_p) holds with B, then $B \geq \mathrm{vol}(X)^{-\frac{1}{m}}$.

Chapter 1

Preliminaries

1.1 Stratified Spaces

In this section we introduce stratified spaces, review some constructions and set up the notation. Throughout the literature there exists various definitions of what a stratified space is, and they are not always consistent with each other (see for example [Ban07],[BHS91],[Mat73], [Pfl01],[Ver84] to name just a few). In this chapter we follow the approach developed in [ALMP12]. There the authors gave a thorough description of smoothly stratified spaces and their resolution into manifolds with corners with iterated fibration structure, which we will describe in this section. We choose to follow this approach, because this definition naturally generalises the simpler situations of manifolds with conical and edge singularities and also allow us to resolve our singular space into a manifold with corners with iterated fibration structure, a notion that is fairly well understood. We begin by giving the definition of a stratified space:

Definition 1.1.1. A stratified space X is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata $\mathfrak{G} = \{Y_\alpha\}$, where each Y_α is a smooth, open, connected manifold, with dimension depending on the index α. We assume the following:

- If $Y_\alpha,\ Y_\beta \in \mathfrak{G}$ and $Y_\alpha \cap \overline{Y_\beta} \neq \emptyset$, then $Y_\alpha \subseteq \overline{Y_\beta}$.
- Each stratum Y is endowed with a set of 'control data' $T_Y,\ \pi_Y$ and ρ_Y; here T_Y is a neighborhood of Y in X which retracts onto Y, $\pi_Y : T_Y \to Y$ is a fixed continuous retraction and $\rho_Y : T_Y \to [0, 2)$ is a proper 'radial function' in this tubular neighborhood such that $\rho_Y^{-1}(0) = Y$. Furthermore, we require that if $Z \in \mathfrak{G}$ and $Z \cap T_Y \neq \emptyset$, then $(\pi_y, \rho_Y) : T_Y \cap Z \to Y \times [0, 2)$, is a proper smooth submersion.

- If $W, Y, Z \in \mathfrak{S}$ and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$, then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.
- If $Y, Z \in \mathfrak{S}$, then $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$, $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subseteq \overline{Z}$, $Y = Z$ or $Z \subseteq \overline{Y}$.
- For each $Y \in \mathfrak{S}$, the restriction $\pi_Y : T_Y \to Y$ is a locally trivial fibration with fiber the cone $C(L_Y)$ over some other stratified space L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(\mathcal{U}) \to \mathcal{U} \times C(L_Y)$ and the transition functions are stratified isomorphisms of $C(L_Y)$ which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in \mathcal{U}$.

If in addition we let X_j be the union of all strata of dimensions less than or equal to j, and require that

$$X = X_m \supseteq X_{m-1} = X_{m-2} \supseteq X_{m-3} \supseteq \cdots \supseteq X_0$$

and $X \setminus X_{m-2}$ is dense in X, then we say that X is a stratified pseudomanifold of dimension m.

The depth of a stratum Y is the largest integer k such that there is a chain of pairwise distinct strata $Y = Y_k, \ldots, Y_0$ with $Y_j \subseteq \overline{Y_{j-1}}$ for $1 \leq j \leq k$. A stratum of maximal depth is always a closed manifold. The maximal depth of any stratum in X is called the depth of X as a stratified space. We refer to the dense open stratum of a stratified pseudomanifold X as its regular set, and the union of all other strata as the singular set,

$$\mathrm{reg}(X) := X \setminus \mathrm{sing}(X) \text{ where } \mathrm{sing}(X) = \bigcup_{Y \in \mathfrak{S},\ \mathrm{depth}\, Y > 0} Y.$$

If X and X' are two stratified spaces, a stratified isomorphism between them is a homeomorphism $F : X \to X'$ which carries the open strata of X to the open strata of X' diffeomorphically and such that $\pi'_{F(Y)} \circ F = F \circ \pi_Y$, $\rho'_Y = \rho_{F(Y)} \circ F$ for all $Y \in \mathfrak{S}(X)$.

In the rest of this chapter we will restrict our attention to compact stratified pseudomanifolds, which we will always denote by X, unless otherwise stated.

1.1.1 Examples

Closed Manifolds: The simplest example of a stratified pseudomanifold is a closed manifold M, i.e. compact with $\partial M = \emptyset$. It consists of only one stratum $M = \mathrm{reg}(X) = \overline{\mathrm{reg}(X)} = X$. It is a stratified pseudomanifold of depth 0.

Manifolds with isolated singularities: The simplest non-trivial example of a stratified pseudomanifold is a manifold with isolated singularities. It is a topological space $(X, \{p_1, \ldots, p_n\} \subseteq X)$ with the property that $X \setminus \{p_1, \ldots, p_n\}$ is a differentiable manifold and for every p_i, there exists an open neighborhood $p_i \in U_i \subseteq X$ and a homeomorphism $\phi_i : U_i \to C(L_i)$ that restricts to a diffeomorphism $\phi_i : U_i \setminus \{p_i\} \to (0, \varepsilon) \times L_i$, where L_i is a closed manifold called the link. By rescaling, we can assume that $\varepsilon = 2$. In this case, $X^{\mathrm{reg}} = X \setminus \{p_1, \ldots, p_n\}$ is the regular part $\mathrm{reg}(X)$, which is dense in X, and the singular strata are simply $Y_1 = \{p_1\}, \ldots, Y_n = \{p_n\}$. It is a stratified space of depth 1. For each $Y_i = \{p_i\}$ the control data are simply: $T_{Y_i} = U_i$, $\pi_{Y_i} : T_{Y_i} \to \{p_i\}$ and $\rho_{Y_i} : T_{Y_i} \to [0, 2)$ defined by $\rho(p) = r$, where $r = \mathrm{pr}_1(\phi_i(p)) = \mathrm{pr}_1(r, z)$ where $(r, z) \in (0, \varepsilon) \times L_i$.

Simple Edge Spaces: Another example of a stratified space of depth 1, is a manifold with edge singularities. Let $\tilde{X}$ be a compact manifold with connected boundary $\partial\tilde{X}$. Suppose that the boundary has a fibration $\pi : \partial\tilde{X} \to Y$ with fibre L. Then the space $X = \tilde{X}/_\sim$, where $p \sim q \Leftrightarrow p, q \in \partial\tilde{X}$ with $\pi(p) = \pi(q)$, is a stratified pseudomanifold with strata $\mathrm{reg}(X) = \tilde{X} \setminus \partial\tilde{X}$ and Y. If $\partial\tilde{X}$ is not connected, then we will end up with different Y_i, $i = 1, \ldots, n$. The case where $Y_i = \{p_i\}$ for $i = 1, \ldots, n$ is exactly the case of a manifold with isolated singularities. If we allow strata of codimension 1 with $\pi = Id$, then we simply obtain a manifold with boundary. In this case, the control data for each fibration $\pi : \partial\tilde{X} \to Y$ with fibre L, are simply $T_Y = \partial\tilde{X} \times [0, 2)/_\sim$, $\pi_Y : T_Y \to Y$ defined by $\pi_Y([p, r]) = \pi(p)$ and $\rho_Y([p, r]) = r$. In this case $[p, r] \sim [q, s]$ iff $r = s = 0$ and $\pi(p) = \pi(q)$.

1.1.2 Resolution of Stratified Spaces

One reason that stratified spaces are difficult objects to work with from an analytical point of view, is that they don't have a differentiable structure that is compatible with the stratification. For example, let X be a compact manifold with an isolated singularity at $p \in X$ and U and open neighborhood around p. Then in local coordinates $(r, z) \in \phi(U)$ around p, there is not a clear meaning for a function to be smooth up to $r = 0$, even though $r = 0$ corresponds to p which belongs to X. This problem (among others) is resolved by introducing the notion of the resolution of a stratified space. In this section we discuss this notion and give precise definitions. The material here can be found in [ALMP12],[Joy12],[Mel96].

Definition 1.1.2. Let $\tilde{X}$ be a paracompact Hausdorff topological space. An $m-$dimensional chart with corners on $\tilde{X}$ is a pair (U,ϕ), with $U \subseteq \tilde{X}$, where ϕ is a homeomorphism between U and an open subset of $\mathbb{R}^k_+ \times \mathbb{R}^{m-k}$. (Here $\mathbb{R}_+ = [0,\infty)$). For notational simplicity we will set $\mathbb{R}^k_+ \times \mathbb{R}^{m-k} = \mathbb{R}^m_k$.

Let now $0 \in U' \subseteq \mathbb{R}^n$, $0 \in V' \subseteq \mathbb{R}^m$ open sets and $U = \mathbb{R}^n_{k_1} \cap U'$ and $V = \mathbb{R}^m_{k_2} \cap V'$. Then we say that $f : U \to V$ is smooth if it extends to a smooth function between open neighborhoods of $U,\ V$. We say that $f : U \to V$ is a diffeomorphism if it is a homeomorphism and $f,\ f^{-1}$ are smooth. Notice that for $k_1 = k_2 = 0$, we can simply take $U,\ V$ as open neighborhoods. In the case of boundaries (or corners), this notion of smoothness, implies smoothness up to the boundary (or corner) and vice versa (see Theorem 1.4.1 in [Mel96]). If $(U,\phi),\ (V,\psi)$ are $m-$dimensional charts on $\tilde{X}$ with corners, then we call them compatible if both $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ and $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ are diffeomorphisms in the above sense. Now we give the first definition of manifold with corners.

Definition 1.1.3. An $m-$dimensional manifold with corners is a paracompact Hausdorff topological space $\tilde{X}$, together with a collection $\{(U_i,\phi_i)\}_{i\in I}$ of $m-$dimensional corner charts such that:

- $\{(U_i,\phi_i)\}_{i\in I}$ is a cover of $\tilde{X}$.
- Each two charts are compatible and the collection $\{(U_i,\phi_i)\}_{i\in I}$ is the maximal collection with respect to this compatibility condition.

An important notion that comes together with the manifolds with corners, is the notion of the boundary face. Let $\tilde{X}$ be a manifold with corners. For $\lambda \leq k$, set

$$\partial_\lambda \mathbb{R}^m_k = \{x \in \mathbb{R}^m_k : x_i = 0 \text{ for exactly } \lambda \text{ of the first k indices}\}$$

Now, define

$$\partial_\lambda \tilde{X} = \{p \in M : \phi(p) \in \partial_\lambda \mathbb{R}^m_k \text{ for } p \text{ in a chart } (U,\phi)\}$$

So $\tilde{X}$ consists of the interior and the boundary part, which in turn consists of the different depth corners. We set

$$\partial^\lambda \tilde{X} = \overline{\partial_\lambda \tilde{X}} = \bigcup_{r\geq\lambda} \partial_r \tilde{X}$$

A boundary hypersurface is the closure of a connected component of $\partial_1 \tilde{X}$. A boundary face of codimension $k \geq 1$ is the closure of a connected component of $\partial_k \tilde{X}$. We denote the set of boundary hypersurfaces (resp. boundary faces) by $\mathcal{M}_1(M)$ (resp. $\mathcal{M}_k(M)$). The hypersurfaces will be denoted by $\{H_a\}_{a \in A}$ and the boundary faces by $H_\alpha = H_{a_1} \cap \cdots \cap H_{a_k}$ for $\alpha = (a_1, \ldots, a_k)$. From now on, whenever we refer to a manifold with corners we will assume that:

Each boundary face is an embedded submanifold.

This allow us to define boundary defining functions ρ_a for each hypersurface H_a, i.e. $\rho_a : \tilde{X} \to \mathbb{R}$ such that

$$\rho_a \geq 0,\ H_a = \{\rho_a = 0\},\ d\rho_a \neq 0 \text{ on } H_a.$$

Remark 1.1.1. We note here that the assumption about the boundary faces being embedded submanifolds is not an integral part of the definition of a manifold with corners. In [Joy12] this is not assumed, while in [Mel96] manifolds with corners satisfy this assumption, and manifolds satisfying Definition 1.1.3 are called t-manifolds. An example of a t-manifold that is not a manifold with corners is the teardrop (see [Gri17] page 11).

Definition 1.1.4. (see [ALMP12], [AM11]) An iterated fibration structure on the manifold with corners $\tilde{X}$ consists of the following data:

- Each H_a is the total space of a fibration $f_a : H_a \to Y_a$, where both the fibre L_a and base Y_a are themselves manifold with corners
- If two boundary hypersurfaces meet, i.e. $H_{ab} := H_a \cap H_b \neq \emptyset$, then $\dim L_a \neq \dim L_b$
- If $H_{ab} \neq \emptyset$ as above, and $\dim L_a < \dim L_b$, then the fibration of H_a restricts naturally to H_{ab} (i.e. the leaves of the fibration of H_a which intersect the corner lie entirely within the corner) to give a fibration of H_{ab} with fibres L_a, whereas the larger fibres L_b must be transverse to H_a at H_{ab}. Writing $\partial_a L_b$ for the boundaries of these fibres at the corner i.e. $\partial_a L_b := F_b \cap H_{ab}$, then H_{ab} is also the total space of a fibration with fibres $\partial_a L_b$. Finally, we assume that the fibres L_a at this corner are all contained in the fibres $\partial_a L_b$, and in fact that each fibre $\partial_a L_b$ is the total space of a fibration with fibres L_a.

The reason for introducing the notion of iterated fibration structure is that it reflects the structure of a stratified pseudomanifold in the language of manifolds with corners. Before stating a precise proposition that illustrates this connection, we show how it works in the simple case of stratified pseudomanifolds of depth 1. Suppose for simplicity that we have a manifold $\tilde{X}$ with boundary $\partial\tilde{X}$, together with a fibration $\pi : \partial\tilde{X} \to Y$ with fiber L. Then by collapsing each fiber L along Y to a point, we obtain a continuous map

$$\beta : \tilde{X} \to \tilde{X}/_{\sim},$$

which is the identity on $\tilde{X} \setminus \partial\tilde{X}$. This map β is called the blowdown map. The base of the fibration on the boundary of $\tilde{X}$ corresponds to a singular stratum of $X = \tilde{X}/_{\sim}$. More precisely, the following is true.

Proposition 1.1.1. If $\tilde{X}$ is a compact manifold with corners with an iterated fibration structure, then there is a stratified pseudomanifold X obtained from $\tilde{X}$ by successively blowing down the connected components of the fibers of each hypersurface boundary of $\tilde{X}$ in order of increasing fiber dimension. The corresponding blowdown map will be denoted by $\beta : \tilde{X} \to X$.

Proof. We demonstrate the idea in the case $\tilde{X}$ is a manifold with boundary $\partial\tilde{X}$. By the collar neighborhood theorem, we find an open neighborhood U of the boundary, such that it admits a product decomposition $U = \partial\tilde{X} \times [0, 2) = \{\rho < 2\}$, where $\rho : \tilde{X} \to [0, \infty)$ is a boundary defining function. The fibration $\tilde{\pi} : \partial\tilde{X} \to Y$ with fibre F extends to a fibration on $U = \partial\tilde{X}$ with fiber $F \times [0, 2)$. Then, by collapsing each fiber F to a point at $\rho = 0$ gives us a new space $\tilde{X}/_{\sim}$ and a fibration $\pi_Y : U/_{\sim} \to Y$ with fiber $C(F)$. $X/_{\sim}$ is a simple edge space and the control data of Y are simply $(T_Y = U/_{\sim}, \pi_Y, \rho)$. For a more detailed proof and a proof of the general case see Proposition 2.3 in [ALMP12] $\square$

The inverse process of the above, is to resolve a stratified pseudomanifold into a manifold with corners with an iterated fibration structure. One may do this by successively blowing up the strata in order of decreasing depth. As mentioned before, the reason for doing this is that from a differential point of view, manifolds with corners are better understood. This process of blowing up, leaves the regular part as it is, and roughly replaces the strata Y_i with boundary faces together with fibrations with base Y_i. More precisely, we have the following proposition:

Proposition 1.1.2. Let X be a stratified pseudomanifold. Then, there exists a manifold with corners with an iterated fibration structure $\tilde{X}$, and a blowdown map $\beta : \tilde{X} \to X$ with the following properties:

- There is a bijective correspondence $Y \leftrightarrow \tilde{X}_Y$ between the strata $Y \in \mathfrak{S}$ of X and the boundary hypersurfaces of $\tilde{X}$ which blow down to these strata.

- $\beta : \mathrm{int}(\tilde{X}) \to \mathrm{reg}(X)$ is a diffeomorphism.
- β is also a smooth fibration of the interior of each boundary hypersurface $\tilde{X}_Y$ with base the corresponding stratum Y and fibre the regular part of the link of Y in X. Moreover, there is a compactification of Y as a manifold with corners $\tilde{Y}$ such that the extension of β to all of $\tilde{X}_Y$ is a fibration with base $\tilde{Y}$ and fibre $\tilde{L}_Y$. Finally, each fiber $\tilde{L}_Y \subseteq \tilde{X}_Y$ is a manifold with corners with an iterated fibration structure and the restriction of β to it is the blowdown onto the stratified pseudomanifold $\overline{Y}$.

Proof. Again, we demonstrate the idea of the proof. Let X be a stratified space with a unique maximal stratum Y of depth k. Let (T_Y, π_Y, ρ) be the control data and define $S_Y = \rho^{-1}(1)$. S_Y is the total space of a fibration, with fiber L_Y and base Y. Essentially, we need to blow up Y at $\rho = 0$. In order to do this, define $R_Y : T_Y \setminus Y \to S_Y$ by $R_Y(y, z, r) = (y, z, 1)$. This is well defined, since we assume that each trivialization preserves the radial variable. Then define a new space

$$X_1 = ((X \setminus Y) \times \{1\}) \sqcup (S_Y \times (-2, 2)) \sqcup ((X \setminus Y) \times \{-1\})/_\sim,$$

where $(p, \varepsilon) \sim (R_Y(p), \rho(p))$ if $\varepsilon\rho(p) > 0$. Define $X' = (X \times \{1\}) \sqcup (X \times \{-1\})/_\sim$, where $(p, \varepsilon) \sim (p', \varepsilon')$ iff $p = p' \in Y$. Then we have that

$$X_1 \setminus (S_Y \times \{0\}) = X' \setminus Y.$$

If we choose $\tilde{X}$ to be the closure in X_1 of a connected component of $X_1 \setminus (S_Y \times \{0\})$, one can see that we actually obtain a manifold with corners of codimension at most $k - 1$, with an iterated fibration structure. For the details and the rest of the proof we refer to Proposition 2.5 in [ALMP12] □

Example 1.1.1. The simplest example is a manifold X with an isolated singularity at $p \in X$. It's resolution is a manifold with boundary $\tilde{X}$, together with a fibration $\pi : \partial\tilde{X} \to \{p\}$. The blowdown map $\beta : \tilde{X} \to X$ collapses the boundary to the point p and it is the identity on the interior.

Example 1.1.2. Similar to the above, in the case of simple edge spaces, the resolution space is again a manifold with boundary $\tilde{X}$, with a fibration $\pi : \partial\tilde{X} \to Y_i$. The blowdown map $\beta : \tilde{X} \to X$ collapses each fiber of π to a point and it is the identity on the interior. The case of manifolds with boundary corresponds to the fibration $\pi : \partial\tilde{X} \to \partial\tilde{X}$.

1.2 Geometry of Stratified Pseudomanifolds

1.2.1 Iterated Edge Metrics

So far we talked about the topology of a stratified pseudomanifold. However, we are ultimately interested in making analysis on them, and therefore we have to talk about Riemannian metrics. A Riemannian metric in this case is defined on the regular part of the stratified pseudomanifold. We will focus on the so called incomplete, iterated edge metrics. This class of metrics consists of Riemannian metrics, that when expressed in local coordinates take a specific form near the singular strata. Some of the material here can be found in [ALMP12] and [BG17]. We begin with the following

Definition 1.2.1. Let X be a stratified pseudomanifold and let g be a Riemannian metric on $\mathrm{reg}(X)$. If $\mathrm{depth}(X) = 0$, that is X is a smooth manifold, an iterated edge metric is understood to be any smooth Riemannian metric on X. Suppose now that $\mathrm{depth}(X) = k$ and that the definition of iterated edge metric is given in the case $\mathrm{depth}(X) \leq k-1$; then, we call a smooth Riemannian metric g on $\mathrm{reg}(X)$ an iterated edge metric if it satisfies the following properties:

- Let Y be a stratum of X such that $Y \subseteq X_i \setminus X_{i-1}$. For each $q \in Y$, there exists an open neighborhood V_Y of q in Y such that

  $$\phi : \pi_Y^{-1}(V_Y) \to V_Y \times C(L_Y)$$

  is a stratified isomorphism. In particular,

  $$\phi : \pi_Y^{-1}(V_Y) \cap \mathrm{reg}(X) \to V_Y \times \mathrm{reg}(C(L_Y))$$

  is a smooth diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, V_Y) such that g restricted on $\pi_Y^{-1}(V_Y) \cap \mathrm{reg}(X)$ satisfies

  $$(\phi^{-1})^*\big(g_{|_{\pi_Y^{-1}(V_Y)\cap \mathrm{reg}(X)}}\big) = dr^2 + h_{V_Y} + r^2 g_{L_Y} + k = g_0 + k,$$

  where h_{V_Y} is the restriction on V_Y of a Riemannian metric h_Y defined on Y. g_{L_Y} is a smooth family of bilinear tensors parametrized by $y \in Y$, that restricts to an iterated edge metric on $\mathrm{reg}(L_Y)$, and k is a $(0,2)-$tensor satisfying $|k|_{g_0} = O(r^\gamma)$ for some $\gamma > 0$, where $|\cdot|_{g_0}$ is the Frobenious norm.

Remark 1.2.1. As we will see, the condition that $|g - g_0|_{g_0} = O(r^\gamma)$ for some $\gamma > 0$ implies that g and g_0 are quasi-isometric. That is very helpful in a variety of situations, because g_0 is easier to handle. For example, they produce equivalent gradient norms, i.e. $\exists\, C > 0$ such that $1/C|\nabla^g u| \leq |\nabla^{g_0} u| \leq C|\nabla^g u|$.

The above definition is inductive. In order to understand better how an iterated edge metric looks like on a coordinate system, we work as follows: Take $q_1 \in Y_1$. By definition and Remark 1.2.1 there exists an open neighborhood $V_1 \subseteq Y_1$ of q_1 and an open neighborhood U_1 of Y_1, where the metric on U_1 takes the form $g = h_{Y_1} + dr_1^2 + r_1^2 g_{L_{Y_1}}$, where L_{Y_1} is a stratified space of depth $\leq k-1$. Pick $q_2 \in Y_2$, where Y_2 is a stratum of L_{Y_1}, and as before we find $q_2 \in V_2 \subseteq Y_2 \subseteq U_2$ such that on U_2, $g_{L_{Y_1}}$ takes the form $g_{L_{Y_1}} = h_{V_2} + dr_2^2 + r_2^2 g_{L_{Y_2}}$, where L_{Y_2} is a stratified space of depth $\leq k-2$. By continuing this procedure, we arrive at local neighborhoods of the form

$$\mathcal{U}_{q_1 \dots q_s} = V_1 \times C(V_2 \times C(V_3 \times \cdots \times V_{s-1} \times C(V_s))), \tag{1.2.1}$$

where the metric takes the form

$$h_1 + dr_1^2 + r_1^2(h_2 + dr_2^2 + r_2^2(h_3 + \cdots + r_{s-1}^2 h_s)). \tag{1.2.2}$$

We will use $y_1, r_1, \dots, y_{s-1}, r_{s-1}, y_s$ to denote the local coordinates of the covers (1.2.1) with the understanding that $y_1 = (y_1^1, \dots, y_1^{\dim Y_1}), \dots, y_s = (y_s^1, \dots, y_s^{\dim Y_s})$ which we will abbreviate in order not to overburden the notation. When the space has depth 1 or when we are not interested in the particular form that coordinates take on the stratified space L_{Y_1} we will use the symbols y, r, z instead of y_1, r, y_2.

Proposition 1.2.1. Let M be a manifold, $U \subseteq M$ open and g, g_0 Riemannian metrics on M. Suppose furthermore that there exists a function $r : M \to (0, +\infty)$ such that for each $\delta > 0$, the set $\{r \geq \delta\}$ is compact and $U = \{r < 2\}$. Let $g_{|U},\ g_{0|U},\ k : U \to T^*U \otimes T^*U$ be $(0,2)-$tensors, satisfying $g = g_0 + k$, with $|k|_{g_0} = O(r^\gamma)$. Then g and g_0 are quasi-isometric.

Proof. For $x \in U$, consider the inner product space

$$\Big(T^*U_x \otimes T^*U_x,\ g_{0_x} \otimes g_{0_x}\Big).$$

Since $k : TU_x \otimes TU_x \to \mathbb{R}$ is a linear operator, we have the inequality $\|k\|_{op} \leq \|k\|_2$ which implies that for small enough $r > 0$ with $r < \delta$, we can obtain

$$\|k\|_{op} < \varepsilon < 1.$$

By the definition of the operator norm $\|\cdot\|_{op}$, we obtain

$$|k(\frac{v}{|v|_{g_0}}, \frac{v}{|v|_{g_0}})| \leq \varepsilon \Rightarrow |k(v,v)| \leq \varepsilon g_0(v,v).$$

Thus, we have that

$$(1-\varepsilon)g_0 \leq g \leq (1+\varepsilon)g_0. \tag{1.2.3}$$

The part $\{r > \delta/2\}$ can be covered by a finite number of open sets U_i, where we have

$$\frac{1}{2}g_{0|U_i} \leq g_{|U_i} \leq 2g_{0|U_i}$$

because both metric g, g_0 are quasi-isometric with the Euclidean on U_i, and hence g and g_0 are quasi-isometric. Then by taking a partition of unity, we see that they are quasi-isometric in $\{r > \delta/2\}$. Combining that with (1.2.3) finishes the proof. □

It is straightforward to conclude, that on a compact stratified pseudomanifold X, if g and g_0 as above, then g and g_0 are quasi-isometric.

The existence of iterated edge metrics is discussed in [ALMP12]. More precisely we have:

Proposition 1.2.2. Let X be a compact stratified pseudomanifold. Then there exists an iterated edge metric on X.

Proof. See [ALMP12] Proposition 3.1. □

Example 1.2.1. Let X be a stratified pseudomanifold with an isolated singularity, $p \in U \subseteq X$ and $\phi : U \setminus \{p\} \to (0,\varepsilon) \times L$ a diffeomorphism. Then an iterated edge metric g on $\mathrm{reg}(X)$ satisfies

$$(\phi^{-1})^*(g_{|U}) = dr^2 + r^2 g_L + k = g_0 + k,$$

with $|k|_{g_0} = O(r^\gamma)$ for some $\gamma > 0$. The manifold $(\mathrm{reg}(X), g)$ is called manifold with conical singularity. Similarly, on a simple edge space, the metric g in local coordinates $\phi : U \setminus Y \to Y \times (0,\varepsilon) \times L$ takes the form

$$(\phi^{-1})^*(g_{|U}) = h_Y + dr^2 + r^2 g_L + k = g_0 + k,$$

where h_Y is a metric on Y, and g_L is a metric on L, depending smoothly on $y \in Y$.

1.2.2 Iterated Edge Vector Fields

Let now X be a compact stratified pseudomanifold with an iterated edge metric g. According to Proposition 1.1.2, there exists a compact manifold with corners $\tilde{X}$ with an iterated fibration structure, and a blowdown map

$$\beta : \tilde{X} \to X,$$

which is a diffeomorphism from $\mathrm{int}(\tilde{X})$ to $\mathrm{reg}(X)$. As before, denote by $\{H_a\}_{a\in A}$ the boundary hypersurfaces of $\tilde{X}$ and by x_a their boundary defining functions. Finally, set $\rho = \prod_{a\in A} x_a$ to be the total boundary defining function and define the metric $\tilde{g} = \frac{g}{\rho^2}$.

Proposition 1.2.3. The space $(\mathrm{int}(\tilde{X}), \tilde{g})$ is complete.

Proof. By Theorem 2 in [Gor73], completeness is equivalent to finding a function $f : \mathrm{int}(\tilde{X}) \to \mathbb{R}$ with bounded gradient. The function $f(p) = log(\rho(p))$ satisfies this assumption, since it is proper, smooth and with bounded gradient with respect to $\tilde{g}$. $\square$

We define now the space of iterated edge vector fields by

$$V_{ie}(X) = \{V \in C^\infty(\tilde{X}, T\tilde{X}) : \; q \to \tilde{g}_q(V, V) \text{ is bounded}\}.$$

In the local coordinates systems $\mathcal{U}_{q_1\dots q_s}$ in (1.2.1) these vector fields take the form

$$r_k \dots r_{s-1}\partial_{y_k}, r_k \dots r_{s-1}\partial_{r_k}, \partial_{y_s} \tag{1.2.4}$$

for $k = 1, \dots, s-1$. $V_{ie}(X)$ is a Lie algebra with respect to the usual bracket, and on $\mathrm{int}(\tilde{X})$, $T\tilde{X}$ is $C^\infty(\tilde{X})-$spanned by $V_{ie}(X)$. Apart from iterated edge vector fields, there is also another natural class of vector fields on $\mathrm{int}(\tilde{X})$ which we will call them incomplete iterated edge vector fields and we will denote them by $V_{iie}(X)$. They are defined by the relation

$$V_{iie}(X) = \rho^{-1} V_{ie}(X). \tag{1.2.5}$$

Even though they are related by a conformal factor, they don't share the same properties. One of the most important differences is that

- $V_{iie}(X)$ is not a Lie algebra. For example, if depth of $\tilde{X}$ is 1, the corresponding iie-vector fields are $\partial_y, \partial_r, \frac{\partial_z}{r}$ and we have $[\partial_r, \frac{\partial_z}{r}] = -\frac{\partial_z}{r^2}$ which is not a $C^\infty(\tilde{X})-$span of $V_{iie}(X)$.

Associated with the notion of iterated edge vector fields is the class of iterated edge differential operators. For $k \in \mathbb{N}$, we define

$$\mathrm{Diff}^k_{ie}(X) = \Big\{ \sum_{\alpha_1,\dots,\alpha_i} a_{\alpha_1,\dots,\alpha_i} X_{a_1} \dots X_{a_i} : \ \alpha_1 + \dots + \alpha_i \leq k, \\ a_{\alpha_1,\dots,\alpha_i} \in C^\infty(\tilde{X}), \ X_{\alpha_j} \in V_{ie}(X) \Big\}.$$

Similarly as before, we define the class of incomplete iterated edge differential operators by the relation

$$\mathrm{Diff}^k_{iie}(X) = \rho^{-k} \mathrm{Diff}^k_{ie}(X).$$

The reason for introducing the notion of iterated edge vector fields, iterated edge differential operators and their incomplete counterparts is that the geometric operators associated to g are described in terms of these vector fields and their repetitive application. For example the (positive) Laplace operator on functions has the local description

$$\Delta = -\sum_{ij} \frac{1}{\sqrt{\det(g)}} \partial_i \big(g^{ij} \sqrt{\det(g)} \partial_j \big).$$

where ∂_i, ∂_j differentiate along the x_i, x_j respectively with respect to a local chart $\phi(p) = (x_1(p), \dots, x_m(p))$ and g^{ij} is the inverse of the metric g in local coordinates. If we take now an iterated edge metric g and assume that $g = g_0$, then the Laplace operator in spaces of depth 1 near the singular stratum where $g = g_0 = h + dr^2 + r^2 g_L(y)$ with $\dim(L) = f$, takes the form:

$$\Delta = -\partial_r^2 - \frac{f}{r}\partial_r + \frac{\Delta_{g_L(y)}}{r^2} + \Delta_h$$

It is clear that $\Delta \in \mathrm{Diff}^2_{iie}(X)$. Similarly, in higher depths, $\Delta \in \mathrm{Diff}^2_{iie}(X)$. In the case where $|g - g_0|_{g_0} = |k|_{g_0} = O(r^\gamma)$ the Laplacian takes the form:

$$\Delta_g = \Delta_{g_0} + R,$$

where $R \in \rho^\gamma \mathrm{Diff}^2_{iie}(X)$. The proof is rather complicated and we give the details on the Appendix (Proposition A.0.1).

1.3 Weighted Sobolev Spaces

In this section we will discuss the notion of iterated edge Sobolev spaces, i.e. Sobolev spaces that are defined in terms of iterated edge vector fields. For

$1 \le p < \infty$ and $k \in \mathbb{N}$, we define

$$H_{ie}^{k,p}(X) = \big\{u \in L^p(X, \mathrm{dvol}(g)) : X_1 \dots X_l u \in L^p(X, \mathrm{dvol}(g)) \\ \text{for } l \le k,\ X_i \in V_{ie}(X)\big\},$$

where $X_1 \dots X_l u$ are applied in the distributional sense. The norm on $H_{ie}^k(X)$ is denoted by $\| \cdot \|_{ie,k,p}$ and it is defined by

$$\|u\|_{ie,k,p} = \big(\sum_{l=0}^{k} \sum_{i_1,\dots,i_l} \|X_{i_1} \dots X_{i_l} u\|_{L^p}^p \big)^{\frac{1}{p}}$$

where the sum is taken over all the possible combinations of indices. Similarly, for $p = \infty$, we obtain the space $H_{ie}^{k,\infty}(X)$ with the norm

$$\|u\|_{ie,k,\infty} = \max_{l=0,\dots,k} \|X_{i_1} \dots X_{i_l} u\|_\infty$$

where $i_1, \dots, i_l$ run over all the possible combinations of indices. We can also define weighted versions of $H_{ie}^{k,p}$. Take ρ to be the total boundary defining function and $\delta > 0$. Then we define

$$\rho^\delta H_{ie}^{k,p}(X) = \{\rho^\delta u :\ u \in H_{ie}^{k,p}(X)\},$$

with norm

$$\|u\|_{ie,k,p,\delta} = \|\rho^{-\delta} u\|_{ie,k,p}$$

for $u \in \rho^\delta H_{ie}^{k,p}(X)$.

Remark 1.3.1. If $U \subseteq \tilde{X}$ has a positive distance from the boundary, i.e. $\exists\, C > 0$ such that $\rho_a(p) > C$ for $a \in A$ and $p \in U$ (here A is the index set for each hypersurface $\{H_a\}_{a \in A}$), then the iterated edge vector fields are ordinary vector fields, and therefore the norm $\| \cdot \|_{ie,k,p,\delta}$ is equivalent to the usual Sobolev norm, defined by taking ordinary derivatives. That means, $\exists\, C > 0$ which depends on $U, \tilde{X}, k, p, \delta$, such that

$$\frac{1}{C}\|u\|_{k,p} \le \|u\|_{ie,k,p,\delta} \le C\|u\|_{k,p}.$$

Let now $\psi : \mathbb{R} \to [0,1]$ be a function, such that it is 1 for $|x| \ge 2$ and 0 for $|x| \le 1$. We define the cutoffs $\psi_a = \psi \circ \rho_a$ for $a \in A$ and $\lambda = \prod_{a \in A} \psi_a$. For $\varepsilon > 0$, we define $\lambda_\varepsilon(\cdot) = \lambda(\frac{\cdot}{\varepsilon})$ which converges pointwise to 1 as $\varepsilon \to 0$.

Then the set $A_n = \lambda_{\frac{1}{n}}^{-1}(1)$ for $n \in \mathbb{N}$ is an exhaustion sequence of $\tilde{X}$, with the property that

$$\partial_{r_1}^{\beta_1} \dots \partial_{r_k}^{\beta_k} \big(\lambda_{\frac{1}{n}}(r_1, \dots, r_k)\big) \leq C \frac{1}{r_1^{\beta_1}} \dots \frac{1}{r_k^{\beta_k}}. \tag{1.3.1}$$

with $\beta = (\beta_1, \dots, \beta_k)$ and $C = C(|\beta|)$, near a neighborhood of the form $\mathbb{R}^n_k$. Now, if $p \in \tilde{X}$ belongs in a corner of codimension $k \in \mathbb{N}$, i.e. $p \in H_{a_1} \cap \cdots \cap H_{a_k}$, then there exists a chart (U, ϕ) around p, $V \subseteq H_{a_1} \cap \cdots \cap H_{a_k}$ relatively open together with a chart $\phi_k : V \to \phi_k(V) \subseteq \mathbb{R}^{n-k}$ and a positive $\varepsilon > 0$ such that $\phi(p) \in \phi(U) = [0, \varepsilon)^k \times \phi_k(V)$ and $\phi(p) = (0, \dots, 0, \phi_k(p))$. Take the function $\lambda_k : \phi_k(V) \to [0, 1]$ which is 1 near $\phi_k(p)$ and extend it to $[0, \varepsilon)^k \times \phi_k(V)$ by simply setting $\lambda_k(r_1, \dots, r_k, y) = \lambda_k(y)$. Then the function $\chi_n := (1 - \lambda_{\frac{1}{n}}) \cdot (\lambda_k \circ \phi)$ is 1 in a neighborhood of p, with the $r_1, \dots, r_k-$derivatives behaving like (1.3.1) and the $y-$derivatives being bounded. Since now $\tilde{X}$ is compact, we can find a finite cover $\{U_1, \dots, U_d\}$ and a subordinate partition of unity $\{\chi^1, \dots, \chi^d\}$, associated to this cover, which has the property that remains bounded, under application of iterated edge vector fields, i.e. for all $k \in \mathbb{N}$ there exists $C_k > 0$, such that for $j = 1, \dots, d$ we have $|X_1 \dots X_k(\chi^j)| \leq C_k$ where $X_1, \dots, X_k \in V_{ie}(X)$.

Proposition 1.3.1. Let $\tilde{X}$ be a compact manifold with corners with an iterated fibration structure. Let $p \in [1, \infty)$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$. Then $C_c^\infty(\mathrm{int}(\tilde{X}))$ is dense in $\rho^\delta H^{k,p}_{ie}(\tilde{X})$ with respect to the norm $\| \cdot \|_{ie,k,p,\delta}$.

Proof. Let $u \in \rho^\delta H^{k,p}_{ie}(\tilde{X})$. For $n \in \mathbb{N}$ consider $u - \lambda_{\frac{1}{n}} u$, where $\lambda_{\frac{1}{n}}$ as above. Then we have $u - \lambda_{\frac{1}{n}} u = \sum_{j=1}^d (1 - \lambda_{\frac{1}{n}}) \chi^j u$ and it suffices to estimate the term

$$\|(1 - \lambda_{\frac{1}{n}}) \chi^j u\|_{ie,k,p,\delta}. \tag{1.3.2}$$

By applying Leibniz rule we end up with terms of the form

$$\|X_1 \dots X_i (1 - \lambda_{\frac{1}{n}}) X_{i+1} \dots X_{i+\mu}(\chi^j) X_{i+\mu+1} \dots X_\nu(\rho^{-\delta} u)\|_{L^p}.$$

with $\nu \leq k$, $X_1 \dots X_\nu \in V_{ie}(X)$. The terms $X_1 \dots X_i(1 - \lambda_{\frac{1}{n}})$ and $X_{i+1} \dots X_{i+\mu}(\chi^j)$ are bounded by constants depending on i, μ and the first term converges to 0 as $n \to \infty$. The last term belongs to $L^p(X)$, thus by using Lebesgue's Dominated Convergence theorem we conclude that the term (1.3.2) converges to 0. $\lambda_{\frac{1}{n}} \rho^{-\delta} u$ has compact support and by using Remark 1.3.1, since it belongs to the ordinary Sobolev space $H^{k,p}$, we can approximate it by a smooth function that has compact support, and that concludes the proof. $\square$

Proposition 1.3.2. Let $\tilde{X}$ be a compact manifold with corners of dimension $m > 1$ with an iterated fibration structure. Let $p \in [1, m)$, $k, k' \in \mathbb{N}$ and $\delta, \delta' \in \mathbb{R}$, with $k' > k$, $\delta' > \delta$. Then

$$\rho^{\delta'} H_{ie}^{k',p}(X) \hookrightarrow \rho^{\delta} H_{ie}^{k,p}(X)$$

is a compact embedding.

Proof. Let $M > 0$ such that $\forall\ l \in \mathbb{N}$ we have $\|u_l\|_{ie,k',p,\delta'} \leq M$. For $n \in \mathbb{N}$, pick the exhaustion $A_n = \lambda_{\frac{1}{n}}^{-1}(1)$ as above. Then write $u_l - u_{l'} = (1-\lambda_{\frac{1}{n}})(u_l - u_{l'}) + \lambda_{\frac{1}{n}}(u_l - u_{l'}) = \sum_j \left(\chi^j(1-\lambda_{\frac{1}{n}})(u_l - u_{l'})\right) \circ \phi_j^{-1} + \sum_j \left(\chi^j \lambda_{\frac{1}{n}}(u_l - u_{l'})\right) \circ \phi_j^{-1}$. Concerning the first term we have

$$\begin{aligned} &\|\chi^j(1-\lambda_{\frac{1}{n}})(u_l - u_{l'}) \circ \phi_j^{-1}\|_{ie,k,p,\delta} \\ &= \|\rho^{\delta'-\delta}\chi^j(1-\lambda_{\frac{1}{n}})(\rho^{-\delta'}u_l - \rho^{-\delta'}u_{l'}) \circ \phi_j^{-1}\|_{ie,k,p}. \end{aligned} \tag{1.3.3}$$

Notice that on a compact manifold with corners, boundary defining functions are bounded and therefore we have $|X_1 \ldots X_i(\rho^{\delta})| \leq C\rho^{\delta}$ for $X_1, \ldots, X_i \in V_{ie}(X)$ where $C > 0$ depends on $\tilde{X}, i, \delta$. Note also that $X_1 \ldots X_i(\chi^j)$ and $X_1 \ldots X_i(1-\lambda_{\frac{1}{n}})$ are bounded, and since by assumption $\rho^{-\delta'}u_l$ are uniformly bounded in the $\|\cdot\|_{ie,k,p}$ norm (because $k < k'$), then we obtain that (1.3.3) converges to 0 as $n \to \infty$, independently of l, l'. Pick now successively $\varepsilon = 1, \frac{1}{2}, \ldots, \frac{1}{N}, \ldots$. For $\varepsilon = \frac{1}{N}$ there exists $n_N \in \mathbb{N}$ such that, for each $j = 1, \ldots, d$ we have

- $\forall n, l, l' \geq n_N$: $\|\chi^j(1-\lambda_{\frac{1}{n}})(u_l - u_{l'})\|_{ie,k,p,\delta} < \frac{1}{N}$ (by above)

Pick now $n = N$ and apply the classical Rellich-Kondrachov theorem on a compact manifold $\Omega_N \subseteq \tilde{X}$ with smooth boundary, that contains $\operatorname{supp}(\lambda_{\frac{1}{N}})$ and has a positive distance from the total boundary, which we can do due to Remark 1.3.1. Then there exists $n'_N \in \mathbb{N}$ and a subsequence of $\{u_l\}$ which we will denote by $\{u_{l_N}\}_{l \in \mathbb{N}}$, such that

- $\forall l_N, l'_N \geq n'_N$: $\|\chi^j \lambda_{\frac{1}{N}}(u_l - u_{l'})\|_{ie,k,p,\delta} < \frac{1}{N}$.

Apply this argument for $N+1$ on the sequence $\{u_{l_N}\}_{l \in \mathbb{N}}$ and so on. Then by picking one term from each chosen subsequence, we obtain the sequence $\{u_{l_1}, u_{l_2}, \ldots, u_{l_N}, \ldots\}$ which is Cauchy in $\rho^{\delta} H_{ie}^{k,p}(X)$ and thus we are done. $\square$

Chapter 2

Sobolev Spaces on Simple Edge Spaces

In this chapter we are mainly focusing on Sobolev spaces on compact stratified pseudomanifolds of depth 1, namely simple edge spaces. Our main aim is to prove the validity of the classical functional inequalities on these spaces, namely Sobolev inequality, Poincare inequality and Sobolev-Poincare inequality. In addition we develop some tools in order to examine the optimality of the constants in the Sobolev inequality.

2.1 Sobolev Spaces

In this section, we recall some basic facts and definitions about Sobolev spaces on open manifolds.

Let (M, g) be an open manifold with metric g. We say that f is equivalent to g $(f \sim g)$, if and only if $f(x) = g(x)$ almost everywhere with respect to the measure μ coming from the Riemannian structure. Then for $p \in [1, \infty)$, we denote by $L^p(M) = L^p$ the space of the equivalence classes of measurable functions $f : M \to \mathbb{C}$, such that

$$\|f\|_{L^p} = \Big(\int_M |f|^p dvol_g \Big)^{\frac{1}{p}} < \infty.$$

For $p = \infty$, we define $L^\infty(M)$ as the space of the equivalence classes of measurable functions $f : M \to \mathbb{C}$, such that

$$\|f\|_{L^\infty} = \operatorname*{ess\,sup}_M |f(x)| < \infty.$$

For $p \in (1, \infty)$, L^p is a reflexive Banach space, and for $p = 2$, $L^2(M)$ is a Hilbert space with inner product

$$< f, g >= \int_M f\overline{g} dvol_g.$$

For $k \in \mathbb{N}$ and $p \in [1, \infty)$ we define

$$W^{k,p}(M) = \big\{ u : M \to \mathbb{C} : \exists\ \nabla^i u \text{ distributionally,} \\ \text{and } \nabla^i u \in L^p(M, T^*M^{\otimes i}) \text{ for } i = 0, 1, \dots, k \big\}$$

with norm

$$\|u\|_{W^{k,p}} = \big(\sum_{i=0}^{k} \int_M |\nabla^i u|^p_{T^*M^{\otimes i}} dvol_g \big)^{\frac{1}{p}}.$$

By adopting the Einstein summation, in local coordinates we have that

$$|\nabla^i u|^p_{T^*M^{\otimes i}} := \big(g_{T^*M^{\otimes i}}(\nabla^i u, \nabla^i u) \big)^{\frac{p}{2}} = \big(g^{\mu_1 \nu_1} \dots g^{\mu_i \nu_i} (\nabla^i u)_{\mu_1 \dots \mu_i} (\nabla^i u)_{\nu_1 \dots \nu_i} \big)^{\frac{p}{2}}.$$

For example, $(\nabla u)_\mu = \frac{\partial u}{\partial \mu}$ and $(\nabla^2 u)_{\mu\nu} = \frac{\partial^2 u}{\partial \mu \partial \nu} - \Gamma^k_{\mu\nu} \frac{\partial u}{\partial k}$. Moreover, we define

$$W_0^{k,p}(M) = \overline{C_c^\infty(M)}^{\|.\|_{W^{k,p}}},$$

i.e., the completion of smooth, compactly supported functions on M with respect to the norm $\|.\|_{W^{k,p}}$. Concerning the Sobolev space $W^{k,p}(M)$, we have the following Meyers-Serrin type theorem

Proposition 2.1.1. Let (M, g) be an open manifold and let $W^{k,p}(M)$ be defined as above. Then the space $W^{k,p}(M) \cap C^\infty(M)$ is dense in $W^{k,p}(M)$ with respect to the norm $\| \cdot \|_{W^{k,p}}$.

Proof. See Theorem 2.9 in [GGP17]. □

2.2 Construction of Cut-Off Functions

Now let X be a compact stratified pseudomanifold of arbitrary depth. In this section we show how to obtain for this space sequences of cut-off functions. We begin by giving a precise

Definition 2.2.1. Let (M, g) be an open manifold, $p \in [1, \infty)$ and $k \in \mathbb{N}$ and let $\{\chi_n\} \subseteq C_c^\infty(M)$. We call $\{\chi_n\}$ a sequence of $(k, p)-$cut-offs if the following properties hold:

- $\forall n \in \mathbb{N}$ we have $0 \le \chi_n \le 1$.
- For every $K \subseteq M$ compact, $\exists\ n_0 \in \mathbb{N}$ such that, $\forall n \ge n_0$ we have $\chi_n|_K = 1$.
- $\forall j = 1, \dots, k : \int_M |\nabla^j \chi_n|^p_{T \otimes^j M} d\mu_g \to 0$, as $n \to \infty$.

In this section we will prove the existence of $(k,p)-$cut-off functions on stratified pseudomanifolds for $k = 1$ and $k = 2$ under some assumptions on p and the iterated edge metric g of $\mathrm{reg}(X)$. But, before doing so, we need some preliminary lemma's.

Lemma 2.2.1. Let X be a stratified pseudomanifold of dimension m, with an iterated edge metric g_0, that near each singular stratum Y, under the trivialisation ϕ as in Definition 1.2.1, takes the form

$$g_0 = h_{V_Y} + dr^2 + r^2 g_{L_Y}$$

where g_{L_Y} is a tensor parametrized by $y \in Y$ such that for each $y \in Y$ it restricts on an iterated edge metric $g_{L_Y}(y)$ on L_Y. Then the Christoffel symbols Γ^k_{ij} in coordinates r, y, z take the form

- For $k = r$.

$$\begin{aligned} \Gamma^r_{rr} &= 0, & \Gamma^r_{ry} &= 0, & \Gamma^r_{rz} &= 0, \\ \Gamma^r_{yy'} &= 0, & \Gamma^r_{yz} &= 0, & \Gamma^r_{zz'} &= -r g_{L,zz'}. \end{aligned}$$

- For $k = z$.

$$\begin{aligned} \Gamma^z_{rr} &= 0, & \Gamma^z_{ry} &= 0, & \Gamma^z_{rz'} &= \frac{\delta^z_{z'}}{r}, \\ \Gamma^z_{yy'} &= 0, & \Gamma^z_{yz'} &= \frac{1}{2}\sum_{\tilde{z}} (\partial_y g_{z'\tilde{z}} g^{z\tilde{z}}), & \Gamma^z_{\tilde{z}z'} &= \Gamma^z_{\tilde{z}z'}(g_L). \end{aligned}$$

- For $k = y$

$$\begin{aligned} \Gamma^y_{rr} &= 0, & \Gamma^y_{ry'} &= 0, & \Gamma^y_{rz} &= 0, \\ \Gamma^y_{\tilde{y}y'} &= \Gamma^y_{\tilde{y}y'}(h_{V_Y}), & \Gamma^y_{y'z} &= 0, & \Gamma^y_{zz'} &= -\frac{1}{2}\sum_{y'} (\partial_{y'} g_{zz'}) h^{y'y}. \end{aligned}$$

Proof. The metric g_0 in a local neighborhood of a singular stratum Y is $g_0 = dr^2 + h_{V_Y} + r^2 g_{L_Y}(y)$. Then the proposition is obtained by using the formula $\Gamma^k_{ij} = \frac{1}{2}\sum_m (\partial_i g_{mj} + \partial_j g_{mi} - \partial_m g_{ij}) g^{km}$. $\square$

We will also need the following lemma:

Lemma 2.2.2. Let X and g_0 as before and let $u : \mathrm{reg}(X) \to \mathbb{R}$, that near a singular stratum Y in local coordinates r, y, z, is a function of either r, y or z. Then the norm of the second order covariant derivative of u, namely $|\nabla^2 u|_{T^* \otimes T^*}$ takes the form

- If $u = u(r)$, then

$$|\nabla^2 u|^2 = |\partial_r^2 u|^2 + m \frac{|\partial_r u|^2}{r^2}.$$

- If $u = u(z)$, then

$$\begin{aligned}|\nabla^2 u|^2 = &\frac{|(\nabla^L)^2 u|^2}{r^4} + 2\frac{|\nabla^L u|^2}{r^4} \\ &+ 2 \sum_{z,z',y,y'} g^{zz'} h^{yy'} \Big[\frac{1}{2} \sum_{z_1,\tilde{z}} (\partial_y g_{z_1 z}) g^{\tilde{z} z_1} \frac{\partial u}{\partial \tilde{z}}\Big] \Big[\frac{1}{2} \sum_{z_2,\tilde{z}'} \partial_{y'} (g_{z' z_2}) g^{\tilde{z}' z_2} \frac{\partial u}{\partial \tilde{z}'}\Big].\end{aligned}$$

- If $u = u(y)$, then

$$\begin{aligned}|\nabla^2 u|^2 = &|(\nabla^Y)^2 u|^2 \\ &+ \sum_{z,\tilde{z},z',\tilde{z}'} g^{z\tilde{z}} g^{z'\tilde{z}'} \Big[\sum_{y,\tilde{y}} (\partial_{\tilde{y}} g_{zz'}) h^{\tilde{y}y} \frac{\partial u}{\partial y}\Big] \Big[\sum_{y',\tilde{y}'} (\partial_{\tilde{y}'} g_{\tilde{z}\tilde{z}'}) h^{\tilde{y}'y'} \frac{\partial u}{\partial y'}\Big].\end{aligned}$$

Proof. The proof makes use of the formula

$$|\nabla^2 u|^2 = \sum_{i,j,k,l} g^{ik} g^{jl} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_c \Gamma_{ij}^c \frac{\partial u}{\partial x_c}\Big)\Big(\frac{\partial^2 u}{\partial x_k \partial x_l} - \sum_d \Gamma_{kl}^d \frac{\partial u}{\partial x_d}\Big) \tag{2.2.1}$$

where $(x_1, \ldots, x_m)$ are local coordinates. Now, by using the fact that the metric g_0 is the direct sum of a warped product metric and another metric, we see that terms of the form g^{ij} where either $i = r, j \in \{y_1, \ldots, y_{\dim Y}\}$, either $i = r, j \in \{z_1, \ldots, z_{\dim L}\}$ or $i \in \{y_1, \ldots, y_{\dim Y}\}, j \in \{z_1, \ldots, z_{\dim L}\}$ are cancelled. That allow us to consider only the cases when $i, k \in \{r, y, z\}$ and $j, l \in \{r, y, z\}$, which due to symmetry are only 6 cases. Then the proof consists of distinguishing the cases $u = u(r), u(z), u(y)$ and using Lemma 2.2.1 on the formula (2.2.1). $\square$

The reason for employing Lemma 2.2.2 is that some constructions in this subsection will be of product type near the singular area and we would like to know how the first and second order covariant derivative behaves. A first application of this consideration allow us to obtain

Proposition 2.2.1. Let X be a stratified pseudomanifold with metric g_0 and $\{U_\alpha\}_{\alpha\in A}$ an open cover. Then, there exists a subordinated partition of unity λ_α such that

- $\mathrm{supp}(\lambda_\alpha)\subseteq U_\alpha$.
- $\sum_\alpha \lambda_\alpha = 1$.
- $\exists\, C_\alpha > 0$ such that for each $\alpha \in A$: $|\nabla\lambda_\alpha|,\ |\nabla^2\lambda_\alpha| \leq C_\alpha$.

Proof. Clearly, it suffices to see what happens near the singular strata. If $p \in \mathrm{reg}(X)$ then set ψ_a to be a function that is 1 in a neighborhood U_a of p. If $p \in Y$, then each open set that contains p is of the form $V_Y \times C(L_Y)$, with V_Y an open neighborhood of p in Y. Pick now $\lambda \in C_c^\infty([0,2))$ such that $\lambda = 1$ near 0, and $\psi \in C_c^\infty(V_Y)$ that is 1 near the point $p \in V_Y$. Then define $\psi_{V_Y}(y,r,z) = \lambda(r)\psi(y)$. We easily see that

$$|\nabla\psi_{V_Y}| \leq |\lambda'(r)\psi(y)| + |\lambda(r)||\nabla\psi(y)| \leq C.$$

Concerning the second order differential, we see that

$$|\nabla^2(\lambda(r)\psi(y)| \leq |\nabla^2\lambda(r)||\psi(y)| + 2|\nabla\lambda(r)||\nabla\psi(y)| + |\lambda(r)||\nabla^2\psi(y)|.$$

We see that according to our construction and Lemma 2.2.2, that every term can be bounded by a constant, apart from the problematic term

$$\sum_{z,\tilde{z},z',\tilde{z'}} g^{z\tilde{z}}g^{z'\tilde{z'}}\Big[\sum_{y,\tilde{y}}(\partial_{\tilde{y}}g_{zz'})h^{\tilde{y}y}\frac{\partial u}{\partial y}\Big]\Big[\sum_{y',\tilde{y'}}(\partial_{\tilde{y'}}g_{\tilde{z}\tilde{z'}})h^{\tilde{y'}y'}\frac{\partial u}{\partial y'}\Big]. \tag{2.2.2}$$

This term needs a closer examination. Recall that $g = h_Y + dr^2 + r^2 g_{L_Y}$, where g_{L_Y} is an iterated edge metric of lower depth. If we want to bound (2.2.2) we just have to check the terms

$$g^{z\tilde{z}}g^{z'\tilde{z'}}(\partial_{\tilde{y}}g_{zz'})(\partial_{\tilde{y'}}g_{\tilde{z}\tilde{z'}}). \tag{2.2.3}$$

because $h^{\tilde{y}y}$ and $h^{\tilde{y'}\tilde{y}}$ are bounded due to the compactness of Y. Recall, that $g^{zz'} = \frac{g_L^{zz'}}{r^2} = \frac{g_L(dz,dz')}{r^2}$ and $g_{zz'} = r^2 g_{Lzz'}$. We see now that the r's cancel out and we are left with the terms

$$g_L^{z\tilde{z}}g_L^{z'\tilde{z'}}(\partial_{\tilde{y}}g_{Lzz'})(\partial_{\tilde{y'}}g_{L\tilde{z}\tilde{z'}}).$$

If L_Y is a compact manifold, then these terms are bounded, due to the compactness of L_Y, and thus (2.2.2) is bounded by a constant. If L_Y has higher depth, then we need a further analysis. In this case we have that in a local neighborhood of L_Y, $g_L = h_{Y_2} + dr_2^2 + r_2^2 g_{L_2}(y_2, y)$. We write $z \in \{r_2, y_2^1, \dots, y_2^{\dim Y_2}, y_3^1, \dots, y_3^{\dim L_2}\}$ and see that $g_L^{zz'}$ is either bounded (if $z, z' \in \{r_2, y_2^1, \dots, y_2^{\dim Y_2}\}$) or $g_L^{y_3, y_3'} = \frac{g_{L_2}^{y_3 y_3'}}{r_2^2}$. Since g_L is of product type, the only case of interest is when $z, z', \tilde{z}, \tilde{z'} \in \{y_3^1, \dots, y_3^{\dim L_2}\}$. But then again as above the r_2's cancel and we obtain

$$g_{L_2}^{z\tilde{z}} g_{L_2}^{z'\tilde{z'}} (\partial_{\tilde{y}} g_{L_2 zz'})(\partial_{\tilde{y'}} g_{L_2 \tilde{z}\tilde{z'}}).$$

If now L_2 is a compact manifold, this is bounded. If it is stratified space, we follow the same procedure and see that eventually (2.2.2) is bounded. That proves the fact that there exists $C > 0$ such that

$$|\nabla \psi_{V_Y}|,\ |\nabla^2 \psi_{V_Y}| \leq C.$$

Since we have this, by taking $\{U_\alpha\}$ an open cover of X, wlog we can assume that each U_α comes with a local coordinate chart ϕ_α which if it intersects a singular stratum Y, then near this stratum maps U_α into $V_{Y\alpha} \times C(L_Y)$. Then define simple $\psi_\alpha = \psi_{Y_{V_a}} \circ \phi_\alpha^{-1}$ and finally

$$\rho_\alpha = \frac{\psi_\alpha}{\sum_\beta \psi_\beta}$$

Then $\{\rho_a\}_{\alpha \in A}$ satisfies all the required properties. $\square$

We have seen that on a local neighborhood of a singular stratum Y, where the metric takes the form $g_0 = h_Y + dr^2 + r^2 g_L(y)$, an important role is played by the $y-$derivatives of the metric $g_{L,y}$. For this reason it is reasonable to form the following assumption, which we will state precisely when we use it:

Assumption 2.2.1. Let X be a stratified pseudomanifold of dimension m. Let g be an iterated edge metric, that near each singular stratum Y takes the form $g = g_0 + k$ with $|k|_{g_0}$, $|\nabla^g(k)|_{g_0} = O(r^\gamma)$ for some $\gamma > 0$, and that $g_0 = h_Y + dr^2 + r^2 g_{L_Y}$ with $g_{L_Y} : Y \to T^*L_Y \otimes T^*L_Y$ a smooth tensor. Then we assume that

$$g_{L_Y} \textit{ is independent of } y \in Y.$$

With this assumption we obtain that g_{L_Y} is constant along $y \in Y$ and therefore each $y-$derivative vanishes. The assumption $|\nabla(g_0)|_{g_0} = O(r^\gamma)$ gives the equivalence of the second order Sobolev space defined by g and g_0 (see Lemma 3.4 in [Pac13]). Therefore, when we consider the metric g instead of g_0, Lemma 2.2.2 is true up to a constant and Proposition 2.2.1 is true as it is. Now we are able to state the first main result of this section.

Theorem 2.2.1. Let X be a compact stratified pseudomanifold of dimension m, endowed with an iterated edge metric on $\text{reg}(X)$ and let $k = 1$ or $k = 2$. Suppose that for every singular stratum Y of X we have the condition

$$\text{codim}(Y) = m - i > kp, \text{ where } i = \dim(Y). \tag{2.2.4}$$

Then

- For $k = 1$, if $\text{depth}(X) = 1$, or the strata Y with $\text{depth}(Y) > 1$ satisfy Assumption 2.2.1, then X admits a sequence of $(1, p)-$cut-offs.
- For $k = 2$, if $\text{depth}(X) = 1$, or the strata Y with $\text{depth}(Y) > 1$ satisfy Assumption 2.2.1, then X admits a sequence of $(2, p)-$cut-offs.

Remark 2.2.1. This theorem for $k = 1$ is Theorem 3.4 in [BG17]. Here we will repeat and expand their argument, in order to cover also the case $k = 2$.

Before giving the proof of this theorem, we state and prove the following proposition, which is fundamental for our construction.

Proposition 2.2.2. Let $p \in [1, \infty)$, $k \in \mathbb{N}$ and $m - i > kp$ for some integers $m > i \geq 0$. Then, there exists a sequence of functions $\{g_n\} \subseteq C_c^\infty\big((0, 2]\big)$, $n \in \mathbb{N}$ with the following properties

- $\forall n \in \mathbb{N}$ we have $0 \leq g_n \leq 1$.
- For every $K \subset\subset (0, 2]$, $\exists\ n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ we have $g_n|_K = 1$.
- For $j = 1, 2$ we have that $\int_0^2 |g_n^{(j)}(r)|^p r^{m-i-1} dr \to 0$, as $n \to \infty$.

Proof. Let $\phi \in C_c^\infty\big((0, 2]\big)$ such that $0 \leq \phi \leq 1$, $\phi(r) = 0$ for $0 \leq r \leq 1$ and $\phi(r) = 1$ for $3/2 \leq r \leq 1$. Then for $n \in \mathbb{N}$ define $g_n : (0, 2] \to \mathbb{R}$ by $g_n(r) = \phi(nr)$. It is straightforward to verify the first two properties. For $1 \leq j \leq k$ we have

$$\begin{aligned}
\int_0^2 |g_n^{(j)}(r)|^p r^{m-i-1} dr &= \int_0^2 |\phi^{(j)}(nr)|^p n^{jp} r^{m-i-1} dr \\
&\leq C_j \int_{1/n}^{3/2n} n^{jp} r^{m-i-1} dr \\
&\leq \frac{C_j}{m-i} \Big(\big(\frac{3}{2}\big)^{m-i} - 1 \Big) n^{jp} \big(\frac{1}{n}\big)^{m-i}.
\end{aligned}$$

Since $m - i > kp \geq jp$, this converges to 0 as $n \to \infty$.

□

Now we can give the proof of the theorem

Proof. The proof goes by induction on the depth of the stratified pseudomanifold X and a partition of unity argument. Let X be a stratified pseudomanifold of depth $l \in \mathbb{N}$ and dimension m, Y a singular stratum of dimension i and $p \in Y$. Then by definition, there exists $U \subseteq X$, $V_Y \subseteq Y$ and an isometry

$$\phi : U \to V_Y \times C(L_Y),$$

where $C(L_Y)$ is the cone over a stratified pseudomanifold L_Y, with depth $\leq l-1$. The variables in $V_Y \times C(L_Y)$ are y, r, z respectively.

- If depth$(X) = 1$, L_Y has depth 0 and it is a compact manifold without boundary. Then trivially, $b_n(y,r,z) = b_n(z) = 1$ is a $(k,p)-$cut-off for L_Y. We then define $g_n(y,r,z) = g_n(r)$ and set $\chi_n = g_n b_n$. It is easy to see that $\{\chi_n\}$ is a $(k,p)-$cut-off in the neighborhood $V_Y \times C(L_Y)$. So the theorem, after gluing with a suitable partition of unity as done below, is proved for stratified pseudomanifolds of depth 1.

- Suppose now that the theorem holds for all stratified pseudomanifolds of depth $< l$ and depth$(X) = l$. Then depth$(L_Y) \leq l-1$ and let $b_n(y,r,z) = b_n(z)$ a $(k,p)-$cut-off for L_Y. As we can see from Proposition 2.2.2, for $n \in \mathbb{N}$, $j = 1, \ldots, k : |g_n^{(j)}|_\infty \leq C_{n,j} < \infty$. Set $C_n = \max_{j=1,\ldots,k} C_{n,j}$. Thus w.l.o.g. we can choose b_n in such a way, that for $j = 1, \ldots, k$ we have

$$\|\big(\nabla^{L_Y}\big)^{(j)} b_n\|_p \leq \frac{1}{nC_n}.$$

  We then set $\chi_n = g_n b_n$ and easily see that

  - $0 \leq \chi_n \leq 1$.
  - For every $K \subset\subset \operatorname{reg}(V_Y \times C(L_Y))$, $\exists\ n_0$ such that $\forall n \geq n_0$ we have $\chi_n|_K = 1$.

  For $j = 1, \ldots, k$, we have

$$\begin{aligned}\|\nabla^j \chi_n\|_{L^p(U)} &= \|\nabla^j (g_n b_n)\|_{L^p(U)} \\ &\leq \sum_{a=0}^{j} C_a \|\nabla^{j-a} g_n \nabla^a b_n\|_{L^p(U)}.\end{aligned}$$

Since $k = 1, 2$ we have to check the cases $j = 1, 2$. For $j = 1$, we obtain the terms $\|\nabla(g_n)b_n\|$, $\|g_n\nabla(b_n)\|$, and for $j = 2$ the terms $\|\nabla^2(g_n)b_n\|$, $\|\nabla(g_n) \otimes \nabla(b_n)\|$, $\|g_n\nabla^2(b_n)\|$. By looking at Lemma 2.2.2, we see that each of these terms is bounded by terms of the form

$$\|\frac{g^{(\alpha)}(r)}{r^{2-\alpha}}(\nabla^{L_Y})^\beta b_n(z)\|_{L^p},$$

where $\alpha, \beta \in \{0, 1, 2\}$ with $1 \leq \alpha + \beta \leq 2$. If $\beta = 0$ the term is bounded by

$$\big(\operatorname{vol}(V_Y)\operatorname{vol}(L_Y)\int_{\frac{1}{n}}^{\frac{3}{2n}} n^{\alpha p} r^{m-i-1-(2-\alpha)p} dr\big)^{1/p} \to 0 \text{ as } n \to \infty,$$

since $m - i > 2p$. If $\beta > 0$, then the term is bounded by

$$\operatorname{vol}(V_Y)^{1/p}|g_n^{(\alpha)}|_\infty\big(\int_0^2 r^{m-i-1-(2-\alpha)p} dr\big)^{1/p}\|\big(\nabla^{L_Y}\big)^\beta b_n\|_{L^p(L_Y)} \to 0$$

as $n \to \infty$, since $m - i > 2p \geq (2 - \alpha)p$, and $g_n^{(\alpha)} \leq C_n$.

Thus, $\{\chi_n\} = \{g_n b_n\}$ is a sequence of $(k, p)-$cut-offs in the neighborhood $U = V_Y \times C(L_Y)$. Finally, the sequence of functions $\{\tilde{\chi_n}\}$ defined by

$$\tilde{\chi}_n = \sum_i \rho_i \chi_{n,U_i} + \rho_{int},$$

is a sequence of $(k, p)-$cut-offs. In order to see this, we take $1 \leq j \leq k$ and we calculate:

$$\|\nabla^j \tilde{\chi}_n\|_{L^p} \leq \|\sum_i \nabla^j(\rho_i)\chi_{n,U_i} + \nabla^j \rho_{int}\|_{L^p} + \|\sum_{i,\ a+\beta \leq j,\ \beta > 0} C_{a,\beta}\nabla^a(\rho_i)\nabla^\beta(\chi_{n,U_i})\|_{L^p}.$$

The first term converges locally uniformly to 0 as $n \to \infty$ and it is bounded since $\sum_i \nabla^j(\rho_i)\chi_{n,U_i} + \nabla^j\rho_{int}$ is bounded and L^p-integrable. Thus by Lebesgue's theorem we conclude that it converges to 0. For the second term, we use that χ_{n,U_i} is a $(k, p)-$cut-off, and that $\nabla^a \rho_i$ is bounded, thus it also converges to 0. □

Remark 2.2.2. For Theorem 2.2.1, we based the construction on Proposition 2.2.2, which gives the condition $m - i > kp$ for $k = 1, 2$. For this choice of functions, this condition is sharp. However, in the case depth$(X) = 1$, we are able to obtain the sharper condition $m-i\geq kp$ for $k=1,k=2,p>1$. For $k=1$, this is Theorem 3.4 in [BG17]. For $k=2$, the sequence of functions is

$$f_n(r)=\begin{cases} 0 & 0\leq r\leq \varepsilon'_n \\ \left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n}\frac{\varepsilon_n(\varepsilon_n-2\varepsilon'_n)}{2\varepsilon'_n}\left(\frac{r}{\varepsilon'_n}-1\right) & \varepsilon'_n\leq r\leq 2\varepsilon'_n \\ \left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n-1}(\varepsilon_n-r) & 2\varepsilon'_n\leq r\leq \varepsilon_n \\ 0 & \varepsilon_n\leq r\leq 2. \end{cases}$$

where $\varepsilon'_n=e^{-n^4}$, $\varepsilon_n=\frac{1}{n^2}$. This sequence belongs toou $W_0^{2,p}((0,2])$. For a proof, we refer to the Appendix, Proposition B.0.1

2.2.1 A Density Theorem for $W^{1,p}(X)$

Now, by using the construction of the weak cut-off functions, we are able to prove a density result about the Sobolev spaces $W^{1,p}(\mathrm{reg}(X))$. Before stating the precise result we prove the following intermediate proposition:

Proposition 2.2.3. Let (M,g) be a Riemannian manifold and $p\in[1,\infty)$. Then the space $\{u\in W^{1,p}(M):\ \|u\|_\infty<\infty\}$ is dense in $W^{1,p}(M)$ in the $\|\cdot\|_{W^{1,p}}$-norm.

Proof. Let $u\in W^{1,p}(M)$. By Proposition 2.1.1 we can assume that $u\in C^\infty(M)$. Then, let $a_n\to\infty$ be regular values of u and define $u_n=\max(-a_n,\min(u,a_n))$. Then $u_n\in W^{1,p}(M)$ and it is easily seen that $\|u_n-u\|_{W^{1,p}}\to 0$ as $n\to\infty$. □

Therefore, we have the following Theorem:

Theorem 2.2.2. Let X be a compact stratified pseudomanifold of dimension m, endowed with an iterated edge metric on $\mathrm{reg}(X)$. Suppose that for every singular stratum Y of X we have the condition

$$\mathrm{codim}(Y)=m-i>p,\ \text{where } i=\dim(Y). \tag{2.2.5}$$

Then,

$$W^{1,p}(\mathrm{reg}(X))=W_0^{1,p}(\mathrm{reg}(X)).$$

Proof. The inclusion $W_0^{1,p}(\mathrm{reg}(X))\subseteq W^{1,p}(\mathrm{reg}(X))$ is obvious. For the converse, let $u\in W^{1,p}(\mathrm{reg}(X))$. According to Proposition 2.2.3, we can assume that $u\in C^\infty(\mathrm{reg}(X))\cap L^\infty(X)\cap W^{1,p}(\mathrm{reg}(X))$. Since $m-i\geq p$, we obtain from Theorem 2.2.1, a sequence $\{\chi_n\}$ of $(1,p)-$cut-offs. We set $u_n = \chi_n u$ and we calculate:

$$\begin{aligned}\|u-u_n\|_{W^{1,p}} &= \|u-u_n\|_{L^p} + \|\nabla(u) - \chi_n\nabla(u) - \nabla(\chi_n)u\|_{L^p}\\ &\leq \|u-u_n\|_{L^p} + \|\nabla(u) - \chi_n\nabla(u)\|_{L^p} + \|\nabla(\chi_n)u\|_{L^p}.\end{aligned}$$

The first two terms converge to 0 by Lebesgue's Theorem, and the last term is bounded by $\|\nabla(\chi_n)\|_{L^p} \cdot \|u\|_{L^\infty} \to 0$ as $n \to \infty$ since χ_n is a $(1,p)-$cut-off and u is bounded. $u_n \in C_c^\infty(\mathrm{reg}(X))$ and that concludes the proof. □

2.3 A Hardy Inequality

In this section, we prove a Hardy-type inequality for simple edge spaces. As a corollary, we will identify the Sobolev spaces we defined on this chapter, with the version of Sobolev spaces defined in Chapter 1. Apart from that, Hardy inequality playes a crucial role in proving the validity of the Sobolev embedding which we will show in the next section. We begin, by proving a weighted Hardy inequality on the real half-line. More precisely, we have:

Proposition 2.3.1. Let $p \geq 1$, $f \in \mathbb{N}$ with $p \neq f+1$. Then for every $u \in C_c^\infty((0,+\infty))$ we have

$$\int_0^\infty \frac{|u(r)|^p}{r^p} r^f dr \leq \left|\frac{p}{f+1-p}\right|^p \int_0^\infty |\partial_r u|^p r^f dr. \tag{2.3.1}$$

Proof. We have

$$\begin{aligned}\int_0^\infty \frac{|u|^p}{r^p} r^f dr &= \int_0^\infty |u|^p \frac{(r^{f+1-p})'}{f+1-p} dr\\ &= -\frac{p}{f+1-p}\int_0^\infty |u|^{p-1}\mathrm{sgn}(u)(\partial_r u) r^{f+1-p} dr\\ &\leq \left|\frac{p}{f+1-p}\right| \int_0^\infty |u|^{p-1}|\partial_r u| r^{f+1-p} dr.\end{aligned}$$

For $p = 1$ the statement has been proved. For $p > 1$ we split $r^{f+1-p} = r^{\frac{f(p-1)}{p}+(1-p)} r^{\frac{f}{p}}$ and apply Hölder Inequality with $\frac{p-1}{p} + \frac{1}{p} = 1$. Then we obtain

$$\int_0^\infty \frac{|u|^p}{r^p} r^f dr \leq \frac{p}{|f+1-p|}\left(\int_0^\infty \frac{|u|^p}{r^p} r^f dr\right)^{1-\frac{1}{p}} \left(\int_0^\infty |\partial_r u|^p r^f dr\right)^{\frac{1}{p}}.$$

Taking powers of p finishes the proof. □

Using the above proposition, we easily obtain a Hardy-type inequality for model simple edge spaces. More precisely we have

Proposition 2.3.2. Let (L, g_L), (Y, h) be manifolds of dimension f, d respectively, $1 \leq p < \infty$ with $p \neq f+1$ and let $Y \times C(L) = Y \times (0, +\infty) \times L$ with metric $g_0 = h + dr^2 + r^2 g_L$, where $g_L : Y \to T^*M \otimes T^*M$ a smooth tensor that restricts to a Riemannian metric on each $y \in Y$. Then for $u \in C_c^\infty(Y \times C(L))$ one has

$$\int_{Y\times C(L)} \frac{|u|^p}{r^p} dvol_{g_0} \leq \left| \frac{p}{f+1-p} \right|^p \int_{Y\times C(L)} |\nabla^{g_0} u|^p dvol_{g_0}.$$

Proof. The volume form on $Y \times C(L)$ is $r^f dr dvol_h dvol_{g_L}$ and for simplicity we denote it by $r^f drdydz$. Thus, we have

$$\begin{aligned}
\int_{Y\times C(L)} \frac{|u(r,y,z)|^p}{r^p} r^f drdydz &= \int_L \int_Y \int_0^\infty \frac{|u(r,y,z)|^p}{r^p} r^f drdydz \\
&\leq \left| \frac{p}{f+1-p} \right|^p \int_L \int_Y \int_0^\infty |\partial_r u(r,y,z)|^p r^f drdydz \\
&\leq \left| \frac{p}{f+1-p} \right|^p \int_L \int_Y \int_0^\infty |\nabla^{g_0} u(r,y,z)|^p r^f drdydz \\
&= \left| \frac{p}{f+1-p} \right|^p \int_L \int_Y \int_0^\infty |\nabla^{g_0} u(r,y,z)|^p dvol_{g_0}.
\end{aligned}$$

where in the first inequality we applied Proposition 2.3.1 and in the second inequality the fact that $|\partial_r u|^p \leq |\nabla^{g_0} u|^p = \left(|\partial_r u|^2 + \frac{|\nabla^{g_L} u|^2}{r^2} + |\nabla^h u|^2\right)^{p/2}$. □

2.3.1 Equality of Sobolev Spaces

As an application of Hardy inequality we can obtain the following corollary:

Corollary 2.3.1. Let X be a compact simple edge space and $p \in [1, \infty)$ that satisfies the condition $p \neq \dim(X) - \dim(Y)$ for every singular stratum Y of X. Then we have the equality

$$\rho H_{ie}^{1,p}(X) = W_0^{1,p}(X).$$

Proof. We only have to show it near a singular stratum Y, since on the interior these spaces induce equivalent norms. Near a singular stratum, the space looks like $V_Y \times C(L_Y)$, where L_Y is a compact manifold, and also $\rho = r$. Note now, that we can always find an open cover of the singular area, such that the norms $|\nabla^{L_Y} u|^2$, $|\nabla^Y u|^2$ are comparable to $\sum_{i=1}^{\dim L_Y} |\partial_{z_i} u|^2$, $\sum_{i=1}^{\dim Y} |\partial_{y_i} u|^2$

respectively (See also section 2.4). Note also, that since $C_c^\infty(\text{reg}(X))$ is dense in both spaces, we can consider that every function is in $C_c^\infty(\text{reg}(X))$. If $u \in W_0^{1,p}(X)$, then

$$\begin{aligned}
\|\frac{u}{r}\|_{ie,1,p}^p &= \int \big(|\frac{u}{r}|^p + |r\partial_r \frac{u}{r}|^p + \sum_{i=1}^{\dim L_Y} |\partial_{z_i} \frac{u}{r}|^p + \sum_{i=1}^{\dim Y} |\partial_{y_i} u|^p\big) \\
&\leq C \int \big(|\partial_r u|^p + |\frac{u}{r}|^p + \sum_{i=1}^{\dim L_Y} |\partial_{z_i} \frac{u}{r}|^p + \sum_{i=1}^{\dim Y} |\partial_{y_i} u|^p\big) \\
&\leq C \int \big(|\partial_r u|^p + \sum_{i=1}^{\dim L_Y} |\partial_{z_i} \frac{u}{r}|^p + \sum_{i=1}^{\dim Y} |\partial_{y_i} u|^p\big) \\
&\leq \|u\|_{W^{1,p}(X)}^p < \infty,
\end{aligned}$$

where in the second inequality we used Hardy inequality. Thus $W^{1,p}(X) \subseteq \rho H_{ie}^{1,p}(X)$. Concerning the other inclusion, for $ru \in \rho H_{ie}^{1,p}(X)$ we have

$$\begin{aligned}
\|ru\|_{W^{1,p}}^p &= \int \big(|ru|^p + |\partial_r(ru)|^p + \sum_{i=1}^{\dim L_Y} |\partial_{z_i} \frac{ru}{r}|^p + \sum_{i=1}^{\dim Y} |\partial_{y_i} ru|^p\big) \\
&\leq C\|u\|_{ie,1,p}^p.
\end{aligned}$$

Thus $\rho H_{ie}^{1,p}(X) \subseteq W^{1,p}(X)$. □

2.4 Geometry of Simple Edge Spaces

In this section, we explore in more detail the geometry of simple edge spaces. As we will see, we can choose a finite cover of any singular stratum Y of X, such that on the regular part near Y, an iterated edge metric is equivalent to the Euclidean. We will use this in the next section in order to obtain a Sobolev inequality on simple edge spaces. We first begin with a Lemma:

Lemma 2.4.1. Let (L, g_L) be a compact manifold without boundary, with $\dim(L) = n$. Then, for every $\varepsilon > 0$, there exists a finite cover of L by charts (U_i, ϕ_i), such that each U_i can be embedded into S^n, through a map $f_i : U_i \to f(U_i) \subseteq S^n$, and on U_i we have

$$(1-\varepsilon) f_i^*(g_{S^n}) \leq g_L \leq (1+\varepsilon) f_i^*(g_{S^n}) \qquad (2.4.1)$$

as bilinear forms, and where S^n is the standard n-sphere.

Proof. Let $1 > \varepsilon > 0$, $p \in L$ and let (U_p, ϕ) to be normal coordinates around p. We denote them by $x_1(p), \dots, x_n(p)$. By shrinking U, we can assume that $\phi : U \to B(0,\delta)$ is a diffeomorphism for each $1 > \delta > 0$. If $v = \sum_i a_i \partial_{x_i}$ in local coordinates, then we have that

$$(1-\varepsilon)\sum_i a_i^2 \leq g_L(v,v) \leq (1+\varepsilon)\sum_i a_i^2. \tag{2.4.2}$$

Since $\delta > 0$ is arbitrary, we can assume that $\delta < 1$. Then we consider the map $f : B(0,\delta) \to S^n$ which is defined by

$$f(x_1,\dots,x_n) = (x_1,\dots,x_n,\sqrt{1-(x_1^2+\dots+x_n^2)}).$$

Denote by $\partial_{x_i}^L, \partial_{x_i}^{S^n}$ the vectors fields with regard to these two maps in L and S^n respectively. Then these are related by $f_*(\phi_*(\partial_{x_i}^L)) = \partial_{x_i}^{S^n}$. A simple computation shows that

$$g_{S^n}(\partial_{x_i}^{S^n}, \partial_{x_j}^{S^n}) = \delta_{ij} + \frac{x_i x_j}{(1-(x_1^2+\dots+x_n^2))}, \tag{2.4.3}$$

since $g_{S^n} = i^*(g_{\mathbb{R}^{n+1}})$. By taking $v = \sum_i a_i \partial_{x_i}^{S^n}$ and making $\delta > 0$ small enough, by unfolding definitions and using (2.4.3) we obtain that

$$(1-\varepsilon)\sum_i a_i^2 \leq g_{S^n}(v,v) \leq (1+\varepsilon)\sum_i a_i^2. \tag{2.4.4}$$

Combining (2.4.2) and (2.4.4) and setting $f_i = f \circ \phi$, we obtain that

$$(1-\varepsilon)f_i^*(g_{S^n}) \leq g_L \leq (1+\varepsilon)f_i^*(g_{S^n}).$$

Since L is compact, we can find a finite cover of L by (U_i, ϕ_i) that each embeds into S^n through an f_i and that concludes the proof. $\square$

Lemma 2.4.2. Let (L, g_L) a compact manifold with $\dim(L) = n-1$ and for $a > 0$ consider the straight cone of L, i.e. the manifold $C_a(L) = \big((0,a) \times L, dr^2 + r^2 g_L\big)$. Then, for every $\varepsilon > 0$, there exists a finite cover of $C_a(L)$ with charts (V_i, ψ_i) such that each V_i embeds into $\mathbb{R}^n$ through f_i and on each V_i we have

$$(1-\varepsilon)f_i^*(\delta_{kl}) \leq dr^2 + r^2 g_L \leq (1+\varepsilon)f_i^*(\delta_{kl}),$$

as bilinear forms, and where δ_{kl} stands for the Euclidean metric $g_{\mathbb{R}^n}(\partial_{x_k}, \partial_{x_l})$.

Proof. Take $1 > \varepsilon > 0$. By Lemma 2.4.1 we obtain a finite cover of L by (U_i, ϕ_i) such that each U_i embeds to S^n through an f_i. This yields the cover (V_i, ψ_i) for $C_a(L)$ which is defined by $V_i = C_a(U_i)$, $\psi_i = (id, \phi_i)$. Then by defining $F_i = (id, f_i)$ we see that $C_a(U_i)$ embeds into $C_a(S^n)$ which we identify with $B(0,a) \setminus \{0\} \subseteq \mathbb{R}^n$ through polar coordinates $\lambda : C_a(S^n) \to B(0,a) \setminus \{0\}$. By setting $f_i = \lambda \circ F_i$, we obtain that

$$(1-\varepsilon) f_i^*(\delta_{kl}) \leq dr^2 + r^2 g_L \leq (1+\varepsilon) f_i^*(\delta_{kl}),$$

and that concludes the proof. □

Lemma 2.4.3. Let (Y, h) a compact Riemannian manifold without boundary with $\dim(Y) = b$, and L be a compact manifold with $\dim(L) = n$. Let, $g_L : Y \to T^*M \times T^*M$, be a smooth tensor, such that for every $y \in Y$, it restricts to a Riemannian metric on L. Then, for every $\varepsilon > 0$ there exists a finite covering with charts $\big(W_i, \psi_i\big)$, of

$$\Big(Y \times (0,a) \times L, h + dr^2 + r^2 g_L\Big),$$

such that each W_i embeds into $\mathbb{R}^{1+b+n}$ through a map $f_i : W_i \to f_i(W_i) \subseteq \mathbb{R}^{1+b+n}$ and on W_i we have

$$(1-\varepsilon) f_i^*(\delta_{kl}) \leq h + dr^2 + r^2 g_L \leq (1+\varepsilon) f_i^*(\delta_{kl}),$$

as bilinear forms, where δ_{kl} stands for the Euclidean metric $g_{\mathbb{R}^{1+b+n}}(\partial_{x_k}, \partial_{x_l})$.

Proof. Let $\varepsilon > 0$ and fix $y_0 \in Y$. Since L is compact, by Lemma 2.4.2, we can find a finite cover $\big(U_i,\ \phi_i\big)$ of L, such on U_i we have

$$(1-\varepsilon)\delta_{kl} \leq g_{L,y_0} \leq (1+\varepsilon)\delta_{kl} \tag{2.4.5}$$

as bilinear forms, where δ_{kl} stands for the Euclidean metric. Let now $y \in V_{y_0} \subseteq Y$, Y_{y_0} open and let $v \in T_{U_i}L$ with $v = \sum_k a_i \frac{\partial}{\partial x_i}$, where $x_1, \cdots, x_n$ are the coordinates on $\phi_i(U_i)$. Then we have $g_{L,y}(v,v) = g_{L,y_0}(v,v) + g_{L,y}(v,v) - g_{L,y_0}(v,v)$. We estimate

$$\begin{aligned}
|g_{L,y}(v,v) - g_{L,y_0}(v,v)| &= |\sum_{k,l} \big(g_{L,y_{kl}} - g_{L,{y_0}_{kl}}\big) a_k a_l| \\
&\leq \sum_{k,l} |\big(g_{L,y_{kl}} - g_{L,{y_0}_{kl}}\big)| \frac{a_k^2 + a_l^2}{2} \\
&\leq \frac{\varepsilon}{2n} \sum_{k,l} \big(a_k^2 + a_l^2\big) \\
&= \frac{\varepsilon}{2n} 2n |v|^2 = \varepsilon \sum_k a_k^2.
\end{aligned} \tag{2.4.6}$$

for $y \in V'_{y_0}$, since $g_L : Y \to T^*M \otimes T^*M$ is a smooth tensor. Now, we are in position to apply Lemma 2.4.1 and Lemma 2.4.2 on the family $g_{L,y}$, with $y \in V'_{y_0}$. Then, we obtain that $C_a(U_i)$ embeds into $C_a(S^n)$ through F_i, which is diffeomorphic to $B(0,a) \setminus \{0\}$ under polar coordinates λ, and that for $y \in V_{y_0}$, $v \in T_{U_i}$, we have

$$(1-2\varepsilon)(\lambda \circ F_i)^*(\delta_{kl}) \leq dr^2 + r^2 g_{L,y} \leq (1+2\varepsilon)(\lambda \circ F_i)^*(\delta_{kl})$$

as bilinear forms, where δ_{kl} stands for the Euclidean metric. Then, by taking a possibly smaller neighborhood V''_{y_0} of $y_0 \in Y$, and normal coordinates ϕ_{y_0} at y_0, we obtain that

$$(1-\varepsilon)\delta_{kl} \leq h \leq (1+\varepsilon)\delta_{kl}.$$

as bilinear forms. Then by taking the charts $\left(V''_{y_0} \times U_{i,y_0},\ \phi_{y_0} \times \phi_i\right)_{i=1}^{N_{y_0}}$ we see that $V''_{y_0} \times C_a(U_{i,y_0})$ embeds to $\phi_{y_0}(V''_{y_0}) \times (B(0,a) \setminus \{0\}) \subseteq \mathbb{R}^{1+b+n}$ through $f_i := \phi_{y_0} \times (\lambda \circ F_i)$ and on $V''_{y_0} \times C_a(U_{i,y_0})$

$$(1-\varepsilon)f_i^*(\delta_{kl}) \leq h + dr^2 + r^2 g_L \leq (1+\varepsilon)f_i^*(\delta_{kl})$$

as bilinear forms. Since Y is compact, if we repeat the above procedure for each $y \in Y$, we can find a finite family

$$\{V_{y_j} \times U_{i,j},\ \phi_{y_j} \times f_{i,j}\}_{j=1,\cdots,N,\ i=1,\cdots N_j}.$$

such that on each cover $V_{y_i} \times U_{i,j}$ we have

$$(1-\varepsilon)f_{i,j}^*(\delta_{kl}) \leq h + dr^2 + r^2 g_L \leq (1+\varepsilon)f_{i,j}^*(\delta_{kl})$$

as bilinear forms, where δ_{kl} stands for the Euclidean metric. That concludes the proof. $\square$

On a simple edge space, the metric has the form

$$g = g_0 + k, \tag{2.4.7}$$

where $|k|_{g_0} = O(r^\gamma)$ for $\gamma > 0$, r the radial variable of the cone, and $g_0 = h + dr^2 + r^2 g_L$. Since g and g_0 are quasi isometric, Lemma 2.4.3 applies to g, but not necessarily with constants $(1+\varepsilon)^{-1}$, $(1+\varepsilon)$. In order to obtain constants like these, one should restrict to small neighborhoods around the singular strata.

Now let X be a compact simple edge space. For simplicity we assume that it has only one stratum Y of depth 1. (In the case where we have more than one singular strata, we proceed in the same way). As a consequence, there exists a neighborhood $U \subseteq X$, compact manifold L and a locally trivial fibration

$$\phi : U \to Y \times C_2(L),$$

such that

$$(\phi^{-1})^*(g)_{|U} = g_0 + k,$$

where $g_0 = h + dr^2 + r^2 g$ and $|k|_{g_0} = O(r^\gamma)$ for some $\gamma > 0$. Since $Y,\ L$ are compact, one can find finite covers $U_j,\ V_{i(j),j}$ respectively, and thus $\phi^{-1}\big(U_j \times C_2(V_{i,j})\big)$ is an open cover for U. According to the previous considerations and Proposition 1.2.1, one can choose this open cover such that each open set of this covers embeds into $\mathbb{R}^m$ through an f and the metric there is equivalent with the Euclidean, i.e.

$$\frac{1}{4} f^*(\delta_{ij}) \leq g \leq 4 f^*(\delta_{ij}). \tag{2.4.8}$$

From now on, whenever we refer to this cover, we will just write $U_j \times C_2(V_i)$ instead of $U_j \times C_2(V_{i(j),j})$. Since X is compact, on then can find a finite cover M_λ such that $X \setminus U \subseteq \bigcup_\lambda M_\lambda$. Moreover, one can choose this cover so that (2.4.8) holds. Define now the projection

$$\begin{aligned} &\pi : Y \times C_2(L) \to Y \times L, \\ &\pi(y, r, z) = (y, z). \end{aligned} \tag{2.4.9}$$

Then if $\chi_i,\ \psi_j$ are partitions of unity associated to the cover $U_i,\ V_j$ respectively, then $\pi^*(\chi_i \psi_j)$ is a partition of unity, associated with the cover $U_i \times C_2(V_j)$. For simplicity, we set $\rho_{ij}(y, r, z) = \pi^*\big(\chi_i(y)\psi_j(z)\big)$. An important observation is that for $u \in C^\infty(U_i \times C_2(V_j))$ we have

$$|\nabla^{g_0} u|^2 = |\partial_r u|^2 + \frac{|\nabla^{g_L} u|^2}{r^2} + |\nabla^h u|^2,$$

and thus, for ρ_{ij} we have the bound

$$|\nabla^{g_0} \rho_{ij}| \leq \frac{C}{r}. \tag{2.4.10}$$

Remark 2.4.1. A novel difference between the partitions of unity ρ_{ij} we considered here, and the partitions of unity we considered in Proposition 2.2.1 is that the former are not bounded. The reason why this happens is that we defined them in open subsets $V_j \subseteq L$. But the partitions of unity in Proposition 2.2.1 are independent of the z-variable, for $z \in L$, therefore they are bounded.

2.5 Functional Inequalities on Simple Edge Spaces

This, together with the next section constitute the core part of Chapter 2. Here we are establishing functional inequalities on compact simple edge spaces. To do so, we heavily rely on the identification of the singular neighborhood near a singular stratum with the Euclidean as shown in section 2.4. We note here that in our case the differential of the partitions of unity are not bounded. For this reason we frequently employ the version of Hardy inequality which we proved before (see Proposition 2.3.2). To begin with, we note that the Sobolev inequality holds in this context:

Proposition 2.5.1. (Sobolev Embedding) Suppose X is a compact simple edge space of dimension $m > 1$. Then, there exists $A,\ B > 0$ such that for all $u \in C_c^\infty(\mathrm{reg}(X))$ we have

$$\Big(\int_X |u|^{\frac{m}{m-1}} dvol_g\Big)^{\frac{m-1}{m}} \leq A\int_X |\nabla u| dvol_g + B\int_X |u| dvol_g. \tag{2.5.1}$$

Proof. We show the proof in the case where we have only one singular stratum $Y \subseteq U$. In the case where we have more, we can apply the same procedure in every stratum. Take $\phi_1 : \mathrm{reg}(X) \to [0,1]$, with $\mathrm{supp}(\phi_1) \subseteq U$ and $\phi_1 = 1$ for $r \leq 1$, $\phi_1 = 0$ for $r \geq 3/2$. Set $\phi_2 = 1 - \phi_1$. Then we have

$$\|u\|_{\frac{m}{m-1}} \leq \|\phi_1 u\|_{\frac{m}{m-1}} + \|\phi_2 u\|_{\frac{m}{m-1}}. \tag{2.5.2}$$

Concerning $\phi_1 u$ we have

$$\|\phi_1 u\|_{\frac{m}{m-1}} = \|\sum_{ij} \rho_{ij}\phi_1 u\|_{\frac{m}{m-1}} \leq \sum_{ij} \|\rho_{ij}\phi_1 u\|_{\frac{m}{m-1}}.$$

Recall, that each neighborhood $U_{ij} = V_i \times C_2(U_j)$ can be embedded through an embedding f by Lemma 2.4.3 to $\mathbb{R}^b \times C_2(S^n)$ which we identify with a subset of $\mathbb{R}^m$ through cylindrical coordinates λ. Therefore we obtain

$$\begin{aligned}
\|\rho_{ij}\phi_1 u\|_{\frac{m}{m-1}} &= \left(\int_{V_i \times C_2(U_j)} |\rho_{ij}\phi_1 u|^{\frac{m}{m-1}} dvol(g_0) \right)^{\frac{m-1}{m}} \\
&\leq \left(C \int_{V_i \times C_2(U_j)} |\rho_{ij}\phi_1 u|^{\frac{m}{m-1}} (\lambda \circ f)^*(dx) \right)^{\frac{m-1}{m}} \\
&= \left(C \int_{(\lambda \circ f)(V_i \times C_2(U_j))} |(\rho_{ij}\phi_1 u) \circ (\lambda \circ f)^{-1}|^{\frac{m}{m-1}} dx \right)^{\frac{m-1}{m}} \\
&\leq C \int_{(\lambda \circ f)(V_i \times C_2(U_j))} |\nabla(\rho_{ij}\phi_1 u) \circ (\lambda \circ f)^{-1}| dx \\
&= C \int_{V_i \times C_2(U_j)} |\nabla(\rho_{ij}\phi_1 u)| (\lambda \circ f)^*(dx) \\
&\leq C \int_{V_i \times C_2(U_j)} |\nabla(\rho_{ij}\phi_1 u)| dvol(g_0),
\end{aligned}$$

where on the first and the third inequality we used Lemma 2.4.3 and on the second inequality the Sobolev inequality on $\mathbb{R}^m$ with $m = 1 + b + n$, since $(\lambda \circ f)(V_i \times C_2(U_j)) \subseteq \mathbb{R}^m$. By (2.4.10) and Proposition 2.3.2, we obtain that the former is bounded by

$$C_1 \int_{U_i \times C_2(V_j)} |\nabla(\phi_1 u)| dvol_{g_0} \leq C_1 \int_X |\nabla u| dvol_{g_0} + C_2 \int_X |u| dvol_{g_0}. \quad (2.5.3)$$

Concerning $\phi_2 u$ one procceeds exactly as above with respect to a cover of $X \setminus U$. The difference is that the partitions of unity are uniformly bounded, thus combining it with (2.5.3) one obtains the required result. □

By classical means (see [Heb96] Lemma 3.1), one also obtains that for $1 \leq p < m$,

$$W_0^{1,p}(X) \hookrightarrow L^{\frac{mp}{m-p}}(X) \quad (2.5.4)$$

continuously. Furthermore, by using Theorem 2.2.2, we obtain that if $1 \leq p \leq m - \dim(Y)$ for every singular stratum Y of X and $p < m$, then

$$W^{1,p}(X) = W_0^{1,p}(X) \hookrightarrow L^{\frac{mp}{m-p}}(X).$$

Remark 2.5.1. The Sobolev inequality with exponent $p = 2$ holds in the more general case of compact stratified pseudomanifolds. For a proof, using different methods, see [ACM14].

Moreover in this setting, the classical Rellich-Kondrachov theorem is true.

Proposition 2.5.2. Let X be a compact simple edge space of dimension $m > 1$, and let p, q satisfy $1 \leq p < m$, $p \neq m - \dim(Y)$ for every singular stratum Y of X, and $q < p^* = \frac{mp}{m-p}$. Then the embedding

$$W_0^{1,p}(X) \hookrightarrow L^q(X)$$

is compact.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(X)$ such that

$$\|u_n\|_{W_0^{1,p}} \leq M < \infty.$$

Since $u_n \in W_0^{1,p}$, there exists $\tilde{u}_n \in C_c^\infty(\mathrm{reg}(X))$ such that $\|u_n - \tilde{u}_n\|_{W^{1,p}} < \frac{1}{n}$. Notice that $\|\tilde{u}_n\|_{W^{1,p}} \leq M + 1$. If we find a convergent subsequence of $\{\tilde{u}_n\}$ that converges to $v \in L^q$, which we denote again by $\tilde{u}_n$, then $u_n \to v$ in L^q, because $\|u_n - v\|_{L^q} \leq \|u_n - \tilde{u}_n\|_{L^q} + \|\tilde{u}_n - v\|_{L^q}$. Notice that the second terms converges to 0, and for the first term we have

$$\begin{aligned}\|u_n - \tilde{u}_n\|_{L^q} &\leq C\|u_n - \tilde{u}_n\|_{L^{p^*}} \\ &\leq C\|u_n - \tilde{u}_n\|_{W^{1,p}} \\ &\leq \frac{C}{n} \to 0.\end{aligned}$$

Therefore, we can assume that $\{u_n\} \subseteq C_c^\infty(\mathrm{reg}(X))$. As before, we can find covers $U_i \times C_2(V_j)$ of $\cup_{Y\in\mathrm{sing}(X)} Y$, such that each cover is embedded though an embedding f into $\mathbb{R}^b \times C_2(S^n)$ which we identify with a subset of $\mathbb{R}^m$ through λ. Then, as in the proof of Proposition 2.5.1 we have

$$\begin{aligned}&\|(\rho_{ij}\phi_1 u_n) \circ (\lambda \circ f)^{-1}\|^p_{W^{1,p}_{B(0,1)}} \\ &= \int_{B(0,1)} |\nabla(\rho_{ij}\phi_1 u_n) \circ (\lambda \circ f)^{-1}|^p dx + \int_{B(0,1)} |(\rho_{ij}\phi_1 u_n) \circ (\lambda \circ f)^{-1}|^p dx \\ &\leq C\Big(\int_{(\lambda\circ f)^{-1}(B(0,1))} |\nabla(\rho_{ij}\phi_1 u_n)|^p dvol_{g_0} + \int_{(\lambda\circ f)^{-1}(B(0,1))} |(\rho_{ij}\phi_1 u_n)|^p dvol_{g_0}\Big) \\ &\leq C\Big(\int_{(\lambda\circ f)^{-1}(B(0,1))} |\frac{\phi_1 u_n}{r}|^p dvol_{g_0} + \|\phi_1 u_n\|^p_{W^{1,p}(X)}\Big) \\ &\leq C\|u_n\|^p_{W^{1,p}(X)} \leq C(M+1),\end{aligned}$$

where on the first inequality we used Lemma 2.4.3, on the second we used that $|\nabla(\rho_{ij}) \leq \frac{C}{r}$ and on the last we used Hardy inequality, i.e. Proposition

2.3.2. Then by Rellich-Kondrachov theorem, for $q < \frac{mp}{m-p}$, in each cover $(\lambda \circ f)(U_i \times C_2(V_j)) \subseteq B(0,1)$ we find a convergent subsequence of u_n in L^q. For the interior part we apply the classical Rellich-Kondrachov theorem and thus, after passing to a subsequence we obtain the desired result. $\square$

Remark 2.5.2. We can utilize Theorem 2.2.2 again. Under the hypotheses of Proposition 2.5.2 and by assuming furthermore that $1 \leq p < m - \dim(Y)$ for every singular stratum Y of X, we obtain that for $1 \leq q < p^*$, the embedding

$$W^{1,p}(X) = W_0^{1,p}(X) \hookrightarrow L^q(X)$$

is compact.

From now on, we assume that we have the condition $1 \leq p < m - \dim(Y)$ for every singular stratum Y of X. So, there is no distinction between $W^{1,p}(X)$ and $W_0^{1,p}(X)$. With this condition, by using Proposition 2.5.2 and the fact that a compact simple edge space has finite volume, one can prove the following version of Poincare Inequality

Proposition 2.5.3. (Poincare Inequality) Let X be a connected, compact simple edge space of dimension $m > 1$ and let $1 \leq p < m$, $p < m - \dim(Y)$ for every singular stratum Y of X. Then there exists a constant $C > 0$ such that for $u \in W^{1,p}(X)$ we have

$$\|u - u_X\|_p \leq C\|\nabla u\|_p, \tag{2.5.5}$$

where $u_X = \frac{1}{\mathrm{vol}(X)} \int_X u(x) dvol_g$.

Proof. Suppose that (2.5.5) is not true. Then, for every $k \in \mathbb{N}$ there exists $u_k \in W^{1,p}(X)$ such that

$$\|u_k - (u_k)_X\|_p > k\|\nabla u_k\|_p.$$

Set $v_k = \frac{u_k - (u_k)_X}{\|u_k - (u_k)_X\|_p} \in W^{1,p}(X)$. Then by hypothesis we have that

$$\|\nabla v_k\|_p < \frac{1}{k}, \quad \|v_k\|_p = 1.$$

Since $\|v_k\|_{W^{1,p}}$ is uniformly bounded, we can apply Proposition 2.5.2 and obtain a subsequence, which we denote again by v_k, that converges strongly in $L^p(X)$, since $p < p^*$. Thus there exists $v \in L^p(X)$ such that $v_k \to v$ strongly in $L^p(X)$. Then we have that $\|v\|_p = 1$, $v_X = 0$ and $\|\nabla v_k\|_p \to 0$. Pairing v against a test function $\phi \in C_c^\infty(X)$ gives

$$\begin{aligned} -\int_X v\nabla\phi &= -\lim_{k\to\infty}\int_X v_k\nabla\phi \\ &= \lim_{k\to\infty}\int_X \nabla v_k\phi \\ &= 0. \end{aligned}$$

That is $\nabla v = 0$, thus v is constant and since $v_X = 0$ then it is 0, which is a contradiction. □

Using the Sobolev Embedding, one can prove a stronger version of Poincare inequality, namely the Sobolev-Poincare inequality:

Proposition 2.5.4. (Sobolev-Poincare Inequality) Let X be a connected, compact simple edge space of dimension $m > 1$ and let $1 \leq p < m$, $p < m - \dim(Y)$ for every singular stratum Y of X. Then there exists a constant $C > 0$ such that for $u \in W^{1,p}(X)$ we have

$$\|u - u_X\|_{p^*} \leq C\|\nabla u\|_p, \tag{2.5.6}$$

where $u_X = \frac{1}{\mathrm{vol}(X)}\int_X u(x)dvol_g$.

Proof. The proof is a combination of Sobolev Embedding (2.5.4) and Poincare Inequality. For details, see [Heb96] Proposition 3.9. □

2.6 Optimization of Constants

In this section we are focusing on obtaining optimal constants of Sobolev inequalities. In order to do so, we use the cut-off functions introduced in section 2.2. We obtain optimal results concerning the constants of the L^p norms of the functions in the embeddings $W_0^{2,p} \hookrightarrow W_0^{1,p^*}$ and $W_0^{1,p} \hookrightarrow L^{p^*}$. To be more precise, in the previous section we proved the Sobolev embedding on a compact simple edge space X, i.e.

$$\|u\|_{p^*} \leq A\|\nabla u\|_p + B\|u\|_p \tag{I_p}$$

with $u \in W_0^{1,p}(X)$ and $1 \leq p < \dim(X) = m$.

2.6.1 The Embedding $W_0^{1,p} \hookrightarrow L^{p^*}$.

The construction of $(1,p)-$cut-off functions allow us to prove some optimal results concerning the constant $B > 0$. To be more precise, we set

$$B_p(X) = \inf\{B > 0 : \text{such that } \exists\ A > 0 \text{ such that } (I_p) \text{ holds}\}.$$

Two questions that are of interest are

- Compute $B_p(X)$.
- Does there exist an $A > 0$ such that (I_p) holds with $B = B_p(X)$?

This questions are part of the so called AB-programm, which consists of finding the optimal constants for various functional inequalities, such as the Sobolev inequality and Sobolev-Poincare inequality (for more details, see [Heb96] and [DH02]). In this section we answer these questions in the case of compact simple edge spaces with the condition $1 \leq p < m - \dim(Y)$ for every singular stratum Y of X. For this reason, we can apply Theorem 2.2.1 along with Remark 2.2.2. The condition $1 \leq p \leq m - \dim(Y)$ for every singular stratum Y of X, guarantees the existence of a sequence of cut-off functions $\{\chi_n\}_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathrm{reg}(X))$ such that

- $0 \leq \chi_n \leq 1$.
- $\forall$ compact $K \subseteq \mathrm{reg}(X)$, $\exists\ n_0 \in \mathbb{N}$ such that $\forall\ n \geq n_0$ we have $\chi_{n_{|K}} = 1$.
- $\int_X |\nabla \chi_n|^p dv_g \to 0$.

Now, plug χ_n in (I_p). Since $vol(X) < \infty$, using the properties of these cut-offs and Lebesgue's dominated convergence theorem, we obtain by taking $n \to \infty$

$$\mathrm{vol}(X)^{\frac{1}{p^*}} \leq B\,\mathrm{vol}(X)^{\frac{1}{p}},$$

which gives a lower bound for B, i.e.

$$B \geq \mathrm{vol}(X)^{-\frac{1}{m}}. \tag{2.6.1}$$

This gives that $\forall\ p$ with $1 \leq p \leq m - \dim(Y)$ for every singular stratum Y of X, we have $\mathrm{vol}(X)^{-\frac{1}{m}} \leq B_p(X)$. Now using a Sobolev-Poincare inequality and the fact that $\mathrm{vol}(X) < \infty$, one can see that $B_p(X)$ is attainable. More precisely, by Sobolev-Poincare we have that for $1 \leq p < m$, $p < m - \dim(Y)$ for every singular stratum Y of X, there exists $C > 0$ such that

$$\|u - u_X\|_{p^*} \leq C\|\nabla u\|_p,$$

for $u \in W_0^{1,p}(X)$. Using triangle inequality we obtain that

$$\begin{aligned}\|u\|_{p^*} &\leq C\|\nabla u\|_p + \|u_X\|_{p^*} \\ &\leq C\|\nabla u\|_p + \operatorname{vol}(X)^{\frac{1}{p^*}-1}\int_X |u| \\ &\leq C\|\nabla u\|_p + \operatorname{vol}(X)^{\frac{1}{p^*}-1}\|u\|_p \operatorname{vol}(X)^{1-\frac{1}{p}} \\ &= C\|\nabla u\|_p + \operatorname{vol}(X)^{-\frac{1}{m}}\|u\|_p.\end{aligned}$$

Combining this with the lower bound (2.6.1) we obtain the following Theorem.

Theorem 2.6.1. Let X be a connected, compact simple edge space of dimension $m > 1$. Then if $1 \leq p < m - \dim(Y)$ for every singular stratum Y of X, there exists $A > 0$ such that for every $u \in W^{1,p}(X)$ we have

$$\|u\|_{p^*} \leq A\|\nabla u\|_p + \operatorname{vol}(X)^{-\frac{1}{m}}\|u\|_p. \qquad (I_{p,B_{opt}})$$

Moreover, the constant $\operatorname{vol}(X)^{-\frac{1}{m}}$ is optimal, in the sense that if there exists a $B > 0$ such that (I_p) holds with B, then $B \geq \operatorname{vol}(X)^{-\frac{1}{m}}$.

2.6.2 The Embedding $W_0^{2,p} \hookrightarrow W_0^{1,p^*}$

Now we focus on the embedding $W^{2,p} \hookrightarrow W^{1,p^*}$. Recall, that this embedding is obtained by the embedding $W^{1,p} \hookrightarrow L^{p^*}$ and the Kato inequality

$$|\nabla|\nabla^k u|| \leq |\nabla^{k+1}u|,$$

for $u \in C^\infty(\operatorname{reg}(X))$. Therefore we obtain positive constants $A, B, C > 0$, such that for every $u \in C_c^\infty(\operatorname{reg}(X))$ we have:

$$\|\nabla u\|_{L^{p^*}} + \|u\|_{L^{p^*}} \leq A\|\nabla^2 u\|_{L^p} + B\|\nabla u\|_{L^p} + C\|u\|_{L^p}. \qquad (2.6.2)$$

with $1 \leq p < m$. Now, by Theorem 2.2.1, the condition $2p^* < m - \dim(Y)$ for every singular stratum Y, implies the existence of $(2, p^*)-$cut-offs. Notice now by using Hölder inequality and $\operatorname{vol}(X) < \infty$, that if we have a sequence $\{\chi_n\}_{n\in\mathbb{N}}$ of $(2, p^*)-$cut-offs, then this is a sequence of $(1, q)$ and $(2, q)-$cut-offs, for $q \leq p^*$. Thus, by plugging it in (2.6.2) and letting $n \to \infty$, we obtain

$$C \geq \operatorname{vol}(X)^{-\frac{1}{m}}.$$

Similarly as before, we apply (2.5.6) and Kato inequality on the function $v = |\nabla u|$, since $v \in W^{1,p}$ and $1 \leq p < p^* < m - \dim(Y)$ for every singular stratum Y of X and we obtain

$$\||\nabla u| - |\nabla u|_X\|_{p^*} \leq C\|\nabla^2 u\|_p.$$

On the other hand we have

$$\begin{aligned}\||\nabla u|_X\|_{p^*} &= \mathrm{vol}(X)^{\frac{1}{p^*}-1}\int|\nabla u| \\ &\le \mathrm{vol}(X)^{\frac{1}{p^*}-1}\|\nabla u\|_p\,\mathrm{vol}(X)^{1-\frac{1}{p}} \\ &= \mathrm{vol}(X)^{-\frac{1}{m}}\|\nabla u\|_p.\end{aligned}$$

Finally, we use Theorem 2.6.1 and by adding the inequalities we obtain

$$\|\nabla u\|_{p^*}+\|u\|_{p^*}\le C\|\nabla^2 u\|_p+(A+\mathrm{vol}(X)^{-\frac{1}{m}})\|\nabla u\|_p+\mathrm{vol}(X)^{-\frac{1}{m}}\|u\|_p.$$

Therefore, we have the following

Theorem 2.6.2. Let X be a connected, compact simple edge space of dimension $m>1$. Then if $p\in[1,\infty)$, with $1\le 2p^*<m-\dim(Y)$ for every singular stratum Y of X, then there exists $A,B>0$ such that

$$\|\nabla u\|_{p^*}+\|u\|_{p^*}\le A\|\nabla^2 u\|_p+B\|\nabla u\|_p+\mathrm{vol}(X)^{-\frac{1}{m}}\|u\|_p. \qquad (I_{p,2,Opt})$$

Moreover, the constant $\mathrm{vol}(X)^{-\frac{1}{m}}$ is optimal, in the sense that if there exists a $C>0$ such that (2.6.2) holds with $C>0$, then $C\ge\mathrm{vol}(X)^{-\frac{1}{m}}$.

Part II

Heat Kernel Asymptotics

Heat Kernel Asymptotics

Let $K : (0, \infty) \times M \times M \to \mathbb{R}$ be the fundamental solution of the heat equation, namely

$$(\partial_t - \Delta_x)K(t, x, y) = 0$$
$$\int_M K_t(x, y)f(y)dvol(y) \to_{t \to 0} f(x),$$

where Δ_x is the Laplace operator with respect to x, $f \in C_c^\infty(M)$, and associate with K the so called heat trace, which is defined for $t > 0$ by

$$H_t = \int_M K(t, x, x)dvol(x).$$

The topic of finding full or partial asymptotic expansions as $t \to 0$ and computing explicity (some of) the coefficients of this expansion in a variety of different domains and manifolds is a well studied topic, active for more than 70 years. In his famous article [Kac66], Mark Kac popularized this subject by showing that the coefficients of this asymptotic expansion were directly linked to the geometry of the domain. More precisely, using the locality property of the heat kernel, he showed that in the case of planar domains, the first 2 terms were determined by the volume of the domain and the length of the boundary. He showed furthermore, that if the domain had corners, then these would be seen in the asymptotic expansion. Then by finding the contribution of a corner to the expansion and by flattening the corner, he conjectured that in the case of a smooth boundary, the third term would be $-\frac{1}{6}h(M)$, where $h(M)$ is the number of holes of the domain.

In the case of a compact manifold M without boundary, already from 1949, Minaksishundaram and Pleijel ([MP49]) had proved that there exists a full asymptotic expansion of the heat trace of the form

$$\int_M K(t, x, x)dvol(x) \sim_{t \to 0} a_0 t^{-n/2} + a_1 t^{-n/2+1} + ... \tag{HE}$$

Their initial motivation was to prove the analytic continuation of the zeta function, which for the Laplace operator $\Delta : L^2(M) \to L^2(M)$ is defined for $s >> 0$ by

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}$$

where λ_i are the eigenvalues of Δ (see [Gil95] Chapter 1 for details). Their proof makes use of normal coordinates and a first approximation of the heat kernel, which is based on the fact that locally, for small times it should be similar to the Euclidean. Then by making a specific ansatz that the heat kernel should asymptotically be like

$$(4\pi t)^{-n/2} e^{-\frac{d^2(x,y)}{4t}} (u_0(x,y) + u_1(x,y)t + \dots),$$

and by imposing the condition that for every $N \in \mathbb{N}$

$$(\partial_t - \Delta_x)\big((4\pi t)^{-n/2} e^{-\frac{d^2(x,y)}{4t}} u_N(x,y) t^N\big) = 0,$$

they ended up with a recursive system of differential equations, which can be explicity solved. The solutions $u_j(x,y)$ are depending on the metric and the derivatives of the metric. That already shows the dependency of the heat trace coefficients on the geometry of the underlying manifold. At last, in this paper also a_0 is computed to be $\frac{\mathrm{vol}(M)}{(4\pi)^{n/2}}$.

In the case of a compact manifold with C^∞-boundary it was MacKean and Singer ([MS67]) who proved that there exists a full heat trace expansion of Laplace type operators. More specifically, they took up the task of finding explicit formulas for the coefficients of this expansion and verify the conjecture of Mark Kac. Their method consisted of using a first approximation of the heat kernel, and then by constructing a Levy Parametrix, they managed to prove the existence of a full asymptotic expansion. After that they introduced a number of ways for finding explicit formulas for some of the coefficients. These ways range from algebraic arguments for the Laplace operator, to careful term by term examination of each part of the parametrix. A novel difference that brings in the presence of the boundary is that the order of t in the asymptotic expansion increases by $\sqrt{t}$ in each step, rather than t as in the boundaryless case. If we denote by $a_{i/2}$ the coefficient of $t^{-n/2+i/2}$, then the Theorem of MacKean and Singer for the Dirichlet Laplacian, states that $a_0 = \mathrm{vol}(M), a_{1/2} = -\frac{\sqrt{4\pi}}{4} \mathrm{vol}(\partial M)$ and $a_1 = \frac{\chi(M)}{6}$, where $\chi(M)$ is the Euler characteristic of M.

The task of proving the existence of full asymptotic expansions of the heat kernel on the diagonal of the form

$$K(t,x,x) \sim_{t\to 0} a_0(x)t^{-n/2} + a_1(x)t^{(-n+1)/2} + \dots$$

or of the heat trace as in (HE) and calculating coefficients developed further in a variety of situations. Here we present an overview which in no case is exhaustive. In the case of a closed manifold, for P a differential operator, the terms a_0, a_2, a_4, a_6 are known (see [Gil95], Chapter 4). For the Laplacian Δ there exist formulas for $a_i(x)$ in local coordinates for all $i \in 2\mathbb{N}$ (see [Pol00]). In the case of manifolds with smooth boundary, there are also asymptotic expansions of the heat trace for different boundary value problems (see [Gil95],[Gru86],[Gre71] to name just a few).

Apart from compact manifolds with or without boundary, similar results have been obtained in a broad class of singular manifolds. Here we give a small account for the particular class of smoothly stratified spaces. More precisely, in the case of manifolds with cone-like singularities or in the case of iterated edge spaces an expansion like (HE) does not hold. In the heat trace expansion of the Laplacian one has logarithmic terms and terms coming from the cross section of the cone (see [AGR17],[BS87],[Che83],[GKM13],[Les97], to name just a few).

In the case of domains (or manifolds) of arbitrary dimension with irregular boundaries, not that much has been said. To be more precise, the results that have been obtained concerning heat trace asymptotics, are mainly about two term asymptotic expansions, or are complete asymptotic expansions for low dimensions (for surfaces with corners). For example, when D is a polygonal region in $\mathbb{R}^2$ with angles $\gamma_1, \dots, \gamma_k$ where $0 < \gamma_i \leq 2\pi$, then the Theorem of van den Berg and Srisatkunarajah ([vdBS88]) states that the Dirichlet heat trace satisfies

$$H_t = \frac{\mathrm{vol}(D)}{4\pi t} - \frac{\mathrm{vol}(\partial D)}{8\sqrt{\pi t}} + \sum_{i=1}^{k} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i} + R(t),\ t \to 0,$$

where $R(t) = Ce^{-ct}$. A similar formula holds when $D \subseteq \mathbb{R}^n$ with Lipschitz boundary. More precisely, in this case, for the Dirichlet heat trace, we get as $t \to 0$, that

$$H_t = (4\pi t)^{-\frac{n}{2}}\left(|D| - \frac{\sqrt{\pi t}}{2}|\partial D| + o(t^{1/2})\right).$$

For a proof see [Bro93]. See also [vdB87], for D satisfying an $R-$smoothness condition. In the case of surfaces with corners, there is a recent preprint, that obtains a full heat trace expansion as $t \to 0$ (see [NRS19]). More precisely let H_t denote the heat trace for the Dirichlet, Neumann or Robin boundary conditions. Then H_t has a polyhomogeneous conormal expansion in $\sqrt{t}$ (see Theorem 6.3 in [NRS19]).

Statement of our results

In this chapter we are dealing with the problem of deriving a complete asymptotic expansion of the heat kernel with Dirichlet (or Neumann) boundary conditions on a general manifold with corners satisfying the following assumption:

Assumption 2.6.1. Let M be a manifold of dimension n with corners of codimension at most k. Suppose furthermore that we have a Riemannian metric, i.e. $g \in C^\infty(T^*M \otimes T^*M)$. Then we assume that at every boundary face $A_i \in \mathcal{M}_i$ of codimension i, $i = 1, \dots, k$, there exist local coordinates such that

$$g_{|A_i} \sim dx_1^2 + \cdots + dx_i^2 + g_{A_i},$$

where $\sim$ is meant as an isometry.

Assumption 2.6.1 is quite strong. It essentially means that the different boundary hypersurfaces meet in a right angle. However, this allow us to begin the construction of the heat kernel by considering the product of the heat kernels as model solutions.

In order to construct the heat kernel with Dirichlet (or Neumann) boundary conditions we follow the considerations in [Mel93] as expositated in [Gri04]. We follow closely the method that was introduced in the latter by making sure that each step can generalise to the situation with corners, and by taking care of the more complicated spaces and the new phenomena that arise. The idea of this method is the following:

The heat kernel on $\mathbb{R}^n$ is $(4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}}$. As a function of the variables $t > 0$, $x, y \in \mathbb{R}^n$ it is smooth, but it has a singularity as $|x-y|,\ t \to 0$. In order to understand better this singularity, it is convenient to consider the heat kernel as a product of a power of $\sqrt{t}$ with a smooth function of $\sqrt{t}$, $\frac{|x-y|}{\sqrt{t}}$. The heat kernel then can be rewritten in the form

$$(4\pi t)^{-n/2} e^{-\frac{|x-y|^2}{4t}} = (4\pi t)^{-n/2} K(\sqrt{t}, \frac{x-y}{\sqrt{t}}, y) \tag{BUHK}$$

with K smooth in $[0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Now, we are able to see that the asymptotics on the diagonal $x = y$ are of order $t^{-n/2}$.

Of course, on a compact manifold without boundary, things are not that simple, because most of the times we don't have an explicit formula for the heat kernel. For this reason, if we want to find it's asymptotics as $t \to 0$, we have to construct an approximate solution u_N, which we can make better as the order N goes to ∞. This approximate solution is called a parametrix and it is carefully constructed in [MP49]. This parametrix can give an asymptotic expansion up to order N, and it can also be used to construct the fundamental solution of the Heat equation (for details see [BGV04]).

Another approach ([Mel93],[Gri04]) is to exploit (BUHK). More precisely, since the manifold is locally Euclidean, the heat kernel for x close to y and small $t > 0$ should be very close to the Euclidean heat kernel. For this reason, it is wise to assume that locally it will have a property like (BUHK). The goal now is to define a calculus of functions that have the property (BUHK) and examine it's properties (symbol, mapping properties of $\partial_t - \Delta_x$, short exact sequence, composition formula). Then, similar as in the case before, a first approximation K_1 that belong in this calculus, and a Volterra series argument will give the actual heat kernel, together with all the information about it's asymptotics as $t \to 0$. For compact manifolds without boundary this is done in these two cited documents, although with some differences between them. It is the generalisation of the second approach that we follow in this chapter.

Chapter 3

A Heat Calculus for Manifolds with Corners

3.1 The Heat Kernel On A Model Corner

Before giving the details of our calculus, we motivate our construction by considering the case of the half-space $\mathbb{R}^{n-1} \times \mathbb{R}_+$. The Dirichlet heat kernel on the half space $\mathbb{R}^{n-1} \times \mathbb{R}_+$ with coordinates $t > 0$, $(y', y_n), (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$ is given by

$$K_t(x', x_n, y', y_n) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2}{4t}} (e^{-\frac{|x_n - y_n|^2}{4t}} - e^{-\frac{|x_n+y_n|^2}{4t}}) \tag{3.1.1}$$

(3.1.1) can be seen in two ways: Either as $K_t^n - K_t^{n,ref}$, where K_t^n is the heat kernel on $\mathbb{R}^n$ and $K_t^{n,ref}$ the reflected heat kernel on $\mathbb{R}^n$ along the axis x_n, or as the product of K_t^{n-1} and the Dirichlet heat kernel K_t^{Dir} on $\mathbb{R}_+$. The first consideration is an application of the method of images. We take the heat kernel on $\mathbb{R}^n$ that solves the heat equation, but doesn't satisfy the Dirichlet boundary condition on $\{x_n = 0\}$. Then we consider the reflected heat kernel, defined by $K_t^{n,ref}(t, x', x_n, y', y_n) := K_t^n(t, x', x_n, y', -y_n)$ which also solves the heat equation. Neither of these satisfy the Dirichlet boundary condition, but their combination $K_t^n - K_t^{n,ref}$ does. Thus, if we restrict $K_t^n - K_t^{n,ref}$ to $\{x_n \geq 0\}$, it is a solution of the heat equation with Dirichlet boundary condition. The second consideration is based on the fact that we are working on a product space. One of the properties of the heat equation is that if we have two spaces A, B and two heat kernels K_A, K_B, then $K_A \cdot K_B$ is a solution of the heat equation on $A \times B$. By noticing that the Dirichlet boundary condition is a requirement only for x_n, we conclude that $K_t^{n-1} \cdot K_t^{Dir}$ is the Dirichlet heat kernel on $\mathbb{R}^{n-1} \times \mathbb{R}_+$.

Let us now consider the case of a model corner $\mathbb{R}^n_k$ and the Dirichlet heat kernel. Since this is a product space, by following the second consideration we see that the Dirichlet heat kernel with coordinates $t > 0$, $(x^k, x_1, \dots, x_k), (y^k, y_1, \dots, y_k) \in \mathbb{R}^n_k$ is given by

$$\begin{aligned} &K_t(x^k, x_1, \dots, x_k, y^k, y_1, \dots, y_k) = \\ &(4\pi t)^{-\frac{n}{2}} e^{-\frac{|x^k-y^k|^2}{4t}} (e^{-\frac{|x_1-y_1|^2}{4t}} - e^{-\frac{|x_1+y_1|^2}{4t}}) \dots (e^{-\frac{|x_k-y_k|^2}{4t}} - e^{-\frac{|x_k+y_k|^2}{4t}}). \end{aligned} \tag{3.1.2}$$

Following the idea in [Gri04], we should understand some key properties and symmetries of this model heat kernel and incorporate them into a calculus that is eventually going to contain the true heat kernel of a manifold with corners. By looking at (3.1.2), we see that the heat kernel on the model corner

- is translation invariant in directions that are tangent to the corner, i.e. $K_t(x^k+z, x_1, \dots, x_k, y^k+z, y_1, \dots, y_k) = K_t(x^k, x_1, \dots, x_k, y^k, y_1, \dots, y_k)$ for $z \in \mathbb{R}^{n-k}$.
- is a product of $t^{-n/2}$ and a smooth function of

$$\sqrt{t},\ X^k = \frac{x^k - y^k}{\sqrt{t}},\ \xi_1 = \frac{x_1}{\sqrt{t}},\ \eta_1 = \frac{y_1}{\sqrt{t}}, ...,\ \xi_k = \frac{x_k}{\sqrt{t}},\ \eta_k = \frac{y_k}{\sqrt{t}}. \tag{3.1.3}$$

We now expand (3.1.2) in order to understand it more thorough. We have

$$(3.1.2) = \sum (-1)^\mu (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x^k-y^k|^2}{4t}} e^{-\sum_{i=1}^{\nu} \frac{|x_{\lambda_i}-y_{\lambda_i}|^2}{4t}} e^{-\sum_{i=1}^{\mu} \frac{|x_{\kappa_i}+y_{\kappa_i}|^2}{4t}} \tag{3.1.4}$$

where the sum is taken for $\nu + \mu = k$ and $\{\lambda_1, ..., \lambda_\nu, \kappa_1, ..., \kappa_\mu\} = \{1, ..., k\}$. If in each summand we gather the term $|x^k - y^k|^2$ with the λ_i terms and we denote them by $x',\ y'$ for simplicity, then it becomes

$$(-1)^\mu (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x'-y'|^2}{4t}} e^{-\sum_{i=1}^{\mu} \frac{|x_{\kappa_i}+y_{\kappa_i}|^2}{4t}}. \tag{3.1.5}$$

(3.1.5) will be called the $\mu-$reflection term after setting $X' = \frac{x'-y'}{\sqrt{t}}, \xi_{\kappa_i} = \frac{x_{\kappa_i}}{\sqrt{t}}, \eta_{\kappa_i} = \frac{y_{\kappa_i}}{\sqrt{t}}$, we see that it satisfies $O\big((\sum_{i=1}^{\mu}(\xi_{\kappa_i} + \eta_{\kappa_i}) + |X'|)^{-\infty}\big)$ as $\xi_{\kappa_i},\ \eta_{\kappa_i},\ |X'| \to \infty$. Thus, additionally, (3.1.2) satisfies:

- In a corner of codimension k, it is a sum of $\mu-$reflection terms, with $\mu = 0, \dots, k$. Each $\mu-$reflection term is a sum of functions with variables $\xi_{\kappa_1}, \eta_{\kappa_1}, \dots, \xi_{\kappa_\mu}, \eta_{\kappa_\mu}, X^{\mu,\kappa}$, where $\kappa_1, \dots, \kappa_\mu$ run over all the possible choices of $\{1, \dots, k\}$, and each one decays like $O\big((\sum_{i=1}^{\mu}(\xi_{\kappa_i} + \eta_{\kappa_i}) + |X^{\mu,\kappa}|)^{-\infty}\big)$ as $\xi_{\kappa_i},\ \eta_{\kappa_i},\ |X^{\mu,\kappa}| \to \infty$.

The same behaviour holds for the interior ($k = 0$) and near the smooth parts of the boundary ($k = 1$). Notice lastly, that

- for $x \neq y$, the heat kernel and its derivatives are $O(t^{\infty})$ as $t \to 0$.

Now that we saw the main properties of the heat kernel on a model corner, we are ready to define the calculus.

3.2 The Heat Calculus

Definition 3.2.1. Let M be a manifold with corners of dimension n and let $a \leq 0$. The corner calculus, which we denote by $\Psi^a_{H,cor}(M)$ is the space of functions $A : (0, +\infty) \times M \times M \to \mathbb{R}$, such that

(a) A is smooth.

(b) If $x \neq y$ then $\partial_t^{\beta} \nabla^{\gamma}_{X,Y} A(t, x, y) = O(t^{\infty})$, as $t \to 0$ and $X, Y \in \mathcal{X}(M)$. Moreover we assume that for $y \in M$ fixed, and $x \in M$ such that $d(x, y) \geq c > 0$, the bound is independent of $d(x, y)$.

(c) If $p \in M \setminus \partial^1 M$, there exists a local coordinate chart (U, ϕ) around p and $\tilde{A} \in C^{\infty}([0, \infty)_{\sqrt{t}} \times \mathbb{R}^n \times U)$ and $M \in C^{\infty}((0, +\infty) \times M \times M)$ such that for $x, y \in U$ we have

$$A(t, x, y) = t^{-\frac{n+2}{2} - a} \tilde{A}(t, \frac{\phi(x) - \phi(y)}{\sqrt{t}}, y) + M(t, x, y)$$

with $D^{\beta}_{\sqrt{t},X,y} \tilde{A}(t, X, y) = O(|X|^{-\infty})$ as $|X| \to \infty$ and $D^{\beta}_{t,x,y} M(t, x, y) = O(t^{\infty})$ as $t \to 0$. This estimate is uniform for bounded $t > 0$ and $y \in U$.

(d) $\forall\ x \in \partial_k M$, there exists a coordinate chart (U, ϕ) around p and functions

$$\begin{aligned}
&\tilde{A}^{dir} \in C^{\infty}([0, \infty)_{\sqrt{t}} \times \mathbb{R}^n \times U), \\
&\tilde{A}^{i-ref} \in C^{\infty}([0, \infty)_{\sqrt{t}} \times \mathbb{R}^{n-i} \times \mathbb{R}^{2i}_{+} \times U) \\
&M \in C^{\infty}((0, +\infty) \times M \times M)
\end{aligned}$$

with $i = 1, ..., k$ such that for $x, y \in U$ we have

$$\begin{aligned}
A(t, x, y) = t^{-\frac{n+2}{2} - a} \Big(&\tilde{A}^{dir}(t, X, y) \\
&+ \sum_{i=1}^{k} \sum_{\kappa} (-1)^i \tilde{A}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y^{i,\kappa}) \Big) \\
&+ M(t, x, y),
\end{aligned} \tag{3.2.1}$$

where $\kappa_1, \dots, \kappa_i$ run over all the possible choices of $\{1, \dots, k\}$. Here $\phi = (\phi^1, \dots, \phi^{n-k}, \phi_1, \dots, \phi_k)$. More presicely we have that $X = \frac{\phi(x)-\phi(y)}{\sqrt{t}}$, $X^{i,\kappa} = (\frac{\phi^1(x)-\phi^1(y)}{\sqrt{t}}, \dots, \frac{\phi^{n-k}(x)-\phi^{n-k}(y)}{\sqrt{t}}, \frac{x_{\lambda_1}-y_{\lambda_1}}{\sqrt{t}}, \dots, \frac{x_{\lambda_{k-i}}-y_{\lambda_{k-i}}}{\sqrt{t}})$ for a specific permutation λ such that $\lambda = \{1, \dots, k\} \setminus \{\kappa_1, \dots, \kappa_i\}$, $\xi_j = \frac{\phi_j(x)}{\sqrt{t}}$, and $\eta_j = \frac{\phi_j(y)}{\sqrt{t}}$ for $j = 1, \dots, k$. Furthermore, the terms $\tilde{A}^{dir}$ and $\tilde{A}^{i-ref}$ satisfy $\tilde{A}^{dir} = O(|X|^{-\infty})$ as $|X| \to \infty$ and $\tilde{A}^{i-ref} = O\big((\sum_{j=1}^{i}(\xi_{\kappa_j} + \eta_{\kappa_j}) + |X^i|)^{-\infty}\big)$ as $(\sum_{j=1}^{i}(\xi_j + \eta_j) + |X^i|) \to \infty$ respectively, together with derivatives, uniformly for bounded $t > 0$, $y \in U$, and for M we have $D^{\beta}_{t,x,y}M(t,x,y) = O(t^{\infty})$, as $t \to 0$. Finally, we set

$$\begin{aligned}&\tilde{A}^{cor}(t, X^k, \xi_1, \eta_1, \dots, \xi_k, \eta_k, y)\\ &:= \big(\tilde{A}^{dir}(t,X,y) + \sum_{i=1}^{k}\sum_{\kappa}(-1)^i \tilde{A}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y^{i,\kappa})\big)\end{aligned}$$

Remark 3.2.1.

- The notation $f \in C^{\infty}([0,+\infty)_{\sqrt{t}})$ means that $f(t)$ is a smooth function of $\sqrt{t}$. Whenever we refer to $\tilde{A}$ we will denote it by $\tilde{A}(t)$ and we will mean that it is a smooth function of $\sqrt{t}$.

- Each A^{i-ref} is defined on a bundle with base manifold a boundary face of codimension i, which contains the corner of codimension k. Therefore, the $y^{i,k}$ variable at the end is in fact the variable for this boundary face. In order to keep the notation more simple, we will denote it with the generic y and thought the chapter whenever we examine a term of this form we will assume that y denotes the variable of the boundary face. This will be further analysed below. (Among else, see the example after Lemma 3.2.1)

- Last but important is the exact change of notation: Throught the text we will interchange between A and $\tilde{A}$ for $A \in \Psi^a_{H,cor}(M)$. For this reason, instead of always referring to the chart (U, ϕ), we will denote the coordinates of a boundary face of codimension k by x^k, y^k and the boundary face coordinate functions by $x_1, y_1, \dots, x_k, y_k$. When we are on the interior x, y will be used both for points on the manifold and for coordinates systems around these points.

Definition 3.2.2. Let M as before and $A \in \Psi^a_{H,cor}(M)$. In a local coordinate system U around $y \in U$, we define the principal symbol by

$$\Phi^{int}_a(A)(X,y) := \tilde{A}(0,X,y)$$

for y in the interior, and by

$$\Phi_a^{cor}(A)(X^k,\xi_1,\eta_1,...,\xi_k,\eta_k,y) := \tilde{A}^{cor}(0,X^k,\xi_1,\eta_1,...,\xi_k,\eta_k,y)$$

for y in a corner of codimension $k \in \mathbb{N}$.

We gave a definition of the principal symbol based on local coordinates. However, the principal symbol can be defined globally and as we will see, it is independent of local coordinates. More precisely, it is a section of a bundle, which we briefly describe now. Let A_k be a boundary face of codimension $k \in \mathbb{N}$, i.e. $A_k \in \mathcal{M}_k(M)$. Let $p \in A_k$ and consider T_pA_k. There is a natural action of T_pA_k to $T_pM \times T_pM$ defined by

$$\begin{aligned} T_pA_k \times \big(T_pM \times T_pM\big) &\to T_pM \times T_pM, \\ v \cdot (u,w) &\to (u+v, w+v). \end{aligned}$$

We form the bundle

$$E^{A_k} \to A_k \text{ with fiber } E_p^{A_k} = \frac{T_pM \times T_pM}{T_pA_k},$$

and take the inward part, defined by

$$E_+^{A_k} \to A_k \text{ with fiber } E_{+,p}^{A_k} = \frac{T_p^{A_k,+}M \times T_p^{A_k,+}M}{T_pA_k},$$

where $T_p^{A_k,+}M$ are the vectors that are tangent to A_k and those which are pointing at the interior of the manifold. The fiber $E_{+,p}^{A_k}$ can be described in local coordinates by $(X^k,\xi_1,\eta_1,\dots,\xi_k,\eta_k,y)$ with $\xi_1,\eta_1,\dots,\xi_k,\eta_k \geq 0$ and where y are the local coordinates in A_k. Finally, the dimension of the fiber is simply $n+k$.

Another important aspect here is the relation of these bundles between different boundary faces. Let $F \in \mathcal{M}_k(M)$, $G \in \mathcal{M}_l(M)$ with $k \geq l$ and F submanifold of G. Then over F we define the map

$$\begin{aligned} \beta^{FG} : E_+^F &\to E_+^G, \\ [u,v]_F &\to [u,v]_G. \end{aligned}$$

This map is well defined since $T_pF \subseteq T_pG$. In local coordinates $x^k, x_1, ..., x_k$, β^{FG} takes the form

$$\begin{aligned} &\beta^{FG}(X^k,\xi_1,\eta_1,\dots,\xi_k,\eta_k,y) \\ &= (X^k,\xi_{i_1}-\eta_{i_1},\dots,\xi_{i_{k-l}}-\eta_{i_{k-l}},\xi_{j_1},\eta_{j_1},\dots,\xi_{j_l},\eta_{j_l},y) \end{aligned}$$

where $F = \{x_1 = \cdots = x_k = 0\}$, $G = \{x_{j_1} = \cdots = x_{j_l} = 0\}$ and $\{i_1,\dots,i_{k-l},j_1,\dots,j_l\} = \{1,\dots,k\}$.

Definition 3.2.3. Let M be a compact, connected manifold of dimension n, with corners of codimension at most $k \in \mathbb{N}$. For $i \in \{1, \dots, k\}$, let $A_i \in \mathcal{M}_i(M)$ be a boundary face of codimension i and consider the fibration $E_+^{A_i} \to A_i$. We define

$$C^\infty_{fib}(E_+^{A_i}) = \big\{\phi : E_+^{A_i} \to \mathbb{C} :\ \phi \text{ has rapid decay on the fibers}\big\}.$$

Consider now all the boundary faces F_j that have codimension $j \leq i-1$, with $A_i \cap F_j \neq \emptyset$ and the fibrations $E_+^{F_j} \to F_j$. Then we define

$$\begin{aligned} C^\infty_{cor}(E_+^{A_i}) = \Big\{ & \phi : E_+^{A_i} \to \mathbb{C} : \exists\ \phi_{A_i}^{dir} \in C^\infty_{fib}(T_{A_i}M), \\ & \phi_{F_1}^{1-ref} \in C^\infty_{fib}(E_+^{F_1}) \text{for all } F_1 \text{ s.t. } F_1 \cap A_i \neq \emptyset, \dots, \\ & \phi_{F_{i-1}}^{(i-1)-ref} \in C^\infty_{fib}(E_+^{F_{i-1}}) \text{for all } F_{i-1} \text{ s.t. } F_{i-1} \cap A_i \neq \emptyset, \\ & \phi^{i-ref} \in C^\infty_{fib}(E_+^{A_i}),\ \text{such that} \\ & \phi = \beta^* \phi_{A_i}^{dir} - \sum_{F_1 \cap A_i \neq \emptyset} \beta^*_{A_i F_1} \phi_{F_1}^{1-ref} + \dots + \\ & (-1)^{i-1} \sum_{F_{i-1} \cap A_i \neq \emptyset} \beta^*_{A_i F_{i-1}} \phi_{F_{i-1}}^{1-ref} + \phi_{A_i}^{i-ref} \Big\}. \end{aligned}$$

Remark 3.2.2. The local coordinates on $E_+^{A_i}$ are $X^i, \xi_1, \eta_1, ..., \xi_i, \eta_i, y_i$ and rapid decay along the fibers for a function $\phi \in C^\infty_{fib}(E_+^{A_i})$ means that $\forall\ N \in \mathbb{N}$

$$\phi = O\big((|X^i| + \sum_{\lambda=1}^{i} (\xi_\lambda + \eta_\lambda))^{-N}\big),\ \text{as } |X^i| + \sum_{\lambda=1}^{i} (\xi_\lambda + \eta_\lambda) \to \infty,$$

together with derivatives.

The definition of the space $C^\infty_{cor}(E_+^{A_i})$ seems complicated at first. In order to understand why it makes sense to define it that way, we should see it in parallel with point (d) of Definition 3.2.1. If we take $A \in \Psi^a_{H,cor}(M)$, $p \in A_i \in \mathcal{M}_i(M)$, and consider the local representation of A near p, then it is straightforward to see that it's symbol lies in $C^\infty_{cor}(E_+^{A_i})$. Therefore, the space $C^\infty_{cor}(E_+^{A_i})$ is important because for a function $A \in \Psi^a_{H,cor}(M)$, it carries it's principal symbol over the component A_i, and it is independent of local coordinates. More precisely, we have the following lemma:

Lemma 3.2.1. Let M be a compact manifold with corners of dimension n, and let $a \leq 0$. Then

(a) If (c) and (d) in Definition 3.2.1 hold in one coordinate system, then they hold in any.

(b) $\Phi_a^{int}(A)$ is defined invariantly as a function of $C^\infty_{fib}(TM)$.

(c) For $A_k \in \mathcal{M}_k$, the corner leading term $\Phi_a^{cor}(A)$ is defined invariantly as a function of the bundle $E_+^{A_k} \to A_k$, with $\Phi_a^k(A) \in C^\infty_{cor}(E_+^{A_k})$.

Proof. We first prove (b). Let $p \in \operatorname{int}(M)$, $(U,x), (\tilde{U}, \tilde{x})$ coordinate charts around p, and $\psi(x) = \tilde{x}$, $\psi = (\psi_1, \ldots, \psi_n)$ a change of coordinates. Suppose that (c) of Definition 3.2.1 holds for $\tilde{x}$ with the function $\overline{\tilde{A}}(t, \tilde{X}, \tilde{y})$. For $x,\ y \in U$, we have that there exists a smooth matrix valued function $h : U \times U \to \mathbb{R}$ such that

$$\psi(x) - \psi(y) = h(x,y)(x-y), \text{ with } h(y,y) = d\psi_{|_y}. \tag{3.2.2}$$

Let $\chi \in C^\infty_c(U)$ be a cut-off function $\chi \in C^\infty_c(U)$ which is one near p, and set

$$\tilde{A}(t,X,y) = \overline{\tilde{A}}(t, h(y + X\sqrt{t}, y)X, \psi(y))\chi(y + X\sqrt{t}). \tag{3.2.3}$$

Then $\tilde{A}$ satisfies (c) of Definition (3.2.1). Evaluating (3.2.3) at $t = 0$ we get that

$$\tilde{A}(0,X,y) = \overline{\tilde{A}}(0, d\psi_{|_y}X, \psi(y)),$$

which shows that $\Phi_a^{int}(A)$ behaves as a function on TM. That proves (b) and the first part of (a). Let now $p \in A_k \in \mathcal{M}_k$ and $\psi(x) = \tilde{x},\ \psi = (\psi^k, \psi_1, ..., \psi_k)$ a change of coordinates around p. Suppose that (d) holds for $\tilde{x}$ with functions

$$\overline{\tilde{A}}^{cor}(t, \tilde{X^k}, \tilde{\xi_1}, \tilde{\eta_1}, ..., \tilde{\xi_k}, \tilde{\eta_k}, \tilde{y}). \tag{3.2.4}$$

For $x,\ y \in U$, following the same considerations as before, we obtain the formula

$$\psi^k(x) - \psi^k(y) = h^k(x,y)(x^k - y^k) + \sum_{j=1}^k h_j(x,y)(x_j - y_j),$$

where $h^k(x,y) = \int_0^1 \frac{\partial(\psi_1,\ldots,\psi_{n-k})}{\partial(x_1,\ldots,x_n)}(tx + (1-t)y)dt$ and $h_j(x,y)$ is a vector defined by $\big(\int_0^1 (\partial_{x_j}\psi^1, \ldots, \partial_{x_j}\psi^{n-k})(tx + (1-t)y)dt\big)^t$. Note also that $h^k(y,y) = d\psi^k_{|y}$

and $h_j(y,y) = d_j\psi_{|y}$. Since ψ sends hypersurfaces to hypersurfaces, wlog we can assume that $\psi_i(x) = x_i\phi_i(x)$ for $i = 1, \ldots, k$. We then set

$$\tilde{A}^{cor}(t, X^k, \xi_1, \eta_1, ..., \xi_k, \eta_k, y) =$$
$$\overline{\tilde{A}}^{cor}(t, h^k X^k + \sum_{j=1}^{k} h_j(\xi_j - \eta_j), \phi_1(x)\xi_1, \phi_1(y)\eta_1, ..., \phi_k(x)\xi_k, \phi_k(y)\eta_k, \psi^i(y))\chi, \tag{3.2.5}$$

where h^i, h_j are evaluated at (x,y) with $x = (y^k + \sqrt{t}X^k, \sqrt{t}\xi_1, \ldots, \sqrt{t}\xi_k)$, $y = (y^k, \sqrt{t}\eta_1, \ldots, \sqrt{t}\eta_k)$ and χ is evaluated at x. $\tilde{A}^{i-ref}$ has the required decay properties. Evaluating (3.2.5) at $t = 0$ we obtain

$$\tilde{A}^{cor}(0, X^k, \xi_1, \eta_1, ..., \xi_k, \eta_k, y) =$$
$$\overline{\tilde{A}}^{cor}(0, d\psi^k_{|y} X^k + \sum_{j=1}^{k} d_j\psi_{|y}(\xi_j - \eta_j), \{d_i\psi_i(x)\xi_i, d_i\psi_i(y)\eta_i\}_{i=1}^k, \psi^k(y))\chi.$$

This is the same rule for the change of variables for functions defined on $E_+^{A_k}$. That proves (c) and the second part of (a). □

Lemma 3.2.1 says that the interior symbol, as well as the corner symbols are well defined, independent of coordinates and they belong to the spaces $C^\infty_{fib}(TM)$, $C^\infty_{cor}(E_+^{A_k})$ respectively. The question we deal with next is what form does the total symbol takes for a function $A \in \Psi^a_{H,cor}(M)$, or similarly, which collection of functions and what kind of compatibility conditions should they obey in order to obtain a function $A \in \Psi^a_{H,cor}(M)$ such that has a total symbol this collection of functions. We begin with a simple example and then give the general description.

Suppose we have a 2-dimensional manifold M, with two hypersurfaces H_1, H_2 and two codimension-2 corners A_1, A_2 with $H_1 \cap H_2 = A_1 \cup A_2$. Suppose we have symbols that are:

- On the interior: ϕ^{int}
- On H_1 : $\beta^*_{int}\phi^{dir}_{H_1} - \phi^{1-ref}_{H_1}$
- On H_2 : $\beta^*_{int}\phi^{dir}_{H_2} - \phi^{1-ref}_{H_2}$
- On A_1 : $\beta^*_{int}\phi^{dir}_{A_1} - \left(\beta^*_{A_1H_1}\phi^{1-ref}_{A_1,H_1} + \beta^*_{A_1H_2}\phi^{1-ref}_{A_1,H_2}\right) + \phi^{2-ref}_{A_1}$
- On A_2 : $\beta^*_{int}\phi^{dir}_{A_2} - \left(\beta^*_{A_2H_1}\phi^{1-ref}_{A_2,H_1} + \beta^*_{A_2H_2}\phi^{1-ref}_{A_2,H_2}\right) + \phi^{2-ref}_{A_2}$

Now we want to define a function such that on the interior it has the symbol ϕ^{int}, on H_1 the symbol $\beta_1^*\phi_{H_1}^{dir} - \phi_{H_1}^{1-ref}$ and so on. An obvious choice is to define locally near A_1 (the other cases are similar or easier) the function

$$\begin{aligned}A(t, x'', y'', x_1, y_1, x_2, y_2) = \\ t^{-\frac{n+2}{2}-a}\Bigg(&\phi^{int}(\frac{x-y}{\sqrt{t}}, y) - \phi_{H_1}^{1-ref}(\frac{x''-y''}{\sqrt{t}}, \frac{x_2-y_2}{\sqrt{t}}, \frac{x_1}{\sqrt{t}}, \frac{y_1}{\sqrt{t}}, y'', y_2) \\ &- \phi_{H_2}^{1-ref}(\frac{x''-y''}{\sqrt{t}}, \frac{x_1-y_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}}, \frac{y_2}{\sqrt{t}}, y'', y_1) \\ &+ \phi_{A_1}^{2-ref}(\frac{x''-y''}{\sqrt{t}}, \frac{x_1}{\sqrt{t}}, \frac{y_1}{\sqrt{t}}, \frac{x_2}{\sqrt{t}}, \frac{y_2}{\sqrt{t}}, y'')\Bigg).\end{aligned}$$

If we compute the symbol in the interior, by using the fact that $\phi_{H_1}^{1-ref}, \phi_{H_2}^{1-ref}$, $\phi_{A_2}^{2-ref}$ decay rapidly, we see that it is simply ϕ^{int}. But if we try to compute it on H_1 we need to assume that $\phi^{int}_{|T_{H_1}M} = \phi^{dir}_{H_1}$. Working in a similar way, we must assume

- On H_1: $\phi^{int}_{|T_{H_1}M} = \phi^{dir}_{H_1}$
- On H_2: $\phi^{int}_{|T_{H_2}M} = \phi^{dir}_{H_2}$
- On A_1: $\phi^{int}_{|T_{A_1}M} = \phi^{dir}_{A_1}$, $\beta^*_{A_1H_1}(\phi_{H_1}^{1-ref}) = \beta^*_{A_1H_1}(\phi_{A_1,H_1}^{1-ref})$, $\beta^*_{A_1H_2}(\phi_{H_2}^{1-ref}) = \beta^*_{A_1H_2}(\phi_{A_1,H_2}^{1-ref})$.

With this assumptions, the above choice is indeed a function of $\Psi^a_{H,cor}(M)$ with symbol as we required. This motivates the following definition:

Definition 3.2.4. Let M be a compact, connected manifold of dimension n, with corners of codimension at most k and $a \leq 0$. Let $a_1 = |\mathcal{M}_1|, \ldots, a_k = |\mathcal{M}_k|$. Then

(a) Let $F \in \mathcal{M}_k, G \in \mathcal{M}_l,\ k > l$ and F submanifold of G, and let $\phi \in C^\infty_{cor}(E_+^F), \psi \in C^\infty_{cor}(E_+^G)$. We say that ϕ, ψ are compatible if for all $A \in \mathcal{M}_m(M)$ with $m \leq j$ and $F \subseteq G \subseteq A$ we have

$$\beta^*_{FA}(\psi_{GA}^{m-ref}) = \beta^*_{FA}(\phi_{FA}^{m-ref})$$

where ψ_{GA}^{m-ref} is defined as the part of ψ which is the m-reflection term coming from embedding F into A. (See Definition 3.2.3).

(b) We define

$$S_{H,cor}(M) = \Big\{(\phi^{int}, \phi_{H_1^1}, \dots, \phi_{H_1^{a_1}}, \dots, \phi_{H_k^1}, \dots, \phi_{H_k^{a_k}}) \in C^\infty_{fib}(TM) \times \prod_{j=1}^{a_1} C^\infty_{cor}(E_+^{H_1^j}) \dots, \times \prod_{j=1}^{a_k} C^\infty_{cor}(E_+^{H_k^j}),$$
$$\text{such that: For each boundary face H we have } \phi^{int}_{|T_H M} = \phi^{dir}_H$$
$$\text{and for each } H_j \in \mathcal{M}_j(M), H^i \in \mathcal{M}_i(M), \text{with } i > j \geq 1$$
$$\text{we have that } \phi_{H_i},\ \phi_{H^j} \text{ are compatible whenever } H^i \subseteq H_j \neq \emptyset$$
$$\text{where } \phi_{H_i} \in C^\infty_{cor}(E_+^{H_i}), \phi_{H_j} \in C^\infty_{cor}(E_+^{H^j})\Big\}.$$

Remark 3.2.3. Compatibility can be seen as follows: As we have seen from the model heat kernel and the Definition of the heat calculus, the heat kernel on a corner F of codimension i is defined with respect to the lower codimension corners A with the property $F \subseteq A$. If we have $F \subseteq G$ with symbols ϕ, ψ respectively, compatibility means that the m-reflection components of both ϕ, ψ should be equal when restricted to F, for every $m \leq \operatorname{codim}(G)$, and that should happen for every pair of corners F, G such that $F \subseteq G$.

After having clarified how the symbols look like, we are able to obtain a short exact sequence for the calculus. More precisely, we have the following

Proposition 3.2.1. Let M be a compact, connected manifold of dimension n with corners of codimension at most k and let $a \leq 0$. Then there is a short exact sequence

$$0 \to \Psi^{a-1/2}_{H,cor} \to \Psi^a_{H,cor}(M) \xrightarrow{\Phi_a} S_{H,cor}(M) \to 0. \tag{3.2.6}$$

Proof. Essentially the surjectivity part is done in the example above. The general case requires just heavier notation. The injectivity part is proved by using Taylor expansion w.r.t $\sqrt{t}$ around 0. $\square$

Now we state and prove the following composition formula for functions in our calculus. Here dz will denote the Riemannian volume form.

Proposition 3.2.2. Let $A \in \Psi^a_{H,cor}(M),\ B \in \Psi^\beta_{H,cor}(M)$ with $a, b < 0$. We define

$$A * B(t, x, y) = \int_0^t \int_M A(t-s, x, z) B(s, z, y) dz ds. \tag{3.2.7}$$

Then $A * B \in \Psi^{a+b}_{H,cor}(M)$.

Proof. On a corner of codimension $k \in \mathbb{N}$, for $x, y \in U$, where U is a coordinate neighboorhood we have

$$\begin{aligned}
&\tilde{A}^{cor}(t, X^k, \xi_1, \eta_1, ..., \xi_k, \eta_k, y) \\
&= t^{-\frac{n+2}{2}-a}\left(\tilde{A}^{dir}(t, X, y) + \sum_{i=1}^{k}\sum_{\kappa}(-1)^i \tilde{A}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{B}^{cor}(t, X^k, \xi_1, \eta_1, ..., \xi_k, \eta_k, y) \\
&= t^{-\frac{n+2}{2}-a}\left(\tilde{B}^{dir}(t, X, y) + \sum_{i=1}^{k}\sum_{\kappa}(-1)^i \tilde{B}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y)\right),
\end{aligned}$$

where κ is taken over all possible permutations as in Definition 3.2.1. On U, if we consider the parts $\tilde{A}^{cor}$ and $\tilde{B}^{cor}$, (3.2.7) consists of sums of 3 different terms, namely

- $I_1 = \tilde{A}^{dir}(t, X, y)\tilde{B}^{dir}(t, X, y)$
- $I_2 = \tilde{A}^{dir}(t, X, y)\tilde{B}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y)$
- $I_3 = \tilde{A}^{i-ref}(t, X^{i,\kappa}, \xi_{\kappa_1}, \eta_{\kappa_1}, ..., \xi_{\kappa_i}, \eta_{\kappa_i}, y)$
  $\cdot \tilde{B}^{j-ref}(t, X^{j,\kappa'}, \xi_{\kappa'_1}, \eta_{\kappa'_1}, ..., \xi_{\kappa'_j}, \eta_{\kappa'_j}, y)$

We will show that the theorem holds for I_3, which is the most general form and thus it implies the same for I_1 and I_2. Take $\chi \in C_c^\infty(U)$ that is 1 near $x, y \in U$. First we consider the part

$$\int_0^t \int_M A(t-s, x, z)B(s, z, y)\chi(z)dzds.$$

Since $x, y \in U$ we can take the local formulas for A^i, B^j, thus we obtain

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k_+} &\tilde{A}^i(t-s, \frac{x^k - z^k}{\sqrt{t-s}}, \frac{(x_1, z_1, ..., x_k, z_k)}{\sqrt{t-s}}, z) \\
&\cdot \tilde{B}^j(s, \frac{z^k - y^k}{\sqrt{s}}, \frac{(z_1, y_1, ..., z_k, y_k)}{\sqrt{s}}, z) \\
&\cdot \chi(z^k, z_1, ..., z_k)(t-s)^{-\frac{n+2}{2}-a}s^{-\frac{n+2}{2}-\beta}dz^k dz_1...dz_k ds. \quad (3.2.8)
\end{aligned}$$

Set $\frac{z^k-y^k}{\sqrt{t}} = Z^k$, $\frac{z_\lambda}{\sqrt{t}} = \zeta_\lambda$, $\frac{x_\lambda}{\sqrt{t}} = \xi_\lambda$, $\frac{y_\lambda}{\sqrt{t}} = \eta_\lambda$, $\frac{x^k-y^k}{\sqrt{t}} = X^k, \sigma = \frac{s}{t}$ for $\lambda = 1, ..., k$. Then (3.2.8) becomes a function of $t^{-\frac{n+2}{2}-a-\beta}$ times

$$\int_0^1 \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k_+} \tilde{A}^i(t(1-\sigma), \frac{X^k - Z^k}{\sqrt{1-\sigma}}, \{\frac{\xi_{a_\lambda} - \zeta_{a_\lambda}}{\sqrt{1-\sigma}}\}_{\lambda=1}^{k-i}, \{\frac{\xi_{\beta_\lambda}, \zeta_{\beta_\lambda}}{\sqrt{1-\sigma}}\}_{\lambda=1}^{i})$$
$$\cdot \tilde{B}^j(t\sigma, \frac{Z^k}{\sqrt{\sigma}}, \{\frac{\zeta_{\gamma_\lambda} - \eta_{\gamma_\lambda}}{\sqrt{\sigma}}\}_{\lambda=1}^{k-j}, \{\frac{\zeta_{\delta_\lambda}, \eta_{\delta_\lambda}}{\sqrt{\sigma}}\}_{\lambda=1}^{j})$$
$$\cdot (1-\sigma)^{-\frac{n+2}{2}-a} \sigma^{-\frac{n+2}{2}-\beta} \chi dZ^k d\zeta_1 ... d\zeta_k d\sigma. \tag{3.2.9}$$

There is a number $\mu \in \{0, \dots, \min(k-i, k-j)\}$, such that μ of the a_λ and γ_λ are the same. W.l.o.g. we assume that $a_1 = \gamma_1, ..., a_\mu = \gamma_\mu$. Rewrite $\frac{\xi_{a_\lambda}-\zeta_{a_\lambda}}{\sqrt{1-\sigma}} = \frac{\xi_{a_\lambda}-\eta_{a_\lambda}}{\sqrt{1-\sigma}} - \frac{\zeta_{a_\lambda}-\eta_{a_\lambda}}{\sqrt{1-\sigma}}$, leave $\frac{\zeta_{\gamma_\lambda}-\eta_{\gamma_\lambda}}{\sqrt{\sigma}}$ as it is for $\lambda = 1, ..., \mu$ and write them as $\frac{X^\mu - Z^\mu}{\sqrt{1-\sigma}}$ and $\frac{Z^\mu}{\sqrt{\sigma}}$ respectively. For simplicity, we write the first $n-k+\mu$ terms in $\tilde{A}^i, \tilde{B}^j$ as $\frac{X^{k+\mu}-Z^{k+\mu}}{\sqrt{1-\sigma}}$, $\frac{Z^{k+\mu}}{\sqrt{\sigma}}$ respectively. Now, by hypothesis for $t < c$, y bounded and $N \in \mathbb{N}$ with $N \geq n-1$, there exists $M > 0$ s.t.

$$|A^i| \lesssim_N (|\frac{X^{k+\mu} - Z^{k+\mu}}{\sqrt{1-\sigma}}| + \sum_{\lambda=\mu+1}^{k-i} |\frac{\xi_{a_\lambda} - \zeta_{a_\lambda}}{\sqrt{1-\sigma}}| + \sum_{\lambda=1}^{i} \frac{\xi_{\beta_\lambda} + \zeta_{\beta_\lambda}}{\sqrt{1-\sigma}})^{-N}, \text{ for } |\Box| \geq M \tag{3.2.10}$$

where $\Box = \left(\frac{X^{k+\mu}-Z^{k+\mu}}{\sqrt{1-\sigma}}, \{\frac{\xi_{a_\lambda}-\zeta_{a_\lambda}}{\sqrt{1-\sigma}}\}_{\lambda=\mu+1}^{k-i}, \{\frac{\xi_{\beta_\lambda}}{\sqrt{1-\sigma}}, \frac{\zeta_{\beta_\lambda}}{\sqrt{1-\sigma}}\}_{\lambda=1}^{i} \right)$, and

$$|B^j| \lesssim_N (|\frac{Z^{k+\mu}}{\sqrt{\sigma}}| + \sum_{\lambda=\mu+1}^{k-j} |\frac{\zeta_{\gamma_\lambda} - \eta_{\gamma_\lambda}}{\sqrt{\sigma}}| + \sum_{\lambda=1}^{j} \frac{\zeta_{\delta_\lambda} + \eta_{\delta_\lambda}}{\sqrt{\sigma}})^{-2N}, \text{ for } |\triangle| \geq M \tag{3.2.11}$$

where $\triangle = \left(\frac{Z^{k+\mu}}{\sqrt{\sigma}}, \{\frac{\zeta_{\gamma_\lambda}-\eta_{\gamma_\lambda}}{\sqrt{\sigma}}\}_{\lambda=\mu+1}^{k-j}, \{\frac{\zeta_{\delta_\lambda}}{\sqrt{\sigma}}, \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}}\}_{\lambda=1}^{j} \right)$. We split the integral in the parts $\sigma \leq 1/2$ and $\sigma \geq 1/2$. By the cut-off we integrate on the area $|(y^k + \sqrt{t}Z^k, \sqrt{t}\zeta_1, \dots, \sqrt{t}\zeta_k)| \leq K$ and wlog we can assume that $K > M$. Let now

$$X = \left(X^{k+\mu}, \{\xi_{a_\lambda}\}_{\lambda=\mu+1}^{k-i}, \{\eta_{\gamma_\lambda}\}_{\lambda=\mu+1}^{k-j}, \{\xi_{\beta_\lambda}\}_{\lambda=1}^{i}, \{\eta_{\delta_\lambda}\}_{\lambda=1}^{j} \right) \text{ with } |X| \geq K > M.$$

The part $\sigma \leq 1/2$. In order to get the estimate we break the integral (3.2.9) into $I_1 + I_2$ where I_1 is the integral over the set $\{Z : \ |\Box| \geq |X|\}$ and I_2 is the integral over the set $\{Z : \ |\Box| \leq |X|\}$.

- The part I_1: We have

$$I_1 = \int_0^{1/2}\int_{|\Box|\geq|X|} = \int_0^{1/2}\int_{\{|\Box|\geq|X|\}\cap\{|\triangle|\leq K\}} + \int_0^{1/2}\int_{\{|\Box|\geq|X|\}\cap\{|\triangle|\geq K\}}$$
$$:= P_1 + P_2.$$

By assumption we have that for $\tilde{A}^i \leq_N |\Box|^{-N} \leq |X|^{-N}$. So, for P_1 we obtain

$$P_1 \leq C|X|^{-N}\int_0^{1/2}\int_{\{|\triangle\leq K\}} \tilde{B}^j(t\sigma,\triangle,y)(1-\sigma)^{-\frac{n+2}{2}-\alpha}\sigma^{-\frac{n+2}{2}-\beta}$$
$$\cdot\,\chi(y+\sqrt{t}Z)dZd\sigma.$$

Set $\frac{Z^{k+\mu}}{\sqrt{\sigma}}, \frac{\zeta_{\gamma_\lambda}-\eta_{\gamma_\lambda}}{\sqrt{\sigma}} = W^{k+\mu}, w_{\gamma_\lambda}$ and $\frac{\zeta_{\delta_\lambda}}{\sqrt{\sigma}} = w_{\delta_\lambda}$. Then, $dZ = \sigma^{n/2}dW$ and P_1 becomes

$$\begin{aligned} P_1 &\leq C|X|^{-N} \\ &\cdot \int_0^{1/2}\int_A \tilde{B}^j(t\sigma, W^{k+\mu}, w_{\gamma_\lambda}, w_{\delta_\lambda}, \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}}, y)\chi(y+\sqrt{t\sigma}W)dW\sigma^{-1-\beta}d\sigma \\ &\leq C|X|^{-N}, \end{aligned}$$

with $A = \{|W^{k+\mu}| + w_{\gamma_\lambda} + w_{\delta_\lambda} + \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}} \leq K\}$. We obtain this estimate because $\tilde{B}^j$ is bounded by C_K, $\beta < 0$ and since $|W| \leq K$ we integrate in a bounded area, thus the whole integral is bounded by $C_K > 0$. For the integral P_2 we have again that $\tilde{A}^i \leq C|\Box|^{-N} \leq C|X|^{-N}$. We make the same change of variables and obtain:

$$\begin{aligned} P_2 &\leq C|X|^{-N}\int_0^{1/2}\int_{\{|\triangle|\geq K\}} \tilde{B}^j(t\sigma, W^{k+\mu}, w_{\gamma_\lambda}, w_{\delta_\lambda}, \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}}, y)\sigma^{-1-\beta}dWd\sigma \\ &\leq C|X|^{-N}\int_0^{1/2}\int_{\{|\triangle|\geq M\}}\left(\frac{1}{|\triangle|}\right)^{2N}dW\sigma^{-1-\beta}d\sigma \\ &\text{with } |\triangle| = |W^{k+\mu}| + |w_{\gamma_\lambda}| + |w_{\delta_\lambda}| + |\frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}}|. \end{aligned}$$

Make the change of variables $\tilde{W}^{k+\mu}, \tilde{w}_{\gamma_\lambda}, \tilde{w}_{\delta_\lambda} = W^{k+\mu}, w_{\gamma_\lambda}, w_{\delta_\lambda} + \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}}$. Then we have $d\tilde{W} = dW$ and P_2 is smaller than

$$P_2 \leq C|X|^{-N}\int_{\{\tilde{W}\geq M\}}\left(\frac{1}{|\tilde{W}|}\right)^{2N}dW\sigma^{-1-\beta}d\sigma.$$

For $N > n-1$ this is bounded by $C_M > 0$. For $N' \leq n-1$ this follows from the bigger N since for $|X| \geq M > 1$ we have $|X|^{-N} \leq |X|^{-N'}$.

- The part I_2: We have that $\{|\square| \leq |X|\} \subseteq \{|\triangle| \geq c \cdot |X|\}$, for $c > 0$ (see Proposition C.0.1). Thus
$$I_2 = \int_0^{1/2} \int_{\{|\square| \leq |X|\}} \leq \int_0^{1/2} \int_{\{|\triangle| \geq c|X|\}}$$
We have that $\tilde{A}^i(t(1-\sigma), \square, y)$ is bounded for $t \leq c$ and y bounded. Then
$$I_2 \leq C \int_0^{1/2} \int_{\{|\triangle| \geq c|X|\}} B(t\sigma, \triangle, y) \sigma^{-\frac{n+2}{2} - \beta} dZ d\sigma.$$
Make the change of variables $\frac{Z^{k+\mu}}{\sqrt{\sigma}}, \frac{\zeta_{\gamma_\lambda} - \eta_{\gamma_\lambda}}{\sqrt{\sigma}}, \frac{\zeta_{\delta_\lambda}}{\sqrt{\sigma}} = W^{k+\mu}, w_{\gamma_\lambda}, w_{\delta_\lambda}$ and obtain
$$\int_0^{1/2} \int_{|W^{k+\mu}| + |w_{\gamma_\lambda}| + w_{\delta_\lambda} + \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}} \geq M} B(t\sigma, \triangle, y) \sigma^{-1-\beta} dW d\sigma$$
$$\leq \int_0^{1/2} \int_{|W^{k+\mu}| + |w_{\gamma_\lambda}| + w_{\delta_\lambda} + \frac{\eta_{\delta_\lambda}}{\sqrt{\sigma}} \geq M} \frac{1}{|\triangle|^N} \frac{1}{|\triangle|^N} dW \sigma^{-1-\beta} d\sigma.$$
Since $|\triangle| \geq c|X| \geq M$ we have $\frac{1}{|\triangle|^N} \leq \frac{c^N}{|X|^N}$. Thus by making the same change of variables as in the last step of I_1 and distinguishing between $N > n-1$ and $N \leq n-1$ we finally obtain that the integral is smaller than $C_{N,M}|X|^{-N}$.

The part $\sigma \geq 1/2$ is treated in a similar manner. If we now consider on (3.2.7) the part $\tilde{A}^{cor} M(s, z, y)$ where M is $O(t^\infty)$, after a simple change of variables we can see that it is $O(t^\infty)$. The part $1 - \chi(Z)$ produces terms of order $O(t^\infty)$, since $1 - \chi(z)$ has support away from x, y and by (a) of definition 3.2.1, we obtain an integral of the form
$$\int_0^t \int_M O((t-s)^N) O(s^N) dz ds$$
for arbitrary $N \in \mathbb{N}$. Concerning the $O(t^\infty)$ decay of $(A * B)(t, x, y)$ for $x \neq y$ as $t \to 0$ we see from
$$A * B(t, x, y) = \int_0^t \int_M A(t-s, x, z) B(s, z, y) dz ds$$
that we have to examine 3 cases. If z is away from x, y then the decay is obvious. If z is close to x, then we write A as $\tilde{A}^{cor}$, make a change of variables $\frac{z-y}{\sqrt{t-s}} = Z$ and end up with a term $(t-s)^{-1-a}$ which is integrable. Then using the decay of B we see that this term satisfies $O(t^\infty)$. Lastly, if z is close to y we treat this term in a similar manner. $\square$

Proposition 3.2.3. Let M be a compact manifold with corners of dimension n that satisfies Assumption 2.6.1, $a \leq -1$ and $A \in \Psi^a_{H,cor}(M)$. Then

a) $(\partial_t - \Delta_x)A \in \Psi^{a+1}_{H,cor}(M)$.

b) On a corner of codimension $k \in \mathbb{N}$, in local coordinates $X^k, \xi_1, \eta_1, \ldots, \xi_k, \eta_k, y^k$, we have

$$\begin{aligned}&\Phi_{a+1}((\partial_t - \Delta_x)A)\\ &= \big(- \frac{n+2}{2} - a - \frac{1}{2} X^k \partial_{X^k} - \Delta^y - \frac{1}{2}\sum_{i=1}^k (\xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i})\big)\Phi_a(A).\end{aligned}$$

Proof. Let $A \in \Psi^a_{H,cor}(M)$, $y \in A_k$ with $A_k \in \mathcal{M}_k$ and $x, y \in U$ a coordinate neighborhood around y. Then in local coordinates we have $A(t,x,y) = t^{-\frac{n+2}{2}-a}\tilde{A}(t, \frac{x^k-y^k}{\sqrt{t}}, \frac{x_1}{\sqrt{t}}, \frac{y_1}{\sqrt{t}}, \ldots, \frac{x_k}{\sqrt{t}}, \frac{y_k}{\sqrt{t}}, y)$ and recall that $\tilde{A}$ is a smooth function of $\sqrt{t}$. Set $l = \frac{n+2}{2} + a$ and compute:

$$\begin{aligned}\partial_t A(t,x,y) &= -lt^{-l-1}\tilde{A} + t^{-l}\frac{1}{2\sqrt{t}}\partial_{\sqrt{t}}\tilde{A} - t^{-l}\frac{x^k - y^k}{2t^{3/2}}\partial_{X^k}\tilde{A}\\ &\quad - \sum_{i=1}^k \big(\frac{x_i}{2t^{3/2}}\partial_{\xi_i}\tilde{A} + \frac{y_i}{2t^{3/2}}\partial_{\eta_i}\tilde{A}\big)\\ &= t^{-l-1}\big(- l - \frac{1}{2}X^k\partial_{X^k} - \frac{1}{2}\sum_{i=1}^k \big(\xi_i\partial_{\xi_i} + \eta_i\partial_{\eta_i}\big)\big)\tilde{A} + R,\end{aligned}$$

where $R_1 \in \Psi^{a+1/2}_{H,cor}(M) \subseteq \Psi^{a+1}_{H,cor}(M)$. For the Laplacian, we use that $\partial_x = \frac{1}{\sqrt{t}}\partial_{X^k}$ if $x = x^k$ or $\partial_x = \frac{1}{\sqrt{t}}\partial_{\xi_i}$ if $x = x_i$, $i \in \{1, \ldots, k\}$. Thus the Laplacian maps A to $\Psi^{a+1}_{H,cor}(M)$. Concerning the term $g^{ij}(x)$ we proceed as follows: Set $X = \frac{x-y}{\sqrt{t}}$ and apply Taylor's theorem at $g^{ij}(x)$ around y. Then we obtain

$$g^{ij}(x) = g^{ij}(y) + \sqrt{t}X^k h^{ij}_k(y) + \sum_{i=1}^k \sqrt{t}\xi_i h_i(y) + a(t,x,y),$$

where $a(t,x,y)$ is smooth and $O(t)$. Concerning the first order terms of the Laplacian, they are in $\Psi^{a+1/2}(M)$ and by taking $t = 0$ we obtain the formula

$$\begin{aligned}&\Phi_{a+1}((\partial_t - \Delta_x)A)\\ &= \big(- \frac{n+2}{2} - a - X^k\partial_{X^k} - \Delta^y - \sum_{i=1}^k (\xi_i\partial_{\xi_i} + \eta_i\partial_{\eta_i})\big)\Phi_a(A),\end{aligned}$$

where $\Delta^y = \sum_{i,j\in\{1,\ldots,n-k\}} g^{ij}(y)\partial^2_{X^iX^j} + \sum_{i=1}^k \partial^2_{\xi_i}$. □

Up until now, the calculus we created takes into account the presence of the boundary and the corners. At this point, in order to actually produce the Dirichlet heat kernel, we need to refine it in order to include the boundary conditions. Therefore, we define

Definition 3.2.5. Let M as before, $a \leq 0$. Then we define

$$\Psi^a_{H,cor,bc}(M) = \{A \in \Psi^a_{H,cor}(M) : \; A(t,x,y) = 0 \text{ for } x \in \partial^1 M\}.$$

It is immediate to see, that composition is preserved in this calculus, and that also, there exists a short exact sequence as in Proposition 3.2.6, namely

Proposition 3.2.4. Let M be a compact, connected manifold of dimension n with corners of codimension at most k and let $a \leq 0$. Then there is a short exact sequence

$$0 \to \Psi^{a-1/2}_{H,cor,bc} \to \Psi^a_{H,cor,bc}(M) \xrightarrow{\Phi_a} S_{H,cor,bc}(M) \to 0, \tag{3.2.12}$$

where

$$S_{H,cor,bc}(M) = \Big\{ \Phi \in S_{H,cor}(M) : \Phi(X^i, \xi_1, \eta_1, \dots, \xi_k, \eta_k, y) = 0$$
$$\text{for } \xi_1 = 0 \text{ or} \dots \xi_i = 0 \text{ near a corner of codimension } i \Big\}$$

In this refined calculus, we have the following proposition.

Proposition 3.2.5. Let $A \in \Psi^{-1}_{H,cor,bc}(M)$. Then $A : C^\infty(M) \to C^\infty([0,\infty)_{\sqrt{t}} \times M)$, where Af is defined by

$$Af(t,x) = \int_M A(t,x,y) f(y) dy.$$

Furthermore, for $x \in \partial^1 M$ we have $Af(t,x) = 0$ and for $x \in M \setminus \partial^1 M$ we have

$$Af(0,x) = \int_{T_x M} \Phi_{-1}(A)(X,x) dX.$$

Proof. Concerning the first statement, for x away from y, $A(t,x,y) = O(t^\infty)$ thus extends up to $t = 0$. For x close to y, make the change of variables $\frac{x-y}{\sqrt{t}} = X$ and the statement follows easily. Concerning the second statement if x away from y, then this part contributes nothing, since it is $O(t^\infty)$. Thus we can consider what happens if U is a coordinate neighborhood around y and $x \in U$. Then, since $x \in M \setminus \partial^1 M$, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} t^{-n/2}\tilde{A}(t, \frac{x-y}{\sqrt{t}}, y)f(y)\chi(y)dy \\ &= \int_{\mathbb{R}^n} \tilde{A}(t, X, x-\sqrt{t}X)f(x-\sqrt{t}X)dX,\end{aligned}$$

and the claim follows from Lebesgue's Theorem. □

Remark 3.2.4. Notice that $A \in \Psi^{-1}$ is sharp. For $\beta < -1$, by the same means we obtain that $Af(0,x) = 0$.

In this context, we also have Duhammel's Principle, namely

Proposition 3.2.6. Let $K_1 \in \Psi^{-1}_{H,cor}(M)$ such that $(\partial_t - \Delta_x)K_1 \in \Psi^{-1/2}_{H,cor}(M)$. Let also $S \in \Psi^{\beta}_{H,cor}(M)$, with $\beta < 0$. Then for $t > 0$, $x, y \in M$ we have

$$(\partial_t - \Delta_x)\big[K_1 * S\big](t,x,y) = S(t,x,y) + \big[\big(\partial_t - \Delta_x\big)K_1 * S\big](t,x,y). \tag{3.2.13}$$

Furthermore, if K_1 is in $\Psi^{a}_{H,cor,bc}(M)$, then

$$\lim_{t\to 0} K_1 * S(t,x,y) = 0.$$

Proof. We will show this for $x, y \in \mathrm{int}(M)$. For x, y in some boundary face the proof is essentially identical. Firstly we have

$$K_1 * S(t,x,y) = \int_0^t \int_M K_1(t-s,x,z)S(s,z,y)dzds.$$

For $0 < s < t$ set $B(t,s,x,y) = \int_M K_1(t-s,x,z)S(s,z,y)dz$. Since $0 < s < t$, by using (a) of Definition 3.2.1 and the compactness of M, we obtain an upper bound $C(t,s)$ of $\partial_t(K_1(t-s,x,z)S(s,z,y))$ and $\Delta_x(K_1(t-s,x,z)S(s,z,y))$. Notice that for $0 \leq \mu \leq 1$ we can still find an upper bound $\tilde{C}(t,s)$ of $\partial_t K_1(t-s+\mu,x,z)$, independent of μ. Therefore, by using Leibniz's theorem we can exchange differentiation and integration and obtain

$$\partial_t \int_M K_1(t-s,x,z)S(s,z,y)dz = \int_M \partial_t K_1(t-s,x,z)S(s,z,y)dz \tag{3.2.14}$$

and

$$\Delta_x \int_M K_1(t-s,x,z)S(s,z,y)dz = \int_M \Delta_x K_1(t-s,x,z)S(s,z,y)dz. \tag{3.2.15}$$

These two formulas are useful if the t and x derivatives can pass inside the integral. And for this we need to estimate $\partial_t B, \Delta_x B$. Concerning the $t-$derivative we have

$$\begin{aligned}\partial_t B(t,s,x,y) &= (\partial_t - \Delta_x)B(t,s,x,y) + \Delta_x B(t,s,x,y)\\ &= \int_M (\partial_t - \Delta_x)K_1(t-s,x,z)S(s,z,y)dz + \Delta_x B(t,s,x,y),\end{aligned}$$

by using (3.2.14) and (3.2.15). Concerning the first term, by hypothesis we have that $(\partial_t - \Delta_x)K_1 \in \Psi^{-1/2}_{H,cor}(M)$. For s close to t, by using (c) of Definition 3.2.1, and performing the change of variables $Z = \frac{x-z}{\sqrt{t-s}}$ we see that the whole term is $C(t)O((t-s)^{-1/2})$. For s close to 0, by the same means, with the change of variables $\frac{z-y}{\sqrt{s}} = Z$ we obtain the bound $C(t)O(s^{-1-\beta})$. In total, the term $(\partial_t - \Delta_x)B(t,s,x,y)$ is

$$C(t)O((t-s)^{-1/2})O(s^{-1-\beta}) \tag{3.2.16}$$

which is integrable w.r.t to s. Notice, that if instead we evalue the term K_1 on $t-s+h$, for $0 < h < c$, the bounds stay the same since $(t+h-s)^{-1/2} \leq (t-s)^{-1/2}$. Concerning the second term we work as follows

- In the regime $s \in (0, t/2]$, we have that $t - s \geq t/2$. Thus, $\Delta_x K_1(t-s,x,z)$ is bounded by $C(t)$. Since s is small, take z close to y, apply (c) of Definition 3.2.1 and after a change of variables we obtain a term which is $O(s^{-1-\beta})$. By formula (3.2.15) Δ_x goes inside, thus in total, the second term in this regime is $C(t)O(s^{-1-\beta})$.

- In the regime $s \in (t/2, t)$, the problem lies in the singularity of K_1 in the diagonal at $t - s = 0$. In particular, the problematic term is

$$\int_M K_1(t-s,x,z)S(s,z,y)\chi(z)dz$$

  where $\chi(z) = 1$ near x. By using (c) of Definition 3.2.1, we obtain

$$\begin{aligned}&\int_M (t-s)^{-n/2}\tilde{K}_1(t-s, \frac{x-z}{\sqrt{t-s}}, z)S(s,z,y)\chi(z)dz\\ &= \int_{|Z|\leq C/\sqrt{t-s}} \tilde{K}_1(t-s, Z, x-\sqrt{t-s}Z)S(s, x-\sqrt{t-s}Z, y)\\ &\qquad \cdot \chi(x-\sqrt{t-s}Z)dZ.\end{aligned}$$

  Take now Δ_x. The derivatives of S w.r.t to x are bounded by $C(t)$ since $s \geq t/2$. Since $x - \sqrt{t-s}Z$ stays in a bounded area, the derivatives of

$\tilde{K_1}$ w.r.t x are $\min\{C, O(|Z|^{-N})\}$ for some $N \in \mathbb{N}$ with $N > n$. Finally the derivatives of the metric and χ w.r.t x are bounded by a constant, thus by bounding each term, and letting integration in the whole of $\mathbb{R}^n$, in total we get

$$\Delta_x \int_M K_1(t-s,x,z)S(s,z,y)\chi(z)dz \leq C(t) \tag{3.2.17}$$

By using now the two bounds (3.2.16) and (3.2.17) we see that $\partial_t B(t,s,x,y)$ is smaller than $C(t)O((t-s)^{-1/2})O(s^{-1-\beta})$ which is integrable. Notice again that if we substitute $t-s$ with $t-s+h$, then the bounds stay the same. Thus, in total $\partial_t, \Delta_x B$ are bounded by functions that are s-integrable, therefore we can apply Leibniz rule and conclude the proof. Concerning the last statement, simply apply Proposition 3.2.5. □

With these Propositions, we are able now to prove the existence of the Dirichlet heat kernel. This construction we will allow us also to identify it's asymptotics as $t \to 0$. More precisely, we have the following proposition:

Proposition 3.2.7. Let M be a compact, connected manifold of dimension n with corners of codimension at most k. Furthermore, let $K_1 \in \Psi^{-1}_{H,cor,bc}(M)$ such that $R = (\partial_t - \Delta_x)K_1 \in \Psi^{-1/2}_{H,cor,bc}(M)$ and denote by R^N the $N-$fold product $R * \cdots * R$. Then we have the following:

(a) The series

$$K = K_1 - K_1 * R + \cdots + (-1)^n K_1 * R^N + \dots \tag{3.2.18}$$

converges in the $C^\infty((0,\infty) \times M \times M)$ topology.

(b) K is a heat kernel.

(c) $K \in \Psi^{-1}_{H,cor,bc}(M)$ and the series (3.2.18) is an asymptotic series, meaning that for every $N \in \mathbb{N}$ we have that $K = K_N + O(t^{N-n/2})$, where $K_N \in \Psi^{-1-N/2}_{H,cor,bc}(M)$.

Proof. Since the above is a global statement that depends on the properties of the calculus we just constructed, the proof is the same as in [Gri04]. We reproduce it here for convenience. Concerning (a), pick $N/2 \geq n/2+1$ and form $S = R^N = R * \cdots * R$. By the properties of the calculus and the fact that $N/2 \geq n/2+1$, we see that for $(t,x,y) \in [0,c) \times M \times M$, $S(t,x,y)$ is bounded by a constant $C > 0$. Pick now $m \in \mathbb{N}$ and form $S^m = R^{mN}$. Then we estimate

$$\begin{aligned} S^m(t,x,y) &= \int_0^t \cdots \int_0^{t_2} \int_{M^{m-1}} S(t - t_{m-1}, x, z_1) \dots S(t_1, z_{m-1}, y) dz_i dt_j \\ &\leq C^m \operatorname{vol}(M)^{m-1} \frac{t^{m-1}}{(m-1)!}, \end{aligned}$$

where $dz_i dt_j = dz_1 \dots dz_{m-1} dt_1 \dots dt_m$. This is a convergent series. Now, for $i = N, \dots, 2N-1$ let's estimate $K_1 * R^{i+mN} = K_1 * R^i * R^{mN}$. By the composition properties of the calculus, we have that $K_1 * R^i \in \Psi_{H,cor}^{-1-i/2}(M) \subseteq \Psi_{H,cor}^{-1-N/2}(M)$ and for each i is bounded by C_i. By picking $C_N = \max_{i=N,\dots,2N-1} C_i$, we can obtain $K_1 * R^i \leq C_N$. Thus by the previous calculation, we obtain

$$\begin{aligned} K_1 * R^{i+mN}(t,x,y) &= \int_0^t \int_M K_1 * R^i(t-s,x,z) R^{mN}(s,z,y) dz ds \\ &\leq C_N C^m \operatorname{vol}(M)^m \frac{t^m}{m!}. \end{aligned}$$

We conclude that the series (3.2.18) is absolutely convergent for $(t,x,y) \in (0,c) \times M \times M$. If we want to show convergence in $C^l((0,\infty) \times M \times M)$ we simply take $N \geq n/2 + l + 1$ and do a similar argument. Concerning (b), it is now straightforward by using Duhamel's Principle, i.e. Proposition 3.2.6. Finally, concerning (c) we assume that p belongs to a codimension 1 boundary face (the case of higher codimension is treated similarly). Then for U a coordinate neighborhood around p and $(x', x_n), (y', y_n) \in U = U' \times [0, \varepsilon)$ we set

$$\begin{aligned} &\tilde{K}^{cor}(t, X', \xi, \eta, y', y_n) \\ &= t^{n/2} K(t, y' + \sqrt{t}X', \sqrt{t}\xi, y', \sqrt{t}\eta) \cdot \chi(y' + \sqrt{t}X', \sqrt{t}\xi) \chi(y', \sqrt{t}\eta), \end{aligned}$$

where $\chi = 1$ in a possibly smaller neighborhood of U. By setting $X' = \frac{x'-y'}{\sqrt{t}}, \xi = \frac{x_n}{\sqrt{t}}, \eta = \frac{y_n}{\sqrt{t}}$ we see that this actually fulfills property (d) of Definition 3.2.1. Concerning the decay as $|X'| + \xi_n + \eta_n \to \infty$ we work as follows. For $l, N \in \mathbb{N}$, we write $K = K_N + R_N$, where $K_N \in \Psi_{H,cor}^{-1}(M)$ and $R_N = O(t^{N+n/2}) \in C^l$ as $t \to 0$. By definition, K_N and it's derivatives up to order l is $O((|X'| + \xi + \eta)^{-N})$. The term $R_N \chi(y' + \sqrt{t}X', \sqrt{t}\xi)\chi(y', \sqrt{t}\eta)$ and it's derivatives up to order l has the same decay, since it is supported on $|X'| + \xi + \eta \leq \frac{C}{\sqrt{t}}$. $\square$

3.3 Proof Of Main Theorem

Now finally we are in position to state and prove the main Theorem of this chapter concerning the construction and the asymptotics of the Dirichlet heat kernel. In particular we have

Theorem 3.3.1. Let M be a compact, connected manifold of dimension n, with corners of codimension at most k. Suppose furthermore, that M satisfies Assumption 2.6.1. Then, there exists a Dirichlet heat kernel $K \in \Psi^{-1}_{H,cor,bc}(M)$. The heat trace $H_t = \int_M K(t,x,x)dx$ admits a complete asymptotic expansion as $t \to 0$ in powers of $\sqrt{t}$, i.e.

$$H_t \sim a_0 t^{-n/2} + a_{1/2} t^{-n/2-1/2} + \dots$$

In particular

$$H_t \sim \frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}} - \sum_{H_i \in \partial_1 M} \frac{\mathrm{vol}_{n-1}(H_i)}{(4\pi t)^{n/2}} \frac{\sqrt{\pi}}{2} \sqrt{t} + O(\sqrt{t}).$$

Proof. We begin by constructing an approximate heat kernel. We do so by defining $(\phi^{int}, \phi_{H_1^1}, \dots, \phi_{H_1^{a_1}}, \dots, \phi_{H_k^1}, \dots, \phi_{H_k^{a_k}})$ by

$$\phi^{int}(X,p) = (4\pi)^{-n/2} e^{\frac{-|X|^2_{g(p)}}{4}}, \ p \in M$$

$$\phi_{H_1^1}([v,w]_{H_1^1}, p) = (4\pi)^{-n/2}\left(e^{\frac{|v-w|^2_{g(p)}}{4}} - e^{\frac{|v-w^*|^2_{g(p)}}{4}}\right), \ p \in H_1^1$$

...

$$\phi_{H_k^{a_k}}([v,w]_{H_k^{a_k}}, p)$$
$$= (4\pi)^{-n/2}\left(e^{\frac{|v-w|^2_{g(p)}}{4}} - \sum_{i=1}^{k} e^{\frac{|v-w^{*i}|^2_{g(p)}}{4}} + \dots + (-1)^k \sum_{c_1,\dots,c_k} e^{\frac{|v-w^{*c_1,\dots,c_k}|^2_{g(p)}}{4}}\right),$$

for $p \in H_k^{a_k}$. Here, for $X = (X_1, \dots, X_n)$, $|X|_{g(p)}$ is defined by $|X|^2_{g(p)} = \sum_{ij} g_{ij}(p) X_i X_j$.

$* : T_pM \to T_pM$ is defined as follows: If p belongs to a codimension $l \leq k$ boundary face A_l, by assumption, there exists a coordinate system, such that the metric at this point is $g(p) = dx_1^2 + \dots + dx_l^2 + g_{A_l}$. So, each vector has the form $w = w_1 + \dots + w_l + w_{A_l}$. Then for $1 \leq i \leq l$ we define $w^{*i} = w_1 + \dots - w_i + \dots + w_l + w_{A_l}$. Similarly, for $c_1, \dots, c_i$ different between them we set $w^{*c_1,\dots,c_i} = w_1 + \dots - w_{c_1} + \dots - w_{c_i} + \dots + w_l + w_{A_l}$. This precisely express the idea of the reflection terms. By this assumption on the metric, we obtain that $|w^{*a_1,\dots,a_l}| = |w^{*b_1,\dots,b_{l'}}|$ for $l, l' \leq k$ and $a's$, $b's$ different between them, and by the binomial theorem, that the term is 0 for $v = 0$. Finally, by construction, these terms satisfy the compatibility condition of Definition 3.2.4. Therefore $(\phi^{int}, \phi_{H_1^1}, \dots, \phi_{H_1^{a_1}}, \dots, \phi_{H_k^1}, \dots, \phi_{H_k^{a_k}}) \in S_{H,cor,bc}(M)$ and by Proposition 3.2.4, there exists $K_1 \in \Psi^{-1}_{H,cor,bc}(M)$, such that

$$\Phi_{-1}(K_1) = (\phi^{int}, \phi_{H_1^1}, \dots, \phi_{H_1^{a_1}}, \dots, \phi_{H_k^1}, \dots, \phi_{H_k^{a_k}})$$

Now notice that by Proposition 3.2.3 and since the symbol $\Phi_{-1}(K_1)$ has a product structure (compare with (3.1.2)), we obtain that $\Phi_0((\partial_t - \Delta_x)K_1) = 0$, thus $(\partial_t - \Delta_x)K_1 \in \Psi^{-1/2}_{H,cor,bc}(M)$. To see this more clearly suppose that $p \in \mathrm{int}(M)$. Then $(-n/2 - \frac{1}{2}X\partial_X - \Delta_y)e^{-\frac{|X|^2_{g(p)}}{4}} = 0$. Suppose now that $p \in H_1 \in \mathcal{M}_1(M)$. Observe that $(-1/2 - \partial_\xi^2 - \frac{1}{2}\xi\partial_\xi - \frac{1}{2}\eta\partial_\eta)(e^{-\frac{|\xi-\eta|^2}{4}} - e^{-\frac{|\xi+\eta|^2}{4}}) = 0$. Therefore, since the symbol in local coordinates is the product $e^{-\frac{|X'|^2_{g(p)}}{4}} \cdot (e^{-\frac{|\xi-\eta|^2}{4}} - e^{-\frac{|\xi+\eta|^2}{4}})$, by Proposition 3.2.3 we are done. The other cases when $p \in H_k \in \mathcal{M}_k(M)$ follow similarly. Now, we are able to apply Proposition 3.2.7, and finally obtain the Dirichlet heat kernel.

Define now for $t > 0$, $H_t = \int_M K(t,x,x)dx$ the heat trace, where $K(t,x,x)$ is the Dirichlet heat kernel. This integral makes sense for every $t > 0$ and it is $O(t^{-n/2})$ since $K \in \Psi^{-1}_{H,cor,bc}(M)$. The method of constructing the Dirichlet heat kernel by the Volterra series allow us to obtain a complete asymptotic expansion as $t \to 0$ in powers of $\sqrt{t}$. In order to see this, we use (d) in Definition 3.2.1. Set $x = y$, and when we integrate a term $\tilde{A}^{i-ref}(t, \frac{x^i - x^i}{\sqrt{t}}, \frac{x_1}{\sqrt{t}}, \frac{x_1}{\sqrt{t}}, \dots, \frac{x_i}{\sqrt{t}}, \frac{x_i}{\sqrt{t}}, x^i)$ just make the change of variables $\frac{x_j}{\sqrt{t}} = \xi_j$ for $j = 1, \dots, i$. That produces a product of $\sqrt{t}^i$ times a function which is a smooth function of $\sqrt{t}$ because of the rapid decay (See also reference 22, page 17 at [Gri04]). So, we obtain $a_0, a_{1/2}, a_1, \dots$ such that

$$H_t \sim a_0 t^{-n/2} + a_{1/2} t^{-n/2+1/2} + a_1 t^{-n/2+1} + \dots \tag{3.3.1}$$

where $\sim$ means asymptotically, i.e. for every $t < c$, $N \in \mathbb{N}$, there exists $C > 0$ such that for $0 < t < c$ we have

$$|H_t - \sum_{i=0}^{N} a_{i/2} t^{-n/2+i/2}| \leq C t^{-n/2+N/2+1/2}.$$

By looking closer at K, we see that we can obtain more information about the terms a_0 and $a_{1/2}$. Concerning the term a_0, we can see that it is only determined by $K_1(t,x,x)$, since $K_1 * R^k$ are all $\Psi^{-1-\beta}_{H,cor}(M)$ for $\beta > 0$. By integrating $K_1(t,x,x)$, we obtain

$$t^{-n/2} \cdot \Big(\frac{\mathrm{vol}(M)}{(4\pi)^{n/2}} - \sum_{H_i \in \partial_1 M} \frac{\mathrm{vol}_{n-1}(H_i)}{(4\pi)^{n/2}} \frac{\sqrt{\pi}}{2} \sqrt{t} \\ + \text{ higher order terms } t^{i/2},\ i = 2, \dots, k \Big).$$

The higher order terms are coming from calculating the contributions from the corners of codimension 2 and greater and their coefficients can be computed explicitly. Concerning the term $a_{1/2}$, it is also determined by $K_1 * R(t,x,x)$, thus in total we get that

$$H_t \sim \frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}} - \sum_{H_i \in \partial_1 M} \frac{\mathrm{vol}_{n-1}(H_i)}{(4\pi t)^{n/2}} \frac{\sqrt{\pi}}{2} \sqrt{t} + O(t^{-n/2+1/2}).$$

□

3.4 Further Extensions

In the course of the previous sections, we constructed the heat kernel for Dirichlet boundary conditions. We did so, by examining how the solution looks like in a model case. Then by identifying the symmetries of this local problem, we laid the foundations of a calculus which we showed that it contains the actual solution of the Dirichlet problem. By taking a careful look at this construction, we see that the main properties of the calculus (definition, composition formula, short exact sequence, mapping properties) are true, even if we slightly change the model solution. This happens for everything except Proposition 3.2.4. In different boundary value problems this space will be different.

In this section we sketch how the above method can be used to prove similar results for the Neumann Laplacian, or more general, for Laplace type operators with Dirichlet or Neumann boundary conditions. By Laplace type operators we mean second order differential operators with principal symbol

$$\sigma_2(\xi,p) = \sum_{i,j} g_{T_p^*M}(\xi_i,\xi_j) = \sum_{i,j} g^{ij}(p)\xi_i\xi_j, \text{ for } \xi \in T_p^*M \tag{3.4.1}$$

For the Neumann condition, instead of (3.1.2) one would use

$$K_1^N(t,x^k,y^k,x_1,y_1,...,x_k,y_k) = \\ (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x^k-y^k|^2_{h(y)}}{4t}} \big(e^{-\frac{|x_1-y_1|^2}{4t}} + e^{-\frac{|x_1+y_1|^2}{4t}}\big) ... \big(e^{-\frac{|x_k-y_k|^2}{4t}} + e^{-\frac{|x_k+y_k|^2}{4t}}\big) \tag{3.4.2}$$

We have $K_1^N \in \Psi_{H,cor}^{-1}(M)$ with $\Phi_0\big((\partial_t - \Delta_x)K_1^N\big) = 0$, thus $K_1^N \in \Psi_{H,cor}^{-1/2}(M)$. Then the Volterra series gives a heat kernel K^N, satisfying the initial condition

$$\int_M K^N(t,x,y)f(y)dy \to f(x), \text{ for } t \to 0$$

and $\partial_\nu K^N(t,x,y) = 0$ for $x, y \in \partial^1 M$, where ∂_ν is the normal derivative. The outer normal vector ν is defined on every smooth boundary hypersurface by $\partial_\nu = -\partial_{x_i}$ where x_i is the boundary defining function of the hypersurface. It is not defined on the corners, but ∂_{x_i} are defined, and K^N satisfies $\partial_{x_i} K^N = 0$ on the corner that the boundary hypersurfaces meet. Theorem 3.3.1 holds as it is. But the second term changes. More specifically, we have

Theorem 3.4.1. Let M be a compact manifold with corners, satisfying (2.6.1). Then there exists a complete asymptotic expansion of the Neumann Trace as $t \to 0$ of the form

$$H_t^N \sim_{t\to 0} t^{-\frac{n}{2}}\big(a_0 + a_{1/2}\sqrt{t} + a_1 t + ...\big). \tag{3.4.3}$$

In particular

$$H_t \sim \frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}} + \sum_{H_i \in \partial_1 M} \frac{\mathrm{vol}_{n-1}(H_i)}{(4\pi t)^{n/2}} \frac{\sqrt{\pi}}{2}\sqrt{t} + O(t^{-n/2+1/2}).$$

In the case of a Laplace type operator $P = \sum_{ij} g^{ij}\partial_{x_i}\partial_{x_j} + \sum_j b_j(x)\partial_{x_i}$ with Dirichlet or Neumann conditions, one can apply the same method. To get a heat kernel, one takes $K_1^D(t,x,y) =$ (3.1.2) for Dirichlet boundary conditions ($K_1^N(t,x,y) =$ (3.4.2) for Neumann boundary conditions respectively). Then Proposition 3.2.6 shows that

$$\Phi_0\big((\partial_t - P_x)K_1^{D,N}\big) = 0$$

since the principal symbol looks only at the leading term, thus $K_1^{D,N}(t,x,y) \in \Psi_{H,cor}^{-1/2}(M)$. Then Proposition 3.2.7 gives a heat kernel and a full asymptotic expansion as before. We can also treat mixed boundary value problems. Let M as before and $H_1, ..., H_m$ the boundary hypersurfaces. Consider the boundary value problem for $K \in C^\infty((0,\infty) \times M \times M)$

$$\begin{aligned} &\partial_t K = \Delta_x K \text{ for } x, y \in \mathrm{int}(M) \\ &K = 0, \ \textit{on } H_{i_1}, ..., H_{i_k} \\ &\partial_\nu K = 0, \ \textit{on } H_{j_1}, ..., H_{j_{m-k}} \\ &\lim_{t\to 0} \int_M K(t,x,y)f(y)dy = f(x) \end{aligned} \tag{3.4.4}$$

with $\{i_1, ..., i_k, j_1, ..., j_{m-k}\} = \{1, ..., m\}$. In order to construct the heat kernel for this boundary value problem, we define $K_1(t, x, y)$ to be the Dirichlet or Neumann heat kernel, depending on either $y \in H_{i_\lambda}$ or $y \in H_{j_\lambda}$. On the boundary faces we take care of the combatibility conditions. For simplicity we assume the that the depth is 2. If we have Dirichlet condition on $H_i = \{x_i = 0\}$ and Neumann condition on $H_j = \{x_j = 0\}$ then define on the intersection

$$K_1(t, x, y) = K_1^{Y^2}(t, x^2, y^2) K_1^D(t, x_i, y_i) K_1^N(t, x_j, y_j) \tag{3.4.5}$$

and so on. Then the Volterra series yields a heat kernel for the boundary value problem (3.4.4), with a complete asymptotic expansion

$$H_t \sim_{t\to 0} t^{-n/2}\big(a_0 + a_{1/2}t^{1/2} + ...\big) \tag{3.4.6}$$

The same considerations apply as before. However we have an interesting phenomenon. A careful computation for the term $a_{1/2}$ shows that the contribution from each hypersurface is $-(4\pi)^{-n/2}\frac{\sqrt{\pi}}{2}\,\mathrm{vol}_{n-1}(H_i)$ for Dirichlet boundary condition and $(4\pi)^{-n/2}\frac{\sqrt{\pi}}{2}\,\mathrm{vol}_{n-1}(H_i)$ for Neumann boundary condition. Thus we have

Corollary 3.4.1. Let M as before and consider the boundary value problem (3.4.4). Then the heat kernel for this problem exists and it lies in $\Psi^{-1}_{H,cor}$. Moreover if $\sum_{\lambda=1}^{k} \mathrm{vol}_{n-1}(H_{i_\lambda}) = \sum_{\lambda=1}^{m-\lambda} \mathrm{vol}_{n-1}(H_{j_\lambda})$ then

$$H_t = t^{-n/2}\big(\frac{\mathrm{vol}(M)}{(4\pi)^{n/2}} + O(\sqrt{t})\big)$$

as $t \to 0$.

Appendix A

The Laplacian for the Metric $g = g_0 + k$

In this appendix we see how the Laplacian looks like in local coordinates near a singular stratum with the metric $g = g_0 + k$, where g, g_0, k are as in Definition 1.2.1. We will denote the local coordinates by r, y, z where r is the radial variable, $y \in V_Y$ and z are local coordinates in L_Y. According to the Definition 1.2.1 we have that $|k|_{g_0} = O(r^\gamma)$, where $|\cdot|_{g_0}$ is the Frobenius norm coming from g_0. If we take U to be the singular neighborhood around the stratum Y as in Proposition 1.2.1 and equip $T^*U \otimes T^*U$ with the inner product $g_0 \otimes g_0$, then for each $p \in U$ it becomes a Hilbert space. Thus we can apply the inequality

$$\|T\|_{op} \leq \|T\|_F$$

for T a linear map. With this inequality we conclude that $|k|_{op} = O(r^\gamma)$. In local coordinates we have that $k = \sum_{ij} k_{ij} dx^i \otimes dx^j$ and for $v, w \in T^*U$ with $\|v\|_{g_0} = \|w\|_{g_0} = 1$ we get

$$|k(v, w)| \leq |k|_{op} = O(r^\gamma).$$

By choosing $v, w \in \{\partial_r, \partial_y, \frac{\partial_z}{r}\}$ we see that $\|v\|_{g_0} = \|w\|_{g_0} = 1$ and we obtain that

$$\begin{aligned} &k_{rr} = O(r^\gamma),\ k_{ry} = O(r^\gamma),\ k_{rz} = O(r^{\gamma+1}) \\ &k_{yy'} = O(r\gamma),\ k_{yz} = O(r^{\gamma+1}),\ k_{zz'} = O(r^{\gamma+2}) \end{aligned}$$

with the obvious understanding that $k_{rr} = k(\partial_r, \partial_r),\ k_{yy'} = k(\partial_y, \partial_{y'})$ etc. In this way we obtain the matrix

$$k = (k_{ij}) = \begin{pmatrix} k_{rr} & k_{rz} & k_{ry} \\ k_{zr} & k_{zz'} & k_{zy} \\ k_{yr} & k_{yz} & k_{yy'} \end{pmatrix}$$

whose elements are described by

$$k = (k_{ij}) = \begin{pmatrix} O(r^\gamma) & O(r^{\gamma+1}) & O(r^\gamma) \\ O(r^{\gamma+1}) & O(r^{\gamma+2}) & O(r^{\gamma+1}) \\ O(r^\gamma) & O(r^{\gamma+1}) & O(r^\gamma) \end{pmatrix}$$

as $r \to 0$. Thus the metric $g = g_0 + k$ on a local basis near a singular stratum Y is

$$g = (g_{ij}) = \begin{pmatrix} 1 + O(r^\gamma) & O(r^{\gamma+1}) & O(r^\gamma) \\ O(r^{\gamma+1}) & r^2 g_L + O(r^{\gamma+2}) & O(r^{\gamma+1}) \\ O(r^\gamma) & O(r^{\gamma+1}) & h + O(r^\gamma) \end{pmatrix} \tag{A.0.1}$$

as $r \to 0$. Recall that the Laplacian in local coordinates is written in the form

$$\Delta = -\sum_{ij} \frac{1}{\sqrt{\det(g)}} \partial_i \big(g^{ij} \sqrt{\det(g)} \partial_j \, \cdot \, \big),$$

and therefore we have to look at $\det(g)$ and g^{ij}.

- **det(g)**: We use the fact that the determinant of an $n \times n$ matrix $A = (a_{ij})$ is a sum with terms of the form

  $$a_{i_1 j_1} \dots a_{i_n j_n} \text{ where } \{i_1, \dots, i_n\} = \{j_1, \dots, j_n\} = \{1, \dots, n\}.$$

  So, if we look at (A.0.1) we will see that it is a sum with terms of the form

  $$\begin{aligned} &\big[(1 + O(r^\gamma))^{\alpha_1} O(r^{\gamma+1})^{\alpha_2} O(r^\gamma)^{\alpha_3}\big] \\ \times &\big[O(r^{\gamma+1})^{\beta_1} (r^2 g_L + O(r^{\gamma+2}))^{\beta_2} O(r^{\gamma+1})^{\beta_3}\big] \\ \times &\big[O(r^\gamma)^{\gamma_1} O(r^{\gamma+1})^{\gamma_2} (h + O(r^\gamma))^{\gamma_3}\big] \end{aligned}$$

  where $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_1 - \alpha_1 + \beta_2 - \alpha_2 + \beta_3 - \alpha_3 = \dim(L_Y)$ and $\gamma_1 - (\alpha_1 + \beta_1) + \gamma_2 - (\alpha_2 + \beta_2) + \gamma_3 - (\alpha_3 + \beta_3) = \dim(Y)$. A careful case by case analysis shows that each term is at least $O(r^{2\dim(L_Y)}) O(r^\gamma)$ apart from the terms that constitute the metric g_0. Thus we conclude that

  $$\det(g) = \det(g_0) + r^{2\dim(L_Y)} O(r^\gamma).$$

- $\mathbf{g^{ij}}$: For finding the inverse matrix we will use the formula $A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$, where $\mathrm{adj}(A) = ((-1)^{k+l} \det(A_{kl}))$ with A_{kl} being the matrix A without the k-row and l-column. If we now remove the k-row and l-column from the matrix (g_{ij}), we see that is of similar form, and therefore we can apply a calculation in a similar spirit to find the entries of $G = \mathrm{adj}(g)$. In this way we obtain that

$$G_{rr} = {G^0}_{rr} + r^{2\dim(L_Y)} O(r^\gamma),\ G_{ry} = r^{2\dim(L_Y)} O(r^\gamma),$$
$$G_{rz} = r^{2\dim(L_Y)} O(r^\gamma), G_{yz} = r^{2\dim(L_Y)} O(r^\gamma),$$
$$G_{yy'} = {G^0}_{yy'} + r^{2\dim(L_Y)} O(r^\gamma),\ G_{zz'} = {G^0}_{zz'} + r^{2\dim(L_Y)} O(r^\gamma).$$

where $G^0 = \mathrm{adj}(g_0)$.

Thus we obtain that

$$\begin{aligned} g^{-1} = \frac{\mathrm{adj}(g)}{\det(g)} &= \frac{\mathrm{adj}(g_0) + r^{2\dim(L_Y)} O(r^\gamma)}{\det(g_0) + r^{2\dim(L_Y)} O(r^\gamma)} \\ &= \frac{\mathrm{adj}(g_0)}{\det(g_0)} \frac{\det(g_0)}{\det(g_0) + r^{2\dim(L_Y)}} + \frac{r^{2\dim(L_Y) O(r^\gamma)}}{\det(g_0) + r^{2\dim(L_Y)} O(r^\gamma)} \end{aligned}$$

and we conclude that

$$g^{ij} = g_0^{ij}(1 + O(r^\gamma)) + O(r^\gamma).$$

From this point we obtain by writing explicitly the Laplacian in local coordinates that is Δ_{g_0} plus smooth (up to $r = 0$) multiple of terms of the form

$$O(r^{\gamma-1})\partial_r,\ O(r^\gamma)\partial_r,\ \partial_r^2,$$
$$O(r^\gamma)\partial_y,\ O(r^\gamma)\partial_r\partial_y, O(r^\gamma)\partial_y\partial_y',$$
$$O(r^\gamma)\partial_z,\ O(r^\gamma)\partial_z\partial_{z'},$$
$$O(r^{\gamma-2})\partial_z,\ O(r^{\gamma-2})\partial_z\partial_{z'}$$

In this way, we obtain the following:

Proposition A.0.1. Let X be a compact stratified space of depth 1. Then near a singular stratum Y with metric $g = g_0 + k$ as in Definition 1.2.1, the Laplace operator takes the form

$$\Delta_g = \Delta_{g_0} + R$$

where $R \in r^\gamma \mathrm{Diff}^2_{iie}(X)$.

Appendix B

A Refined Cut-off

Proposition B.0.1. Let $m - i \geq 2p$ and let $p \in (1, +\infty)$. Then, there exists a sequence of functions $\{g_n\} \subseteq W_0^{2,p}\big((0,2]\big)$, $n \in \mathbb{N}$ with the following properties

- $0 \leq g_n \leq 1$ $\forall n \in \mathbb{N}$
- $g_n \to 1$ a.e.
- $\int_0^2 |g_n'(r)|^p r^{m-i-1} dr \to 0$
- $\int_0^2 \frac{|g_n'(r)|^p}{r^p} r^{m-i-1} dr \to 0$
- $\int_0^2 |g_n''(r)|^p r^{m-i-1} dr \to 0$

Proof. For $n \in \mathbb{N}$, let $\varepsilon_n = \frac{1}{n^2}$ and $\varepsilon_n' = e^{-n^4}$. Then we define $f_n : (0,2] \to \mathbb{R}$ by

$$f_n(r) = \begin{cases} 0 & 0 \leq r \leq \varepsilon_n' \\ \left(\frac{2\varepsilon_n'}{\varepsilon_n}\right)^{\varepsilon_n} \frac{\varepsilon_n(\varepsilon_n - 2\varepsilon_n')}{2\varepsilon_n'} \left(\frac{r}{\varepsilon_n'} - 1\right) & \varepsilon_n' \leq r \leq 2\varepsilon_n' \\ \left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n - 1} (\varepsilon_n - r) & 2\varepsilon_n' \leq r \leq \varepsilon_n \\ 0 & \varepsilon_n \leq r \leq 2 \end{cases}$$

We can easily see that f_n is continuous with compact support away from 0 and is weakly differentiable. Thus $f_n \in W^{1,p}\big((0,2], r^a dr\big)$ for $a \geq 0$ and $p \in [1, \infty)$. Now we define $g_n : (0,2] \to \mathbb{R}$ by

$$g_n(r) = \frac{\int_0^r f_n(s) ds}{\int_0^{\varepsilon_n} f_n(s) ds}$$

If we set $c_n = \int_0^{\varepsilon_n} f_n(s)ds$ a straightforward calculation then shows that

$$g_n(r) = \frac{1}{c_n}\begin{cases} 0 & 0 \le r \le \varepsilon'_n \\ a_n\left(\frac{r^2}{2\varepsilon'_n} - r + \frac{\varepsilon'_n}{2}\right) & \varepsilon'_n \le r \le 2\varepsilon'_n \\ \left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n}\varepsilon_n\left(1-\frac{r}{\varepsilon_n+1}\right) - \beta_n + \int_0^{2\varepsilon'_n} f_n(s)ds & 2\varepsilon'_n \le r \le \varepsilon_n \\ c_n & \varepsilon_n \le r \le 2 \end{cases}$$

where

$$a_n = \left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n}\frac{\varepsilon_n(\varepsilon_n - 2\varepsilon'_n)}{2\varepsilon'_n},\ \beta_n = \left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n}\varepsilon_n\left(1 - \frac{2\varepsilon'_n}{\varepsilon_n+1}\right)$$

and

$$c_n = \varepsilon_n\left(1 - \frac{\varepsilon_n}{\varepsilon_n+1}\right) - \beta_n + a_n\frac{\varepsilon'_n}{2}$$

By definition, we have that

- $0 \le g_n \le 1$
- $g_n(r) \to 1$ a.e. in $(0,2]$

We want to show that the L^p norms of g'_n and g''_n w.r.t to $r^{m-i-1}dr$ go to 0. For this we have

$$\|g'_n\|_p^p \le \int_{\varepsilon'_n}^{2\varepsilon'_n} \left|\frac{a_n}{c_n}\left(\frac{r}{\varepsilon'_n} - 1\right)\right|^p r^{m-i-1}dr + \int_{2\varepsilon'_n}^{\varepsilon_n}\left|\left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n-1}\frac{\varepsilon_n - r}{c_n}\right|^p r^{m-i-1}dr$$

The first integral is bounded by

$$\begin{aligned} &\frac{a_n^p}{c_n^p}\frac{2^{m-i}-1}{m-i}(\varepsilon'_n)^{m-i} \\ &\le \frac{(2\varepsilon'_n)^{-p}}{\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{-\varepsilon_n}\varepsilon_n\left(1-\frac{\varepsilon_n}{\varepsilon_n+1}\right) - \varepsilon_n\left(1-\frac{2\varepsilon'_n}{\varepsilon_n+1}\right) + \frac{\varepsilon_n(\varepsilon_n-2\varepsilon'_n)}{4}}\frac{2^{m-i}-1}{m-i}(\varepsilon'_n)^{m-i} \end{aligned}$$

We have that

$$\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{-\varepsilon_n}\varepsilon_n\left(1-\frac{\varepsilon_n}{\varepsilon_n+1}\right) = \frac{1}{2^{1/n^2}}\frac{1}{(n^2)^{1/n^2}}e^{n^2}\frac{1}{n^2}\left(1 - \frac{\frac{1}{n^2}}{\frac{1}{n^2}+1}\right) \to \infty$$

and the other two terms in the denominator go to 0. Since $m-i \geq 2p > p$ and $\varepsilon'_n \to 0$ we have that the first integral goes to 0 as $n \to \infty$. Concerning the second integral, we have that it is bounded by

$$\begin{aligned}&\left(\frac{(\varepsilon_n)^{1-\varepsilon_n}}{c_n}\right)^p \int_{2\varepsilon'_n}^{\varepsilon_n} r^{p\varepsilon_n - p + m - i - 1} dr \\ &\leq \left(\frac{(\varepsilon_n)^{1-\varepsilon_n}}{c_n}\right)^p \frac{(\varepsilon_n)^{p\varepsilon_n - p + m - i} - (2\varepsilon'_n)^{p\varepsilon_n - p + m - i}}{p\varepsilon_n - p + m - i}\end{aligned}$$

Now it is easy to see that $\frac{\varepsilon_n}{c_n} \to 1$, $\frac{1}{\varepsilon_n^{\varepsilon_n}} \to 1$ (since $n^{1/n} \to 1$) and lastly that since $m-i > 2p > p$ we have that $\frac{(\varepsilon_n)^{p\varepsilon_n - p+m-i} - (2\varepsilon'_n)^{p\varepsilon_n - p + m - i}}{p\varepsilon_n - p + m - i} \to 0$. Thus we have that

$$\|g'_n\|_p^p \to 0 \tag{B.0.1}$$

as $n \to \infty$. Taking into account the above calculations and that $p > 1$, it is straightforward to see that also

$$\|\frac{g'_n}{r}\|_p \to 0, \text{ as } n \to \infty.$$

Concerning g''_n, a simple calculation yields

$$g''_n(r) = \frac{f'_n(r)}{c_n} = \frac{1}{c_n}\begin{cases} 0 & 0 \leq r \leq \varepsilon'_n \\ \frac{a_n}{\varepsilon'_n} & \varepsilon'_n \leq r \leq 2\varepsilon'_n \\ \frac{\varepsilon_n - 1}{\varepsilon_n}\left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n - 2}\big(\varepsilon_n - r\big) - \left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n - 1} & 2\varepsilon'_n \leq r \leq \varepsilon_n \\ 0 & \varepsilon_n \leq r \end{cases}$$

Thus we have that

$$\begin{aligned}\|g''_n\|_p^p \leq &\int_{\varepsilon'_n}^{2\varepsilon'_n} |\frac{a_n}{c_n\varepsilon'_n}|^p r^{m-i-1} dr \\ &+ \int_{2\varepsilon'_n}^{\varepsilon_n} |\frac{\varepsilon_n - 1}{c_n\varepsilon_n}\left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n - 2}\big(\varepsilon_n - r\big) - \frac{1}{c_n}\left(\frac{r}{\varepsilon_n}\right)^{\varepsilon_n - 1} |^p r^{m-i-1} dr\end{aligned}$$

Concerning the first integral, it is equal to

$$\begin{aligned}\frac{\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n p}}{c_n^p} \int_{\varepsilon'_n}^{2\varepsilon'_n} \frac{\big|\varepsilon_n(\varepsilon_n - 2\varepsilon'_n)\big|^p}{\big(2(\varepsilon'_n)^2\big)^p} r^{m-i-1} dr &\leq \frac{\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n p}}{c_n^p} \int_{\varepsilon'_n}^{2\varepsilon'_n} \frac{1}{\big(2(\varepsilon'_n)^2\big)^p} r^{m-i-1} dr \\ &\leq \frac{\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{\varepsilon_n p}}{c_n^p} \frac{2^{m-i} - 1}{2^p(m-i)} (\varepsilon'_n)^{m-i-2p}\end{aligned}$$

Following the previous considerations, that $\frac{1}{\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{-\varepsilon_n} c_n} \to 0$, and the fact that $m-i \geq 2p$ we can see that the last term goes to 0. Concerning the second integral, we use the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ and we treat each term separately. The first term is bounded by

$$\int_{2\varepsilon'_n}^{\varepsilon_n} \frac{r^{(\varepsilon_n-2)p+m-i-1}}{c_n^p \varepsilon_n^{(\varepsilon_n-2)p}} dr = \left(\frac{\varepsilon_n}{c_n}\right)^p \varepsilon_n^{p-\varepsilon_n} \frac{\left((\varepsilon_n)^{(\varepsilon_n-2)p+m-i} - (2\varepsilon'_n)^{(\varepsilon_n-2)p+m-i}\right)}{(\varepsilon_n-2)p+m-i}$$

which goes to 0, because it is bounded by $O(\varepsilon_n^{p-1})$ which converges to 0 as $n \to \infty$ because $p > 1$. Now the second term is

$$\int_{2\varepsilon'_n}^{\varepsilon_n} \frac{1}{c_n^p}\left(\frac{r}{\varepsilon_n}\right)^{(\varepsilon_n-1)p} r^{m-i-1} dr = \int_{2\varepsilon'_n}^{\varepsilon_n} \frac{\varepsilon_n^p}{c_n^p}\left(\frac{1}{\varepsilon_n^{\varepsilon_n}}\right)^p r^{(\varepsilon_n-1)p+m-i-1} dr$$

and is easily seen to go to 0, as $n \to \infty$.

□

Appendix C

A Technical Lemma

Proposition C.0.1. Let $\square$, $\triangle$ and X be as in Proposition 3.2.2. Then

$$|\square| \leq \frac{|X|}{2} \Rightarrow |\triangle| \geq \frac{|X|}{2}$$

Proof.

$$\begin{aligned}|\square| \leq \frac{|X|}{2} &\Rightarrow |\frac{X^{k+\mu} - Z^{k+\mu}}{\sqrt{1-\sigma}}| + \sum_{\lambda=\mu+1}^{k-i} |\frac{\xi_{a_\lambda} - \zeta_{a_\lambda}}{\sqrt{1-\sigma}}| + \sum_{\lambda=1}^{i} \frac{\xi_{\beta_\lambda} + \zeta_{\beta_\lambda}}{\sqrt{1-\sigma}} \leq \frac{|X|}{2} \\ &\Rightarrow \sqrt{2}\left(|X^{k+\mu}| - |Z^{k+\mu}| + \sum_{\lambda=\mu+1}^{k-i} (\xi_{a_\lambda} - \zeta_{a_\lambda}) + \sum_{\lambda=1}^{i} (\xi_{\beta_\lambda} + \zeta_{\beta_\lambda})\right) \leq |X| \end{aligned} \tag{C.0.1}$$

since $\frac{2}{\sqrt{1-\sigma}} \geq \sqrt{2}$. (C.0.1) implies that

$$\begin{aligned}&(\sqrt{2}-1)\left(|X^{k+\mu} + \sum_{\lambda=\mu+1}^{k-i} \xi_{a_\lambda} + \sum_{\lambda=1}^{i} \xi_{\beta_\lambda}\right) \\ \leq &\sqrt{2}\left(|Z^{k+\mu}| + \sum_{\lambda=\mu+1}^{k-i} \zeta_{a_\lambda} + \sum_{\lambda=1}^{i} \zeta_{\beta_\lambda}\right) + \sum_{\lambda=\mu+1}^{k-j} \eta_{\gamma_\lambda} + \sum_{\lambda=1}^{j} \eta_{\delta_\lambda}.\end{aligned}$$

Add on both sides the term $(\sqrt{2}-1)\left(\sum_{\lambda=\mu+1}^{k-j} \eta_{\gamma_\lambda} + \sum_{\lambda=1}^{j} \eta_{\delta_\lambda}\right)$, and we obtain:

$$(\sqrt{2}-1)|X| \leq \sqrt{2}\left(|Z^{k+\mu}| + \sum_{\lambda=\mu+1}^{k-i} \zeta_{a_\lambda} - \sum_{\lambda=1}^{i} \zeta_{\beta_\lambda} + \sum_{\lambda=\mu+1}^{k-j} \eta_{\gamma_\lambda} + \sum_{\lambda=1}^{j} \eta_{\delta_\lambda}\right). \tag{C.0.2}$$

Notice now that for the indices $n-k+\mu,\dots,n$ we have the two partitions

$$a_{\mu+1},\dots,a_{k-i},\beta_1,\dots,\beta_i \text{ and}$$
$$\gamma_{\mu+1},\dots,\gamma_{k-j},\delta_1,\dots,\delta_j$$

which represent them, and by definition $a\neq\gamma$. Thus $\{a_{\mu+1},\dots,a_{k-i}\}\subseteq\{\delta_1,\dots,\delta_j\}$ and $\{\gamma_{\mu+1},\dots,\gamma_{k-j}\}\subseteq\{\beta_1,\dots,\beta_i\}$. So in (C.0.2) the sums of a_λ and γ_λ can be substituted by the sums of β_λ and δ_λ respectively. Since $\{a_{\mu+1},\dots,a_{k-i}\}\subseteq\{\delta_1,\dots,\delta_j\}$ and $\{\gamma_{\mu+1},\dots,\gamma_{k-j}\}\subseteq\{\beta_1,\dots,\beta_i\}$, this may be a subset. But since ζ and η are positive numbers, we can complete the missing terms and eventually obtain the inequality

$$(\sqrt{2}-1)|X|\leq\sqrt{2}\left(|Z^{k+\mu}|+\sum_{\lambda=\mu+1}^{k-j}(\eta_{\gamma_\lambda}-\zeta_{\gamma_\lambda})+\sum_{\lambda=1}^{i}(\eta_{\delta_\lambda}+\zeta_{\delta_\lambda})\right).$$

Since $\sigma\leq 1/2$ we obtain $(\sqrt{2}-1)|X|\leq|\triangle|$. $\square$

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