Some Aspects in Cosmological Perturbation Theory and $f(R)$ Gravity

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In memoriam: My father Ruperto and my sister Cecilia

Abstract

General Relativity, the currently accepted theory of gravity, has not been thoroughly tested on very large scales. Therefore, alternative or extended models provide a viable alternative to Einstein's theory. In this thesis I present the results of my research projects together with the Grupo de Gravitación y Cosmología at Universidad Nacional de Colombia; such projects were motivated by my time at Bonn University.

In the first part, we address the topics related with the metric $f(R)$ gravity, including the study of the boundary term for the action in this theory. The Geodesic Deviation Equation (GDE) in metric $f(R)$ gravity is also studied. Finally, the results are applied to the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime metric and some perspectives on use the of GDE as a cosmological tool are commented.

The second part discusses a proposal of using second order cosmological perturbation theory to explore the evolution of cosmic magnetic fields. The main result is a dynamo-like cosmological equation for the evolution of the magnetic fields. The couplings between the perturbations in the metric and the magnetic fields are present in the dynamo equation, opening a new perspective in the amplification of magnetic fields at early stages of the universe expansion.

The final part of this work is in the field of stellar kinematics in galaxies. It is a project that started at Sternwarte-Bonn Institut some years ago. Here we study the stellar and gas kinematics in HCG 90. Furthermore, we analyze the rotation curves and velocity dispersion profiles for the galaxies in the core of the group. Some possible future applications of the work are discuss.

Contents

Declaration

I hereby declare that I have produced this thesis without the prohibited assistance of third parties and without making use of other than those specified; notions taken over directly or indirectly from other sources have been identified as such.
This thesis has not previously been presented in identical or similar form to any other German or foreign examination board.
All the ideas and results presented in this work are mainly product of the author's work as a head of the research group: Gravitación y Cosmología at Universidad Nacional de Colombia-Bogotá and without assistance of any other group or researcher.
A significant part of this work has been published in peer-reviewed journals:

- Boundary term in metric $f(R)$ gravity: fields equations in the metric formalism. A. Guarnizo, **L. Castañeda**, J.M. Tejeiro. General Relativity and Gravitation, Volume 42, Issue 11, 2010.

- Geodesic Deviation equation in $f(R)$ gravity. A. Guarnizo, **L. Castañeda**, J.M. Tejeiro. General Relativity and Gravitation, Volume 43, Issue 10, 2011.

- Evolution of magnetic fields through cosmological perturbation theory. H. Hortúa, **L. Castañeda**, J.M. Tejeiro. Physical Review D 87, 103531, 2013.

- Kinematics in Hickson Compact Group 90. **Leonardo Castañeda**, Michael Hilker . Proceedings of Science. Baryons in Dark Matter Halos. Novigrad, Croatia 2004.

As an extension of some ideas presented in this thesis, there is some work in progress:

- Contrasting formulations of cosmological perturbations in a magnetic FLRW cosmology. **L. Castañeda**, Héctor J. Hortúa. arXiv:1403.7789. To be submitted.

CHAPTER 1

Introduction

The last century was one of the most important in the develop of our understanding about physics and astronomy. Two of the most important theories in physics were developed during the first decades offering new perspectives and a solid framework to explain natural phenomena in a very precise language, they are: *Quantum Mechanics* (QM) and *General Relativity* (GR). These theories have opened a new window to build up a new paradigm about our cosmological conception:*The Modern Cosmological Model*.

Together with the theoretical advance, important observational facts as the expansion of the universe, the discovery of the Cosmic Microwave Background Radiation (CMB) and its temperature fluctuations give us a set of principles to formulate the cosmological model from different perspectives. Fields of research as particle physics, thermodynamics, statistical physics, gravitation among others have proportioned us a promising model for the universe known as ***Big Bang Cosmology***. For an excellent review about the cosmological model and its relationship with different branches of physics and mathematics see [1]. Today, researches in cosmology claim that we are in the era of *precision cosmology*. From important tools as cosmological numerical simulations and satellites we know about important facts about the universe as its composition, the rate of actual expansion and the large scale structure. A model called Lambda-Cold-Dark-Matter (ΛCDM) seems to be one of the most serious candidates to fit both theory and observations. However, there are some aspects about the cosmological model that need to be considered. Despite the successful of cosmology during the last decades, the main components of the ΛCDM model remain without a satisfactory physical explanation . About 25% of the actual universe is believed to be dominated by a strange kind of non-relativistic matter. Until now we only know the dark components of the universe by its gravitational interaction. Dark matter is a generic name for this component which dominates the external dynamics in galaxies and seems to be the best explanation for multiples images in the strong lensing regime for several well known studied gravitational lens systems [2]. This component gives the name *cold dark matter* to the model. The other dark component known as *dark energy* dominates about 70% of the model and it provides an explanation for the actual accelerated expansion of the universe. Only 5% of the matter content in the universe is explained by the actual theories in physics. Our cosmological model needs to find the explanation for about 95% of its content. There are several proposals in the literature not only with the goal of searching a satisfactory explanation for the cosmological model but also for fundamental physics issues, as an example, one of the most active field of research is the idea of modify GR with cosmological, physical and mathematical purposes.

Besides the success and problems of the ΛCDM model, there are still also deep physical and mathematical reasons to study modifications to GR and QM. One of the most important is to achieve a model for the *Quantum Gravity* problem. There are different approaches to the problem as *String Theory* and *Loop Quantum Gravity* with some success and a lot of challenges [3–5].

For our purposes, the first part of this thesis deals with some classical aspects of a specific class of modification to GR known in the literature as metric $f(R)$ gravity. More specifically, the scientific topics covered in this thesis are

- The formulation of an action for metric $f(R)$ gravity including the proper boundary conditions in terms only of the metric and geometrical variables, without using the scalar-tensor equivalence.
- The generalization of the Geodesic Deviation Equation (GDE) in metric $f(R)$ gravity and some cosmological applications.
- A new approach to the cosmological dynamo equation in the framework of GR using cosmological perturbation theory up to second order in a gauge invariant form.
- Some aspects in the stellar and gas kinematics in Hickson Compact Group90.

Each chapter is a broader perspective of the refereed publications made by the author with the *Grupo de Gravitación y Cosmología* under his entire direction and responsibility. The work about Kinematics in HCG90 is also supported by two short publications cited in the chapter. There have been several works from the author coming from the topics covered in this thesis [6].

CHAPTER 2

Field equations and variational principles

2.1 Introduction

During the last decades there has been an increasing interest to explore alternatives in order to unify Gravitational Theories with the Standard Model of Particle Physics [3, 5], and many efforts are addressed in this direction. A widely set of alternatives and approaches from different disciplines in physics and mathematics are explored for this goal, but until now we do not have a complete and successful framework to realize this task [5]. One of these alternatives are The Grand-Unified Theories. These theories appeared just after the advent of General Relativity (hereafter GR), very well known cases are the Weyl's work [5, 7] and Kaluza-Klein models [3]. With these proposals, the field called *gauge theory* started. Nowadays, the gauge theories are the main point for contemporary version of unified theories. As a motivation, we should mention that any attempt to formulate quantum field theory on a curved spacetime leads to modifying the Einstein-Hilbert action with terms containing non-linear invariants of the curvature tensor or non-minimal couplings between matter and curvature [3]. Historically, the Einstein-Hilbert action for gravity has been modified for different reasons and one of the first modifications is the Brans-Dicke gravity model, where a *non-minimal coupling* between matter and gravity is allowed [5]. Brans-Dicke gravity and its generalizations are called *scalar-tensor* theories and they play a very important role in differents fields as cosmology, astrophysics and mathematics. We refer the reader to the references [3, 5] for a detailed review about historical and modern aspects of scalar-tensor theories, as we should comment, the methodology and results for the Brans-Dicke model are from the excellent [5] book.

The scalar-tensor theories are also a very rich scenario for mathematical issues. Low energy limits in string theory share some mathematical aspects with scalar-tensor theories of gravity [3, 5, 8] and it has motivated an extensive research in models beyond Einstein GR for gravity. On the other hand, there is not only modifications to the Einstein-Hilbert action through non-minimal coupling, another very active field of research consists in generalizations of the Einstein-Hilbert action from Lagrangian made up from geometrical-curvature invariants [3, 5]. High order theories of gravity known also as *Extended Theories of Gravity* (ETGs) also offer a broad landscape for gravity. The most natural generalization for the Einstein-Hilbert action is instead of work only with the Ricci scalar R, a general scalar function $f(R)$ is introduced. There is a very important point to stress here, there exists an equivalence between scalar-tensor theories of gravity and $f(R)$ gravity in the metric and Palatini formulations [3, 5, 9]. We focus mainly in the treatment of the boundary problem in metric $f(R)$ gravity.

The central aspect of this chapter is to present the results obtained in our work [10] where we obtained the field equations for the metric $f(R)$ gravity theory, including the proper boundary contribution to the action in terms of quantities coming from the derivative $f'(R) \equiv \frac{\mathrm{d}f(R)}{\mathrm{d}R}$ of the $f(R)$ function on the boundary $\partial\mathcal{V}$, for the spacetime bounded region where the dynamical equations are defined (see fig 2.1). The boundary term is needed in order to have a well-posed mathematical problem in $f(R)$ gravity. There has been an extensive discussion about the initial-boundary Cauchy problem for GR and ETGs. The very important problem of initial value for a viable $f(R)$ theory of gravity was studied in [11] and it is also deeply discussed in [3]. In this work, we adopt the following criteria in order to establish a well-posed mathematical problem [9]: *If the action for the field theory is restricted to those field configurations consistent with the boundary and initial data,*

$$S = \int_{\mathcal{V}} \mathrm{d}^4x \mathcal{L}([\phi], x)\Big|_{\{\phi^i \text{ consistent with data on } \partial\mathcal{V}\}}, \tag{2.1}$$

the unique solution to the Euler-Lagrange equations (B.59) should be the only extremum of the action (2.1). For a gravity theory, the initial-boundary conditions are setting on $\partial\mathcal{V}$ (see figure 2.1) for the metric or any other gravity variable. In our work, the equations for $f(R)$ gravity in the bulk were obtained using elementary variational principles instead of the usual treatment in an equivalent scalar-tensor approach. The problem of the boundaries in general relativity was studied in a seminal work by Hawking et al. [12], where the so called *Gibbons-York-Hawking* term was added to the Einstein-Hilbert action to avoid fix both $\delta g_{\alpha\beta} = \partial_\sigma(\delta g_{\alpha\beta}) = 0\Big|_{\partial\mathcal{V}}$. Only the former is fixed and the boundary problem in GR is solved [9]. However, there is a lack of importance in almost all the specialized literature related with the correct treatment of the boundary conditions for higher order gravity theories and field theories [13]. Our approach shows in a different way that with the proper boundary term in the action and with only a formulation in terms of the $f(R)$ function and its first derivative on the boundary, instead of the use of scalar fields. We get a well-posed mathematical problem, because the equations on the bulk remain in the standard form studied in [11] where its initial problem was well established. We should point out that the geometrical restriction $\delta R\Big|_{\partial\mathcal{V}} = 0$ is a condition we recover in our treatment. It is a common feature of scalar-tensor theories with boundaries [9] where the same mathematical condition is written as $\delta\phi\Big|_{\partial\mathcal{V}}=0$.

The mathematical and physical consequences of the boundaries in any field theory are of remarkable importance as is pointed out in [9] where boundaries for different field theories and the case of the scalar-tensor gravity is particularly studied.

We start with some comments about the cosmological and other motivations to study models beyond GR. A detailed exposition of the field equations in GR and metric $f(R)$ gravity together with the proper mathematical treatment of the boundaries is included. We compare the boundary term with the Gibbons-York-Hawking in GR and the boundary term in metric $f(R)$ gravity. As a final topic, the equivalence between Brans-Dicke gravity and metric $f(R)$ gravity is discussed including the boundary contribution in scalar-tensor gravity.

2.2 Extended theories of gravity

One of the biggest achievements in physics in the last century, was the formulation of Einstein General Relativity. Nowadays, the Einstein theory provides one of the most coherent and precise descriptions of spacetime, gravity and matter at macroscopic level [3, 14]. However, besides all the very well known effort in the unification program dealing with GR and Quantum Mechanics, the theoretical interest in ETGs began almost since the formulation of GR. People have tried to work with the idea of generalized

GR to include, in a general scheme, first the electromagnetic interaction and more recently the other interactions [3, 5]. One pioneering work was the program carried out by Th. Kaluza and O. Klein [5], with their five-dimensional theory which enables to describe the electromagnetic field on the geometrical side of the field equations.

There are excellent bibliographical references and textbooks appropriate to follow the development of the so called *Extended Theories of Gravity* (ETGs). Give an extensive review about these theories is out of the scope for this thesis. The interested reader can see the wonderful texts [3, 5, 13] and references therein.

As general features of the ETGS we find that the geometry can couple non-minimally to some scalar field and derivatives of the metric of the order higher than second may appear in the field equations [5]. One important aspect about these ETGs is related with the proper boundary conditions in a Lagrangian formulation. In this chapter, we will mention how to face this problem adding proper surface boundary terms and specially in the case of $f(R)$ gravity, our work [10] shows that it is possible to have a well behaved mathematical problem without using the scalar-tensor equivalence.

In this thesis, a subclass of ETGs known as $f(R)$ gravity in the metric formalism is considered. The gravity equations are obtained in the metric formalism with appropriate boundary conditions. This chapter follows basically our work [10]. This work exhibits important consequences about the role of boundaries in metric $f(R)$ gravity and how to get a well-posed mathematical problem within these theories. The set of equations for metric $f(R)$ gravity is employed in the rest of the thesis mainly in cosmological applications.

2.2.1 Cosmological and other motivations

Together with the physical and mathematical richness of the ETGs, these theories also offer a very promising scenario to be tested in both astrophysical and cosmological context. Today, the successful of the *Standard Cosmological Model* relies on two very unknown components: *Dark Matter* and *Dark Energy* [1, 4, 15]. Recent observations of the Type Ia supernovae and other cosmological probes as the localization of the Doppler peaks in the spectrum of CMB anisotropies reveal two very important features of the cosmological model:

- The universe is currently in a phase of accelerated expansion and this acceleration is explained within the cosmological theory by a fluid with negative pressure. Any kind of ordinary known matter does not have this property and to overcome this problem a variety of models in the literature are explored starting from scalar fields (*quintessence*), cosmological constant (Λ), K-essence, brane-world models, two-scalar model and many other possibilities are advocated to solve this puzzle, for an extensive review see [3, 5].

- The other important consequence derived from observations is that the universe is spatially flat and when we join it with the abundances of primordial chemical elements calculated from the theory of Big-Bang Nucleosynthesis (BBN) the result is a model of the universe roughly dominated by the dark components at 95%.

The aim of this thesis in not going deep in the observational problems of the "ΛCDM" as is also recognized the standard cosmological model, but explores the possibility offered by $f(R)$ gravity in the cosmological context as a potentially candidate theory to achieve some understanding and a different way to solve problems for cosmology [16–19]. Even the big and wonderful success of the scalar fields in the inflationary scenario, they do not rule out the possibility to explore ETGs as theories beyond GR. For example, in the infrared limit the acceleration of the universe could be a signal of a breakdown of

Einstein's theory and ETGs can be considered as alternatives. In fact, there are another very different ways to approach the cosmic acceleration problem, one is the modification of the "Friedmann-Einstein" equations. The next chapter is dedicated to cosmology in $f(R)$ gravity together with the generalization of the *geodesic deviation equation* in the context of EGTs. The rest of the present chapter deals with the mathematical problem for the field equations in the context of metric $f(R)$ gravity.

2.3 Field equations on $\mathcal{M}$

One of the postulates in GR are the field equations. The Einstein field equations for GR are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.2}$$

where the $T_{\mu\nu}$ is the stress-energy tensor and G the gravitational constant. The set of equations can be obtained from the Einstein-Hilbert Lagrangian, including the appropriate boundary term [20]

$$S = \frac{1}{2\kappa}(S_{EH} + S_{GYH}) + S_M, \tag{2.3}$$

where

$$S_{EH} = \int_{\mathcal{V}} d^4x\, \sqrt{-g}R, \tag{2.4}$$

$$S_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\, \varepsilon\, \sqrt{|h|}K, \tag{2.5}$$

here $\mathcal{V}$ is a hypervolume on $\mathcal{M}$ (see 2.1), $\partial\mathcal{V}$ its boundary , h the determinant of the induced metric, K is the trace of the extrinsic curvature of the boundary $\partial\mathcal{V}$, and ε is equal to $+1$ if $\partial\mathcal{V}$ is timelike and -1 if $\partial\mathcal{V}$ is spacelike (it is assumed that $\partial\mathcal{V}$ is nowhere null, see figure B.1). Coordinates x^α are used for the finite region $\mathcal{V}$ and y^α for the boundary $\partial\mathcal{V}$[1].

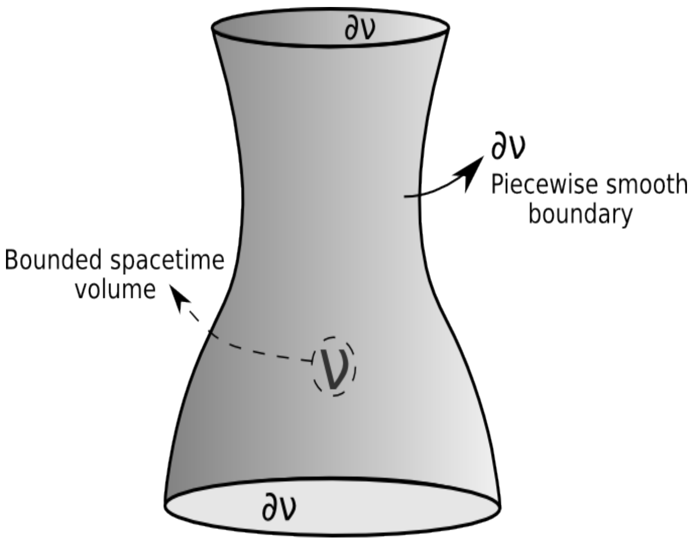

Figure 2.1: Spacetime bounded region

[1] For details see Appendix B.1

Now we will obtain the Einstein field equations varying the action with respect to $g^{\alpha\beta}$. We fixed the variation with the condition

$$\delta g_{\alpha\beta}\Big|_{\partial\mathcal{V}} = 0, \tag{2.6}$$

i.e., the variation of the metric tensor vanishes in the boundary $\partial\mathcal{V}$. We use the results [21, 22]

$$\delta g_{\alpha\beta} = -g_{\alpha\mu}g_{\beta\nu}\delta g^{\mu\nu}, \qquad \delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}, \tag{2.7}$$

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}, \tag{2.8}$$

$$\delta R^{\alpha}_{\beta\gamma\delta} = \nabla_\gamma(\delta\Gamma^{\alpha}_{\delta\beta}) - \nabla_\delta(\delta\Gamma^{\alpha}_{\gamma\beta}), \tag{2.9}$$

$$\delta R_{\alpha\beta} = \nabla_\gamma(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_\beta(\delta\Gamma^{\gamma}_{\gamma\alpha}). \tag{2.10}$$

We give a detailed review for the variation principles in GR following [23], [21] and [22],. The variation of the Einstein-Hilbert term gives

$$\delta S_{EH} = \int_{\mathcal{V}} d^4x\,(R\delta\sqrt{-g} + \sqrt{-g}\,\delta R). \tag{2.11}$$

Now with $R = g^{\alpha\beta}R_{\alpha\beta}$, we have that the variation of the Ricci scalar is

$$\delta R = \delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}\delta R_{\alpha\beta}. \tag{2.12}$$

using the Palatini's identity (2.10) we can write [22]:

$$\begin{aligned}\delta R &= \delta g^{\alpha\beta}R_{\alpha\beta} + g^{\alpha\beta}(\nabla_\gamma(\delta\Gamma^{\gamma}_{\beta\alpha}) - \nabla_\beta(\delta\Gamma^{\gamma}_{\alpha\gamma})),\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + \nabla_\sigma(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})),\end{aligned} \tag{2.13}$$

where we have used the metric compatibility $\nabla_\gamma g_{\alpha\beta} \equiv 0$ and relabeled some dummy indices. Inserting this results for the variations in expression (2.11) we have:

$$\begin{aligned}\delta S_{EH} &= \int_{\mathcal{V}} d^4x\,(R\delta\sqrt{-g} + \sqrt{-g}\,\delta R),\\ &= \int_{\mathcal{V}} d^4x\Big(-\frac{1}{2}Rg_{\alpha\beta}\sqrt{-g}\,\delta g^{\alpha\beta} + R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + \sqrt{-g}\nabla_\sigma(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma}))\Big),\\ &= \int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\Big)\delta g^{\alpha\beta} + \int_{\mathcal{V}} d^4x\,\sqrt{-g}\nabla_\sigma(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})).\end{aligned} \tag{2.14}$$

Denoting the divergence term with δS_B,

$$\delta S_B = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,\nabla_\sigma(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})), \tag{2.15}$$

we define

$$V^\sigma = g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma}), \tag{2.16}$$

then the boundary term can be written as

$$\delta S_B = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\,\nabla_\sigma V^\sigma. \tag{2.17}$$

Using Gauss-Stokes theorem [21, 22] (B.55):

$$\int_{\mathcal{V}} d^n x \sqrt{|g|}\nabla_\mu A^\mu = \oint_{\partial\mathcal{V}} d^{n-1}y\, \varepsilon \sqrt{|h|} n_\mu A^\mu, \tag{2.18}$$

where n_μ is the unit normal to $\partial\mathcal{V}$. Using this we can write (2.17) in the following boundary term

$$\delta S_B = \oint_{\partial\mathcal{V}} d^3y\, \varepsilon \sqrt{|h|} n_\sigma V^\sigma, \tag{2.19}$$

with V^σ given in (2.16). The variation $\delta\Gamma^\sigma_{\beta\alpha}$ is obtained by using that $\Gamma^\sigma_{\beta\alpha}$ is the Christoffel symbol $\{{}^{\sigma}_{\beta\alpha}\}$:

$$\Gamma^\alpha_{\beta\gamma} \equiv \left\{{}^{\alpha}_{\beta\gamma}\right\} = \frac{1}{2} g^{\alpha\sigma}[\partial_\beta g_{\sigma\gamma} + \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\beta\gamma}], \tag{2.20}$$

getting

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \delta\left(\frac{1}{2} g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}]\right),\\ &= \frac{1}{2}\delta g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}] + \frac{1}{2} g^{\sigma\gamma}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})]. \end{aligned} \tag{2.21}$$

From the boundary conditions $\delta g_{\alpha\beta} = \delta g^{\alpha\beta} = 0$ the variation (2.21) gives:

$$\delta\Gamma^\sigma_{\beta\alpha}\Big|_{\partial\mathcal{V}} = \frac{1}{2} g^{\sigma\gamma}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})], \tag{2.22}$$

and

$$V^\mu\Big|_{\partial\mathcal{V}} = g^{\alpha\beta}\left[\frac{1}{2} g^{\mu\gamma}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})]\right] - g^{\alpha\mu}\left[\frac{1}{2} g^{\nu\gamma}\partial_\alpha(\delta g_{\nu\gamma})\right], \tag{2.23}$$

we can write

$$\begin{aligned} V_\sigma\Big|_{\partial\mathcal{V}} = g_{\sigma\mu}V^\mu\Big|_{\partial\mathcal{V}} &= g_{\sigma\mu}g^{\alpha\beta}\left[\frac{1}{2} g^{\mu\gamma}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})]\right] - g_{\sigma\mu}g^{\alpha\mu}\left[\frac{1}{2} g^{\nu\gamma}\partial_\alpha(\delta g_{\nu\gamma})\right],\\ &= \frac{1}{2}\delta^\gamma_\sigma g^{\alpha\beta}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})] - \frac{1}{2}\delta^\alpha_\sigma g^{\nu\gamma}[\partial_\alpha(\delta g_{\nu\gamma})],\\ &= g^{\alpha\beta}[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})]. \end{aligned} \tag{2.24}$$

We now evaluate the term $n^\sigma V_\sigma\big|_{\partial\mathcal{V}}$ by using for this that

$$g^{\alpha\beta} = h^{\alpha\beta} + \varepsilon n^\alpha n^\beta, \tag{2.25}$$

then

$$\begin{aligned} n^\sigma V_\sigma\Big|_{\partial\mathcal{V}} &= n^\sigma(h^{\alpha\beta} + \varepsilon n^\alpha n^\beta)[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})],\\ &= n^\sigma h^{\alpha\beta}[\partial_\beta(\delta g_{\sigma\alpha}) - \partial_\sigma(\delta g_{\beta\alpha})], \end{aligned} \tag{2.26}$$

where we use the antisymmetric part of $\varepsilon n^\alpha n^\beta$ with $\varepsilon = n^\mu n_\mu = \pm 1$. To the fact $\delta g_{\alpha\beta} = 0$ in the boundary we have $h^{\alpha\beta}\partial_\beta(\delta g_{\sigma\alpha}) = 0$ [21]. Finally we get

$$n^\sigma V_\sigma\Big|_{\partial\mathcal{V}} = -n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha}). \tag{2.27}$$

Thus the variation of the Einstein-Hilbert term is:

$$\delta S_{EH} = \int_{\mathcal{V}} d^4x\, \sqrt{-g}\Big(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\Big)\delta g^{\alpha\beta} - \oint_{\partial\mathcal{V}} d^3y\, \varepsilon\, \sqrt{|h|}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \tag{2.28}$$

Now we consider the variation of the Gibbons-York-Hawking boundary term:

$$\delta S_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\, \varepsilon\, \sqrt{|h|}\delta K. \tag{2.29}$$

Using the definition of the trace of extrinsic curvature [21]:

$$\begin{aligned} K &= \nabla_\alpha n^\alpha, \\ &= g^{\alpha\beta}\nabla_\beta n_\alpha, \\ &= (h^{\alpha\beta} + \varepsilon n^\alpha n^\beta)\nabla_\beta n_\alpha, \\ &= h^{\alpha\beta}\nabla_\beta n_\alpha, \\ &= h^{\alpha\beta}(\partial_\beta n_\alpha - \Gamma^\gamma_{\beta\alpha}n_\gamma), \end{aligned} \tag{2.30}$$

the variation is

$$\begin{aligned} \delta K &= -h^{\alpha\beta}\delta\Gamma^\gamma_{\beta\alpha}n_\gamma, \\ &= -\frac{1}{2}h^{\alpha\beta}g^{\sigma\gamma}[\partial_\beta(\delta g_{\sigma\alpha}) + \partial_\alpha(\delta g_{\sigma\beta}) - \partial_\sigma(\delta g_{\beta\alpha})]n_\gamma, \\ &= -\frac{1}{2}h^{\alpha\beta}[\partial_\beta(\delta g_{\sigma\alpha}) + \partial_\alpha(\delta g_{\sigma\beta}) - \partial_\sigma(\delta g_{\beta\alpha})]n^\sigma, \\ &= \frac{1}{2}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \end{aligned} \tag{2.31}$$

This comes from the variation $\delta\Gamma^\gamma_{\beta\alpha}$ evaluated in the boundary, and the fact that $h^{\alpha\beta}\partial_\beta(\delta g_{\sigma\alpha}) = 0$, $h^{\alpha\beta}\partial_\alpha(\delta g_{\sigma\beta}) = 0$. Then we have for the variation of the Gibbons-York-Hawking boundary term:

$$\delta S_{GYH} = \oint_{\partial\mathcal{V}} d^3y\, \varepsilon\, \sqrt{|h|}h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \tag{2.32}$$

We see that this term exactly cancel the boundary contribution of the Einstein-Hilbert term. Now, if we have a matter action defined by:

$$S_M = \int_{\mathcal{V}} d^4x\, \sqrt{-g}\mathcal{L}_M[g_{\alpha\beta}, \psi], \tag{2.33}$$

where ψ denotes the matter fields. The variation of this action takes the form:

$$\begin{aligned} \delta S_M &= \int_{\mathcal{V}} d^4x\, \delta(\sqrt{-g}\mathcal{L}_M), \\ &= \int_{\mathcal{V}} d^4x\Big(\frac{\partial\mathcal{L}_M}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g} + \mathcal{L}_M\delta\sqrt{-g}\Big), \\ &= \int_{\mathcal{V}} d^4x\, \sqrt{-g}\Big(\frac{\partial\mathcal{L}_M}{\partial g^{\alpha\beta}} - \frac{1}{2}\mathcal{L}_M g_{\alpha\beta}\Big)\delta g^{\alpha\beta}, \end{aligned} \tag{2.34}$$

as usual, defining the stress-energy tensor by:

$$T_{\alpha\beta} \equiv -2\frac{\partial \mathcal{L}_M}{\partial g^{\alpha\beta}} + \mathcal{L}_M g_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}}, \tag{2.35}$$

then:

$$\delta S_M = -\frac{1}{2}\int_{\mathcal{V}} d^4x\, \sqrt{-g} T_{\alpha\beta}\delta g^{\alpha\beta}, \tag{2.36}$$

imposing the total variations remains invariant with respect to $\delta g^{\alpha\beta}$. Finally the equations are writing as:

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = 0, \Longrightarrow R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{2.37}$$

which corresponds to Einstein field equations in geometric units $c = 1$.

2.4 Variational principle in metric $f(R)$ gravity

In this section we introduce the boundary term in metric $f(R)$ gravity and we express it in a new form which allow us to perform a novel analysis of the boundary problem. The general action can be written as [9, 10]:

$$S_{mod} = \frac{1}{2\kappa}(S_{met} + S'_{GYH}) + S_M, \tag{2.38}$$

with the bulk term

$$S_{met} = \int_{\mathcal{V}} d^4x\, \sqrt{-g} f(R), \tag{2.39}$$

and the Gibbons-York-Hawking like boundary term [9, 10, 24]

$$S'_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\, \varepsilon \sqrt{|h|} f'(R) K, \tag{2.40}$$

with $f'(R) = df(R)/dR$. Again, S_M represents the action associated with all the matter fields (2.33). We fixed the variation to the condition

$$\delta g_{\alpha\beta}\Big|_{\partial\mathcal{V}} = 0. \tag{2.41}$$

First, the variation of the bulk term is:

$$\delta S_{met} = \int_{\mathcal{V}} d^4x\, (f(R)\delta\sqrt{-g} + \sqrt{-g}\,\delta f(R)), \tag{2.42}$$

and the functional derivative of the $f(R)$ term can be written as

$$\delta f(R) = f'(R)\delta R. \tag{2.43}$$

Using the expression for the variation of the Ricci scalar:

$$\delta R = \delta g^{\alpha\beta} R_{\alpha\beta} + \nabla_\sigma (g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma})), \tag{2.44}$$

where the variation of the term $g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})$ is given in 2.5. With this result the variation of the Ricci scalar becomes

$$\begin{aligned}\delta R &= \delta g^{\alpha\beta}R_{\alpha\beta} + \nabla_\sigma(g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})),\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + g_{\mu\nu}\nabla_\sigma\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\sigma\nabla_\gamma(\delta g^{\sigma\gamma}),\\ &= \delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta}).\end{aligned} \tag{2.45}$$

Here we define $\Box \equiv \nabla_\sigma\nabla^\sigma$ and relabeled some indices. Putting the previous results together in the variation of the modified action (2.42):

$$\begin{aligned}\delta S_{met} &= \int_{\mathcal{V}} d^4x\,(f(R)\delta\sqrt{-g} + \sqrt{-g}\,f'(R)\delta R),\\ &= \int_{\mathcal{V}} d^4x\Big(-f(R)\frac{1}{2}\sqrt{-g}\,g_{\alpha\beta}\delta g^{\alpha\beta} + f'(R)\sqrt{-g}\big(\delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})\big)\Big),\\ &= \int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(f'(R)\big(\delta g^{\alpha\beta}R_{\alpha\beta} + g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta})\big) - f(R)\frac{1}{2}\,g_{\alpha\beta}\delta g^{\alpha\beta}\Big).\end{aligned} \tag{2.46}$$

Now we will consider the next integrals:

$$\int_{\mathcal{V}} d^4x\,\sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}), \qquad \int_{\mathcal{V}} d^4x\,\sqrt{-g}f'(R)\nabla_\alpha\nabla_\beta(\delta g^{\alpha\beta}). \tag{2.47}$$

We shall see that these integrals can be expressed differently performing integration by parts. For this we define the next quantities:

$$M_\tau = f'(R)g_{\alpha\beta}\nabla_\tau(\delta g^{\alpha\beta}) - \delta g^{\alpha\beta}g_{\alpha\beta}\nabla_\tau(f'(R)), \tag{2.48}$$

and

$$N^\sigma = f'(R)\nabla_\gamma(\delta g^{\sigma\gamma}) - \delta g^{\sigma\gamma}\nabla_\gamma(f'(R)). \tag{2.49}$$

The combination $g^{\sigma\tau}M_\tau + N^\sigma$ is

$$g^{\sigma\tau}M_\tau + N^\sigma = f'(R)g_{\alpha\beta}\nabla^\sigma(\delta g^{\alpha\beta}) - \delta g^{\alpha\beta}g_{\alpha\beta}\nabla^\sigma(f'(R)) + f'(R)\nabla_\gamma(\delta g^{\sigma\gamma}) - \delta g^{\sigma\gamma}\nabla_\gamma(f'(R)), \tag{2.50}$$

in the particular case $f(R) = R$, the previous combination reduces to the expression (2.16) with equation (2.73). The equations (2.48) and (2.49) are the main results to use in our method. The quantities M_τ and N^σ allow us to write the variation of the bulk term (2.46) in the following way (for details see 2.6):

$$\begin{aligned}\delta S_{met} = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - f(R)\frac{1}{2}\,g_{\alpha\beta}\Big)\delta g^{\alpha\beta}\\ + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n^\tau M_\tau + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n_\sigma N^\sigma.\end{aligned} \tag{2.51}$$

In the next section we will work out with the boundary contribution from (2.51), and show how this terms cancel with the variations of the S'_{GYH} action.

2.4.1 Boundary terms in $f(R)$ gravity

We express the quantities M_τ and N^σ calculated in the boundary $\partial\mathcal{V}$. Is convenient to express them in function of the variations $\delta g_{\alpha\beta}$. Using the equation (2.7) in (2.48) and (2.49) yields :

$$M_\tau = -f'(R)g^{\alpha\beta}\nabla_\tau(\delta g_{\alpha\beta}) + g^{\alpha\beta}\delta g_{\alpha\beta}\nabla_\tau(f'(R)), \tag{2.52}$$

and

$$N^\sigma = -f'(R)g^{\sigma\mu}g^{\gamma\nu}\nabla_\gamma(\delta g_{\mu\nu}) + g^{\sigma\mu}g^{\gamma\nu}\delta g_{\mu\nu}\nabla_\gamma(f'(R)). \tag{2.53}$$

To evaluate this quantities in the boundary we use the fact that $\delta g_{\alpha\beta}|_{\partial\mathcal{V}} = \delta g^{\alpha\beta}|_{\partial\mathcal{V}} = 0$, then the only terms not vanishing are the derivatives of $\delta g_{\alpha\beta}$ in the covariant derivatives. Hence we have

$$M_\tau\Big|_{\partial\mathcal{V}} = -f'(R)g^{\alpha\beta}\partial_\tau(\delta g_{\alpha\beta}), \tag{2.54}$$

and

$$N^\sigma\Big|_{\partial\mathcal{V}} = -f'(R)g^{\sigma\mu}g^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu}), \tag{2.55}$$

We now compute $n^\tau M_\tau\big|_{\partial\mathcal{V}}$ and $n_\sigma N^\sigma\big|_{\partial\mathcal{V}}$ which are the terms in the boundary integrals (2.51)

$$\begin{aligned} n^\tau M_\tau\Big|_{\partial\mathcal{V}} &= -f'(R)n^\tau(\varepsilon n^\alpha n^\beta + h^{\alpha\beta})\partial_\tau(\delta g_{\alpha\beta}), \\ &= -f'(R)n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\alpha\beta}), \end{aligned} \tag{2.56}$$

where we rename the dummy index τ. In the other hand

$$\begin{aligned} n_\sigma N^\sigma\Big|_{\partial\mathcal{V}} &= -f'(R)n_\sigma(h^{\sigma\mu} + \varepsilon n^\sigma n^\mu)(h^{\gamma\nu} + \varepsilon n^\gamma n^\nu)\partial_\gamma(\delta g_{\mu\nu}), \\ &= -f'(R)n^\mu(h^{\gamma\nu} + \varepsilon n^\gamma n^\nu)\partial_\gamma(\delta g_{\mu\nu}), \\ &= -f'(R)n^\mu h^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu}) \\ &= 0, \end{aligned} \tag{2.57}$$

where we have used that $n_\sigma h^{\sigma\mu} = 0$, $\varepsilon^2 = 1$ and the fact that the tangential derivative $h^{\gamma\nu}\partial_\gamma(\delta g_{\mu\nu})$ vanishes. With this results the variation of the action S_{met} becomes:

$$\begin{aligned} \delta S_{met} = &\int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - f(R)\frac{1}{2}\,g_{\alpha\beta}\Big)\delta g^{\alpha\beta} \\ &- \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\,\sqrt{|h|}f'(R)n^\sigma h^{\alpha\beta}\partial_\sigma(\delta g_{\alpha\beta}). \end{aligned} \tag{2.58}$$

We proceed with the boundary term S'_{GYH} in the total action. The variation of this term gives

$$\begin{aligned} \delta S'_{GYH} &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\,\sqrt{|h|}(\delta f'(R)K + f'(R)\delta K), \\ &= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\,\sqrt{|h|}(f''(R)\delta R\,K + f'(R)\delta K). \end{aligned} \tag{2.59}$$

Using the expression for the variation of K, equation (2.31), we can write

$$\delta S'_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}\Big(f''(R)\delta R\,K + \frac{1}{2}f'(R)h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma\Big),$$
$$= 2\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f''(R)\delta R\,K + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}f'(R)h^{\alpha\beta}\partial_\sigma(\delta g_{\beta\alpha})n^\sigma. \tag{2.60}$$

We see that the second term in (2.60) cancels the boundary term in the variation (2.58), and in addition we need to impose $\delta R = 0$ in the boundary. Similar argument is given in [9]. Finally, with the variation of the matter action, given in (2.36), the total variation of the action of modified $f(R)$ gravity is:

$$\delta S_{mod} = \frac{1}{2\kappa}\int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - \frac{1}{2}f(R)\,g_{\alpha\beta}\Big)\delta g^{\alpha\beta}$$
$$- \frac{1}{2}\int_{\mathcal{V}} d^4x\,\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta}. \tag{2.61}$$

Imposing that this variation becomes stationary we have:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{mod}}{\delta g^{\alpha\beta}} = 0 \Longrightarrow f'(R)R_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) - \frac{1}{2}f(R)\,g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{2.62}$$

which are the field equations in the metric formalism of $f(R)$ gravity.

It is very important to mention that we have got the result (2.62) without using the Euler-Lagrange equations (B.59) directly. We have shown in this section that only with the condition $\delta S'_{mod} = 0$ and with the metric $g_{\alpha\beta}$ *as the only degree of freedom in the theory*, we are able to have the set of equations (2.62) with the conditions $\delta g_{\alpha\beta}\big|_{\partial\mathcal{V}} = 0$ and an additional geometrical restriction $\delta R\big|_{\partial\mathcal{V}} = 0$. The geometrical restriction seems not be a problem when it is interpreted in the framework of scalar-tensor theories. However, within the pure metric $f(R)$ gravity this is a very strong geometrical condition which can be used to test important issues as the Weak Equivalence Principle [25].

2.5 Evaluation of the term $g^{\alpha\beta}(\delta\Gamma^{\sigma}_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^{\gamma}_{\alpha\gamma})$

There are some important results that deserve some attention. We compute explicitly the quantities involved in our method.

We already have calculated the variation $\delta\Gamma^{\sigma}_{\beta\alpha}$:

$$\delta\Gamma^{\sigma}_{\beta\alpha} = \frac{1}{2}\delta g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}] + \frac{1}{2}g^{\sigma\gamma}[\partial_\beta(\delta g_{\gamma\alpha}) + \partial_\alpha(\delta g_{\gamma\beta}) - \partial_\gamma(\delta g_{\beta\alpha})], \tag{2.63}$$

writing the partial derivatives for the metric variations with the expression for the covariant derivative:

$$\nabla_\gamma\delta g_{\alpha\beta} = \partial_\gamma\delta g_{\alpha\beta} - \Gamma^{\sigma}_{\gamma\alpha}\delta g_{\sigma\beta} - \Gamma^{\sigma}_{\gamma\beta}\delta g_{\alpha\sigma}, \tag{2.64}$$

and also using that we are working in a torsion-free manifold i.e., the symmetry in the Christoffel symbol

$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, we can write:

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \frac{1}{2}\delta g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}] + \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\gamma\alpha}) + \nabla_\alpha(\delta g_{\gamma\beta}) - \nabla_\gamma(\delta g_{\beta\alpha}) + \Gamma^\lambda_{\beta\alpha}\delta g_{\gamma\lambda} + \Gamma^\lambda_{\alpha\beta}\delta g_{\lambda\gamma}],\\ &= \frac{1}{2}\delta g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}] + g^{\sigma\gamma}\Gamma^\lambda_{\beta\alpha}\delta g_{\gamma\lambda} + \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\gamma\alpha}) + \nabla_\alpha(\delta g_{\gamma\beta}) - \nabla_\gamma(\delta g_{\beta\alpha})],\end{aligned} \tag{2.65}$$

using equation (2.7) in the second term:

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \frac{1}{2}\delta g^{\sigma\gamma}[\partial_\beta g_{\gamma\alpha} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}] - \delta g^{\mu\nu}g^{\sigma\gamma}g_{\gamma\mu}g_{\lambda\nu}\Gamma^\lambda_{\beta\alpha} + \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\gamma\alpha}) + \nabla_\alpha(\delta g_{\gamma\beta}) - \nabla_\gamma(\delta g_{\beta\alpha})],\\ &= \delta g^{\sigma\nu}g_{\lambda\nu}\Gamma^\lambda_{\beta\alpha} - \delta g^{\mu\nu}\delta^\sigma_\mu g_{\lambda\nu}\Gamma^\lambda_{\beta\alpha} + \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\gamma\alpha}) + \nabla_\alpha(\delta g_{\gamma\beta}) - \nabla_\gamma(\delta g_{\beta\alpha})],\\ &= \delta g^{\sigma\nu}g_{\lambda\nu}\Gamma^\lambda_{\beta\alpha} - \delta g^{\sigma\nu}g_{\lambda\nu}\Gamma^\lambda_{\beta\alpha} + \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\gamma\alpha}) + \nabla_\alpha(\delta g_{\gamma\beta}) - \nabla_\gamma(\delta g_{\beta\alpha})].\end{aligned} \tag{2.66}$$

Then we have

$$\delta\Gamma^\sigma_{\beta\alpha} = \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(\delta g_{\alpha\gamma}) + \nabla_\alpha(\delta g_{\beta\gamma}) - \nabla_\gamma(\delta g_{\beta\alpha})], \tag{2.67}$$

and similarly

$$\delta\Gamma^\gamma_{\alpha\gamma} = \frac{1}{2}g^{\sigma\gamma}[\nabla_\alpha(\delta g_{\sigma\gamma})]. \tag{2.68}$$

However it is convenient to express the previous result in function of the variations $\delta g^{\alpha\beta}$, we again use (2.7):

$$\begin{aligned}\delta\Gamma^\sigma_{\beta\alpha} &= \frac{1}{2}g^{\sigma\gamma}[\nabla_\beta(-g_{\alpha\mu}g_{\gamma\nu}\delta g^{\mu\nu}) + \nabla_\alpha(-g_{\beta\mu}g_{\gamma\nu}\delta g^{\mu\nu}) - \nabla_\gamma(-g_{\beta\mu}g_{\alpha\nu}\delta g^{\mu\nu})],\\ &= -\frac{1}{2}g^{\sigma\gamma}[g_{\alpha\mu}g_{\gamma\nu}\nabla_\beta(\delta g^{\mu\nu}) + g_{\beta\mu}g_{\gamma\nu}\nabla_\alpha(\delta g^{\mu\nu}) - g_{\beta\mu}g_{\alpha\nu}\nabla_\gamma(\delta g^{\mu\nu})],\\ &= -\frac{1}{2}[\delta^\sigma_\nu g_{\alpha\mu}\nabla_\beta(\delta g^{\mu\nu}) + \delta^\sigma_\nu g_{\beta\mu}\nabla_\alpha(\delta g^{\mu\nu}) - g_{\beta\mu}g_{\alpha\nu}g^{\gamma\sigma}\nabla_\gamma(\delta g^{\mu\nu})],\\ &= -\frac{1}{2}[g_{\alpha\gamma}\nabla_\beta(\delta g^{\sigma\gamma}) + g_{\beta\gamma}\nabla_\alpha(\delta g^{\sigma\gamma}) - g_{\beta\mu}g_{\alpha\nu}\nabla^\sigma(\delta g^{\mu\nu})],\end{aligned} \tag{2.69}$$

where we write $\nabla^\sigma = g^{\sigma\gamma}\nabla_\gamma$. In a similar way:

$$\delta\Gamma^\gamma_{\alpha\gamma} = -\frac{1}{2}g_{\mu\nu}\nabla_\alpha(\delta g^{\mu\nu}). \tag{2.70}$$

Now we compute the term $g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma})$

$$
\begin{aligned}
g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) = &-\frac{1}{2}\Big([g^{\alpha\beta}g_{\alpha\gamma}\nabla_\beta(\delta g^{\sigma\gamma}) + g^{\alpha\beta}g_{\beta\gamma}\nabla_\alpha(\delta g^{\sigma\gamma}) - g^{\alpha\beta}g_{\beta\mu}g_{\alpha\nu}\nabla^\sigma(\delta g^{\mu\nu})] \qquad (2.71)\\
&- [g^{\alpha\sigma}g_{\mu\nu}\nabla_\alpha(\delta g^{\mu\nu})]\Big),\\
= &-\frac{1}{2}\Big([\delta^\beta_\gamma\nabla_\beta(\delta g^{\sigma\gamma}) + \delta^\alpha_\gamma\nabla_\alpha(\delta g^{\sigma\gamma}) - \delta^\alpha_\mu g_{\alpha\nu}\nabla^\sigma(\delta g^{\mu\nu})] - [g_{\mu\nu}g^{\alpha\sigma}\nabla_\alpha(\delta g^{\mu\nu})]\Big),\\
= &-\frac{1}{2}\Big([\nabla_\gamma(\delta g^{\sigma\gamma}) + \nabla_\gamma(\delta g^{\sigma\gamma}) - g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu})] - [g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu})]\Big),\\
= &-\frac{1}{2}\Big(2\nabla_\gamma(\delta g^{\sigma\gamma}) - 2g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu})\Big), \qquad (2.72)
\end{aligned}
$$

then we have,

$$g^{\alpha\beta}(\delta\Gamma^\sigma_{\beta\alpha}) - g^{\alpha\sigma}(\delta\Gamma^\gamma_{\alpha\gamma}) = g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\gamma(\delta g^{\sigma\gamma}). \qquad (2.73)$$

2.6 Integrals with M_τ and N^σ

Taking the covariant derivative in M_σ:

$$
\begin{aligned}
\nabla^\tau M_\tau &= \nabla^\tau(f'(R)g_{\alpha\beta}\nabla_\tau(\delta g^{\alpha\beta})) - \nabla^\tau(\delta g^{\alpha\beta}g_{\alpha\beta}\nabla_\tau(f'(R))),\\
&= \nabla^\tau(f'(R))g_{\alpha\beta}\nabla_\tau(\delta g^{\alpha\beta}) + f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \nabla^\tau(\delta g^{\alpha\beta})g_{\alpha\beta}\nabla_\tau(f'(R)) - \delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)),\\
&= f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)). \qquad (2.74)
\end{aligned}
$$

Here we have used the metric compatibility $\nabla^\tau g_{\alpha\beta} = 0$, integrating this expression

$$\int_{\mathcal{V}} d^4x\, \sqrt{-g}\nabla^\tau M_\tau = \int_{\mathcal{V}} d^4x\, \sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) - \int_{\mathcal{V}} d^4x\, \sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)), \qquad (2.75)$$

using again the Gauss-Stokes theorem (B.55), the first integral can be written as a boundary term:

$$\oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n^\tau M_\tau = \int_{\mathcal{V}} d^4x\, \sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta})) - \int_{\mathcal{V}} d^4x\, \sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)), \qquad (2.76)$$

then we can write:

$$\int_{\mathcal{V}} d^4x\, \sqrt{-g}f'(R)g_{\alpha\beta}\Box(\delta g^{\alpha\beta}) = \int_{\mathcal{V}} d^4x\, \sqrt{-g}\delta g^{\alpha\beta}g_{\alpha\beta}\Box(f'(R)) + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n^\tau M_\tau. \qquad (2.77)$$

In a similar way, taking the covariant derivative of N^σ:

$$
\begin{aligned}
\nabla_\sigma N^\sigma &= \nabla_\sigma(f'(R)\nabla_\gamma(\delta g^{\sigma\gamma})) - \nabla_\sigma(\delta g^{\sigma\gamma}\nabla_\gamma(f'(R))),\\
&= \nabla_\sigma(f'(R))\nabla_\gamma(\delta g^{\sigma\gamma}) + f'(R)\nabla_\sigma\nabla_\gamma(\delta g^{\sigma\gamma}) - \nabla_\sigma(\delta g^{\sigma\gamma})\nabla_\gamma(f'(R)) - \delta g^{\sigma\gamma}\nabla_\sigma\nabla_\gamma(f'(R)),\\
&= f'(R)\nabla_\sigma\nabla_\beta(\delta g^{\sigma\beta}) - \delta g^{\sigma\beta}\nabla_\sigma\nabla_\beta(f'(R)), \qquad (2.78)
\end{aligned}
$$

integrating:

$$\int_{\mathcal{V}} d^4x\, \sqrt{-g}\nabla_\sigma N^\sigma = \int_{\mathcal{V}} d^4x\, \sqrt{-g}f'(R)\nabla_\sigma\nabla_\beta(\delta g^{\sigma\beta}) - \int_{\mathcal{V}} d^4x\, \sqrt{-g}\delta g^{\sigma\beta}\nabla_\sigma\nabla_\beta(f'(R)), \qquad (2.79)$$

using again the Gauss-Stokes theorem we can write:

$$\int_{\mathcal{V}} d^4x\ \sqrt{-g} f'(R)\nabla_\sigma\nabla_\beta(\delta g^{\sigma\beta}) = \int_{\mathcal{V}} d^4x\ \sqrt{-g}\delta g^{\sigma\beta}\nabla_\sigma\nabla_\beta(f'(R)) + \oint_{\partial\mathcal{V}} d^3y\,\varepsilon\sqrt{|h|}n_\sigma N^\sigma. \tag{2.80}$$

In the next sections, we summarize some aspects of the Brans-Dicke theory. We have stressed that our main result does not depend on the equivalence between scalar-tensor theory and $f(R)$ gravity. However, the boundary term we have used has its equivalence in the scalar-tensor framework and even more important, the scalar-tensor approach provides a natural way to deal with the boundary problem in more general theories than metric $f(R)$ gravity.

2.7 Brans-Dicke gravity

This thesis focus on a specific class of extended theories of gravity, namely $f(R)$ gravity with R the Ricci scalar. However, in different aspects of fundamental physics, cosmology and mathematics the Brans-Dicke theory of gravity plays an important role as one of the fundamental prototypes for alternatives theories to GR [3, 5]. The action in the Jordan's frame (set of variables for gravity $\left(\phi, g_{\mu\nu}\right)$) is [5]

$$S_{BD} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega_{BD}}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right] + S^m(g_{\mu\nu}, \Psi, \nabla_\mu\Psi\dots), \tag{2.81}$$

with $S^m(g_{\mu\nu}, \Psi)$ the matter-action that depends on the matter fields $(\Psi, \nabla_\mu\Psi\dots)$ but the matter-action is independent of ϕ. It is a very important point [3, 5] even when it is possible through a conformal transformation take the action (2.81) to the Einstein's frame where the fields for the action in the gravity sector are $(\hat{R}(\phi), \hat{g}_{\mu\nu}(\phi))$. In this scheme the physical interpretation of the field equations and the conservation laws face physical difficulties. In the other hand, ω_{BD} is the only dimensionless free parameter of the theory. The field equations for the Brans-Dicke theory are

$$\frac{\delta S_{BD}}{\delta g^{\mu\nu}} = 0, \tag{2.82}$$

and

$$\frac{\delta S_{BD}}{\delta\phi} = 0. \tag{2.83}$$

We can proceed in two mathematically equivalent ways, we can use (B.59) or employ the methodology in (2.4). We will see that both procedures are equivalent with the care that when we use (B.59) the equations are computed directly for the bulk and the boundaries have been setting with the criteria of a well-posed mathematical problem.

2.7.1 Variation respect to $g^{\mu\nu}$

First we consider the equation (2.82) respect to the metric tensor $g_{\mu\nu}$. The variation for (2.81) is [5]

$$\begin{aligned}\delta S_{\mathrm{BD}} = \frac{1}{16\pi}\int d^4x\Big[\delta(\sqrt{-g}\phi R) - \frac{\omega_{BD}}{\phi}\delta(\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)\Big] \\ -\frac{1}{16\pi}\int d^4x\Big[\delta(\sqrt{-g}V(\phi))\Big].\end{aligned} \tag{2.84}$$

The equation (2.84) has the following contributions:

$$\int d^4x\Big[\delta(\sqrt{-g}V(\phi))\Big] = -\frac{1}{2}\int d^4x\,\sqrt{-g}g_{\mu\nu}V(\phi)\delta g^{\mu\nu}. \tag{2.85}$$

The next term in (2.84) gives the contribution

$$\frac{1}{16\pi}\int d^4x\Big[\frac{\omega}{\phi}\delta(\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi)\Big] = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\Big[\frac{\omega_{BD}}{\phi}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi)\Big]\delta g^{\mu\nu}. \tag{2.86}$$

The term in the Brans-Dicke action (2.81) with non-minimal coupling (ϕR) must be treated carefully. For this purpose and following [5], the Ricci scalar (B.37) becomes

$$R = g^{\mu\nu}(\partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma_\mu) + \Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\rho\sigma}, \tag{2.87}$$

where, the quantities Γ_μ and C^λ are defined as [5]

$$\begin{aligned} \Gamma_\mu \equiv \Gamma^\rho_{\rho\mu} &= \frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}} = \frac{1}{2}g^{\rho\sigma}\partial_\mu g_{\rho\sigma}, \\ C^\lambda \equiv g^{\mu\nu}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\mu\nu}\Big[g^{\lambda\sigma}\{g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\}\Big] = -\partial_\sigma g^{\lambda\sigma} - g^{\lambda\nu}\Gamma_\nu. \end{aligned} \tag{2.88}$$

The last result is a consequence of $\left(g^{\lambda\sigma}g_{\mu\sigma}\right)_{,\nu} = 0$. The previous results allow us to split the Ricci scalar as

$$R = g^{\mu\nu}\Big(\partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma_\mu\Big) + \Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\sigma\rho}. \tag{2.89}$$

Einstein himself employed an action coming from the Ricci scalar with the property that it is first order in the metric (see equation (B.63)) to obtain the field equations with proper boundary conditions [9], for a more detailed explanation see (B.2). From the results (2.88) and (2.89) we have [5]

$$\sqrt{-g}g^{\mu\nu}(\partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma_\mu) = \partial_\lambda\Big(\sqrt{-g}G^\lambda\Big) - 2\sqrt{-g}\Big(\Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\rho\sigma}\Big), \tag{2.90}$$

where

$$G^\lambda = C^\lambda - g^{\lambda\mu}\Gamma_\mu. \tag{2.91}$$

The first term in (2.81) is now written as

$$\int d^4x\,\sqrt{-g}\phi R = \int \phi\Big\{\partial_\lambda(\sqrt{-g}G^\lambda) - \sqrt{-g}\Big(\Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\rho\sigma}\Big)\Big\}d^4x, \tag{2.92}$$

or using (B.55) together with the identity $\nabla_\lambda A^\lambda = \frac{1}{\sqrt{-g}}\partial_\lambda A^\lambda$

$$\begin{aligned} &\int \phi\Big\{\partial_\lambda(\sqrt{-g}G^\lambda) - \sqrt{-g}\Big(\Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\rho\sigma}\Big)\Big\}d^4x \\ &= -\int \sqrt{-g}\Big\{\partial_\lambda\phi G^\lambda + \phi\Big(\Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\rho\sigma}\partial_\lambda g^{\rho\sigma}\Big)\Big\}d^4x + \oint \phi G^\alpha \mathrm{d}\Sigma_\alpha. \end{aligned} \tag{2.93}$$

The variation in the first term in (2.93) gives the contribution for the Brans-Dicke theory different from the Einstein tensor. The second term in (2.93) is the usual GR part and there is a contribution from the boundary. As we see, this contribution makes necessary the introduction of a boundary term for the Brans-Dicke gravity[2]. For an example of Brans-Dicke gravity plus boundary conditions applied to thermodynamics of black holes see [26]. The equation of motion in the bulk are directly computed from (B.59), we first consider the term in (2.93)

$$-\int d^4x\delta\Big(\sqrt{-g}\partial_\lambda\phi G^\lambda\Big), \tag{2.94}$$

it provides the usual field equations system

$$\frac{\delta\Big(\sqrt{-g}G^\lambda\partial_\lambda\phi\Big)}{\delta g_{\sigma\rho}} = \partial_\lambda\phi\frac{\partial}{\partial g_{\sigma\rho}}\Big(\sqrt{-g}G^\lambda\Big) - \partial_\mu\left[\partial_\lambda\phi\frac{\partial\Big(\sqrt{-g}G^\lambda\Big)}{\partial\partial_\mu g_{\sigma\rho}}\right]. \tag{2.95}$$

In order to gain some insight about (2.95), we can compute the quantities in the local Lorentz frame [5] [3]

$$\frac{\partial\Big(\sqrt{-g}G^\lambda\Big)}{\partial g_{\sigma\rho}} = \frac{\partial\Big(\sqrt{-g}(-\partial_\delta g^{\lambda\delta} - g^{\lambda\nu}g^{\alpha\beta}\partial_\nu g_{\alpha\beta})\Big)}{\partial g_{\sigma\rho}} = 0, \tag{2.96}$$

and from (2.88) we get

$$-\partial_\mu\left(\partial_\lambda\phi\frac{\partial\Big(\sqrt{-g}G^\lambda\Big)}{\partial\partial_\mu g_{\sigma\rho}}\right) = -\sqrt{-g}\partial_\mu\left(\partial_\lambda\phi\frac{\partial G^\lambda}{\partial\partial_\mu g_{\sigma\rho}}\right). \tag{2.97}$$

The equation (2.97) together with the definitions for Γ_μ and C^λ brings the desired result [5]

$$\frac{\delta}{\delta g_{\rho\sigma}}\Big(\sqrt{-g}G^\lambda\partial_\lambda\phi\Big) = -\sqrt{-g}\Big(\nabla^\rho\nabla^\sigma - g^{\rho\sigma}\Box\Big)\phi. \tag{2.98}$$

The equation (2.98) holds for the curved spacetime. The total variation for (2.81) is [5]

$$\int d^4x\sqrt{-g}\Big\{\phi G_{\mu\nu} - \Big(\nabla_\mu\phi\nabla_\sigma\phi - g_{\mu\nu}\Box\phi\Big) - \frac{\omega_{BD}}{\phi}\Big[\nabla_\mu\phi\nabla_\sigma\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi\Big] + \frac{1}{2}g_{\mu\nu}V(\phi)\Big\}\delta g^{\mu\nu} = 0 \tag{2.99}$$

where

$$\frac{\delta\Big(\phi(\Gamma_\lambda C^\lambda + \frac{1}{2}\Gamma^\lambda_{\alpha\beta}\partial_\lambda g^{\alpha\beta})\Big)}{\delta g_{\rho\sigma}} = \phi\sqrt{-g}G_{\rho\sigma}. \tag{2.100}$$

The equation (2.100) was derived in the modified $f(R)$ gravity section (2.3). The procedure is equivalent to take the equation (B.59). The equation for the Brans-Dicke gravity considering the energy-momentum tensor finally becomes

$$\begin{aligned} G_{\mu\nu} = \frac{8\pi}{\phi}T^m_{\mu\nu} + \frac{\omega_{BD}}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi\Big) \\ +\frac{1}{\phi}\Big(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\Big) - \frac{V}{2\phi}g_{\mu\nu}. \end{aligned} \tag{2.101}$$

[2] Generally, surface terms are dropped out with arguments that the fields vanishes at these points.
[3] In the local Lorentz frame the derivatives of Γ are non zero.

The trace for (2.101) is

$$R = \frac{-8\pi T^m}{\phi} + \frac{\omega_{BD}}{\phi^2}\nabla_\mu\phi\nabla^\mu\phi + \frac{3\Box\phi}{\phi} + \frac{2V}{\phi}. \tag{2.102}$$

The full system for the Brans-Dicke gravity is completed with the variation of (2.81) respect to the scalar ϕ. It is consider in the next section.

2.7.2 Variation with respect to ϕ

The variation for (2.81) is

$$\frac{\delta S_{BD}}{\delta\phi} = 0. \tag{2.103}$$

Now, we write (2.103) as

$$\frac{\delta S_{BD}}{\delta\phi} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\Big[\delta(\phi R) - \delta\Big(\frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\Big) - \delta\Big(V(\phi)\Big)\Big]. \tag{2.104}$$

Working with the right hand side of (2.104) each contribution becomes

$$\begin{aligned}\int d^4x\sqrt{-g}\Big[\delta(\phi R)\Big] &= \int d^4x\sqrt{-g}R\delta\phi,\\ \int d^4x\sqrt{-g}\Big[\delta\Big(\frac{\omega_{BD}}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\Big)\Big] &= \int d^4x\sqrt{-g}\Big[-\frac{\omega_{BD}}{\phi^2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\delta\phi + 2\frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi\Big],\\ \int d^4x\sqrt{-g}\delta\Big(V(\phi)\Big) &= \int d^4x\sqrt{-g}\frac{\mathrm{d}V}{\mathrm{d}\phi}\delta\phi.\end{aligned} \tag{2.105}$$

To go further with (2.105) using the identitities

$$\begin{aligned}g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi &= g^{\mu\nu}\nabla_\mu\Big(\delta\phi\nabla_\nu\phi\Big) - \Big(g^{\mu\nu}\nabla_\mu\nabla_\nu\phi\Big)\delta\phi.\\ \nabla_\mu\Big(\frac{\omega_{BD}}{\phi}(\nabla_\nu\phi)\delta\phi\Big) &= -\frac{\omega_{BD}}{\phi^2}\Big(\nabla_\mu\phi\nabla_\nu\phi\Big)\delta\phi + \frac{\omega_{BD}}{\phi}\nabla_\mu\Big(\delta\phi\nabla_\nu\phi\Big).\end{aligned} \tag{2.106}$$

The equation (2.106) makes possible an integration by parts of (2.105) and with the Gauss-Stokes together with the boundary condition $\delta\phi = 0\big|_{\mathcal{V}}$ the variation for the action (2.81) respect to the scalar ϕ is

$$\int d^4x\sqrt{-g}\Big[R - \frac{\omega_{BD}}{\phi^2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2\frac{\omega_{BD}}{\phi}\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi}\Big]\delta\phi = 0, \tag{2.107}$$

in other words, the field equation from the scalar ϕ for the Brans-Dicke theory is

$$R - \frac{\omega_{BD}}{\phi^2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2\frac{\omega_{BD}}{\phi}\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0, \tag{2.108}$$

with $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$. One can eliminate R from (2.108) with the expression (2.102)

$$\Box\phi = \frac{1}{2\omega_{BD} + 3}\left(8\pi T^m + \phi\frac{\mathrm{d}V}{\mathrm{d}\phi} - 2V\right). \tag{2.109}$$

From (B.59) the equation (2.108) is easily obtained as

$$\frac{\delta S_{BD}}{\delta\phi} = \frac{\partial}{\partial\phi}\Big(\phi R - \frac{\omega_{BD}}{\phi} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V\Big) - \partial_\rho\Big(-\frac{\partial\big(\frac{\omega_{BD}}{\phi} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\big)}{\partial\partial_\rho\phi}\Big), \tag{2.110}$$

which reproduces exactly the required result (2.108). However, this procedure does not allow to set the boundary condition $\delta\phi = 0$ contrary to our method as we can see from (2.105).

The action (2.81) and the field equations suggest that the scalar field ϕ could be identified with the inverse of the effective gravitational coupling

$$G_{eff}(\phi) = \frac{1}{\phi}, \tag{2.111}$$

and it is a function of the spacetime location, with the restriction $\phi > 0$ to ensure a positive gravitational coupling. The dimensionless Brans-Dicke parameter ω_{BD} is a free parameter of the theory. The larger the value of ω_{BD} the closer Brans-Dicke theory to GR [3, 14]. For experimental and solar systems bounds on ω_{BD} see [5, 14]. One of the main points to include the Brans-Dicke theory in this thesis is due to the fact that metric and Palatini $f(R)$ gravities are equivalent to scalar-tensor theories with the derivative of the function $f(R)$ playing the role of the Brans-Dicke scalar [3].

2.8 Equivalence between $f(R)$ and scalar-tensor gravity

One of the most interesting and intriguing points about models beyond GR is the *equivalence* between some kinds of theories [3, 4]. In the case of metric and Palatini $f(R)$ gravity, there is an equivalence with the scalar-tensor theory [3, 4]. The equivalence is based on the derivative of $f(R)$ acting as the Brans-Dicke scalar field (scalaron) [3, 4]. We only summarize the metric case, for details see ([3]). We start with the action for $f(R)$ gravity (2.39) and we set $\phi \equiv R$. The action (2.39) is rewritten in the form

$$S_{mod} = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\Big(\psi(\phi)R - V(\phi)\Big) + S^m, \tag{2.112}$$

when $f''(\phi) \neq 0$, in (2.112) we can set

$$\begin{aligned}\psi &= f'(\phi),\\ V(\phi) &= \phi f'(\phi) - f(\phi).\end{aligned} \tag{2.113}$$

and the equivalence is obtained from (2.113)

$$S_{mod} = \frac{1}{16\pi}\int \mathrm{d}^4x\,\sqrt{-g}\,(\psi(\phi)R - \phi f'(\phi) + f(\phi)) = \frac{1}{16\pi}\int \sqrt{-g}\mathrm{d}^4x f(R) + S^m. \tag{2.114}$$

where we should use the fundamental theorem of auxiliary fields [9, 27] to recover the right side of (2.114) in terms of $f(R)$. Moreover, we can see that *vice-versa* is also valid. From the variation respect to ϕ of (2.112), which leads to

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi}R - \frac{\mathrm{d}V}{\mathrm{d}\phi} = \big(R - \phi\big)f'' = 0, \tag{2.115}$$

and it implies $\phi = R$ when $f'' \neq 0$. The action (2.112) has the Brans-Dicke 2.81 form

$$S = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\Big[\psi R - \frac{\omega_{BD}}{2}\nabla^\mu\psi\nabla_\mu\psi - U(\psi)\Big] + S^m \tag{2.116}$$

with Brans-Dicke field ψ, Brans-Dicke parameter $\omega_{BD} = 0$, and potential $U(\psi) = V[\phi(\psi)]$. There are several important consequences about a Brans-Dicke theory with $\omega_{BD} = 0$, for example, in the weak-field limit we can get Yukawa-like corrections to the Newtonian potential. Another interesting application of the theory is in the context of *dilaton gravity*. For deep study about scalar-tensor theories and their generalizations see [3–5, 28].

2.8.1 Boundary contribution in scalar-tensor gravity

The action for a general scalar-tensor theory of gravity is [3, 9]

$$S = \int_{\mathcal{V}} d^4x\,\sqrt{-g}\Big(F(\phi)R - \frac{1}{2}\lambda(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - U(\phi)\Big), \tag{2.117}$$

and the generalized boundary term [9]

$$S_{GYH} = 2\oint_{\partial\mathcal{V}} d^3y\,\sqrt{|h|}F(\phi)K, \tag{2.118}$$

with the boundary conditions $\delta g_{\alpha\beta}|_{\mathcal{V}} = 0$ and $\delta\phi|_{\mathcal{V}} = 0$.[4] The generalized field equations are directly from (2.81) setting the proper boundary condition $\delta g_{\alpha\beta}\big|_{\mathcal{V}} = 0$. For the variation respect to $g^{\mu\nu}$ the field equations from (2.117) are

$$2F(\phi)\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big) + 2\Box F(\phi)g_{\mu\nu} - 2\nabla_\mu\nabla_\nu F(\phi) = T^{\phi}_{\mu\nu} + T^{mat}_{\mu\nu}, \tag{2.119}$$

with $\delta g_{\mu\nu} = 0$ on the boundary and

$$T^{\phi}_{\mu\nu} = \lambda(\phi)\Big(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\phi\nabla^\alpha\phi - g_{\mu\nu}U(\phi)\Big). \tag{2.120}$$

which is the natural generalization to (2.101). In the case of the scalar field, the variation respect to ϕ leads

$$\lambda(\phi)\Box\phi + \frac{1}{2}\lambda'(\phi)\nabla_\alpha\phi\nabla^\alpha\phi - U'(\phi) + F'(\phi)R = 0, \tag{2.121}$$

in this case $\delta\phi|_{\mathcal{V}} = 0$ and the equation (2.121) is the generalization to (2.108).

2.8.2 Boundary term for metric $f(R)$ gravity

The boundary term for metric $f(R)$ gravity from (2.113) and the equivalence in (2.114) is

$$S_{GYH} = 2\oint_{\partial\mathcal{V}} \sqrt{|h|}d^3y\, f'(\phi)K \equiv 2\oint_{\partial\mathcal{V}} \sqrt{|h|}d^3y\, f'(R)K. \tag{2.122}$$

We have shown that is possible to start with an action coming only from $f(R)$ gravity and adding the boundary term (2.122) without the equivalence between $f(R)$ gravity and scalar-tensor theory.

[4] Notice that F plays the role of ψ in (2.112).

Moreover, we have derived in [10] the field equations for metric $f(R)$ gravity with the appropriate boundary contribution only from geometrical contributions adding a boundary contribution as (2.122) fixing only $\delta g_{\alpha\beta} = 0\big|_{\partial\mathcal{V}}$, but we got the constraint $\delta R|_{\partial\mathcal{V}} = 0$ similar to [24] without scalar fields. Some authors [9, 25] argued that given the equivalence between $f(R)$ and the scalar-tensor theory the curvature R encodes the scalar degree of freedom of the theory and the condition $\delta R|_{\partial\mathcal{V}} = 0$ is equivalent to $\delta\phi|_{\partial\mathcal{V}} = 0$. However, our proposal of write the action for metric $f(R)$ gravity with the action given by (2.38) has been widely accepted as a methodology to work in metric $f(R)$ gravity without mention the equivalence with scalar-tensor theories but with proper boundary conditions.

2.8.3 Higher order gravities

Generalizations of modify gravity including actions as

$$S \approx \int \sqrt{-g}\mathrm{d}^n x\, F(R, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}, \ldots \nabla_\nu R \nabla^\nu R, \ldots), \tag{2.123}$$

are widely known in the literature [9, 28]. We can think that the models derived from (2.123) are quite diverse, but in fact, as is studied in the very important paper [28], the models coming from (2.123) are essentially equivalent to various scalar-tensor theories [3, 9, 28]. As is shown in [28] the transition from a theory with higher order derivatives in the metric field to scalar-tensor theories involves an elimination of higher derivatives of the metric, usually they are replaced by a set of scalar fields that encode the physical degrees of freedom of the theory. The equivalence between the theory given by (2.123) is a scalar-tensor theory, typically a Brans-Dicke like theory in the Jordan frame [5, 9, 28]. In these theories the matter fields are minimally coupled to a potential for the scalar fields. The boundary term for an action like (2.123) can be obtained from the GYH term when we write (2.123) in the Einstein frame and after we can recover the boundary contribution in the Jordan frame through a conformal transformation and using the equivalence with the scalar-tensor theory. For higher derivative Lagrangians, where second and higher order derivatives of the metric appear and they cannot be removed by adding total derivatives to the action, the equations of motion are at least of fourth order and the GYH term obtained from the procedure described above are not generally sufficient to get a well-posed mathematical problem only setting $\delta g_{\alpha\beta}|_{\partial\mathcal{V}} = 0$. As we found in our work [10], for metric $f(R)$ gravity, the well-posed mathematical problem demands $\delta R|_{\partial\mathcal{V}} = 0$. This point also opens a new issue when we are dealing with actions like (2.123). Should we use the equivalence between higher order derivatives Lagrangians with scalar-field theories to build up the boundary contribution? Are possible higher order derivatives theories only with metric degrees of freedom? Some aspects related with deep mathematical problems in gravity theories can be check in [12, 16–20, 24, 28, 29]. For further discussion about mathematical motivations in field theories see [15, 30–40].

2.8.4 Some remarks

We have obtained the field equations in the metric formalism of $f(R)$ gravity by using the direct results from variational principles. The modified action in the metric formalism of $f(R)$ gravity plus a Gibbons-York-Hawking like boundary term must be written as:

$$S_{mod} = \frac{1}{2\kappa}\Big[\int_{\mathcal{V}} d^4x\, \sqrt{-g}\Big(f(R) + 2\kappa\mathcal{L}_M[g_{\alpha\beta}, \psi]\Big) + 2\oint_{\partial\mathcal{V}} d^3y\, \varepsilon\sqrt{|h|} f'(R) K\Big], \tag{2.124}$$

with $f'(R) = df(R)/dR$ and $\mathcal{L}_M$ the Lagrangian associated with all the matter fields. From the quantities M_σ and N^σ, defined in (2.48) and (2.49) respectively, we recovered GR plus Gibbons-York-Hawking boundary term in the particular case $f(R) = R$. We see that including the boundary term, we have a well behaved mathematical problem setting both, $\delta g_{\alpha\beta} = 0$ and $\delta R = 0$ in $\partial\mathcal{V}$.

We must emphasize that ETGs are one class of the most well established and deeply studied alternative theories of gravity and they naturally arise in different contexts of physics and mathematics. For example, dimensionally reduced effective theories of higher dimensionality such Kaluza-Klein and string models [41], it motivates to study EGTs within the context of cosmology and astrophysics, as well, they bring some interesting mathematical and physical problems as the one we addressed in our work [10]. After the publication there have been a number of citations and papers covering very different aspects concerning to physical and mathematical aspects related with the boundary term coming only from the (2.38). The topics cover from the Weak Equivalence Principle to superconductors in $f(R)$ gravity. In all the citations our proposal takes importance for several physical applications and mathematically there are still open problems arising from our proposal which should be investigated. As a final remark, several works including PhD thesis from important schools have adopted our approach to study the boundary problem in metric $f(R)$ gravity as can be easily checked in the quotations to our work.

CHAPTER 3

Covariant dynamics of the cosmological models: covariant "1+3" formalism

3.1 Introduction

In this chapter we describe the basic tools for the dynamics of the cosmological models in the 1 + 3 formalism following [1]. After a short review about the main aspects of the cosmological dynamics in both GR and metric $f(R)$ gravity, we present our contribution about the **Geodesic Deviation Equation** (GDE) in metric $f(R)$ gravity. Our work [42] is one of the first known attempts to generalize the GDE to ETGs. We also discuss the GDE in the $1+3$ formalism [43]. As an example, the equation for the angular diameter distance in the Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmologies is derived. For definitions and useful equations in the "1+3 formalism" see Appendix B.

3.2 Cosmology in "1+3" language

It is very common to find in the cosmological literature the expression: *The spacetime manifold is covered with comoving coordinates*. Even in the case of "homogeneous and isotropic solutions" for a gravity theory as GR, this is a not trivial task [1]. Deep ideas as the **Weyl principle** and strong assumptions as the **Copernican Principle** are the fundamental blocks to write the field equations in any gravity theory. When we consider the perturbed universe the situation is worse, due to the general covariance of the gravity theory, the cosmological perturbation theory differs from other branches of physics because in this theory *the perturbed object is the spacetime itself*. There has been a very important effort to describe the dynamics of the cosmological models in a coordinate independent way. Ehlers in his famous paper gave important clues for this task [44]. Others authors as G.F.R. Ellis and his collaborators have used this framework with success in GR specially for cosmological purposes. In this chapter, we will review the main aspects of the "1+3 covariant formalism". The main goal is to discuss the results obtained in our work [42] and complemented in [45]. The extension to the covariant approach for $f(R)$ gravity is also reviewed from [43].

To describe the spacetime manifold is convenient to use comoving coordinates adapted to the fundamental world lines. The construction of the coordinates locally follows the rules [1].

- Choose a surface S that intersects each world line once, in general, this surface is not unique. Label each world line where it intersects this surface; as the surface is three-dimensional, three labels y^i, $(i = 1, 2, 3)$, are needed to label all the world lines.

- Extend this labeling off the surface S by maintaining the same labeling for the world lines at later and earlier times. Thus, y^i are comoving coordinates: the value of the coordinate is maintained along the world lines.

- Define a time coordinate t along the fluid flow lines (it must be a increasing function along each flow line).

The set $\left(t, y^i\right)$ are **comoving coordinates adapted to the flow lines.** In general, the surfaces $\left(t = \text{constant}\right)$ are not orthogonal to the fundamental world lines. A particular choice for the temporal coordinate is $s = \tau + s_0$, where τ is the proper time measured along the fundamental world lines from S (positive to the future of S, negative to the past) and s_0 an arbitrary constant. The set of coordinates $\left(s, y^i\right)$ are called "normalized comoving coordinates." For homogeneous and isotropic solutions as FLRW such normalized comoving coordinates are used. It is important to notice that the $1+3$ formalism naturally enables to keep the discussion about the cosmological models quite general, assumptions as *the cosmological principle* are not necessary as a fundamental condition.

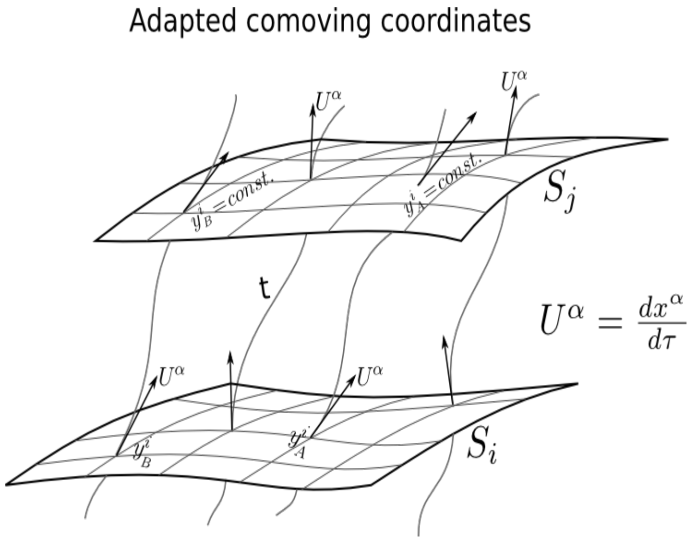

Figure 3.1: comoving coordinates

3.3 The Covariant approach

The matter components in the universe define a physically motivated choice of preferred motion. It is usual to choice the CMB frame, where the radiation dipole vanishes, as the natural reference frame in cosmology. But it is not the only choice, other useful frame is where the total momentum density of of all components vanishes [1]. Each frame defines a preferred 4−velocity field u^a with preferred world lines. The 4−velocity field is timelike and normalized by the condition $g_{ab}u^a u^b = -1$. The preferred world lines are given in terms of *local coordinates* $x^\mu = x^\mu(\tau)$, with τ the proper time along the world lines, the preferred 4−velocity is the unit timelike vector (see figure3.1)

$$u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, \tag{3.1}$$

and from the spacetime interval ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$) invariant

$$\tau = \int \left[-(\mathrm{d}x^a/\mathrm{d}\tau)\left(\mathrm{d}x^b/\mathrm{d}\tau\right)g_{ab}\right]^{1/2}\mathrm{d}\tau = \int \left(-u^a u_a\right)d\tau, \tag{3.2}$$

we find the normalization of the 4−velocity field as

$$u^a u_a = -1. \tag{3.3}$$

The above considerations are quite powerful when we should deal with the cosmological perturbation theory as we will show in the next chapters. In normalized comoving coordinates $u^a = \left(\frac{\mathrm{d}s}{\mathrm{d}\tau}, \frac{\mathrm{d}y^i}{\mathrm{d}\tau}\right) = \delta^a_0$. In the next section, the kinematical quantities associated with the fluid flow lines are employed to defined preferred rest frames at each point in $\mathcal{M}$.

3.3.1 Space-Time Splitting

Given any 4− velocity vector field u^a, the projection operator h_{ab} is defined by

$$h_{ab} \equiv g_{ab} + u_a u_b, \tag{3.4}$$

where g_{ab} is the metric tensor and

$$h_{ab}u^b = g_{ab}u^b + u_a u_b u^b = u_a - u_a = 0, \tag{3.5}$$

it means that h_{ab} projects into the instantaneous rest-space of an observer with 4−velocity u^a [1, 46]. The vector field u^a induces two differential operations for a general tensor from the covariant derivate along the preferred world lines. The first is the general *time derivative* along the fluid flow lines

$$\dot{T}^{abcd}{}_{efg} \equiv u^l \nabla_l T^{abcd}{}_{efg}, \tag{3.6}$$

and the second is *the fully projected three-dimensional covariant derivative*

$$\overline{\nabla}_l T^{abc}{}_{ef} \equiv h_l{}^p h^a{}_r h^b{}_m h^c{}_s h_e{}^q h_f{}^g \nabla_p T^{rsm}{}_{qg}. \tag{3.7}$$

The projector (3.4) satisfaces the following algebra,

$$h^a{}_b h^b{}_c = g^a{}_b h^b{}_c + u^a u_b h^b{}_c = h^a_c, \tag{3.8}$$

and

$$h^a{}_a = g^a{}_a + u^a u_a = 4 - 1 = 3. \tag{3.9}$$

The vector field u^a defines a natural decomposition for tensor fields, the metric itself can be write as

$$g_{\perp ab} = h_a^c h_b^d g_{cd} = h_{ab}, \tag{3.10}$$

perpendicular to u^a and the parallel part by

$$g_{\|ab} = U_a^{\ c} U_b^{\ d} g_{cd} = (-u_a u^c)(-u_b u^d) g_{cd} = u_a u_b u_c u^c = -u_a u_b = U_{ab}, \tag{3.11}$$

with $U_{ab} \equiv -u_a u_b$. These results also imply

$$g_{ab} = g_{\perp ab} + g_{\|ab} = h_{ab} + U_{ab}. \tag{3.12}$$

The equation (3.12) allow us to write any vector field as

$$V_a = g_{ab} V^b = h_{ab} V^b - u_a u_b V^b, \tag{3.13}$$

where we can see the natural orthogonal decomposition for the vector field. The equation (3.12) exhibits an important result, the invariant

$$ds^2 = g_{ab} dx^a dx^b = h_{ab} dx^a dx^b - u_a u_b dx^a dx^b = h_{ab} dx^a dx^b - (u_a dx^a)^2 \,, \tag{3.14}$$

defines the "spatial separation" between two events by $\left(h_{ab} dx^a dx^b\right)^{1/2}$ and a "time separation" $(u_a dx^a)$ for an observer moving with 4−velocity u^a. Once one can see that h_{ab} and u^a separate any physical tensor field into "space" and "time" parts corresponding to the way an observer moving with 4−velocity u^a would measure these fields [1, 46]. The $1+3$ covariant approach to cosmology makes use of this fact choosing in each point the "average velocity in the universe" [1, 46]. In other words, the $1+3$ formalism is as a *covariant Lagrangian* approach.

3.4 The kinematical quantities

One can decompose the covariant derivative of u_a using (3.12), from the identity

$$u_{a;b} = g_a^{\ c} g_b^{\ d} u_{c;d} = g_c^a u_{c;d} h_b^{\ d} - g_a^{\ c} u_{c;d} u^d u_b, \tag{3.15}$$

and using (3.4) the expression (3.15) is written as

$$g_a^{\ c} u_{c;d} h_b^{\ d} = \left(h_a^c - u^c u_a\right) u_{c;d} h_b^d = h_a^c h_b^d u_{c;d} - u_a u^c u_{c;d} h_b^d, \tag{3.16}$$

the last term in the right side of (3.16) is null due to the normalization condition $u_c u^c = -1$ and $\nabla_d (u_c u^c) = 2u^c u_{c;d} = 0$. Finally we can write 3.15 as

$$u_{a;b} = h_a^{\ c} h_b^{\ d} u_{c;d} - \dot{u}_a u_b. \tag{3.17}$$

The equation (3.17) is the natural spatial and time decomposition for the first covariant derivative for u_a. We can go further and define

$$u_{a;b} \equiv \sigma_{ab} + \frac{1}{3} h_{ab} \Theta + \omega_{ab} - \dot{u}_a u_b, \tag{3.18}$$

where $\sigma_{ab} = \sigma_{(ab)}$, $\sigma_{ab} u^b = 0$ and $\sigma^a_{\ a} = 0$ is the shear tensor. The vorticity $\omega_{ab} = \omega_{[ab]}$ and satisfaces $\omega_{ab} u^b = 0$. From (3.18), the volume-expansion is $u^a_{;a} = \Theta$. The temporal term in (3.18) is $\dot{u}_a u_b$ where $\dot{u}_a$ is the "acceleration vector"; it represents the effects of non-gravitational forces and satisfaces $u_a \dot{u}^a = 0$. The definitions for each component are

$$\omega_{ab} \equiv h_a^c h_b^d u_{[c;d]}, \tag{3.19}$$

$$\sigma_{ab} \equiv h^c_a h^d_b u_{(c;d)} - \frac{1}{3} u^c_{;c} h_{ab}. \tag{3.20}$$

From 3.19 the *vorticity vector* ω^a is defined by

$$\omega_a := -\frac{1}{2} \mathrm{curl}\, u_a = \frac{1}{2} \eta_{abc} \omega^{bc}. \tag{3.21}$$

From the Einstein field equations for GR or for any ETGs, we have in general the dynamics for the gravity variables in a gravity theory. But there are some results coming from differential geometry that they are valid independently of the gravity theory. We will use some of these results to give the equation of motion for the kinematical quantities in (3.18).

3.5 Ricci tensor identities

From the "1+3 decomposition" for the 4−velocity and employing the Riemann's identities (see Appendix B) we can derive the following result [47]

$$u^c \left[\nabla_c, \nabla_d\right] u_a = R_{abcd} u^b u^c, \tag{3.22}$$

and using $\nabla_d \left(u^c \nabla_c u_a\right) = u^c \nabla_d \nabla_c u_a + \nabla_d u^c \nabla_c u^a$ the equation (3.22) becomes

$$\left(\nabla_d u_a\right)^{\cdot} - \nabla_d \dot{u}_a + \left(\nabla_d u^c\right)\left(\nabla_c u_a\right) = R_{abcd} u^b u^c. \tag{3.23}$$

Taking the contraction from (3.23) finally we get

$$\left(\nabla_a u^a\right)^{\cdot} - \nabla_a \dot{u}^a + \left(\nabla^a u^c\right)\left(\nabla_c u_a\right) = -R_{bc} u^b u^c. \tag{3.24}$$

The (3.24) in terms of the shear and vorticity is

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\left(\sigma^2 - \omega^2\right) - \nabla_a \dot{u}^a = -R_{bc} u^b u^c, \tag{3.25}$$

where we use the result

$$\left(\nabla_d u^c\right)\left(\nabla_c u_a\right) = 2\left(\sigma^2 - \omega^2\right) + \frac{1}{3}\Theta^2, \tag{3.26}$$

with $2\sigma^2 = \sigma^{ac}\sigma_{ca}$, $-2\omega^2 = \omega^{ac}\omega_{ca} = -\omega^{ac}\omega_{ac}$. Sometimes is useful to express $\nabla_a \dot{u}^a = \overline{\nabla}_a u^a + \dot{u}_a \dot{u}^a$. The (3.25) is the master equation to study the evolution of the cosmological models in the "1+3 formalism". Another important result coming also from the Riemann tensor is [1]

$$\nabla_c \nabla_d u_a - \nabla_d \nabla_c u_a = R_{abcd} u^b, \tag{3.27}$$

and taking the spatial projection of (3.27) we get in two equivalent forms

$$h_a^{\ b} \nabla_c \left(\sigma^c_{\ b} + \omega^c_{\ b}\right) - \frac{2}{3} \overline{\nabla}_a \Theta - \left(\omega_{ab} + \sigma_{ab}\right) \dot{u}^b = h_a^{\ b} R_{bd} u^d, \tag{3.28}$$

$$\overline{\nabla}^b \sigma_{ab} - \mathrm{curl}\, \omega_a - \frac{2}{3} \overline{\nabla}_a \Theta + 2\eta_{abc} \omega^b \dot{u}^c = R_{<a>b} u^b. \tag{3.29}$$

The equation (3.23) gives also important constrains about the vorticity, from

$$R_{a[bcd]} u^a = 0 \implies u_{[b;cd]=0}, \tag{3.30}$$

or in terms of the kinematical variables, the divergence of the vorticity vector is

$$h_a^{\ b}\nabla_b\omega^a = \omega_a\dot{u}^a, \tag{3.31}$$

and the vorticity propagation

$$\dot{\omega}^{\langle e\rangle} = -\frac{2}{3}\Theta\omega^e + \sigma^{ed}\omega_d - \frac{1}{2}\mathrm{curl}\,\dot{u}^e. \tag{3.32}$$

The results summarized in this section are general for any spacetime because they are derived only from the Riemann tensor properties. The only special issue was the decomposition of the $\nabla_b u_a$ tensor.

3.6 The energy-momentum tensor

In the “1+3 covariant approach” a general fully relativistic fluid measured by an observer with 4−velocity is described by the symmetric tensor:

$$T_{ab} = \rho u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \Pi_{ab}. \tag{3.33}$$

with $\rho = T_{ab}u^a u^b$ the relativistic energy density, $p = \frac{1}{3}h^{ab}T_{ab}$ the relativistic pressure, $q_a = -h_a^c T_{cb}u^b$ the relativistic momentum density (diffusion and heat conduction) and $\Pi_{ab} = h^{\ c}_{(a}h^{\ d}_{b)}T_{cd} - \frac{1}{3}h_{ab}h^{cd}T_{cd}$ is the anisotropic (traceless) stress tensor due to effects such as viscosity or magnetic fields. The 10 components of T_{ab} are represented by two scalar fields (ρ, p), the three components of the vector q_a and five components of the tensor Π_{ab} [1].

3.6.1 The conservation laws

The Bianchi identity

$$\nabla_b T^{ab} = 0, \tag{3.34}$$

in terms of the “1+3 covariant” frame determines the rate of change of the relativistic energy along the world lines [1, 6]

$$u_a\nabla_b T^{ab} = \dot{\rho} + (\rho + p)\Theta + \Pi^{ab}\sigma_{ab} + \overline{\nabla}_a q^a + 2\dot{u}_a q^a = 0, \tag{3.35}$$

for the temporal part and the orthogonal projection is

$$h_{ac}\nabla_b T^{ab} = \dot{q}_{\langle c\rangle} + \frac{4}{3}\Theta q_c + (\rho + p)\dot{u}_c + \overline{\nabla}_c p + \overline{\nabla}^b\Pi_{cb} + \dot{u}^a\Pi_{ca} + \left(\sigma_{cb} + \eta_{cbd}\omega^d\right)q^b = 0, \tag{3.36}$$

where we have used $\Pi^{ab} = h^a_{\ d}h^b_{\ f}\Pi^{df}$. The equation (3.36) describes the acceleration caused by various pressure contributions and it is the generalization for the Euler equation in classical fluid dynamics.

3.7 Einstein field equations in the covariant language

The Einstein Field Equations can be split using the $U^a_{\ b}$ and $h_a^{\ b}$ operators are:

$$R_{ab}u^a u^b - \frac{1}{2}R g_{ab}u^a u^b = 8\pi G T_{ab}u^a u^b + \Lambda g_{ab}u^a u^b, \tag{3.37}$$

and from the normalization for the four-velocity $u^a u_a = -1$ we have

$$R_{ab}u^a u^b + \frac{1}{2}R = 8\pi G T_{ab}u^a u^b - \Lambda. \tag{3.38}$$

To go further, the right side of (3.38) and using as the source the energy momentum tensor for a general fluid

$$T_{ab}u^a u^b = \rho u_a u_b u^a u^b + q_a u_b u^a u^b + u_a q_b u^a u^b + p h_{ab} u^a u^b + \Pi_{ab} u^a u^b, \tag{3.39}$$

and from the properties $q_a = h_a^{\ c} q_c = q_{\langle a \rangle}$, $\Pi_{ab} = h_a^{\ c} h_b^{\ d} \Pi_{cd} = \Pi_{\langle ab \rangle}$ and replacing the trace for the energy momentum tensor $T_a^a = -\rho + 3p$ the equation (3.38) finally is

$$R_{ab}u^a u^b = 4\pi G\big(\rho + 3p\big) - \Lambda. \tag{3.40}$$

The equation (3.40) is one of the fundamental equations to study the dynamics of the cosmological models. From the properties of the projectors, the rest of the field equations are:

$$R_{ab}u^a h^b_{\ c} = -8\pi G q_c, \tag{3.41}$$

and

$$R_{ab}h^a_{\ c} h^b_{\ d} = \big(4\pi G(\rho - p) + \Lambda\big)h_{cd} + 8\pi G \Pi_{cd}. \tag{3.42}$$

The full set (3.40), (3.41), (3.42) are the field equations in GR in the "Covariant Approach" for a relativistic fluid as a source.

3.7.1 The Ehlers-Raychaudhuri equation

The equation (3.40) can be transformed using the Bianchi identity [1, 48]

$$-R_{ab}u^a u^b = \dot{\Theta} + \frac{1}{3}\Theta^2 + 2\left(\sigma^2 - \omega^2\right) - \overline{\nabla}_a \dot{u}^a - \dot{u}^a \dot{u}_a, \tag{3.43}$$

as

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\big(\sigma^2 - \omega^2\big) - \overline{\nabla}_a \dot{u}^a - \dot{u}^a \dot{u}_a + 4\pi G\,(\rho + 3p) - \Lambda = 0. \tag{3.44}$$

The equation (3.44) is the dynamics for the kinematical quantities and contains all the possible cosmological models including shear (σ_{ab}) and vorticity (ω_{ab}). Several important features can be studied from (3.44). For a general fluid, we can define a representative "length scale" $l(\tau)$ with τ the proper time along the fluid flow lines. The length is defined by

$$\frac{\dot{l}(\tau)}{l} \equiv \frac{1}{3}\Theta. \tag{3.45}$$

The equation (3.45) determines $l(\tau)$ up a constant along the world lines. The change of volume along the fluid flow is characterized by $V \propto l^3$. In the very special case of the FLRW spacetime, the length l corresponds to the scale-factor a. The *Hubble parameter* is generalized as

$$H \equiv \frac{\dot{l}}{l} = \frac{1}{3}\Theta, \tag{3.46}$$

and the *deceleration generalized parameter*

$$q \equiv -\left(\frac{\ddot{l}}{l}\right)\frac{1}{H^2}. \tag{3.47}$$

The equation (3.44) in terms of l is

$$\frac{3\ddot{l}}{l} = -2\left(\sigma^2 - \omega^2\right) + \overline{\nabla}_a \dot{u}^a + \dot{u}^a \dot{u}_a - 4\pi G\left(\rho + 3p\right) + \Lambda. \tag{3.48}$$

The equation (3.48) clearly shows the effect from the different contributions in the $\ddot{l}(\tau)$. While the shear, energy density and pressure act making the matter tend to the collapse, the vorticity and a positive cosmological constant tend to make the matter to expand. The acceleration terms are of indefinite sign. The equation (3.48) applies to different astrophysical and cosmological scenarios when the matter source is a fluid [29]. Solutions to (3.48) can be found in [29] for a cosmological model with non vanish vorticity (Gödel's universe) with rotation and in models for relativistic stars [1, 29]. The equation (3.44) is also the fundamental tool to study the singularity theorems [23, 29]. The equation (3.44) is naturally extended for alternative gravity theories, the special case of $f(R)$ gravity is treated somewhere in this chapter.

One of the most important analysis that can be done is to evaluate the equation (3.44) today, it is done denoting the values for the parameters with the subscript ($_0$) and it implies to choose a initial value t_0 when the coordinates are set. We can rewrite the (3.44) equation as follows [1]

$$q_0 = \frac{2}{3}\left(\frac{{\sigma_0}^2}{H_0^2} - \frac{{\omega_0}^2}{H_0^2}\right) - \frac{\left(\overline{\nabla}_a \dot{u}^a + \dot{u}_a \dot{u}^a\right)_0}{3H_0^2} + \frac{\Omega_0}{2} + \frac{3\Omega_{p0}}{2} - \Omega_{\Lambda 0}, \tag{3.49}$$

where the definitions for (3.49) are $q_0 = \frac{1}{H_0^2}\left(\frac{\ddot{l}}{l}\right)_0$, the Hubble parameter $\Theta = 3\left(\frac{\dot{l}}{l}\right)_0 = 3H_0$ and l the scale factor. For the densities $\Omega_0 = \frac{8\pi G\rho_0}{3H_0^2}$, $\Omega_{p0} = \frac{8\pi G p_0}{3H_0^2}$ and $\Omega_{\Lambda 0} = \frac{\Lambda}{3H_0^2}$. For some observational aspects of (3.49) see [1] and references therein. In the chapter 4 we will explore the role of the shear and vorticity in the context of perturbation theory in the *1+3 formalism*. The rest of this chapter is dedicated to a special class of cosmological models.

3.8 The constraint equations

The 1+3 formalism gives the full set of equations for the variables $\left\{\dot{u}^a, \omega_a, \Theta, \sigma_{ab}, \rho, q_a, p, \Pi_{ab}, E_{ab}, H_{ab}\right\}$ for a given gravity theory. Now, we can write the equation (3.41) in terms of the kinematical variables from (3.28) [1]

$$\overline{\nabla}^b \sigma_{ab} - \mathrm{curl}\, \omega_{\langle a\rangle} - \frac{2}{3}\overline{\nabla}_a \Theta + 2\eta_{abc}\omega^b \dot{u}^c + 8\pi G q_{\langle a\rangle} = 0, \tag{3.50}$$

and clearly (3.50) shows that the heat flow q^a controls the spatial gradients of $\{\Theta, \sigma, \omega\}$ in GR [1, 46]. However, it is mandatory to comment that the 1 + 3 formalism has some special features when we face the resolution of the full set of equations. The equations (3.50), (3.31) and (3.32) show the importance of the vorticity ω_a in the theory.

3.8.1 Vorticity-free ($\omega_a = 0$) equations

When $\omega = 0$ there is a unique family of surfaces orthogonal to u^a, and in the case of GR the equations (B.52) and (B.53) are [48]

$$\begin{aligned} {}^3R_{ab} = \overline{\nabla}_{\langle a}\dot{u}_{b\rangle} - l^{-3}(l^3\sigma)^{\cdot}_{\langle ab\rangle} + 8\pi G\Pi_{ab} + \dot{u}_{\langle a}\dot{u}_{b\rangle} \\ +\frac{2}{3}\Big(\sigma^2 + \frac{1}{3}\Theta^2 + \Lambda + 8\pi G\rho\Big)h_{ab}. \end{aligned} \tag{3.51}$$

clearly the equation (3.51) shows how the matter tensor affects the Ricci curvature of the three-dimensional space and closes the full set of equations together with (3.41) and (3.48) for GR. Taking the trace from (3.51)

$$^3R = 2\Big(2\sigma^2 - \frac{1}{3}\Theta^2 + \Lambda + 8\pi G\rho\Big) \tag{3.52}$$

which is a generalized Friedmann equation [1]. In the three-dimensional case (${}^3C^{abcd} = 0$) and the Ricci tensor and the Ricci scalar characterized the ${}^3R^{abcd}$, it means that when $\omega = 0$ the Riemann tensor for the three-space is fully described by the shear, expansion, acceleration, energy density and anisotropic pressure.
When $\omega \neq 0$ the situation is very difficult and give a set of equations analogous to (3.51) and (3.52) is a not trivial task. For some examples see [1]. We point out the problem related with the vorticity due to the fact that in several cosmological and physical situations the vorticity plays an important role. It has motivated several works including our study [49] in the generation and evolution of cosmic magnetic fields which will be the topic in next chapters. In our work we use the standard cosmological perturbation theory in the metric approach up to second order instead of the covariant 1 + 3 formalism and we study the importance of the vorticity in the cosmological dynamo equation.

3.8.2 Bianchi identities

The Riemann tensor defines a traceless four-rank tensor field

$$C_{abcd} \equiv R_{abcd} - \frac{1}{2}\Big(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\Big) + \frac{R}{6}\Big(g_{ac}g_{bd} - g_{ad}g_{bc}\Big), \tag{3.53}$$

with $C^a_{bad} = 0$. The equation (3.53) is the *Weyl tensor*. The Weyl tensor can be split using the projector U_{ab} into an *electric* E_{ab} and *magnetic* part H_{ab},

$$E_{ab} \equiv C_{aebd}u^e u^d, \tag{3.54}$$

and

$$H_{ab} \equiv \frac{1}{2}\eta_{ae}{}^{fg}C_{fgbd}u^e u^d. \tag{3.55}$$

The set of equations (3.54, 3.55) represents completely the Weyl tensor [46]. The names electric and magnetic obey to a close analogy between the Maxwell equations in electrodynamics and the equations we can get from the Bianchi identities

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6}g^{c[a}R^{;b]} := J^{abc}, \tag{3.56}$$

where the *current* J^{abc} obeys $\nabla_c J^{abc} = 0$. Following the same procedure as with the Einstein equations we can split (3.56) respect to u^a and h^b_a. Using the Einstein field equations combined with the results from section 3.4 we get [1]

$$\begin{aligned}\overline{\nabla}^b E_{ba} &= \eta_{abc}\sigma^b_d H^{dc} - 3H_{ab}\omega^b + \frac{1}{3}\overline{\nabla}_a\rho \\ &-\frac{1}{2}\overline{\nabla}^b\Pi_{ba} - \frac{1}{3}\Theta q_a + \frac{1}{2}\sigma_{ab}q^b + \frac{3}{2}\eta_{abc}\omega^b q^c,\end{aligned} \tag{3.57}$$

$$\begin{aligned}\dot{E}_{\langle ab\rangle} = {} & \operatorname{curl} H_{ab} + 2\dot{u}^c\eta_{cd(a}H_{b)}{}^d - \Theta E_{ab} + 3\sigma_{c\langle a}E^c{}_{b\rangle} - \omega_{c\langle a}E^c{}_{b\rangle} \\ & -\frac{1}{2}\overline{\nabla}_{\langle a}q_{b\rangle} - \dot{u}_{\langle a}q_{b\rangle} - \frac{1}{2}\dot{\Pi}_{\langle ab\rangle} - \frac{1}{6}\Theta\Pi_{ab} - \frac{1}{2}\sigma^c{}_{\langle a}\Pi_{b\rangle c} \\ & -\frac{1}{2}\omega^c{}_{\langle a}\Pi_{b\rangle c} - \frac{1}{2}\big(\rho + p\big)\sigma_{ab},\end{aligned} \tag{3.58}$$

$$\begin{aligned}\overline{\nabla}^b H_{ba} &= -\eta_{abc}\sigma^b_d E^{dc} + 3E_{ab}\omega^b \\ &+\big(\rho + p\big)\omega_a - \frac{1}{2}\eta_{abc}\sigma^b_d\Pi^{dc} - \frac{1}{2}\Pi_{ab}\omega^b - \frac{1}{2}\operatorname{curl} q_a,\end{aligned} \tag{3.59}$$

$$\begin{aligned}\dot{H}_{\langle ab\rangle} = {} & -\operatorname{curl} E_{ab} - 2\dot{u}^c\eta_{cd(a}E_{b)}{}^d - \Theta H_{ab} + 3\sigma_{c\langle a}H^c{}_{b\rangle} - \omega_{c\langle a}H^c{}_{b\rangle} \\ & +\frac{1}{2}\operatorname{curl}\Pi_{ab} + \frac{1}{2}\sigma^c{}_{(a}\eta_{b)cd}q^d - \frac{3}{2}\omega_{\langle a}q_{b\rangle},\end{aligned} \tag{3.60}$$

where this set of equations has the same functional form of the Maxwell equations. However, there is an important comment, to derive the full set of equations, the Einstein field equations are employed, it will be a difference when we consider a different theory as $f(R)$ gravity.

The equation (3.23) could be rewritten in terms of the Weyl tensor (3.53) taking its contraction with η^{cde} and two very important consequences are derived

$$H_{ab} = \operatorname{curl}\sigma_{ab} + \overline{\nabla}_{\langle a}\omega_{b\rangle} + 2\dot{u}_{\langle a}\omega_{b\rangle}, \tag{3.61}$$

$$E_{ab} - \frac{1}{2}R_{\langle ab\rangle} = -\dot{\sigma}_{\langle ab\rangle} - \frac{2}{3}\Theta\sigma_{ab} + \overline{\nabla}_{\langle a}\dot{u}_{b\rangle} + \dot{u}_{\langle a}\dot{u}_{b\rangle} - \omega_{\langle a}\omega_{b\rangle} - \sigma_{c\langle a}\sigma_{b\rangle}{}^c. \tag{3.62}$$

The set of equations (3.50),(3.30)and (3.61) are the *constraint equations*. Any solution in GR should obeys (3.38) together with the constrain equations. It is very important in the context of cosmological perturbation theory. Now we only discuss a class of solutions for the cosmological model: The FLRW solution.

3.9 FLRW cosmologies

FLRW cosmologies are models for universes which are everywhere isotropic about the fundamental velocity. It implies that there is a group of isometries G_3 acting about each point in the spacetime which leaves the fundamental velocity invariant [1, 14]. This is the a very special and restrictive case, however, with very important consequences. Isotropic observations of every fundamental observer all the times imply that isotropic universes are *spatially homogeneous*: all the physical properties are the same everywhere on spacelike surfaces orthogonal to the fluid lines. The homogeneity implies a G_3 group of isometries acting transitively on these surfaces [1, 14]. The high degree of symmetry makes the FLRW unrealistic models for the observed universe, but there is a very important fact: *realistic models of the observed universe are provided by perturbed FLRW universes* [1]. These *almost FLRW models are the standard models of cosmology at the present time*. The theory of cosmological perturbation theory uses this important remarkable observational feature. The justification for assuming the properties of isotropy and homogeneity from observations relies on:

- Average on large scales larger than any astrophysical object (galaxy clusters), and
- allow our peculiar velocity relative to the mean motion of matter in the universe (in practice, relative to the cosmic microwave background radiation.)

The last condition sets the coordinates for the model. Once the two points are considered, on cosmological scales, there is not preferred direction and if we consider we are not in a special point, it means, the universe is isotropic for every observer and the consequence is that we reach one of the most powerful principles: *The Cosmological Principle*. During this thesis, we use the FLRW geometries as a background for the cosmological perturbation theory. There are more possibilities as the homogeneous but anisotropic family of *Bianchi Models* and exact isotropic but inhomogeneous family of solutions as the *Lemaitre-Tolman-Bondi* (LTB) which allow a wider class of the cosmological models, for further analysis see [50, 51].

3.9.1 Splitting of the FLRW spacetimes

The standard FLRW solutions arise from setting $\sigma_{ab} = \omega^a = 0 = \dot{u}^a$. In this case, there is normalized proper time t which is a potential for the four-velocity field [1]

$$u_a = -t_{,a} \tag{3.63}$$

and as a direct consequence of (3.63), the four velocity field is orthogonal to the surfaces of spatial homogeneity t = const. The isotropy forces to $\Pi_{ab} = q_a = 0$ in (3.33), which implies $p = p(t)$ or one can see anisotropy from $\overline{\nabla}_a p$. The null spatial pressure gradient implies $\dot{u}^a = 0$ from (3.36) and (3.41) implies $\overline{\nabla}_a \Theta = 0$. The complete characterization of the FLRW spacetimes is

$$\rho = \rho(t), \quad p = p(t), \quad \Theta = \Theta(t) \tag{3.64}$$

or equivalently

$$\overline{\nabla}_a \rho = \overline{\nabla}_a p = \overline{\nabla}_a \Theta = 0, \tag{3.65}$$

which is the condition for the spatial homogeneity of these universes on the t = const surfaces [1]. The spacetime geometry for the FLRW family is given by

$$u_{a;b} = \frac{1}{3}\Theta(t) h_{ab} = \frac{\dot{a}}{a} h_{ab}, \tag{3.66}$$

with $a(t) = l(t)$ is the scale factor, the solution of (3.66) is

$$a(t) \propto \exp\Big[\int^t \frac{1}{3}\Theta(t')\mathrm{d}t' \Big]. \tag{3.67}$$

In normalized comoving coordinates (t, x^i), the four velocity field

$$u^\mu = \delta^\mu_0 \tag{3.68}$$

and from the orthogonality condition the metric tensor is

$$ds^2 = -dt^2 + a^2(t) f_{ij}(x^k) \mathrm{d}x^i \mathrm{d}x^j, \tag{3.69}$$

where the set of functions f_{ij} describes three-dimensional spaces of constant curvature k. Finally, the FLRW metric is conveniently expressed as a function of the normalized comoving coordinates (t, r, θ, ϕ) as

$$ds^2 = -dt^2 + a^2(t)\Big(dr^2 + f_k^2(r)\Big(d\theta^2 + sin^2\theta \mathrm{d}\phi^2\Big)\Big) \tag{3.70}$$

with k the spatial constant curvature. We can choose $k = \pm 1, 0$ and the comoving coordinate r dimensionless and hence the scale factor $a(t)$ has dimension length. Other possibility in (3.70) is to take $a(t)$ as a dimensionless parameter and consequently the comoving coordinate r has length dimension and k has dimension (length)$^{-2}$ and the function $f_k(r)$ becomes

$$f_k(r) = \begin{cases} k^{-1/2} sin(\sqrt{k}r) & \text{for } k > 0 \\ (-k)^{-1/2} sinh(\sqrt{-k}r) & \text{for } k < 0 \\ r & \text{for } k = 0 \end{cases} \tag{3.71}$$

We can normalize the current value of the scale factor as $a_0 = 1$. There are more possibilities for the comoving normalized coordinates, a very useful set is to keep the spatial coordinates in (3.70) but instead to use t the conformal time $\left(\tau := \int^t \frac{dt'}{a(t')}\right)$ is employed.

A very important consequence of the symmetries in (3.70) is that a FLRW universe is conformally flat (i.e. the Weyl tensor (3.53) is null $\{C_{abcd} \equiv 0\}$).

3.9.2 The Hubble's law

The scale factor $a(t)$ in (3.70) governs how all the spatial distances change in the evolution of a FLRW universe. We consider two different observers in y^i_A and y^i_B spatial comoving coordinates and choosing a $C(x^i(\lambda))$ curve joining the observers, the distance between the observers along the C in $t = t_1$ is

$$d(t_1) = a(t_1) \int_A^B \left(f_{ij} \frac{\mathrm{d}x^i(\lambda)}{\mathrm{d}\lambda} \frac{\mathrm{d}x^j(\lambda)}{\mathrm{d}\lambda}\right)^{1/2}, \tag{3.72}$$

and (3.72) holds for any time t. The speed of motion in the surface $t = \text{const}$ is

$$v(t) := \dot{d}(t) = \frac{\dot{a}}{a} d(t) = H(t)d(t), \tag{3.73}$$

which is the *Hubble's law* and may be interpreted as an exact law of recession in the surfaces $t = \text{const}$ [1].

3.10 Dynamics of FLRW universes

The first model for a $f(R)$ function is GR with cosmological constant. In this case $f(R) = R - 2\Lambda$ and the field equations correspond to the Einstein gravity given by [1]

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 4\pi G\,(\rho + 3p) - \Lambda = 0, \tag{3.74}$$
$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) + \Lambda,$$

where we use $\frac{1}{3}\Theta = H = \frac{\dot{a}}{a}$. The energy (3.35) implies

$$\dot{\rho} + \Theta\left(p + \rho\right) = 0, \qquad (3.75)$$
$$\dot{\rho} + 3H\left(p + \rho\right) = 0.$$

In this special case, when $\dot{a} \neq 0$, we can integrate (3.74) multiplying by $\dot{a}a$ and using $\dot{\left(a^2\rho\right)} = -\dot{a}a(\rho+3p)$ from (3.75), the result is

$$\dot{a}^2 - 4\pi G a^2 \rho - \Lambda = \text{const}, \qquad (3.76)$$

where const is the spatial three-curvature. This result is also a consequence of the Gauss equation (B.52), finally we have

$$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}, \qquad (3.77)$$

which is the *Friedmann equation*. From (3.74) we notice that "static universe" ($\Theta = 0$) is only possible in GR with the condition

$$4\pi G\left(\rho + 3p\right) = \Lambda, \qquad (3.78)$$

which implies $\Lambda > 0$. Besides the very well known results from the FLRW solutions, the set of equations (3.74), (3.75) and (3.77) needs the specification of a suitable state equation for matter. The kinematical properties of the FLRW family imply a stress tensor with the form of a perfect fluid ($\Pi_{ab} = q_a = 0$) in equation (3.33) but it does not specify the for any specific form for the state equation. A general possibility is to include dissipative effects in the form $p = p(\rho, s)$ with s the entropy of the matter, it means, the fluid does not need to have a perfect fluid equation of state [1]. For most of the cosmological applications, a state equation of the form

$$w(t) = \frac{p(t)}{\rho(t)} \qquad (3.79)$$

seems to be a good description for the observations. The equation (3.79) closes the system for the cosmological models. The equation (3.77) in terms of the dimensionless parameters is

$$\Omega_m = \frac{8\pi G\rho}{3H^2}, \qquad (3.80)$$
$$\Omega_\Lambda = \frac{\Lambda}{3H^2},$$
$$\Omega_k = -\frac{k}{a^2H^2}$$

and it takes a singular form known as the *cosmic sum rule*:

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1, \qquad (3.81)$$

with Ω_m the contribution for all the matter fields. With (3.79), we write the Raychaudhuri-Ehlers equation (3.44) for the FLRW family as

$$3\frac{\ddot{a}}{a} = -4\pi G\left(1 + 3\,w(t)\right)\rho + \Lambda, \qquad (3.82)$$

and the deceleration parameter

$$q = \frac{3}{2}\Omega_m\Big(w(t) + \frac{1}{3}\Big) - \Omega_\Lambda. \tag{3.83}$$

For some applications, (3.77) is normalized defining $y := \frac{a}{a_0} = \frac{1}{1+z}$, where z is the *cosmological redshift*. As an example, a universe with non-interacting matter (Ω_M) and radiation ($\Omega_m = \Omega_M + \Omega_r$), the equation (3.77) is

$$\dot{y}^2 = H_0^2\Big[\Omega_{M0}y^{-1} + \Omega_{r0}y^{-2} + \Omega_\Lambda y^2 + \Omega_{k0}\Big]. \tag{3.84}$$

The equation (3.84) has been object of deep studies. The references [1, 22, 23] have a very comprehensive analysis for (3.84) and some consequences as the role of initial conditions, Big Bang singularities are found in [29]. We will use the results of the FLRW cosmological models to study the geodesic deviation equation in the context of $f(R)$ gravity. In the next chapters, the general results are considered in the frame of cosmological perturbation theory.

3.11 Cosmological dynamics in metric $f(R)$ gravity

The field equations in metric $f(R)$ gravity were studied in the chapter 2 and the equation of motion are (2.62)

$$f'(R)\,R_{\alpha\beta} - \frac{f(R)}{2}\,g_{\alpha\beta} + g_{\alpha\beta}\Box f'(R) - \nabla_\alpha\nabla_\beta f'(R) = \kappa T_{\alpha\beta},$$

where $\Box = \nabla_\sigma\nabla^\sigma$, $f'(R) = df(R)/dR$; the energy-momentum tensor is defined by

$$T_{\alpha\beta} \equiv -2\frac{\partial\mathcal{L}_M}{\partial g^{\alpha\beta}} + \mathcal{L}_M g_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\alpha\beta}},$$

being $\mathcal{L}_M$ the Lagrangian for all the matter fields, we also have the conservation equation $\nabla_\alpha T^{\alpha\beta} = 0$. Contracting with $g^{\alpha\beta}$ we have for the trace of the field equations

$$f'(R)\,R - 2f(R) + 3\Box f'(R) = \kappa T, \tag{3.85}$$

The Ricci tensor from (2.62) is

$$R_{\alpha\beta} = \frac{1}{f'(R)}\Big[\kappa T_{\alpha\beta} + \frac{f(R)}{2}\,g_{\alpha\beta} - g_{\alpha\beta}\Box f'(R) + \nabla_\alpha\nabla_\beta f'(R)\Big], \tag{3.86}$$

and from (3.85)

$$R = \frac{1}{f'(R)}\Big[\kappa T + 2f(R) - 3\Box f'(R)\Big]. \tag{3.87}$$

It is possible to write the field equations in $f(R)$ gravity, in the form of Einstein equations with an effective energy-momentum tensor [52]

$$\begin{aligned} G_{\alpha\beta} &\equiv R_{\alpha\beta} - \frac{1}{2}R\,g_{\alpha\beta} \\ &= \frac{\kappa T_{\alpha\beta}}{f'(R)} + g_{\alpha\beta}\frac{[f(R) - Rf'(R)]}{2f'(R)} + \frac{[\nabla_\alpha\nabla_\beta f'(R) - g_{\alpha\beta}\Box f'(R)]}{f'(R)}, \end{aligned} \tag{3.88}$$

or

$$G_{\alpha\beta} = \frac{\kappa}{f'(R)}(T_{\alpha\beta} + T^{eff}_{\alpha\beta}), \tag{3.89}$$

with

$$T^{eff}_{\alpha\beta} \equiv \frac{1}{\kappa}\Big[\frac{[f(R) - Rf'(R)]}{2} g_{\alpha\beta} + [\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box]f'(R)\Big]. \tag{3.90}$$

which could be interpreted as an fluid composed by curvature terms. In order to give the general equations in metric $f(R)$ gravity in a very close way as is done for GR, we can rewrite (2.62) using the results (B.78) [48]

$$R_{\mu\nu} = \frac{1}{f'}\Big[\kappa T_{\mu\nu} + \frac{g_{\mu\nu}}{2} f + f'''\{\nabla_\mu R\nabla_\nu R - g_{\mu\nu}\nabla_\sigma R\nabla^\sigma R\} + f''\{\nabla_\mu\nabla_\nu R - g_{\mu\nu}\Box R\}\Big], \tag{3.91}$$

$$R = \frac{1}{f'}\Big[2f - 3f''\Box R - 3f'''\nabla_\rho R\nabla^\rho R + \kappa T\Big] \tag{3.92}$$

and from (3.92)

$$\nabla_\mu R = \frac{3f''\nabla_\mu(\Box R) + 6f'''(\nabla_\mu\nabla_\nu R)\nabla^\nu R - \kappa\nabla_\mu T}{f' - f''R - 3f'''\Box R - 3f''''\nabla_\rho R\nabla^\rho R}. \tag{3.93}$$

The set of equations (3.91), (3.92) and (3.93) are some fundamental results to give the general kinematics and dynamics in $f(R)$ gravity. Our main goal is to use these results in the geodesic deviation equation. We should mention our work [42] was the only one known addressing the problem of the geodesic deviation equation in $f(R)$ gravity. After our publication, different papers have been published, specially [43] where the GDE for homogeneous cosmologies in the "1+3 formalism" is studied and some numerical solutions are shown.

3.12 Dynamics in metric $f(R)$ gravity

Using the results from the previous section and with (B.79) the Raychaudhuri equation in $f(R)$ gravity [1] [48]

$$\begin{aligned} \dot\Theta + \frac{1}{3}\Theta^2 + 2\left(\sigma^2 - \omega^2\right) - \nabla_a \dot u^a = -R_{bc}u^b u^c = \\ -\frac{1}{f'}\Big[-\frac{1}{2}f + f''h^{\mu\nu}\nabla_\mu\nabla_\nu R + f'''h^{\mu\nu}\nabla_\mu R\nabla_\nu R + \kappa T_{\mu\nu}u^\mu u^\nu\Big]. \end{aligned} \tag{3.94}$$

We should notice that the equation (3.94) uses the fact that the Ricci identities is a result coming from differential geometry and not from the dynamics on the manifold. The rest of the contractions in $f(R)$ gravity are [48]

$$R_{\mu\rho}h^\mu{}_\nu u^\rho = \frac{1}{f'}\Big(f''h^\nu{}_\mu(\nabla_\nu\nabla_\rho R)u^\rho + f'''\dot R h^\nu{}_\mu\nabla_\nu R + \kappa T_{\mu\rho}h^\nu{}_\mu u^\rho\Big), \tag{3.95}$$

$$\begin{aligned} R_{\rho\sigma}h^\rho{}_\mu h^\sigma{}_\nu = \frac{1}{f'}\Big(\frac{1}{2}fh_{\mu\nu} + f''h^\rho_\mu h^\sigma_\nu(\nabla_\rho\nabla_\sigma R - \Box R) \\ + f'''h^\rho_\mu h^\sigma_\nu(\nabla_\rho R\nabla_\sigma R - h_{\rho\sigma}\nabla_\lambda R\nabla_\lambda R) + \kappa T_{\rho\sigma}h^\rho{}_\mu h^\sigma{}_\nu\Big). \end{aligned} \tag{3.96}$$

for the vorticity propagation and With the same steps as in the case of GR the constraint and Bianchi identities are generalized as [48]

$$h^\alpha{}_{[\mu}h^\beta{}_{\nu]}(\dot\omega_{\alpha\beta} + \nabla_\alpha\dot u_\beta) - 2\sigma^\rho{}_{[\mu}\omega_{\nu]\rho} + \frac{2}{3}\Theta\omega_{\mu\nu} = 0, \tag{3.97}$$

[1] $(\mu, \nu) = 0, \dots 3$

and for the shear in $f(R)$ gravity

$$\begin{aligned} h^{\alpha}_{(\mu}h^{\beta}_{\ \nu)}\Big(\dot{\sigma}_{\alpha\beta} - \nabla_{\alpha}\dot{u}_{\beta}\Big) - \dot{u}_{\mu}\dot{u}_{\nu} + \omega_{\mu}\omega_{\nu} + \sigma_{\mu\rho}\sigma^{\rho}_{\ \nu} + \frac{2}{3}\Theta\sigma_{\mu\nu} \\ +\frac{1}{3}h_{\mu\nu}\Big(\nabla_{\rho}\dot{u}^{\rho} - \omega^{2} - 2\sigma^{2}\Big) + E_{\mu\nu} \\ -\frac{1}{2f'}\Big(h^{\alpha}_{\ \mu}h^{\beta}_{\ \nu} - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\Big)\Big(f''\nabla_{\alpha}\nabla_{\beta}R + f'''\nabla_{\alpha}R\nabla_{\beta}R + \kappa T_{\alpha\beta}\Big) = 0. \end{aligned} \tag{3.98}$$

The (0ν) equations and the rest of the constraints in $f(R)$ gravity are [48]

$$\begin{aligned} h^{\mu}_{\ \nu}\Big(\frac{2}{3}\nabla^{\nu}\Theta - h^{\rho}_{\ \sigma}\sigma^{\nu\sigma}\Big) + \epsilon^{\mu\nu\rho\sigma}u_{\sigma}\Big(\nabla_{\nu}\omega_{\rho} - 2\omega_{\nu}\dot{u}_{\rho}\Big) \\ +\frac{1}{f'}[f''h^{\mu}_{\ \nu}(\nabla^{\mu}\nabla_{\rho}R)u^{\rho} + f'''\dot{R}h^{\mu}_{\ \nu}\nabla^{\nu}R + \kappa T^{\nu}_{\ \rho}h^{\mu}_{\ \nu}u^{\rho}] = 0, \end{aligned} \tag{3.99}$$

and the constraints

$$h^{\mu}_{\ \nu}\nabla_{\mu}\omega^{\nu} = \dot{u}_{\mu}\omega^{\mu}, \tag{3.100}$$

together with

$$H_{\mu\nu} = 2u_{(\mu}\omega_{\nu)} + h^{\alpha}_{\ (\mu}h^{\beta}_{\ \nu)}(\nabla^{\gamma}\omega_{\alpha}^{\ \delta} + \nabla^{\gamma}\sigma_{\alpha}^{\ \delta})\epsilon_{\beta\gamma\delta\epsilon}u^{\epsilon}. \tag{3.101}$$

the Bianchi identities for $f(R)$ gravity are [48]

$$\begin{aligned} h^{\mu}_{\ \alpha}h^{\gamma}_{\ \beta}\nabla_{\gamma}E^{\alpha\beta} = \epsilon^{\mu\alpha\beta\gamma}\sigma_{\alpha\delta}H^{\delta}_{\ \beta}u_{\gamma} - 3H^{\mu}_{\ \alpha}\omega^{\alpha} \\ +\frac{\kappa}{f'}h^{\mu}_{\ \alpha}[h^{\beta}_{\gamma}\nabla^{[\alpha}T^{\gamma]} - \frac{1}{3}\nabla^{\alpha}T] \\ +\frac{1}{2f'}f''h^{\mu}_{\ \alpha}\Big[\dot{R}R^{\alpha}_{\ \beta}u^{\beta} - \Big(R_{\beta\gamma}u^{\beta}u^{\gamma} + \frac{1}{3}R\Big)\nabla^{\alpha}R \\ -\nabla^{\alpha}(\Box R) - (\nabla^{\alpha}\nabla_{\beta}R)^{\cdot}u^{\beta} + h^{\beta}_{\ \gamma}\nabla^{\alpha}(\nabla_{\beta}\nabla^{\gamma}R)\Big], \end{aligned} \tag{3.102}$$

$$\begin{aligned} h^{\mu}_{\ \alpha}h^{\gamma}_{\ \beta}\dot{E}^{\alpha\beta} = 3E^{(\mu}_{\ \ \alpha}\sigma^{\nu)\alpha} + E^{(\mu}_{\ \ \alpha}\omega^{\nu)\alpha} - \Theta E^{\mu\nu} - h^{\mu\nu}E_{\alpha\beta}\sigma^{\alpha\beta} \\ -2H^{(\mu}_{\ \ \alpha}\epsilon^{\nu)\alpha\beta\gamma}\dot{u}_{\beta}u_{\gamma} + h^{(\nu}_{\ \ \alpha}\epsilon^{\nu)\beta\gamma\delta}\nabla_{\beta}H^{\alpha}_{\ \gamma}u_{\delta} \\ +\frac{\kappa}{2f'}\Big[h^{(\mu}_{\ \ \alpha}h^{\nu)}_{\ \ \beta}\nabla^{\alpha}T^{\beta}_{\ \gamma}u^{\gamma} + h^{(\mu}_{\ \ \alpha}h^{\nu)}_{\ \ \beta}(T^{\alpha\beta})^{\cdot} + \frac{1}{3}\dot{T}h^{\mu\nu}\Big] \\ +\frac{1}{2f'}f''h^{(\mu}_{\ \ \alpha}h^{\nu)}_{\ \ \beta}\Big[\dot{R}\Big(R^{\alpha\beta} - \frac{1}{3}Rg^{\alpha\beta}\Big) - \nabla^{\alpha}RR^{\beta}_{\ \gamma}u^{\gamma} + \nabla^{\alpha}(\nabla_{\gamma}\nabla^{\beta}R)u^{\gamma} - (\nabla^{\alpha}\nabla^{\beta}R)^{\cdot}\Big], \end{aligned} \tag{3.103}$$

$$\begin{aligned} h^{\mu}_{\ \alpha}h^{\gamma}_{\ \beta}\nabla_{\gamma}H^{\alpha\beta} = -\epsilon^{\mu\alpha\beta\delta}\sigma_{\alpha\delta}E^{\delta}_{\ \beta}u_{\gamma} + 3E^{\mu}_{\ \alpha}\omega^{\alpha} + \frac{\kappa}{2f'}\epsilon^{\mu\alpha\beta\gamma}\nabla_{\alpha}T^{\delta}_{\ \beta}u_{\gamma}u_{\delta} \\ +\frac{1}{2f'}f''\epsilon^{\mu\alpha\beta\gamma}[\nabla_{\alpha}\nabla_{\beta}\nabla^{\delta}R - \nabla_{\alpha}RR^{\delta}_{\ \beta}]u_{\gamma}u_{\delta}, \end{aligned} \tag{3.104}$$

$$
\begin{aligned}
h^{\mu}_{\ \alpha}h^{\gamma}_{\ \beta}\dot{H}^{\alpha\beta} = 3H^{(\mu}_{\ \ \alpha}\sigma^{\nu)\alpha} + H^{(\mu}_{\ \ \alpha}\omega^{\nu)\alpha} - \Theta H^{\mu\nu} - h^{\mu\nu}H_{\alpha\beta}\sigma^{\alpha\beta} \\
+2E^{(\mu}_{\ \ \alpha}\epsilon^{\nu)\alpha\beta\gamma}\dot{u}_{\beta}u_{\gamma} - h^{(\nu}_{\ \ \alpha}\epsilon^{\nu)\beta\gamma\delta}\nabla_{\beta}E^{\alpha}_{\ \gamma}u_{\delta} \\
+\frac{\kappa}{2f'}h^{(\mu}_{\ \ \alpha}\epsilon^{\nu)\beta\gamma\delta}\nabla_{\beta}T^{\alpha}_{\ \gamma}u_{\delta} \\
+\frac{1}{2f'}f''h^{(\mu}_{\ \ \alpha}\epsilon^{\nu)\beta\gamma\delta}[\nabla_{\beta}RR^{\alpha}_{\gamma} - \nabla_{\beta}\nabla_{\gamma}\nabla^{\alpha}R]u_{\delta}.
\end{aligned}
\tag{3.105}
$$

The dynamics of any solution in $f(R)$ gravity should obey the field equations together with the *constraints* and *evolution equations* from Bianchi identities. The full set of equations are the fundamental point in the treatment of the cosmological perturbation theory.

3.12.1 The Gauss equation in $f(R)$ gravity

In the case of vorticity free solutions in $f(R)$ gravity, the generalization of (3.51) and (3.52) for spacetime configurations with an hypersurface orthogonal to u^a. The 3−Riemann curvature tensor [48]

$$
\begin{aligned}
{}^3R_{\mu\nu\rho\sigma} = {}^3R_{\nu\rho}h_{\nu\sigma} - {}^3R_{\mu\sigma}h_{\nu\rho} + {}^3R_{\nu\sigma}h_{\mu\rho} \\
-{}^3R_{\nu\rho}h_{\mu\sigma} - \frac{1}{2}{}^3R\Big(h_{\mu\rho}h_{\nu\sigma} - h_{\mu\sigma}h_{\nu\rho}\Big),
\end{aligned}
\tag{3.106}
$$

and now using the fields equations (2.62), the trace from (3.106) [48]

$$
\begin{aligned}
{}^3R = 2\sigma^2 - \frac{2}{3}\Theta^2 + \frac{1}{f'}\Big[f + f''\Big(2h^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R - 3\Box R\Big) \\
+f'''\Big(2\dot{R}^2 - \nabla_{\nu}\nabla^{\mu}R\Big) + \kappa\Big(2T_{\mu\nu}u^{\mu}u^{\nu} + T\Big)\Big],
\end{aligned}
\tag{3.107}
$$

and

$$
\begin{aligned}
{}^3R_{\langle\mu\nu\rangle} \equiv {}^3R_{\mu\nu} - \frac{1}{3}{}^3Rh_{\mu\nu} = h^{\alpha}_{\ (\mu}h^{\beta}_{\ \nu)}\Big(\nabla_{\alpha}\dot{u}_{\beta} - \dot{\sigma}_{\alpha\beta}\Big) \\
-\Theta\sigma_{\mu\nu} + \dot{u}_{\mu}\dot{u}_{\nu} - \frac{1}{3}h_{\mu\nu}\nabla_{\alpha}\dot{u}^{\alpha} \\
+\frac{1}{f'}\Big(h^{\alpha}_{\ \mu}h^{\beta}_{\ \nu} - \frac{1}{3}h_{\mu\nu}h^{\alpha\beta}\Big)\Big[f''\nabla_{\alpha}\nabla_{\beta}R + f'''\nabla_{\alpha}\nabla_{\beta}R + \kappa T_{\alpha\beta}\Big].
\end{aligned}
\tag{3.108}
$$

The equations (3.107) and (3.108) are the generalizations to (3.52) and (3.51) in $f(R)$ gravity for a general energy momentum tensor $T_{\mu\nu}$ which naturally could include contributions from scalar and magnetic fields.

3.12.2 FLRW in metric $f(R)$ gravity

The FLRW models are the very special case $\Big\{\omega_{ab} = \sigma_{ab} = \dot{u}^a = q^a = C^a_{\ bcd} = 0\Big\}$. The non-null field equations for the FLRW metric from 3.70 for metric $f(R)$ gravity are

$$
H^2 + \frac{k}{a^2} = \frac{1}{3f'(R)}\Big[\kappa\rho_m + \frac{(Rf'(R) - f(R))}{2} - 3Hf''(R)\dot{R}\Big],
\tag{3.109}
$$

and

$$2\dot{H}+3H^2+\frac{k}{a^2}=-\frac{1}{f'(R)}\Big[\kappa p_m+2Hf''(R)\dot{R}+\frac{(f(R)-Rf'(R))}{2}+f''(R)\ddot{R}+f'''(R)\dot{R}^2\Big], \tag{3.110}$$

where we have used the definition for the Hubble parameter $\frac{\Theta}{3}=H\equiv\frac{\dot{a}}{a}$. We see that R is only a function of time. From the FLRW metric the expression for the Ricci scalar is

$$R=6\Big[\frac{\ddot{a}}{a}+\Big(\frac{\dot{a}}{a}\Big)^2+\frac{k}{a^2}\Big]=6\Big[\dot{H}+2H^2+\frac{k}{a^2}\Big]. \tag{3.111}$$

The energy momentum tensor directly from (3.35) obeys

$$\dot{\rho}_m+(\rho_m+p)\Theta=\dot{\rho}_m+3H(\rho_m+p)=0. \tag{3.112}$$

To close the system of equations (3.109), (3.110) and (3.112) we should assume a state equation for the matter components as

$$p_m=w_m\rho_m, \tag{3.113}$$

where ω_m is the barotropic equation of state that describes each type of fluid.

To go further with our analysis we consider the FLRW models the Raychaudhuri-Ehlers equation

$$\dot{\Theta}+\frac{1}{3}\Theta^2=-\frac{1}{f'}\Big(\kappa\rho_m-\frac{1}{2}f-\dot{R}f''\Theta\Big). \tag{3.114}$$

and the constraint equations and Bianchi identities are trivial in the FLRW case. As it is mention in [43] the equation (3.114) can be obtained from (3.109) and (3.110). It implies that any solution of this system is automatically a solution of (3.114).

We can use the equations (3.89) and (3.90) and write the field equations in metric $f(R)$ gravity in a close form to GR

$$G_{ab}=\kappa T_{ab}, \tag{3.115}$$

with the energy momentum tensor in the form

$$T_{ab}\equiv\frac{T^{mat}_{ab}}{f'}+T^{curv}_{ab}, \tag{3.116}$$

where $T^{\rm mat}_{ab}$ is the standard tensor for the matter content and

$$T^{curv}_{ab}=\frac{1}{\kappa f'}\Big[\frac{1}{2}(f-Rf')g_{ab}+\nabla_a\nabla_b f'-g_{ab}\nabla_c\nabla^c f'\Big]. \tag{3.117}$$

The Bianchi identities $G^{ab}_{\;\;;b}=0$ give us

$$\nabla^b T_{ab}=\nabla^b\Big(\frac{T^{mat}_{ab}}{f'}\Big)+\nabla^b T^{\rm curv}_{ab}=0, \tag{3.118}$$

and with the assumption that $\nabla^b T^{mat}_{ab} = 0$ the equation (3.118) is

$$\begin{aligned} \nabla^b T_{ab} &= -\frac{1}{f'^2} T^{mat}_{ab} \nabla^b f' + \frac{1}{f'} \nabla^b T^{mat}_{ab} + \nabla^b T^{curv}_{ab} = 0, \\ &= -\frac{f''}{f'^2} T^{mat}_{ab} R^{;b} + \nabla^b T^{curv}_{ab} = 0, \end{aligned} \tag{3.119}$$

where we have used $\nabla^b f' = f'' R^{;b}$. The equation (3.119) implies

$$\nabla^b T^{curv}_{ab} = \frac{f''}{f'^2} T^{mat}_{ab} R^{;b}. \tag{3.120}$$

For a more detailed exposition about conservation laws in $f(R)$ gravity see [43, 53]. From (B.83) we can define for FLRW geometries [2]

$$\rho^R := T^{curv}_{ab} u^a u^b = \frac{1}{f'}\Big[\frac{1}{2}(Rf' - f) - \Theta f'' \dot{R}\Big], \tag{3.121}$$
$$p^R = \frac{1}{3} T^{curv}_{ab} h^{ab} = \frac{1}{f'}\Big[\frac{f - Rf'}{2} + f''(\ddot{R} + \frac{2}{3}\Theta \dot{R}) + f''' \dot{R}^2\Big].$$

Even when the main goal of this chapter is to present our results [42] and give the basis for the topic in cosmological perturbation theory, the next section presents the geodesic deviation equation (GDE) in the 1 + 3 formalism in a complete general way. We specialize the general result for the GDE to FLRW geometries and for this case the 1 + 3 formalism coincides with the 3 + 1 decomposition. We compare both and discuss some aspects related to the GDE as a cosmological probe.

3.13 Geodesic deviation equation

One way to study the spacetime curvature is through the GDE [54]. This equation is a powerful tool to study several problems in some branches of physics and mathematics [55], as is the case of gravitational lensing. To start, we consider a pair of two neighbour curves taken from a congruence of geodesics (autoparallels) with tangent vector field V^a, it means

$$\frac{DV^b}{D\lambda} := V^a \nabla_a V^b = 0, \tag{3.122}$$
$$\frac{\mathrm{d}x^a(\lambda)}{\mathrm{d}\lambda} = V^a(\lambda)$$

where (3.122) including the geodesics curves. The congruence defines naturally a vector field X^a namely *Jacobi field* defined by

$$X^b(\lambda) := x^b_1(\lambda) - x^b_2(\lambda), \tag{3.123}$$

with $x^a_1(\lambda)$ and $x^b_2(\lambda)$ autoparallels. The "Jacobi field" is defined by the condition [23, 29, 56]

$$\frac{DX^b}{D\lambda} := V^a \nabla_a X^b = X^a \nabla_a V^b. \tag{3.124}$$

[2] See appendix B

We compute the second covariant derivative along the parallel field V^a of the Jacobi field X^b and using (3.124)

$$\frac{D^2X^a}{D\lambda^2} = \left(X^b V^c \nabla_c \nabla_b V^a\right) + \nabla_b V^a \nabla_c V^b X^c. \tag{3.125}$$

The equation (3.125) in terms of the Riemann tensor is

$$X^b V^c \nabla_c \nabla_b V_a = R_{adcb} V^d X^b V^c + V^c X^b \nabla_b \nabla_c V_a, \tag{3.126}$$

and with this result, the equation (3.125) is

$$\frac{D^2X^a}{D\lambda^2} = -R^a_{\ dbc} V^d X^b V^c + \nabla_c\left(V^b \nabla_b V^a\right) X^c \tag{3.127}$$

where we have used the identity

$$\nabla_c\left(V^b \nabla_b V^a\right) X^c = \nabla_b V^a \nabla_c V^b X^c + V^b X^c \nabla_b \nabla_c V^a. \tag{3.128}$$

When the vector field is autoparallel $V^a \nabla_a V^b = 0$ the equation (3.125) is

$$\frac{D^2X^a}{D\lambda^2} = -R^a_{\ dbc} V^d X^b V^c, \tag{3.129}$$

which describes the relative acceleration between autoparallels [3] and is commonly known as the Geodesic Deviation Equation. The equation (3.127) allow us to describe the case when the curves are not autoparallels, as in the case when matter is in presence of extra non-gravitational forces. We emphasize in this thesis in 3.130 for null $(g_{ab}k^a k^b) = 0$ and timelike $(g_{ab}V^a V^b < 0)$ vector fields.

3.13.1 GDE in the "1+3 formalism"

One of the most interesting cases in (3.130) is for null geodesics [1, 43, 57]

$$\frac{D^2X^a}{D\lambda^2} = -R^a_{\ dbc} k^d X^b k^c, \tag{3.130}$$

with $k_a k^a = 0$. The null geodesic are solutions for Maxwell's equations in the WKB approximation or geometrical-optics approximation. The vector field k_a satisfaces

$$\begin{aligned} k_a k^a = g^{ab} \frac{\partial S}{\partial x^a} \frac{\partial S}{\partial x^b} &= 0, \\ k^a \nabla_a k^b &= 0. \end{aligned} \tag{3.131}$$

where S is the invariant face of the radiation field. The wave vector k^a which is solution to the family of curves $\frac{\mathrm{d}x^a}{\mathrm{d}\lambda} = k^a$ is decomposed as

$$k^a = g^{ab} k_b = h^{ab} k_b - u^a u^b k_b = \left(-k_b u^b\right)\left(e^a + u^a\right) \tag{3.132}$$

with

$$e^a := \frac{h^{ab} k_b}{(-u^b k_b)} = \frac{k^{<a>}}{(-u^b k_b)}, \tag{3.133}$$

[3] In Riemannian spaces, geodesics

here e^a is the propagation direction of the light-ray as measured by u^a and $\nu := -u_a k^a$ is the photon frequency in the u^a-frame. In the literature [2] the GDE has been studied in terms of the Sachs's formalism and it will be consider in next sections. Here we summarized the most important steps to write (3.130) in the "1 + 3-covariant formalism" [1, 56]. The four velocity u^a and the propagation vector k^a define the "screen space", which is the two-dimensional space orthonormal to both and it is spanned by two space-like orthonormal vectors e^a_I with the conditions

$$e_{Ia}e^a_J = \delta_{IJ} \qquad (3.134)$$
$$u^a e_{Ia} = e^a e_{Ia} = 0,$$

with $\{I \in 1, 2\}$. The metric tensor in the screen space is given by

$$s^{ab} := h^{ab} - e^a e^b, \qquad (3.135)$$

which projects in the space orthonormal to e^a and obeys $s^a_a = 2$. The kinematics for a congruence of null geodesics in absence of non-gravitational interactions defines in analogy with u^a the kinematical variables

$$\hat{\Theta}_{ab} := s^c_a s^d_b \nabla_c k_d = \hat{\sigma}_{ab} + \frac{1}{2}\hat{\Theta} s_{ab}, \qquad (3.136)$$

with $\hat{\Theta}$ *the scalar expansion* of the null congruence and $\hat{\sigma}_{ab}$ *the shear*. In the geometric-optics limit of the Maxwell's equations, the wave-vector $k_a = \nabla_a S$ and it implies $\nabla_{(c}k_{d)} = \nabla_c k_d$. The expansion and the shear fulfills [1, 56]

$$g^{ab}\hat{\Theta}_{ab} = \hat{\Theta} = \nabla_a k^a \qquad (3.137)$$
$$s^{ab}\hat{\sigma}_{ab} = 0.$$

The projector s_{ab} enables to define the *projected Jacobi field* as

$$\hat{X}^a := s^a_d X^d = g^a_d X^d + u^a u_d X^d - e^a e_d X^d = X^a - \frac{X_c u^c}{k_c u^c} k^a, \qquad (3.138)$$

where we used $k_d X^d = 0$ in order to have the connecting X^a on the null surface defined by k^a. Choosing the affine parameter λ properly along the geodesics, the evolution for the Jacobi field is

$$\frac{D\hat{X}^a}{D\lambda} = \hat{X}^c \nabla_c k^a, \qquad (3.139)$$

where the condition $k^a \nabla_a k^b = 0$ is employed. There are other important consequences from the null expansion $\hat{\Theta}$. The proper congruence-area δA in the u^a frame is proportional to the square of the Jacobi field and its rate expansion is [1, 56]

$$\frac{D}{D\lambda}\delta A = \frac{1}{2}\hat{\Theta}\delta A \qquad (3.140)$$

and the Raychaudhuri equation is in analogy with (3.25)

$$\frac{D}{D\lambda}\hat{\Theta} = -\frac{1}{2}\hat{\Theta}^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{ab}k^a k^b. \qquad (3.141)$$

The shear evolves along the null congruence as

$$\frac{D}{D\lambda}\hat{\sigma}_{ab} = -\hat{\Theta}\hat{\sigma}_{ab} - C_{acbd}k^c k^d. \tag{3.142}$$

The physical meaning of (3.139) in terms of the kinematical quantities can be seen from

$$s_{ba}\frac{D}{D\lambda}\hat{X}^a = s_{ba}\hat{X}^c\nabla_c k^a = \frac{1}{2}\hat{\Theta}\hat{X}_b + \sigma_{bd}\hat{X}^d. \tag{3.143}$$

Now, we can write the GDE for the projected Jacobi field from (3.139) and again using the Ricci identity $2\nabla_{[a}\nabla_{b]}k_c = R^d_{\ abc}k_d$

$$s^c_a\frac{D}{D\lambda}\left(s_{cb}\frac{D}{D\lambda}\hat{X}^b\right) = s^c_a\hat{X}^b k^d k^e R_{bdce}, \tag{3.144}$$

which is the same (3.130) but in the 1 + 3 language. In terms of the Weyl tensor (3.149) is [1, 56]

$$s^c_a\frac{D}{D\lambda}\left(s_{cb}\frac{D}{D\lambda}\hat{X}^b\right) = s^c_a\hat{X}^b k^d k^e C_{bdce} - \frac{1}{2}\hat{X}_a R_{bc}k^b k^c. \tag{3.145}$$

The equation (3.145) is the GDE in the 1 + 3 covariant approach. For a deep discussion about mathematical aspects of the GDE see [58, 59].

3.14 Geodesic Deviation Equation in $f(R)$ Gravity

The general expression for the GDE in $f(R)$ gravity is given by (3.130)

$$\frac{D^2X^a}{D\lambda^2} = -R^a_{\ dbc}V^d X^b V^c,$$

The right hand side of the GDE in terms of the energy-momentum tensor and the Weyl curvature could be written as

$$\begin{aligned} R^\alpha_{\ \beta\gamma\delta}V^\beta\eta^\gamma V^\delta = C^\alpha_{\ \beta\gamma\delta}V^\beta\eta^\gamma V^\delta + \frac{1}{2f'}\Big[&\kappa(T_{\delta\beta}\delta^\alpha_\gamma - T_{\gamma\beta}\delta^\alpha_\delta + T_\gamma^{\ \alpha}g_{\beta\delta} - T_\delta^{\ \alpha}g_{\beta\gamma}) + f(\delta^\alpha_\gamma g_{\delta\beta} - \delta^\alpha_\delta g_{\gamma\beta}) \\ &+ (\delta^\alpha_\gamma\mathcal{D}_{\delta\beta} - \delta^\alpha_\delta\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}_\gamma^{\ \alpha} - g_{\beta\gamma}\mathcal{D}_\delta^{\ \alpha})f'\Big]V^\beta\eta^\gamma V^\delta \\ &- \frac{1}{6f'}\Big(\kappa T + 2f - 3\Box f'\Big)(\delta^\alpha_\gamma g_{\delta\beta} - \delta^\alpha_\delta g_{\gamma\beta})V^\beta\eta^\gamma V^\delta, \end{aligned} \tag{3.146}$$

being $R^\alpha_{\ \beta\gamma\delta}$ the Riemann curvature tensor, η^α the deviation vector between geodesics of tangent vector field V^α, $\mathcal{D}_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta - g_{\alpha\beta}\Box$, $\Box = \nabla_\sigma\nabla^\sigma$, $f' = f'(R) = df(R)/dR$, $T_{\alpha\beta}$ the energy-momentum tensor and T its trace. The contribution of the operators $\mathcal{D}_{\alpha\beta}$ could be further simplified as [45]

$$\begin{aligned} &(\delta^\alpha_\gamma\mathcal{D}_{\delta\beta} - \delta^\alpha_\delta\mathcal{D}_{\gamma\beta} + g_{\beta\delta}\mathcal{D}_\gamma^{\ \alpha} - g_{\beta\gamma}\mathcal{D}_\delta^{\ \alpha})f'V^\beta\eta^\gamma V^\delta \\ &\quad = (\nabla_\delta\nabla_\beta f')V^\beta\eta^\alpha V^\delta - 2\epsilon(\Box f')\eta^\alpha - (\nabla_\gamma\nabla_\beta f')V^\beta\eta^\gamma V^\alpha + \epsilon(\nabla_\gamma\nabla^\alpha f')\eta^\gamma, \end{aligned} \tag{3.147}$$

with $\epsilon = V^\alpha V_\alpha$, and using $\eta_\alpha V^\alpha = 0$. The contributions for the GDE are

$$R^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = C^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta + \frac{1}{2f'}\Bigg[\kappa(T_{\delta\beta}V^\beta\eta^\alpha V^\delta - T_{\gamma\beta}V^\beta\eta^\gamma V^\alpha + \epsilon T_\gamma{}^\alpha\eta^\gamma) - \epsilon\Big(\frac{\kappa T}{3} - \frac{f}{3} + \Box f'\Big)\eta^\alpha \\ + (\nabla_\delta\nabla_\beta f')V^\beta\eta^\alpha V^\delta - (\nabla_\gamma\nabla_\beta f')V^\beta\eta^\gamma V^\alpha + \epsilon(\nabla_\gamma\nabla^\alpha f')\eta^\gamma\Bigg]. \quad (3.148)$$

which is the general expression for any metric, and any energy-momentum content in the framework of metric $f(R)$ gravity. The equation (3.148) was one of the our results in [42].

3.15 GDE in FLRW geometries

The GDE for homogeneous and isotropic FLRW spacetimes in the 1 + 3 formalism involves the contractions [43]

$$\frac{D^2X^a}{D\lambda^2} = -R^a_{bcd}V^bX^cV^d = \\ -\Bigg[\frac{1}{2}(g^a{}_cR_{bd} - g^a{}_dR_{bc} + g_{bd}R^a{}_c - g_{bc}R^c_d) - \frac{R}{6}(g^a{}_cg_{bd} - g^a_dg_{bc})\Bigg]V^bX^cV^d, \quad (3.149)$$

for FLRW spacetimes one can choose $\eta_a = \eta_{\langle a\rangle}$, $E = -V_au^a$, $\epsilon = V_aV^a$ and $X_au^a = X_aV^a = 0$ for timelike or null [4] congruences. The contraction in the right side of (3.149) becomes

$$R^a_{bcd}V^bX^cV^d = \frac{1}{2}\Big(R_{bd}V^bV^dX^a - R_{bc}V^bX^cV^a + \epsilon R^a_cX^c\Big) - \frac{R}{6}\epsilon X^a, \quad (3.150)$$

for details see appendix B. We can write the GDE in terms of the energy momentum tensor as

$$\frac{D^2X^a}{D\lambda^2} = -\frac{1}{2f'}\Bigg[\Big\{\kappa(\rho_m + p_m) - \frac{1}{3}\Theta\dot{f}' + \ddot{f}'\Big\}E^2 + \epsilon\Big\{\frac{\kappa\rho_m}{3} + \kappa p_m + \frac{f}{3} + \frac{1}{3}\Theta\dot{f}' + \ddot{f}'\Big\}\Bigg]X^a. \quad (3.151)$$

The equation (3.151) was obtained in our work [42] using the 3 + 1 formalism in cosmology. In the 1 + 3 formalism it was obtained in [43]. In the next sections we will write (3.151) in a more convenient way and we show that for FLRW the two formalism are equivalent.

In the following section we explicitly show the steps in order to find the GDE in $f(R)$ gravity using FLRW metric, our purpose is compare with the results from GR in the limit case $f(R) = R - 2\Lambda$.

3.16 Geodesic Deviation Equation in FLRW spacetimes: coordinate method

Now we give a brief discussion about the GDE following [21, 57, 60] in order to give a more intuitive description about the topic (see figure 3.2). Let be γ_0 and γ_1 two neighbor geodesics with an affine parameter v. We introduce between the two geodesics a entire family of interpolating geodesics s, and collectively describe these geodesics with $x^\alpha(v, s)$, figure The vector field $V^\alpha = \frac{dx^\alpha}{dv}$ is tangent to the geodesic. The family s has $\eta^\alpha = \frac{dx^\alpha}{ds}$ like it's tangent vector field. Thus, the acceleration for this vector field is given by [21, 23]

[4] In the null case, we choose for a general spacetime the X_a according to (3.138)

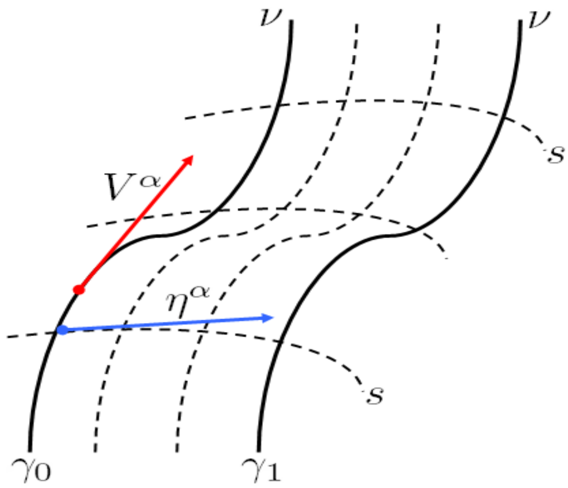

Figure 3.2: Geodesic Deviation Equation

Thus, the acceleration for this vector field is given as we showed in (3.130)

$$\frac{D^2\eta^\alpha}{D\nu^2} = -R^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta.$$

Here $\frac{D}{D\nu}$ corresponds to the covariant derivative a long the curve. We want to relate the geometrical properties of the space-time (Riemann and Ricci tensors) with the matter fields through field equations. For this we write the Riemann tensor as [23],[61]

$$R_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\delta\beta} - g_{\alpha\delta}R_{\gamma\beta} + g_{\beta\delta}R_{\gamma\alpha} - g_{\beta\gamma}R_{\delta\alpha}) - \frac{R}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\gamma\beta}), \tag{3.152}$$

with $C_{\alpha\beta\gamma\delta}$ the Weyl tensor. In the case of standard cosmology with the FLRW metric we have the line element

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\Big], \tag{3.153}$$

where $a(t)$ is the scale factor and k the spatial curvature of the universe. In this case the Weyl tensor $C_{\alpha\beta\gamma\delta}$ vanishes, and for the energy momentum tensor we have

$$T_{\alpha\beta} = (\rho + p)u_\alpha u_\beta + pg_{\alpha\beta}, \tag{3.154}$$

being ρ the energy density and p the pressure, the trace is

$$T = 3p - \rho. \tag{3.155}$$

The standard form of the Einstein field equations in GR (with cosmological constant) is

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{3.156}$$

then we can write the Ricci scalar R and the Ricci tensor $R_{\alpha\beta}$ using (3.154)

$$R = \kappa(\rho - 3p) + 4\Lambda, \tag{3.157}$$

$$R_{\alpha\beta} = \kappa(\rho + p)u_\alpha u_\beta + \frac{1}{2}[\kappa(\rho - p) + 2\Lambda]g_{\alpha\beta}, \tag{3.158}$$

from these expressions the right side of equation (3.152) is written as [60]

$$R^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \left[\frac{1}{3}(\kappa\rho + \Lambda)\epsilon + \frac{1}{2}\kappa(\rho + p)E^2\right]\eta^\alpha, \tag{3.159}$$

with $\epsilon = V^\alpha V_\alpha$ and $E = -V_\alpha u^\alpha$. This equation is known as *Pirani equation* [62]. The equation (3.159) is trivially obtained from (3.151 for GR including cosmological constant using $f(R) = R - 2\Lambda$. The GDE and some solutions for spacelike, timelike and null congruences has been studied in detail in [60], which gives some important result concerning cosmological distances also showed in [61]. Our purpose here is to extend these results from the modified field equations in metric $f(R)$ gravity.

3.16.1 Geodesic Deviation Equation for the FLRW universe

Using the FLRW metric as background we have

$$R_{\alpha\beta} = \frac{1}{f'(R)}\left[\kappa(\rho + p)u_\alpha u_\beta + \left(\kappa p + \frac{f(R)}{2}\right)g_{\alpha\beta} + \mathcal{D}_{\alpha\beta}f'(R)\right], \tag{3.160}$$

$$R = \frac{1}{f'(R)}\left[\kappa(3p - \rho) + 2f(R) - 3\Box f'(R)\right], \tag{3.161}$$

with these expressions the Riemann tensor could be written as

$$R^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f'}\Bigg[\left(\kappa(\rho + p)E^2 + \frac{\epsilon}{3}(\kappa(\rho + 3p) + f - 3\Box f')\right)\eta^\alpha$$
$$+ (\nabla_\delta\nabla_\beta f')V^\beta\eta^\alpha V^\delta - (\nabla_\gamma\nabla_\beta f')V^\beta\eta^\gamma V^\alpha + \epsilon(\nabla_\gamma\nabla^\alpha f')\eta^\gamma\Bigg], \tag{3.162}$$

with $E = -\eta_\alpha V^\alpha$, and $\eta_\alpha u^\alpha = 0$. For the FLRW case the covariant derivatives are

$$\nabla_0\nabla_0 f' = f''\ddot{R} + f'''\dot{R}^2, \qquad \nabla_i\nabla_j f' = -Hg_{ij}f''\dot{R}, \qquad \Box f' = -f''(\ddot{R} + 3H\dot{R}) - f'''\dot{R}^2, \tag{3.163}$$

since in this case the four-velocity is $u^\alpha = (1, 0, 0, 0)$ from the orthogonality conditions we get $E = -V_\alpha u^\alpha = -V_0$, $\eta_\alpha u^\alpha = \eta_0 u^0 = 0$ (thus the deviation vector just have non-vanishing spatial components $\eta^0 = 0$), and $\eta_\alpha V^\alpha = \eta_i V^i$. The GDE reduces to

$$R^\alpha_{\beta\gamma\delta}V^\beta\eta^\gamma V^\delta = \frac{1}{2f'}\left[\left(\kappa(\rho+p)+f''(\ddot{R}-H\dot{R})+f'''\dot{R}^2\right)E^2+\left(\frac{\kappa\rho}{3}+\kappa p+\frac{f}{3}+f''(\ddot{R}+H\dot{R})+f'''\dot{R}^2\right)\epsilon\right]\eta^\alpha. \tag{3.164}$$

We have already defined the following quantities (3.121)

$$\rho_{eff} = \frac{1}{f'}\left[\kappa\rho + \frac{Rf' - f}{2} - 3Hf''\dot{R}\right], \qquad p_{eff} = \frac{1}{f'}\left[\kappa p + \frac{f - Rf'}{2} + f''(\ddot{R} + 2H\dot{R}) + f'''\dot{R}^2\right], \tag{3.165}$$

equation (3.164) could be written in a more compact form and for the FLRW equivalent to the results found in [43]

$$R^{\alpha}_{\beta\gamma\delta}V^{\beta}\eta^{\gamma}V^{\delta} = \frac{1}{2}\left[(\rho_{eff} + p_{eff})E^2 + \frac{1}{3}\left(\rho_{eff} + 3p_{eff} + R\right)\epsilon\right]\eta^{\alpha}, \tag{3.166}$$

and finally we can write the GDE as

$$\frac{D^2\eta^{\alpha}}{D\nu^2} = -\frac{1}{2}\left[(\rho_{eff} + p_{eff})E^2 + \frac{1}{3}\left(\rho_{eff} + 3p_{eff} + R\right)\epsilon\right]\eta^{\alpha}, \tag{3.167}$$

with $\frac{D}{D\nu}$ the covariant derivative along the curve. The equation (3.167) was obtained in a very special reference frame. However, to show the equivalence for FLRW where the $1 + 3$ formalism is equivalent to the $3 + 1$ methodology, we use the result (3.151) and the result arises naturally with the special case $\Theta = 3H$

$$\frac{D^2X^a}{D\lambda^2} = -\frac{1}{2f'}\left[\left\{\kappa(\rho_m + p_m) - H\dot{R}f'' + \ddot{R}f'' + f'''\dot{R}^2\right\}E^2 + \epsilon\left\{\frac{\kappa\rho_m}{3} + \kappa p_m + \frac{f}{3} + H\dot{R}f'' + \ddot{R}f'' + f'''\dot{R}^2\right\}\right]X^a, \tag{3.168}$$

and where we use the general identities $\dot{f}' = \dot{R}f''$ and $\ddot{f}' = \ddot{R}f'' + f'''\dot{R}^2$. The equation (3.168) is the generalization for the *Pirani* equation in $f(R)$ gravity. As we expect the GDE induces only a change in the magnitude of the deviation vector η^{α}, which also occurs in GR. This result is inferred from the form of the metric, which describes an homogeneous and isotropic universe. For anisotropic universes, like Bianchi I, the GDE also induces a *change in the direction* of the deviation vector, as shown in [63].

3.16.2 GDE for fundamental observers

As a very simple but representative example of the GDE (3.168) we consider the Hubble's flow as a bundle of geodesics and we compute the GDE for this situation. In this case we have V^{α} as the four-velocity of the fluid u^{α}. The affine parameter ν matches with the proper time of the fundamental observer $\nu = t$. Because we have temporal geodesics then $\epsilon = -1$ and also the vector field are normalized $E^2 = 1$, thus from (3.151)

$$R^{\alpha}_{\beta\gamma\delta}u^{\beta}\eta^{\gamma}u^{\delta} = \frac{1}{2f'(R)}\left[\frac{2\kappa\rho}{3} - \frac{f(R)}{3} - 2Hf''(R)\dot{R}\right]\eta^{\alpha}, \tag{3.169}$$

if the deviation vector is $\eta_{\alpha} = \ell e_{\alpha}$, isotropy implies

$$\frac{De^{\alpha}}{Dt} = 0, \tag{3.170}$$

and

$$\frac{D^2\eta^{\alpha}}{Dt^2} = \frac{\mathrm{d}^2\ell}{\mathrm{d}t^2}e^{\alpha}, \tag{3.171}$$

using this result in the GDE (3.152) with (3.169) gives

$$\frac{\mathrm{d}^2\ell}{\mathrm{d}t^2} = -\frac{1}{2f'(R)}\left[\frac{2\kappa\rho}{3} - \frac{f(R)}{3} - 2Hf''(R)\dot{R}\right]\ell. \tag{3.172}$$

In particular with $\ell = a(t)$ we have

$$\frac{\ddot{a}}{a} = \frac{1}{f'(R)}\left[\frac{f(R)}{6} + Hf''(R)\dot{R} - \frac{\kappa\rho}{3}\right]. \tag{3.173}$$

This equation is exactly the equation (3.114) and obviously it could obtained as a particular case of the generalized Raychaudhuri equation given in (3.94) [48]. Is possible to obtain the standard form of the modified Friedmann equations [64] from this Raychaudhuri equation giving

$$H^2 + \frac{K}{a^2} = \frac{1}{3f'(R)}\Big[\kappa\rho + \frac{(Rf'(R) - f(R))}{2} - 3Hf''(R)\dot{R}\Big], \tag{3.174}$$

and

$$2\dot{H} + 3H^2 + \frac{K}{a^2} = -\frac{1}{f'(R)}\Big[\kappa p + 2Hf''(R)\dot{R} + \frac{(f(R) - Rf'(R))}{2} + f''(R)\ddot{R} + f'''(R)\dot{R}^2\Big]. \tag{3.175}$$

The set of equations (3.174) and (3.175) are the field equations for the FLRW spacetimes in metric $f(R)$ gravity. These equations are also studied in ETGs theory from the equivalence mentioned in section (2.8) in the previous chapter. The equations using such equivalence are studied in [65]. We notice again that the geometrical method to get the GDE is based on geometrical analysis on manifold theory, for this reason, the extension to theories as ETGs follows the same methodology as we used for $f(R)$ gravity. The extension to theories as in the case of (2.119) is work in progress by the author.

3.17 GDE for null vector fields

Now we consider the GDE for null vector fields past directed. In this case we have $V^\alpha = k^\alpha$, $k_\alpha k^\alpha = 0$ and $k^\alpha = \frac{dx^\mu}{d\nu}$ with ν the affine-parametrization. Then equation (3.168) reduces to

$$\frac{D^2X^a}{D\lambda^2} = -\frac{1}{2f'}\Big[\{\kappa(\rho_m + p_m) - H\dot{R}f'' + \ddot{R}f'' + f'''\dot{R}^2\}E^2\Big]X^a, \tag{3.176}$$

or in terms of the (3.121)[5]

$$\frac{D^2\eta^\alpha}{D\lambda^2} = -R^\alpha_{\beta\gamma\delta}k^\beta\eta^\gamma k^\delta = -\frac{1}{2}(\rho_{eff} + p_{eff})E^2\,\eta^\alpha, \tag{3.177}$$

that could be interpreted as the *Ricci focusing* in $f(R)$ gravity. Writing $\eta^\alpha = \eta e^\alpha$, $e_\alpha e^\alpha = 1$, $e_\alpha u^\alpha = e_\alpha k^\alpha = 0$ and choosing an aligned base parallel propagated $\frac{De^\alpha}{D\nu} = k^\beta\nabla_\beta e^\alpha = 0$, the equation (3.177) becomes a scalar equation for the norm η. In the case of GR discussed in [60], all families of past-directed null geodesics ($k^0 = \frac{dt}{d\nu}$<0) experience focusing, provided $\kappa(\rho + p) > 0$, and for a fluid with equation of state $p = -\rho$ (cosmological constant) there is no influence in the focusing [60]. From (3.176) the focusing condition for $f(R)$ gravity is

$$\frac{1}{f'}\Big[\kappa(\rho_m + p_m) - H\dot{R}f'' + \ddot{R}f'' + f'''\dot{R}^2\Big] \geq 0, \tag{3.178}$$

The equation (3.178) is the weak energy condition for $f(R)$ gravity analogous to GR in the FLRW metric. Some more restricted conditions to the energy conditions in $f(R)$ can be found in [29, 53]. A similar condition to (3.178) over the function $f(R)$ was established in order to avoid the appearance of ghosts [4, 66] with the extra condition $f'(R) > 0$.

[5] Here we change the notation from X^a to η^α

3.17.1 From ν to redshift z

We want to write the equation (3.177) in function of the cosmological redshift parameter z. For this the differential operators is

$$\frac{\mathrm{d}}{\mathrm{d}\nu} = \frac{\mathrm{d}z}{\mathrm{d}\nu}\frac{\mathrm{d}}{\mathrm{d}z}, \tag{3.179}$$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\nu^2} &= \frac{\mathrm{d}z}{\mathrm{d}\nu}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}}{\mathrm{d}\nu}\right), \\ &= \left(\frac{\mathrm{d}\nu}{\mathrm{d}z}\right)^{-2}\left[-\left(\frac{\mathrm{d}\nu}{\mathrm{d}z}\right)^{-1}\frac{\mathrm{d}^2\nu}{\mathrm{d}z^2}\frac{\mathrm{d}}{\mathrm{d}z} + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\right].\end{aligned} \tag{3.180}$$

In the case of null geodesics

$$(1+z) = \frac{a_0}{a} = \frac{E}{E_0} \quad\longrightarrow\quad \frac{\mathrm{d}z}{1+z} = -\frac{\mathrm{d}a}{a}, \tag{3.181}$$

with a the scale factor, and $a_0 = 1$ the present value of the scale factor. Thus for the past directed case

$$\mathrm{d}z = (1+z)\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\nu}\,\mathrm{d}\nu = (1+z)\frac{\dot{a}}{a}E\,\mathrm{d}\nu = E_0 H(1+z)^2\,\mathrm{d}\nu. \tag{3.182}$$

Then we get

$$\frac{\mathrm{d}\nu}{\mathrm{d}z} = \frac{1}{E_0 H(1+z)^2}, \tag{3.183}$$

and

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}z^2} = -\frac{1}{E_0 H(1+z)^3}\left[\frac{1}{H}(1+z)\frac{\mathrm{d}H}{\mathrm{d}z} + 2\right], \tag{3.184}$$

writing $\frac{\mathrm{d}H}{\mathrm{d}z}$ as

$$\frac{\mathrm{d}H}{\mathrm{d}z} = \frac{\mathrm{d}\nu}{\mathrm{d}z}\frac{\mathrm{d}t}{\mathrm{d}\nu}\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{1}{H(1+z)}\frac{\mathrm{d}H}{\mathrm{d}t}, \tag{3.185}$$

we use ($k^0 = \frac{\mathrm{d}t}{\mathrm{d}\nu} = E_0(1+z)$) and the minus sign comes from the condition of past directed geodesic $k^0 < 0$. Now, from the definition of the Hubble parameter H

$$\dot{H} \equiv \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{a}}{a} = \frac{\ddot{a}}{a} - H^2, \tag{3.186}$$

and using the Raychaudhuri equation (3.173)

$$\dot{H} = \frac{1}{f'(R)}\left[\frac{f(R)}{6} + Hf''(R)\dot{R} - \frac{\kappa\rho}{3}\right] - H^2, \tag{3.187}$$

then

$$\frac{\mathrm{d}^2\nu}{\mathrm{d}z^2} = -\frac{3}{E_0 H(1+z)^3}\left[1 + \frac{1}{3H^2 f'(R)}\left(\frac{\kappa\rho}{3} - \frac{f(R)}{6} - Hf''(R)\dot{R}\right)\right]. \tag{3.188}$$

Finally, the operator $\frac{\mathrm{d}^2\eta}{\mathrm{d}\nu^2}$ is

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\nu^2} = (EH(1+z))^2\left[\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \frac{3}{(1+z)}\left[1 + \frac{1}{3H^2 f'(R)}\left(\frac{\kappa\rho}{3} - \frac{f(R)}{6} - Hf''(R)\dot{R}\right)\right]\frac{\mathrm{d}\eta}{\mathrm{d}z}\right], \tag{3.189}$$

and the GDE (3.177) reduces to

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \frac{3}{(1+z)}\left[1 + \frac{1}{3H^2 f'(R)}\left(\frac{\kappa\rho}{3} - \frac{f(R)}{6} - Hf''(R)\dot{R}\right)\right]\frac{\mathrm{d}\eta}{\mathrm{d}z} = -\frac{\left[\kappa(\rho_m + p_m) - H\dot{R}f'' + \ddot{R}f'' + \dot{R}^2 f'''\right]}{2H^2(1+z)^2 f'(R)}\eta. \tag{3.190}$$

Given the important fact

$$\nabla_a T^{ab}_{\text{mat}} = 0 = \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \tag{3.191}$$

the energy density ρ_m and the pressure p_m considering could be written in the following way

$$\rho(z) = \rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4, \qquad p(z) = \frac{1}{3}\rho_{r0}(1+z)^4, \tag{3.192}$$

where we have used $p_m = 0$ and $p_r = \frac{1}{3}\rho_r$. Thus the GDE could be written as

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \mathcal{P}(H,R,z)\frac{\mathrm{d}\eta}{\mathrm{d}z} + \mathcal{Q}(H,R,z)\eta = 0, \tag{3.193}$$

with

$$\mathcal{P}(H,R,z) = \frac{3}{(1+z)}\left[1 + \frac{1}{3H^2 f'(R)}\left(\frac{\kappa\rho}{3} - \frac{f(R)}{6} - Hf''(R)\dot{R}\right)\right], \tag{3.194}$$

$$\mathcal{Q}(H,R,z) = \frac{\left[\kappa(\rho_m + p_m) - H\dot{R}f'' + \ddot{R}f'' + \dot{R}^2 f'''\right]}{2H^2(1+z)^2 f'(R)}, \tag{3.195}$$

and H given by the modified field equations (3.174)

$$\begin{aligned} H^2 &= \frac{1}{3f'(R)}\left[\rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4 + \frac{(Rf'(R) - f(R))}{2} - 3Hf''(R)\dot{R}\right] - \frac{k}{a^2}, \\ H^2 &= \frac{1}{3f'(R)}\left[\rho_{m0}(1+z)^3 + \rho_{r0}(1+z)^4 + \rho_{DE}\right] - k(1+z)^2, \end{aligned} \tag{3.196}$$

where

$$\rho_{DE} \equiv \left[\frac{(Rf'(R) - f(R))}{2} - 3Hf''(R)\dot{R}\right]. \tag{3.197}$$

In order to solve (3.193) it is necessary to write R and H in function of the redshift. First we define the operator

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}z} = -(1+z)H\frac{\mathrm{d}}{\mathrm{d}z}, \tag{3.198}$$

then the Ricci is [19]

$$\begin{aligned} R &= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right], \\ &= 6\left[2H^2 + \dot{H} + \frac{k}{a^2}\right], \\ &= 6\left[2H^2 - (1+z)H\frac{dH}{dz} + k(1+z)^2\right], \end{aligned}$$

if we want $H = H(z)$ is necessary to fix the form of $H(z)$ or either a specific form of the $f(R)$ function. This point has been studied in [19] and the method to fix the form of $H(z)$ and find the form of the

$f(R)$ function by observations is given in [67]. The $f(R)$ models offer a wide possibility to explain fundamental problems in cosmology, as we mention the actual acceleration phase one of the problems addressed by the ETGs [68–71].

There is a method employed by [43] using a dynamical system to solve the Friedmann equations together with (3.190). It has some advantages to explore general features of the problem. However, our method is equivalent, we can write(3.190) using the transformation between the affine parameter ν and the redshift z, $\frac{\mathrm{d}}{\mathrm{d}\nu} \longrightarrow \frac{\mathrm{d}}{\mathrm{d}z}$ and (3.121), the equation (3.190) is written as [43]

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} + \frac{(7+3w_{eff})}{2(1+z)}\frac{\mathrm{d}\eta}{\mathrm{d}z} + \frac{3(1+w_{eff})}{2(1+z)^2}\eta = 0, \tag{3.199}$$

with $w_{eff} = \frac{p_{eff}}{\rho_{eff}}$. This is the equation [43, eq. (39)]. In the particular case $f(R) = R - 2\Lambda$, implies $f'(R) = 1$, $f''(R) = 0$. The expression for Ω_{DE} reduces to

$$\rho_{DE} = \left[\frac{(R - R + 2\Lambda)}{6}\right] = \frac{\Lambda}{3}, \tag{3.200}$$

then the quantity ρ_{DE} generalizes the Dark Energy parameter. The Friedmann modified equation (3.196) reduces to the well know expression in GR

$$H^2 = H_0^2[\Omega_{m0}(1+z)^3 + \Omega_{r0}(1+z)^4 + \Omega_\Lambda + \Omega_k(1+z)^2], \tag{3.201}$$

the expressions $\mathcal{P}$, and Q reduces to

$$\mathcal{P}(z) = \frac{4\Omega_{r0}(1+z)^4 + (7/2)\Omega_{m0}(1+z)^3 + 3\Omega_{k0}(1+z)^2 + 2\Omega_\Lambda}{(1+z)(\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_\Lambda)}, \tag{3.202}$$

$$Q(z) = \frac{2\Omega_{r0}(1+z)^2 + (3/2)\Omega_{m0}(1+z)}{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_\Lambda}. \tag{3.203}$$

and the GDE for null vector fields is

$$\begin{aligned}\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} &+ \frac{4\Omega_{r0}(1+z)^4 + (7/2)\Omega_{m0}(1+z)^3 + 3\Omega_{k0}(1+z)^2 + 2\Omega_\Lambda}{(1+z)(\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_\Lambda)}\frac{\mathrm{d}\eta}{\mathrm{d}z} \\ &+ \frac{2\Omega_{r0}(1+z)^2 + (3/2)\Omega_{m0}(1+z)}{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2 + \Omega_\Lambda}\eta = 0.\end{aligned} \tag{3.204}$$

The Mattig relation in GR is obtained in the case $\Omega_\Lambda = 0$ and writing $\Omega_{k0} = 1 - \Omega_{m0} - \Omega_{r0}$ which gives [60]

$$\frac{\mathrm{d}^2\eta}{dz^2} + \frac{6 + \Omega_{m0}(1+7z) + \Omega_{r0}(1+8z+4z^2)}{2(1+z)(1+\Omega_{m0}z + \Omega_{r0}z(2+z))}\frac{\mathrm{d}\eta}{\mathrm{d}z} + \frac{3\Omega_{m0} + 4\Omega_{r0}(1+z)}{2(1+z)(1+\Omega_{m0}z + \Omega_{r0}z(2+z))}\eta = 0, \tag{3.205}$$

then, the equation (3.193) give us a generalization of the Mattig relation in $f(R)$ gravity.

3.17.2 The angular diameter distance D_A

In a spherically symmetric space-time, like in FLRW universe, the magnitude of the deviation vector η is related with the proper area dA of a source in a redshift z by $d\eta \propto \sqrt{dA}$, and from this, the definition of the angular diametral distance D_A could be written as [2]

$$D_A = \sqrt{\frac{\mathrm{d}A}{\mathrm{d}\Omega}}, \tag{3.206}$$

with $d\Omega$ the solid angle. Thus the GDE in terms of the angular diametral distance is then

$$\frac{\mathrm{d}^2 D_A^{f(R)}}{\mathrm{d}z^2} + \frac{(7+3w_{eff})}{2(1+z)}\frac{\mathrm{d}D_A^{f(R)}}{\mathrm{d}z} + \frac{3(1+w_{eff})}{2(1+z)^2} D_A^{f(R)} = 0. \tag{3.207}$$

where we denote the angular diametral distance by $D_A^{f(R)}$, to emphasize that any solution of the previous equation needs a specific form of the $f(R)$ function, or either a form of $H(z)$. This equation satisfies the initial conditions (for $z \geq z_0$)

$$D_A^{f(R)}(z, z_0)\Big|_{z=z_0} = 0, \tag{3.208}$$

$$\frac{\mathrm{d}D_A^{f(R)}}{\mathrm{d}z}(z, z_0)\Big|_{z=z_0} = \frac{H_0}{H(z_0)(1+z_0)}, \tag{3.209}$$

being $H(z_0)$ the modified Friedmann equation (3.196) evaluated at $z = z_0$. The conditions (3.208) and (3.209) keep the functional form because they come from the geometrical properties of the FLRW models. The proper distances and the definition for the Hubble parameter as $H = \frac{\dot{a}}{a}$ allow to write the boundary conditions for (3.207) [2].

3.18 Is it possible a Dyer-Roeder like Equation in $f(R)$ Gravity?

Finally we get an important relation that is a tool to study cosmological distances also in inhomogeneous universes. The Dyer-Roeder equation gives a differential equation for the diametral angular distance d_A as a function of the redshift z [72]. The standard form of the Dyer-Roeder equation in GR can be given by [57],[73]

$$(1+z)^2 \mathcal{F}(z)\frac{\mathrm{d}^2 D_A}{\mathrm{d}z^2} + (1+z)\mathcal{G}(z)\frac{\mathrm{d}D_A}{\mathrm{d}z} + \mathcal{H}(z)D_A = 0 \tag{3.210}$$

with

$$\mathcal{F}(z) = H^2(z) \tag{3.211}$$

$$\mathcal{G}(z) = (1+z)H(z)\frac{\mathrm{d}H}{\mathrm{d}z} + 2H^2(z) \tag{3.212}$$

$$\mathcal{H}(z) = \frac{3\tilde{\alpha}(z)}{2}\Omega_{m0}(1+z)^3 \tag{3.213}$$

with $\tilde{\alpha}(z)$ is the *smoothness parameter*, which gives the character of inhomogeneities in the energy density [73]. When we consider α = const the physical interpretation for the clumpiness parameter is clear as is pointed out in [2]. There have been some studies about the influence of the smoothness parameter $\tilde{\alpha}$ in the behavior of $D_A(z)$ [57, 74]. In order to obtain the Dyer-Roeder like equation in $f(R)$ gravity we follow [57]. First, we note that the terms containing the derivatives of $D_A^{f(R)}$ in equation (3.207) come from the transformation $\frac{\mathrm{d}}{\mathrm{d}v} \longrightarrow \frac{\mathrm{d}}{\mathrm{d}z}$ and the term with only $D_A^{f(R)}$ comes from the Ricci focusing (3.176). In this point there is a big difference between GR and $f(R)$ as we can notice in (3.177). Besides the matter density ρ_m there is a geometrical contribution coming from geometrical terms in the Ricci focusing term. As a first attempt to consider inhomogeneous contributions we follow [57, 72, 73] and we introduce a mass-fraction $\tilde{\alpha}$ (smoothness parameter) of matter in the universe, and then we replace only this contribution in the Ricci focusing $\rho \longrightarrow \tilde{\alpha}\rho$. It could be a danger to estimate distances from (3.190) due to the fact that the terms in the Ricci focusing depend on more than ρ_m, the contributions from the $f(R)$ function make a difference that should be investigated with a different approach. Thus, the problem for distances in inhomogeneous cosmologies should be treated with tools as (3.145). This equation is a fundamental key in applications as gravitational lensing [56] and being the equation general could be applied to more general spacetimes as LTB models [1] in a more clear way.

3.19 An *Alternative* Derivation

To see the direct equivalence for the FLRW between the coordinate and the covariant methods, the results for the angular diametral distance could be also obtained from the focusing equation (see [42, 45] and references therein):

$$\frac{\mathrm{d}^2 D_A}{dv^2} = -\left(|\sigma|^2 + \frac{1}{2}R_{\alpha\beta}k^\alpha k^\beta\right)D_A, \tag{3.214}$$

being σ the shear, k^α a null vector. From the field equations in $f(R)$ gravity we can write

$$R_{\alpha\beta} = \frac{1}{f'}\left[\kappa T_{\alpha\beta} + \frac{f}{2}g_{\alpha\beta} + \nabla_\alpha\nabla_\beta f' - g_{\alpha\beta}\Box f'\right], \tag{3.215}$$

for the specific case of the FLRW universe ($\sigma = 0$), the previous expression gives

$$R_{\alpha\beta}k^\alpha k^\beta = \frac{1}{f'}\left[\kappa(\rho + p) + f''(\ddot{R} - H\dot{R}) + f'''\dot{R}^2\right]E^2; \tag{3.216}$$

and then

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}v^2} = -\frac{1}{2f'}\left[\kappa(\rho + p) + f''(\ddot{R} - H\dot{R}) + f'''\dot{R}^2\right]E^2 D_A, \tag{3.217}$$

which has the same form as equation (3.190) with $\eta \propto D_A$.

3.20 Conclusions and Discussion

In this chapter, we have generalized the GDE equation in metric $f(R)$ gravity and we have studied the equivalence for the FLRW models between the 3+1 formalism [42] and the covariant 1+3 approach [43] for the GDE. We have summarized the covariant formalism for the cosmological models. Our general expression (3.199) contains all the information about the dynamics of the models in the (w_{eff}) parameter. As we expect, the analytic solutions for (3.199) $f(R)$ are not trivial. Some numerical solutions are shown in [43], but this point deserves a deep study. Further analysis for the optical Sachs equations including the shear contribution, and distance-redshift relations in $f(R)$ gravity is in progress from our previous work [42, 57]. In principle, the formulation given in 3.13.1 makes the extension of the GDE to more general spacetimes more clear. For perturbed FLRW models in GR and applications to weak gravitational lensing of the $1 + 3$ formalism see [56].

In the previous chapter 2.2.1 we have mentioned some cosmological reasons to study $f(R)$ as a viable model for gravity. It is important to mention some important points related with this topic [1]. Maybe one the most actual open questions coming from the cosmological data is the late-time acceleration of the universe from $z \approx 1$. Nonlinear effects is a viable way to explain this phenomena without dark energy or cosmological constant. The other possibility to explain the late acceleration of the universe is to

modify the gravity equations (see previous chapter). There are some proposal to explain the acceleration phase with modifications to the Friedmann equations such as $f(H^2) = \frac{8\pi G\rho}{3}$ or $H^2 = \frac{8\pi G g(\rho)}{3}$, but with these ideas is only possible to compute the background dynamics and not the perturbations, it is not possible with to explain the structure formation.
We can think in a general modification to gravity (section 2.8.3 previous chapter) to solve the acceleration problem, but as is well known, GR has a unique status as a four-dimensional theory where gravity is mediated by a massless spin-2 particle, and the field equations are of the second order. Any gravity modification will produce equation of motion at least of fourth order, and gravity is then also carried by spin-0, spin-1 massless fields. However,in order to avoid ghost in the theory, the only acceptable low-energy generalization of GR are $f(R)$ gravities, with $f''(R) \neq 0$ [1]. This is a very strong reason to study $f(R)$ gravity as one of the most promising models. Cosmological perturbation theory in $f(R)$ gravity could be studied in a fully nonlinear regime with the $1 + 3$ presented in this chapter.
In $f(R)$ gravity theories, the gravitational interaction is mediated by a spin-0 scalar as well the spin-2 field. As we point out in section 2.8 the equivalence between $f(R)$ gravity and a type Brans-Dicke gravity causes some problems with the spin-0 field in the solar system. The equivalence indicates that $f(R)$ gravity is a Brans-Dicke theory with $\omega_{BD} = 0$ whilst probes as binary pulsar demand $\omega_{BD} > 40.000$ [1]. To avoid the solar system/binary pulsar problem one can use the potential in ETGs and the equivalence with $f(R)$ theory increasing the mass of the spin-0 field near to massive objects and keeping the ultra-light mass on cosmological scales. This is the *chameleon mechanism*. Other possibility is to explore our alternative approach [10] where we have demonstrated that it is viable to have the equations of motion only from metric degree of freedom.

CHAPTER 4

Cosmological Perturbation Theory and Cosmic magnetic fields

4.1 Introduction

The origin of galactic and extra-galactic magnetic fields is an unsolved problem in modern cosmology. One sentence to summarize our understanding about cosmic magnetic fields is due to A. Vilenkin 2009 [75]:
There is much to be learned about cosmic magnetic fields. We have a rather sketchy information about the field distribution on the largest scales, and the origin of the magnetic fields remains a mystery.

In this chapter we present our work [49] where we link the cosmological perturbation theory up to second order in a gauge invariant language, applied to the problem of cosmic magnetic fields.
We explore the idea that the cosmic magnetic fields emerged from a small field, *a seed*, which was produced in the early universe (phase transitions, inflation, ...) and it evolves in time. The problem of how the seed appears is not discussed. This *magnetogenesis* problem is an active field of research and we recommend the excellent reviews [75, 76].

Cosmological perturbation theory offers a natural way to study the evolution of primordial magnetic fields. The dynamics for this field in the cosmological context is described by a cosmic dynamo like equation, through the dynamo term. In this chapter we get the perturbed Maxwell's equations and compute the energy momentum tensor up to second order in perturbation theory in terms of gauge invariant quantities. Two possible scenarios are discussed, first we consider a FLRW background without magnetic field and we study the perturbation theory introducing the magnetic field as a perturbation. In the second scenario, we consider a magnetized FLRW and build up the perturbation theory from this background. We compare the cosmological dynamo like equation in both scenarios.

As a complementary part of this chapter, in appendix C the 1 + 3 formalism and the gauge invariant cosmological perturbation theory using the cosmic magnetic fields is studied as an example [6].

4.2 Cosmological perturbation theory: linear regime

Since the phenomenal paper by Lifshitz in 1946 [77], there has been an enormous progress in understanding how to study properly the theory of cosmological perturbations. The main aspects for linear theory are very well established nowadays and there are a wide variety of bibliography sources, as an example [78–81]. The extension of the relativistic cosmological perturbation theory to non-linear regime is also a very active field of research and involves some interesting aspects in mathematics and physics. The non linear regime contains physics hidden at the linear one, it has been our motivation to develop our proposal about the *cosmological dynamo*. From very influential works [82, 83], higher order perturbation theory is understood today not as a simple *relabeling* of the spacetime but a phenomenological scenario with important results as the fluid vorticity as source of primordial magnetic fields [49] and non-Gaussianity signatures in the CMB setting for perturbations from inflation [83]. There are several groups and different approaches to handle the problem. One of the most important methods is called *gauge invariant perturbation theory*, which is mainly due to K. Nakamura [82], but many other important authors have contributed to the topic [83–87]. For this first section, we introduce the main elements for the linear cosmological perturbation in the metric-based approach which was introduced by Lifshitz [77] and for further references about the methodology see [81, 88].

4.2.1 Mathematical background

In cosmological perturbation theory we deal with two spacetimes, one is an idealized universe, usually the FLRW is employed not only because it is an exact solution of the GR equations but from observations it is realized that when a enough large scale is reached the universe looks homogeneous and isotropic [1]. The idealized universe is described by a FLRW metric (3.70)

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x^k)dx^i dx^j = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{4.1}$$

where γ_{ij} is the metric on the 3−manifold Σ. To cover the spacetime manifold in (4.1) we choose the $3+1$ splitting [80]. For an extensive review about the $3+1$ formalism see the review [89]. In the $3+1$ formalism to each t the space is a spacelike hypersurface Σ_t. The greek indices run from $0, ...3$ and the latin $1, .., 3$. The right side in (4.1) is written in comoving spherical coordinates $\{r, \theta, \phi\}$ and k is the spatial curvature, which is constant. For linear cosmological perturbation theory, the main assumption is that we can describe the *perturbed real world* by a metric tensor which is a *small deviation* from the background (4.1). This sentence has a deep physical meaning and the mathematical effort to formulate the problem properly has a long history in cosmology. In one of the most influential works in the topic, Bardeen in 1980 [78] found the method to formulate the problem of linear perturbation in a new language known as *gauge invariant variables*. After the Bardeen's work, very important results to improve the theory have been developed. The mathematical aspects to deal with the perturbed Einstein field equations are in the work by Kodama & Sasaki in 1984 [88].

The mathematical and physical precise meaning of *small deviation* for a perturbation of any variable is found in [90, 91]. We should describe first some quantities in the idealized universe which we will call background.

Sometimes it is useful introduce a vector base to write tensor fields, for the metric (4.1) one can write

$$\boldsymbol{\omega} = \omega_i \mathbf{e}^i, \tag{4.2}$$

and the a second rank tensor

$$\boldsymbol{h} = h_{ij}\mathbf{e}^i \otimes \mathbf{e}^j. \tag{4.3}$$

There are several results with this notation. The 3−metric tensor γ_{ij} defines the algebra for the 3−fields as

$$A^i = \gamma^{ij} A_j, \qquad (4.4)$$
$$\gamma^{ij}\gamma_{ik} = \delta^i_k,$$
$$\boldsymbol{\gamma}(\mathbf{e}_i, \mathbf{e}_j) = \gamma_{ij}.$$

The 3−vector basis avoid the use of a particular spatial coordinate system. The 3−metric induces the definition of a covariant derivative $\boldsymbol{\nabla}_i$ on Σ_t even in the case when $k \neq 0$. With the condition $\nabla_i \gamma_{lk} = 0$, the connection coefficients Γ^i_{jk} take the form

$$\Gamma^k_{ij} = \frac{1}{2}\gamma^{kl}\left(\gamma_{il,j} + \gamma_{lj,k} - \gamma_{ij,k}\right). \qquad (4.5)$$

The 3−Laplacian (Laplace-Beltrami) operator is defined by

$$\boldsymbol{\nabla}^2 \equiv \gamma^{ik}\boldsymbol{\nabla}_i\boldsymbol{\nabla}_k \qquad (4.6)$$

and the set of functions Y solution of

$$\left(\boldsymbol{\nabla}^2 + k^2\right) Y = 0, \qquad (4.7)$$

where $-k^2$ represents an eigenvalue for the Laplace-Beltrami operator. This set of functions is the natural basis to expand any scalar function on Σ. The $\{k^2\}$ takes continues values larger or equal to $\{(n-2)^2|k|\}$ for $\{k \leq 0\}$ and $\{k^2 = l(l+n+1)k \quad (l = 0, 1, 2, ..)\}$ for $k > 0$. It has been obtained in [88] and Nakamura [84]. The submanifold Σ is Riemannian and the covariant derivatives have the usual algebra

$$\left[\boldsymbol{\nabla}_i, \boldsymbol{\nabla}_j\right] A^k = {}^{(3)}R^k{}_{lij} A^l, \qquad (4.8)$$
$$\left[\boldsymbol{\nabla}_i, \boldsymbol{\nabla}_j\right] h^{kl} = {}^{(3)}R^k{}_{nij} h^{nl} + {}^{(3)}R^l{}_{nij} h^{kn},$$

where

$$\left[\boldsymbol{\nabla}_i, \boldsymbol{\nabla}_j\right] = \boldsymbol{\nabla}_i\boldsymbol{\nabla}_j - \boldsymbol{\nabla}_j\boldsymbol{\nabla}_i. \qquad (4.9)$$

The Riemann tensor for constant curvature is

$${}^{(3)}R^l{}_{nij} = k\left(\delta^l_i \gamma_{nj} - \delta^l_j \gamma_{ni}\right). \qquad (4.10)$$

From (4.10) the Ricci tensor and Ricci scalar are given by

$${}^{(3)}R_{nj} = {}^{(3)}R^l{}_{nlj} = k\left(\delta^l_l \gamma_{nj} - \delta^l_j \gamma_{nl}\right) = k\left(3\gamma_{nj} - \gamma_{nj}\right) = 2k\gamma_{nj}, \qquad (4.11)$$
$${}^{(3)}R =^{(3)} R^l_l = 6k.$$

From the above relations, some useful results can be derived, the lapĺacian for a scalar field is given by

$$\boldsymbol{\nabla}^2\phi = \gamma^{ij}\partial_i\partial_j\phi - \gamma^{ij}\Gamma^l_{ij}\partial_l\phi, \qquad (4.12)$$

which using $\gamma^{ij}\Gamma^l_{ij} = -\frac{1}{\gamma^{1/2}}\partial_k\left(\gamma^{1/2}\gamma^{lk}\right)$, with $\gamma \equiv \det\{\gamma_{ij}\}$, takes the form

$$\boldsymbol{\nabla}^2\phi = \frac{1}{\gamma^{1/2}}\partial_i\left(\gamma^{1/2}\gamma^{ij}\partial_j\phi\right). \qquad (4.13)$$

There are at least two more important results, the divergence for a 3−vector field **v** is generalized by

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{1}{\gamma^{1/2}} \partial_i \left(\gamma^{1/2} v^i \right), \tag{4.14}$$

and the curl for the 3−vector field

$$\boldsymbol{\nabla} \times \boldsymbol{v} = \epsilon^{ijk} \left(\partial_i v_j \right) \boldsymbol{e}_k, \tag{4.15}$$

with $\epsilon^{ijk} \equiv \gamma^{-1/2}[ijk]$ is the three-dimensional Levi-Civita tensor, with $[ijk] = +1$ for an even permutation of $\{123\}$, $[ijk] = -1$ for an odd permutation, and 0 for any two equal indices. The factor $\gamma^{1/2}$ makes the Levi-Civita a well defined 3−tensor density.

In the next section, we will employ the nomenclature *scalar, vector and tensor* in the perturbation variables, this terminology refers how the quantities transform respect to the symmetry group of the submanifold with metric γ_{ij}. In most of this chapter $\{\gamma_{ij} = \delta_{ij}\}$ meaning we focus in a flat ($k = 0$) background.

4.3 The perturbed metric tensor

One of the most important results from the Kodama & Sasaki work [88] is that the metric tensor for the *real universe* is written as:

$$ds^2 = a^2(\tau) \left\{ -(1 + 2\psi)\, d\tau^2 + 2\omega_i d\tau dx^i + \left[(1 - 2\phi)\, \gamma_{ij} + 2h_{ij} \right] dx^i dx^j \right\}, \tag{4.16}$$

with $\gamma^{ij} h_{ij} = 0$[1] and $\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)}$. The metric tensor (4.16) has the new ingredients for the cosmological perturbation theory. Two 3−scalar fields $\phi(\mathbf{x}, \tau)$ and $\psi(\mathbf{x}, \tau)$, one 3−vector $\boldsymbol{\omega}(\mathbf{x}, \tau)$ and one symmetric, traceless second-rank 3−tensor $h_{ij}(\mathbf{x}, \tau)$. One of the most clear treatment for the physics and notation related with the metric (4.16) is given by Bertschinger in his lectures [79] but are the same definitions are founded in Kodama & Sasaki [88]. From (4.16) there are 10 new fields $\{1 + 1 + 3 + 5\}$ corresponding to $\{\phi, \psi, \omega, \mathbf{h}\}$. We summarize some important results in order to have a clear idea when second order perturbation theory is introduced.

4.3.1 Decomposition of perturbations

The excellent review by Nakamura [82] is one of the most influential works in the field of non-linear cosmological perturbation theory. The metric (4.16) is the most general form for the metric perturbations [80, 81] for a FLRW background. However, there are some aspects that should be addressed in order to use (4.16) as the most general metric for cosmological perturbations. The first aspect in (4.16) is how many degrees of freedom we have for the metric tensor field. In four dimensions, due to the symmetry of the Einstein Field Equations, in principle (4.16) has ten new fields, it means, two scalar fields (including the trace for h_{ij} which in principle can be absorbed in the ϕ definition), three in the vector and due to the symmetry in h_{ij} and with the traceless condition we have five degrees, it means that (4.16) has ten independent components. Due to the general covariance, only six fields can be physical degrees of freedom, in fact, we can transform the coordinates $\{\tau, x^i\}$ without change any physical quantity, as for example the spacetime interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. It is one way to study the so call *gauge problem* in

[1] Not confuse h_{ij} with the projector h_{ab}.

cosmology. We perform a coordinate transformation in $p \in \mathcal{M}$ using two coordinate charts related by

$$\begin{aligned} \hat{\tau} &= \tau + \alpha(\mathbf{x}, \tau), \\ \hat{x}^i &= x^i + \gamma^{ij}\nabla_j\beta(\mathbf{x}, \tau) + \epsilon^i(\mathbf{x}, \tau), \\ \text{with} \quad & \nabla \cdot \epsilon^i = 0. \end{aligned} \tag{4.17}$$

The coordinate freedom leads to ambiguity in the perturbations. The definition of a perturbation quantity is given by

$$\delta\Gamma := \Gamma\Big|_{\mathcal{M}} - \Gamma\Big|_{\mathcal{M}_0}, \tag{4.18}$$

where $\Gamma_{|\mathcal{M}}$ is the value of the quantity in the real universe and $\Gamma_{|\mathcal{M}_0}$ is the value in the fictitious background. The main point with the definition (4.18) is that the difference depends on the transformation (4.18). Any identification between the real universe and the fictitious one makes(4.18) totally depended of the identification map between the two manifolds. In other words, the identification between $\mathcal{M}_0$ and $\mathcal{M}$ once a coordinate system is fixed in $\mathcal{M}_0$ makes the equation (4.18) depending on the identification map. The fact of *keeping the background coordinates fixed* is known in cosmology as a *gauge transformation*. We should notice than when we split any physical quantity as

$$\mathbf{T}(\tau, x^i) = \mathbf{T}(\tau) + \delta\mathbf{T}(\tau, x^i), \tag{4.19}$$

this process in non-covariant [83]. A coordinate transformation (4.17) relabel the events in the manifold, but a due to the general covariance of a theory as GR or ETGs the splitting non-covariant. As we will see after, there is another possibility to study this problem. Due to the general transformation law between coordinates charts in a manifold, for example the metric tensor components in an event $p \in \mathcal{M}$ in two different coordinate charts are related by

$$g'_{\mu\nu}(x'^{\lambda}(p)) = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}g_{\sigma\lambda}(x^{\lambda}(p)), \tag{4.20}$$

where $x'^{\mu} = x'^{\mu}(x^{\lambda})$ is a general, smooth differentiable function. The problem to find the functional form for the coordinate transformation is studied with the *infinitesimal coordinate transformation* ($x'^{\mu} = x^{\mu} + \xi^{\mu}$), with ξ^{μ} the generator for the transformation. It is called in mathematics *gauge transformation*. We are dealing here with the theory of *linear perturbation theory*, but there are several extensions to higher orders in cosmological perturbation theory [83, 86], and including Maxwell equations [49], where the main tool is the *gauge transformation to higher order* and where ξ^{μ} is used as the *gauge vector field* and the information for the higher orders is encoded on it. In the equation (4.17) the functions (α, β) and the vector ϵ^i are the generators of the gauge transformation. The identification between the background and the physical space induces a coordinate transformation on the physical spacetime, it is called the passive approach and is the most used method in the literature. In the section 4.4 we deal with the gauge problem in cosmology.

The main assumption in linear perturbation theory is that we use $\boldsymbol{\gamma}$ as the 3−metric in the perturbed hypersurface of constant τ, even when the perturbation metric functions are present, the argument is such that any metric perturbation field is small and quadratic perturbation terms are neglected. It will be not true in high order perturbation theory as we will see in the next sections where the problem is studied using a more powerful method known as the *active approach*.

The metric decomposition (4.16) allow us to generate the different contributions as

$$\omega = \omega_{\parallel} + \omega_{\perp}, \tag{4.21}$$

with

$$\nabla \times \omega_{||} = \nabla \cdot \omega_{\perp} = 0. \tag{4.22}$$

The equation (4.22) implies $\omega_{||} = -\nabla\omega$ and only the transverse part $\omega_{\perp}$ represents a vector perturbation.

The tensor h_{ij} can be generated as

$$h(\mathbf{x}) = h_{\perp} + h_{||} + h_T, \tag{4.23}$$

with

$$\begin{aligned} h_{||} &:= D_{ij}h, \\ h_{ij,\perp} &:= \frac{1}{2}(\nabla_i h_j + \nabla_j h_i), \\ \nabla_i h^i{}_{jT} &= 0. \end{aligned} \tag{4.24}$$

with $D_{ij} \equiv \nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2$. The divergences for $h_{||}$

$$\nabla \cdot h_{||} = \nabla^j\nabla_i\nabla_j h - \frac{1}{3}\gamma_{ij}\nabla^j\nabla^2 h, \tag{4.25}$$

and from the Riemann tensor

$$\nabla^j\nabla_i\nabla_j h = \gamma^{kj}R_{jmki}\nabla^m h + \gamma^{kj}\nabla_i\nabla_k\nabla_j h = R_{mi}\nabla^m h + \nabla_i\nabla^2 h, \tag{4.26}$$

in the case of constant curvature,

$$\nabla \cdot h_{||} = \frac{2}{3}\nabla\left(\nabla^2 + 3k\right)h. \tag{4.27}$$

In the same way, the divergence for $h_{\perp}$ is

$$\nabla \cdot h_{\perp} = \frac{1}{2}\left(\nabla^2 + 2k\right)h, \tag{4.28}$$

are longitudinal and transverse vectors, respectively [79]. The important conclusion is that the most general perturbations of the FLRW metric at each point may be decomposed in four scalar $\{\psi, \phi, \omega, h\}$, two vector parts each having two degrees of freedom $\{\omega_{\perp}, h_{\perp}\}$ and one tensor part h_T with three degrees of freedom. The classification is a very powerful tool to distinguish between gauge modes (coordinate artifacts) from physical quantities. In principle one can write the field equations (GR or any ETG) from (4.16). However, is more clear to choose an appropriate coordinate system though (4.17) with an important requirement, a gauge choice should not eliminate physical phenomena.

The decomposition (4.24) was performed by Lifshitz [77] and it has been a fundamental key in cosmological perturbation theory. We will use the natural extension in the case of the second order case.

We can proceed to write the field equations for the metric (4.16), but given the freedom in the gauge choosing, it is common to write the fields equations in a particular gauge, for example two very common are the Poisson, which will be discussed in section 4.11 and the synchronous gauge introduced by Lifschitz [77]. We must care that our gauge allows all the physical degree of freedom [79]. There are several important consequences when we use a particular gauge, for example one very important feature of the Poisson gauge is that it gives the relativistic cosmological generalization of the *Newtonian gravity* and under the consideration of slow motion and inside the Hubble's horizon, the Poisson gauge coordinates reduce to the Eulerian coordinates used in Newtonian cosmology [79].

4.3.2 The energy-momentum tensor

In the chapter 3, we described the source for the FLRW model in GR or in ETG as a perfect fluid allowing some general equation of state which respects the spacetime symmetries, for details see section 3.6. Also a more general energy momentum tensor describing phenomena as bulk viscosity, thermal conduction and other physical situations was introduced (3.33). In a coordinate system the tensor for a general fluid (or a sum of uncoupled components as neutrinos,baryons, dark matter) takes the form

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu} + \Sigma^{\mu\nu}, \tag{4.29}$$

where ρ, p are the proper energy density and pressure measured in the fluid rest frame. The tensor $\Sigma^{\mu\nu}$ obeys $\Sigma^{\mu}_{\nu}u^{\nu} = 0$ and $\Sigma^{\mu}_{\mu} = 0$. When we choose (4.29) instead of (3.33), the quantity ρu^{μ} carries any heat conduction and p includes any bulk viscosity. One can separate the heat conduction and bulk viscosity with terms $q^{\mu}u^{\nu}$ and from (4.29) the expression in (3.33) is recovered. When scalar fields or electromagnetic fields are present, the energy momentum becomes

$$T^{\mu\nu} = \sum_i T_i^{\mu\nu}, \tag{4.30}$$

where $T_i^{\mu\nu}$ denotes each component. Also, the total energy momentum satisfaces

$$T^{\mu\nu}_{;\mu} = 0, \tag{4.31}$$

but no necessarily each component. The tensor (4.30) for a general perturbation contains sources as the anisotropic part generated by scalar fields or in our case of interest magnetic fields.

In the next sections the gauge problem in cosmology is addressed in a general geometrical way using the Lie derivative as a fundamental tool. Some notation will be changed without change any definition made in this section. The definitions are followed easily, the purpose is to keep our notation in [49].

4.4 The gauge problem in perturbation theory

Cosmological Perturbation Theory helps us to find approximate solutions of the Einstein field equations through small deviations from an exact solution [92]. In this theory one works with two different space-times, one is the real space-time $(\mathcal{M}, g_{\alpha\beta})$ which describes the perturbed universe and the other is the background space-time $(\mathcal{M}_0, g^{(0)}_{\alpha\beta})$ which is an idealization and is taken as reference to generate the real space-time. Then, the perturbation of any quantity Γ (e.g., energy density $\mu(x,t)$, 4-velocity $u^{\alpha}(x,t)$, magnetic field $B^i(x,t)$ or metric tensor $g_{\alpha\beta}$) is the difference between the value that the quantity Γ takes in the real space-time and the value in the background at a given point [2]. In order to determine the perturbation in Γ, we must have a way to compare Γ (tensor on the real space-time) with $\Gamma^{(0)}$ (being $\Gamma^{(0)}$ the value on $\mathcal{M}_0$). This requires the assumption to identify points of $\mathcal{M}$ with those of $\mathcal{M}_0$. This is accomplished by assigning a mapping between these space-times called *gauge choice* given by a function $\mathcal{X} : \mathcal{M}_0(p) \longrightarrow \mathcal{M}(\bar{p})$ for any point $p \in \mathcal{M}_0$ and $\bar{p} \in \mathcal{M}$, which generate a pull-back

$$\mathcal{X}^* : \begin{array}{c} \mathcal{M} \\ T^*(\overline{p}) \end{array} \longrightarrow \begin{array}{c} \mathcal{M}_0 \\ T^*(p) \end{array}, \tag{4.32}$$

[2] This difference should be taken in the same physical point.

thus, points on the real and background space-time can be compared through of $\mathcal{X}$. Then, the perturbation for Γ is defined as

$$\delta\Gamma(p) = \Gamma(\bar{p}) - \Gamma^{(0)}(p). \tag{4.33}$$

We see that the perturbation $\delta\Gamma$ is completely dependent of the gauge choice because the mapping determines the representation on $\mathcal{M}_0$ of $\Gamma(\bar{p})$. However, one can also choose another correspondence $\mathcal{Y}$ between these space-times so that $\mathcal{Y} : \mathcal{M}_0(q) \to \mathcal{M}(\overline{p})$, $(p \neq q)$.[3] In the literature a change of this identification map is called *gauge transformation*. The freedom to choose between different correspondences is due to the general covariance in General Relativity, which states that there is no preferred coordinate system in nature [86, 93]. Hence, this freedom will generate an arbitrariness in the value of $\delta\Gamma$ at any space-time point p, which is called *gauge problem* in the general relativistic perturbation theory and has been treated by [82, 94]. This problem generates unphysical degrees of freedom to the solutions in the theory and therefore one should fix the gauge or build up non dependent quantities of the gauge.

4.4.1 Gauge transformations and gauge invariant variables

To define the perturbation to a given order, it is necessary to introduce the concept of Taylor expansion on a manifold and thus the metric and matter fields are expanded in a power series. Following [91, 95, 96], is considered a family of four-dimensional submanifolds $\mathcal{M}_\lambda$ with $\lambda \in \mathbb{R}$, embedded in a 5-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$. Each submanifold in the family represents a perturbed space-time and the background space-time is represented by the manifold $\mathcal{M}_0$ ($\lambda = 0$). On these manifolds we consider that the Einstein field and Maxwell's equations are satisfied

$$\mathtt{E}[g_\lambda, T_\lambda] = 0 \quad \text{and} \quad M[F_\lambda, J_\lambda] = 0; \tag{4.34}$$

each tensor field Γ_λ on a given manifold $\mathcal{M}_\lambda$ is extended to all manifold $\mathcal{N}$ through $\Gamma(p,\lambda) \equiv \Gamma_\lambda(p)$ to any $p \in \mathcal{M}_\lambda$ likewise the above equations are extended to $\mathcal{N}$.[4] We used a diffeomorphism such that the difference in the right side of equation (4.33) can be done. Is introduced an one-parameter group of diffeomorphisms $\mathcal{X}_\lambda$ which identifies points in the background with points in the real space-time labeled with the value λ. Each $\mathcal{X}_\lambda$ is a member of a flow $\mathcal{X}$ on $\mathcal{N}$ and it specifies a vector field X with the property $X^4 = 1$ everywhere (transverse to the $\mathcal{M}_\lambda$)[5] then points which lie on the same integral curve of X have to be regarded as the same point [82]. Therefore, according to the above, one gets a definition for the tensor perturbation

$$\Delta\Gamma_\lambda \equiv \mathcal{X}^*_\lambda \Gamma|_{\mathcal{M}_0} - \Gamma_0. \tag{4.35}$$

At higher orders the Taylor expansion is given by [95],

$$\Delta^X\Gamma_\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\delta^{(k)}_X\Gamma - \Gamma_0 = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!}\delta^{(k)}_X\Gamma, \tag{4.36}$$

where

$$\delta^{(k)}_X\Gamma = \left[\frac{\mathrm{d}^k\mathcal{X}^*_\lambda\Gamma}{\mathrm{d}\lambda^k}\right]_{\lambda=0,\mathcal{M}_0}. \tag{4.37}$$

[3] This is the active approach where transformations of the perturbed quantities are evaluated at the same coordinate point.

[4] In (4.34), g_λ and T_λ are the metric and the matter fields on $\mathcal{M}_\lambda$, similarly F_λ and J_λ are the electromagnetic field and the four-current on $\mathcal{M}_\lambda$.

[5] Here we introduce a coordinate system x^α through a chart on $\mathcal{M}_\lambda$ with $\alpha = 0,1,2,3$, thus, giving a vector field on $\mathcal{N}$, which has the property that $X^4 = \lambda$ in this chart, while the other components remain arbitrary.

Now, rewriting equation (4.35) we get

$$\mathcal{X}_\lambda^*\Gamma|_{\mathcal{M}_0} = \Gamma_0 + \lambda\delta^{(1)}_{\mathcal{X}}\Gamma + \frac{\lambda^2}{2}\delta^{(2)}_{\mathcal{X}}\Gamma + O(\lambda^3), \tag{4.38}$$

Notice in the equations (4.37) and (4.38) the representation of Γ on $\mathcal{M}_0$ is splitting in the background value Γ_0 plus $O(k)$ perturbations in the gauge $\mathcal{X}_\lambda$. Therefore, the k-th order $O(k)$ in Γ depends on gauge $\mathcal{X}$. With this description the "perturbations are fields lie in the background". The first term in equation (4.35) admits an expansion around $\lambda = 0$ given by [95]

$$\mathcal{X}_\lambda^*\Gamma|_{\mathcal{M}_0} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_X^k\Gamma|_{\mathcal{M}_0} = \exp\left(\lambda\mathcal{L}_X\right)\Gamma|_{\mathcal{M}_0}, \tag{4.39}$$

where $\mathcal{L}_X\Gamma$ is the Lie derivative of Γ with respect to a vector field X that generates the flow $\mathcal{X}$. If we define $\mathcal{X}_\lambda^*\Gamma|_{\mathcal{M}_0} \equiv \Gamma_\lambda^{\mathcal{X}}$ and proceeding in the same way for another gauge choice $\mathcal{Y}$, using equations (4.35)-(4.39), the tensor fields $\Gamma_\lambda^{\mathcal{X},\mathcal{Y}}$ can be written as

$$\Gamma_\lambda^{\mathcal{X}} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{X}}\Gamma = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_X^k\Gamma|_{\mathcal{M}_0}, \tag{4.40}$$

$$\Gamma_\lambda^{\mathcal{Y}} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{Y}}\Gamma = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_Y^k\Gamma|_{\mathcal{M}_0}, \tag{4.41}$$

if $\Gamma_\lambda^{\mathcal{X}} = \Gamma_\lambda^{\mathcal{Y}}$ for any arbitrary gauge $\mathcal{X}$ and $\mathcal{Y}$, from here it is clear that Γ is totally gauge invariant. It is also clear that Γ is gauge invariant to order $n \geqslant 1$ if only if satisfy $\delta^{(k)}_{\mathcal{Y}}\Gamma = \delta^{(k)}_{\mathcal{X}}\Gamma$, or in other way

$$\mathcal{L}_X\delta^{(k)}\Gamma = 0, \tag{4.42}$$

for any vector field X and $\forall k < n$. To first order ($k = 1$) any scalar that is constant in the background or any tensor that vanished in the background are gauge invariant. This result is known as *Stewart-Walker Lemma* [91], i.e., equation (4.42) generalizes this Lemma. However, when Γ is not gauge invariant and there are two gauge choices $\mathcal{X}_\lambda$, $\mathcal{Y}_\lambda$, the representation of $\Gamma|_{\mathcal{M}_0}$ is different depending of the used gauge. To transform the representation from a gauge choice $\mathcal{X}_\lambda^*\Gamma|_{\mathcal{M}_0}$ to another $\mathcal{Y}_\lambda^*\Gamma|_{\mathcal{M}_0}$ as with the map $\Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0$ given by

$$\Phi_\lambda \equiv \mathcal{X}_{-\lambda}\circ\mathcal{Y}_\lambda \Rightarrow \Gamma_\lambda^{\mathcal{Y}} = \Phi_\lambda^*\Gamma_\lambda^{\mathcal{X}}, \tag{4.43}$$

as a consequence, the diffeomorphism Φ_λ induce a pull-back Φ_λ^* which changes the representation $\Gamma_\lambda^{\mathcal{X}}$ of Γ in a gauge $\mathcal{X}_\lambda$ to the representation $\Gamma_\lambda^{\mathcal{Y}}$ of Γ in a gauge $\mathcal{Y}_\lambda$. Now, following [97] and using the Baker-Campbell-Hausdorff formula [98], one can generalize equation (4.39) to write $\Phi_\lambda^*\Gamma_\lambda^{\mathcal{X}}$ in the following way

$$\Phi_\lambda^*\Gamma_\lambda^{\mathcal{X}} = \exp\left(\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_{\xi_k}\right)\Gamma_\lambda^{\mathcal{X}}, \tag{4.44}$$

where ξ_k is any vector field on $\mathcal{M}_\lambda$. Substituting (4.44) in (4.43), we have explicitly that

$$\Gamma_\lambda^{\mathcal{Y}} = \Gamma_\lambda^{\mathcal{X}} + \lambda\mathcal{L}_{\xi_1}\Gamma_\lambda^{\mathcal{X}} + \frac{\lambda^2}{2}\left(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2}\right)\Gamma_\lambda^{\mathcal{X}} + O(\lambda^3). \tag{4.45}$$

Replacing (4.40) and (4.41) into (4.45), the relations to first and second order perturbations of Γ in two different gauge choices are given by

$$\delta^{(1)}_{\mathcal{Y}}\Gamma - \delta^{(1)}_{\mathcal{X}}\Gamma = \mathcal{L}_{\xi_1}\Gamma_0, \tag{4.46}$$

$$\delta^{(2)}_{\mathcal{Y}}\Gamma - \delta^{(2)}_{\mathcal{X}}\Gamma = 2\mathcal{L}_{\xi_1}\delta^{(1)}_{\mathcal{X}}\Gamma_0 + \left(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2}\right)\Gamma_0, \tag{4.47}$$

where the generators of the gauge transformation Φ are

$$\xi_1 = Y - X \quad \text{and} \quad \xi_2 = [X, Y]\,. \tag{4.48}$$

This vector field can be split in their time and space part

$$\xi^{(r)}_\mu \to \left(\alpha^{(r)}, \partial_i\beta^{(r)} + d^{(r)}_i\right), \tag{4.49}$$

here $\alpha^{(r)}$ and $\beta^{(r)}$ are arbitrary scalar functions, and $\partial^i d^{(r)}_i = 0$. The function $\alpha_{(r)}$ determines the choice of constant time hypersurfaces, while $\partial_i\beta^{(r)}$ and $d^{(r)}_i$ fix the spatial coordinates within these hypersurfaces. The choice of coordinates is arbitrary and the definitions of perturbations are thus gauge dependent. The gauge transformation given by the equations (4.46) and (4.47) are quite general. To first order Γ is gauge invariant if $\mathcal{L}_{\xi_1}\Gamma_0 = 0$, while to second order one must have other conditions $\mathcal{L}_{\xi_1}\delta^{(1)}_{\mathcal{X}}\Gamma_0 = \mathcal{L}^2_{\xi_1}\Gamma_0 = 0$ and $\mathcal{L}_{\xi_2}\Gamma_0 = 0$, and so on at high orders. We will apply the formalism described above to the Robertson-Walker metric, where k does mention the expansion order.

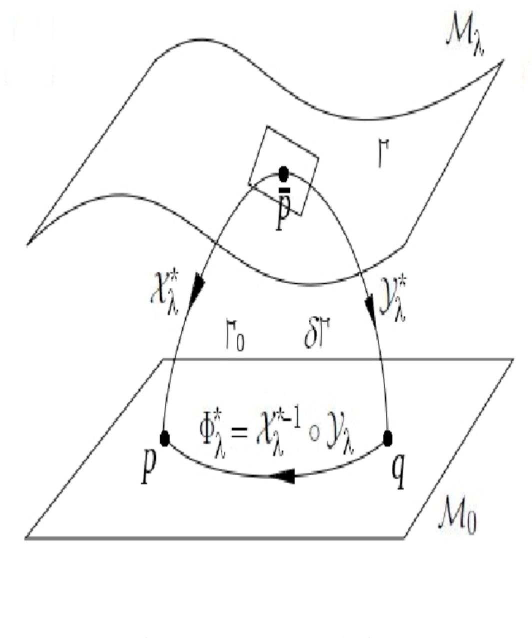

Figure 4.1: Gauge choice

4.5 Cosmic magnetic fields

Magnetic fields have been observed on several scales in the universe. Galaxies and clusters of galaxies contain magnetic fields with strengths of $\sim 10^{-6}$ G [99], fields within clusters are also likely to exist, with strengths of comparable magnitude [100]. There is also evidence of magnetic fields on scales of superclusters [101]. On the other hand, the possibility of cosmological magnetic field has been addressed comparing the CMB quadrupole with one induced by a constant magnetic field (in coherence scales of $\sim 1Mpc$), constraining the field magnitude to $B < 6.8 \times 10^{-9}(\Omega_m h^2)^{1/2}$ Gauss [102]. However, the origin of such large scale magnetic fields is still unknown. These fields are assumed to be increased and maintained by dynamo mechanism, but it needs a *seed* before the mechanism takes place [103]. Astrophysical mechanisms, such as the Biermann battery have been used to explain how the magnetic field is maintained in objects such as galaxies, stars and supernova remnants [104], but they are not likely correlated beyond galactic sizes [105]. It makes difficult to use astrophysical mechanisms to explain the origin of magnetic fields on cosmological scales. In order to overcome this problem, the primordial origin should be found in other scenarios from which the astrophysical mechanism starts. For example, magnetic fields could be generated during primordial phase transitions (such as QCD, the electroweak or GUT), parity-violating processes that generates magnetic helicity or during inflation [106]. Magnetic fields also are generated during the radiation era in regions with non vanishing vorticity. This seed was proposed by Harrison [106]. Magnetic fields generation from density fluctuations in pre-recombination era has been investigated in [106]. The advantage of these primordial processes is that they offer a wide range of coherence lengths (many of which are strongly constrained by Nucleosynthesis [107–109]), while the astrophysical mechanisms produce fields at the same order of the astrophysical size of the object. Recently a lower limit of the large scale correlated magnetic field was found. It constrains models for the origin of cosmic magnetic fields, giving a possible evidence for their primordial origin [110–112].

One way to describe the evolution of magnetic fields is through Cosmological Perturbation Theory and this point gave the start point for our work [49]. As it was mention previous in this chapter, this theory [80] is a powerful tool for understanding the present properties of the large-scale structure of the Universe and their origin. It has been mainly used to predict effects on the temperature distribution in the Cosmic Microwave Background (CMB) [85, 113]. Futhermore, linear perturbation theory combined with inflation suggests that primordial fluctuations of the universe are adiabatic and Gaussian [114]. However, due to the high precision measurements reached in cosmology, higher order cosmological perturbation theory is required to test the current cosmological framework [115, 116].

There are mainly two approaches to studying higher order perturbative effects: one uses nonlinear theory and different manifestations of the separate universe approximation, using the ΔN formalism [117, 118], and the other is the Bardeen approach where metric and matter fields are expanded in a power series [78]. Within the Bardeen approach, a set of variables are determined in such a way that has no gauge dependence. These are known in the literature as *gauge-invariant variables* which have been widely used in different cosmological scenarios [88]. One important result of cosmological perturbation theory is the coupling between gravity and electromagnetic fields, which have shown a magneto-geometrical interaction that could change the evolution of the fields on large scales. An effect is the amplification of cosmic fields. Indeed, large scale magnetic fields in perturbed spatially open FLRW models decay as a^{-1}, a rate considerably slower than the standard a^{-2} [119–122]. The hyperbolic geometry of these open FLRW models leads to the superadiabatic amplification on large scales [123].

The main goal in this chapter is to study the late evolution of magnetic fields that were generated in early stages of the universe. We use the cosmological perturbation theory following the Gauge Invariant formalism to find the perturbed Maxwell equations up to second order, and also we obtain a dynamo like equation written in terms of gauge invariant variables to first and second order. Futhermore, we discuss the importance that both curvature and the gravitational potential plays in the evolution of these fields.

The next section presents the matter equations in the homogeneous and isotropic universe, which is used to generate the first and second order dynamical equations. In section 4.7, we define the first order gauge invariant variables for the perturbations not only in the matter (energy density, pressure, magnetic and electric field) but also in the geometrical quantities (gravitational potential, curvature, shear ..). The first-order perturbation of the Maxwell's equations is reviewed in section 4.8 and together with the Ohm's law allows to find the cosmological dynamo equation to describe the evolution of the magnetic field. The derivation of second-order Maxwell's equations is given in section 4.10, and following the same methodology for the first-order case, we find the cosmological dynamo equation at second order written in terms of gauge invariant variables. In the section 4.12, we use an alternative approximation to the model considering a magnetic field in the FLRW background. It is found that amplification effects of magnetic field appear at first order in the equations, besides of the absence of fractional orders. Also a discussion between both approaches is done. The final section 4.13 is devoted to a discussion of the main results and the connection with future works.

4.6 FLRW background

At zero order (background), the universe is well described by a spatially flat FLRW

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right), \tag{4.50}$$

with $a(\tau)$ the scale factor with τ the conformal time. Hereafter the Greek indices run from 0 to 3, and the Latin ones run from 1 to 3 and a prime denotes the derivative with respect to τ. The Einstein tensor components in this background are given by

$$G^0_0 = -\frac{3H^2}{a^2}, \tag{4.51a}$$

$$G^i_j = -\frac{1}{a^2}\left(2\frac{a''}{a} - H^2\right)\delta^i_j, \tag{4.51b}$$

with $H = \frac{a'}{a}$ the Hubble parameter. We consider the background filled with a single barotropic fluid where the energy momentum tensor is

$$T^{\mu}_{\text{fluid }\nu} = \left(\mu_{(0)} + P_{(0)}\right) u^{\mu}_{(0)} u^{(0)}_{\nu} + P_{(0)}\delta^{\mu}_{\nu}, \tag{4.52}$$

with $\mu_{(0)}$ the energy density and $P_{(0)}$ the pressure. The comoving observers are defined by the four-velocity $u^{\nu} = (a^{-1}, 0, 0, 0)$ with $u^{\nu}u_{\nu} = -1$ and the conservation law for the fluid is

$$\mu'_{(0)} + 3H(\mu_{(0)} + P_{(0)}) = 0. \tag{4.53}$$

To deal with the magnetic field, the space-time under study is the fluid permeated by a weak magnetic field,[6] which is a stochastic field and can be treated as a perturbation on the background [124, 125]. Since the magnetic field has no background contribution, the electromagnetic energy momentum tensor is automatically gauge invariant at first order (see equation (4.46)). The spatial part of Ohm's law which is the proyected current is written by

$$\left(g_{\mu i} + u_{\mu}u_i\right) j^{\mu} = \sigma g_{\lambda i} g_{\alpha\mu} F^{\lambda\alpha} u^{\mu}, \tag{4.54}$$

[6] With the property $B^2_{(0)} \ll \mu_{(0)}$.

where $j^\mu = (\varrho, J^i)$ is the 4-current and $F^{\lambda\alpha}$ is the electromagnetic tensor given by

$$F^{\lambda\alpha} = \frac{1}{a^2(\tau)} \begin{pmatrix} 0 & E^i & E^j & E^k \\ -E^i & 0 & B^k & -B^j \\ -E^j & -B^k & 0 & B^i \\ -E^k & B^j & -B^i & 0 \end{pmatrix}. \tag{4.55}$$

At zero order in equation (4.54) the usual Ohms law is found which gives us the relation between the 3-current and the electric field

$$J_i = \sigma E_i, \tag{4.56}$$

where σ is the conductivity. Under MHD approximation, large scales the plasma is globally neutral and charge density is neglected ($\varrho = 0$) [126, 127]. If the conductivity is infinite ($\sigma \to \infty$) in the early universe [100], then equation (4.54) states that the electric field must vanish ($E_i = 0$) in order to keep the current density finite [128, 129]. However, the current also should be zero ($J_i = 0$) because a nonzero current involves a movement of charge particles that breaks down the isotropy in the background.

4.7 Gauge invariant variables at first order

We write down the perturbations on a spatially flat FLRW. The perturbative expansion at $k-$th order of the matter quantities is given by

$$\mu = \mu_{(0)} + \sum_{k=1}^{\infty} \frac{1}{k!}\mu_{(k)}, \tag{4.57}$$

$$B^2 = \sum_{k=1}^{\infty} \frac{1}{k!} B^2_{(k)}, \tag{4.58}$$

$$E^2 = \sum_{k=1}^{\infty} \frac{1}{k!} E^2_{(k)}, \tag{4.59}$$

$$P = P_{(0)} + \sum_{k=1}^{\infty} \frac{1}{k!} P_{(k)}, \tag{4.60}$$

$$B^i = \frac{1}{a^2(\tau)} \left(\sum_{k=1}^{\infty} \frac{1}{k!} B^i_{(k)} \right), \tag{4.61}$$

$$E^i = \frac{1}{a^2(\tau)} \left(\sum_{k=1}^{\infty} \frac{1}{k!} E^i_{(k)} \right), \tag{4.62}$$

$$u^\mu = \frac{1}{a(\tau)} \left(\delta^\mu_0 + \sum_{k=1}^{\infty} \frac{1}{k!} v^\mu_{(k)} \right), \tag{4.63}$$

$$j^\mu = \frac{1}{a(\tau)} \left(\sum_{k=1}^{\infty} \frac{1}{k!} j^\mu_{(k)} \right), \tag{4.64}$$

where the fields used in above formulas are the average ones (i.e. $B^2 = \langle B^2 \rangle$).[7] We also consider the perturbations about a FLRW background, so that the metric tensor is given by

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{k=1}^{\infty}\frac{1}{k!}\psi^{(k)}\right), \tag{4.65}$$

$$g_{0i} = a^2(\tau)\sum_{k=1}^{\infty}\frac{1}{k!}\omega_i^{(k)}, \tag{4.66}$$

$$g_{ij} = a^2(\tau)\left[\left(1 - 2\sum_{k=1}^{\infty}\frac{1}{k!}\phi^{(k)}\right)\delta_{ij} + \sum_{k=1}^{\infty}\frac{\chi_{ij}^{(k)}}{k!}\right]. \tag{4.67}$$

The perturbations are split into a scalar, transverse vector part, and transverse trace-free tensor

$$\omega_i^{(k)} = \partial_i \omega^{(k)\|} + \omega_i^{(k)\perp}, \tag{4.68}$$

with $\partial^i \omega_i^{(k)\perp} = 0$. Similarly we can split $\chi_{ij}^{(k)}$ as

$$\chi_{ij}^{(k)} = D_{ij}\chi^{(k)\|} + \partial_i \chi_j^{(k)\perp} + \partial_j \chi_i^{(k)\perp} + \chi_{ij}^{(k)\top}, \tag{4.69}$$

for any tensor quantity.[8] Following [131], one can find the scalar *gauge invariant variables* at first order given by

$$\Psi^{(1)} \equiv \psi^{(1)} + \frac{1}{a}\left(\mathcal{S}_{(1)}^{\|} a\right)', \tag{4.70}$$

$$\Phi^{(1)} \equiv \phi^{(1)} + \frac{1}{6}\nabla^2\chi^{(1)} - H\mathcal{S}_{(1)}^{\|}, \tag{4.71}$$

$$\Delta^{(1)} \equiv \mu_{(1)} + (\mu_{(0)})'\, \mathcal{S}_{(1)}^{\|}, \tag{4.72}$$

$$\Delta_P^{(1)} \equiv P_{(1)} + (P_{(0)})'\, \mathcal{S}_{(1)}^{\|}, \tag{4.73}$$

with $\mathcal{S}_{(1)}^{\|} \equiv \left(\omega^{\|(1)} - \frac{(\chi^{\|(1)})'}{2}\right)$ the scalar contribution of the shear. The vector modes are

$$v_{(1)}^{i} \equiv v_{(1)}^{i} + \left(\chi_{\perp(1)}^{i}\right)', \tag{4.74}$$

$$\vartheta_i^{(1)} \equiv \omega_i^{(1)} - \left(\chi_i^{\perp(1)}\right)', \tag{4.75}$$

$$\mathcal{V}_{(1)}^{i} \equiv \omega_{(1)}^{i} + v_{(1)}^{i}. \tag{4.76}$$

Other gauge invariant variables are the 3-current, the charge density and the electric and magnetic fields, because they vanish in the background. The tensor quantities are also gauge invariant because they are null in the background (see equation (4.46)).

[7] This happens because the average evolves exactly like B^2 [130].
[8] With $\partial^i \chi_{ij}^{(k)\top} = 0$, $\chi_i^{(k)i} = 0$ and $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_k\partial^k$.

4.7.1 The Ohm law and the energy momentum tensor

Using (4.54) the Ohm law at first order is

$$J_i^{(1)} = \sigma E_i^{(1)}. \tag{4.77}$$

As the conductivity of the medium finite (real MHD), the electric field and the 3-current are nonzero. Now, the electromagnetic energy momentum tensor is

$$\begin{aligned} T^0_{(em)\,0} &= -\frac{1}{8\pi}\left(B^2_{(1)} + E^2_{(1)}\right), \\ T^i_{(em)0} &= 0, \\ T^0_{(em)i} &= 0, \\ T^i_{(em)\,l} &= \frac{1}{4\pi}\Big[\frac{1}{6}\left(B^2_{(1)} + E^2_{(1)}\right)\delta^i{}_l + \Pi^{i(1)}_{l(em)}\Big], \end{aligned} \tag{4.78}$$

where

$$\Pi^{i(1)}_{l(em)} = \frac{1}{3}\left(B^2 + E^2\right)\delta^i{}_l - B_l B^i - E_l E^i \tag{4.79}$$

is the anisotropic stress tensor that is gauge invariant by definition (4.46). This term is important to constrain the total magnetic energy because it is source of gravitational waves [107–109]. We can see that the electromagnetic energy density appears like a quadratic term in the energy momentum tensor, which means that the electromagnetic field should be regarded as one half order perturbation.[9] Using (4.52) and considering the fluctuations of the matter fields, equations (4.57) and (4.60), the energy momentum tensor for the fluid is given by

$$T^0_{\text{fluid}\,0} = -\Delta^{(1)} + (\mu_{(0)})'\,\mathcal{S}^{||}_{(1)}, \tag{4.80}$$

$$T^i_{\text{fluid}\,0} = (\mu_0 + P_0)\left(\mathcal{V}^i_{(1)} - \vartheta^i_{(1)} - \left(\chi^i_{\perp(1)}\right)'\right), \tag{4.81}$$

$$T^0_{\text{fluid}\,i} = -(\mu_0 + P_0)\,\mathcal{V}^{(1)}_i, \tag{4.82}$$

$$T^i_{\text{fluid}\,j} = \left(\Delta^{(1)}_P - (P_{(0)})'\,\mathcal{S}^{||}_{(1)}\right)\delta^i{}_j + \Pi^{i(1)}_{j(fl)}, \tag{4.83}$$

where $\Pi^{i(1)}_{j(f)}$ is the anisotropic stress tensor [134]. The above equations are written in terms of gauge invariant variables plus terms as $\mathcal{S}^{||}_{(1)}$ that depend of the gauge choice.

4.7.2 The conservation equations

The total energy momentum conservation equation $\mathcal{T}^{\alpha}_{\beta\alpha} = 0$ can be split in each component that is not necessarily conserved independently

$$\mathcal{T}^{\alpha}_{\beta;\alpha} = T^{\alpha(f)}_{\beta;\alpha} + T^{\alpha(E.M)}_{\beta;\alpha} = 0, \tag{4.84}$$

where

$$T^{\alpha(E.M)}_{\beta;\alpha} = F_{\beta\alpha}\,j^{\alpha}. \tag{4.85}$$

[9] Therefore the magnetic field should be split as $B^i = \frac{1}{a(\tau)^2}\left(B^i_{(\frac{1}{2})} + B^i_{(1)} + B^i_{(\frac{3}{2})} + ...\right)$, see [132–134].

Using the equations (4.80) and (4.83), the continuity equation $\mathcal{T}^{\alpha}_{0;\alpha} = 0$ is given by

$$\begin{aligned}\left(\Delta^{(1)}\right)' + 3H\left(\Delta_P^{(1)} + \Delta^{(1)}\right) - 3\left(\Phi^{(1)}\right)'\left(P_{(0)} + \mu_{(0)}\right) + \left(P_{(0)} + \mu_{(0)}\right)\nabla^2 \upsilon^{(1)} - 3H\left(P_{(0)} + \mu_{(0)}\right)' \mathcal{S}^{\|}_{(1)} \\ - \left(\left(\mu_{(0)}\right)' \mathcal{S}^{\|}_{(1)}\right)' + \left(P_{(0)} + \mu_{(0)}\right)\left(-\frac{1}{2}\nabla^2\chi^{(1)} + 3H\mathcal{S}^{\|}_{(1)}\right)' - \left(P_{(0)} + \mu_{(0)}\right)\nabla^2\left(\frac{1}{2}\chi^{\|(1)}\right)' = 0.\end{aligned} \tag{4.86}$$

The Navier-Stokes equation $\mathcal{T}^{\alpha}_{i;\alpha} = 0$ is

$$\left(\mathcal{V}_i^{(1)}\right)' + \frac{\left(\mu_{(0)} + P_{(0)}\right)'}{\left(\mu_{(0)} + P_{(0)}\right)}\mathcal{V}_i^{(1)} + 4H\mathcal{V}_i^{(1)} + \partial_i\Psi^{(1)} + \frac{\partial_i\left(\Delta_P^{(1)} - \left(P_{(0)}\right)'\mathcal{S}^{\|}_{(1)}\right) + \partial_l\Pi^{(1)l}_{(f)i}}{\left(\mu_{(0)} + P_{(0)}\right)} - \partial_i\frac{1}{a}\left(\mathcal{S}^{\|}_{(1)}a\right)' = 0. \tag{4.87}$$

The last equations are written in terms of gauge invariant variables in according to [86, 107–109, 134–136]. It is shown there is not exist contribution of electromagnetic terms to the conservation equations. The energy-momentum tensor of each component is not conserved independently and its divergence has a source term that takes into account the energy and momentum transfer between the components of the photon, electron, proton and the electromagnetic field $T^{\alpha(f)}_{\beta;\alpha} = K_\beta$.

4.8 Maxwell equations and the cosmological dynamo equation

The Maxwell's equations are written as

$$\nabla_\alpha F^{\alpha\beta} = j^\beta, \quad \nabla_{[\gamma}F_{\alpha\beta]} = 0. \tag{4.88}$$

Using (B.90) and the perturbation equations for the metric and electromagnetic fields, the non-homogeneous Maxwell equations are

$$\partial_i E^i_{(1)} = a\varrho_{(1)}, \tag{4.89}$$
$$\epsilon^{ilk}\partial_l B_k^{(1)} = \left(E^i_{(1)}\right)' + 2HE^i_{(1)} + aJ^i_{(1)}, \tag{4.90}$$

and the homogeneous Maxwell equations

$$B'_{k(1)} + 2HB_k^{(1)} + \epsilon_k^{\ ij}\partial_i E_j^{(1)} = 0, \tag{4.91}$$
$$\partial^i B_i^{(1)} = 0, \tag{4.92}$$

written also by [137–140]. Now using the last equations together with the ohm's law (4.77), we get an equation which describes the evolution of magnetic field at first order, this relation is the *dynamo equation*:

$$\left(B_k^{(1)}\right)' + 2HB_k^{(1)} + \eta\left[\nabla\times\left(\nabla\times B^{(1)} - \left(E^{(1)}\right)' - 2HE^{(1)}\right)\right]_k = 0, \tag{4.93}$$

with $\eta = \frac{1}{4\pi\sigma}$ the diffusion coefficient. Equation (4.93) is similar to dynamo equation in MHD but it is in the cosmological context [100]. This equation has one term that depends on η which takes into account the dissipation phenomena of the magnetic field (the electric field in this term in general is dropped if we neglect the displacement current). Notice that η is a expansion parameter (due to σ is large). From equation (4.93) we see that for finite η, the diffusion term should not be neglected. Care should be taken the assumption $\eta = 0$, because it could break at small scales [126, 127]. In the frozen in condition of magnetic field lines, where amplification of the field is not taking account, the last equation has the solution $\mathbf{B} = \frac{\mathbf{B}_0}{a^2(\tau)}$ where $\mathbf{B}_0$ is the actual magnetic field, the actual value of the scale factor $a_0(\tau) = 1$

and **B** is the magnetic field when the scale factor was $a(\tau)$.

4.9 Generalization at second order

Following [91, 95, 96] the variable $\delta^{(2)}\boldsymbol{T}$ defined by

$$\delta^{(2)}_{X}\boldsymbol{T} \equiv \delta^{(2)}_{X}\Gamma - 2L_{X}\left(\delta^{(1)}_{X}\Gamma\right) + L^{2}_{X}\Gamma_{0}; \tag{4.94}$$

is introduced. Inspecting the gauge transformation (4.47) one can see that $\delta^{(2)}\boldsymbol{T}$ is transformed as

$$\delta^{(2)}_{\mathcal{Y}}\boldsymbol{T} - \delta^{(2)}_{X}\boldsymbol{T} = L_{\sigma}\Gamma_{0}, \tag{4.95}$$

with $\sigma = \xi_{2} + [\xi_{1}, X]$ and X is the gauge dependence part in linear order perturbation. The gauge transformation rule (4.95) is identical to the gauge transformation at linear order (4.46). This property is general and is the key to extend this theory to second order

$$L\left[\delta^{2}\boldsymbol{T}\right] = S\left[\delta\boldsymbol{T}, \delta\boldsymbol{T}\right]. \tag{4.96}$$

Notice that first and second order equations are similar, however the last have as sources the coupling between linear perturbations variables. Using the equation (4.95) we arrive to the gauge invariant quantities at second order. This coupling appearing as the quadratic terms of the linear perturbation is due to the nonlinear effects of the Einstein field equations, besides one can classify them again in scalar, vector and tensor modes, where this modes couple with each other. Now, to clarify the physical behavior of perturbations at this order we should obtain the gauge invariant quantities and express these equations of motion in terms of these quantities. The scalar gauge invariants are given by

$$\Psi^{(2)} \equiv \psi^{(2)} + \frac{1}{a}\left(\mathcal{S}^{\parallel}_{(2)}a\right)' + \mathcal{T}^{1}(O^{(2)}), \tag{4.97}$$

$$\Phi^{(2)} \equiv \phi^{(2)} + \frac{1}{6}\nabla^{2}\chi^{(2)} - H\mathcal{S}^{\parallel}_{(2)} + \mathcal{T}^{2}(O^{(2)}), \tag{4.98}$$

$$\Delta^{(2)}_{\mu} \equiv \mu_{(2)} + (\mu_{(0)})'\,\mathcal{S}^{\parallel}_{(2)} + \mathcal{T}^{3}(O^{(2)}), \tag{4.99}$$

$$\Delta^{(2)}_{\varrho} \equiv \varrho^{(2)} + \mathcal{T}^{4}(O^{(2)}), \tag{4.100}$$

$$\Delta^{(2)}_{B} \equiv B^{2}_{(2)} + \mathcal{T}^{4}(O^{(2)}), \tag{4.101}$$

$$\Delta^{(2)}_{E} \equiv E^{2}_{(2)} + \mathcal{T}^{6}(O^{(2)}), \tag{4.102}$$

$$\upsilon^{(2)} \equiv \upsilon^{(2)} + \left(\frac{1}{2}\chi^{\parallel(2)}\right)' + \mathcal{T}^{7}(O^{(2)}), \tag{4.103}$$

with $\mathcal{S}^{\parallel}_{(2)} \equiv \left(\omega^{\parallel(2)} - \frac{(\chi^{\parallel(2)})'}{2}\right) + \mathcal{T}^{8}(O^{(2)})$. The expression for $\mathcal{T}^{8}(O^{(2)})$ is given in 4.13.1. In this case $\mathcal{S}^{\parallel}_{(2)}$ can be interpreted like shear at second order. Again it is showed that it is similar to found at first order but it has a source term which is quadratic in the first order functions of the transformations. The vector modes found are as follows

$$\begin{aligned} \upsilon^i_{(2)} &\equiv v^i_{(2)} + \left(\chi^i_{\perp(2)}\right)' + \mathcal{T}^9(O^{(2)}), && (4.104)\\ \vartheta^{(2)}_i &\equiv \omega^{(2)}_i - \left(\chi^{\perp(2)}_i\right)' + \mathcal{T}^{10}(O^{(2)}), && (4.105)\\ \mathcal{V}^i_{(2)} &\equiv \omega^i_{(2)} + v^i_{(2)} + \mathcal{T}^{11}(O^{(2)}), && (4.106)\\ \Pi^{(2)T}_{ij} &\equiv \Pi^{(2)fl}_{ij} + \Pi^{(2)em}_{ij} + \mathcal{T}^{13}(O^{(2)}), && (4.107)\end{aligned}$$

The electromagnetic fields modes (from $F^{\lambda\alpha}$) are then given by

$$\begin{aligned} \mathcal{E}^{(2)}_i &= E^{(2)}_i + 2\left[\frac{1}{a^2}\left(a^2E^{(1)}_i\alpha^{(1)}\right)' + \left(\xi'_{(1)}\times B^{(1)}\right)_i\right.\\ &+ \left.\xi^l_{(1)}\partial_l E^{(1)}_i + E^{(1)}_l\partial_i\xi^l_{(1)}\right], && (4.108)\\ \mathcal{B}^{(2)}_i &= B^{(2)}_i + 2\left[\frac{\alpha^{(1)}}{a^2}\left(a^2B^{(1)}_i\right)' + \xi^l_{(1)}\partial_l B^{(1)}_i\right.\\ &+ \left.B^{(1)}_i\partial_l\xi^l_{(1)} + \left(E^{(1)}\times\nabla\alpha^{(1)}\right)_i - B^{(1)}_l\partial^l\xi^{(1)}_i\right], && (4.109)\\ \varrho^{(2)}_{(Inv.)} &= \varrho^{(2)} + 2\left[\left(\varrho'_{(1)} - H\varrho^{(1)}\right)\alpha^{(1)} + \xi^i_{(1)}\partial_i\varrho^{(1)}\right.\\ &- \left.\alpha'_{(1)}\varrho^{(1)} - J^i_{(1)}\partial_i\alpha^{(1)}\right], && (4.110)\\ \mathcal{J}^i_{(2)} &= J^i_{(2)} + 2\left[\left((J^i_{(1)})' - H\mathcal{J}^i_{(1)}\right)\alpha^{(1)} + \xi^l_{(1)}\partial_l J^i_{(1)}\right.\\ &- \left.\varrho^{(1)}(\xi^i_{(1)})' - J^l_{(1)}\partial_l\xi^i\right], && (4.111)\end{aligned}$$

which are gauge invariant quantities for electromagnetic fields. All these variables are similar to the quantities obtained at first order, but in second order case appear as sources as $T^k(O^{(2)})$ that depend of the gauge choice and the coupling with terms of first order. The explicit calculation of $\mathcal{T}^k(O^{(2)})$ is shown in [95, 131].

4.9.1 The Ohm law and the energy momentum tensor: second order

Using equations (4.54), (4.61) and (4.62), we get the Ohm law at second order

$$\begin{aligned} \mathcal{J}^{(2)}_i &= 4J^{(1)}_i\Phi^{(1)} + S^1_i(O^{(2)})\\ &+\varrho^{(1)}\upsilon^{(1)}_i + 2\sigma\left(\left(\mathcal{V}_{(1)}\times B^{(1)}\right)_i + \frac{1}{2}\mathcal{E}^{(2)}_i\right.\\ &\left.-2E^{(1)}_i\left(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)}\right) + S^2_i(O^{(2)})\right). && (4.112)\end{aligned}$$

In this case we see that 3-current has a type of Lorentz term and shows coupling between first order terms that affect the evolution of the current. Hereafter the functions $S^n_i(O^{(2)})$ with $n \in \mathbb{Z}$ and i being the component, gives us the gauge dependence. The last equation shows also a coupling between the electric field and terms like $(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)})$ that is associated to tidal forces (this quantity is similar to scalar part of the electric part of Weyl tensor) and the first right hand term between the current and perturbation in the curvature. There exist models where the coupling of the charge particles and the field is important for explaining some phenomena like collapse or generation of magnetic field during recombination period. In this case, the Ohm law shown in (4.112) should be generalized and terms like

Biermann battery and Hall effect should appear. Doing the expansion at second order in the fluid energy momentum tensor, one finds the following expressions

$$\begin{aligned}
T^{0}_{(2)\,0} &= -\frac{\Delta^{(2)}_{\mu}}{2} - (\mu_{(0)} + P_{(0)})\left(\upsilon^{(1)}_{l}\upsilon^{l}_{(1)} + \vartheta^{(1)}_{l}\upsilon^{l}_{(1)}\right) \\
&+ \boldsymbol{S}^{3}(\mathcal{O}^{(2)}),
\end{aligned} \tag{4.113}$$

$$\begin{aligned}
T^{i}_{(2)\,0} &= -(\mu_{(0)} + P_{(0)})\left(\frac{\mathcal{V}^{i}_{(2)} - \vartheta^{i}_{(2)}}{2} + \Psi^{(1)}\upsilon^{i}_{(1)}\right) \\
&- \left(\Delta^{(1)}_{\mu} + \Delta^{(1)}_{P}\right)\upsilon^{i}_{(1)} + \boldsymbol{S}^{i}_{4}\left(\mathcal{O}^{(2)}\right),
\end{aligned} \tag{4.114}$$

$$\begin{aligned}
T^{0}_{(2)\,i} &= -(\mu_{(0)} + P_{(0)})\left(\frac{\mathcal{V}^{(2)}_{i}}{2} - 2\vartheta^{(2)}_{i}\Psi^{(1)} - 2\upsilon^{(1)}_{i}\Phi^{(1)}\right. \\
&+ \left.\upsilon^{j}_{(1)}\chi^{(1)}_{ij} - \upsilon^{(1)}_{i}\Psi^{(1)}\right) - \left(\Delta^{(1)}_{\mu} + \Delta^{(1)}_{P}\right)\mathcal{V}^{(1)}_{i} \\
&+ \boldsymbol{S}^{5}_{i}\left(\mathcal{O}^{(2)}\right),
\end{aligned} \tag{4.115}$$

$$\begin{aligned}
T^{i}_{(2)\,j} &= \frac{1}{2}\Delta^{(2)}_{P}\delta^{i}{}_{j} + \frac{1}{2}\Pi^{i(2)}_{j} + \boldsymbol{S}^{6i}_{j}\left(\mathcal{O}^{(2)}\right) \\
&+ (\mu_{(0)} + P_{(0)})\left(\upsilon^{(1)}_{j}\upsilon^{i}_{(1)} + \vartheta^{(1)}_{j}\upsilon^{i}_{(1)}\right),
\end{aligned} \tag{4.116}$$

similar to [86, 135, 136]. Now considering (4.91) the electromagnetic momentum tensor at second order is

$$T^{0}_{(em)\,0} = -\frac{1}{8\pi}\left(\Delta^{(2)}_{E} + \Delta^{(2)}_{B} + \boldsymbol{S}^{8}(\mathcal{O}^{(2)})\right), \tag{4.117}$$

$$T^{i}_{(em)0} = \frac{1}{4\pi}\left[-\epsilon^{ikm}E^{(1)}_{k}B^{m}_{(1)} + \boldsymbol{S}^{i}_{\boldsymbol{9}}(\mathcal{O}^{(2)})\right], \tag{4.118}$$

$$T^{0}_{(em)i} = \frac{1}{4\pi}\left[\epsilon^{km}_{i}E^{(1)}_{k}B^{m}_{(1)} + \boldsymbol{S}_{10i}(\mathcal{O}^{(2)})\right], \tag{4.119}$$

$$\begin{aligned}
T^{i}_{(em)\,l} &= \frac{1}{4\pi}\left[\frac{1}{6}\left(\Delta^{(2)}_{E} + \Delta^{(2)}_{B} + \boldsymbol{S}^{i}_{4l}(\mathcal{O}^{(1)})\right)\delta^{i}{}_{l}\right. \\
&+ \left.\Pi^{i(2)}_{l(em)} + \boldsymbol{S}^{i}_{11l}\left(\mathcal{O}^{(2)}\right)\right].
\end{aligned} \tag{4.120}$$

Using (4.84) the continuity equation is given by

$$\begin{aligned}
\left(\Delta^{(2)}_{\mu}\right)' &+ 3H\left(\Delta^{(2)}_{P} + \Delta^{(2)}_{\mu}\right) - 3\left(\Phi^{(2)}\right)'(P_{(0)} + \mu_{(0)}) \\
&+ (P_{(0)} + \mu_{(0)})\nabla^{2}\upsilon^{(2)} = -a^{4}\left(2E^{(1)}_{i}J^{i}_{(1)}\right) \\
&- \boldsymbol{S}_{12}\left(\mathcal{O}^{(2)}\right),
\end{aligned} \tag{4.121}$$

and the Navier-stokes equation

$$\begin{aligned}
&\frac{1}{2}\frac{\left[\mu_{(0)}(1+w)\mathcal{V}^{(2)}_{i}\right]'}{\mu_{(0)}(1+w)} + 2H\mathcal{V}^{(2)}_{i} + \frac{1}{2}\frac{\partial_{i}P^{(2)} + 2\partial_{j}\Pi^{j(2)}_{i}}{\mu_{(0)}(1+w)} \\
+ &\frac{1}{2}\partial_{i}\Psi^{(2)} + \boldsymbol{S}^{13}_{i}\left(\mathcal{O}^{(2)}\right) = \frac{a^{4}\left(E^{(1)}_{i}\varrho_{(1)} + \epsilon_{ijk}J^{j}_{(1)}B^{(1)}_{k}\right)}{\mu_{(0)}(1+w)},
\end{aligned} \tag{4.122}$$

where $w = \frac{P_{(0)}}{\mu_{(0)}}$ and S_i^{13} is shown in (4.13.2). Therefore, electromagnetic fields affect the evolution of matter energy density $\Delta_\mu^{(2)}$ and the peculiar velocity $\mathcal{V}_i^{(2)}$ also, these fields influence the large structure formation and can leave imprints on the temperature anisotropy pattern of the CMB [86, 135, 137, 139].

4.10 The Maxwell equations and the cosmological dynamo at second order

Using the (4.77), the non homogeneous Maxwell's equations are

$$\begin{aligned} \partial_i \mathcal{E}^i_{(2)} &= -4E^i_{(1)}\partial_i\left(\Psi^{(1)} - 3\Phi^{(1)}\right) + a\Delta_\varrho^{(2)} \\ &- S_{14}\left(O^{(2)}\right), \end{aligned} \tag{4.123}$$

$$\begin{aligned} \left(\nabla\times\mathcal{B}^{(2)}\right)^i &= 2E^i_{(1)}\left(2\left(\Psi^{(1)}\right)' - 6\left(\Phi^{(1)}\right)'\right) + \left(\mathcal{E}^i_{(2)}\right)' \\ &+ 2H\mathcal{E}^i_{(2)} + 2\left(2\Psi^{(1)} - 6\Phi^{(1)}\right)\left(\nabla\times B_{(1)}\right)^i \\ &+ a\mathcal{J}^i_{(2)} + S^i_{15}\left(O^{(2)}\right). \end{aligned} \tag{4.124}$$

While the homogeneous Maxwell's equations are

$$\frac{1}{a^2}\left(a^2\mathcal{B}_k^{(2)}\right)' + \left(\nabla\times\mathcal{E}_{j(2)}\right)_k = -S_k^{17}\left(O^{(2)}\right), \tag{4.125}$$

$$\partial_i\mathcal{B}^{i(2)} = 0. \tag{4.126}$$

Again the S_k^n terms carry out the gauge dependence. Using the Maxwell equations together with the Ohm law at second order and following the same methodology for the first order case, we get the cosmological dynamo equation that describes the evolution of the magnetic field at second order

$$\begin{aligned} &\left(\mathcal{B}_k^{(2)}\right)' + 2H\left(\mathcal{B}_k^{(2)}\right) + \eta\Bigg[\nabla\times\Bigg(\frac{1}{a}\Big(\left(\nabla\times\mathcal{B}^{(2)}\right) - 2E_{(1)}\left(2\left(\Psi^{(1)}\right)' - 6\left(\Phi^{(1)}\right)'\right) \\ &- \left(\mathcal{E}_{(2)}\right)' - 2H\mathcal{E}_{(2)} - 2\left(\nabla\times B_{(1)}\left(2\Psi^{(1)} - 6\Phi^{(1)}\right)\right) - S_{15}\left(O^{(1)}\right)\Big) - \varrho^{(1)}v^{(1)} + S^1\left(O^{(2)}\right)\Bigg)\Bigg]_k \\ &+ \left(\nabla\times\left[-2\left(\mathcal{V}_{(1)}\times B^{(1)}\right) - 2E^{(1)}\Psi^{(1)} - 2S^2\left(O^{(1)}\right)\right]\right)_k = -S_k^{17}\left(O^{(2)}\right), \end{aligned} \tag{4.127}$$

where the value of $\varrho^{(1)}$ can be found to resolve the differential equation given in a (4.13.2). Thus the perturbations in the space-time play an important role in the evolution of primordial magnetic fields. The equations (4.93) and (4.127) are dependent on geometrical quantities (perturbation in the gravitational potential, curvature, velocity ...). These quantities evolve according to the Einstein field equations (the Einstein field equation to second order are given in [82]). In this way, equation (4.127) tells us how the magnetic field evolves according to the scale of the perturbation. In sub-horizon scale, the density contrast and the geometrical quantities grow. Hence, the dynamo term should amplify the magnetic field. As a comment we point out that in order to solve the dynamo like-equation for the magnetic field is necessary to solve the Einstein field equations to the second order together with the conservation equations.

4.11 Specifying to Poisson gauge

It is possible to fix the four degrees of freedom by imposing gauge conditions. If we impose the gauge restrictions

$$\partial^i \omega_i^{(r)} = \partial^i \chi_{ij}^{(r)} = 0, \tag{4.128}$$

all equations can be written in terms in quantities independent of the coordinates [79]. This gauge is called *Poisson gauge* and it is the gravitational analogue of Coulomb gauge in electromagnetism (see 4.13.1). The perturbed metric in the Poisson gauge reads

$$\begin{aligned} g_{00} &= -a^2(\tau)\left(1+2\psi^{(1)}+\psi^{(2)}\right), \\ g_{ij} &= a^2(\tau)\left[\left(1-2\phi^{(1)}-\phi^{(2)}\right)\delta_{ij}+\frac{2\chi_{ij}^{(1)\top}+\chi_{ij}^{(2)\top}}{2}\right], \\ g_{0i} &= a^2(\tau)\left(\omega_i^{(1)\perp}+\frac{\omega_i^{(2)\perp}}{2}\right), \end{aligned} \tag{4.129}$$

where $\omega^{\|}, \chi^{\|}, \chi_i^{\perp}$ are null. In this case the dynamo equation in the Poisson gauge is given by

$$\begin{aligned} B'_{k(2)} &+ 2HB_k^{(2)}+\eta\Big[\nabla\times(\nabla\times B_{(2)})-\left(\nabla\times E'_{(2)}\right)-2H\left(\nabla\times E^{(2)}\right)-4\left(\Psi'_{(1)}-3\Phi'_{(1)}\right)(\nabla\times E_{(1)}) \\ &- 4\nabla\left(\Psi'_{(1)}-3\Phi'_{(1)}\right)\times E^{(1)}+4\left(\nabla\times\left(\nabla\left(\Psi^{(1)}-3\Phi^{(1)}\right)\times B^{(1)}\right)\right)-4\Big(\Big(\nabla\times\left(\nabla\times B^{(1)}\right) \\ &- \nabla\times E'_{(1)}-2H\left(\nabla\times E^{(1)}\right)\Big)\Phi^{(1)}+\nabla\Phi^{(1)}\times\left(\nabla\times B^{(1)}-E'_{(1)}-2HE_{(1)}\right)\Big)-\left(\nabla\varrho^{(1)}\right)\times v^{(1)} \\ &+ 2\nabla\times\left(\left(\nabla\times B^{(1)}-\frac{1}{a^2}\left(a^2E_{(1)}\right)'\right)\cdot\chi_{(1)}^{\top}\right)-\varrho^{(1)}\left(\nabla\times v^{(1)}\right)\Big]_k-2\left(\nabla\times\left(\left(v^{(1)}+\omega_{(1)}^{\perp}\right)\times B^{(1)}\right)\right)_k \\ &+ 4\left(\left(\nabla\left(\Phi^{(1)}-\frac{\Psi^{(1)}}{2}\right)\times E^{(1)}\right)+\left(\Phi^{(1)}-\frac{\Psi^{(1)}}{2}\right)\left(\nabla\times E^{(1)}\right)\right)_k-2\nabla\times\left(E^{(1)}\cdot\chi_{(1)}^{\top}\right)_k=0, \end{aligned} \tag{4.130}$$

where $E^{(1)}\cdot\chi_{(1)}^{\top}=E_i^{(1)}\chi_{\top(1)}^{ij}$. The last equation is a specific case of the equation (4.127) where we fix the gauge (coordinate fixing). It is important to notice the relevance of the geometrical perturbation quantities in the evolution of the magnetic fields, again we see the influence of the tidal and Lorentz forces in the amplification of the fields. In some sense, the above equation differs from equation (4.127) due to the fact that we fix the adequately the choice of the perturbation functions (we choose a gauge for writing the equation of motion without the presence of unphysical modes) while before we just wrote the equations in terms of gauge invariant quantities, which were built up with the formalism explained in the fist sections, plus terms which have in taken into account the dependence of the gauge and where we need to fix them.

4.12 Weakly magnetized FLRW-background

In this section we work a magnetized FLRW, i.e we allow the presence of a weak magnetic field into our FLRW background with the property $B_{(0)}^2 \ll \mu_{(0)}$ which must to be sufficiently random to satisfy $\langle B_i\rangle = 0$ and $\left\langle B_{(0)}^2\right\rangle = \left\langle B_i^{(0)}B_{(0)}^i\right\rangle \neq 0$ to ensure that symmetries and the evolution of the background remain unaffected. Again we work under MHD approximation, and thus in large scales the plasma is globally neutral, charge density is neglected and the electric field with the current should be zero, thus the only zero order magnetic variable is $B_{(0)}^2$ [125]. The evolution of density magnetic field can be found contracting the induction equation with B_i arriving at

$$\left(B^2_{(0)}\right)' = -4HB^2_{(0)}, \tag{4.131}$$

showing $B^2 \sim a^{-4}$ in the background. Bianchi models are often used to describe the presence of a magnetic field in the universe due to anisotropic properties of this metric. However, as we are dealing with weak magnetic fields, it is worth to assuming the presence of a magnetic field in a FLRW metric as background. Indeed, the authors in [122] found that, although there is a profound distinction between the Bianchi I equations and the FLRW approximation, at the weak field limit, these differences are reduced dramatically, and therefore the linearized Bianchi equations are the same as with the FLRW ones. Under these conditions, we find that to zero order the electromagnetic energy momentum tensor in the background is given by:

$$T^{0}_{(em)\,0} = -\frac{1}{8\pi}B^2_{(0)}, \tag{4.132}$$

$$T^{0}_{(em)\,i} = T^{i}_{(em)\,0} = 0, \tag{4.133}$$

$$T^{i}_{(em)\,l} = \frac{1}{24\pi}B^2_{(0)}\delta^i{}_l. \tag{4.134}$$

The magnetic anisotropic stress is treated as a first-order perturbation due to stochastic properties of the field, therefore it does not contribute to the above equations. We can see in equations (4.52) and (4.132)-(4.134), that fluid and electromagnetic energy-momentum tensor are diagonal tensors, that is, are consistent with the condition of an isotropic and homogeneous background [125]. If we consider the average magnetic density of the background different to zero, the perturbative expansion at $k-$th order of the magnetic density is given by

$$B^2 = B^2_{(0)} + \sum_{k=1}^{\infty}\frac{1}{k!}B^2_{(k)}, \tag{4.135}$$

where at first order we get a gauge invariant term which describes the magnetic energy density

$$\Delta^{(1)}_{mag} \equiv B^2_{(1)} + \left(B^2_{(0)}\right)' \mathcal{S}^{\parallel}_{(1)}; \tag{4.136}$$

one can find that average density of the background field decays as $B^2_{(0)} \sim \frac{1}{a^4(\tau)}$ [141]. At first Gorden we work with finite conductivity (real MHD), in this case the electric field and the current becomes nonzero, therefore using the equation (4.54) and assuming the ohmic current is not neglected, we find the Ohm's law

$$J^{(1)}_i = \sigma\left[E^{(1)}_i + \left(\mathcal{V}^{(1)} \times B^{(0)}\right)_i\right]. \tag{4.137}$$

In the last equation the Lorentz force appears at first order when a magnetic field is consider as a part of the background. Again doing the same procedure described before, but taking a weak magnetic field as a contribution from the background we shall show the implication of this supposition afterwards. The electromagnetic energy momentum tensor at first order is given by

$$
\begin{aligned}
T^{0}_{(em)\,0} &= -\frac{1}{16\pi}F^{2}_{(1)}, &(4.138)\\
T^{i}_{(em)0} &= \frac{1}{4\pi}\Big[B^{2}_{(0)}\vartheta^{i}_{(1)} \\
&- \epsilon^{ikm}E^{(1)}_{k}B^{m}_{(0)} + B^{2}_{(0)}\left(\chi^{i}_{\perp(1)}\right)'\Big] &(4.139)\\
T^{0}_{(em)i} &= \frac{1}{4\pi}\left[\epsilon_{i}^{\,km}E^{(1)}_{k}B^{m}_{(0)}\right], &(4.140)\\
T^{i}_{(em)\,l} &= \frac{1}{4\pi}\left[\frac{1}{12}F^{2}_{(1)}\delta^{i}_{\;l} + \Pi^{i(1)}_{l(em)}\right], &(4.141)
\end{aligned}
$$

where

$$
\begin{aligned}
F^{2}_{(1)} &= 2\Delta^{(1)}_{(mag)} - 8\Phi^{(1)}B^{2}_{(0)} - 2\left(B^{2}_{(0)}\right)'\mathcal{S}^{\parallel}_{(1)} \\
&+ \frac{4}{3}\nabla^{2}\chi^{(1)}B^{2}_{(0)} - 8H\mathcal{S}^{\parallel}_{(1)}B^{2}_{(0)}, &(4.142)
\end{aligned}
$$

and $\Pi^{i(1)}_{l(em)} = \frac{1}{3}\left(\Delta^{(1)}_{mag} + E^{2}\right)\delta^{i}_{\;l} - B_{l}B^{i} - E_{l}E^{i}$ is the anisotropic stress that appears as a perturbation of the background, this term is important to constraining the total magnetic energy because it is a source of gravitational waves [107–109]. The above equations are written in terms of gauge invariant variables plus terms as $\mathcal{S}_{(1)}$ which are gauge dependent. Now, using the above equations (4.89), (4.91), (4.90) and (4.92) with the Ohm's law (4.137), we arrive to the dynamo equation that gives us the evolution of magnetic field to first order

$$
\left(B^{(1)}_{k}\right)' + 2HB^{(1)}_{k} + \eta\left[\nabla\times\left(\nabla\times B^{(1)} - \left(E^{(1)}\right)' - 2HE^{(1)}\right)\right]_{k} + \left(\nabla\times\left(B_{(0)}\times\mathcal{V}_{(1)}\right)\right)_{k} = 0. \qquad (4.143)
$$

When we suppose a weak magnetic field on the background, in the dynamo equation a new term called *dynamo term* appears which could amplify the magnetic field. This term depends of the evolution in $\mathcal{V}_{(1)}$, see (4.87), and also from (4.87), it seems likely when matter and velocity perturbation grow the dynamo term amplifies the magnetic field, this is a difference with the first approach where the dynamo term just appears at second order. For convenience it is better use the Lagrangian coordinates which are comoving with the local Hubble flow. [10] In this picture the magnetic field lines are frozen into the fluid [11] and we obtain the following result

$$
\frac{dB_{i}}{dt} + 2HB_{i} = B_{j}\left(\frac{\partial\mathcal{V}^{(1)}_{i}}{\partial x_{j}} - \frac{1}{3}\delta_{ij}\frac{\partial\mathcal{V}^{(1)}_{k}}{\partial x_{k}}\right) + \frac{2}{3}B_{i}\frac{\partial\mathcal{V}^{(1)}_{j}}{\partial x_{j}}, \qquad (4.145)
$$

where diffusion term will not be considered for the moment. The first term in the right hand side is associated with the shear and the last term describes the expansion of the region where $\mathcal{V}^{(1)}$ is not zero. In the case of a homogeneous collapse, $B \sim \mathcal{V}^{-\frac{2}{3}}$ gives rise to amplification of the magnetic field in places where gravitational collapse takes place. Now we write the equation (4.143) in the Poisson

[10] We use the convective derivative which is evaluated according to the operator formula (i.e $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{V}^{i}_{(1)}\partial_{i}$).

[11] Using the well known identity formula

$$
\nabla\times(a\times b) = a\,(\nabla\cdot b) - b\,(\nabla\cdot a) + (b\cdot\nabla)\,a - (a\cdot\nabla)\,b, \qquad (4.144)
$$

gauge getting the following

$$\begin{aligned}\frac{dB^k_{(1)}}{dt} &+ 2HB^k_{(1)} + \eta\left[-\nabla^2 B^k_{(1)} - \left(\nabla\times\left(\frac{1}{a^2}\frac{d(a^2E_{(1)})}{dt} - \mathcal{V}^i_{(1)}\partial_i E_{(1)}\right)\right)^k - B^k_{(0)}\nabla^2\left(\Psi^{(1)} - 3\Phi^{(1)}\right)\right.\\ &+ \left.\left(B^{(0)}\cdot\nabla\right)\partial^k\left(\Psi^{(1)} - 3\Phi^{(1)}\right) - \left(\nabla\left(\Psi^{(1)} - 3\Phi^{(1)}\right)\cdot\nabla\right)B^k_{(0)}\right] = B^{(0)}_l\sigma^{lk}_{(1)} - \frac{2}{3}B^k_{(0)}\partial_l\mathcal{V}^l_{(1)}, \quad (4.146)\end{aligned}$$

where $\sigma^{lk}_{(1)}$ is the shear found in (4.145). The last term on the left-hand side in (4.146) should vanish due to the background isotropy. The evolution of magnetic field following the last equation is highly dependent of term $\Psi^{(1)} - 3\Phi^{(1)}$. If the perturbations are turned off, one can check that last equation recovers to the dynamo equation found in the literature. It should be noted terms as $\left\langle B^k_{(0)}\right\rangle$ are zero due to statistical field properties, therefore contracting (4.146) with magnetic field $B^{(1)}_k$, we arrive at an equation at second order which we can physically study the evolution of the density magnetic field

$$\begin{aligned}&\frac{d\Delta^{(2)}_{(mag)}}{dt} + 4H\Delta^{(2)}_{(mag)} + 2\eta\left[-B^{(1)}\cdot\nabla^2 B^{(1)} - B^{(1)}\cdot\left(\nabla\times\left(\frac{1}{a^2}\frac{d(a^2E_{(1)})}{dt} - \mathcal{V}^i_{(1)}\partial_i E_{(1)}\right)\right)\right. \quad (4.147)\\ &- \left.\frac{1}{2}\Delta^{(1)}_{(mag)}\nabla^2\left(\Psi^{(1)} - 3\Phi^{(1)}\right) + B^k_{(1)}\left(B^{(0)}\cdot\nabla\right)\partial_k\left(\Psi^{(1)} - 3\Phi^{(1)}\right)\right] = -2\Pi^{(1)}_{ij(em)}\sigma^{ij}_{(1)} - \frac{2}{3}\Delta^{(1)}_{(mag)}\partial_l\mathcal{V}^l_{(1)},\end{aligned}$$

where using equations (4.135) and (4.47) the energy density magnetic field at second order transforms as

$$\begin{aligned}\Delta^{(2)}_{(mag)} &= B^2_{(2)} + B^{2\prime}_{(0)}\alpha_{(2)}\\ &+ \alpha_{(1)}\left(B^{2\prime\prime}_{(0)}\alpha_{(1)} + B^{2\prime}_{(0)}\alpha'_{(1)} + 2B^{2\prime}_{(1)}\right)\\ &+ \xi^i_{(1)}\left(B^{2\prime}_{(0)}\partial_i\alpha^{(1)} + 2\partial_i B^2_{(1)}\right). \quad (4.148)\end{aligned}$$

The parameters α and ξ are set using the Poisson gauge calculated in (4.13.1). The equation (4.147) shows how the field acts as an anisotropic radiative fluid which is important in times where universe is permeated by anisotropic components. In addition, the second term on the right-hand side describes the perturbation at first order in the volume expansion. Equations (4.143) and (4.147) show the important role of a magnetized FLRW model. The set of equations (4.143)-(4.146) directly offers a first estimation of how perturbed four-velocity coupling to magnetic field gives a common dependence of $B \sim \mathcal{V}^{-\frac{2}{3}}$ under an ideal assumption of infinity conductivity. However, for a real MHD a complete solution should be calculated together with the case of (4.147). The right hand side in (4.147) provides new phenomenology about the role of the shear and the anisotropic magnetic stress tensor together with a kinematical effect driven for the last term, reinforcing the claim in [106]. In the paper from Matarrese et al. [106] an estimation of the magnetic field to second order dropping the matter anisotropic stress tensor is given. They are able to compute a solution for the magnetic field, although in our case we suppose the presence of stress and vector modes at first order possibly generated in early stages from the universe.

4.13 Discussion

A problem in modern cosmology is to explain the origin of cosmic magnetic fields. The origin of these fields is still in debate but they must affect the formation of large scale structure and the anisotropies in the cosmic microwave background radiation (CMB) [142–144]. We can see this effect in (4.121) where the evolution of $\Delta_\mu^{(2)}$ depends on the magnetic field. In this work we show that the perturbed metric plays an important role in the global evolution of magnetic fields.

From our analysis, we wrote a dynamo like equation for cosmic magnetic fields to second order in perturbation theory in a gauge invariant form. We get the dynamo equation from two approaches. First, using the FLRW as a background space-time and the magnetic fields as a perturbation. The results are equations (4.93) and (4.127) to second order. The second approach a weakly magnetic field was introduced in the background space-time and due to it's statistical properties which allow us to write down the evolution of magnetic field equations (4.143) and (4.147) and fluid variables in accordance with [125]. We observe that essentially, the functional form is the same in the two approaches, the coupling between geometrical perturbations and fields variables appear as sources in the magnetic field evolution giving a new possibility to explain the amplification of primordial cosmic magnetic fields. One important distinction between both approximations is the fractional order in the fields which appears when we consider the magnetic variables as perturbations on the background at difference when the fields are from the beginning of the background (section 4.12). Although the first alternative is often used in studies of GWs production in the early universe [132–134], the physical explanation of these fractional orders is sometimes confused, while if we consider an universe permeated with a magnetic density from the background, the perturbative analysis is more straightforward. Further studies as anisotropic (Bianchi I) and inhomogeneous (LTB) models should be addressed to see the implications from the metric behavior in the evolution of the magnetic field and relax the assumption in the weakness of the field.

4.13.1 Gauge fixing

For removing the degrees of freedom we fix the gauge conditions as

$$\partial^i \omega_i^{(r)} = \partial^i \chi_{ij}^{(r)} = 0, \tag{4.149}$$

this lead to some functions being dropped

$$\omega^{\|(r)} = \chi_i^{(r)\perp} = \chi^{(r)\|} = 0, \tag{4.150}$$

with the functions defined in equations (4.68) and (4.69). The perturbed metric in the Poisson gauge is given by (4.129) thus, using the last constraints together with equations (5.18)-(5.21) in [95] and following the procedure made in [80, 83], the vector that determines the gauge transformation at first order $\xi_i^{(1)} = \left(\alpha^{(1)}, \partial_i\beta^{(1)} + d_i^{(1)}\right)$ is given by,

$$\alpha^{(1)} \to \omega^{\|}_{(1)} + \beta'_{(1)}, \quad \beta^{(1)} \to -\frac{\chi^{\|(1)}}{2}, \quad d_i^{(1)} \to -\chi_i^{\perp(1)}. \tag{4.151}$$

Now to second order, when we use the results in [95] together with (4.69) we obtain the following transformations

$$\tilde{\chi}^{(2)\|} = \chi^{(2)\|} + 2\beta^{(2)} + \frac{3}{2}\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\|}, \tag{4.152}$$

we get the following transformations

$$\tilde{\chi}^{(2)\|} = \chi^{(2)\|} + 2\beta^{(2)} + \frac{3}{2}\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\|}, \tag{4.153}$$

with

$$
\begin{aligned}
\mathrm{X}^{(2)\|} &= 2\left(\partial^i\partial^j D_{ij}\chi'_{(1)\|} + 2H\partial^i\partial^j D_{ij}\chi^{(1)\|}\right)\alpha^{(1)} \\
+&\quad \frac{2}{a^2}\left(a^2\chi^{(1)}_{ij}\right)'\partial^i\partial^j\alpha^{(1)} + 2\xi^k_{(1)}\partial^i\partial^j\partial_k D_{ij}\chi^{(1)\|} \\
+&\quad 2\partial_k\chi^{(1)}_{ij}\partial^i\partial^j\xi^k_{(1)} + 2\left(-4\partial^i\partial^j\phi_{(1)} + \partial^i\partial^j\alpha^{(1)}\partial_0\right. \\
+&\quad \left.\partial^i\partial^j\xi^k_{(1)}\partial_k + 4H\partial^i\partial^j\alpha^{(1)}\right)\left(\partial_{(j}d^{(1)}_{i)} + D_{ij}\beta^{(1)}\right) \\
+&\quad 2\left(-4\phi^{(1)} + \alpha^{(1)}\partial_0 + \xi^k_{(1)}\partial_k + 4H\alpha^{(1)}\right)\left(\partial^i\partial^j D_{ij}\beta_{(1)}\right) \\
+&\quad 2\left[2\omega^{(1)}_i\partial^i\nabla^2\alpha_{(1)} + 2\partial^j\nabla^2\omega^{\|}_{(1)}\partial_j\alpha^{(1)} - \partial_j\alpha^{(1)}\partial^j\nabla^2\alpha_{(1)}\right. \\
+&\quad \partial^j\nabla^2\beta'_{(1)}\partial_j\alpha^{(1)} + \xi^{(1)\prime}_i\partial^i\nabla^2\alpha^{(1)} - \nabla^2\left[\left(2\omega^k_{(1)} - \partial^k\alpha_{(1)}\right.\right. \\
+&\quad \left.\left.\left.\xi'_{(1)}\right)\partial_k\frac{\alpha_{(1)}}{3}\right]\right] + 2\left[2\partial^i\partial^j\left(D_{ij}\chi^{\|}_{(1)} + \partial_i\chi^{\perp}_{k(1)}\right)\partial_j\xi^k_{(1)}\right. \\
+&\quad 2\chi^{(1)}_{ik}\partial^i\nabla^2\xi^k_{(1)} + 2\partial_j\xi^k_{(1)}\partial^j\nabla^2\xi^{(1)}_k + \partial_k\xi^{(1)}_i\partial^i\nabla^2\xi^k_{(1)} \\
+&\quad \left.\partial_j\xi^k_{(1)}\partial^j\nabla^2\beta_{(1)} - \frac{1}{3}\nabla^2\left[\left(2\chi^{(1)}_{kl} + 2\partial_{(l}\xi^{(1)}_{k)}\right)\partial^l\xi^k_{(1)}\right]\right].
\end{aligned}
\tag{4.154}
$$

Now if we fix the poisson gauge, $\tilde{\chi}^{(2)\|} = 0$ we can fix the scalar part of the space gauge

$$
\beta^{(2)} = -\frac{\chi^{(2)\|}}{2} - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\|}. \tag{4.155}
$$

For the vector space part we should know the transformation rule for the vector part

$$
\tilde{\chi}^{(2)\perp}_i = -\partial_i\left(\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\|}\right) + \chi^{(2)\perp}_i + d^{(2)}_i + \nabla^{-2}\mathrm{X}^{(2)\perp}_i, \tag{4.156}
$$

with

$$
\begin{aligned}
\mathrm{X}_i^{(2)\perp} &= 2\Big[\partial^j D_{ij}\chi'^{\parallel}_{(1)} + \nabla^2\chi'^{\perp}_{i(1)} + 2H\left(\partial^j D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi^{\perp}_{i(1)}\right)\Big]\alpha^{(1)} + \\
&\frac{2}{a^2}\Big(a^2\chi^{(1)}_{ij}\Big)'\partial^j\alpha_{(1)} + 2\xi^k_{(1)}\partial_k\Big(\partial^j D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi_i^{\perp(1)}\Big) + 2\partial_k\chi^{(1)}_{ij}\partial^j\xi^k_{(1)} \\
&2\left(-4\partial^j\phi_{(1)} + \partial^j\alpha_{(1)}\partial_0 + \partial^j\xi^k_{(1)}\partial_k + 4H\partial^j\alpha^{(1)}\right)\cdot\left(\partial_{(j}d^{(1)}_{i)} + D_{ij}\beta^{(1)}\right) \\
&+\Big(\alpha^{(1)}\partial_0 - 4\phi^{(1)} + \xi^k_{(1)}\partial_k + 4H\alpha^{(1)}\Big)\cdot\left(\nabla^2 d_i^{(1)} + 2\partial^j D_{ij}\beta^{(1)}\right) \\
&+2\Big(\partial^j\omega_i^{(1)}\partial_j\alpha^{(1)} + \omega_i^{(1)}\nabla^2\alpha^{(1)}\nabla^2\omega^{\parallel}_{(1)}\partial_i\alpha^{(1)} + \omega_j^{(1)}\partial^j\partial_i\alpha^{(1)} - \partial^j\partial_i\alpha^{(1)}\partial_j\alpha^{(1)} \\
&-\nabla^2\alpha^{(1)}\partial_i\alpha^{(1)} + \partial^j\xi'_{i(1)}\partial_j\frac{\alpha^{(1)}}{2} + \xi'_{i(1)}\nabla^2\frac{\alpha^{(1)}}{2} + \frac{1}{2}\xi'_{j(1)}\partial_i\partial^j\alpha^{(1)} \\
&-\frac{1}{3}\partial_i\Big[(2\omega^k_{(1)} - \partial^k\alpha_{(1)} + \xi'_{(1)})\partial_k\alpha_{(1)}\Big]\Big) \\
&2\Big(\chi^{(1)}_{ik}\nabla^2\xi^k_{(1)} + \partial_i\xi^k_{(1)}(\partial^j D_{jk}\chi^{\parallel}_{(1)} + \nabla^2\chi_k^{(1)\perp}) + \chi^{(1)}_{jk}\partial^j\partial_i\xi^k_{(1)} + \partial_j\xi^k_{(1)}\partial^j\partial_i\xi_k^{(1)} \\
&+\nabla^2\xi^k_{(1)}\partial_i\xi_k^{(1)} + \frac{1}{2}\partial^j\partial_k\xi_i^{(1)}\partial_j\xi^k_{(1)} + \frac{1}{2}\partial_k\nabla^2\beta^{(1)}\partial_i\xi^k_{(1)} + \frac{1}{2}\partial_k\xi_j^{(1)}\partial^j\partial_i\xi^k_{(1)} \\
&\nabla^2\beta'_{(1)}\partial_i\frac{\alpha^{(1)}}{2} + \frac{1}{2}\partial_k\xi_j^{(1)}\partial^j\partial_i\xi^k_{(1)} + \nabla^2\beta'_{(1)}\partial_i\frac{\alpha^{(1)}}{2} + \frac{1}{2}\partial_k\xi_i^{(1)}\nabla^2\xi^k_{(1)} \\
&+\partial^j\chi^{(1)}_{ik}\partial_j\xi^k_{(1)} - \frac{1}{3}\partial_i[(2\chi^{(1)}_{kl} + 2\partial_{(l}\xi^{(1)}_{k)})\partial^l\xi^k_{(1)}]\Big).
\end{aligned}
\tag{4.157}
$$

Now we use the condition $\tilde{\chi}_i^{(2)\perp} = 0$ for instance,

$$
d_i^{(2)} = \partial_i\left(\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\parallel}\right) - \chi_i^{(2)\perp} - \nabla^{-2}\mathrm{X}_i^{(2)\perp}. \tag{4.158}
$$

To find the temporal part of the gauge transformation, we use the equation (5.35) in [95] and (4.68). With some algebra, the scalar part transforms like

$$
\tilde{\omega}^{(2)\parallel} = \omega^{(2)\parallel} - \alpha^{(2)} + \beta'_{(2)} + \nabla^{-2}\mathrm{W}^{(2)\parallel}, \tag{4.159}
$$

with

$$
\begin{aligned}
\mathrm{W}^{(2)\parallel} &= -4\left(\partial^i\psi^{(1)}\partial_i\alpha^{(1)} + \psi^{(1)}\nabla^2\alpha^{(1)}\right) + \partial^i\alpha^{(1)}\Big[2\omega_i^{(1)\prime} \\
+ &\quad 4H\omega_i^{(1)} - \partial_i\alpha'_{(1)} + \xi''_{(1)i} - 4H\left(\partial_i\alpha^{(1)} - \xi'_{i(1)}\right)\Big] \\
+ &\quad \alpha^{(1)}\Big[2\nabla^2\omega'_{(1)\parallel} + 4H\nabla^2\omega^{(1)}_{\parallel} - \nabla^2\alpha'_{(1)} + \nabla^2\beta'' \\
- &\quad 4H\left(\nabla^2\alpha^{(1)} - \nabla^2\beta'_{(1)}\right)\Big] + \partial^i\xi^j_{(1)}\left(2\partial_j\omega_i^{(1)}\right. \\
- &\quad \left.\partial_i\partial_j\alpha^{(1)} + \partial_j\xi'_{i(1)}\right) + \xi^j_{(1)}\left(2\partial_j\nabla^2\omega^{\parallel}_{(1)} - \partial_j\nabla^2\alpha^{(1)}\right. \\
+ &\quad \left.\partial_j\nabla^2\beta'_{(1)}\right) + \alpha'_{(1)}\left(2\nabla^2\omega^{\parallel}_{(1)} - 3\nabla^2\alpha^{(1)} + \nabla^2\beta'_{(1)}\right) \\
+ &\quad \partial^i\alpha'_{(1)}\left(2\omega_i^{(1)} - 3\partial_i\alpha^{(1)} + \xi'_{(1)i}\right) + \nabla^2\xi^j_{(1)}\left(2\omega_j^{(1)}\right. \\
- &\quad \left.\partial_j\alpha^{(1)}\right) + \partial_i\xi^j_{(1)}\left(2\partial^i\omega_j^{(1)} - \partial^i\partial_j\alpha^{(1)}\right) + \partial^i\xi^{j\prime}_{(1)}\Big[\partial_j\xi_i^{(1)} \\
+ &\quad 2\chi^{(1)}_{ij} + 2\partial_i\xi_j^{(1)} - 4\phi^{(1)}\delta_{ij}\Big] + \xi^{j\prime}_{(1)}\Big[-4\partial_j\phi^{(1)} \\
+ &\quad 2\left(\partial^i D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi_j^{(1)\perp} + 2\nabla^2\xi_j^{(1)} + \partial^i\partial_j\xi_i^{(1)}\right)\Big],
\end{aligned}
\tag{4.160}
$$

in this way we fix the temporal part of the gauge using $\tilde{\omega}^{(2)\parallel} = 0$ in the last equation finding the follows

$$\alpha^{(2)} = \omega^{\parallel}_{(2)} + \partial_0\beta_{(2)} + \nabla^{-2}\mathrm{W}^{(2)\parallel}. \tag{4.161}$$

Therefore, we found explicitly the set of functions that fix the gauge dependence given by equations (4.151), (4.155), (4.158) and (4.161). Thus, using the above equations we can calculate the gauge dependence in the scalar perturbations at second order that were shown in (4.97)

$$\mathcal{S}^{\parallel}_{(2)} = \omega^{\parallel}_{(2)} - \frac{\chi^{\parallel(2)\prime}}{2} - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\prime}_{\parallel} + \nabla^{-2}\mathrm{W}^{(2)\parallel}, \tag{4.162}$$

which can be interpreted like shear to second order, again we see the last equation is a generalization for the first order scalar shear plus quadratic terms in the perturbed functions.

4.13.2 Density evolution

To find the charge evolution, we use the fact that $j^{\alpha}_{;\alpha} = 0$ therefore, the temporal part of this equation drive us to the charge conservation

$$\varrho^{(1)\prime} + 3H\varrho^{(1)} + \partial_i J^i_{(1)} = 0, \tag{4.163}$$

at first order in the approximation and

$$\begin{aligned}\varrho^{(2)\prime} &+ 3H\varrho^{(2)} + \partial_i J^i_{(2)} + \left(\Psi^{(1)\prime} - 3\Phi^{(1)\prime}\right)\varrho^{(1)} \\ &+ \partial_i\left(\Psi^{(1)\prime} - 3\Phi^{(1)\prime}\right)J^i_{(1)} = 0,\end{aligned} \tag{4.164}$$

to second order. These equations are important for resolving the dynamo equation. In the section 4.9.1 was found the momentum equation at second order, where $\boldsymbol{S}^{13}_i$ is given by

$$\begin{aligned}\boldsymbol{S}^{13}_i &= \frac{\left[\Delta^{(1)}\left(1+c_s^2\right)\mathcal{V}^{(1)}_i\right]'}{\mu_{(0)}\left(1+w\right)} + 4H\frac{\Delta^{(1)}\left(1+c_s^2\right)\mathcal{V}^{(1)}_i}{\mu_{(0)}\left(1+w\right)} \\ &+ 2\frac{\left[\mu_{(0)}\left(1+w\right)\chi^{(1)}_{ij}v^j_{(1)}\right]'}{\mu_{(0)}\left(1+w\right)} + 8H\chi^{(1)}_{ij}v^j_{(1)} - 8H\Phi^{(1)}v^{(1)}_i \\ &- \frac{\left[\mu_{(0)}\left(1+w\right)\left(\omega^{(1)}_i + \mathcal{V}^{(1)}_i\right)\right]'\Psi^{(1)}}{\mu_{(0)}\left(1+w\right)} + \frac{\Delta^{(1)}\left(1+c_s^2\right)\partial_i\Psi^{(1)}}{\mu_{(0)}\left(1+w\right)} \\ &- \omega^{(1)}_i\Psi'_{(1)} - 4H\Psi^{(1)}\left(\mathcal{V}^{(2)}_i + \omega^{(1)}_i\right) + v^j\left(\partial_j v^{(1)}_i\right) \\ &- 3\Phi'_{(1)}\left(\mathcal{V}^{(1)}_i + v^{(1)}_i\right) - v^j_{(1)}\partial_{[i}\omega^{(1)}_{j]} - 2\Psi^{(1)}\partial_i\Psi^{(1)} \\ &- 2\Phi^{(1)}\left[\mu_{(0)}\left(1+w\right)v^{(1)}_i\right]' + \frac{\left(\partial_j\Psi^{(1)} + H\omega^{(1)}_j\right)\Pi^{j(1)}_i}{\mu_{(0)}\left(1+w\right)} \\ &+ \mathcal{V}^{(1)}_i\partial_\alpha\partial^\alpha v_{(1)} - \frac{6\partial_j\Phi^{(1)}\Pi^{j(1)}_i - \frac{1}{2}\partial_j\chi^{k(1)}_i\Pi^{j(1)}_k}{\mu_{(0)}\left(1+w\right)},\end{aligned} \tag{4.165}$$

where $w = \frac{P_{(0)}}{\mu_{(0)}}$ is the state equation ($w = 0$ for dust and $w = 1/3$ for radiation era) and $c_s^2 = \frac{P_{(1)}}{\mu_{(1)}}$ the adiabatic sound speed. Using the expression for the momentum exchange among particles and the momentum conservation, we obtain the following equations for protons, electrons and photons during radiation era

$$
\begin{aligned}
& \mu_{(0)}^{(p)}\left[\mathcal{V}_i^{(2)(p)\prime}+H\mathcal{V}_i^{(2)(p)}+\partial_i\Psi^{(2)}+2\boldsymbol{S}_i^{13(p)}\right]+\partial_i\Delta_P^{(2)(p)} \\
+ \quad & \frac{4}{3}\partial_i\nabla^2\Pi_{(p)}^{(2)}=a^4\left(E_i^{(1)}\varrho_{(1)}^{(p)}+\epsilon_{ij}^{k}J_{(1)}^{j(p)}B_k^{(1)}\right)+K_i^{(ep)},
\end{aligned}
\tag{4.166}
$$

$$
\begin{aligned}
& \mu_{(0)}^{(e)}\left[\mathcal{V}_i^{(2)(e)\prime}+H\mathcal{V}_i^{(2)(e)}+\partial_i\Psi^{(2)}+2\boldsymbol{S}_i^{13(e)}\right] \\
+ \quad & \partial_i\Delta_P^{(2)(e)}+\frac{4}{3}\partial_i\nabla^2\Pi_{(e)}^{(2)} \\
= \quad & a^4\left(E_i^{(1)}\varrho_{(1)}^{(e)}+\epsilon_{ij}^{k}J_{(1)}^{j(e)}B_k^{(1)}\right)-K_i^{(ep)}+K_i^{(\gamma)},
\end{aligned}
\tag{4.167}
$$

$$
\begin{aligned}
& \frac{4}{3}\mu_{(0)}^{(\gamma)}\left[\mathcal{V}_i^{(2)(\gamma)\prime}+\partial_i\Psi^{(2)}+2\boldsymbol{S}_i^{13(\gamma)}\right]+\partial_i\Delta_P^{(2)(\gamma)} \\
+ \quad & \frac{4}{3}\partial_i\nabla^2\Pi_{(\gamma)}^{(2)}=-K_i^{(\gamma)}.
\end{aligned}
\tag{4.168}
$$

CHAPTER 5

Kinematics in Hickson Compact Group 90

5.1 Introduction

Understand the main aspects about galaxy groups is one of the most important issues in modern astronomy and cosmology. About 50% of the galaxies in the universe live in galaxy groups. One important sample of nearby galaxy groups is called *Hickson Compact Groups*. Due to their peculiar properties like high mass density profiles and very low velocity dispersion, Hickson Groups are the perfect laboratories to study galaxy-galaxy interactions, gas effects on galaxy evolution and dark matter at intermediate scales.

We use long-slit and MXU spectroscopy obtained in May 2002 at VLT FORS2 Facility Grism 1400V. We employ a truncated Gauss-Hermite line-of-sight velocity distribution (LOSVD) for the stellar component and from this function the main kinematical quantities: rotation curves, velocity dispersion and Hermite coefficients h_3 and h_4 for galaxies in the core of HCG 90 at two P.A. (72°, 132°) were obtained. Emission lines H_β and O_{III} are used to study the ionized gas kinematics.

5.2 Galaxy groups

Galaxy Groups are the most common and gravitationally bounded structures in the universe. More than half of galaxies are located in groups [145]. Galaxy groups show a wide range of masses and sizes. From pairs, triplets to groups contained a few hundred galaxies. There have been several studies about group of galaxies, one of them is the pioneer work by Hickson [146]. Hickson Catalog consists in a sample of 100 galaxy groups selected by visual searching from the first Palomar Observatory Sky Survey (POSS). The increasing sensitivity and bigger available dataset make possible to improve the selection criteria, survey completeness and reach fainter groups. It opens the possibility to explore groups at different cosmological ages and environments [147]. The first studies were mainly focused on low redshift groups, but since some years ago, the situation is changing. Features of the galaxy groups in earlier cosmological epochs will be a strong test to the hierarchical ΛCDM paradigm and galaxy formation theory.

New Surveys as the 2dFGRS and SDSS [145] have increased our understanding about galaxy groups. Some studies target groups as laboratories for astrophysical studies: *gas physics, galaxy formation, interactions and morphology dependence, dark matter at small scales, AGN activity.* Contrary to earlier claims, galaxy groups are good tracers of the large-scale structure in the universe as galaxy clusters [148] and they provide the link between our local neighbourhood and the universe at the largest scales [149].

5.2.1 Hickson compact groups

Hickson Compact Groups of galaxies (hereafter HCGs) are small systems in a dense and isolated configuration on the sky [146, 150]. Their galaxy content lies between three to seven major galaxies. The physical and dynamical properties of HCGs are really peculiar. Together with the centers of rich clusters of galaxies they are the densest regions known in the universe; but opposite to galaxy clusters, where the velocity dispersion of the galaxies reaches typically $1000\,\mathrm{km\,s^{-1}}$, the galaxy velocity dispersion in HCGs is much lower, its at the same order of internal velocity dispersion of the stars in galaxies. The mean galaxy velocity dispersion in the Hickson sample is about $200\,\mathrm{km\,s^{-1}}$ [146]. The peculiar features mentioned above suggest that HCGs are perfect laboratories to study galaxy-galaxy interactions, effects of the intragroup medium on galaxy transformations and morphology-kinematics correlations. The physics in compact groups may provide analogs to hierarchical galaxy formation in the early universe [151]. Given the abundance of galaxy groups, they can be used as a bridge in an intermediate scale between isolated galaxies and cluster of galaxies. In particular, studies of the dark matter distribution in groups will give important clues about the nature and distribution of dark matter in the universe [152].

The sample of compact groups in the Hickson Catalog shows that HCGs are living in the nearby universe. The mean redshift of the sample is $\bar{z} \approx 0.030$. But from cosmological studies, is expected to find galaxy groups at higher redshifts. Taken from the COSMOS facility at Royal Observatory of Edinburgh a new sample of Compact Groups has been obtained by Iovino and collaborators [153, 154]. In Iovino's work, a substantial refinement about the group membership, compactness and isolation was applied. After several and different membership tests the group got a new southern catalog of compact groups (SCG). Adding redshift information to the SCG survey, galaxy evolution in compact groups was studied in [155]. The role of galaxy-groups in galaxy formation theory is still an active field of research and many important questions remain unsolved [145, 154]. Specially compact groups appear to be a dynamical paradox. Strong galaxy interactions combined with low velocity dispersion make difficult to explain why HCGs are located cosmologically at very low redshift. Many more fossil groups and big elliptical are expected to be the product of merging in HCGs, however, fossil groups are more related to poor clusters rather than end-products of HCGs [149].

HCGs have been also studied at different wavelengths and their physical properties explored at different scales, from strong galaxy interactions, ionized gas to extended X-ray properties [156]. Important research about the infrared properties of compact groups and their star formation activity is addressed in [151]. In the next section, we will describe our galaxy group HCG90 in the context of HGCs.

5.2.2 HCG90 in the context of HCGs

Hickson Compact Group 90 is a group of four bright galaxies, two early type galaxies, NGC 7173 (HCG90C) and NGC7176(HCG90B), and two late-type galaxies, NGC7172 (HCG90A, which is not included in our dataset) and NGC7176 (HC90D) [157]. The group was catalogued by Hickson in 1982 [146]. HCG90 is at a distance of roughly $33\,\mathrm{Mpc}h^{-1}$ [1] [158]. The mean systemic velocity of the group was estimated to be $2643\,\mathrm{km\,s^{-1}}$ with a galaxy velocity dispersion of $\sigma_V \approx 200\,\mathrm{km\,s^{-1}}$ [146]. It is common to find in the literature the description of HCG90 as a dense core of three interacting galaxies (HCG90B,HCG90C,HCG90D) in a region of $6' \times 6'$ (see figure 5.1), which is surrounded by an extended loose group [159, 160].

[1] h is the dimensionless Hubble's constant $\frac{H_0}{100\,\mathrm{km/s/\,Mpc}}$

The fourth member, NGC7172 is a Seyfert2 galaxy located 6′ north from the core. NGC7172 has been identified as a strong X-ray source [161]. Most of the studies done on HCG90 are concentrated in the core of the group [157, 159, 162], and the purpose of this chapter is to extend the previous work [162] in stellar kinematics.

As we mentioned above, the redshift for the sample in HCGs is very low. For HGC90, the redshift estimated from NGC7172 is $z \approx 0.009$ [146]. One of the open questions about HCGs is their dark matter content. Due to the high mass density and low velocity dispersion, the crossing-time is smaller than the Hubble-time and the most natural scenario is that they will merge in a short time, after a few crossings. The final structure expected is a fossil group with probably a big elliptical galaxy at the center. One possible explanation, why the HCGs are still in such dynamical stage of not completed merging, supported by numerical simulations, is that HCGs are embedded in a common dark matter halo [163, 164], but on the other hand it is also known that the dark matter content of HCGs is poorer in comparison with clusters of galaxies and loose groups [162].

For long time the reality of HCGs as a gravitationally bounded structures was questioned [165–167]. From dynamical properties, several authors pointed out that a large fraction of HCGs are a superposition of galaxies along the line of sight and they did not belong to a bounded structure. But X-ray observations of the intragroup gas in HCGs showed that they are really physically gravitationally bounded structures [168]. For an extensive review about X-ray properties in HCGs see [169].

5.2.3 X-ray gas in HCG 90

In order to study the hot gas in HCGs some assumptions must to be done. The most common scenario is to treat the gas in hydrostatic equilibrium in the underlaying gravitational potential of the group. In general, the state of the gas in gravitationally bounded structures as galaxy clusters and groups is not in hydrostatic equilibrium and its description is difficult, however, a simple model has been used since time ago [170] for this task. The main feature of this model is to set a single temperature for the gas and looking for a tracer of the gravitational potential. All the physics can be described from the properties of such tracer. Usually the galaxy velocity dispersion σ_{v} is used as tracer of the gravitational potential. Further assumptions as spherical symmetry, isotropic velocity dispersion distribution can be tested observationally to refine the model. The most simple model is an uniparameter β model, with $\beta \equiv \frac{\sigma_{\mathrm{v}}^2}{(\mathrm{kT}/\mu \mathrm{m_p})}$ being the ratio between the kinetic energy of the group to the thermal energy in the gas. The X-ray emission comes mainly from thermal-bremsstrahlung and line-recombination depending on the gas density and gravitational potential well.

In a pioneer work [169] a sample of galaxy groups including some HCGs was studied. From this study, results are consistent with a projected gas profile in HCGs with $\Sigma_{\mathrm{gas}}(s) = \Sigma^0_{\mathrm{gas}}(1 + (s/R_c)^2)^{(0.5-3\beta)}$, a core radius of ($4 \leq R_c \leq 30\,h^{-1}$kpc) and slope ($0.38 \leq \beta \leq 0.92$) [169]. In general the emission is not centered in any particular galaxy, which likely means that the groups are not quite evolved structures in a massive, concentrated dark matter halo [164]. Moreover, in a study by Osmond & Ponman [171]) using data from the *ROSAT* satellite, they have studied a sample of 60 groups including some HCGs and particularly HCG90 extending previous work in X-ray for galaxy groups. High resolution spectra allow them in some cases to decompose the X-ray emission in two principal components:

- From the brightest galaxy in the group, comparing the ratio between luminosities ($L_{\mathrm{X}}/L_{\mathrm{BCG}}$), this feature is common from evolved groups.
- From the entire group, including the halo-group potential. However, the gas physics in galaxy groups is richer than a simple isothermal plasma, interesting phenomena as shock fronts and complex gas structures have been detected in HCGs, a typical example is HGC62 [151].

In the case of HGC90, diffuse X-ray emission from the intragroup gas is detected especially in the group's core. The X-ray contours are centered around the three galaxies in the core. There is not clear limit for the size of such emission region, X-ray emission is detected up 2′ from the core. From the plasma model for the gas, a more extended emission region is expected, unfortunately in this case seems to be at the same level that X-ray background [159]. The best fit model has a plasma temperature of $\mathrm{T} = (7.8 \pm 0.4) \times 10^6\mathrm{K}$, meaning a X-ray luminosity of $L_X \approx 0.7 \times 10^{41}\mathrm{erg/s}$. For a complete information about X-ray properties in HCG90 see [159, 171].

5.2.4 The environment of HCG90

Observationally several samples of galaxy groups have been studied [152, 154, 172]. Mulchaey & Zabludoff [173] for a combined sample of nine X-ray detected groups including HCG90, found a trend of constant velocity dispersion with increasing radius of the group, suggesting that groups are embedded in a common dark matter halo. Together with the four bright-big galaxy, several candidates belonging to HCG90 have been identified . From spectroscopic observations of a circular region with diameter $d = 1°$ centered on the group, five new members were identified [160].
The enlarged system has a velocity dispersion of $\sigma_\mathrm{v} \approx 166\,\mathrm{km\,s^{-1}}$. If a larger region is considered ($1.5° \times 1.5°$) the member number increases, 19 members can be identified at a mean velocity of $\overline{V} \approx 2600\,\mathrm{km\,s^{-1}}$ with a velocity dispersion of $\sigma_V \approx 190\,\mathrm{km\,s^{-1}}$ [157, 159].

Warm gas has been also used by [157] to study the interaction and evolutionary status of HCG90. Deep images and X-ray data from the *Chandra X-Ray Observatory* were used in [159]) to study the common envelope of diffuse-light around the core in HCG90. The results coming from this study are quite controversial, while the X-ray emission from the intragroup medium indicates that HCG90 is a non-evolving group, the common stellar envelope around the core shows a very narrow red colour distribution, suggesting that HCG90 is an old group in merging process. On the other hand, particular features in the group mislead and make difficult disentangle its evolutionary status, it is the case of a sample of young star clusters studied in [174].

In the present chapter we present the stellar kinematical analysis in the core of HCG90. Our high quality data allow us to study the internal kinematics in the core and compare it with the previous work [162]. In section 5.3 we describe the dataset and the reduction steps. The kinematical analysis from the MXU-data is discussed in 5.4 and the long-slit together with the ionized gas kinematics from $\mathrm{O_{III}}$ and H_β emission is in 5.5.2. Some conclusions and annotations are given in section 5.6.

5.3 Observations and data reduction

The dataset and the steps for the reduction are described in this section. For most of the steps a computer-script was developed.

5.3.1 Observations

The observations were carried out in May 2002 with the VLT-FORS2 facility at the European Southern Observatory, Chile. The spectroscopical dataset consists in a MXU mask (see figure 5.1) and two longslit in the core of HCG90 (see figure5.5). The wavelength spectral range is between 4500 – 5750Å. The Grism 1400V was used [2]. The grism has a dispersion of $0.62\,\text{Å}\,\mathrm{pix^{-1}}$. The velocity resolution in this case is $\approx 50\,\mathrm{km\,s^{-1}}$. The main features of the observations are shown in Table 5.1 and Table 5.2.

[2] For a detailed description see http://www.eso.org/sci/facilities/paranal/instruments/fors/inst/grisms.html

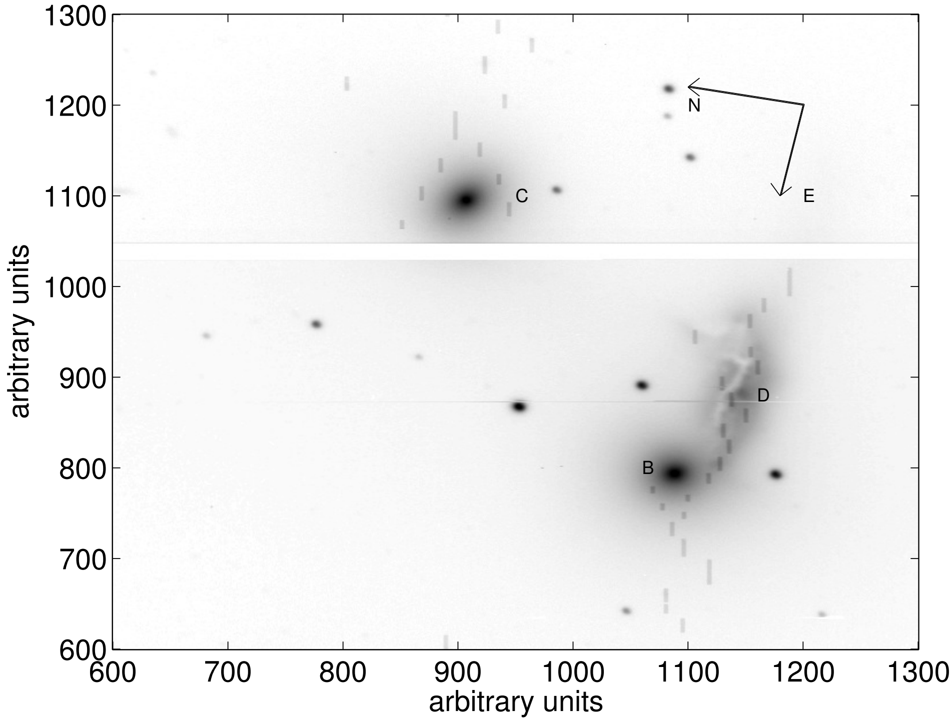

Figure 5.1: MXU mask in HCG90. The plot shows the core of HCG90 and its orientation on the sky.

Table 5.1: MXU-mask

Date	Exposure time	Seeing
2002-05-13	1200 s	0.98

A set of lamp calibration spectra, dome flat-field images were obtained during the same observation run.

5.3.2 Data reduction

The information about the slit is in Table 5.1 and Table 5.2. The standard data reduction was performed within the IRAF-package [3]. The mask-exposures were bias corrected and combined using a cosmic-ray removal algorithm. After bias subtraction and bad pixels removal, the combined images were response calibrated using the dome-flat spectra averaged perpendicularly to the dispersion direction. The MXU science spectra together with the calibration lamp spectra were reduced using the task *apall* in the twodspec package. The spectra were wavelength calibrated using the lamp reference spectra within onedspec package. We checked the quality of our calibration keeping a small value in the residuals and fitting low order polynomial functions. The sky subtraction was performed carefully with the skytweak routine.

For the long-slit spectra the 2D wavelength calibration was performed after correction for spatial

[3] Image Reduction and Analysis Facility (IRAF) package. Distributed by the National Optical Astronomy Observatories (NOAO), which is operated by AURA (Association of Universities for research in Astronomy) under cooperative agreement with the National Science Foundation

Table 5.2: Long-slit

Date	Slit-length	Exposure-time	P.A	Seeing
2002-05-10	460″	1800 s	132°	1.63
2002-05-10	485″	3600 s	72°	1.38

distortions and illumination; in this case we used the twodspec.longslit IRAF package. The sky was subtracted as an average sky spectral image. All spectra (science and templates) were rebinned to a dispersion of 1Å pix^{-1} before to start the kinematical analysis

In the next section after a briefly description of each member in HCG90, we will focus on the results from the MXU spectroscopy. As an example, the figure 5.2 represents a typical spectra from our data-set. The most prominent absorption features are used for the kinematical analysis. The emission from ionized gas is clearly seen H_β and O_{III} at 4861 Å and 5007 Å respectively.

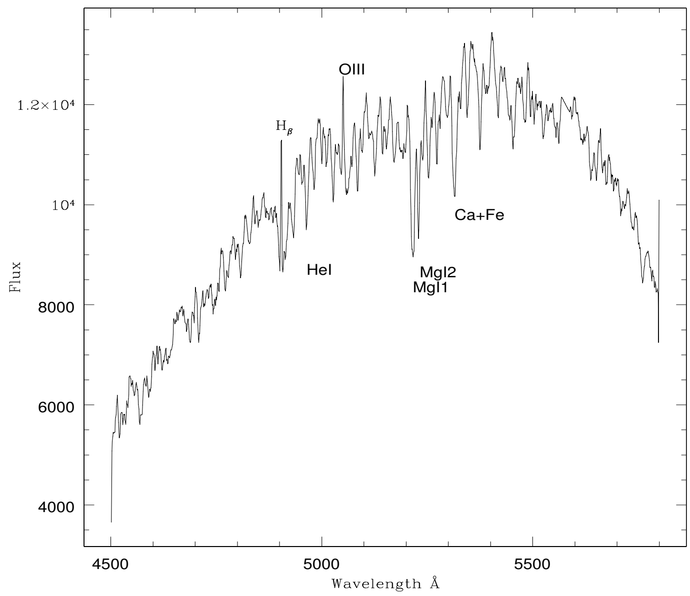

Figure 5.2: Typical MXU spectrum from the region between HCG90B and HCG90D

5.4 Kinematical analysis

For completeness of the chapter we will describe briefly each galaxy in HCG90 before to proceed with the kinematical analysis. Descriptions of the HCG90 have been written by many authors [157, 159, 162], we refer the reader at these papers for a detailed description of the group.
Table 5.3 summarizes the identification and location for each galaxy in HCG90

Group Members

The main morphological and kinematical features for the galaxies in the core of HCG90 are discussed briefly.

Table 5.3: Core of HCG90

Galaxy	α h min s	δ (° ′ ″)	Type	Angular Size arcminute (′)
NGC 7173	22:02:03.38	-31:58:26.92	E pec	1.2
NGC 7174	22:02:06.82	-31:59:36.53	Sab pec	2.3
NGC 7176	22:02:08.45	-31:59:29.52	E pec	1.0

NGC7173 (HCG90C)

HCG90C has been classified by Hickson [150] as an E0 galaxy and as an E + pec galaxy in the RC3 catalog [157]. Its low ellipticity ($\epsilon = 0.26$) makes difficult to determine the P.A of its major axis (P.A. is measured from north to east). P.A has been observed to change from 60° in the inner part to 130° in the outer part [157]. The strong interaction in the core of the group makes difficult to define the P.A for the galaxy after some angular distance from the group center. The estimated systemic velocity using absorption lines is 2696 km s^{-1} ± 24 km s^{-1} [160].

NGC7176 (HCG90B)

HCG90B has been classified as an E0 galaxy by Hickson [150] and as a E + pec galaxy in the RC3 [157]. This galaxy shows similar features as HCG90C, it is an almost a round object with a P.A. of 55° ± 5° reported in [175]. The systemic velocity is 2525 km s^{-1} ± 29 km s^{-1} [160].

NGC7174 (HCG90D)

HCG90D is a very disturbed disk galaxy classified as an irregular galaxy in [176] and as an early-type spiral by the RC3. The systemic velocity for this galaxy has been estimated to be 2659 km s^{-1} ± 9 km s^{-1} [160].

5.4.1 Kinematical analysis from MXU data

For the kinematical analysis with the MXU-mask we used the cross-correlation method within the IRAF package *rv* using the task *fxcor*. The routine uses the cross correlation method developed by Tonry & Davies [177]. The science spectra and the template are rebinned in logarithmic scale, emission lines and continuum subtracted. Both spectra are Fourier transformed and filtered to remove variations coming from the continuum subtraction. The relative velocity ($V_{\rm rel}$) between the galaxy and the template is calculated from the highest peak in the cross-correlation. For all our spectra the Tonry (R) parameter is between 12-15. Before the correlation is done, and in order to get the correct heliocentric radial velocity, kinematical corrections must to be done, for this purpose we used the *astutil* IRAF task.

A systematical shift from the oxygen sky-line (5577Å) was removed. The Heliocentric radial velocity is obtained from $V_{\rm Helio} = V_{\rm rel} + V_{\rm tem} + V_{\rm hc}$ with $V_{\rm tem}$ the template velocity and $V_{\rm hc}$ the heliocentric correction for the object.

The Heliocentric radial velocities obtained from the MXU mask after several combinations of stellar templates from the Table 5.4. It is important to note that even the mismatch between the stellar and the galaxy spectra is reduced combining stellar spectra, the kinematical results do not change appreciably. For cross correlation we use the main absorption features in the galaxy spectrum. The velocity field is shown in 5.3 as a function of the distance from HCG90B. The region between HCG90B and HCG90D does not show a big disturbance expected whether a strong interaction between the galaxies are taken place, however, the high signal to noise ratio in the spectra makes feasible that the smoothness in the rotation curve between the galaxies is a real feature. This smoothness is difficult to explain by a superposition of two uncorrelated motions, as was pointed out in [178]. Compact groups of galaxies seem to

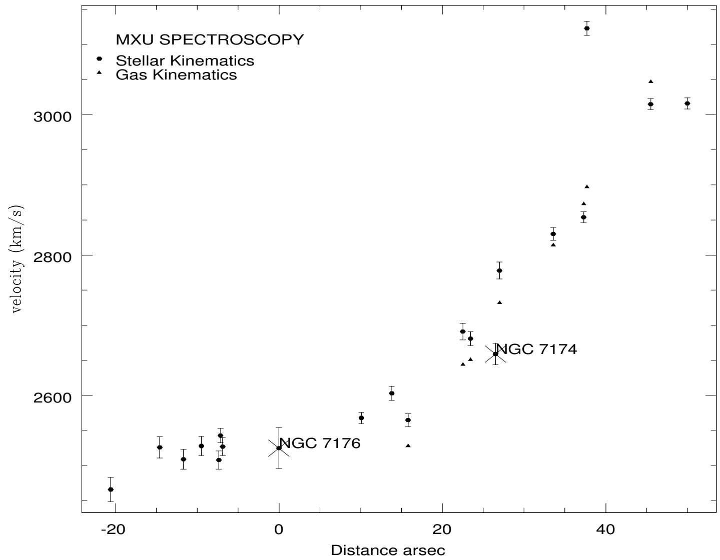

Figure 5.3: Heliocentric radial velocities in the MXU-mask

have peculiar dynamical properties, from previous work [157, 179] warm gas is used to trace the ionized gas in the group. The analysis gives as a result that an interaction between these two galaxies is taken place with the perturbed disk galaxy (HCG90D) being the host for the warm gas of the system [157]. Some new insights from our data appear to confirm the interaction, but this point must to be studied with new spectroscopic data and from UV− spectra. In a previous work by Longo et al. [162] from long-slit spectra, they found a U-shape in the velocity dispersion profile, meaning a possible heating of the system due to tidal interactions between the galaxies. For long time, based specially in numerical simulations, different shapes have been used to characterize kinematical properties in interacting galaxies and stellar remnants [180]. From our dataset a similar shape is seen in the velocity dispersion profile along the long-slit at P.A.72° (see figure 5.6), with a quite smooth rotation curve along this direction. The gas shows a shift respect to the stellar velocity in HCG90B and it is also clear the in HCG90D. Clearly, the disk galaxy is the gas reservoir for the system. Plana et al. [157], from their data analysis argue that is possible to have a much faster response to the gravitational interaction from the gas than from the stars and these features in the gas kinematics should arise from this response.

The high galaxy density in HCG90 forces us to think that all the three galaxies are in a strong interaction and the final product will be a fossil group with a elliptical galaxy at the center of the potential well. However, as we mentioned before, from the stellar kinematics only is difficult to have clues for the interaction. As a particular feature in the gas kinematics from the MXU spectra in HCG90C, a disagreement between gas and stellar kinematics is founded (see figure 5.4). This feature is also shown in the long-slit analysis in the next section. This contradictory result with the warm gas kinematics from Plana et al. [157] encourage us to think for a very deep study in HCG90, specially related with its gaseous content. There is at least one possibility for the unexpected result. From the Plana's work the results is clear: HCG90C has a decoupled gas component, the stellar and gas axes make an angle of 60°. Its origin may come from the external accreted gas by the group and/or due to a much closer interaction with the other galaxies [181]. Our MXU mask in HCGC is in the outher part (see figure 5.1) and the gas emission coming from the stars should have the signs from the interaction between HCG90C with

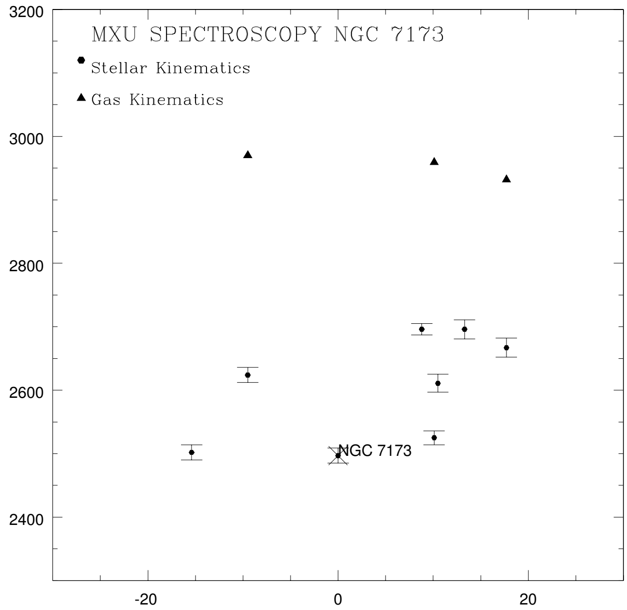

Figure 5.4: Stellar and gas kinematics in HCG90C. The velocity in the vertical axis is in km s^{-1}. From the plot is clear that the stellar and gas components are kinematically decoupled.

and the disk galaxy [4]. Several signs of tidal interactions are seen for this group member, a long-slit connecting HCG90C with HCG90D should give clues about the peculiar features in the gas kinematics.

5.5 Long Slit Spectroscopy

Two long-slit spectra in the same spectral range as MXU were obtained during the observation run. The slits positions are shown in figure 5.5.

The longer exposure time in the slit connecting HCG90B with HCG90D allows us to extract more kinematical information from this spectra. As in the case of MXU data, prominent absorption lines were used for the kinematics. The high quality of the spectra together with a rebbining compromise between signal to noise and spatial resolution make possible go beyond the usual Gaussian parametrization for the LOSVD.

5.5.1 Templates

To perform an accuracy kinematics, is indeed to choose the stellar template as closer to the galaxy spectra as much as possible. A method was developed by Rix & White [182] to minimize the *template mismatch* adding several stellar templates. This method was improved by Cappellari & Emsellem [183] and their tools to perform stellar kinematics deal with the template-mismatch problem properly. The purpose is that the broadened template reproduces the galaxy spectrum as close as possible [184].
For our kinematical analysis the star spectra cited in table 5.4 were used and we worked with several combinations of them. The stars used as a templates were observed in the same run, with the same

[4] Longo et al. [162] have data from a slit connecting HCG90C and HCG90D missing in this work.

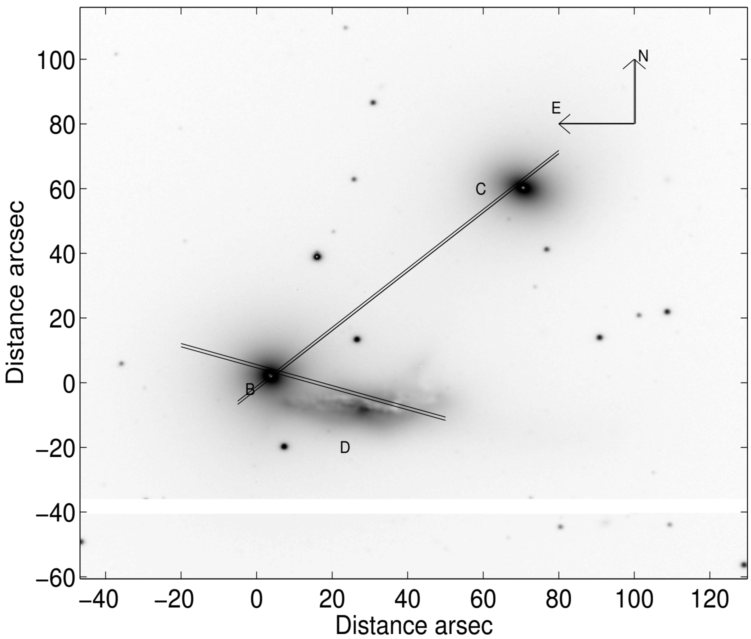

Figure 5.5: long-slit kinematics

instrumental setup and the reduction was done in the same way as the galaxy spectra. It makes the analysis more straightforward and confident. After kinematical corrections the heliocentric radial velocities were obtained.

Table 5.4: Stellar Templates

Template	Spectral Type	Metallicity	Radial Velocity $km\,s^{-1}$
HD 192718	F8	-0.74	-109.0
HD 193901	F7V	-1.1	-172.0
HD 196892	F8V(F6V)	-1.0	-30.0
HD 204543	KII (G0)	-1.8	-98.0

5.5.2 Long-Slit Kinematics

Complex systems as elliptical galaxies have been object of intensive studies. Modern N-Body simulations together with star-orbits and information from stellar populations are in the basis for modeling such systems. In addition to the theoretical basis, important information from the spectra must to be incorporated in the model. The spectra observed in a point from a galaxy is a composition of stellar populations present on it. One of the most difficult tasks is try to recover from the spectra the main kinematical and dynamical properties for the galaxies. The spectra contain the kinematical information along the line of sight and together with the photometric light profiles are the most common sets of information used in the models.

For long time a Gaussian parametric distribution function for the LOSVD has been employed, even knowing that physics behind of such parametrizations is not well understood [185]. The Gaussian distribution function is characterized by a mean velocity $\overline{\mathrm{V}}$, velocity dispersion σ_{v} and an amplitude γ as a free parameters. For elliptical galaxies, this model has shown to be a good guess even the collisionless nature of the system. Model spiral galaxies with Gaussian distribution functions can be a big mistake given the complexity of such systems: Active star formation, dust and multiphase interestelar medium, waves and collective excitations make the Gaussian distribution a poor guess to describe the stellar distribution in spiral galaxies. To the date only a few attempts to model spiral galaxies are known [186]. For our purpose, following Van der Marel & Franx [185], the LOSVD is parametrized by a truncated Gauss-Hermite series. The galaxy spectrum is a convolution between the LOSVD and a stellar template

$$G(u) = \int \mathrm{d}v_{\mathtt{los}}\, L(v_{\mathtt{los}})\, S\,(u - v_{\mathtt{los}}), \tag{5.1}$$

with $u = c\ln\lambda$ and S a representative stellar template. For a more general LOSVD, the function

$$L(\omega) = \left(\frac{\gamma}{\sqrt{2\pi\sigma}}\right)\left(1 + \sum_{n=3}^{N} h_n H_n(\omega)\right)e^{-\omega^2/2}, \qquad \omega = \frac{V - \overline{V}}{\sigma_v}, \tag{5.2}$$

is employed. This parametrization is an extension of the traditional assumption to write down the LOSVD as a simple Gaussian profile with dispersion σ, amplitude γ (line-strength) and centered at the mean velocity $\overline{V}$. The even/odd coefficients h_n in the expansion are a measurement of the symmetric/asymmetric deviations from the Gaussian profile respectively. The lowest coefficients h_3 and h_4 are proportional to the coefficients of skewness and kurtosis of the distribution respectively. The Hermite polynomial H_3, H_4 expressed explicitly are

$$\begin{aligned} H_3(\omega) &= \frac{1}{\sqrt{3}}\left(2\omega^3 - 3\omega\right), \\ H_4(\omega) &= \frac{1}{\sqrt{24}}\left(4\omega^4 - 12\omega^2 + 3\right). \end{aligned} \tag{5.3}$$

The physical interpretation for the coefficients in 5.2 is still matter of discussion, even in the case of isolated galaxies. In elliptical galaxies, asymmetries arise from different scenarios, e.g. the superposition of two ordered motions with different kinematical parameters as in the case of slowly rotating bulge and a more rapidly rotating disc component [187]. Positive values of h_3 correspond to a distribution skewed towards velocities *lower* than the systemic velocity and conversely negative h_3 values correspond to a distribution skewed towards velocities *greater* than the systemic velocity [187]. In rapid rotating cores, large h_3 values are expected whilst h_4 is related with the velocity dispersion anisotropy in the galaxy core, it takes into account deviations from a Gaussian [188]. For $h_4 \geq 0$ the corresponding distribution is more peaked than a Gaussian at small velocities. Conversely, distribution less peaked than a Gaussian will have $h_4 \leq 0$. Higher moments are more difficult to interpret, but they are still useful for identifying structures in the residuals of the model fits.

In the context of interacting galaxies, Hermite moments have been used to study kinematical properties of merger remnants. Following ideas from Barnes [189–191], different models for low luminosity elliptical galaxy formation from disks progenitors have been explored by Cretton et al. [192, 193]) and the h_3 profile seems to be a good tracer to characterize remnants. The LOSVD is now parametrized by a set $\{\overline{V}, \sigma_v, \gamma, h_3, ...h_n\}$. In practise, the method works in the pixel space ($\ln\lambda$) and after convolve a stellar template with an initial guess of $L(\omega)$, the best set of parameters (γ, V, σ_v) and errors are recovered minimizing the model galaxy spectra and the observed spectra in the χ^2 sense that is pointed out in van der Marel et al. [185].

In order to compare the results given by the Gauss-Hermite method, we use also the Cross Correlation Method [177] to estimate radial velocities, and the results agree with the van der Marel method. One important point to drive right conclusions in the kinematical analysis, we should minimize the mismatch between the spectra. In our case, we tested several templates to get at least the correct trend for the two first coefficients in the expansion. The results for the long-slit spectroscopic dataset are in the following figures:

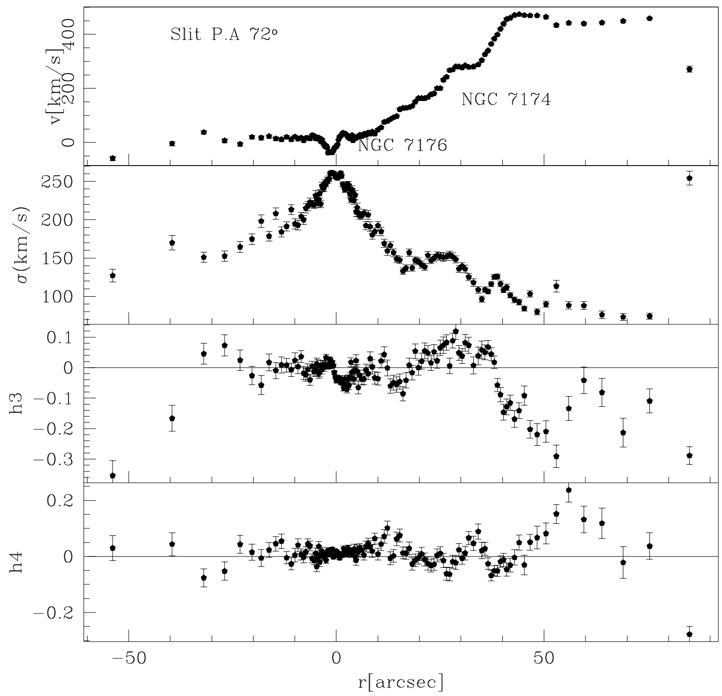

Figure 5.6: Kinematical quantities from the long-slit at 72° position angle

The figure 5.6 shows the rotation curve (upper panel), the velocity dispersion (middle panel) and Hermite coefficients (lower panels) for the slit connecting HCG90B and HCG90D.

In this case, we rebbined the spectra to a high signal to noise ratio (S/N $\geq$ 50) for all the templates. The region between the two galaxies is completely covered and the exposure time allows us to derive some more strong conclusions. The rotation curve connecting HCG90B and HCG90D is incredibly smooth and there is not evidence for uncorrelated motions [194]. The distance between the two galaxy centers is $\approx 25''$ meaning a physical separation of $\approx 5.7 \times 10^3$ pc[5] in projection. The rotation curve had been normalized to the HCG90B's systemic velocity. We can analyze three different regions in the curve. The left side shows the rotation for the elliptical galaxy dominated in the inner region for a depression $r \leq 5''$. We analyze in detail this feature in the next section (see figure 5.8). The most plausible explanation for this feature is a kinematically decoupled core in the galaxy [188]. The shape of the bump in the nucleus and the shape of the rotation curve in the outer parts force us to think that most

[5] We assume $h = 0.7$

natural interpretation for the core is a co-rotating one [188]. The outer part, between $5 \leq r \leq 20''$ which belongs to this galaxy shows a increasing rotation curve reaching a maximum value of $\approx 160\,\mathrm{km\,s^{-1}}$. A more clear evidence from a decoupled core comes from the velocity dispersion profile. There is a depression at the center, the two peaks at different locations and with different high are a strong sign of a two component system. It is also evident from the middle panel in figure 5.6 and stressed in the behavior of h_3, the sign change in this Hermite moment marks a departure of the core from the mean velocity in the LOSVD. All the kinematical features are in agreement with a co-rotating component (see section about HCG90B). Whilst the central part of the galaxy is kinematically dominated by the decoupled component, the right side in the rotation curve define the system as a fast rotator, for the entire galaxy the parameter ($\frac{v}{\sigma} \approx 0.7$). It was reported by Longo et al. [162] and it indicates a quite important percentage of rotational support for the galaxy (see figure 5.6). After a drop in the velocity dispersion for the elliptical galaxy at $r \approx 20''$ there is again a quite smooth transition in the velocity dispersion profile, even whether there is no a completely clear evidence for the interaction between HCG90B and HCG90D from the rotation curve, the S and U shapes in the velocity dispersion are a more evident insights for strong gravitational interaction [180]. These shapes have been explored for a set of input parameters in the simulations, as an example, the U-shape is a direct evidence for tidal coupling and tidal friction. There are arguments against the tidal friction in this small systems, specially to the small gravitational potential for the stellar component, however, the diffuse red star light around the group [159] indicates that all the core is in interaction and tidal forces and dynamical friction must play an important role there. In a recent study by Coziol et al. [149]) from a sample of 25 galaxies in Compact Groups have shown evidence for tidal interactions and mergers and their relation with galaxy morphology. On the other hand, several authors keep the Hermite parametrization to study interacting systems [193]. From their simulations some conclusions are very clear. Specially h_3 is a good indicator for merger remnants, features as sign changes are common in this parameter. In our case, even we evidence a trend in the Hermite coefficient to behave in the transition zone as a typical interacting pair, with sign change for h_3, is more difficult to reach a final conclusion about it.

The last part in curve shows the rotation curve for the disk galaxy. After the transition zone, the curve looks quite symmetrical respects to disk's center. After reach a maximum the curve shows a constant value in the very outer part ($r \geq 15$ arcsec). The velocity dispersion peaks at the center with a value of $150\,\mathrm{km\,s^{-1}} \pm 10\,\mathrm{km\,s^{-1}}$ and the asymmetry around the peak can be interpreted as a heating due to the tidal interaction with HCG90B [178].

Unfortunately, our signal for the slit connecting the two elliptical galaxies HCG90B and HCG90C (P.A 132°) has a shorter exposure time and almost all the region between the galaxies is missing in our study. The kinematical information for the slit P.A. 132° is plotted in figure 5.7. The distance between the galaxies is $\approx 22\,\mathrm{kpc}$ in projection. The rotation curve is shown in the very inner part. In the case of HCG90B the curve is very flat in this region as already it was found in Longo et al. [162] and the core is kinematically dominated by the decoupled component. The h_3 curve also shows the change from negative values to positive ones expected from the composite system. The features in the h_4 Hermite moment also shows a low value, natural from the wider velocity dispersion distribution. Along this direction there is also a particular kinematical feature in the rotation curve of HCG90C, there is a peak in the very inner part and h_3 also peaks negatively in this region. For this data seems that kind of decoupled core is also present in this galaxy. However, a better dataset must to be used to addressed this point. The narrow velocity dispersion is supported by the h_4 distribution. Next section is dedicated to analyze each galaxy separately.

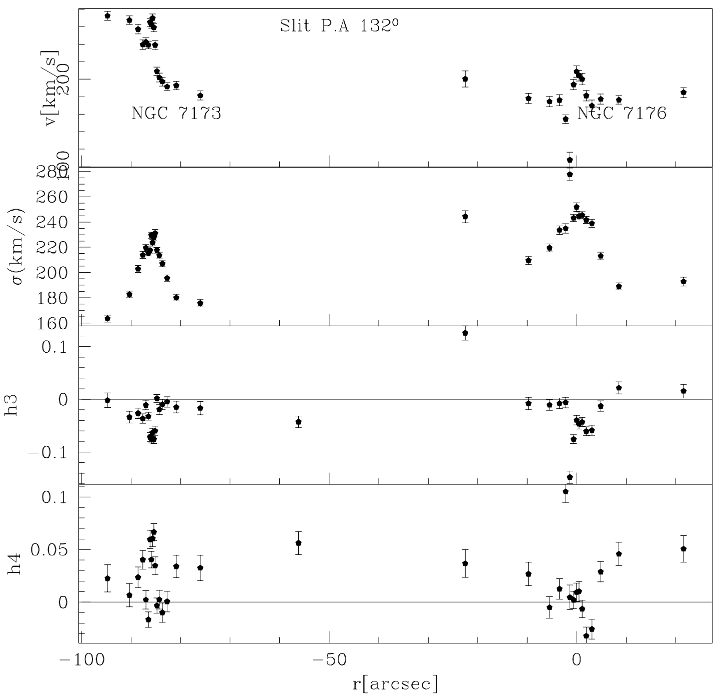

Figure 5.7: Kinematical quantities from the long-slit at P.A. 132°

5.5.3 Velocity dispersion profiles

The velocity dispersion profiles deserve a more detailed analysis. We analyze each galaxy in some detail. During the last years the knowledge about elliptical galaxies has increased. Old pictures which consider the shape of the galaxies as a consequence of pure rotation are reevaluated. Nowadays is known that anisotropic stellar pressure coming from velocity dispersion and low rotation are also common features in elliptical galaxies. Elliptical galaxies are also interesting for galaxy formation theories [193] and they are the dominant population in big structures as galaxy groups and clusters. In the case of HCG90 the core is dominated by small elliptical galaxies with particular kinematical properties. Is well known that big elliptical galaxies, coming from disk progenitors have decoupling kinematical systems, however, in the case of small elliptical inside of galaxy groups this picture is less favored. As a interesting result, a kinematical decoupled cores seem to be present in this work opening a new scenario to see the effects of tidal fields and dynamical friction and stressing the results found in [149].

- HCG90B

The figure 5.8 shows the rotation curve and the velocity dispersion profile along the slit at P.A. 72°. Longo et al. ([162]) have studied from long-slit data the kinematics for this galaxy. The velocity dispersion profile is typical from a elliptical galaxy, the maximum peak at $245\,\mathrm{km\,s^{-1}} \pm 10\,\mathrm{km\,s^{-1}}$ in perfect agreement with [162, 195], with a value for the central velocity dispersion of $242\,\mathrm{km\,s^{-1}}$ and $245\,\mathrm{km\,s^{-1}}$ respectively. The rotation curve shows a co-rotation in the inner part at $r \leq 5''$, it is symmetrical respect to the center and with amplitude of $50\,\mathrm{km\,s^{-1}}$. The doubled peak in the velocity dispersion profile seems to be a clear evidence for a composite system, even the difference is small (less than $20\,\mathrm{km\,s^{-1}}$) the depression is clear at the center. The rotation curve reaches a maximum value of $160\,\mathrm{km\,s^{-1}} \pm 15\,\mathrm{km\,s^{-1}}$

and as we mentioned, the galaxy is mostly rotation-supported [162].

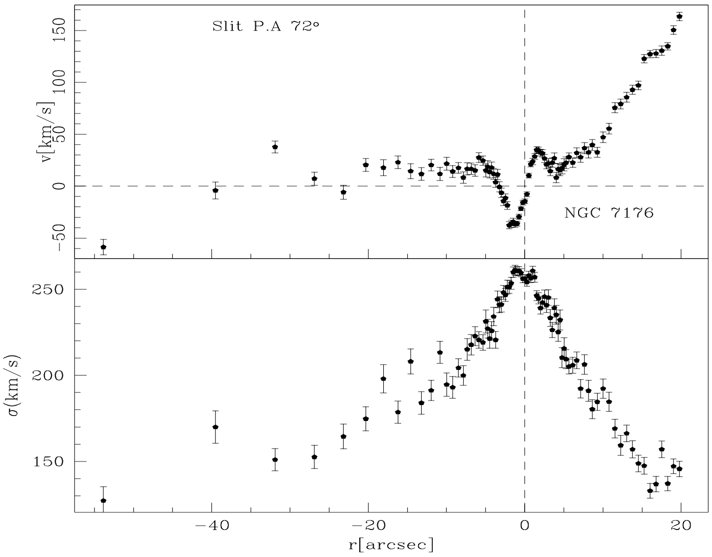

Figure 5.8: HCG90B Rotation curve and velocity dispersion profile at P.A. 72° Kinematically decoupled components are likely at $r \leq 5''$

The figure 5.9 shows the kinematics along the slit at P.A. 132° for HCG90B. The data does not allow us to study the outer part of the galaxy, due mainly to the low S/N ratio. The velocity pattern shows again the decoupled component at the center and the lower panel contain velocity dispersion for the composite system. From our dataset, no clear evidence for ionized gas was found in the galaxy. Our data does not reach the external part of the galaxy and the slit seems to be shifted from the kinematical center, however, from the rotation curves the galaxy has its stellar-axis at P.A. about 130° as it was noticed in [157] from the Longo's work.

- HCG90C

As in the last two figures, figure 5.10 contains the rotation curve and velocity dispersion for HCG90C. The velocity dispersion maximum is at $225\,\mathrm{km\,s^{-1}} \pm 15\,\mathrm{km\,s^{-1}}$ in concordance with $215\,\mathrm{km\,s^{-1}} \pm 32\,\mathrm{km\,s^{-1}}$ from previous studies. In the outer right part ($r \geq 12''$) which lies in the direction to HCG90B there is a clear increasing and asymmetry in the velocity dispersion profile, it may be due to a heating of the system from the interaction with the other galaxy. It has been the argument given in the Longo's work to claim the interaction between HCG90B and HCG90C.

In the very inner part, a peculiar increasing in the rotation curve is noticed, however, even most of the features expected from a decoupled component seem to be present, nevertheless, the low signal makes difficult give a conclusion.

- HCG90D

This galaxy is the most peculiar of the group. The morphology shows a very disturb disc galaxy with signs of interaction with the two early type galaxies in the core. From figure 5.11 a very smooth transition between the velocity field in HCG90B and HCG90D is observed. The rotation curve for the galaxy is symmetrical and there is a constant trend in the very outer part. The maximum along the slit

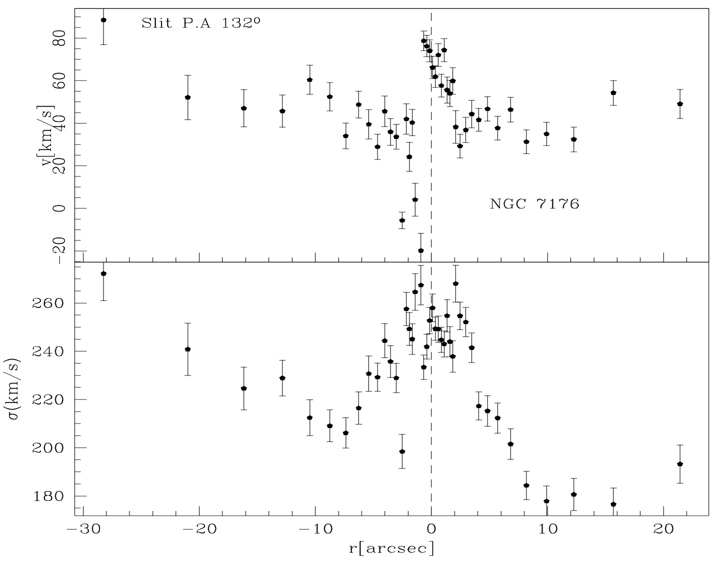

Figure 5.9: Rotation curve and velocity dispersion at P.A. 132°. The right panel is in the direction of HCG90C. The rotation velocity curve is dominated by a decoupled component at the center. In our case, the rotation curve is shown in the inner part of the galaxy.

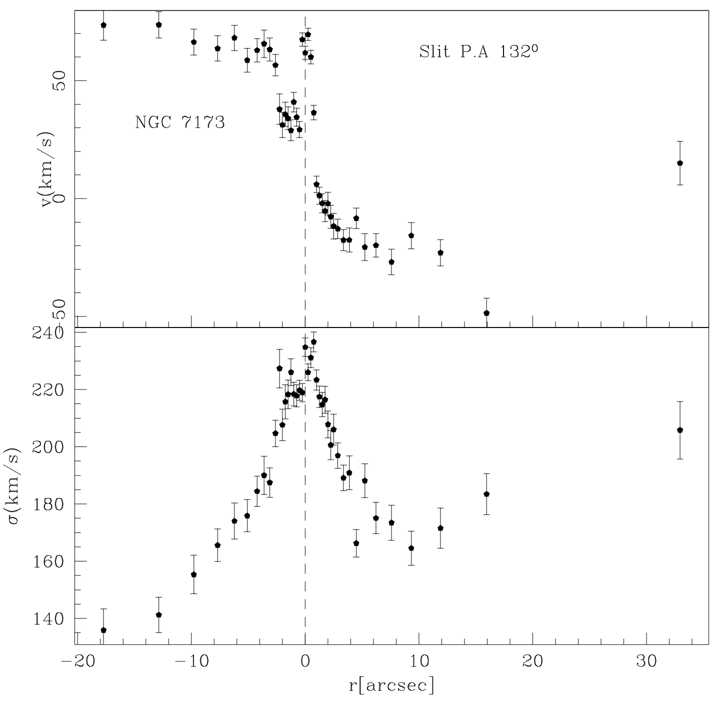

Figure 5.10: Kinematical quantities from the long-slit at P.A.132° for HCG90C

is higher $\approx 250\,\mathrm{km\,s^{-1}} \pm 15\,\mathrm{km\,s^{-1}}$ but very close to the value $227\,\mathrm{km\,s^{-1}}$ found by Longo et al. [162] at 68°. The velocity dispersion profile peaks around $154\,\mathrm{km\,s^{-1}}$. The velocity dispersion profile has ripples, the U-shape claimed by [162] is seen in the region $10'' \leq r \leq 20''$. From the velocity field is more difficult to see whether an interaction between HCG90B and HCG90D is taken place. On the other hand, given the shape of the velocity dispersion and its special smooth transition, with no scatter expected from two uncorrelated motions, makes the interaction picture the most natural explanation. From the ionized gas [157] reached the conclusion that the system is in interaction and it is favored due to the prograde nature of this two-galaxy system. It is important to comment, the LOSVD in the case of a spiral galaxy is not expected to be as simpler as 5.2. However, here we maintain the same function for the kinematical analysis closely to many other works [196].

5.5.4 Velocity field of the ionized gas

The warm gas in the system has been studied in detail in [157] and here we show the main features from our dataset.

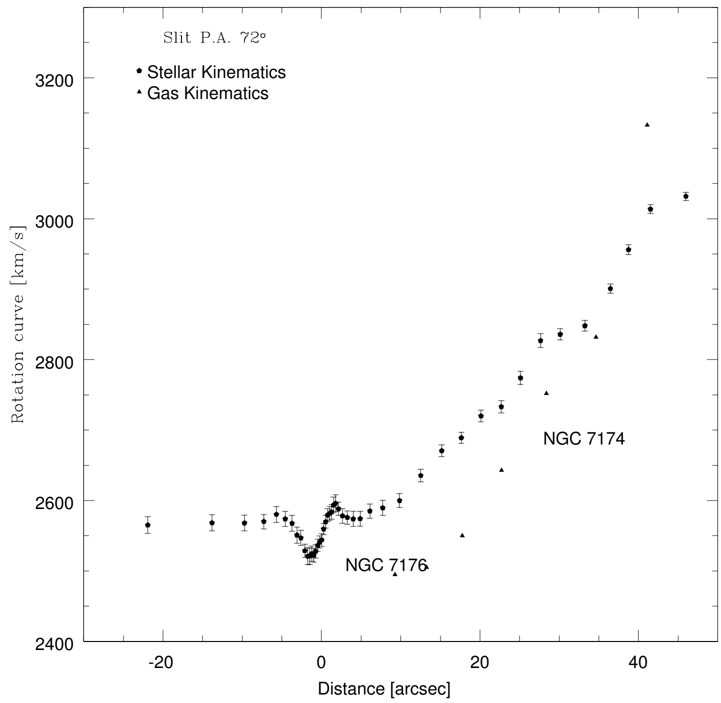

Figure 5.11: Gas kinematicas from the long-slit at P:A: 72°

The velocity field for the ionized gas is studied from the most prominent emission lines, as in the MXU mask, we use H_β and O_{III} separately to get the values. In the case of the slit at P.A 72° 5.11 the ionized gas is in the region between the two galaxies is more prominent in the disk galaxy, meaning that HCGD is hosting the gas. There is a clear shift between the gas and the stellar component seen also in the MXU data (figure 5.3). The gas velocity curve does not show the ripples found in the velocity dispersion profile (U-shapes). High resolution velocity maps in Plana's [157] work indicate a clear evidence for the stellar and warm gas velocity decoupling in the system. However, the most important feature from these maps is the gas-bridge between the galaxies and its no axisymmetric motions. The

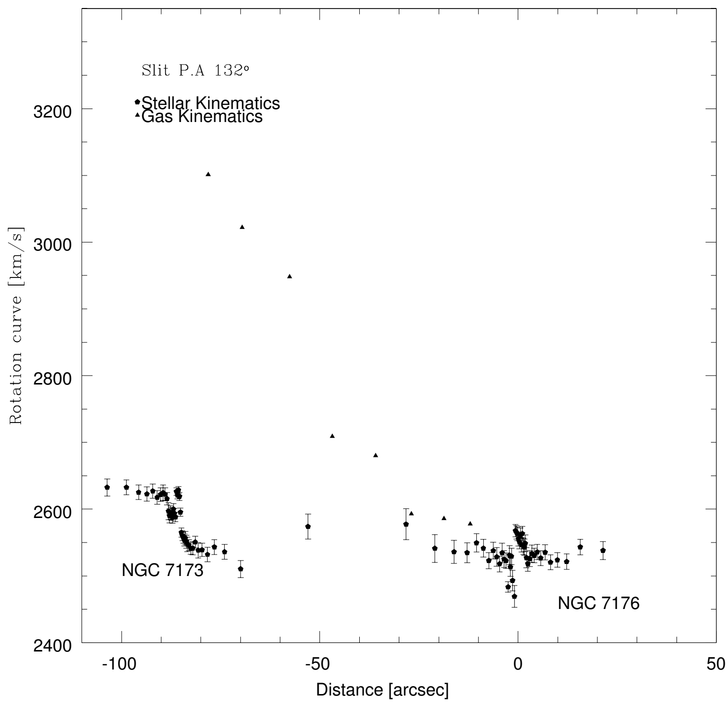

Figure 5.12: Gas Kinematics from the long-slit at P.A.132°

twist features belonging to the contours in the map (see [157] figure 9) could come from an interaction between these galaxies. In our case the ionized gas comes from the stellar component. In the case of slit connecting the two early type galaxies in the group HCGB and HCGC the situation is different as is shown in figure 5.12. The slit exposure time is shorter than the previous one, but as we mentioned before from the MXU spectra we can see a lower gas signal in this direction. It may be not surprising, early type galaxies are poor systems in their gas content. Further studies must to be addressed to clarify the gas content and kinematics for this galaxy.

5.6 Some remarks

We have presented a detailed kinematical analysis in HCG90 from the dataset which is described in the text. This chapter is already different from the rest of the thesis, and in some sense it is contained in two short publications made by the author [197, 198], some points have a broader explanation.

CHAPTER 6

Summary & Outlook

6.1 Summary Outlook

In this thesis we have studied some aspects of Modified Gravity in particular the proper treatment of the boundary conditions in the action for metric $f(R)$ gravity. We have introduced the proper boundary conditions in the extended action for metric $f(R)$ gravity using only the metric degree of freedom of the theory instead of the current scalar-tensor equivalence with ETGs. We have obtained the field equations with the condition $\delta g_{\mu\nu}|_{\partial\mathcal{V}} = 0$. In a similar way that the GYH term solves the boundary problem in GR, our boundary proposal solves the problem in metric $f(R)$ gravity. We have studied the equivalence of ETGs with scalar-tensor theories and some aspects as the equivalence between Brans-Dicke gravity and metric $f(R)$ was introduced. One central point in our results, common with any higher order derivatives gravity theories is the extra geometrical restriction we get in our boundary proposal, the geometrical condition $\delta R = 0|_{\partial\mathcal{V}}$. This aspect instead to be a defect of the proposal is a wonderful problem in all the higher order theories of gravity. In the equivalence with the scalar-tensor, this condition is encoded in $\delta\phi|_{\partial\mathcal{V}} = 0$. In our proposal, the conditions reflects more deep mathematical and physical insights. A simple example where boundary conditions and the Weak Equivalence Principle are faced in a concrete example of self-gravitating bodies is shown in [25]. The main aspect about the boundary conditions is whether or not, the theory as $f(R)$ generates an extra-scalar degree of freedom in the theory. With our work, the derivative $\frac{\mathrm{d}f}{\mathrm{d}R}$ seems to carry all the information of the scalar aspect of the theory. It is mandatory to continue with the mathematical aspects of $f(R)$ gravity. In general, the equation of motion in ETGs are of higher order than second order coming from GR and it opens a vast field of research in both physics and mathematics.

In this thesis and at least in the known literature by the author, we generalized for the first time the GDE to metric $f(R)$ gravity. The GDE is one of the most powerful tools to study geometrical aspects in manifold theory. In gravity, very important aspects as singularities of the spacetime and curvature properties are studied with the GDE. One important aspect is related with the example shown in the thesis when the GDE is employed in the particular case of the FLRW cosmologies. We have introduced two methods, the coordinate method where the 3 + 1 slicing of the FLRW is employed. In the 3 + 1 formalism, the spacetime is sliced into spacelike surfaces with normal vector n^α and a proper set of coordinates $\{x^0, x^i\}$ specifying also the extra-temporal coordinate x^0. We addressed the problem of the cosmological distances for a generic cosmological model in metric $f(R)$ gravity. The result is in a new parameter w_{eff} [43] which contains all the information about the $f(R)$ function. Further numerical analysis in necessary to consider our results as a standard tool to test cosmological models and make restrictions about the functional form of $f(R)$ gravity. As a complement for the thesis, a complete framework for cosmological models in the 1 + 3 formalism is introduced. In the 1 + 3 formalism, instead of slicing the spacetime, we use a timelike vector field u^a to threading the spacetime. We have summarized the main aspects of the WKB approximation for geometrical optics in the 1 + 3 formalism and the optical Sachs's equations are presented. We have demonstrated that for the FLRW spacetime the 1 + 3 and the 3 + 1 approaches are equivalent.

The best summary about chapter about cosmic magnetic fields is given by the comments from the referee of our publication [49]:

This is an interesting study of the evolution of seed magnetic fields in the universe using cosmological perturbation theory. Two different models are considered: one with standard FLRW metric background and magnetic field as perturbation, and second, where magnetic field is already present in the background model.

The work takes on an extensive task and derive nicely energy-momentum tensor to second order using gauge invariant quantities. The authors derive gauge invariant form of equation that couples metric perturbations with magnetic fields that can be effectively interpreted as the cosmological dynamo equation. Results are novel and intriguing, opening new prospects for the analysis of the magnetic field amplification at early stages of the universe expansion.

New research coming from our analysis about magnetic fields has been recently published [199]. The next step is to consider the possibility to incorporate our dynamo proposal in a MHD-code. The PENCIL code is one of the best candidates to start with this task [1]. The main goal is to introduce our cosmological dynamo proposal in the code. For this purpose, several projects for students in the *Grupo de Gravitación y Cosmología* have started.

A very intriguing question about the *equivalence* between 1 + 3 and gauge invariant formalisms related with cosmological perturbation theory is work in progress, for further information about the topic see [6].

The last chapter of the thesis is about the stellar and gas kinematics in the core of HCG90. As it was mentioned, this chapter is the first attempt by the author to write a PhD thesis in astronomy. We have obtained the rotation curves and velocity dispersion profiles for the three galaxies in the core of HCG90 from our spectroscopic data set. Unfortunately the photometric data for the group are useless for any scientific purpose. Very important consequences about of the matter distribution could be addressed with the kinematical information.

Fortunately the situation respect to available data to perform kinematical analysis has change in time. Our purpose with this topic is to write a proposal to carry a deep study of the kinematical properties of galaxy groups. A very important question to be studied is about the dark matter distribution in galaxy groups. Nowadays is not clear if each galaxy in a galaxy group is embedded in a dark matter halo or there is a common dark matter halo hosting all the galaxies. The models for the three spatial distribution of dark matter are still a problem in galactic dynamics and combined techniques as kinematics, dynamics and gravitational lensing have proved be a viable way to face the problem . Working with ideas as combined stellar templates and high resolution rotation curves together with velocity dispersion profiles could give us understanding about the distribution of baryonic and dark matter content in galaxy groups. It should be mention that even it is not completely surprise for the author, while this thesis is written, important changes as well known groups have started to report results using numerical $N-$body simulations in $f(R)$ gravity at cosmological scales, it is also important to think about how these theories work at galactic scales. Many open problem are now in the Gravity Group at Universidad Nacional de

[1] http://pencil-code.nordita.org/

Colombia, some of them coming from ideas exposed in the thesis.

APPENDIX A

Notation and conventions

For this work, the metric signature is $(-,+,+,+)$. The spacetime coordinates are x^μ with $\mu = 0,1,2,3$ and latin indices x^i (i= $1,2,3$) label the spatial coordinates. In a general basis the tensor indices are denoted by a,b, $\cdots = 0,1,2,3$; $i, j, \cdots = 1,2,3$ whilst in a coordinate basis by $\mu, \nu, \cdots = 0,1,2,3$; $i, j, \cdots = 1,2,3$. The most used conventions and symbols are:

G	Gravitation constant (6.67×10^{-11} m^2/kgs^2)
c	Light velocity (3.00×10^8 km s^{-1})
$\hbar$	Planck constant (1.05×10^{-27}erg-sec)
$g_{\alpha\beta}$	Metric tensor
g	Metric tensor determinant
$R_{\alpha\beta\gamma\delta}$	Riemann tensor
$R_{\alpha\beta}$	Ricci tensor
R	Ricci Scalar
$\Gamma^\alpha_{\beta\gamma}$	Christoffel symbol
$G_{\alpha\beta}$	Einstein tensor
∂_α	Partial derivative x^α
$_{,\alpha}$	Partial derivative x^α
∇_α	Covariant derivative
$\Box$	d'Alambertiano $\Box = \nabla^\gamma \nabla_\gamma$
$\frac{D}{D\nu}$	Covariant derivative along a curve
$A_{(\alpha\beta)}$	Symmetrization
$A_{[\alpha\beta]}$	Antisymetrization
$\dot{}$	Temporal derivative
$T_{\alpha\beta}$	Energy-Momentum tensor
$a(t)$	Dimensionless scala factor
t	Cosmic time
η	Conformal time
z	Redshift
H	Hubble parameter
k	Spatial curvature

ρ	Energy density
p	Pressure
Λ	Cosmological constant
d_L	Luminosity distance
D_A	Angular diameter distance

For the Einstein Field equations, we should choose one of the following conventions

$$\eta^{\mu\nu} = [s1] \times (-1, 1, 1, 1)$$

$$R^{\mu}_{\alpha\beta\gamma} = [s2] \times \left[\Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\mu}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha}\right]$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = [s3] \times \kappa T_{\mu\nu}$$

The $[s3]$ sign is related with the convention

$$R_{\mu\nu} = [s2] \times [s3] \times R^{\alpha}_{\mu\alpha\nu}.$$

Here, we choose $(+, +, +)$. The κ factor depends which units for the coordinates (x^{μ}) is used. Usually when $c = 1$ is straightforward to write $(\kappa = 8\pi G)$, otherwise one should write $\left(\kappa = \frac{8\pi G}{c^2}\right)$ or $\left(\kappa = \frac{8\pi G}{c^4}\right)$ when time or length units are employed for physical quantities.

APPENDIX **B**

Basic definitions

In this appendix, we introduce the notation and conventions used in this thesis from [1]. Most of the definitions and mathematical tools are the standard ones, but the purpose is to make the text self-consistent.

B.1 Basic definitions

Definition 1 (Spacetime manifold) *The mathematical model for the spacetime, i.e. the collection for all the physical events is a pair* $(\mathcal{M}, \mathbf{g})$*, with* $\mathcal{M}$ *a four-dimensional connected Hausdorff* C^∞ *manifold and* $\mathbf{g}$ *a Lorentz metric with signature* $(-,+,+,+)$ *on* $\mathcal{M}$*.*

Definition 2 *Let* $\mathcal{F}$ *the set of functions on* $\mathcal{M}$

$$\mathcal{F} : f \to R \tag{B.1}$$

$$f|_{p\in\mathcal{M}} := f(x^\alpha(p)), \tag{B.2}$$

with $\{x^\alpha \in R^4\}$ *the set of coordinates of* $p \in \mathcal{M}$*.*

Definition 3 (Tangent vector) *A tangent vector in* $p \in \mathcal{M}$ *is a differential directional operator acting on* $\mathcal{F}$*. The tangent vector* $\mathbf{V}$ *obeys the usual differentiation rules of sums and products. The set of tangent vectors at* p *is called the tangent space* $T_p(\mathcal{M})$ *and is a vector space with the usual rules of linear combinations. Writing in a coordinate basis*

$$\mathbf{V}(f) = V^\alpha \frac{\partial f(x^\beta)}{\partial x^\alpha}. \tag{B.3}$$

In (B.3) the Einstein summation convention is used: an repeated index exactly twice in a product, once as superscript and once a subscript is to be summed over all its possible values.

Definition 4 (Maps between manifolds) *Let* h *be a function between a* $m-$*dimensional manifold* $\mathcal{M}$

and a n−dimensional manifold N

$$h : \mathcal{M} \to \mathcal{N}, \tag{B.4}$$
$$p \in \mathcal{M} \to q \in \mathcal{N}, \tag{B.5}$$
$$h(p) = q. \tag{B.6}$$
$$\tag{B.7}$$

The equation (B.4) induces a map h_* between tangent spaces

$$h_* : T_p\mathcal{M} \to T_q\mathcal{N}, \tag{B.8}$$
$$V \in T_p \to h_*V \in T_q, \tag{B.9}$$
$$h_*V(f) := V(f(h(p))), \ \forall f \in \mathcal{N}. \tag{B.10}$$

The equation (B.8) defines the push-forward-map.

Definition 5 ($1-$forms or covariant vector) *The dual-space vector $T^*_p\mathcal{M}$ defines the $1-$ forms or covariant vector as the set of the linear operators on $T_p\mathcal{M}$*

$$\omega \in T^*_p\mathcal{M}, \tag{B.11}$$
$$\forall \mathbf{V} \in T_p\mathcal{M}, \tag{B.12}$$
$$\omega(\mathbf{V}) \to R. \tag{B.13}$$

The equation (B.4) defines a natural map h^* as

$$h^* : \ T^*_q\mathcal{N} \to T^*_p(\mathcal{M}), \tag{B.14}$$
$$\omega(h_*\mathbf{V}) := (h^*\omega)(\mathbf{V}) \tag{B.15}$$

denoted as the pullback.

Definition 6 *A tensor of type (p, q) is an operator acting linearly on each of a number of copies of a vector space V and its dual V^*, giving a real number $T(\mathbf{v}_1, ..., \mathbf{v}_q, \omega^1, ...\omega^q)$.*

Definition 7 *The metric tensor is a T^0_2 tensor with the properties*

$$g(X, Y) = g(Y, X) = g_{\alpha\beta}X^\alpha Y^\beta \quad (X, Y) \in T_p\mathcal{M} \tag{B.16}$$
$$g(X, \alpha Y + \beta Z) = \alpha g(X, Y) + \beta g(X, Z) \quad (X, Y, Z) \in T_p\mathcal{M}, \ (\alpha, \beta \in R). \tag{B.17}$$

Definition 8 *The metric determinant is defined by*

$$g \equiv det[g_{\alpha\beta}], \tag{B.18}$$

defines the invariant volume element

$$\sqrt{-g}d^4x. \tag{B.19}$$

Definition 9 (Lie Derivatives) *The manifold structure admits a naturally differential operator called*

"Lie differentiation". Let V a smooth vector field on $\mathcal{M}$. The vector field V defines the operator

$$\mathcal{L}_{\mathbf{V}} f = f_{,a} V^a, \tag{B.20}$$

$$\left(\mathcal{L}_{\mathbf{V}} W\right)^d = [V, W]^d = \left(V^c W^d_{,c} - W^c V^d_{,c}\right), \tag{B.21}$$

$$\left(\mathcal{L}_{\mathbf{V}} T\right)_{ab} = T_{ab,c} V^c + T_{cb} V^c_{,a} + T_{ac} V^c_{,b}. \tag{B.22}$$

The equation (B.20) is the fundamental tool to study the cosmological perturbation theory in the active approach where in the extended manifold $\mathcal{N}$ the higher orders are defined in a natural way, for details see chapter 4.

Definition 10 (Covariant derivative) *The covariant derivative ∇_c is a differential operation obeying the Leibniz rule and*

- *The covariant derivative for a scalar function f*

  $$\nabla_a f = \partial_a f \equiv f_{,a}, \tag{B.23}$$

  with $f \in F(M)$ and, $F(M)$, the set of scalar functions defined on $\mathcal{M}$.

- *For vectors fields, the covariant derivative defines the connection Γ in the following form:*

  $$\nabla_a V^b = \partial_a V^b + \Gamma^b_{ca} V^c, \tag{B.24}$$

  and for the 1*−forms field ω_b*

  $$\nabla_b \omega_c = \partial_b \omega_c - \Gamma^d_{cb} \omega_d. \tag{B.25}$$

- *In the general case, for a tensor field*

  $$\begin{aligned}\nabla_d T^{abc\dots\ e}{}_{fg\dots h} = T^{abc\dots\ e}{}_{fgh,d} + \Gamma^a_{rd} T^{rbc\dots e}{}_{fg\dots h} \cdots + \Gamma^e_{rd} T^{abc\dots\ r}{}_{fgh} \\ - \Gamma^r_{fd} T^{abc\dots\ e}{}_{rg\dots h} \cdots - \Gamma^r_{hd} T^{abc\dots\ e}{}_{fgr} \equiv T^{abc\dots\ e}{}_{fg\dots h;d}\end{aligned} \tag{B.26}$$

- *The covariant derivative naturally induces a map between vector fields $\mathbf{V}$ and differential operators by*

  $$\nabla_{\mathbf{V}} T^{ab\dots c}_{kl\dots m} = T^{ab\dots c}_{kl\dots m;r} V^r. \tag{B.27}$$

  When V^r is the tangent vector to a curve with parameter λ, for any quantity Q

  $$\frac{DQ}{D\lambda} = Q_{;b} V^b. \tag{B.28}$$

- *The equation (B.27) for the special case*

  $$V^a_{;b} V^b = 0, \tag{B.29}$$

  selects a special family of curves on M named geodesics. *In components (B.29) takes the form*

  $$\frac{d^2 x^a}{d\lambda^2} + \Gamma^a_{bc} \frac{dx^b}{d\lambda} \frac{dx^c}{d\lambda} = 0, \tag{B.30}$$

  with $V^a = \frac{dx^a}{d\lambda}$.

- *The connection adds an additional tensor on M, the* torsion, T^a_{bc} *defined by*

$$T^a_{bc} \equiv \Gamma^a_{bc} - \Gamma^a_{cb}. \tag{B.31}$$

- *The* Levi-Civita *connection is torsion free $T^a_{bc} = 0$, or equivalently $\Gamma^a_{bc} = \Gamma^a_{cd}$. In this thesis, we work under this condition.*

- *The metric tensor g_{ab} satisfaces the metricity condition, $\nabla_c g_{ab} = 0$, and for the torsion free condition defines the* Christoffel symbols*:*

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(g_{bd,c} + g_{dc,b} - g_{bc,a} \right), \tag{B.32}$$

The non-commutation of the covariant derivatives defines the curvature, R^d_{abc}, or *Riemann tensor* given by

$$\left[\nabla_b, \nabla_c \right] V^d \equiv \nabla_b \nabla_c V^d - \nabla_c \nabla_b V^d = -R^d_{abc} V^a. \tag{B.33}$$

The equation (B.33) in components is

$$R^d_{abc} = \Gamma^d_{ac,b} - \Gamma^d_{ab,c} + \Gamma^d_{eb} \Gamma^e_{ac} - \Gamma^d_{ec} \Gamma^e_{ab}, \tag{B.34}$$

where we used

$$\begin{aligned} \nabla_b \nabla_c V^d = \partial_b \partial_c V^d + \Gamma^d_{eb,c} V^e + \Gamma^d_{eb} V^e_{,c} - \partial_e V^d \Gamma^e_{bc} \\ - \Gamma^f_{bc} \Gamma^d_{ef} V^e + \partial_b V^e \Gamma^d_{ec} + \Gamma^d_{fc} \Gamma^f_{eb} V^e, \end{aligned} \tag{B.35}$$

together with the torsion free condition. From (B.33) the *Ricci tensor* is defined as

$$R_{ac} \equiv R^d_{bdc} = \Gamma^d_{ac,d} - \Gamma^d_{ad,c} + \Gamma^d_{ed} \Gamma^e_{ac} - \Gamma^d_{ec} \Gamma^e_{ad}, \tag{B.36}$$

and the *Ricci scalar*

$$R \equiv g^{ac} R_{ac} = g^{ac} \Gamma^d_{ac,d} - g^{ac} \Gamma^d_{ad,c} + g^{ac} \Gamma^d_{ed} \Gamma^e_{ac} - g^{ac} \Gamma^d_{ec} \Gamma^e_{ad}. \tag{B.37}$$

B.1.1 Hypersurfaces

One of the most used concepts when we are dealing with boundary problems in a field theory is the "hypersurface" in a spacetime. Mathematically a hypersurface Σ in an $n-$dimensional $\mathcal{M}$ manifold is an $n-1$dimensional submanifold. We can cover Σ by a set of local coordinates $(u^1, ..., u^{n-1}, y)$ with the condition for a family of non-intersecting surfaces $y = \text{const}$. Given the mathematical structure in $\mathcal{M}$, several natural definitions an extensions can be defined on Σ. There is a injection map

$$i : \Sigma \to \mathcal{M} \tag{B.38}$$

$$p \in \Sigma \to i(p) \equiv p \in \mathcal{M}, \tag{B.39}$$

which identifies a point on Σ with itself in $\mathcal{M}$. Also a map between vector and tensor fields are defined by i, named i^*. One particular case is the *three-dimensional metric* induced by the four-dimensional metric tensor g

$$i^* g = h, \tag{B.40}$$

with h the metric on Σ. The tensor field h is called *first fundamental form* of Σ. We restrict our analysis to $n = 4$, but natural extensions to $n-$dimensions can be easily made with mathematical careful. In the local coordinates (u^1, u^2, u^3), the components for h are

$$h_{ij} = g_{ab} e^a_i e^b_j \qquad \text{with } \{i, j = 1, 2, 3\}, \tag{B.41}$$

where e^a_j are the components for the basis vectors $\{e^a\}$. The set $\{i, j\}$ only denotes the dimension of the submanifold, in this case does not mean spatial components, because we are not restricted to spacelike surfaces. There are two important mathematical facts about hypersurfaces. When exist a non null unit vector n^a, with $n_a n^a = \epsilon$ orthogonal to Σ, we can classify the hypersurfaces as

$$\begin{aligned} \epsilon = -1 &\implies \quad \Sigma \text{ is spacelike}, \\ \epsilon = 1 &\implies \quad \Sigma \text{ is timelike}. \end{aligned} \tag{B.42}$$

Given a set of coordinates $\{x^\alpha\}$ in $\mathcal{M}$, the hypersurfaces can be generated by a function $\Phi(x^\alpha)$ with the condition (see fig B.1)

$$\Phi(x^\alpha) = \text{constant}, \tag{B.43}$$

and it induces a set of parametric equations $x^\alpha = x^\alpha(u_i)$. The normal vector n^α is then the gradient to the Φ function

$$n^\alpha = \frac{\epsilon \Phi_{,\alpha}}{\left| g^{\alpha\beta} \Phi_{,\alpha} \Phi_{,\beta} \right|^{1/2}}, \tag{B.44}$$

where n^α points in the direction of increasing Φ. The unit vector n^α enables a definition for the four-

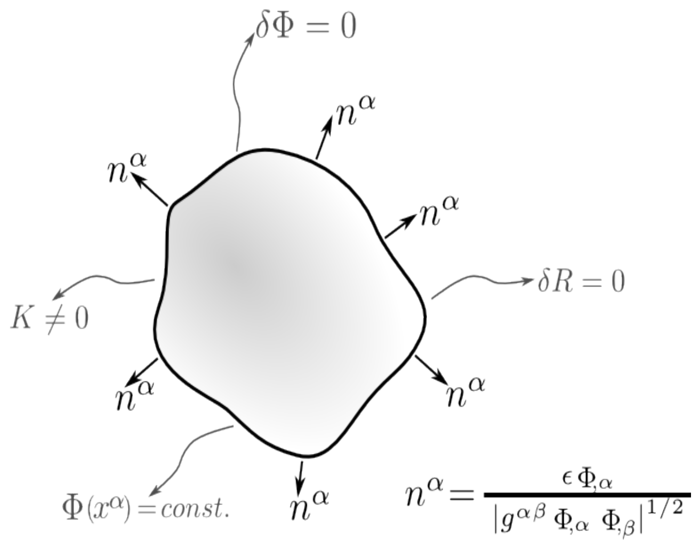

Figure B.1: Hypersurface in $\mathcal{M}$

dimensional projector

$$h_{ab} = g_{ab} - \frac{n_a n_b}{n_a n^a}, \tag{B.45}$$

which projects geometrical objects from $\mathcal{M}$ to Σ. Other very important consequence from the vector field n^α is the *second fundamental form* or *extrinsic curvature* on Σ. Taking the map

$$\mathbf{K} := i^* (\nabla \mathbf{n}), \tag{B.46}$$

is a second rank tensor with components in (u^1, u^2, u^3)

$$K_{ij} = e^a_i e^b_j n_{a;b} \tag{B.47}$$

and is symmetric because the connection in the covariant derivative is; it can be calculate entirely on Σ. When n^α is non null, $\mathbf{K}$ can be considered as a four-dimensional tensor using the projector (B.45)

$$K_{ab} = h^c_a h^d_b n_{c;d}. \tag{B.48}$$

When the normal vector n^α is null, one can get a projection from M to Σ using a non-zero vector field $\mathbf{l}$ not lying in Σ, it can be chosen with the condition $\mathbf{n}(\mathbf{l}) = 1$ and building a projector tensor $P^a_b = \delta^a_b - l^a n_b$, which maps any vector V^b to

$$P^a_b V^b = \delta^a_b V^b - l^a n_b V^b = V^a - l^a(n_b V^b), \tag{B.49}$$

in Σ, because

$$V^a n_a - l^a(n_b V^b) n_a = 0. \tag{B.50}$$

For some physical applications [1](cosmology, gravitational collapse, ...), the case when Σ is spacelike is of fundamental interest. Denoting $|$ the covariant derivative on Σ, the Riemann tensor obeys the usual rule for each vector orthogonal to n^α $(n^\alpha V_\alpha = 0)$

$$^3R_{ijkl} V^i = V_{j|kl} - V_{j|lk}, \tag{B.51}$$

and using the projector (B.45) for the second covariant derivative together with (B.47), we get the Gauss equation

$$^3R_{ijkl} = R_{ijkl} - K_{ik}K_{jl} + K_{jk}K_{li}, \tag{B.52}$$

showing the relation between the three-dimensional curvature with the four-dimensional curvature through the extrinsic curvature. Another important result for a spacelike surface is

$$R^a_{jkm} n_a = K_{jm|k} - K_{jk|m}, \tag{B.53}$$

which is known in the literature as the Codazzi equation.

The trace for (B.48) gives the important relation

$$K := g^{ab} K_{ab} = h^{ab} K_{ab}. \tag{B.54}$$

While the induced metric h_{ab} deals with the intrinsic properties of the hypersurface, the tensor K_{ab} characterizes the extrinsic aspects, it means, the way in which the hypersurface is embedded in the spacetime manifold.

B.1.2 Gauss-Stokes theorem

Theorem 1 (Gauss theorem) *Let $\mathcal{V}$ a finite region of the spacetime $\mathcal{M}$, bounded by a closed hypersurface $\partial\mathcal{V}$ with not restrictions about the signature of the hypersurface (see figure B.1). For any vector field A^α defined within $\mathcal{V}$,*

$$\int_{\mathcal{V}} A^\alpha_{;\alpha} \sqrt{-g} d^4x = \oint_{\partial\mathcal{V}} A^\alpha d\Sigma_\alpha, \tag{B.55}$$

with $d\Sigma_\alpha := \epsilon n_\alpha d\Sigma$ when n^α is non null and the "surface element" $d\Sigma \equiv |h|^{1/2} d^3u$. The equation (B.55) is the Gauss-Stokes theorem*. For an excellent review and proof in a general case see [29]. It is important to notice that $\partial\mathcal{V}$ may have sections that are timelike, spacelike or null, for the last case, one should choose the proper volume- orientation.*

B.2 A note on the variational principle in field theories

Variational methods play a big role in physics and mathematics and gravity is not the exception. Field theories, including GR and extended theories have been formulated in terms of the *Lagrangian* and *Hamiltonian* approaches. One of the motivations for our work [10] is the generally missing fact of a well-posed mathematical problem. Indeed, we have to include the boundary conditions in any field theory in order to have a well established set of equations for the variables in the theory. Despite the mathematical formal requirement to have a well posed mathematical problem, boundaries are also necessary in order to account for deep physical reasons, as an example, the GHY term in GR is required to define properly the path integral in GR [12]. In black hole theory, the entropy comes from entirely from the boundary term when the semiclassical approach is used. The direct way to get the field equations in a field theory is start from the local Lagrangian density in the form

$$\mathcal{L} = \mathcal{L}([\phi], x^\alpha), \tag{B.56}$$

with $[\phi] \equiv \phi, \partial_a\phi, ...$ denotes the field dependence for the Lagrangian density and x^a are the coordinates for the spacetime. The action S is introduced formally as

$$S = \int \sqrt{-g} \mathrm{d}^4x \mathcal{L}([\phi], x^\alpha). \tag{B.57}$$

One then get the equation of motion varying the equation (B.57) respect to ϕ^i

$$\delta S = 0, \tag{B.58}$$

and when the boundaries are ignored due to the fact that physical boundaries do not exist or their effects can be ignored in the local dynamics, the equation (B.58) could be integrated by part and ignoring all the boundary contributions is equivalent to

$$\frac{\delta\mathcal{L}([\phi], x^\alpha)}{\delta\phi^i} = \frac{\partial\mathcal{L}([\phi], x^\alpha)}{\partial\phi^i} - \partial_\mu \frac{\partial\mathcal{L}([\phi], x^\alpha)}{\partial(\partial_\mu\phi^i)} + \partial_\mu\partial_\nu \frac{\partial^S \mathcal{L}([\phi], x^\alpha)}{\partial(\partial_\mu\partial_\nu\phi^i)} + ... = 0, \tag{B.59}$$

known in the literature as the **Euler-Lagrange equations**. The notation in (B.59) for high-order derivatives respect to the fields is

$$\frac{\partial^S}{\partial(\partial_{\mu_1}...\partial_{\mu_k}\phi^i)} \partial_{\nu_1}..\partial_{\nu_k}\phi^j = \delta^j_i \delta^{\mu_1}_{(\nu_1}...\delta^{\mu_k}_{\nu_k)}, \tag{B.60}$$

the symmetrization avoids overcounting multiple derivatives in (B.59). The formal solution to (B.59) offers the possibility to find symmetries and conserved Noether quantities from the Lagrangian density [9, 200]. However, without boundary conditions formally the system (B.59) (has for a consistent Lagrangian density) many solutions. The boundary conditions should bring the *true solution* for the theory. The boundary conditions must work properly in both directions, setting the true solution for (B.59) and making the system wellposed. The sentence *well posed* must be treated carefully, we follow [9] and the condition for a well posed problem is based in the existence of a class of boundary conditions such that given a choice within this class, there exist a unique solution for (B.59) compatible with such choice. The Euler-Lagrange system for a wide range of applications is hyperbolic demanding the specification of all possible choices of initial conditions and velocities for the fields, and all possible choices of spatial boundary conditions at all times.

To gain some insight about of a well-formulated problem in GR or any ETGs the mathematical-requirement can be described as: Using the 3 + 1 set of equations of motion for the theory, a well-formulated problem consists in take the equations of motion to the form [3]

$$\partial_t \mathbf{u} + M^i \nabla_i \mathbf{u} = \mathbf{S}(\mathbf{u}), \tag{B.61}$$

where $\mathbf{u}$ denotes the fundamental variables $\{h_{ij}, K_{ij}, \dots\}$ of the Arnowitt-Deser-Misner (ADM) splitting, M^i is the characteristic matrix of the system and $\mathbf{S}(\mathbf{u})$ is the source and it contains the fundamental variables but not their derivatives. The initial value formulation is well-posed if the system (B.61) is symmetric hyperbolic (M^i is symmetric) and strongly hyperbolic (i.e., $M^i s_i$ has a real set of eigenvalues and a complete set of eigenvectors s_i, and obeys some boundedness conditions). In the case of metric $f(R)$ gravity the ADM system analogous to (B.61) is given in the chapter 3 of [3]. One can use the GYH term in the ETGs and analyze the connection with the initial-value problem. For a special case of the Holst action and applications to Loop Quantum Gravity see [201].

The action (B.57) encodes more information than be a formal device to get the equation of motion for a theory. The main aspect in the formulation of a field theory is that *among all the possible field configurations given boundary conditions, the true configuration extremizes the action [9]*. For a bounded region $\mathcal{V}$ with a general boundary with timelike region $\mathcal{V}$ representing the *spatial boundary* and spacelike edges Σ where we set the generic initial conditions. The criteria for a well-posed problem is: *when we restrict the field configurations consistent with the boundary condition* $\{\phi(\mathcal{V}), \phi(\Sigma)\}$, *the unique solution to the (B.59) system should be the extremum for the action*

$$S[\phi(\mathcal{V})] = \int_{\mathcal{V}} d^4x\, \sqrt{-g}\mathcal{L}(\phi, x^\alpha). \tag{B.62}$$

For a complete revision and examples from classical mechanics to quantum field theory with some comments about GR see [9, 200] and references therein. We only emphasize that there is a well established and powerful procedure in field theory where the variational methods together with the correct treatment of the boundaries achieves the goal for the well posed mathematical problem.

Einstein Field Equations are a postulate in GR and they can be derived from the Einstein-Hilbert action. However, the Einstein-Hilbert action is not the only one which gives the bulk equations. As we showed, a GYH term produces the same set of equations but makes the action well posed. But a modification for the Einstein-Hilbert action with a GYH term is not unique, as is point out in [9]. Einstein himself used instead of a Lagrangian with the Ricci scalar R (B.37) an action of the form

$$H = g^{\alpha\beta}(\Gamma^\nu_{\mu\alpha}\Gamma^\mu_{\nu\beta} - \Gamma^\mu_{\mu\nu}\Gamma^\alpha_{\alpha\beta}) = R - \nabla_\alpha A^\alpha, \tag{B.63}$$

where we must use $\nabla_\alpha g^{\beta\delta} = 0$ and the four-vector

$$A^\alpha = g^{\mu\nu}\Gamma^\alpha_{\mu\nu} - g^{\alpha\mu}\Gamma^\nu_{\mu\nu}. \tag{B.64}$$

The four-vector (B.64) divergence is

$$\nabla_\alpha A^\alpha = g^{\mu\nu}\Gamma^\alpha_{\mu\nu,\alpha} - g^{\alpha\mu}\Gamma^\mu{}_{\mu\nu,\alpha} + g^{\mu\nu}{}_{,\alpha}\Gamma^\alpha_{\mu\nu} - g^{\alpha\mu}{}_{,\alpha}\Gamma^\nu_{\mu\nu} + g^{\mu\nu}\Gamma^\alpha{}_{\alpha\delta}\Gamma^\delta_{\mu\nu} - g^{\delta\mu}\Gamma^\nu_{\mu\nu}\Gamma^\alpha_{\alpha\delta}. \tag{B.65}$$

and combining (B.65) with the metric compatibility expressed as

$$g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu_{\alpha\sigma}g^{\sigma\nu} - \Gamma^\nu_{\alpha\sigma}g^{\mu\sigma}, \tag{B.66}$$

the Lagrangian (B.63) is recovered. When we vary the equation (B.63) it reproduces the Einstein equations with the boundary condition $\delta g^{\mu\nu}\big|_{\partial\mathcal{V}} = 0$. The modified Einstein-Hilbert action from (B.63) is now

$$\int_{\mathcal{V}} \sqrt{-g}\mathrm{d}^4xH = \int_{\mathcal{V}} \sqrt{-g}\mathrm{d}^4x(R - \nabla_\alpha A^\alpha), \tag{B.67}$$

and using (B.55) the boundary contribution arises as

$$\int_{\mathcal{V}} \sqrt{-g}\mathrm{d}^4xH = \int_{\mathcal{V}} \sqrt{-g}\mathrm{d}^4xR - \oint_{\partial\mathcal{V}} A^\alpha n_\alpha \sqrt{|h|}\mathrm{d}^3y, \tag{B.68}$$

and employing (B.45) the contraction

$$A^\alpha n_\alpha = -2K + 2h^{\alpha\beta}\partial_\beta n_\alpha - n^\mu h^{\nu\sigma}\partial_\nu g_{\sigma\mu}, \tag{B.69}$$

gives the GYH term for GR plus a function

$$2h^{\alpha\beta}\partial_\beta n_\alpha - n^\mu h^{\nu\sigma}\partial_\nu g_{\sigma\mu}, \tag{B.70}$$

clearly showing that the GYH term is not the only choice which reproduces the Einstein field equations in the bulk. The Lagrangian (B.63) only contains first order terms in the metric. The variation with respect to $g^{\mu\nu}$ of (B.70) vanishes with the boundary condition $\delta g_{\alpha\beta}|_{\partial\mathcal{V}} = 0$. The induced metric on the boundary has six independent components as unconstrained variables. The gauge invariance of the theory, as is the case of GR, leaves only two independent components, it means, the boundary term does not affect the number of freedom of the theory. In $f(R)$ gravity or in scalar-tensor theory there are more constraints on the boundary. In the scalar-tensor theory, the extra boundary condition is in the scalar field $\delta\phi\big|_{\partial\mathcal{V}} = 0$. When we consider an action only coming only from geometric invariants, as we described in ([10]), the constraint $\delta R\big|_{\partial\mathcal{V}} = 0$ is claimed to carry the scalar degree of freedom of the theory [9, 24].

B.3 "1+3 definitions"

The "1+3" formalism requires some definitions. We summarize the most used in this thesis, for a complete reference see [1].

Definition 11 (Alternating volume tensor) *The totally skew pseudotensor*

$$\eta^{abcd} \equiv \eta^{[abcd]} \quad , \eta^{0123} = (-g)^{-1/2} \tag{B.71}$$

or equivalently, the equation (B.71) defines the alternating volume tensor

$$\eta_{abcd} = -\sqrt{-g}\delta^0_{\ [a}\delta^1_{\ b}\delta^2_{\ c}\delta^3_{\ d]}. \tag{B.72}$$

The tensor (B.71) is preserved under parallel transport, it means

$$\eta^{abcd}_{\quad\ ;e} = 0. \tag{B.73}$$

Definition 12 (Projected symmetric tracefree (PSTF) parts) *The projector h_{ab} enables the following definitions*

$$V_{\langle a\rangle} \equiv h_a^{\ b} V_b, \tag{B.74}$$

and for a second rank tensor

$$S_{\langle ab\rangle} \equiv \left[h_{(a}^{\ \ c}h_{b)}^{\ \ d} - \frac{1}{3}h_{ab}h^{cd}\right]S_{cd}. \tag{B.75}$$

Definition 13 (Covariant spatial curl) *The tensor (B.71) generalizes*

$$\begin{aligned} curl\, V_a &= \eta_{abc}\overline{\nabla}^b V^c, \\ curl\, S_{ab} &= \eta_{cd(a}\overline{\nabla}^c S_{b)}^{\ \ d}, \\ \eta_{abc} &\equiv \eta_{abcd}u^d. \end{aligned} \tag{B.76}$$

Definition 14 (Covariant divergence) *The covariant divergence is defined by*

$$\begin{aligned} divV &= \overline{\nabla}^a V_a, \\ (divS)_a &= \overline{\nabla}^b S_{ab}. \end{aligned} \tag{B.77}$$

It is very important to mention that the operations (B.76) and (B.77) are defined as operators on a 3*−manifold only if the vorticity vanishes. When the vorticity is non-zero, they are operators in the tangent hyperplane at each point and not on a manifold [1, 46].*

B.3.1 Useful identities

In order to write the field equations in $f(R)$ gravity, we can use the following identities

$$\begin{aligned} \nabla_\mu f'(R) &= f''(R)\nabla_\mu R, \\ \nabla_\nu\nabla_\mu f' &= f'''\nabla_\nu R\nabla_\mu R + f''\nabla_\nu\nabla_\mu R, \\ \Box f' &= f'''\nabla_\sigma R\nabla^\sigma R + f''\Box R. \end{aligned} \tag{B.78}$$

The contraction of equation (3.91) is

$$R_{\mu\nu}u^\mu u^\nu = \frac{1}{f'}\Big[-\frac{1}{2}f + f''h^{\mu\nu}\nabla_\mu\nabla_\nu R + f'''h^{\mu\nu}\nabla_\mu R\nabla_\nu R + \kappa T_{\mu\nu}u^\mu u^\nu\Big], \tag{B.79}$$

where we have used $u^\mu u^\nu = h^{\mu\nu} - g^{\mu\nu}$ and $g^{\mu\nu}u_\mu u_\nu = -1$. There are some results needed to deal with the GDE.
The first derivative is

$$\nabla_b f' = U_b^c\nabla_c f' + h_b^c\nabla_c f' = -u_b\dot{f}' + \overline{\nabla}_b f'. \tag{B.80}$$

The second covariant derivative for general spacetimes [43]

$$\nabla_a \nabla_b f' = -\dot{f}' \left(\frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab} - u_a \dot{u}_b \right) + u_a u_b \ddot{f}', \tag{B.81}$$

and

$$\nabla^a \nabla_a f' = \Box f' = g^{ab} \nabla_a \nabla_b f' = -\dot{f}' \Theta - \ddot{f}'. \tag{B.82}$$

B.3.2 Curvature energy momentum stress tensor

We can define the energy momentum tensor associate to the curvature as

$$T^{\text{curv}}_{ab} = \frac{1}{f'} \left(\frac{f - Rf}{2} g_{ab} + \nabla_a \nabla_b f' - g_{ab} \Box f' \right) \tag{B.83}$$

and from (B.83) an effective energy density as

$$\rho^R := T^{\text{curv}}_{ab} u^a u^b = \frac{1}{f'} \left[\frac{Rf' - f}{2} - \theta \dot{f}' \right], \tag{B.84}$$

and the pressure

$$p^R := \frac{1}{3} T^{\text{curv}}_{ab} h^{ab} = \frac{1}{f'} \left[\frac{f - Rf'}{2} + f''(\ddot{R} + \frac{2}{3}\Theta \dot{R}) + \dot{R}^2 f''' \right]. \tag{B.85}$$

B.4 Electrodynamics in the $1 + 3$ formalism

The $1 + 3$ splitting can also be applied to the electromagnetic field. The Faraday tensor $F_{ab} = F_{[ab]}$ defines the electric field for an observer with 4−velocity u^a as [1, 46]

$$E_a := F_{ab} u^b = E_{<a>}, \tag{B.86}$$

and the magnetic field,

$$B_a = \frac{1}{2} \eta_{acd} F^{cd} = F^*_{ab} u^b = B_{<a>}, \tag{B.87}$$

with F^*_{ab} the dual. The Faraday tensor is given by

$$F_{ab} = 2u_{[a} E_{b]} + \eta_{abc} B^c. \tag{B.88}$$

A charge particle with mass m and electric charge e experiences a Lorentz force

$$V^a \nabla_a V^b = (e/m) F^{bc} V_c. \tag{B.89}$$

B.4.1 Maxwell equations

The Maxwell equations in curved backgrounds are

$$\nabla_b F^{ab} = J^a, \quad \nabla_{[a} F_{bc]} = 0, \tag{B.90}$$

When a 4−velocity is chosen the 4−current J^a is written as

$$J^a = h^a_{\ b} J^b - u^a u_b J^b = j^a + \mu u^a, \tag{B.91}$$

where the tri-current is $j^a := h^a_{\ b} J^b$ and the charge density $\mu := -(u_b J^b)$. In the $1+3$ formalism the Maxwell equations (B.90) are

$$\overline{\nabla}_a E^a = \mu - 2\omega_a B^a, \tag{B.92}$$
$$\overline{\nabla}_a B^a = 2\omega_a E^a, \tag{B.93}$$
$$\dot{E}_{<a>} = \Big(\sigma_{ab} + \eta_{abc}\omega^c - \frac{2}{3}\Theta h_{ab}\Big)E^b + \eta_{abc}\dot{u}^b B^c + \mathrm{curl}\, B_a - j_a, \tag{B.94}$$
$$\dot{B}_{<a>} = \Big(\sigma_{ab} + \eta_{abc}\omega^c - \frac{2}{3}\Theta h_{ab}\Big)B^b - \eta_{abc}\dot{u}^b E^c + \mathrm{curl}\, E_a. \tag{B.95}$$

The equation (B.90) implies

$$\begin{aligned}\nabla_a J^a &= \nabla_a \nabla_b F^{ab} = \frac{1}{2}(\nabla_a \nabla_b F^{ab} - \nabla_b \nabla_a F^{ab}) \\ &= \frac{1}{2}\Big(- R_{bae}{}^{b} F^{ea} + R_{abe}{}^{b} F^{ea}\Big) = R_{ae} F^{ea} = 0.\end{aligned} \tag{B.96}$$

The equation (B.96) in the $1+3$ language becomes

$$\dot{\mu} + \mu\Theta + \overline{\nabla}_a j^a + \dot{u}^a j_a = 0. \tag{B.97}$$

The set of equations (B.92) is equivalent to (B.90). The energy-momentum tensor for the electromagnetic field is

$$T^{ab}_{\mathrm{em}} = -F^{ac} F_c^{\ b} - \frac{1}{4} g_{ab} F_{cd} F^{cd}, \tag{B.98}$$

and (B.98) together with (B.90) implies

$$\nabla_b T^{ab}_{\mathrm{em}} = -F^{ab} J_b. \tag{B.99}$$

B.4.2 Contractions in the GDE: $1+3$ formalism

The GDE gives the following contributions

$$\Big[\frac{1}{2}(g^a_{\ c} R_{bd} - g^a_{\ d} R_{bc} + g_{bd} R^a_{\ c} - g_{bc} R^c_{d}) - \frac{R}{6}(g^a_{\ c} g_{bd} - g^a_{d} g_{bc})\Big] V^b X^c V^d, \tag{B.100}$$

or

$$g^a_{\ c} R_{bd} V^b X^c V^d = R_{bd} V^b X^a V^d, \tag{B.101}$$
$$g^a_{\ d} R_{bc} V^b X^c V^d = R_{bc} V^b X^c V^a, \tag{B.102}$$
$$g_{bd} R^a_{\ c} V^b X^c V^d = R^a_{\ c} V_d V^d X^c = \epsilon R^a_{\ c} X^c, \tag{B.103}$$
$$g_{bc} R^a_{\ d} V^b X^c V^d = R^a_{\ c} V_c X^c V^d = 0, \tag{B.104}$$
$$-\frac{R}{6} g^a_{\ c} g_{bd} V^b X^c V^d = -\frac{R}{6} g_{bd} V^b X^a V^d = -\frac{R}{6}\epsilon X^a, \tag{B.105}$$
$$\frac{R}{6} g^a_{\ b} g_{bc} V^b X^c V^d = \frac{R}{6} g_{bc} V^b X^c V^a = \frac{R}{6} V_c X^c V^a = 0. \tag{B.106}$$

In terms of the energy momentum tensor the equation (B.101)

$$
\begin{aligned}
R_{bd}V^bX^aV^d &= \frac{1}{f'}\Big[\kappa T_{bd} + \frac{f}{2}g_{bd} - g_{bd}\Box f' + \nabla_b\nabla_d f'\Big]V^bX^aV^d = \\
= \frac{1}{f'}\Big[\{\kappa(\rho_m + p_m) &- \frac{1}{3}\Theta\dot{f}' + \ddot{f}'\}E^2 + \epsilon\{\kappa p_m + \frac{f}{2} + \frac{2}{3}\Theta\dot{f}' + \ddot{f}'\}\Big]X^a,
\end{aligned}
\tag{B.107}
$$

the rest of the contractions are

$$
R_{bc}V^bX^cV^a = \nabla_b\nabla_c f'V^bX^cV^a = 0, \tag{B.108}
$$

$$
R^a_{\ c}X^c = \frac{1}{f'}\Big[\kappa p_m + \frac{f}{2} + \ddot{f}' + \frac{2}{3}\Theta\dot{f}'\Big]X^a, \tag{B.109}
$$

$$
\frac{R}{6}X^a = \frac{1}{6f'}\Big[\kappa(3p_m - \rho_m) + 2f + 3\dot{f}'\Theta + 3\ddot{f}'\Big]X^a. \tag{B.110}
$$

B.4.3 Contractions in GDE: coordinate method

Expanding explicitly the contractions for the GDE in the coordinate method we get

$$
\begin{aligned}
(\nabla_\delta\nabla_\beta f')V^\beta V^\delta &= (\nabla_0\nabla_0 f')V^0V^0 + (\nabla_i\nabla_j f')V^iV^j, \\
&= (f''\ddot{R} + f'''\dot{R}^2)E^2 - Hf''\dot{R}g_{ij}V^iV^j, \\
&= (f''\ddot{R} + f'''\dot{R}^2)E^2 - Hf''\dot{R}V_jV^j, \\
&= (f''\ddot{R} + f'''\dot{R}^2)E^2 - Hf''\dot{R}(\epsilon - V_0V^0), \\
&= (f''\ddot{R} + f'''\dot{R}^2)E^2 - Hf''\dot{R}(\epsilon + E^2),
\end{aligned}
\tag{B.111}
$$

$$
\begin{aligned}
(\nabla_\gamma\nabla_\beta f')V^\beta\eta^\gamma V^\alpha &= (\nabla_0\nabla_0 f')V^0\cancelto{0}{\eta^0}V^\alpha + (\nabla_i\nabla_j f')V^i\eta^jV^\alpha, \\
&= -Hf''\dot{R}g_{ij}V^i\eta^jV^\alpha, \\
&= -Hf''\dot{R}\cancelto{0}{V_j\eta^j}V^\alpha, \\
&= 0.
\end{aligned}
\tag{B.112}
$$

$$
\begin{aligned}
(\nabla_\gamma\nabla^\alpha f')\eta^\gamma &= (\nabla_\gamma g^{\alpha\sigma}\nabla_\sigma f')\eta^\gamma, \\
&= g^{\alpha\sigma}(\nabla_\gamma\nabla_\sigma f')\eta^\gamma, \\
&= g^{\alpha 0}(\nabla_0\nabla_0 f')\cancelto{0}{\eta^0} + g^{\alpha j}(\nabla_i\nabla_j f')\eta^i, \\
&= -Hf''\dot{R}g^{\alpha j}g_{ij}\eta^i, \\
&= -Hf''\dot{R}\delta^\alpha_j\eta^j, \\
&= -Hf''\dot{R}\eta^\alpha.
\end{aligned}
\tag{B.113}
$$

Then we see that in comparison with our results in [42], both formulations for the FLRW spacetime agree. The method used in [43], which relies in the $1+3$ decomposition, gives the contribution in a more straightforward way but we have shown the equivalence for the GDE. However, the 1 + 3 formulation of the GDE offers advantages for more general spacetimes than FLRW.

APPENDIX C

Some useful results

In this appendix, we describe some results used in the framework of *Covariant Nonlinear and Gauge-Invariant approach* (CGI) [1].

C.1 $1+3$ quantities in CPT

The fundamental difference between the CGI approach and the standard CPT is that the CGI starts from the fully nonlinear equations, rather than from the background. One full nonlinear exact set of equations is found and so we can choose a physically motivated $1+3$ splitting of the key variables to describe the dynamics of the system. In the following sections some results specially from [1, 85] are shown.

The comoving-volume expansion gradient is defined as

$$Z_a \equiv a\overline{\nabla}_a\Theta, \tag{C.1}$$

we can see

$$Z_{\langle a\rangle} = ah_a^{\ b}\overline{\nabla}_b\Theta = ah_a^{\ b}h_b^{\ d}\nabla_d\Theta = ah_a^{\ d}\nabla_d\Theta = Z_a. \tag{C.2}$$

We can obtain the equation of motion for (C.1)

$$\dot{Z}_a = (a\overline{\nabla}_a\Theta)^{\cdot} = \dot{a}\nabla_a\Theta + a(\overline{\nabla}_a\Theta)^{\cdot}, \tag{C.3}$$

and from the identity

$$(\overline{\nabla}_a\Theta)^{\cdot} = \overline{\nabla}_a\dot{\Theta} + (\dot{u}^b\overline{\nabla}_b\Theta)u_a + \dot{u}_a\dot{\Theta} - \frac{1}{3}\Theta\overline{\nabla}_a\Theta - \sigma_{ab}\overline{\nabla}^b\Theta + \eta_{abc}\omega^b\overline{\nabla}^c\Theta, \tag{C.4}$$

it implies

$$\dot{Z}_a = a\overline{\nabla}_a\dot{\Theta} + \dot{u}^bZ_bu_a + a\dot{u}_a\dot{\Theta} - \sigma_{ab}Z^b + \eta_{abc}\omega^bZ^c \tag{C.5}$$

or for the projection

$$\dot{Z}_{\langle a\rangle} = a\overline{\nabla}_a\dot{\Theta} + a\dot{u}_a\dot{\Theta} - \sigma_{ab}Z^b + \eta_{abc}\omega^bZ^c. \tag{C.6}$$

We can go further in (C.6) and try to see the effect of use a $f(R)$ gravity theory. For this purpose we take advantage of the Raychaudhuri equation and together with the conservation equations

$$a\dot{u}_a\dot{\Theta} = a\dot{u}_a\Big(- \frac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) + \overline{\nabla}_b\dot{u}^b + \dot{u}_b\dot{u}^b - R_{de}u^du^e\Big) \tag{C.7}$$

and for $f(R)$ gravity

$$a\dot{u}_a\dot{\Theta} = a\dot{u}_a\Big(- \frac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) + \overline{\nabla}_b\dot{u}^b + \dot{u}_b\dot{u}^b - \frac{1}{f'}\big[- \frac{f}{2} + f''h^{de}\nabla_d\nabla_e R + f'''h^{de}\nabla_d R\nabla_e R + \kappa T_{ed}u^eu^d\big]\Big). \tag{C.8}$$

To write (C.1) close to the literature [1] we use

$$a\overline{\nabla}_a\big(- \frac{1}{3}\Theta^2\big) = -\frac{2}{3}\Theta Z_a, \tag{C.9}$$

$$- 2a\overline{\nabla}_a(\sigma^2 - \omega^2) = -2a\overline{\nabla}_a(\sigma^2 - \omega^2), \tag{C.10}$$

$$a\overline{\nabla}_a(\dot{u}_b\dot{u}^b) = 2a\dot{u}_b\overline{\nabla}_a\dot{u}^b. \tag{C.11}$$

The final expression for (C.1) is

$$\begin{aligned}\dot{Z}_{\langle a\rangle} = &-\frac{2}{3}\Theta Z_a - 2a\overline{\nabla}_a(\sigma^2 - \omega^2) + 2a\dot{u}_b\overline{\nabla}_a\dot{u}^b + a\overline{\nabla}_a\overline{\nabla}_b\dot{u}^b\\ &+a\dot{u}_a\Big(- \frac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) + \overline{\nabla}_b\dot{u}^b + \dot{u}_b\dot{u}^b - \frac{1}{f'}\big[- \frac{f}{2} + f''h^{de}\nabla_d\nabla_e R + f'''h^{de}\nabla_d R\nabla_e R + \kappa T_{ed}u^eu^d\big]\Big)\\ &-a\overline{\nabla}_a\Big[\frac{1}{f'}\Big\{ - \frac{f}{2} + f''h^{de}\nabla_d\nabla_e R + f'''h^{de}\nabla_d R\nabla_e R + \kappa T_{ed}u^eu^d\Big\}\Big]\\ &-(\sigma_{ba} + \omega_{ba})Z^b.\end{aligned} \tag{C.12}$$

Now, we can check the density gradient, for GR the results are

$$\Delta_a \equiv \frac{a\overline{\nabla}_a\rho}{\rho}. \tag{C.13}$$

The equation of motion for (C.13) is

$$\dot{\Delta}_a = \Big(\frac{\dot{a}}{a}\Delta_a\Big) - \frac{\dot{\rho}}{\rho}\Delta_a + \frac{a}{\rho}\Big(\overline{\nabla}_a\rho\Big)^{\cdot}. \tag{C.14}$$

The terms in (C.14) are

$$\Big(\frac{\dot{a}}{a}\Delta_a\Big) = \frac{1}{3}\Theta\Delta_a, \tag{C.15}$$

$$\frac{\dot{\rho}}{\rho}\Delta_a = -\Big[\frac{(\rho + p)\Theta + \Pi^{cd}\sigma_{cd} + \overline{\nabla}_c q^c + 2\dot{u}_c q^c}{\rho}\Big]\Delta_a, \tag{C.16}$$

where we should notice that in (C.16) the conservation laws for the energy momentum of matter obeys $T^{ab}_{\text{mat};b} = 0$. If we assume a state equation

$$p = \omega_m\rho, \tag{C.17}$$

the equation (C.16) becomes

$$\frac{\dot{\rho}}{\rho}\Delta_a = -\Big[\frac{(1+\omega_m)\rho\Theta + \Pi^{cd}\sigma_{cd} + \overline{\nabla}_c q^c + 2\dot{u}_c q^c}{\rho}\Big]\Delta_a. \tag{C.18}$$

The term

$$\frac{a}{\rho}(\nabla_a \rho)^{\cdot} = \frac{a}{\rho}\Big\{\overline{\nabla}_a\dot{\rho} + (\dot{u}^b\overline{\nabla}_b\rho)u_a + \dot{u}_a\dot{\rho} - \frac{1}{3}\Theta\overline{\nabla}_a\rho - \sigma_{ab}\overline{\nabla}^b\rho + \eta_{abc}\omega^b\overline{\nabla}^c\rho\Big\}, \tag{C.19}$$

together with the conservation laws and (C.17) (i.e. ω_m = const) write (C.14) as

$$\begin{aligned}\dot{\Delta}_a = \omega_m\Theta\Delta_a - (1+\omega_m)Z_a + \frac{a\Theta}{\rho}[\dot{q}_{<a>} + \frac{4}{3}\Theta q_a]\\ +\frac{a\Theta}{\rho}\Big[\sigma_{ba} + \omega_{ba}\Big]q^b + \frac{a\Theta}{\rho}\overline{\nabla}^b\Pi_{ab}\\ -\frac{a}{\rho}\overline{\nabla}_a\Big(\Pi^{cd}\sigma_{cd} + 2\dot{u}_c\dot{q}^c\Big) - \frac{a}{\rho}\overline{\nabla}_a\overline{\nabla}_c q^c - \Big(\sigma_{ba} + \omega_{ba}\Big)\Delta^b\\ +\frac{a}{\rho}\Theta\dot{u}^b\Pi_{ab} + \frac{1}{\rho}\Big(\sigma^{cd}\Pi_{cd} + 2\dot{u}_c q^c + \overline{\nabla}^c q_c\Big)\Big(\Delta_a - a\dot{u}_a\Big)\\ +\Big(\frac{a}{\rho}\dot{u}^b\overline{\nabla}_b\rho\Big)u_a,\end{aligned} \tag{C.20}$$

or equivalently

$$\begin{aligned}\dot{\Delta}_{\langle a\rangle} = h_a^{\ b}\dot{\Delta}_b = \omega_m\Theta\Delta_a - (1+\omega_m)Z_a + \frac{a\Theta}{\rho}[\dot{q}_{\langle a\rangle} + \frac{4}{3}\Theta q_a]\\ +\frac{a\Theta}{\rho}\Big[\sigma_{ba} + \omega_{ba}\Big]q^b + \frac{a\Theta}{\rho}\overline{\nabla}^b\Pi_{ab}\\ -\frac{a}{\rho}\overline{\nabla}_a\Big(\Pi^{cd}\sigma_{cd} + 2\dot{u}_c\dot{q}^c\Big) - \frac{a}{\rho}\overline{\nabla}_a\overline{\nabla}_c q^c - \Big(\sigma_{ba} + \omega_{ba}\Big)\Delta^b\\ +\frac{a}{\rho}\Theta\dot{u}^b\Pi_{ab} + \frac{1}{\rho}\Big(\sigma^{cd}\Pi_{cd} + 2\dot{u}_c q^c + \overline{\nabla}^c q_c\Big)\Big(\Delta_a - a\dot{u}_a\Big).\end{aligned} \tag{C.21}$$

The equation (C.20) or (C.21) are the starting point for the analysis of how inhomogeneities behave in this framework.

C.1.1 Magnetized fluids

Magnetized fluids are important in several applications in cosmology. One of the results in this work is associated with the evolution of magnetic fields using cosmological perturbation theory in GR.

However, one can check if we can use the 1 + 3 formalism to study the presence of magnetic fields in this language in GR and its extension to $f(R)$ gravity. For this purpose, we can see from (C.12) the contribution coming from the matter fields in T^{ab}. The inclusion of the magnetic fields is made considering

$$T^{ab} = T^{ab}_{\mathrm{mat}} + T^{ab}_{em}. \tag{C.22}$$

One of the most widely used physical assumptions is to work in the ideal MHD limit which is expressed in the Ohm's law in the rest frame of the fluid

$$j^a = \sigma E^a \tag{C.23}$$

with the condition $\sigma \to \infty$. The ideal MHD condition implies $E^a \to 0$ and the conservation laws

$$\nabla_b (T^{ab}_{\rm mat} + T^{ab}_{\rm mag}) = 0, \tag{C.24}$$

for a perfect fluid and an electromagnetic fields are

$$\dot{\rho} + \Theta(\rho + p) = E^a j_a = 0, \tag{C.25}$$

and

$$(\rho + p)\dot{u}^a + \overline{\nabla}_a p = \mu E_a + \eta_{abc} j^b B^c = \eta_{abc} j^b B^c. \tag{C.26}$$

The right side of (C.26) is the Lorentz force density. The equations (C.25) and (C.26) are slightly different written in [85] but are equivalent. To write the expressions for the quantities (C.13) and (C.6) including the magnetic fields we should use in (C.8) the energy-momentum tensor

$$\tilde{T}^{ab} = (\rho + \frac{1}{2}B^2)u^a u^b + (p + \frac{1}{6}B^2)h^{ab} + \Pi^{ab}_{\rm mag}. \tag{C.27}$$

The propagation equations for Z_a

$$\begin{aligned} \dot{Z}_{\langle a \rangle} = -\frac{2}{3}\Theta Z_a - 2a\overline{\nabla}_a(\sigma^2 - \omega^2) + 2a\dot{u}_b\overline{\nabla}_a\dot{u}^b + a\overline{\nabla}_a\overline{\nabla}_b\dot{u}^b \\ +a\dot{u}_a\Big(-\frac{1}{3}\Theta^2 - 2(\sigma^2 - \omega^2) + \overline{\nabla}_b\dot{u}^b + \dot{u}_b\dot{u}^b - \frac{1}{f'}[-\frac{f}{2} + f'' h^{de}\nabla_d\nabla_e R + f''' h^{de}\nabla_d R\nabla_e R + \kappa\tilde{T}_{ed}u^e u^d]\Big) \\ -a\overline{\nabla}_a\Big[\frac{1}{f'}\Big\{-\frac{f}{2} + f'' h^{de}\nabla_d\nabla_e R + f''' h^{de}\nabla_d R\nabla_e R + \kappa\tilde{T}_{ed}u^e u^d\Big\}\Big] \\ -(\sigma_{ba} + \omega_{ba})Z^b. \end{aligned} \tag{C.28}$$

Where $\tilde{T}^{ab}$ includes the matter components and the electromagnetic field. For a perfect fluid with equation of state $p = \omega(\rho)\rho$ and in the ideal MHD limit, the equation (C.28) for GR is [1]

$$\begin{aligned} \dot{Z}_{\langle a \rangle} = -\frac{2}{3}\Theta Z_a - 4\pi G\Big(\rho\Delta_a + B^2\mathcal{B}_a\Big) \\ +12\pi G a\eta_{abc}B^b \mathrm{curl}\, B^c + a\overline{\nabla}_a\overline{\nabla}_b\dot{u}^b \\ +2a\dot{u}^b\overline{\nabla}_a\dot{u}_b + \Big[\frac{1}{2}{}^3R - 3\Big(\sigma^2 - \omega^2\Big) + \overline{\nabla}^b\dot{u}_b + \dot{u}_b\dot{u}^b\Big]a\dot{u}_a \\ -(\sigma_{ba} + \omega_{ba})Z^b + 12\pi G a\Pi^{\rm mag}_{ab}\dot{u}^b - 2a\overline{\nabla}_a(\sigma^2 - \omega^2), \end{aligned} \tag{C.29}$$

with $\mathcal{B}_a$ defined in (C.31) and

$$\begin{aligned} \dot{\Delta}_{<a>} = \omega\Theta\Delta_a - (1 + \omega)Z_a + \frac{a\Theta}{\rho}\eta_{abc}B^b \mathrm{curl}\, B^c \\ +\frac{2a\Theta}{3}\frac{B^2}{\rho}\dot{u}_a - (\sigma_{ba} + \omega_{ba})\Delta^b + \frac{a\Theta}{\rho}\Pi^{\rm mag}_{ab}\dot{u}^b. \end{aligned} \tag{C.30}$$

with $\pi^{ab}_B = -B^{<a}B^{b>}$. Inhomogeneities associated with the magnetic field may be described with the comoving fractional gradient [1]

$$\mathcal{B}_a \equiv \frac{a}{B^2}\overline{\nabla}_a B^2, \tag{C.31}$$

with the same methodology we can compute the equation of motion for (C.31)

$$\dot{\mathcal{B}}_a = \frac{1}{3}\Theta\mathcal{B}_a - \frac{(B^2)^{\cdot}}{B^2}\mathcal{B}_a + \frac{a}{B^2}\left(\overline{\nabla}_a B^2\right)^{\cdot}, \tag{C.32}$$

the contributions in the equation (C.32) in the ideal MHD limit and using the Maxwell equations can be written as [1]

$$(\dot{B})^2 = (B_c B^c)^{\cdot} = 2\dot{B}_{\langle c\rangle}B^{\langle c\rangle} = 2\left(\sigma_{cb}B^b B^c + \eta_{cbd}\omega^d B^b B^c - \frac{4\Theta}{3}B_c B^c\right) = -2\sigma_{cd}\Pi^{cd}_{\mathrm{mag}} - \frac{4\Theta}{3}B^2. \tag{C.33}$$

The equation (C.32) finally becomes (for GR [1])

$$\begin{aligned}\dot{\mathcal{B}}_{<a>} &= \frac{4}{3(1+w)}\dot{\Delta}_{<a>} - \frac{4w\Theta}{3(1+w)}\Delta_a \\ -\frac{4a\Theta}{3\rho(1+w)}\eta_{abc}B^b\mathrm{curl}B^c &- \frac{4}{3}a\Theta\left(1+\frac{2B^2}{3\rho(1+w)}\right)\dot{u}_a - (\sigma_{ba}+\omega_{ba})\mathcal{B}^b \\ +\frac{4}{3(1+w)}(\sigma_{ba}+\omega_{ba})\Delta^b &- \frac{4a\Theta}{3\rho(1+w)}\Pi^B_{ab}\dot{u}^b - \frac{2a}{B^2}\Pi^{bc}_B\bar{\nabla}_a\sigma_{bc} \\ -\frac{2a}{B^2}\sigma^{bc}\bar{\nabla}_a\Pi^B_{bc} &+ \frac{2}{B^2}\pi^{bc}_B\sigma_{bc}\mathcal{B}_a - \frac{2a}{B^2}\sigma_{bc}\Pi^{bc}_B\dot{u}_a.\end{aligned} \tag{C.34}$$

The idea to study the propagation equations to $f(R)$ gravity and see its consequences is work in progress.

Bibliography

[1] Ellis, G. and Maartens, R. and MacCallum, M, *Relativistic Cosmology*, 2012, ISBN: 9780521381154, URL: http://www.cambridge.org.

[2] P. Schneider, J. Ehlers and E. E. Falco, *Gravitational Lenses*, Springer-Verlag, 1999.

[3] S. Capozzielo and V. Faraoni, *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology and Astrophysics*, 2011, ISBN: 9789400701649.

[4] De Felice, A. and Tsujikawa, S., "$f(R)$ Theories", *Living. Rev. Relativity* 88 (2013) 161, URL: http://www.livingreviews.org.

[5] F. Yasunori and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, 2007, ISBN: 0521811597.

[6] Castañeda, L. and Hortua, H., "Contrasting formulations of cosmological perturbations in a magnetic FLRW cosmology" (2013), arXiv:1403.7789v1.

[7] H. Weyl, *Space, Time, Matter*, Dover Books on Physics, 1952.

[8] Minazzoli, O., "γ parameter in Brans-Dicke-like (light-)scalar-tensor theory with a universal scalar-matter coupling", *Phys. Rev. D* 88 (2013) 064050, URL: http://journals.aps.org/prd/.

[9] Dyer, E. and Hinterbichler, K, "Boundary terms, variational principles and higher derivate modified gravity", *Phys. Rev. D* 79 (2009) 024028, URL: http://pdg.lbl.gov.

[10] Guarnizo, A., Castañeda, L., Tejeiro, J.M., "Boundary term in metric $f(R)$ gravity: field equations in the metric formalism", *Gen. Relativ. Gravit.* 42 (2010) 2713, URL: http://link.springer.com.

[11] Lanahan-Tremblay, N. and Faraoni, V., "The Cauchy problem of $f(R)$ gravity", *Classical and Quantum Gravity* 24 (2007) 5667.

[12] Gibbons, G.W. and Hawking, S.W., "Action integrals and partition functions in quantum gravity", *Physical Review D* 15 (1977) 2752–2756.

[13] T. Padmanabhan, *Gravitation*, Cambridge University Press, 2010.

[14] S. Weinberg, *Gravitation and Cosmology*, 1972, ISBN: 0471925675.

[15] T. Sotiriou, "Modified Actions for Gravity: Theory and Phenomenology", PhD Thesis: School for Advanced Studies, 2007, arXiv:0710.4438v1.

[16] Nojiri, S. and Odinstov, S.D.,
" Modified gravity as an alternative for Lambda-CDM cosmology.",
J. Phys. A 40 (2007) 6725–6732, arXiv:hep-th/0610164.

[17] Nojiri, S. and Odinstov, S.D.,
"Introduction to modified gravity and gravitational alternative for dark energy.",
International Journal of Geometric Methods in Modern Physics 04 (2007) 115,
arXiv:hep-th/0601213.

[18] Durrer, R. and Maartens, R., "Dark energy and dark gravity: theory overview.",
Gen. Rel. Grav. 40 (2008) 301328.

[19] Capozziello, S. and Francaviglia, M.,
"Extended theories of gravity and their cosmological and astrophysical applications.",
Gen.Rel.Grav. 40 (2008) 357–420, arXiv:0706.1146v2.

[20] Hawking, S.W. and Horowitz, G.T.,
"The Gravitational Hamiltonian, action, entropy and surface terms.",
Class. Quant. Grav 13 (1996) 1487–1498, arXiv:gr-qc/9501014.

[21] E. Poisson, *A Relativistic's Toolkit-The Mathematics of Black Mechanics*,
Cambridge University Press, 2004.

[22] S. Carroll, *Space and Geometry*, ed. by S. F. Addison Wesley, 2004, ISBN: 0521830915.

[23] R. M. Wald, *General Relativity*, ed. by C. The University of Chicago Press, 1984,
ISBN: 0521830915.

[24] Madsen, M.S. and Barrow,J.D,
"De Sitter ground states and boundary terms in generalized gravity",
Nuc. Phys. B 323 (1989) 242–252,
URL: http://www.journals.elsevier.com/nuclear-physics-b/.

[25] DI Casola, E. and Liberati, S. and Sonego, S, "Weak equivalence principle for self-gravitating bodies: A sieve for purely metric theories of gravity", *Phys. Rev. D* 89 (2014) 084053.

[26] Kim,H. and Kim, Y., "Thermodynamics of black holes in Brans-Dicke gravity",
Nuovo Cimento B 112B (1997) 329–338.

[27] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*,
Princeton University Press, 1992.

[28] Farhoudi, M., "On higher order gravities, their analogy to GR, and dimensional dependent version of Duff's trace anomaly relation", *Gen. Rel. Grav.* 38 (2006) 1261–1284,
arXiv:physics/0509210v2.

[29] S. Hawking and G. Ellis, *The large scale structure of space-time*,
Cambridge University Press, 1973.

[30] Buchdahl, H.A., "Non-linear Lagrangians and cosmological theory.",
Mon. Not. R. astr. Soc. 1970 (150) 1–8.

[31] Barth, N.H., "The fourth-order gravitational action for manifolds with boundaries.",
Class. Quant.Grav. 2 (1985) 497–513.

[32] Nojiri, S. and Odinstov, S. D., "Brane-world cosmology in higher derivative gravity or warped compactification in the next-to-leading order of AdS/CFT correspondence",
Journal of High Energy Physics 49 (2000) 1–22, arXiv:hep-th/0006232.

[33] Nojiri, S. and Odinstov, S. D., "Is brane cosmology predictable?", *General Relativity and Gravitation* 37 (2005) 1419–1425, arXiv:hep-th/0409244.

[34] Fatibene, L. and Ferraris, M. and Fracaviglia, M., "Augmented Variational Principles and Relative Conservation Laws in Classical Field Theory", *International Journal of Geometric Methods in Modern Physics* 02 (2005) 373–392, arXiv:math-ph/0411029v1.

[35] Francaviglia, M. and Raiteri, M., "Hamiltonian, energy and entropy in general relativity with non-orthogonal boundaries", *Classical and Quantum Gravity* 19 (2002) 237–258, arXiv:gr-qc/0107074v1.

[36] Casadio, R. and Gruppuso, A., "On boundary terms and conformal transformations in curved spacetimes", *Int. J. Mod. Phys. D.* 11 (2002) 703–714, arXiv:gr-qc/0107077.

[37] Balcerzak, A. and Dabrowkski, M.P., "Gibbons-Hawking boundary terms and junction conditions for higher-order brane gravity models", *JCAP* 018 (2009), arXiv:0804.0855.

[38] Nojiri, S. and Odinstov, S. D., "Finite gravitational action for higher derivative and stringy gravities", *Physical Review D* 62 (2000) 064018, arXiv:hep-th/9911152.

[39] Nojiri, S. and Odinstov, S. D. and Ogushi, S., "Finite Action in d5 Gauged Supergravity and Dilatonic Anomaly for Dual Quantum Field Theory", *Physical Review D* 62 (2000) 124002, arXiv:hep-th/0001122v4.

[40] Sotiriou, T.P. and Liberati, S., "Metric-affine $f(R)$ theories of gravity", *Annals Phys.* 322 (2007) 935–966, arXiv:gr-qc/00604006v2.

[41] Clifton, T., Ferreira, P.G., Padilla, A., Skordis, C., "Modified gravity and cosmology", *Physics Reports* 513 (2012) 189, URL: http://www.elsevier.com/locate/physrep.

[42] Guarnizo, A. and Castañeda, L and Tejeiro, J.M., "Geodesic deviation equation in $f(R)$ gravity", *General. Rel. Grav.* 43 (2011) 2713–2728, URL: http://link.springer.com.

[43] de la Cruz-Dombriz, A. and Dunsby, P. K. S. and Busti, V. and Kandhal, S., "Tidal forces in $f(R)$ theories gravity ", *Phys. Rev. D.* 89 (2014) 064029.

[44] Ehlers, J., "Beiträge zur relativistischen Mechanik kontinuer Medien", *Akad. Wiss. Lit. Mainz, abh.Math-Nat.Kl* 11 (1961).

[45] Guarnizo, A. and Castañeda, L. and Tejeiro, J.M., "A Brief Comment on Geodesic Deviation Equation in f(R) Gravity " (2014), arXiv:arXiv:1402.3196.

[46] Ellis, G. F. R., "Relativistic Cosmology", *Cargese lectures in physics* 6 (1973) 1–60.

[47] C. Misner, K. Thorne and J. Wheeler, *Gravitation*, W. H. Freeman and Company, 1973.

[48] Rippl, S. van Elst, H. and Tavakol, R. and Taylor, D., "Kinematics and dynamics of $f(R)$ theories of gravity", *Gen. Relativ. Gravit.* 28 (1996) 193.

[49] H. Hortua, L. Castañeda and J. Tejeiro, "Evolution of magnetic fields through cosmological perturbation theory", *Physical Review D* 87 (2013) 103531.

[50] A. Krasinski, *Inhomogeneous Cosmological Models*, Cambridge University Press, 2006.

[51] J. Wainwright and G. F. R. e. Ellis, *Dynamical Systems in Cosmology*, Cambridge University Press, 2005.

[52] Capoziello, S. and Carloni, S. and Troisi, A., "Quintessence without scalar fields", *Recent Res. Dev. Astron. Astrophys.* 1 (2003) 625.

[53] Santos, J. and Alcaniz, J. S. and Reboucas, M. J. and Carvalho, F. C., "Energy conditions in $f(R)$ gravity ", *Phys. Rev. D.* 76 (2007) 083513.

[54] Synge, J. L., " On the Deviation of Geodesics and Null Geodesics, Particularly in Relation to the Properties of Spaces of Constant Curvature and Indefinite Line Element", *Ann. Math.* 35 (1934) 705.

[55] Corda, C., " Interferometric detection of gravitational waves: the definitive test for General Relativity", *Int. J. Mod. Phys. D* 18 (2009) 2275–2282, eprint: arxiv.org/abs/0905.2502.

[56] A. Lewis and A. Challinor, "Weak gravitational lensing of the CMB", *Physics Reports* 429 (2006) 65, DOI: 10.1016/j.physrep.2006.03.002.

[57] L. Castaneda, "Efecto de la constante cosmológica en la probabilidad de lentes gravitaciones", MsC Thesis: Universidad Nacional de Colombia, 2002.

[58] Seitz, S. and Schneider, P. and Ehlers, J., "Light propagation in arbitrary spacetimes and gravitational lens approximation", *Class. Quantum Grav.* 11 (1994) 2345–2373.

[59] Reimberg, P.H.F and Abramo, R., "The Jacobi map for gravitational lensing: the role of the exponential map", *Class. Quantum Grav.* 30 (2013) 13.

[60] Ellis, G. F. R. and Van Elst, H., "Deviation of geodesics in FLRW spacetime geometries" (1997), arXiv:gr-qc/9709060.

[61] Ellis, G. F. R. and Van Elst, H., " Cosmological models (Cargèse lectures 1998)" (1998), arXiv:gr-qc/9812046v5.

[62] Pirani, F. A. E., "On the Physical Significance of the Riemann Tensor", *Acta Phys. Polon.* 15 (1956) 389.

[63] Caceres, D. and Castañeda, L. and Tejeiro, J. M., "Geodesic Deviation Equation in Bianchi Cosmologies", *J. Phys. Conf. Ser.* 229 (2010) 012076.

[64] Sotiriou, T. P. and Faraoni, V., "$f(R)$ Theories of Gravity" (2008), eprint: arXiv:0805.1726v4.

[65] Saez-Gomez, D., "Modified $f(R)$ gravity from scalar-tensor theory and inhomogeneous EoS dark energy", *General. Rel. Grav.* 41 (2009) 1527–15388.

[66] Starobinsky, A., "Disappearing cosmological constant in $f(R)$ gravity", *J. Exp. Theor. Phys. Lett.* 86 (2007) 157–163.

[67] Capoziello, S. and Cadone, V. F. and Troisi, A., "Reconciling dark energy models with $f(R)$ theories", *Phys. Rev. D.* 71 (2005) 043503.

[68] Shojai, F. and Shojai, A., "Geodesic Congruences in the Palatini $f(R)$ Theory", *Phys. Rev. D* 78 (2008) 104011.

[69] Faraoni, V., "$f(R)$ gravity: successes and challenges" (2008), eprint: arXiv:0810.2602.

[70] Borowiec, A. and Godlowksi, W. and Szydlowski, M., " Dark matter and dark energy as a effects of Modified Gravity", *ECONF C0602061* 09 (2006).

[71] Sotiriou, T. P., " 6+1 lessons from $f(R)$ gravity", *J. Phys. Conf. Ser.* 189 (2009) 012039.

[72] Dyer, C. C. and Roeder, R. C., "The Distance-Redshift Relation for Universes with no Intergalactic Medium", *Astrophysical Journal Letters* 174 (1972) 115.

[73] Okamura, T. and Futamase, T., " Distance-Redshift Relation in a Realistic Inhomogeneous Universe", *Prog. Theor. Phys.* 122 (2009) 511–520.

[74] Linder, E. V., "Isotropy of the microwave background by gravitational lensing", *Astronomy and Astrophysics* 206 (1998) 199–203.

[75] L. Widrow et al., "The First Magnetic Fields", *Space Science Reviews* 166 (2012) 37–70.

[76] R. Durrer and A. Neronov, "Cosmological magnetic fields: their generation, evolution and observation", *The Astronomy and Astrophysics Review* 21 (2013) 62.

[77] E. Lifshitz, "On the gravitational instability of the expanding Universe", *J. Phys. (USSR)* 10 (1946) 116.

[78] J. Bardeen, "Gauge-invariant cosmological perturbations", *Physical Review D* 22 (1980) 1882–1905.

[79] Bertschinger, E., "Cosmological dynamics", 96-22249, Massachusetss Institute of Technology, Jan. 1995.

[80] R. Durrer, *The Physics of the Early Universe*, 2005, ISBN: 978-3540227120.

[81] V. Mukhanov, H. Feldman and R. Brandenberger, "Theory of cosmological perturbations", *Physics Reports* 215 (1992) 203–333.

[82] K. Nakamura, "Gauge Invariant Variables in Two-Parameter Nonlinear Perturbations", *Progress of Theoretical Physics* 110 (2003) 723–755.

[83] K. Malik and D. Wands, "Cosmological perturbations", *Physics Reports* 475 (2009) 1–51.

[84] K. Nakamura, "second-order Gauge invariant Cosmological Perturbation Theory: Einstein equations in terms of gauge invariant variables", *Prog.Theor.Phys.* 117 (2007) 17–74.

[85] C. Tsagas, A. Challinor and R. Maartens, "Relativistic cosmology and large-scale structure", *Physics Reports* 465 (2008) 61–147.

[86] K. Nakamura, "Perturbations of matter fields in the second-order gauge-invariant cosmological perturbation theory", *Physical Review D* 80 (2009) 124021.

[87] A. Christopherson et al., "Comparing two different formulations of metric cosmological perturbation theory", *Classical and Quantum Gravity* 28 (2011) 225024.

[88] H. Kodama and M. Sasaki, "Cosmological Perturbation Theory", *Progress of Theoretical Physics Supplement* 78 (1984) 1.

[89] Gourgoulhon, E., “3 + 1 Formalism and Bases of Numerical Relativity” (2007), arXiv:arXiv:gr-qc/0703035v1.

[90] J. Stewart and M Walker, “Perturbations of space-times in general relativity”, *Royal Society (London), Proceedings, Series A* 341 (1974) 49–74.

[91] J. Stewart, “Perturbations of Friedmann-Robertson-Walker cosmological models”, *Classical and Quantum Gravity* 7 (1990) 1169–1180.

[92] M. Bruni and S. Sonego, “Observables and gauge invariance in the theory of nonlinear spacetime perturbations”, *Classical and Quantum Gravity* 16 (1999) 29–36.

[93] R. Sachs, *Gravitational Radiation in Relativity, Groups and Topology*, Gordon and Breach, New York, 1964.

[94] G. Ellis and M. Bruni, “Covariant and gauge-invariant approach to cosmological density fluctuations”, *Physical Review D* 40 (1989) 1804.

[95] M. Bruni et al., “Perturbations of spacetime: gauge transformations and gauge invariance at second order and beyond”, *Classical and Quantum Gravity* 14 (1997) 2585–2606.

[96] K. Nakamura, “Gauge-Invariant Cosmological Perturbation Theory: Current Status”, *Adv. Astron* (2010) 576273.

[97] K. Malik and D. Matravers, “A concise introduction to perturbation theory in cosmology”, *Classical and Quantum Gravity* 25 (2008) 193001.

[98] C. Sopuerta, M. Bruni and L. Gualtieri, “Nonlinear N-parameter spacetime perturbations: Gauge transformations”, *Physical Review D* 70 (2004) 064002.

[99] R. Beck, “Galactic and Extragalactic Magnetic Fields”, *AIP Conference Proceedings*, AIP, Dec. 2008 83; J. Eilek and F. Owen, “Magnetic Fields in Cluster Cores: Faraday Rotation in A400 and A2634”, *The Astrophysical Journal* 567 (2002) 202–220.

[100] M. Giovannini, “The Magnetized Universe”, *Int. J. Mod. Phys. D* 13 (2004) 391–502; L. Widrow, “Origin of galactic and extragalactic magnetic fields”, *Reviews of Modern Physics* 74 (2002) 775–823.

[101] Kojima, K. and Ichiki, K., “Evolution of Cosmological Perturbations in the Presence of Primordial Magnetic Fields” (2009), arXiv:09021367v2; Y. Xu et al., “A Faraday Rotation Search for Magnetic Fields in Large-scale Structure”, *The Astrophysical Journal* 637 (2006) 19–26; J. R. Shaw and A. Lewis, “Constraining primordial magnetism”, *Physical Review D* 86 (2012) 043510; P. Kronberg et al., “A Global Probe of Cosmic Magnetic Fields to High Redshifts”, *The Astrophysical Journal* 676 (2008) 70–79.

[102] R. Durrer, “Cosmic magnetic fields and the CMB”, *New Astronomy Reviews* 51 (2007) 275–280; J. D. Barrow, P. Ferreira and J. Silk, “Constraints on a Primordial Magnetic Field”, *Physical Review Letters* 78 (1997) 3610–3613; D. Yamazaki, K. Ichiki and G. Mathews, “Constraints on the Evolution of the Primordial Magnetic Field from the Small-Scale Cosmic Microwave Background Angular Anisotropy”, *The Astrophysical Journal* 646 (2006) 719–729.

[103] R. Kulsrud and E. Zweibel, “On the origin of cosmic magnetic fields”, *Reports on Progress in Physics* 71 (2008) 046901.

[104] L. Biermann and A. Schluter, “Cosmic Radiation and Cosmic Magnetic Fields. II. Origin of Cosmic Magnetic Fields”, *Physical Review* 82 (1951) 863–868; H. Hanayama et al., “Biermann Mechanism in Primordial Supernova Remnant and Seed Magnetic Fields”, *The Astrophysical Journal* 633 (2005) 941–945; Ichiki, K. and Takahashi, K. and Sugiyama, N. and Hanayana, H. and Ohno, H., “Magnetic Field Spectrum at Cosmological Recombination” (2007), arXiv:astro-ph/0701329; R. Kulsrud et al., “The Protogalactic Origin for Cosmic Magnetic Fields”, *The Astrophysical Journal* 480 (1997) 481–491.

[105] K. Dolag, M. Bartelmann and H. Lesch, “SPH simulations of magnetic fields in galaxy clusters”, *Astronomy and Astrophysics* 348 (1999) 351–363.

[106] D. Grasso and A. Riotto, “On the nature of the magnetic fields generated during the electroweak phase transition”, *Physics Letters B* 418 (1998) 258–265; K. Bamba and J. Sasaki, “Large-scale magnetic fields in the inflationary universe”, *JCAP* 02 (2007) 030; K. Kunze, “Primordial magnetic fields and nonlinear electrodynamics”, *Physical Review D* 77 (2008) 022530; R. Emami, H. Firouzjahi and M. Movahed, “Inflation from charged scalar and primordial magnetic fields?”, *Physical Review D* 81 (2010) 083526; D. Grasso and H. Rubinstein, “Magnetic fields in the early Universe”, *Physics Reports* 348 (2010) 163–266; L. Campanelli and M. Giannotti, “Magnetic helicity generation from the cosmic axion field”, *Physical Review D* 72 (2005) 123001; A. Berera, T. Kephart and S. Wick, “GUT cosmic magnetic fields in a warm inflationary universe”, *Physical Review D* 59 (1999) 043510; R. de Souza and R. Opher, “Origin of primordial magnetic fields”, *Physical Review D* 77 (2008) 043529; A. Diaz-Gil et al., “Magnetic Field Production during Preheating at the Electroweak Scale”, *Physical Review Letters* 100 (2008) 241301; R. Durrer, L. Hollenstein and J. Rajeev-Kumar, “Can slow roll inflation induce relevant helical magnetic fields?”, *JCAP* 03 (2011) 037; E. Harrison, “generation of magnetic fields in the radiation ERA”, *MNRAS* 147 (1970) 279; S. Matarrese et al., “Large-scale magnetic fields from density perturbations”, *Physical Review D* 71 (2005) 043502; L. Hollenstein et al., “Challenges for creating magnetic fields by cosmic defects”, *Physical Review D* 77 (2008) 063517; K. Takahashi, K. Ichiki and N. Sugiyama, “Electromagnetic properties of the early universe”, *Physical Review D* 77 (2008) 124028.

[107] C. Caprini and R. Durrer,
"Gravitational wave production: A strong constraint on primordial magnetic fields",
Physical Review D 65 (2002) 023517.

[108] T. Kahniasvili, A. Tevzadze and B. Ratra,
"Phase Transition Generated Cosmological Magnetic Field at Large Scales",
The Astrophysical Journal 726 (2011) 78.

[109] T. Kahniasvili et al., "Primordial magnetic field limits from cosmological data",
Physical Review D 82 (2010) 083005.

[110] A. Neronov and I. Vovk,
"Evidence for Strong Extragalactic Magnetic Fields from Fermi Observations of TeV Blazars",
Science 328 (2010) 73.

[111] K. Takahashi et al.,
"Evidence for Strong Extragalactic Magnetic Fields from Fermi Observations of TeV Blazars",
MNRAS 410 (2011) 2741–2748.

[112] K. Dolag et al.,
"Lower Limit on the Strength and Filling Factor of Extragalactic Magnetic Fields",
The Astrophysical Journal Letters 727 (2011) 4.

[113] R. Durrer, "Gauge Invariant Cosmological Perturbation Theory: A General Study and It's Application to the Texture Scenario of Structure Formation",
Fundamentals of Cosmic Physics 15 (1994) 209–339.

[114] D. Spergel et al., "First-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters",
The Astrophysical Journal Supplement Series 148 (2003) 175–179.

[115] D. Nitta et al., "CMB anisotropies at second order III: bispectrum from products of the first-order perturbations", *JCAP* 421 (2009) 14.

[116] K. Malik and D. Wands, "Evolution of second-order cosmological perturbations",
Classical and Quantum Gravity 421 (2004) 65.

[117] D. Wands et al.,
"New approach to the evolution of cosmological perturbations on large scales",
Physical Review D 62 (2000) 043527.

[118] Y. Takamizu et al.,
"Non-Gaussianity of superhorizon curvature perturbations beyond delta N formalism",
JCAP 06 (2010) 19.

[119] M. Giovannini and Z. Rezaei,
"Gradient expansion, curvature perturbations, and magnetized plasmas",
Physical Review D 83 (2011) 083519.

[120] M. Bruni, R. Maartens and C. Tsagas,
"Magnetic field amplification in cold dark matter anisotropic collapse",
MNRAS 338 (2003) 785–789.

[121] C. Tsagas, "Electromagnetic fields in curved spacetimes",
Classical and Quantum Gravity 22 (2005) 393–407.

[122] C. Tsagas and R. Maartens, "Magnetized cosmological perturbations", *Physical Review D* 61 (2000) 083519.

[123] J. Barrow and C. Tsagas, "Cosmological magnetic field survival", *MNRAS* 414 (2011) 512–518.

[124] C. Clarkson, R. Maartens and C. Tsagas, "CMB limits om large-scale magnetic fields in an inhomogeneous universe", *Classical and Quantum Gravity* 20 (2003) 1519–1528.

[125] J. Barrow, R. Maartens and C. Tsagas, "Cosmology with inhomogeneous magnetic fields", *Physics Reports* 449 (2007) 131–171.

[126] K. Takahashi et al., "Magnetic Field Generation from Cosmological Perturbations", *Physical Review Letters* 95 (2005) 121301.

[127] M. Giovannini, "Estimating relic magnetic fields from CMB temperature correlations", *Physical Review D* 79 (2009) 121302.

[128] J. Jackson, *Classical Electrodynamics*, Wiley, New York, 1999.

[129] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Pergamon, New York, 1959.

[130] T. Buchert, "On Average Properties of Inhomogeneous Fluids in General Relativity: Perfect Fluid Cosmologies", *General Relativity and Gravitation* 33 (2001) 1381–1405.

[131] C. Pitrou, "Gauge-invariant Boltzmann equation and the fluid limit", *Classical and Quantum Gravity* 24 (2007) 6127–6158.

[132] R. Durrer, P. Ferreira and T. Kahniashvili, "Tensor microwave anisotropies from a stochastic magnetic field", *Physical Review D* 61 (2000) 043001.

[133] C. Caprini, R. Durrer and T. Kahniashvili, "Cosmic microwave background and helical magnetic fields: The tensor mode", *Physical Review D* 69 (2004) 063006.

[134] R. Durrer, *The Cosmic Microwave Background*, Cambridge University Press, 2008.

[135] C. Bonvin and C Caprini, "CMB temperature anisotropy at large scales induced by a causal primordial magnetic field", *JCAP* 05 (2010) 022.

[136] E. Fenu, C. Pitrou and R. Maartens, "The seed magnetic field generated during recombination", *MNRAS* 414 (2011) 2354–2366.

[137] Brown, I.A., "Primordial Magnetic Fields in Cosmology" (2008), arXiv:0812.1781v1.

[138] A. Mack, T. Kahniashvili and A. Kosowsky, "Microwave background signatures of a primordial stochastic magnetic field", *Physical Review D* 65 (2002) 123004.

[139] M. Giovannini and K. Kunze, "Faraday rotation, stochastic magnetic fields, and CMB maps", *Physical Review D* 78 (2008) 023010.

[140] M. Giovannini, "Dark energy, integrated Sachs-Wolfe effect and large-scale magnetic fields", *Classical and Quantum Gravity* 27 (2010) 105011.

[141] K. Subramanian, "Magnetic fields in the early Universe", *Astronomische Nachrichten* 331 (2010) 110.

[142] J. Adams et al.,
"Distortion of the acoustic peaks in the CMBR due to a primordial magnetic field",
Physics Letters B 388 (1996) 253–258.

[143] A. Kosowsky and A. Loeb,
"Faraday Rotation of Microwave Background Polarization by a Primordial Magnetic Field",
Astrophysical Journal 469 (1996) 1.

[144] R. Durrer, T. Kahniashvili and A. Yates,
"Microwave background anisotropies from Alfvén waves",
Physical Review D 58 (1998) 123004.

[145] V. Eke et al., "Galaxy groups in the Two-degree Field Galaxy Redshift Survey: the luminous content of the groups",
Monthly Notices of the Royal Astronomical Society 355 (2004) 769–784.

[146] P. Hickson, "Systematic properties of compact groups of galaxies",
The Astrophysical Journal 255 (1982) 382–391.

[147] A. Iovino et al., "A New Sample of Distant Compact Groups from the Digitized Second Palomar Observatory Sky Survey", *The Astronomical Journal* 125 (2003) 1660–1681.

[148] I. Saviane, V. Ivanov and J. Borissova, *Report on the Conference on Groups of Galaxies in the Nearby Universe ementary particle theory.*
Santiago de Chile: The Messenger, volume 123, 5th–9th Dec. 2006 63.

[149] R. Coziol and I. Plauchu-Frayn., "Evidence for Tidal Interactions and Mergers as the Origin of Galaxy Morphology Evolution in Compact Groups",
The Astronomical Journal 133 (2007) 2630–2642.

[150] P. Hickson, "Atlas of Compact Groups of Galaxies (Special Issue)",
Astrophysical Letters and Communications 29 (1993) 1.

[151] K. Johnson et al., "The Infrared Properties of Hickson Compact Groups",
The Astronomical Journal 134 (2007) 1522–1543.

[152] C. Mendes de Oliveira et al., "Dynamical Effects of Interactions and the Tully-Fisher Relation for Hickson Compact Groups", *The Astronomical Journal* 126 (2003) 2635–2643.

[153] I. Prandoni, A. Iovino and H. T MacGillivray,
"Automated search for compact groups of galaxies in the southern sky",
The Astronomical Journal 107 (1994) 1235–1244.

[154] A. Iovino, "Detecting Fainter Compact Groups: Results from a New Automated Algorithm",
The astronomical Journal 124 (2002) 2471–2489.

[155] A. Pompei and A. Iovino, "Galaxy evolution in a new Southern Compact Groups sample",
Astrophysics and Space Science 285 (2003) 133–136.

[156] J. Mulchaey, "X-ray Properties of Groups of Galaxies",
Annual Review of Astronomy and Astrophysics 38 (2000) 289–335.

[157] H. Plana et al., "The Kinematics of the Warm Gas in the Interacting Hickson Compact Group of Galaxies HCG 90", *The Astronomical Journal* 116 (1998) 2123–2135.

[158] S. Faber et al.,
"Spectroscopy and photometry of elliptical galaxies. VI - Sample selection and data summary",
Astrophysical Journal Supplement Series 69 (1989) 763–808.

[159] P. White et al., "Extraordinary Diffuse Light in Hickson Compact Group 90", *The Astrophysical Journal* 585 (2003) 739–749.

[160] R. de Carvalho et al., "Redshift Survey of Galaxies around a Selected Sample of Compact Groups", *The Astrophysical Journal Supplement Series* 110 (1997) 1–8.

[161] A. Akylas, I. Georgantopoulos and A. Comastri, "BeppoSAX observations of the Seyfert 2 galaxies NGC 7172 and ESO 103-G35", *MNRAS* 324 (2001) 521–528.

[162] G. Longo et al., "Kinematical properties of the members of Hickson group No.90", *Astronomy and Astrophysics* 282 (1994) 418–424.

[163] P. Hickson, "Compact Groups of Galaxies", *Annual Review of Astronomy and Astrophysics* 35 (1997) 357–388.

[164] M. Gómez-Flechoso and R. Domínguez-Tenreiro, "Is the Fast Evolution Scenario for Virialized Compact Groups Really Compelling? The Role of a Dark Massive Group Halo", *The Astrophysical Journal* 549 (2001) L87–L90.

[165] T. Naab, A. Burket and L. Hernquist, "On the Formation of Boxy and Disky Elliptical Galaxies", *The Astrophysical Journal* 523 (1999) L133–L136.

[166] G. Mamon, "Are compact groups of galaxies physically dense?", *Astrophysical Journal* 307 (1986) 426–430.

[167] P. Hickson and H. Rood, "The nature of compact groups of galaxies", *The Astrophysical Journal* 331 (1988) L69–L72.

[168] T. Ponman et al., "A ROSAT survey of Hickson's compact galaxy groups", *Monthly Notices of the Royal Astronomical Society* 283 (1996) 690–708.

[169] J. Mulchaey et al., "The Intragroup Medium in Poor Groups of Galaxies", *Astrophysical Journal* 456 (1996) 80.

[170] A. Cavaliere and R. Fusco-Femiano, "X-rays from hot plasma in clusters of galaxies", *Astronomy and Astrophysics* 49 (1976) 137–144.

[171] J. Osmond and T. Ponman, "The GEMS project: X-ray analysis and statistical properties of the group sample", *Monthly Notices of the Royal Astronomical Society* 350 (2004) 1511–1535.

[172] L. Verdes-Montenegro et al., "The evolution of HCG 31: Optical and high-resolution HI study", *Astronomy and Astrophysics* 430 (2005) 443–464.

[173] J. Mulchaey and A. Zabludoff, "The Properties of Poor Groups of Galaxies. II. X-Ray and Optical Comparisons", *The Astrophysical Journal* 496 (1998) 73–92.

[174] G. Longo, A. Grimaldi and G. Richter, "Discovery of an excess of blue star clusters around Hickson 90", *Astronomy and Astrophysics* 299 (1995) L45.

[175] C. M. de Oliveira, "Evolution of Galaxies in Compact Groups", PhD Thesis: University of British Columbia, 1992.

[176] P. Hickson, E. Kindl and J. Auman, “A photometric catalog of compact groups of galaxies”, *Astrophysical Journal Supplement Series* 70 (1989) 687–698.

[177] J. Tonry and M. Davies, “A survey of galaxy redshifts. I - Data reduction techniques”, *Astronomical Journal* 84 (1979) 1511–1525.

[178] R. Madejski, R. Bender and C. Moellenhoff, “Morphology and kinematics of the pair of interacting elliptical galaxies NGC4782 and NGC4783”, *Astronomy and Astrophysics* 242 (1991) 58–68.

[179] S. Helsdon and T. Ponman, “The intragroup medium in loose groups of galaxies”, *Monthly Notices of the Royal Astronomical Society* 315 (2000) 356–370.

[180] K. Borne et al., “Interacting binary galaxies. 7: Kinematic data for 12 disturbed ellipticals”, *The Astrophysical Journal* 435 (1994) 79–92.

[181] M. Spavone et al., “Hickson 62. I. Kinematics of NGC 4778”, *Astronomy and Astrophysics* 457 (2006) 493–500.

[182] H. Rix and S. White, “Optimal estimates of line-of-sight velocity distributions from absorption line spectra of galaxies - Nuclear discs in elliptical galaxies”, *Monthly Notices of the Royal Astronomical Society* 254 (1992) 389–403.

[183] M. Cappellari and E. Emsellem, “Parametric Recovery of Line-of-Sight Velocity Distributions from Absorption-Line Spectra of Galaxies via Penalized Likelihood”, *The Publications of the Astronomical Society of the Pacific* 116 (2004) 138–147.

[184] T. Statler et al., “Stellar Kinematics of the Double Nucleus of M31”, *The Astronomical Journal* 117 (1999) 894–907.

[185] R. van der Marel and M. Franx, “A new method for the identification of non-Gaussian line profiles in elliptical galaxies”, *Astrophysical Journal* 407 (1993) 525–539.

[186] G. Contopoulos, “Self-consistent models of spiral galaxies”, *Galactic models; Proceedings of the 4th Florida Workshop on Nonlinear Dynamics, Gainesville, FL*, New York Academy of Sciences, Annals, 1st Mar. 1989–2nd Mar. 189 101–113.

[187] C. Halliday et al., “Line-of-sight velocity distributions of low-luminosity elliptical galaxies”, *Monthly Notices of the Royal Astronomical Society* 326 (2001) 473–489.

[188] S. De Rijcke et al., “Dwarf elliptical galaxies with kinematically decoupled cores”, *Astronomy and Astrophysics* 426 (2004) 53–63.

[189] J. Barnes, “Encounters of disk/halo galaxies”, *Nature* 331 (1988) 699–717.

[190] J. Barnes, “Making ellipticals by mergers”, *Nature* 344 (1990) 379–380.

[191] J. Barnes and L. Hernquist, “Transformations of Galaxies. II. Gasdynamics in Merging Disk Galaxies: Addendum”, *The Astrophysical Journal* 495 (1998) 187–188.

[192] N. Cretton et al., “Axisymmetric Three-Integral Models for Galaxies”, *The Astrophysical Journal Supplement Series* 124 (1999) 383–401.

[193] N. Cretton et al., “The Kinematics of 3:1 Merger Remnants and the Formation of Low-Luminosity Elliptical Galaxies”, *The Astrophysical Journal* 554 (2001) 291–297.

[194] E. Barton, B. C. Bromley and M. Geller,
"Kinematic Effects of Tidal Interaction on Galaxy Rotation Curves",
ASP Conference Series 182 (1999).

[195] R. Davies et al.,
"Spectroscopy and photometry of elliptical galaxies. II - The spectroscopic parameters",
Astrophysical Journal Supplement Series 64 (1987) 581–600.

[196] G. Longo et al., "Kinematics of 14 early-type galaxies",
Astronomy and Astrophysics Suppl. 105 (1994) 433–445.

[197] L. Castaneda and M. Hilker, "Kinematics in Hickson Compact Group 90", *Astronomische Nachrichten, Vol. 325, Supplement 1, Short Contributions Presented at the Annual Scientific Meeting of the Astronomische Gesellschaft and the Czech Astronomical Society in Prague*, Astronomical Nachrichten, 20th–25th Sept. 2004 129.

[198] L. Castaneda-Colorado and M. Hilker, "Kinematics in Hickson Compact Group 90",
Proceedings of Baryons in Dark Matter Halos, ed. by R. Dettmar, U. Klein and P. Saluci,
Published by SISSA, Proceedings of Science, 5th–9th Oct. 2004 34.1.

[199] H. Hortua and L. Castañeda, "Power spectrum of the post-inflationary magnetic fields.",
Physical Review D 90 (2014) 123520.

[200] H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*, Addison-Wesley, 2001.

[201] Corichi, A, and Reyes, J. D.,
"Boundary Terms and the 3 + 1 Decomposition of the Holst Action",
Journal of Physics: Conference Series 360 (2012) 012021.

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