# Symmetric products, subgroup lattices and filtrations of global K-theory

**DISSERTATION**

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Bad Nauheim

Bonn, 2016

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 14.06.2016
Erscheinungsjahr: 2016

## Abstract

This thesis consists of two projects in equivariant stable homotopy theory. In the first we study the rational homotopy groups of symmetric products of the $G$-sphere spectrum and show that they are naturally isomorphic to the rational homology groups of certain subcomplexes of the subgroup lattice of $G$. In the second we investigate global equivariant versions of spectrum level filtrations introduced by Arone and Lesh which interpolate between topological/algebraic $K$-theory and the Eilenberg-MacLane spectrum for the integers. We determine the global homotopy type of the subquotients and use this description to obtain algebraic formulas for filtrations of representation rings that arise on 0-th homotopy groups.

# Contents

# Chapter 1

# Introduction

This thesis takes place in the field of algebraic topology, the study of topological spaces and continuous maps via algebraic invariants such as homotopy or homology groups, vector bundles or bordism groups of manifolds.

It is a fundamental phenomenon that many of these invariants stabilize as one lets the dimension go to infinity. For example, given a (reasonable) compact space $X$, the following two invariants eventually become independent of $n$:

- The set of isomorphism classes of $n$-dimensional complex vector bundles over $X$.
- The set of homotopy classes of continuous maps $[\Sigma^n X, \Sigma^n Y]$, where $Y$ is a fixed space and $\Sigma^n$ denotes the $n$-fold suspension.

Moreover, the invariants which lie in the stable range have more structure and better properties: They define generalized cohomology theories in $X$ and are therefore considerably easier to compute. The idea of stable homotopy theory is to disregard the unstable information and package the stable invariants into a new category. The new objects, stable generalizations of spaces, are called *spectra* and correspond to cohomology theories. This approach to homotopy theory has been very successful and has led to considerable advances, both in algebraic and geometric topology.

In this thesis we are concerned with symmetries of these stable objects. We consider spectra that come equipped with an action of a compact Lie or finite group $G$. Such objects are called $G$-spectra, which again assemble to a stable category. In fact, there are several variants on how to define a $G$-spectrum and the $G$-stable homotopy category, and we work in the most structured one (called 'genuine $G$-spectra'), which is stable with respect to all spheres with linear $G$-action and which allows the construction of transfer maps between fixed point spectra. For example, the spectrum representing stable complex vector bundles has an action of the cyclic group $C_2$ via complex conjugation and forms a genuine $C_2$-spectrum. Its spectrum of fixed points represents stable real vector bundles, and the transfer map takes a complex bundle to its underlying real one. This highly structured form of symmetry for spectra has had many applications in topology, an important example being the recent proof of Hill, Hopkins and Ravenel [HHR14] that every framed manifold of dimension $\geq 127$ is framed cobordant to a homotopy sphere. On the other hand, the category of genuine $G$-spectra is also intrinsically interesting: It reflects properties of the group $G$, it is related to representation theory and forms the natural home of cohomology theories for $G$-spaces.

This work is roughly divided into two projects in equivariant stable homotopy theory, the first of which makes up Part II, while the second makes up Parts III and IV. In the following we first give an overview over these two projects and then discuss the results and organization of this thesis in detail.

## Symmetric products and subgroup lattices

Classical, ordinary (co-)homology is represented by the Eilenberg-MacLane spectrum $H\mathbb{Z}$. On the other hand, the sphere spectrum $\mathbb{S}$ represents stable (co-)homotopy, which is very interesting but extremely difficult to compute. For every $n \in \mathbb{N}$ there is a spectrum $Sp^n$, made up of the $n$-th symmetric products $(S^k)^{\times n}/\Sigma_n$ of spheres, which sits in between the two. Together they assemble to a filtration

$$\mathbb{S} = Sp^1 \to Sp^2 \to \ldots \to Sp^\infty \simeq H\mathbb{Z}.$$

This passage from stable homotopy to ordinary homology has been much studied and has interesting properties: For example, the cohomology of the $Sp^n$ realizes the filtration by length of admissible sequences on the Steenrod algebra [Nak58] and the subquotients $Sp^n/Sp^{n-1}$ are suspension spectra related to partition complexes and Tits buildings [AD01].

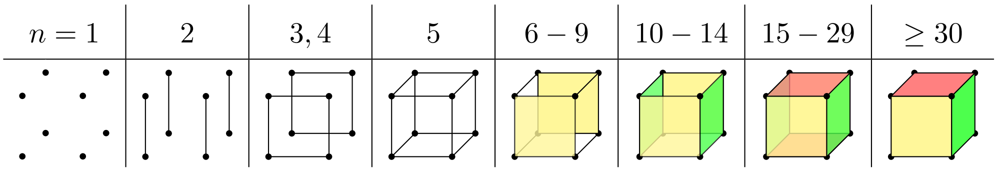
Now we take $G$ to be a finite group. For a finite dimensional real $G$-representation $V$ we denote by $S^V$ its associated representation sphere, i.e., the one-point compactification of $V$. The symmetric products $(S^V)^{\times n}/\Sigma_n$ of representation spheres give rise to a genuine $G$-spectrum, which we denote by $Sp^n_G$. Again one obtains a filtration

$$\mathbb{S}_G = Sp^1_G \to Sp^2_G \to \ldots \to Sp^\infty_G \simeq H\underline{\mathbb{Z}},$$

this time interpolating between the $G$-sphere spectrum and an Eilenberg-MacLane spectrum for the constant Mackey functor $\underline{\mathbb{Z}}$ [dS03].

This project is about the study of one of the central invariants of equivariant symmetric products, their homotopy groups $\pi_k^G(Sp^n_G)$. Of course, a full computation is out of reach, as this would in particular include the stable homotopy groups of spheres $\pi_k(\mathbb{S})$, which despite enormous effort are only known up to $k \sim 60$. However, in stable homotopy theory there are various ways of localization that allow to concentrate on one - hopefully computable - part of a homotopy group and disregard the rest. Perhaps the most drastic such localization is given by rationalization, i.e., tensoring with $\mathbb{Q}$. It still allows one to recover the rank of a maximal free abelian subgroup, but one loses all torsion information. For the non-equivariant symmetric product filtration this localization is too drastic: After tensoring with $\mathbb{Q}$, there is no difference between $\mathbb{S}$ and $H\mathbb{Z}$, and in fact all the $Sp^n$ are equivalent to one another. However, it turns out that the rationalization of their equivariant analogs $Sp^n_G$ still contains interesting information about $G$: We show that the rationalized homotopy groups $\pi_k^G(Sp^n_G) \otimes \mathbb{Q}$ are closely related to the topology of the subgroup lattice of $G$, i.e., the simplicial complex whose vertices are the subgroups of $G$ and whose higher simplices are associated to chains of subgroup inclusions. More precisely, each $Sp^n_G$ is modeled by a certain subcomplex of the subgroup lattice, and one obtains $\pi_k^G(Sp^n_G) \otimes \mathbb{Q}$ by computing the rational homology of this subcomplex.

**Example.** The subgroup lattice of the cyclic group $G = C_{30}$ is a cube, and the subcomplexes which model the $Sp^n_{C_{30}}$ rationally are given by




from which one can read off the following homotopy groups:

| $n$ | 1 | 2 | 3, 4 | 5 | 6 – 9 | 10 – 14 | 15 – 29 | ≥ 30 |
|---|---|---|---|---|---|---|---|---|
| $\pi_2^{C_{30}}(Sp^n_{C_{30}}) \otimes \mathbb{Q}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | 0 |
| $\pi_1^{C_{30}}(Sp^n_{C_{30}}) \otimes \mathbb{Q}$ | 0 | 0 | $\mathbb{Q}^2$ | $\mathbb{Q}^5$ | $\mathbb{Q}^3$ | $\mathbb{Q}$ | 0 | 0 |
| $\pi_0^{C_{30}}(Sp^n_{C_{30}}) \otimes \mathbb{Q}$ | $\mathbb{Q}^8$ | $\mathbb{Q}^4$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

For small $G$ as above this makes it an easy exercise to compute all $\pi_k^G(Sp^n_G) \otimes \mathbb{Q}$ explicitly and we do so in various other examples.

## Filtrations of global $K$-theory

Another important example of a spectrum is given by connective $K$-theory $ku$, which represents the cohomology theory of stable complex vector bundles. In [AL07], Arone and Lesh showed that there exists a sequence of spectra

$$ku = A_0^u \to A_1^u \to A_2^u \to \ldots \to A_\infty^u \simeq H\mathbb{Z},$$

interpolating between connective $K$-theory and the Eilenberg-MacLane spectrum for the integers, with similar properties as the symmetric product filtration considered in the previous section. For example, the subquotients are again suspension spectra, this time related to decomposition lattices of finite dimensional complex vector spaces. In fact, the authors showed that both filtrations arise as special cases of a more general construction, which also produces similar filtrations for real topological $K$-theory $ko$ and for $kR$, the free algebraic $K$-theory of a discrete (reasonable) ring $R$. We call the filtrations constructed this way *complexity filtrations*, based on the usage of that term in [Les00]. In a later paper [AL10], the authors showed that complexity filtrations are linked to filtrations of the $K$-theory spectra themselves, which they call *modified rank filtrations*.

In this project we set up and investigate equivariant versions of both the modified rank and complexity filtration. We work with the following equivariant generalizations of the spectra involved:

- Topological $K$-theory is replaced by equivariant $K$-theory in the sense of Segal ([Seg68a]), the $K$-theory of equivariant vector bundles on $G$-spaces.
- Free algebraic $K$-theory of a discrete ring $R$ is replaced by a $G$-spectrum whose $H$-fixed points (for $H$ a subgroup of $G$) represent the direct sum $K$-theory of $R[H]$-modules that are finitely generated free as $R$-modules, so-called *$R[H]$-lattices*. In particular, $\pi_0^H(kR)$ is the group completion of the monoid of isomorphism classes of $R[H]$-lattices, denoted $\mathrm{Rep}_R(H)$.
- The Eilenberg-MacLane spectrum $H\mathbb{Z}$ is replaced by the Eilenberg-MacLane spectrum for the constant Mackey functor $\underline{\mathbb{Z}}$.

The first and last item make sense for all compact Lie groups, but for non-discrete groups our results have to be taken with a grain of salt, as we discuss later. Equivariant algebraic $K$-theory is only defined for finite groups.

In Part III, which is joint work with Dominik Ostermayr, we lift the modified rank and complexity filtration to the equivariant context and determine the equivariant homotopy type of the subquotients, generalizing results of Arone and Lesh. In Part IV we then apply these results to demonstrate a further similarity to the symmetric product filtration, through the effect on the 0-th equivariant homotopy group $\pi_0^G$. In [Sch14], Schwede showed that, loosely speaking, $\pi_0^G(Sp^n)$ is obtained from the Burnside ring of $G$ by forgetting all information at $G$-sets of order at most $n$. When replacing the sphere spectrum by $ku$, the role of the Burnside ring is played by the representation ring. So, by analogy, this suggests that $\pi_0^G((A_n^u)_G)$ should be obtained from the representation ring of $G$ by forgetting all information at $G$-representations of dimension at most $n$, and this is what we show. A similar result holds for the real and algebraic analogs. These slogans are best made precise by working in the global equivariant framework, i.e., by considering all compact Lie or finite groups simultaneously instead of focusing on a single $G$. We now recall this framework.

## Global equivariant homotopy theory

All the equivariant spectra that have appeared so far are defined uniformly over all compact Lie or at least finite groups, they are not specific to a single group $G$. Moreover, they are related by various restriction and transfer maps. Global equivariant homotopy theory is a framework to capture this rich functoriality and to focus on the constructions and operations that are natural for all $G$. To achieve this, one assembles equivariant spectra for varying compact Lie groups $G$ into one global spectrum. For example, the global sphere spectrum $\mathbb{S}$ is now just one object, but it encodes all the information of $G$-sphere spectra $\mathbb{S}_G$ for all compact Lie groups $G$.

There have been various approaches to formalizing this idea, for example in [LMS86, Chapter 2], [GM97, Section 5] and [Boh14]. We work with a recent model due to Schwede

[Sch15], who showed that a category of global spectra can be obtained by looking at the well-known category of orthogonal spectra from another angle: Every orthogonal spectrum gives rise to a $G$-orthogonal spectrum for any compact Lie group $G$ by endowing it with the trivial $G$-action and, classically, changing from the trivial to a complete $G$-universe, but this change of universe is an equivalence of categories on the point-set level. The fundamental observation used in [Sch15] is that the $G$-homotopy type of such a $G$-orthogonal spectrum with trivial action is not determined by its non-equivariant homotopy type. There are maps of orthogonal spectra that are a non-equivariant stable equivalence but not a $G$-stable equivalence when given the trivial $G$-action. Taking these $G$-homotopy types for varying $G$ into account gives rise to a much finer notion of weak equivalence called *global equivalence* and thereby to the global stable homotopy category, which splits each non-equivariant homotopy type into many global variants. It turns out that all spectra mentioned in the previous sections fit into this framework or its symmetric spectrum analog, which was developed in [Hau15].

The reason for us to work in the setup of global homotopy is twofold. On the one hand it seems to be the right framework to capture the full functoriality of the situation and hence yield the strongest results. On the other hand, and more importantly, it puts us in the position to use universal properties and describe phenomena that are not present over a single group. One essential feature of global spectra is that their equivariant homotopy groups assemble to a so-called *global functor*, meaning they allow restrictions for arbitrary group homomorphisms and transfers for subgroup inclusions. It turns out that various equivariant homotopy groups that we encounter in this thesis carry universal properties when viewed as global functors, but no longer carry them as $G$-Mackey functors if one focuses on a single group $G$. This implies that some of our results would be harder to describe and appear less natural in the non-global context.

## Statement of results

We now describe the content of this thesis in more detail.

### Symmetric products and subgroup lattices

Let $Sp^n(X) = X^{\times n}/\Sigma_n$ denote the $n$-th symmetric product of a space $X$ and $Sp^n = \{Sp^n(S^k)\}$ the spectrum consisting of the various $n$-th symmetric products of spheres. Insertion of basepoints yields maps $Sp^n \to Sp^{n+1}$ and hence the symmetric product filtration

$$\mathbb{S} = Sp^1 \to Sp^2 \to \ldots \to Sp^\infty.$$

All the $Sp^n$ form orthogonal spectra. We think of them as global spectra and write $\pi_*^G(Sp^n)$ for what was denoted $\pi_*^G(Sp^n_G)$ before. We are interested in a computation of the rationalized $\pi_*^G(Sp^n)$ for finite groups $G$.

For this we denote by $L(G)$ the subgroup lattice of $G$, i.e., the nerve of the poset of subgroups of $G$. We define a filtration

$$\emptyset = L(G)_0 \subseteq L(G)_1 \subseteq L(G)_2 \subseteq \ldots \subseteq L(G)_\infty = L(G)$$

on this subgroup lattice by declaring a simplex $H_0 \leq \ldots \leq H_k$ to lie in $L(G)_n$ if and only if the index $[H_k : H_0]$ is at most $n$. The lattice $L(G)$ carries an action by $G$ through conjugation, which preserves the subcomplexes $L(G)_n$. Then the main result is the following, where $H_*(-, \mathbb{Q})$ denotes the ordinary homology of a space with coefficients in $\mathbb{Q}$:

**Theorem A** (Theorem 4.1.1)**.** *For all finite groups $G$ and $n \in \mathbb{N} \cup \{\infty\}$ there are isomorphisms*

$$\pi_*^G(Sp^n) \otimes \mathbb{Q} \cong H_*(L(G)_n, \mathbb{Q})/G.$$

Hence, the process of adding the $n$-th coordinate in the symmetric products of $G$-spheres can be modeled rationally by adding all chains of total index $n$ in the subgroup lattice of $G$. The subcomplexes $L(G)_n$ can have arbitrarily high non-trivial rational homology, though of course for every fixed $G$ the homology is bounded since $L(G)$ is a finite complex.

The 0-th homotopy groups $\pi_0^G(Sp^n)$ were computed previously by Schwede [Sch14], even integrally, as we recall later. However, the following holds:

**Proposition** (Proposition 8.0.18)**.** The only finite groups $G$ for which all $\pi_*^G(Sp^n) \otimes \mathbb{Q}$ are concentrated in degree 0 are the cyclic $p$-groups.

This is perhaps surprising, because $\pi_*^G(Sp^1) \otimes \mathbb{Q}$ and $\pi_*^G(Sp^\infty) \otimes \mathbb{Q}$ are always concentrated in degree 0.

Furthermore, we investigate the global properties of the rationalized $Sp^n_{\mathbb{Q}}$. Theorem A implies that the assignment

$$G \mapsto H_k(L(G)_n, \mathbb{Q})/G$$

must extend to a global functor, and we describe this functoriality on the side of subgroup lattices. In fact, this structure already exists on the level of chains, yielding a chain complex of global functors. We then explain how one can reconstruct the full global homotopy type of the rationalized symmetric products from this global chain complex, which uses an explicit equivalence between the homotopy category of rational global spectra and the derived category of rational global functors due to Wimmer [Wim16]. This yields a stronger version of Theorem A, which allows us to deduce:

**Theorem B** (Theorem 8.0.17)**.** *The rationalized symmetric product $Sp^n_{\mathbb{Q}}$ is not a product of global Eilenberg-MacLane spectra unless $n$ is 1 or $\infty$.*

Over a fixed finite group $G$ this phenomenon is not visible: Every rational $G$-spectrum decomposes as a product of Eilenberg-MacLane spectra. Theorem B can be interpreted as saying that these decompositions cannot be chosen compatibly for all finite groups.

**Remark.** One ingredient in the proof of Theorem A is a global equivariant version of the theorem by Arone and Dwyer [AD01, Theorem 1.11] which relates $Sp^n/Sp^{n-1}$ to the partition complex $\Pi_n$ of the set $\{1, \ldots, n\}$. The equivariant case is Theorem 6.6.1 and might be of independent interest. It also gives rise to a short proof that the integral $\mathcal{F}in$-global Steenrod algebra is given by a single copy of $\mathbb{Z}$ concentrated in degree 0, cf. Theorem 6.6.9.

### Filtrations of global $K$-theory I: Subquotients

All results in this section are joint work with Dominik Ostermayr. For now we write $kX$ to mean either of the following global spectra:

- connective global complex $K$-theory $ku$
- connective global real $K$-theory $ko$
- connective global algebraic $K$-theory $kR$ of a discrete ring $R$ satisfying dimension invariance

All these are defined in work of Schwede ([Sch15] and [Sch16]) and come with a natural global generalization of the modified rank filtration, which we denote by

$$* \to kX^1 \to kX^2 \to \ldots \to kX.$$

We describe the subquotients of these filtrations. For this we let $\mathcal{L}_n^u$ denote the topological poset of proper decompositions of $\mathbb{C}^n$ as an orthogonal sum of subspaces, ordered by refinement. Here, 'proper' means that the trivial decomposition into one summand is excluded. The poset carries a $U(n)$-action by applying the isometry to each summand in the decomposition. Similarly, $\mathcal{L}_n^o$ denotes the $O(n)$-poset of proper decompositions of $\mathbb{R}^n$, and $\mathcal{P}_n^R$ denotes the $GL_n(R)$-poset of proper decompositions of $R^n$ as a direct sum of free submodules. We show:

**Theorem C** (Theorems 11.1.5, 11.2.8 and 13.1.6)**.** *There are global equivalences*

$$\begin{aligned} ku^n/ku^{n-1} &\simeq \Sigma^\infty(E_{gl}U(n)_+ \wedge_{U(n)} |\mathcal{L}_n^u|^\diamond) \\ ko^n/ko^{n-1} &\simeq \Sigma^\infty(E_{gl}O(n)_+ \wedge_{O(n)} |\mathcal{L}_n^o|^\diamond) \\ kR^n/kR^{n-1} &\simeq \Sigma^\infty(E_{gl}GL_n(R)_+ \wedge_{GL_n(R)} |\mathcal{P}_n^R|^\diamond). \end{aligned}$$

The underlying non-equivariant statement of this theorem is due to Arone and Lesh (cf. [AL10, Section 2.2] for the case of topological $K$-theory). The expression $\Sigma^\infty$ denotes the suspension spectrum of a global space (in the framework we use: an orthogonal space) and $(-)^\diamond$ stands for the unreduced suspension. The global spaces that appear here are part of a general construction that takes a based $K$-space $X$ for some topological group $K$ and produces a global space $E_{gl}K_+ \wedge_K X$, its *global homotopy orbits*. Given a compact Lie group $G$, the underlying $G$-homotopy type of this construction is $E_G K_+ \wedge_K X$, where $E_G K$ is a universal space for principal $K$-bundles in $G$-spaces. In particular, the underlying non-equivariant homotopy type agrees with the usual homotopy orbits. However, global homotopy orbits depend on the 'genuine' equivariant homotopy type of $X$ at all compact subgroups of $G$, while the usual homotopy orbits only depend on the 'naive' $K$-homotopy type. So, in a sense, the global subquotients $kX^n/kX^{n-1}$ see even more of the decomposition lattice than the non-equivariant ones.

**Remark** (Global Barratt-Priddy-Quillen Theorem, Theorem 12.0.11)**.** Our methods can also be applied to a rank filtration $k\mathcal{F}in^1 \to k\mathcal{F}in^2 \to \ldots \to k\mathcal{F}in$ of the global

$K$-theory of finite sets $k\mathcal{F}in$, yielding an easy proof of the global Barratt-Priddy-Quillen Theorem: The unit $\mathbb{S} \to k\mathcal{F}in$ is a global equivalence.

We proceed by considering complexity filtrations

$$kX \simeq A_0^X \to A_1^X \to \ldots \to A_\infty^X \simeq Sp^\infty.$$

We define these as suitable homotopy pushouts that involve the modified rank and symmetric product filtration, generalizing the non-equivariant description of [AL10]. In [AL07, Corollary 8.3] it is shown that the $n$-th subquotient of the complexity filtration can be non-equivariantly described as the suspension spectrum of a classifying space for the collection of so-called standard subgroups of $U(n)$ (respectively $O(n)$ or $GL_n(R)$). We denote these collections by $\overline{\mathcal{C}}_n^X$. There are natural global equivariant generalizations of classifying spaces for collections (similarly to the global homotopy orbits discussed above), which we denote by $B_{gl}\overline{\mathcal{C}}_n^X$. Generalizing the non-equivariant statement in [AL07], we then show:

**Theorem D** (Theorems 11.2.3 and 13.2.1)**.** *There are global equivalences*

$$\begin{aligned} A_n^u/A_{n-1}^u &\simeq \Sigma^\infty(B_{gl}\overline{\mathcal{C}}_n^u)^\diamond \\ A_n^o/A_{n-1}^o &\simeq \Sigma^\infty(B_{gl}\overline{\mathcal{C}}_n^o)^\diamond \end{aligned}$$

*and if $R$ is an integral domain with $2 \neq 0$ also*

$$A_n^R/A_{n-1}^R \simeq \Sigma^\infty(B_{gl}\overline{\mathcal{C}}_n^R)^\diamond.$$

The conditions on $R$ were also required in [AL07], and in fact it can be shown that the statement is false in full generality.

**Remark.** The methods we use to obtain these descriptions of the subquotients are quite different from those of [AL07] and [AL10]. While Arone and Lesh perform categorical constructions, we work with an explicit $\Gamma$-space model for connective $K$-theory and decompose it geometrically. It turns out that with this model the quotients are in fact *isomorphic* to suspension spectra of global spaces. Hence the main work lies in examining the global equivariant homotopy type of those and identifying them as geometric models of classifying spaces or lattices.

### Filtrations of global $K$-theory II: Induced filtrations on representation rings

Afterwards, we apply our global description of the filtration subquotients to show another formal similarity between complexity filtrations and the symmetric product filtration. For this we recall a result of Schwede [Sch14] on the symmetric product filtration. On 0-th homotopy, the map $\mathbb{S} \to H\mathbb{Z}$ induces the augmentation from the Burnside ring global functor $A(-) \cong \underline{\pi}_0(\mathbb{S})$ to the constant functor $\mathbb{Z}$, sending a finite $G$-set to its number of elements. Applying $\underline{\pi}_0$ to the symmetric product filtration gives a filtration

$$\underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(Sp^2) \to \ldots \to \underline{\pi}_0(H\mathbb{Z}) \cong \underline{\mathbb{Z}}$$

of this augmentation. Schwede showed that this algebraic filtration allows a compact description when considered in the global context. For this we let $\tau_n^{\Sigma}$ denote the tautological $n$-element $\Sigma_n$-set, thought of as an element in $\pi_0^{\Sigma_n}(\mathbb{S}) \cong A(\Sigma_n)$.

**Theorem** (Schwede, [Sch14, Theorem 3.13])**.** The map $\underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(Sp^n)$ is surjective for all $n \in \mathbb{N}$, with kernel generated as a global functor by the element $(\tau_n^{\Sigma} - n \cdot 1) \in \pi_0^{\Sigma_n}(\mathbb{S})$. In particular,

$$\underline{\pi}_0(Sp^n) \cong A(-)/(\tau_n^{\Sigma} - n \cdot 1)$$

as global functors.

The $\Sigma_n$-set $\tau_n^{\Sigma}$ is the universal $n$-element $G$-set, in the sense that for every $n$-element $G$-set $X$ there is a unique up to conjugacy group homomorphism $\alpha : G \to \Sigma_n$ such that $\alpha^*(\tau_n^{\Sigma}) \cong X$. So dividing out by $(\tau_n^{\Sigma} - n \cdot 1)$ can be loosely interpreted as forgetting all $G$-actions on sets with size at most $n$, though it is in fact more complicated, due to the presence of transfers.

In our case the complexity filtration induces a filtration of the augmentation from the complex representation ring global functor $RU(-)$ (or $RO(-)$ in the case of $ko$ and $\mathrm{Rep}_R(-)$ in the case of $kR$) to the constant functor with value $\mathbb{Z}$, sending a $G$-representation to its dimension or rank. There is a natural replacement for the universal $\Sigma_n$-set $\tau_n^{\Sigma}$ in this context: The $n$-th unitary group $U(n)$ acts tautologically on $\mathbb{C}^n$ (respectively $O(n)$ on $\mathbb{R}^n$ and $GL_n(R)$ on $R^n$), and every $n$-dimensional $G$-representation can be obtained by pulling this representation back along a homomorphism which is unique up to conjugacy. We let these universal representations be denoted by $\tau_n^u \in \pi_0^{U(n)}(ku)$ respectively $\tau_n^o \in \pi_0^{O(n)}(ko)$ and $\tau_n^R \in \pi_0^{GL_n(R)}(kR)$ if $R$ is finite. Then we have:

**Theorem E** (Complexity filtration on $\underline{\pi}_0$, Theorems 15.1.1, 15.1.2 and 15.1.3)**.** *The maps $\underline{\pi}_0(ku) \to \underline{\pi}_0(A_n^u)$, $\underline{\pi}_0(ko) \to \underline{\pi}_0(A_n^o)$ and $\underline{\pi}_0(kR) \to \underline{\pi}_0(A_n^u)$ are surjective for all $n \in \mathbb{N}$. The kernel is generated as a global functor by the elements $(\tau_n^u - n \cdot 1)$, $(\tau_n^o - n \cdot 1)$ respectively $(\tau_n^R - n \cdot 1)$ for finite $R$. In particular, there are isomorphisms of global functors:*

$$\underline{\pi}_0(A_n^u) \cong \underline{\pi}_0(ku)/(\tau_n^u - n \cdot 1)$$
$$\underline{\pi}_0(A_n^o) \cong \underline{\pi}_0(ko)/(\tau_n^o - n \cdot 1)$$
$$\underline{\pi}_0(A_n^R) \cong \underline{\pi}_0(kR)/(\tau_n^R - n \cdot 1)$$

Hence, $\underline{\pi}_0(A_n^X)$ can be interpreted as the representation ring global functor modulo forgetting all group actions on vector spaces/free modules of dimension/rank $n$. This theorem reduces an explicit calculation of $\pi_0^G(A_n^X)$ to an algebraic exercise in representation theory, for which we give examples in Section 16. The reason why $R$ has to be finite in Theorem E is that otherwise the general linear groups are not finite and so are not part of the global theory. There is also a description of $\underline{\pi}_0(A_n^R)$ when $R$ is not finite (Proposition 15.1.4) but it is no longer simplified by the global framework.

Moreover, we compute an algebraic description of the filtration on the representation ring itself that is induced from the modified rank filtration:

**Theorem F** (Modified rank filtration on $\underline{\pi}_0$)**.** *The global functor $\underline{\pi}_0(ku^n)$ (and similarly $\underline{\pi}_0(ko^n)$ and $\underline{\pi}_0(kR^n)$ for finite $R$) is the free global functor on the classes $\tau_1^u, \tau_2^u, \ldots, \tau_n^u$ modulo finitely many universal relations that identify*

- *homotopy-theoretic sums with direct sums of representations*
- *transfers with induction of representations*

*as long as the total dimension is at most $n$.*

A precise formulation is given in Theorems 14.1.3, 14.1.5 and 14.1.6. The proof uses an elementary examination of the fixed points of decomposition lattices of $G$-representations and the construction of an explicit geometric representative of a certain stable map, which allows us to identify its effect on $\underline{\pi}_0$. We note that this is only a filtration in the sense that the colimit gives the representation ring, as the connecting maps are in general neither injective nor surjective. Again there is also a description for non-finite $R$ (Proposition 14.1.7), but it is in general not finitely generated as a global functor.

## Organization

This thesis is organized as follows:

Chapter 2 deals with unstable global homotopy theory based on the models of orthogonal spaces and **I**-spaces, with a focus on two classes of examples: Global classifying spaces associated to a collection of subgroups of a fixed Lie group $K$, and global homotopy orbits of $K$-spaces. In Chapter 3 we move on to the stable version and discuss basic definitions and properties of the global homotopy theory of orthogonal and symmetric spectra, in particular global equivariant homotopy groups and global functors. We then recall models for symmetric products of spheres, global topological $K$-theory, global algebraic $K$-theory and the global $K$-theory of finite sets in this context.

In Part II we discuss the relation between rational symmetric products and subgroup lattices. Chapter 4 contains a statement of the main theorem and how it relates to Schwede's result on $\underline{\pi}_0$. In Chapter 5 we recall basics of rational global homotopy theory, explain how the main theorem can be stated in terms of geometric instead of categorical fixed points and state the stronger version via chain complexes of global functors. In Chapter 6 we then construct explicit maps from the suspension spectra of the $L(G)_n$ into the geometric fixed points of $Sp^n$ and show that they induce equivalences on subquotients, by first showing that the effect on rational homology is split injective and then abstractly checking that the homology groups are of the same dimension. Chapter 7 contains some examples for small $G$. Finally, in Chapter 8 we prove that the $Sp^n_{\mathbb{Q}}$ do not split as products of global Eilenberg-MacLane spectra and that the $\pi_*^G(Sp^n)\otimes\mathbb{Q}$ are only concentrated in degree 0 when $G$ is a cyclic $p$-group.

Part III is concerned with a computation of the global homotopy of the subquotients in modified rank and complexity filtrations, whose global versions are defined in Chapter 9. In Chapter 11 we first focus on the case of global topological $K$-theory and later in Chapter 13 translate the methods to the algebraic case. In between, we give a proof of the global Barratt-Priddy-Quillen theorem (Chapter 12). Part III concludes with two rather technical appendices. In the first we provide a proof that the maps relating the filtration terms are cofibrations, in the second we show that the orthogonal spaces that appear in the filtration quotients are sufficiently cofibrant.

Finally, in Part IV we compute the effect of the modified rank filtrations (Chapter 14) and complexity filtrations (Chapter 15) on $\underline{\pi}_0$ and give examples for various finite groups (Chapter 16).

## Acknowledgements

First of all I thank my adviser Stefan Schwede for introducing me to the world of equivariant homotopy theory and inviting me to take part in his global equivariant project. I am very grateful for his ongoing support and encouragement, which started the first year of my university studies and went on all the way to my PhD. I would also like to thank Dominik Ostermayr for our nice collaboration, which led to Part III of this thesis. Furthermore, I want to thank Greg Arone for his continuing interest in the project, for various helpful discussions and for agreeing to be on the thesis committee. I am grateful to the MIT and in particular Haynes Miller for giving me the opportunity to spend two semesters as a visiting student there. Furthermore, I want to thank Christian Wimmer for answering my questions on rational global homotopy theory and Lennart Galinat, Irakli Patchkoria and Julia Schulz for proofreading a preliminary version.

I thank the GRK 1150 'Homotopy and Cohomology' for its support during the first two years of my PhD and the DFG Priority program SPP 1786 'Homotopy Theory and Algebraic Geometry' for the support during the last year.

Finally, I thank my wife Angela for always being there for me. This thesis is for you.

# Part I

# Global equivariant homotopy theory

# Chapter 2

# Unstable global homotopy theory

In this chapter we discuss two models for unstable global homotopy theory, as developed by Schwede in [Sch15]. We begin with orthogonal spaces, which model the full global theory over all compact Lie groups, and later deal with the more combinatorial **I**-spaces, which only give rise to equivariant homotopy types over finite groups. Both models are needed later on. We further discuss two classes of examples that are central to Part III: Global classifying spaces of collections and global homotopy orbits.

## 2.1 Orthogonal spaces

### 2.1.1 Definition and global equivalences

Let $L_{\mathbb{R}}$ be the topological category of finite dimensional real inner product spaces with linear isometric embeddings.

**Definition 2.1.1.** An *orthogonal space* is a continuous functor from $L_{\mathbb{R}}$ to the category of spaces.

All orthogonal spaces that occur in this thesis have the following additional property:

**Definition 2.1.2.** An orthogonal space $X$ is *closed* if for every linear isometric embedding $\varphi : V \hookrightarrow W$ the structure map $X(\varphi) : X(V) \to X(W)$ is a closed embedding.

Equivariance comes into play as follows: Let $G$ be a compact Lie group and $V$ a real $G$-representation, by which we throughout this thesis mean a real inner product space with a $G$-action through linear isometries. Then the evaluation $X(V)$ of any orthogonal space $X$ on $V$ inherits a $G$-action via the homomorphism $G \to O(V)$. Moreover, if $W$ is another $G$-representation and $\varphi : V \hookrightarrow W$ is a $G$-equivariant inner product map, then the structure map $X(\varphi) : X(V) \to X(W)$ is $G$-equivariant.

The evaluation can be extended further to infinite dimensional $G$-representations via the formula

$$X(W) = \operatorname*{colim}_{f.d. V \subseteq W} X(V)$$

where the colimit is taken over the poset of finite dimensional $G$-subrepresentations $V$ of $W$. In particular this is used to evaluate orthogonal spaces on a complete $G$-universe $\mathcal{U}_G$, a countably infinite dimensional representation in which every finite dimensional representation embeds.

An important invariant for an orthogonal space $X$ is the 0-th homotopy set functor, i.e., the collection of the sets

$$\pi_0^G(X) = \operatorname*{colim}_{f.d.V \subseteq \mathcal{U}_G} \pi_0(X(V)^G)$$

for all compact Lie groups $G$. Given a continuous group homomorphism $\psi : K \to G$ there is an induced restriction map $\psi^* : \pi_0^G(X) \to \pi_0^K(X)$ constructed as follows: Every $G$-fixed point $x \in X(V)$ for some $V$ also represents a $K$-fixed point in $X(\psi^*(V))$, where $\psi^*(V)$ denotes the restricted representation. While $\psi^*(V)$ might not be contained in the chosen $K$-universe $\mathcal{U}_K$, there exists a $K$-embedding $\psi^*(V) \hookrightarrow \mathcal{U}_K$ which we can use to obtain an element $\psi^*([x])$ in $\pi_0^K(X)$. In [Sch15, Proposition I.6.5] it is shown that this element does not depend on the chosen embedding and that, furthermore, inner automorphisms act as the identity.

**Definition 2.1.3.** Let Rep denote the category of compact Lie groups and conjugacy classes of continuous group homomorphisms.

Then the restrictions defined above turn

$$\underline{\pi}_0(X) : G \mapsto \pi_0^G(X)$$

into a contravariant functor from Rep to the category of sets.

**Remark 2.1.4.** If $X$ is closed, $\pi_0^G(X)$ can be naturally identified with $\pi_0(X(\mathcal{U}_G)^G)$, as in this case $\pi_0$ commutes with the colimit.

Taking into account the equivariant evaluations of an orthogonal space leads to a notion of weak equivalence called *global equivalence*. It is easiest to state if the involved orthogonal spaces are closed:

**Definition 2.1.5** (cf. [Sch15, Proposition I.1.14])**.** A morphism $f : X \to Y$ of closed orthogonal spaces is called a *global equivalence* if the induced map

$$f(\mathcal{U}_G)^G : X(\mathcal{U}_G)^G \to Y(\mathcal{U}_G)^G$$

on $G$-fixed points is a weak equivalence of spaces for every compact Lie group $G$ .

The evaluation $X(\mathcal{U}_G)$ should be thought of as *the $G$-space underlying $X$*. In this sense a map of orthogonal spaces is a global equivalence if and only if it is a $G$-weak equivalence on underlying $G$-spaces for all compact Lie groups $G$. For general orthogonal spaces the colimit defining $X(\mathcal{U}_G)$ needs to be replaced by a homotopy colimit (cf. [Sch15, Proposition I.1.6]). In [Sch15, Section I.5] it is also shown that the class of global equivalences takes part in several model structures and hence the localized homotopy category can be dealt with by methods of homotopical algebra.

### 2.1.2 $\mathcal{F}in$-global version

Instead of taking into account information at all compact Lie groups, one can restrict to smaller global families of groups. In particular, one can work with respect to the family $\mathcal{F}in$ of finite groups.

**Definition 2.1.6.** A morphism $f : X \to Y$ of closed orthogonal spaces is a $\mathcal{F}in$-global equivalence if for all finite groups $G$ the map

$$f(\mathcal{U}_G)^G : X(\mathcal{U}_G)^G \to Y(\mathcal{U}_G)^G$$

is a weak equivalence of spaces.

Again, for non-closed orthogonal spaces one needs to work with the homotopy colimit. This notion of equivalence is also part of a model structure, as shown by Schwede in [Sch15, Section I.7].

## 2.2 Global universal and classifying spaces of collections

In this section we discuss a class of examples of orthogonal spaces, so-called *global universal* and *global classifying* spaces, associated to a collection of subgroups of a (not necessarily compact) Lie group $K$.

**Definition 2.2.1.** Let $K$ be a Lie group. A set of closed subgroups of $K$ is called a *collection* if it is closed under conjugation.

A *universal space* $E\mathcal{C}$ for a collection $\mathcal{C}$ is a cofibrant $K$-space with the property that all isotropy groups lie in $\mathcal{C}$ and for every subgroup $H$ in $\mathcal{C}$ the $H$-fixed points $E\mathcal{C}^H$ are contractible. Here and throughout this thesis we say that a $K$-space is *cofibrant* if it is the retract of a $K$-cell complex. Every collection possesses a universal space, which is unique up to $K$-homotopy equivalence. The quotient of such a universal space by the $K$-action is called a *classifying space of* $\mathcal{C}$ and denoted $B\mathcal{C}$.

Given a collection $\mathcal{C}$ of subgroups of a Lie group $K$ together with an additional Lie group $G$, the set of those closed subgroups of $K \times G$ whose intersection with $K \times 1$ lies in $\mathcal{C}$ also forms a collection, which we denote by $\mathcal{C}\langle G\rangle$.

**Example 2.2.2.** An important example is the collection $1_K$ which only contains the trivial subgroup of $K$. A subgroup of $K \times G$ lies in $1_K\langle G\rangle$ if and only if it is the graph $\Gamma(\psi)$ of a continuous group homomorphism $\psi$ from a closed subgroup of $G$ to $K$.

This gives rise to the following global notion:

**Definition 2.2.3.** Let $\mathcal{C}$ be a collection of subgroups of a Lie group $K$. A closed $K$-orthogonal space $X$ is called a *global universal space* for $\mathcal{C}$ if for every compact Lie group $G$ the $(K \times G)$-space $X(\mathcal{U}_G)$ is a universal space for the collection $\mathcal{C}\langle G\rangle$. The quotient of a global universal space by the $K$-action is called a *global classifying space of* $\mathcal{C}$.

A global universal space for $\mathcal{C}$ will be denoted $E_{gl}\mathcal{C}$, a global classifying space $B_{gl}\mathcal{C}$. The following example of a global classifying space is fundamental to global equivariant homotopy theory:

**Example 2.2.4.** A global universal space $E_{gl}1_K$ (respectively global classifying space $B_{gl}1_K$) associated to the collection $1_K$ is called a *global universal space* (respectively *global classifying space) of* $K$. We also use the notation $E_{gl}K$ (respectively $B_{gl}K$) for this global homotopy type. If $K$ is compact, a model for $E_{gl}K$ is given by the $K$-orthogonal space

$$W \mapsto L_{\mathbb{R}}(V, W) \tag{2.2.1}$$

for a fixed *faithful* $K$-representation $V$ (cf. [Sch15, Section I.2]).

The $\mathrm{Rep}^{op}$-functor $\underline{\pi}_0(B_{gl}K)$ is naturally isomorphic to the one which sends a compact Lie group $G$ to the set of conjugacy classes of continuous group homomorphisms from $G$ to $K$, with functoriality through precomposition. This is proved in [Sch14, Theorem 1.7] if $K$ is compact and follows from Corollary 2.3.2 below in the general case. Hence, it is representable if $K$ is compact.

Global classifying spaces of compact Lie groups are the fundamental building blocks of global homotopy theory. As shown in [Sch15, Proposition I.6.12], $B_{gl}K$ represents the functor $\pi_0^K(-)$ in the global homotopy category. In the model (2.2.1) above, a fundamental class is given by the class of the identity of $V$, as an element in $L_{\mathbb{R}}(V, V)^K$.

### 2.2.1 Existence of global universal and classifying spaces

We explain a construction of a global universal space for an arbitrary collection $\mathcal{C}$ of subgroups of a Lie group $K$, hence showing that these always exist.

We first recall a model for non-global universal spaces. For this we think of $\mathcal{C}$ as a topological poset under inclusion, with $K$-action via conjugation. The topology on the object and morphism sets of $\mathcal{C}$ is the finest one which makes this action continuous, i.e., the objects and morphisms are disjoint unions of $K$-orbits. Furthermore, given a subgroup $H$ of $K$, we denote by $H\backslash K$ its space of right cosets and by $|E(H\backslash K)|$ the geometric realization of the nerve of the topological category with object space $H\backslash K$ and exactly one morphism between any two elements. Then the assignment

$$H \mapsto |E(H\backslash K)| \tag{2.2.2}$$

defines a continuous functor from $\mathcal{C}$ to spaces. Every element $k \in K$ induces a conjugation map $|E(H\backslash K)| \to |E(kHk^{-1}\backslash K)|$ via $H\widetilde{k} \mapsto (kHk^{-1})k\widetilde{k} = kH\widetilde{k}$, which turns (2.2.2) into a $K$-functor (in the sense of [JS01] or [DM14]). This implies that the homotopy colimit (by which we mean the topological bar construction)

$$\mathcal{E}\mathcal{C} = \operatorname*{hocolim}_{H \in \mathcal{C}}(|E(H\backslash K)|)$$

inherits a $K$-action. The $K$-space $\mathcal{E}\mathcal{C}$ is a model for a universal space for $\mathcal{C}$.

**Remark 2.2.5.** This model is for example discussed in [AD01, Section 2], but it is presented slightly differently there. The $K$-space $\mathcal{E}\mathcal{C}$ can also be described as the geometric realization of the nerve of the category whose objects are pairs $(\widetilde{H} \in \mathcal{C}, k\widetilde{H} \in K/\widetilde{H})$ and whose morphisms are equivariant maps of orbits which preserve the chosen left cosets. It carries a $K$-action which fixes the first component $\widetilde{H}$ and permutes the cosets in the second component. This category is isomorphic to the category of pairs $(H \in \mathcal{C}, Hk \in H\backslash K)$ with exactly one morphism from $(H, Hk)$ to $(H', H'k')$ if $H$ is contained in $H'$, and no morphism otherwise. The isomorphism sends $(\widetilde{H}, k\widetilde{H})$ to $(k\widetilde{H}k^{-1}, (k\widetilde{H}k^{-1})k)$. This second description leads to the definition we gave above.

To obtain the global version, we now combine the construction of $\mathcal{E}\mathcal{C}$ with a functor called $b$ in [Sch15], which assigns to every space $X$ an orthogonal space $bX$ given by

$$bX(V) = \mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), X).$$

The assignment

$$H \mapsto b(|E(H\backslash K)|) = \mathrm{map}(L_{\mathbb{R}}(-, \mathbb{R}^\infty), |E(H\backslash K)|)$$

yields a $K$-functor from $\mathcal{C}$ to orthogonal spaces. We define $\mathcal{E}_{gl}\mathcal{C}$ as the homotopy colimit of this $K$-functor.

**Proposition 2.2.6.** *The $K$-orthogonal space $\mathcal{E}_{gl}\mathcal{C}$ is a global universal space for $\mathcal{C}$.*

In fact we show something stronger, which will also be useful later on: Each level $(\mathcal{E}_{gl}\mathcal{C})(V)$ is already a universal space for $\mathcal{C}\langle O(V)\rangle$.

*Proof.* We first check that the $K$-isotropy lies in $\mathcal{C}$. A point $x$ in $(\mathcal{E}_{gl}\mathcal{C})(V)$ is represented by a chain of subgroups $H_0 \leq \ldots \leq H_k$ in $\mathcal{C}$, a continuous map $L_{\mathbb{R}}(V, \mathbb{R}^\infty) \to |E(H_0\backslash K)|$ and simplex coordinates $(t_0, \ldots, t_k) \in \Delta^k$. We claim that the isotropy of $x$ is given by $H_0$. To see this, first note that any element of $K$ which fixes $x$ must in particular fix $H_0$ and hence lies in the normalizer of $H_0$. The normalizer of $H_0$ acts on $|E(H_0\backslash K)|$ by multiplication from the left. The isotropy of any point in $|E(H_0\backslash K)|$ under this action is $H_0$, and so it follows that the $N_K(H_0)$-isotropy of the map $L(V, \mathbb{R}^\infty) \to |E(H_0\backslash K)|$ is also $H_0$. So the isotropy of $x$ can be at most $H_0$. Since $H_0$ also fixes the chain $H_0 \leq \ldots \leq H_k$, we are done.

Now let $P \leq K \times O(V)$ be an element of $\mathcal{C}\langle O(V)\rangle$. We want to show that the fixed point space $(\mathcal{E}_{gl}\mathcal{C}(V))^P$ is weakly contractible, for which we need the following notation: Let $L \leq K$ be the intersection of $P$ with $K \times 1$, $L' \leq K$ the image of $P$ under the projection to $K$ and $J$ the image of $P$ under the projection to $O(V)$. For every $j \in J$ let $(\psi(j), j) \in P$ be a preimage under the projection. Then the element $\psi(j) \in K$ lies in $L'$ and is unique up to multiplication with $L$. Finally, we note that $L'$ is contained in the normalizer of $L$.

Now we determine the fixed points $(\mathcal{E}_{gl}\mathcal{C}(V))^P$. They are given by

$$\operatorname*{hocolim}_{H \in \mathcal{C}^P} (\mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), |E(H\backslash K)|))^{\mathrm{Iso}_P(H)}, \tag{2.2.3}$$

where $\mathrm{Iso}_P(H)$ denotes the $P$-isotropy of $H \in \mathcal{C}$. A subgroup $H \in \mathcal{C}$ is $P$-fixed if and only if $L'$ lies in the normalizer of $H$. Given such a subgroup $H$, the $L$-fixed points of $|E(H\backslash K)|$ are empty unless $L$ is contained in $H$, hence so are the fixed points for the larger group $\mathrm{Iso}_P(H)$. If $L$ is contained in $H$, it acts trivially on $|E(H\backslash K)|$. So we find that we can rewrite (2.2.3) as

$$\operatorname*{hocolim}_{\substack{L \leq H \in \mathcal{C} \\ L' \leq N_K(H)}} \left(\mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), |E(H\backslash K)|)\right)^J,$$

where $J$ acts on $|E(H\backslash K)|$ through its morphism to the Weyl group $W_K(H)$ given by $j \mapsto [\psi(j)]$. The $J$-space $L_{\mathbb{R}}(V, \mathbb{R}^\infty)$ is a model for the universal space $EJ$, hence $(\mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), |E(H\backslash K)|))^J$ computes the homotopy fixed points of $|E(H\backslash K)|$. Since $|E(H\backslash K)|$ is non-equivariantly contractible, these homotopy fixed points are weakly contractible. This shows that the fixed points $(\mathcal{E}_{gl}\mathcal{C}(V))^P$ are weakly equivalent to the nerve of the topological poset of those subgroups in $\mathcal{C}$ which contain $L$ and whose normalizer contains $L'$. This topological poset has $L$ as a minimal element and hence its nerve is contractible, which finishes the proof that $(\mathcal{E}_{gl}\mathcal{C})(V)$ is a universal space for $\mathcal{C}\langle O(V)\rangle$.

Hence, the restriction of $(\mathcal{E}_{gl}\mathcal{C})(V)$ along an injective homomorphism $G \hookrightarrow O(V)$ is a universal space for $\mathcal{C}\langle G\rangle$. So we find that in the colimit

$$\operatorname*{colim}_{\text{f.d.} V \subseteq \mathcal{U}_G} (\mathcal{E}_{gl}\mathcal{C})(V)$$

defining $(\mathcal{E}_{gl}\mathcal{C})(\mathcal{U}_G)$, all terms at faithful representations are universal spaces for $\mathcal{C}\langle G\rangle$. It follows that the colimit is also a universal space, which proves the claim. $\square$

**Remark 2.2.7.** Strictly speaking, we still have to show that the levels $(\mathcal{E}_{gl}\mathcal{C})(V)$ are sufficiently $(K \times O(V))$-cofibrant. In fact we are not sure whether this is the case, but it can be corrected easily: For every $K$-orthogonal space $X$ there is a replacement $X' \to X$ up to levelwise $(K \times O(V))$-weak equivalence, for which each evaluation $X'(V)$ is $(K \times O(V))$-cofibrant. For example, $X'$ can be taken to be a $K$-flat replacement of $X$, as is briefly discussed in the next section. It then follows that such a cofibrant replacement of $\mathcal{E}_{gl}\mathcal{C}$ is a global universal space for $\mathcal{C}$.

### 2.2.2 Relation to collection as a poset

There is a forgetful map from $\mathcal{E}_{gl}\mathcal{C}$ to the constant $K$-orthogonal space $|\mathcal{C}|$ which collapses each $b(|E(K/H)|)$ to a point. We now show that the $V$-th level of this map is a weak equivalence on all fixed points of graph subgroups of $K \times O(V)$.

Let $\Gamma(\psi)$ be the graph of a homomorphism $\psi : J \to K$, where $J$ is a closed subgroup of $O(V)$. Then the $\Gamma(\psi)$-fixed points of $\mathcal{E}_{gl}\mathcal{C}$ are given by

$$\operatorname*{hocolim}_{H \in \mathcal{C}^{\mathrm{im}(\psi)}} \left(\mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), |E(H\backslash K)|)\right)^J.$$

As we argued before in the proof of Proposition 2.2.6, the space

$$\left(\mathrm{map}(L_{\mathbb{R}}(V, \mathbb{R}^\infty), |E(H\backslash K)|)\right)^J$$

is weakly contractible and so the projection to the point induces a weak equivalence to

$$|\mathcal{C}|^{\Gamma(\psi)} = \operatorname*{hocolim}_{H \in \mathcal{C}^{\mathrm{im}(\psi)}} *.$$

This implies the following:

**Proposition 2.2.8.** *The forgetful map induces a global equivalence*

$$E_{gl}K \times_K \mathcal{E}_{gl}\mathcal{C} \simeq E_{gl}K \times_K |\mathcal{C}|.$$

Using the uniqueness result proved in the next section, it follows that such a global equivalence exists for any global universal space $E_{gl}\mathcal{C}$. The orthogonal space $E_{gl}K \times_K |\mathcal{C}|$ is called the *global homotopy orbit space of* $|\mathcal{C}|$, which is further discussed in Section 2.3.

### 2.2.3 Uniqueness

One can further show that any two global universal spaces $E_{gl}\mathcal{C}$ and $E'_{gl}\mathcal{C}$ are connected by a zig-zag of morphisms of $K$-orthogonal spaces

$$E_{gl}\mathcal{C} \to Z_1 \leftarrow Z_2 \to \ldots \leftarrow Z_n \to E'_{gl}\mathcal{C}, \tag{2.2.4}$$

where each $Z_i$ is also a global universal space for $\mathcal{C}$. Every map in (2.2.4) necessarily induces $(K \times G)$-homotopy equivalences when evaluated on complete $G$-universes $\mathcal{U}_G$. Hence, in particular, the global homotopy type of global classifying spaces is well-defined.

If the collection $\mathcal{C}$ is closed under intersection (for example, this holds when $\mathcal{C}$ is a family), this is easy to see: In this case the product of two global universal spaces is again a global universal space, and the two projections

$$E\mathcal{C} \leftarrow E\mathcal{C} \times E'\mathcal{C} \to E'\mathcal{C}$$

yield the zig-zag.

In general it is a bit more complicated, and we only give a sketch of the proof. First of all we recall from [Sch15, Section I.4] that for every orthogonal space $X$ there are functorially defined latching spaces $L_n(X)$ which

- only depend on the restriction of $X$ to subspaces of dimension smaller than $n$,
- carry an action of $O(n)$, and
- come with natural $O(n)$-maps $\nu_n(X) : L_n(X) \to X(\mathbb{R}^n)$.

Furthermore, $L_0(X)$ is the empty set. Whenever all maps $\nu_n$ are $O(n)$-cofibrations, $X$ is called *flat*. Similarly, we say that a $K$-orthogonal space is *$K$-flat* if these maps are even $(K \times O(n))$-cofibrations, where the $K$-action on $L_n(X)$ is through functoriality. Every $K$-orthogonal space can be replaced - up to levelwise weak $(K \times O(V))$-equivalence - by a $K$-flat one (which one can see similarly as in the proof of the factorization axiom in [Sch15, Proposition A.3.28]). A $K$-flat orthogonal space is in particular closed, so it follows that the $K$-flat replacement of any global universal space is again a global universal space. Hence, we can restrict ourselves to global universal spaces which are $K$-flat.

We claim that every $K$-flat universal space $E_{gl}\mathcal{C}$ allows a map $f$ to the model $\mathcal{E}_{gl}\mathcal{C}$ constructed in the previous section, finishing the proof of the existence of a zig-zag. For the construction of $f$, we can restrict ourselves to the skeleton of $L_{\mathbb{R}}$ given by the $\mathbb{R}^n$. The definition is then inductively: We let $n \in \mathbb{N}$ and assume $f$ to already be defined on $\mathbb{R}^m$ for all $m < n$. Since $L_n(-)$ only depends on these smaller-dimensional subspaces, one obtains a $(K \times O(n))$-map

$$L_n(E_{gl}\mathcal{C}) \xrightarrow{L_n(f)} L_n(\mathcal{E}_{gl}\mathcal{C})$$

and hence by composition with $\nu_n(\mathcal{E}_{gl}\mathcal{C})$ a map

$$L_n(E_{gl}\mathcal{C}) \to \mathcal{E}_{gl}\mathcal{C}(\mathbb{R}^n).$$

The latching map $\nu_n(E_{gl}\mathcal{C}) : L_n(E_{gl}\mathcal{C}) \to E_{gl}\mathcal{C}(\mathbb{R}^n)$ is a $(K \times O(n))$-cofibration whose relative $K$-isotropy necessarily lies in $\mathcal{C}$, since $E_{gl}\mathcal{C}(\mathbb{R}^n)$ embeds into $E_{gl}\mathcal{C}(\mathcal{U}_G)$ which we assumed to have isotropy in $\mathcal{C}$. Furthermore, by the results of Section 2.2.1 we know that $\mathcal{E}_{gl}\mathcal{C}(\mathbb{R}^n)$ is a universal space for $\mathcal{C}\langle O(n)\rangle$, and so there exists a lift $f_n$ in the diagram

$$\begin{tikzcd} L_n(E_{gl}\mathcal{C}) \arrow[r] \arrow[d, tail, "\nu_n(E_{gl}\mathcal{C})"'] & \mathcal{E}_{gl}\mathcal{C}(\mathbb{R}^n) \\ E_{gl}\mathcal{C}(\mathbb{R}^n) \arrow[ur, dashed, "f_n"'] & \end{tikzcd}$$

and we have extended the map to dimension $n$. This finishes the sketch of the proof.

## 2.3 Global homotopy orbits

Let $K$ be a Lie group, $X$ a $K$-space. There is an associated orthogonal space $E_{gl}K \times_K X$, the *global homotopy orbits* of $X$, defined via

$$(E_{gl}K \times_K X)(V) = (E_{gl}K)(V) \times_K X.$$

This gives rise to a large class of examples. To understand the underlying $G$-spaces of global homotopy orbits, we need the following lemma:

**Lemma 2.3.1.** *Let $K$ and $G$ be Lie groups of which $G$ is compact, and $Y$ be a cofibrant $(K \times G)$-space such that the $K$-action is free. Then there is a natural homeomorphism*

$$(Y/K)^G \cong \coprod_{\langle \alpha: G \to K\rangle} Y^{\Gamma(\alpha)}/C(\alpha)$$

*where $\alpha$ ranges through a set of representatives of conjugacy classes of continuous group homomorphisms from $G$ to $K$, $C(\alpha)$ denotes the centralizer of the image of $\alpha$ and $\Gamma(\alpha) \subseteq K \times G$ is the graph of $\alpha$.*

*Proof.* The statement is the same as [Sch15, Proposition A.1.28], where the Lie group $K$ is also required to be compact. This is not necessary, as one only needs the space of continuous group homomorphisms from $G$ to $K$ modulo conjugation to be discrete to see that the topology on the union of the $Y^{\Gamma(\alpha)}/C(\alpha)$ is indeed that of a disjoint union. For this it suffices that the source $G$ is compact ([CF64, Lemma 38.1]). $\square$

Applying this to $Y = X(\mathcal{U}_G)$ we see:

**Corollary 2.3.2.** *For $K$ a Lie group, $X$ a cofibrant $K$-space and $G$ a compact Lie group there is a natural homeomorphism*

$$(E_{gl}K \times_K X)(\mathcal{U}_G)^G \cong \coprod_{\langle \alpha: G \to K\rangle} EC(\alpha) \times_{C(\alpha)} X^{\mathrm{im}(\alpha)}.$$

This shows that the global homotopy orbits depend on the fixed points $X^H$ for all compact subgroups $H$ of $K$, or more precisely on the functor on the orbit category of $K$ with compact isotropy that is associated to $X$. This stands in contrast to the underlying space of $E_{gl}K \times_K X$, the usual homotopy orbits, which only depend on $X$ up to 'naive' $K$-equivalence.

**Remark 2.3.3.** The unstable global homotopy category is equivalent to the homotopy category of stacks, in the sense introduced in [GH08] (the equivalence follows from the results in [GH08, Section 4.4] together with [Sch15, Theorem 8.37]). In this language, $E_{gl}K \times_K X$ corresponds to the quotient stack $X /\!\!/ K$.

There is also a pointed version of global homotopy orbits, defined as $E_{gl}K_+ \wedge_K X$.

## 2.4 I-spaces

If one restricts to finite groups, there is a more combinatorial model for unstable global homotopy theory, called **I**-spaces. It is needed in Chapter 13 to describe the subquotients in the rank and complexity filtration of global algebraic $K$-theory. Like orthogonal spaces, **I**-spaces were originally considered as a symmetric monoidal model for the usual homotopy theory of spaces. They have the convenient property that commutative monoids correspond to $E_\infty$-spaces (see, for example, [SS12], [SS13] and [Lin13]). In [Sch15, Section I.7], Schwede describes a global equivariant point of view on **I**-spaces, which we quickly recall.

### 2.4.1 Definition and global equivalences

Let **I** denote the category of finite sets and injective maps.

**Definition 2.4.1.** An **I**-*space* is a functor from **I** to the category of spaces.

Equivariance enters in a similar way as for orthogonal spaces, with $G$-representations replaced by $G$-sets. Let $A$ be an **I**-space. By functoriality, if a finite set $M$ comes equipped with an action of a finite group $G$, the evaluation $A(M)$ becomes a $G$-space. Every injection of $G$-sets $M \hookrightarrow N$ induces a $G$-equivariant map $A(M) \to A(N)$. There is a level model structure on $\mathbf{I}$-spaces where the weak equivalences and fibrations are those morphisms that become $G$-weak equivalences respectively $G$-fibrations on evaluations $-(M)$ for all finite groups $G$ and finite $G$-sets $M$ (cf. [Sch15, Proposition I.7.17]).

An $\mathbf{I}$-space $A$ is called *static* if for every injection $M \hookrightarrow N$ of faithful finite $G$-sets the induced map $A(M)^G \to A(N)^G$ is a weak equivalence. A morphism of $\mathbf{I}$-spaces is a *global equivalence* if it induces bijections on all hom-sets into static $\mathbf{I}$-spaces in the level homotopy category. Together with the cofibrations of the level model structure, these form the global model structure for $\mathbf{I}$-spaces introduced in [Sch15, Theorem I.7.19].

For a static $\mathbf{I}$-space $A$, the evaluation $A(M)$ at a faithful finite $G$-set $M$ should be thought of as the *$G$-space underlying $A$*. By the definition of static, its $G$-homotopy type does not depend on the choice of $M$. The $G$-space underlying an arbitrary $\mathbf{I}$-space $A$ is not as easy to describe directly, but it can be defined by first replacing by a globally equivalent static $\mathbf{I}$-space $QA$ and then taking the underlying $G$-space of $QA$. This is the main technical difference to orthogonal spaces, where the underlying $G$-space can always be obtained by forming a (homotopy) colimit along the finite subrepresentations of a complete $G$-universe, cf. Section 2.1.

### 2.4.2 Quillen equivalence to $\mathcal{F}in$-global orthogonal spaces

There is a natural embedding $i : \mathbf{I} \hookrightarrow L_{\mathbb{R}}$ which sends a finite set to its $\mathbb{R}$-linearization together with the inner product for which the canonical basis is orthonormal. This gives rise to a forgetful functor $U$ from orthogonal spaces to $\mathbf{I}$-spaces with left adjoint $L$ given by left Kan extension along $i$.

**Proposition 2.4.2** ([Sch15, Theorem I.7.26 and Proposition I.7.25])**.** *The adjunction*

$$L : \mathbf{I} - spaces \rightleftarrows orth.\ spaces : U$$

*is a Quillen equivalence between the global model structure on $\mathbf{I}$-spaces and the $\mathcal{F}in$-global model structure on orthogonal spaces.*

*Moreover, a morphism $f$ of orthogonal spaces is a $\mathcal{F}in$-global equivalence if and only if $U(f)$ is a global equivalence of $\mathbf{I}$-spaces.*

Using this Quillen equivalence we can transport the global homotopy types introduced in Sections 2.2 and 2.3 to $\mathbf{I}$-spaces: An $\mathbf{I}$-space is a global classifying space for a collection $\mathcal{C}$ if it is globally equivalent to $U(B_{gl}\mathcal{C})$, where $B_{gl}\mathcal{C}$ is a global classifying space for $\mathcal{C}$ in orthogonal spaces.

**Example 2.4.3** (Global classifying spaces for finite groups)**.** Let $G$ be a finite group and $M$ a finite $G$-set. This data gives rise to an $\mathbf{I}$-space $\mathbf{I}(M,-)/G$ whose evaluation on a finite set $N$ is the set of injective maps from $M$ to $N$, modulo the $G$-action by pre-composition. Giving a morphism from $\mathbf{I}(M,-)/G$ to an $\mathbf{I}$-space $A$ is equivalent to picking a $G$-fixed point in the evaluation $A(M)$. By adjunction, this shows that $L$ maps $\mathbf{I}(M,-)/G$ to the orthogonal space $L_{\mathbb{R}}(\mathbb{R}[M],-)/G$ of Example 2.2.4. Since $\mathbf{I}(M,-)/G$ is also cofibrant in the model structure mentioned above, this means that it is a global classifying space for $G$.

This model for $B_{gl}G$ is an example of an $\mathbf{I}$-space for which it is difficult to read off the underlying $H$-spaces directly. For example for $H$ the trivial group, the underlying $H$-space is the usual $BG$. On the other hand, each evaluation $\mathbf{I}(M, N)/G$ is a discrete finite set.

# Chapter 3

# Stable global homotopy theory

In this chapter we give the relevant definitions and results for global homotopy theory based on orthogonal spectra (the stable version of orthogonal spaces) and symmetric spectra (the stable version of **I**-spaces). We also introduce the global spectra which are studied in Parts II, III and IV. Symmetric products of spheres and global topological $K$-theory are modeled on orthogonal spectra, while the model for global algebraic $K$-theory only forms a symmetric spectrum.

## 3.1 Orthogonal spectra, global functors and global equivalences

We quickly give the relevant definitions of orthogonal spectra from the perspective of global equivariance, for details we refer to [Sch15, Chapter III].

### 3.1.1 Definition

**Definition 3.1.1.** An orthogonal spectrum consists of

- a collection of based spaces $X(V)$ for every real inner product space $V$
- based structure maps $\varphi_* : X(V) \wedge S^{W-\varphi(V)} \to S^W$ for every linear isometric embedding $\varphi : V \hookrightarrow W$.

Here, $W - \varphi(V)$ denotes the orthogonal complement of $\varphi(V)$ in $W$. The structure maps have to be unital, associative and 'vary continuously in $\varphi$'. The latter is made precise in the following way: We let $V$ and $W$ be two real inner product spaces and recall that $L_{\mathbb{R}}(V, W)$ denotes the spaces of linear isometric embeddings between them. Then the subspace

$$\{(\varphi, w) \mid w \in (W - \varphi(V))\} \subseteq L_{\mathbb{R}}(V, W) \times W$$

defines a vector bundle over $L_{\mathbb{R}}(V, W)$. Let $Th(V, W)$ denote the Thom space of this bundle. As a set, $Th(V, W)$ is the based union of all $S^{W-\varphi(V)}$, hence the structure maps assemble to a based map

$$X(V) \wedge Th(V, W) \to X(W)$$

and the requirement is that this map be continuous.

A morphism $f : X \to Y$ of orthogonal spectra is a sequence of based maps $f(V) : X(V) \to Y(V)$ which commute with the structure maps.

We note that, in particular, each level $X(V)$ of an orthogonal spectrum carries a based action of $O(V)$.

**Example 3.1.2** (Suspension spectra)**.** Any based orthogonal space $X$ gives rise to an orthogonal spectrum $\Sigma^\infty X$, its *suspension spectrum*, via

$$(\Sigma^\infty X)(V) = X(V) \wedge S^V,$$

where the structure maps are the smash product of the maps $X(\varphi) : X(V) \to X(W)$ with the homeomorphisms $S^V \wedge S^{W-\varphi(V)} \cong S^W$ sending $(v \wedge w)$ to $(\varphi(v) + w)$.

For an unbased orthogonal space $X$ we denote by $\Sigma^\infty_+ X$ the suspension spectrum of the based orthogonal space $X_+$. In particular, $\Sigma^\infty_+ *$ gives the sphere spectrum $\mathbb{S}$.

### 3.1.2 Global homotopy groups and global equivalences

If $V$ comes equipped with an action of a compact Lie group $G$, the evaluation $X(V)$ also inherits a $G$-action through the homomorphism $G \to O(V)$, just like for orthogonal spaces. Furthermore, for any other $G$-representation $W$ and $G$-equivariant linear isometric embedding $\varphi : V \hookrightarrow W$, the structure map

$$\varphi_* : X(V) \wedge S^{W-\varphi(V)} \to X(W)$$

is $G$-equivariant. Fixing a complete $G$-universe $\mathcal{U}_G$ for every compact Lie group $G$, one defines the equivariant homotopy groups of an orthogonal spectrum $X$ via

$$\pi_k^G(X) = \operatorname*{colim}_{f.d.V \subseteq \mathcal{U}_G} [S^{k+V}, X(V)]_*^G$$

where the connecting maps in the colimit system are induced by the equivariant structure maps. For $k < 0$, the colimit system is indexed over all finite dimensional $G$-subrepresentations $V \subseteq \mathcal{U}_G$ which contain a chosen trivial copy of $\mathbb{R}^{-k} \subseteq (\mathcal{U}_G)^G$, in which case the expression $S^{k+V}$ means $S^{V-\mathbb{R}^{-k}}$. Since the space of linear isometric embeddings $\mathbb{R}^{-k} \hookrightarrow (\mathcal{U}_G)^G$ is connected (even contractible), the group $\pi_k^G(X)$ is independent of this choice up to canonical isomorphism.

**Remark 3.1.3.** If $G$ is finite, $\pi_k^G(X)$ can be defined more directly as the colimit

$$\operatorname*{colim}_{n\in\mathbb{N}} [S^{k+n\cdot\rho_G}, X(n \cdot \rho_G)]^G$$

over the values at multiples of the regular representation $\rho_G$ of $G$.

It is an important feature of the global equivariant homotopy theory of orthogonal spectra that for each fixed $k \in \mathbb{N}$ the collection of homotopy groups

$$\underline{\pi}_k(X) = \{\pi_k^G(X)\}_{G \text{ compact Lie group}}$$

has a rich natural functoriality, it forms a so-called *global functor*. Concretely this means that there are

- contravariantly functorial *restriction maps* $\psi^* : \pi_k^G(X) \to \pi_k^K(X)$ for every continuous group homomorphism $\psi : K \to G$, and
- covariantly functorial *transfer maps* $\mathrm{tr}_H^G : \pi_k^H(X) \to \pi_k^G(X)$ for every closed subgroup inclusion $H \leq G$.

The restrictions are defined in the same manner as for orthogonal spaces in Section 2.1. If $\psi$ is the inclusion of a closed subgroup $H$ of $G$, we also use the notation $\mathrm{res}_H^G$ instead of $\psi^*$.

We quickly recall the construction of the transfer in the case where $H$ is of finite index in $G$, as we need it explicitly in Section 14.2:

**Example 3.1.4** (Finite index transfers)**.** Let $X$ be an orthogonal spectrum and $x$ an element of $\pi_0^H(X)$ (the construction for other degrees is similar). Then $x$ is represented by an $H$-map $f : S^V \to X(V)$ for some $H$-representation $V$, which we can without loss of generality assume to be the restriction of a $G$-representation that also allows an embedding of the $G$-set $G/H$. This embedding can be extended to a $G$-equivariant embedding $G/H \times D(V) \hookrightarrow V$ (where $D(V)$ denotes the unit disc in $V$). Collapsing everything outside the interiors of the discs to a point (the 'Thom-Pontryagin construction') gives a $G$-map $S^V \to G/H_+ \wedge S^V$, from which one obtains a representative for the transfer $\mathrm{tr}_H^G(x)$ of $x$ by postcomposing with the map $G/H_+ \wedge S^V \to X(V)$ sending a tuple $([g] \wedge v)$ to $gf(g^{-1}v)$.

Restrictions and transfers satisfy the double coset formula (cf. [Sch15, III.4.15]). Furthermore, the restriction along an inner automorphism of a compact Lie group is always the identity, and transfers along subgroup inclusions with infinite Weyl group are trivial.

For every orthogonal space $X$, the group $\pi_0^G(\Sigma_+^\infty X)$ can be expressed in terms of the $\mathrm{Rep}^{op}$-functor $\underline{\pi}_0(X)$. Every element $[x]$ of $\pi_0^G(X)$ is represented by a point $x \in X(V)^G$ for some $G$-representation $V$ contained in the chosen $G$-universe $\mathcal{U}_G$ and gives rise to an element (also denoted $[x]$) in $\pi_0^G(\Sigma_+^\infty X)$ represented by the $G$-map

$$\begin{array}{ccc} S^V & \to & X(V)_+ \wedge S^V \\ v & \mapsto & x \wedge v. \end{array}$$

This construction commutes with restrictions along group homomorphisms. Based on the tom Dieck-splitting, Schwede shows:

**Proposition 3.1.5** ([Sch15, Proposition III.4.8])**.** *Let $G$ be a compact Lie group and $X$ an orthogonal space. Then the 0-th homotopy group $\pi_0^G(\Sigma_+^\infty X)$ is free with basis $\{\mathrm{tr}_H^G([x])\}$, where $H$ ranges through conjugacy classes of subgroups of $G$ with finite Weyl group $W_G(H)$ and $[x]$ ranges through a set of representatives of $W_G(H)$-orbits of $\pi_0^H(X)$.*

**Example 3.1.6.** Applying this to $X$ a global classifying space $B_{gl}K$ (cf. Example 2.2.4) we see that $\pi_0^G(\Sigma_+^\infty(B_{gl}K))$ has a basis $\{\mathrm{tr}_H^G([\alpha])\}$, where $(H, \alpha : H \to K)$ ranges through $(G \times K)$-conjugacy classes of pairs of a closed subgroup $H$ of $G$ together with a continuous group homomorphism to $K$. If $K$ is compact, we call $\underline{\pi}_0(\Sigma^\infty_+(B_{gl}K))$ the *free global functor in degree $K$*. Giving a morphism of global functors

$$\underline{\pi}_0(\Sigma^\infty_+(B_{gl}K)) \to F$$

to some global functor $F$ is equivalent to specifying an element in $F(K)$.

Equivariant homotopy groups also give rise to the notion of global equivalence:

**Definition 3.1.7.** A morphism $f : X \to Y$ of orthogonal spectra is a *global equivalence* if it induces an isomorphism on $\pi_k^G$ for all integers $k$ and every compact Lie group $G$.

The localization of the category of orthogonal spectra at the class of global equivalences gives rise to the *global stable homotopy category.* There is a model structure on orthogonal spectra with weak equivalences the global equivalences, and so this localization can be studied via homotopical algebra (cf. [Sch15, Section IV.1]).

### 3.1.3 $\mathcal{F}in$-global version

Like for orthogonal spaces, there is also a global theory of orthogonal spectra which only takes finite groups into account. In this case, the collection

$$\{\pi_k^G(X)\}_{G \text{ finite group}}$$

forms a $\mathcal{F}in$-global functor, i.e., the restriction of the structure of a global functor to the class of finite groups. $\mathcal{F}in$-global functors can be interpreted as functors on a $\mathcal{F}in$-global Burnside category defined via $(G, K)$-bisets, cf. [Sch15, Remark III.3.28] and have previously been considered in an algebraic setup by Webb [Web93], where they are called 'inflation functors'.

A map of orthogonal spectra is called a *$\mathcal{F}in$-global equivalence* if it induces isomorphisms on $\pi_*^G$ for all finite groups $G$. The localization of orthogonal spectra at $\mathcal{F}in$-global equivalences yields the $\mathcal{F}in$-global homotopy category. Again, there are model structures available (cf. [Sch15, Section IV.4]).

### 3.1.4 Comparison to $G$-spectra

We quickly recall how the global theory relates to stable equivariant homotopy theory for a fixed compact Lie group $G$.

Every orthogonal spectrum $X$ gives rise to a $G$-orthogonal spectrum $X_G$ indexed on a complete universe, in the sense of Mandell-May [MM02], by equipping it with the trivial action. Morally this means that one forgets all evaluations at representations of groups other than $G$. This is not strictly true, since these can be reconstructed from the $O(V)$-actions, as we argued before. The cleaner statement is that these other evaluations no longer play a role for the homotopy theory and hence do not carry over to the homotopy category.

Given a closed subgroup $H$ of $G$, the homotopy groups $\pi_*^H(X_G)$ of the $G$-orthogonal spectrum agree with the global homotopy groups $\pi_*^H(X)$ of $X$ itself. Since $G$-stable equivalences of $G$-orthogonal spectra are defined as those morphisms which induce isomorphisms on all homotopy groups $\pi_*^H$, it follows that the functor $X \mapsto X_G$ takes global equivalences to $G$-stable equivalences. Hence, every global homotopy type determines a $G$-homotopy type for all compact Lie groups $G$.

This means in particular that every global homotopy type determines a cohomology theory on based $G$-spaces. Using the global homotopy orbits construction of Section 2.3, this cohomology theory can be read off more directly: There are natural isomorphisms

$$[\Sigma^\infty((E_{gl}G)_+ \wedge_G A), X]_*^{\mathrm{gl}} \cong [\Sigma^\infty A, X_G]_*^G \cong X_G^{-*}(A)$$

for all based $G$-spaces $A$. In words, $X$ determines a cohomology theory on based orthogonal spaces, and applying it to the global homotopy orbits of $G$-spaces yields the $G$-cohomology theory associated to the underlying $G$-spectrum $X_G$.

One consequence of this factorization is that the $G$-cohomology theories that arise from global spectra have more structure and special properties: Every map of based orthogonal spaces

$$(E_{gl}G)_+ \wedge_G A \to (E_{gl}G)_+ \wedge_G B$$

induces a map $X_G^*(B) \to X_G^*(A)$. For example, on coefficients this yields the existence of restriction maps for arbitrary group homomorphisms, while general $G$-cohomology theories only allow restrictions for subgroup inclusions and conjugation maps.

## 3.2 Global symmetric products of spheres

In this section we recall the definition and some properties of symmetric products of the sphere spectrum, a class of orthogonal spectra that is central to this thesis. We also recall Schwede's result on the 0-th equivariant homotopy groups of symmetric products.

### 3.2.1 Definition and the Dold-Thom theorem

Let $X$ be a topological space and $n$ a natural number. The $n$-th symmetric product $Sp^n(X)$ of $X$ is defined as $X^{\times n}/\Sigma_n$, the $n$-fold cartesian product modulo the $\Sigma_n$-action which permutes the coordinates. In other words, an element of $Sp^n(X)$ is an unordered $n$-tuple of elements in $X$. If $X$ is based, there are comparison maps $Sp^{n-1}(X) \to Sp^n(X)$ obtained by adding a basepoint to every tuple. This allows one to form the colimit over all $n$, the infinite symmetric product $Sp^\infty(X)$. By a famous theorem of Dold-Thom [DT58], the construction $Sp^\infty(-)$ turns homology into homotopy, i.e., there are natural isomorphisms

$$\pi_*(Sp^\infty(X), *) \cong \widetilde{H}_*(X, \mathbb{Z}) \tag{3.2.1}$$

for every connected CW complex $X$. Moreover, under this isomorphism the map $X \to Sp^\infty(X)$ induces the Hurewicz map on homotopy groups. So one can think of the sequence

$$X = Sp^1(X) \to Sp^2(X) \to Sp^3(X) \to \ldots \to Sp^\infty(X)$$

as a natural filtration of the Hurewicz map.

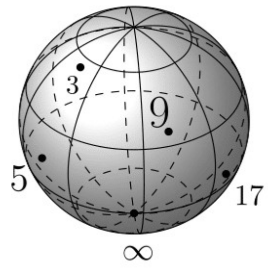



Figure 3.1: An element of $Sp^{34}(S^2)$

**Remark 3.2.1.** Elements of the infinite symmetric product $Sp^\infty(X)$ of a space $X$ can be visualized as configurations in $X$ with labels in the natural numbers, cf. Figure 3.1. The label on a point indicates how often it occurs in the tuple. Since this is not well-defined for the basepoint, its label is set to $\infty$. The topology on this labeled configuration space is such that if two points collide, their corresponding labels are added. Under this identification, a point lies in the subspace $Sp^n(X)$ if and only if the sum of the non-basepoint labels is at most $n$.

### 3.2.2 The orthogonal spectra $Sp^n$

This construction has a stable version: Let $Sp^n$ denote the orthogonal spectrum with $V$-th level $Sp^n(S^V)$, the $n$-th symmetric product of the $V$-sphere, and with structure maps

$$\begin{array}{rcl} Sp^n(S^V) \wedge S^{W-\varphi(V)} & \xrightarrow{\varphi_*} & Sp^n(S^W) \\ [v_1, \ldots, v_n] \wedge w & \mapsto & [\varphi(v_1) \wedge w, \ldots, \varphi(v_n) \wedge w]. \end{array}$$

Inserting a basepoint yields the orthogonal spectrum level symmetric product filtration

$$Sp^1 \xrightarrow{i_1} Sp^2 \xrightarrow{i_2} \ldots \to Sp^\infty.$$

For $n = 1$ and $n = \infty$, the spectra $Sp^n$ are known under different names. The spectrum $Sp^1$ is isomorphic to the sphere spectrum $\mathbb{S}$. In Proposition 3.1.5 we saw that $\pi_0^G(\mathbb{S})$ is a free abelian group with basis $\{\mathrm{tr}_H^G(1)\}$, where $H$ ranges through conjugacy classes of subgroups $H$ of $G$ with finite Weyl group. This abelian group is also known as the *Burnside ring* of $G$ and sometimes denoted $A(G)$. If $G$ is finite, $A(G)$ has a simpler description as the group completion of the commutative monoid of isomorphism classes of finite $G$-sets with disjoint union. In particular, it is a free abelian group with basis the isomorphism classes of transitive $G$-sets $G/H$. The isomorphism to $\pi_0^G(\mathbb{S})$ is given by sending $G/H$ to $\mathrm{tr}_H^G(1)$.

By the Dold-Thom theorem, $Sp^\infty(S^k)$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}, k)$. The naturality of the isomorphism (3.2.1) further implies that the structure maps

$$\pi_k(Sp^\infty(S^k)) \to \pi_{k+1}(Sp^\infty(S^{k+1}))$$

are isomorphisms, and so $Sp^\infty$ is non-equivariantly a model for the Eilenberg-MacLane spectrum $H\mathbb{Z}$. In [dS03] it is shown that something similar is true for finite $G$: The homotopy groups $\pi_*^G(Sp^\infty)$ are a copy of $\mathbb{Z}$ concentrated in degree 0. Moreover, all restriction maps

$$\psi^* : \pi_0^K(Sp^\infty) \to \pi_0^G(Sp^\infty)$$

associated to group homomorphisms $\psi : G \to K$ are isomorphisms, and the transfer maps

$$\mathrm{tr}_H^G : \pi_0^H(Sp^\infty) \to \pi_0^G(Sp^\infty)$$

are given by multiplication with the index $[G : H]$. In other words, on finite groups, $\pi_0(Sp^\infty)$ is the constant $\mathcal{F}in$-global functor with value $\mathbb{Z}$ and $Sp^\infty$ is an Eilenberg-MacLane spectrum for this $\mathcal{F}in$-global functor. This is no longer true for non-discrete compact Lie groups $G$. Neither is $\pi_0^G(Sp^\infty)$ always isomorphic to $\mathbb{Z}$, nor are the higher homotopy groups $\pi_k^G(Sp^\infty)$ always trivial (cf. [Sch15, Construction V.3.21]).

### 3.2.3 Schwede's computation of $\pi_0^G(Sp^n)$

For finite $G$, the map

$$A(G) \cong \pi_0^G(\mathbb{S}) \to \pi_0^G(Sp^\infty) \cong \mathbb{Z}$$

is the augmentation which sends a finite $G$-set to its number of elements. Applying $\pi_0^G(-)$ to the symmetric products gives rise to a filtration

$$\pi_0^G(\mathbb{S}) \to \pi_0^G(Sp^2) \to \pi_0^G(Sp^3) \to \ldots \to \pi_0^G(Sp^\infty)$$

of this augmentation. In [Sch14], Schwede has given the following algebraic description of this filtration: Let $\tau_n^\Sigma \in A(\Sigma_n)$ denote the (isomorphism class of the) $\Sigma_n$-set $\Sigma_n/\Sigma_{n-1}$.

**Theorem 3.2.2** ([Sch15, Theorem 3.13])**.** *The maps*

$$A(-) \cong \underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(Sp^n)$$

*are surjective for all $n \in \mathbb{N}$. As a global functor, the kernel is generated by the single element*

$$\tau_n^\Sigma - n \cdot 1 \in A(\Sigma_n).$$

*In particular, there are isomorphisms of global functors*

$$\underline{\pi}_0(Sp^n) \cong A(-)/(\tau_n^\Sigma - n \cdot 1).$$

The $\Sigma_n$-set $\tau_n^\Sigma$ is isomorphic to $\underline{n} = \{1, \ldots, n\}$ with its tautological $\Sigma_n$-action. It is the universal $n$-element $G$-set, in the sense that for every $n$-element $G$-set $X$ there exists a homomorphism $\psi : G \to \Sigma_n$, unique up to conjugacy, such that $X \cong \psi^*(\tau_n^\Sigma)$. In particular, restricting $(\tau_n^\Sigma - n \cdot 1)$ along this group homomorphism gives the relation $X - n \cdot 1 \in A(G)$. So we see that all $n$-element $G$-sets become identified with $n \cdot 1$ in $\pi_0^G(Sp^n)$. However, this is not all that happens, because in order to determine $\pi_0^G(Sp^n)$ concretely one also needs to apply transfers to such relations for subgroups of $G$. It becomes an algebraic computation in the Burnside ring global functor - more and more complex as the order of $G$ increases - and the result is often not very enlightening. This demonstrates the philosophy behind global homotopy theory: When viewed in the full context, various objects (spectra or their homotopy groups) enjoy natural universal properties which they lose after evaluation at a single group.

In Part II we compute the higher homotopy groups $\pi_*^G(Sp^n)$ after tensoring with $\mathbb{Q}$. In Part IV we are concerned with finding similar global formulas for the effect of the rank and complexity filtrations of global $K$-theory on $\underline{\pi}_0$.

## 3.3 Global topological $K$-theory

We describe a model for global connective complex $K$-theory, as introduced in [Sch15, Section V.6].

For this we recall that a $\Gamma$-space (in the sense of Segal [Seg74]) is a functor from the category of finite based sets to spaces. We write $\underline{n}_+$ for the set $\{1,\ldots,n\}$ with an added basepoint. For simplicity, we further assume that our $\Gamma$-spaces take $\underline{0}_+$ to a point. Every $\Gamma$-space $X$ can be evaluated on based spaces $A$ via the formula

$$\begin{aligned} X(A) &= \int^{\underline{n}_+} X(\underline{n}_+)\times A^{\times n} \\ &= \operatorname{coeq}\left(\coprod_{\underline{n_1}_+,\underline{n_2}_+} \hom(\underline{n_1}_+,\underline{n_2}_+)\times X(\underline{n_1}_+)\times A^{\times n_2} \rightrightarrows \coprod_{\underline{n}_+} X(\underline{n}_+)\times A^{\times n}\right) \end{aligned}$$

with basepoint $X(\underline{0}_+)\times A^{\times 0}\cong *$. Here, $\hom(-,-)$ denotes the morphism sets in the category of finite based sets. Given another based space $B$, there are assembly maps

$$X(A)\wedge B\to X(A\wedge B)$$

which send a tuple $(x,a_1,\ldots,a_n)\wedge b$ to $(x,a_1\wedge b,\ldots,a_n\wedge b)$.

In particular, the evaluations $X(S^V)$ of a $\Gamma$-space $X$ on all spheres form an orthogonal spectrum denoted by $X(\mathbb{S})$, with structure maps

$$\varphi_*: X(S^V)\wedge S^{W-\varphi(V)}\to X(S^V\wedge S^{W-\varphi(V)})\cong X(S^W).$$

The orthogonal spectrum $X(\mathbb{S})$ is called the *spectrum realization of* $X$. The functor $-(\mathbb{S})$ is the left adjoint of a Quillen equivalence between $\Gamma$-spaces and (non-equivariant) connective spectra, i.e., spectra with homotopy groups concentrated in non-negative degrees, cf. [BF78].

Now we come to the model for topological $K$-theory. For every complex inner product space $W$ and a finite based set $A_+$ we define $ku(W,A_+)$ to be the space of tuples $(W_a)_{a\in A}$ of finite dimensional pairwise orthogonal subspaces of $W$ indexed on $A$, or in other words the space

$$\coprod_{(n_a\in\mathbb{N})_{a\in A}}\left(L_{\mathbb{C}}(\bigoplus_{a\in A}\mathbb{C}^{n_a},W)/\prod_{a\in A}U(n_a)\right).$$

Every based map $\alpha : A_+ \to B_+$ induces a map $ku(W, \alpha) : ku(W, A_+) \to ku(W, B_+)$ via

$$ku(W, \alpha)((W_a)_{a \in A}) = (\bigoplus_{a \in \alpha^{-1}(b)} W_a)_{b \in B}.$$

This turns $ku(W, -)$ into a $\Gamma$-space.

For $W = \mathbb{C}^\infty$, the realization $ku(\mathbb{C}^\infty, \mathbb{S})$ is a non-equivariant model for connective topological $K$-theory. However, it does not have the right equivariant homotopy type. In principle, to obtain a model for connective $G$-equivariant $K$-theory, one should use $W = \mathcal{U}_G^{\mathbb{C}}$, a complete complex $G$-universe. However, the resulting orthogonal spectrum has a non-trivial $G$-action and hence does not represent a global homotopy type.

Schwede showed that this issue can be resolved by letting $W$ vary functorially in the levels $V$ of an orthogonal spectrum. For this we let $\mathrm{Sym}(W)$ denote the symmetric algebra on $W$, equipped with an inner product as explained in [Sch15, Proposition V.6.7]. Further, let $V_{\mathbb{C}}$ denote the complexification of a real inner product space $V$. Then the assignment

$$V \mapsto ku(\mathrm{Sym}(V_{\mathbb{C}}), -)$$

defines a functor from $L_{\mathbb{R}}$ to the category of $\Gamma$-spaces, or in other words an orthogonal $\Gamma$-space. From this orthogonal $\Gamma$-space one forms the orthogonal spectrum $ku$ via

$$ku(V) := ku(\mathrm{Sym}(V_{\mathbb{C}}), S^V). \tag{3.3.1}$$

The structure maps are the composites

$$ku(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \wedge S^{W - \varphi(V)} \to ku(\mathrm{Sym}(V_{\mathbb{C}}), S^W) \to ku(\mathrm{Sym}(W_{\mathbb{C}}), S^W),$$

where the first map is the structure map for the spectrum associated to the $\Gamma$-space $ku(\mathrm{Sym}(V_{\mathbb{C}}), -)$, and the second one is induced from the embedding

$$\mathrm{Sym}(\varphi_{\mathbb{C}}) : \mathrm{Sym}(V_{\mathbb{C}}) \hookrightarrow \mathrm{Sym}(W_{\mathbb{C}}).$$

**Remark 3.3.1.** In fact it would suffice to take $ku(V_{\mathbb{C}}, S^V)$ in Definition (3.3.1), in the sense that it would yield the same global homotopy type. Schwede uses the symmetric algebra because it turns $ku$ into a commutative orthogonal ring spectrum (from the point of view of global homotopy theory: an ultracommutative ring spectrum). We do not make use of the highly structured multiplication in this thesis. The reason for using $\mathrm{Sym}(-)$ nonetheless is that for global algebraic $K$-theory it will help us to identify the global homotopy type of a certain associated **I**-space more easily, and we prefer to use uniform definitions for the topological and algebraic case.

**Remark 3.3.2.** The levels $ku(V)$ can be visualized in a similar manner as symmetric products of spheres, cf. Figure 3.2. Every point in $ku(V)$ is represented by a finite configuration $[(W_1, x_1), \dots, (W_k, x_k)]$, where the $x_i$ are points in $S^V$ and the $W_i$ are pairwise orthogonal complex subspaces of $\mathrm{Sym}(V_{\mathbb{C}})$. This representation becomes unique up to a permutation of labels if one requires all the $x_i$ to be distinct elements of $V$ and all the $W_i$ to be non-zero. Again, the topology is such that if two points collide their labels are added up, and all labeled points which move to the basepoint vanish.

In [Sch15, Construction V.6.11], Schwede shows that for finite groups $G$, the orthogonal spectrum $ku$ models the connective cover of global equivariant $K$-theory. In particular, the 0-th $K$-group $ku^0_G(A)$ of a based $G$-space $A$ is given by isomorphism classes of stable equivariant complex vector bundles on $A$, in the sense of Segal [Seg77]. This means that $\pi_0^G(ku)$ is isomorphic to the complex representation ring $RU(G)$. The isomorphism can be made explicit as follows: Let $W$ be a finite dimensional complex $G$-representation. We choose a real $G$-representation $V$ together with a $G$-embedding $\varphi : W \hookrightarrow \mathrm{Sym}(V_{\mathbb{C}})$. Then we obtain a $G$-map

$$S^V \to ku(V)$$

by sending $v$ to the tuple $(\varphi(W), v)$ and let $[W] \in \pi_0^G(ku)$ denote its associated stable class. A different way to view the assignment $W \mapsto [W]$ is given by the following: Let $L(\mathbb{C}^n)$ denote the orthogonal space $L_{\mathbb{C}}(\mathbb{C}^n, \mathrm{Sym}(-_{\mathbb{C}}))$. Then there are maps

$$\alpha_n : \Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \to ku^n$$

which in level $V$ send a pair $(\varphi, v)$ to the configuration $(\varphi(\mathbb{C}^n), v)$. The orthogonal space $L(\mathbb{C}^n)$ is a model for a global universal space $E_{gl}U(n)$ of $U(n)$, which follows from the fact that the space of equivariant linear isometric embeddings of any $G$-representation into a complete complex $G$-universe is contractible (cf. Example 2.2.4). Hence there are isomorphisms

$$\pi_0^G(L(\mathbb{C}^n)/U(n)) \cong \mathrm{Rep}(G, U(n)) \cong \{\text{isom. classes of } n\text{-dim. } G\text{-representations}\}.$$

The assignment $W \mapsto [W]$ is then the composite

$$\pi_0^G(L(\mathbb{C}^n)/U(n)) \to \pi_0^G(\Sigma^\infty_+(L(\mathbb{C}^n)/U(n))) \xrightarrow{(\alpha_n)_*} \pi_0^G(ku).$$

In particular, this shows that $[W]$ only depends on the isomorphism type of $W$ and not on the choice of $V$ and $\varphi$.

The following properties are shown in [Sch15, Corollary V.6.9], and also follow from the results of Chapter 14:

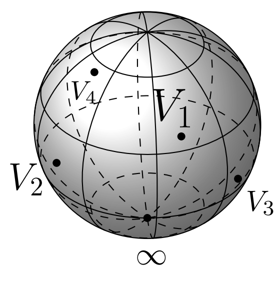



Figure 3.2: An element of $ku(V)$

- The map $W \mapsto [W]$ is additive and induces a homomorphism

$$[-] : RU(G) \to \pi_0^G(ku).$$

- For finite $G$, $[-]$ is an isomorphism.
- $[-]$ takes restrictions of representations to homotopy theoretic restrictions.
- For a finite index subgroup $H$ of $G$ and an $H$-representation $W$, $[-]$ takes the induced representation $\mathrm{Ind}_H^G(W)$ to the homotopy theoretic transfer $\mathrm{tr}_H^G([W])$.

That is, on finite groups, $[-]$ induces an isomorphism from the representation ring global functor $RU(-)$ to $\underline{\pi}_0(ku)$.

The map $[-] : RU(G) \to \pi_0^G(ku)$ above also makes sense for non-discrete compact Lie groups, but in general it fails to be an isomorphism. In particular, this shows that $ku$ does not represent a connective cover of equivariant $K$-theory for non-discrete groups. One problem is that $[-]$ does not take Segal's smooth transfers for infinite index inclusions ([Seg68b]) to homotopy-theoretic transfers, which are added on freely in $\underline{\pi}_0(ku)$. Ultimately, these issues are related to the problem that there is no clean theory of equivariant $\Gamma$-spaces over non-discrete compact Lie groups. Because the rank filtration and also our proofs in Part III rely on point-set level models arising through $\Gamma$-spaces, we work with this version of $ku$ despite these flaws. Interestingly though, it turns out that the description of the filtration quotients in the rank and complexity filtration as well as the global formulas for the behavior on $\underline{\pi}_0$ work for all compact Lie groups, and the universal classes even lie at non-discrete groups. We say a few more words about this in Remark 14.1.4.

There is a natural morphism from $ku$ to the infinite symmetric product $Sp^\infty$, obtained by sending a complex subspace of $\mathrm{Sym}(V_\mathbb{C})$ to its dimension. This morphism is used in the definition of the rank and complexity filtrations in Part III.

### 3.3.1 Global real $K$-theory

There is also a real version of connective global $K$-theory, called $ko$. It is given by

$$ko(V) = ko(\mathrm{Sym}(V), S^V),$$

where $ko(\mathrm{Sym}(V), -)$ is the $\Gamma$-space parametrizing orthogonal real subspaces of $\mathrm{Sym}(V)$, analogously to the complex case. Again, $ko$ represents connective real equivariant $K$-theory on finite groups, but differs in general for non-discrete compact Lie groups.

## 3.4 Global $K$-theory of finite sets

We recall a model for the $K$-theory of finite sets $k\mathcal{F}in$.

For a finite pointed set $A_+$ and a real inner product space $W$ we let $k\mathcal{F}in(W, A_+)$ denote the space of tuples $(M_a)_{a\in A}$ of pairwise orthogonal finite orthonormal systems

$M_a$ of vectors of $W$, or in other words the space

$$\coprod_{(n_a \in \mathbb{N})_{a \in A}} \left( L_{\mathbb{R}}(\bigoplus_{a \in A} \mathbb{R}^{n_a}, W) / \prod_{a \in A} \Sigma_{n_a} \right).$$

For a fixed $W$, the spaces $k\mathcal{F}in(W, A_+)$ carry a $\Gamma$-space structure via

$$k\mathcal{F}in(W, \alpha)((M_a)_{a \in A}) = (\coprod_{a \in \alpha^{-1}(b)} M_a)_{b \in B}.$$

Again we let $W$ vary in order to implement equivariance and obtain an orthogonal $\Gamma$-space

$$(V, A_+) \mapsto k\mathcal{F}in(\mathrm{Sym}(V), A_+).$$

We write $k\mathcal{F}in$ for the realization of this orthogonal $\Gamma$-space.

There is a unit map $\mathbb{S} \to k\mathcal{F}in$ given by the 1-tuple with value $1 \in \mathbb{R} \cong \mathrm{Sym}(0)$. In Chapter 12 we use the modified rank filtration to give a short proof of the global version of the Barratt-Priddy-Quillen theorem: The morphism $\mathbb{S} \to k\mathcal{F}in$ is a global equivalence.

Finally, we note that there is also a morphism $k\mathcal{F}in \to Sp^\infty$ which sends a finite orthonormal system $M_a$ to its number of elements.

## 3.5 Global homotopy theory of symmetric spectra

In this section we quickly recall the definition of another model for global homotopy theory, the category of symmetric spectra. Symmetric spectra are the stable analog of **I**-spaces (cf. Section 2.4) and similarly only model equivariant homotopy types over finite groups. Symmetric spectra were originally introduced in [HSS00] as a symmetric monoidal model for the non-equivariant stable homotopy category. Their global homotopy theory is developed in [Hau15], to which we refer for more details.

Symmetric spectra are a more combinatorial version of orthogonal spectra, with orthogonal groups replaced by symmetric groups.

**Definition 3.5.1** (Symmetric spectrum)**.** A symmetric spectrum is a collection of based spaces $X(M)$ for every finite set $M$, together with associative and unital structure maps

$$\varphi_* : X(M) \wedge S^{N - \varphi(M)} \to X(N)$$

for injections $\varphi : M \hookrightarrow N$.

In this case there is no condition on 'varying continuously in $\varphi$', since the set of injective maps between two finite sets is discrete.

**Example 3.5.2.** Every based **I**-space $A$ gives rise to its suspension symmetric spectrum $\Sigma^\infty A$ defined via $(\Sigma^\infty A)(M) = A(M) \wedge S^M$.

Like for **I**-spaces, the evaluation of a symmetric spectrum $X$ on a finite $G$-set $M$ carries an induced $G$-action through functoriality. For the definition of homotopy groups, we say that a countable $G$-set is a *complete $G$-set universe* if it allows an embedding of every finite $G$-set. Unlike for orthogonal spectra, it turns out that the definition of the homotopy groups depends on the choice of a complete $G$-set universe $\mathcal{U}_G$, so it is part of the notation.

**Definition 3.5.3.** Let $n \in \mathbb{Z}$ be an integer and $\mathcal{U}_G$ a complete $G$-set universe. Then the $n$-th $G$-equivariant homotopy group $\pi_n^{G,\mathcal{U}_G}(X)$ of a symmetric spectrum of spaces $X$ is defined as the colimit

$$\pi_k^{G,\mathcal{U}_G}(X) := \operatorname*{colim}_{\text{finite } M \subseteq \mathcal{U}_G} [S^{k \sqcup M}, X(M)]^G.$$

For negative $k$, one makes the same adjustments as for orthogonal spectra, cf. Section 3.1.

The dependance on the choice of universe is as follows: Every $G$-isomorphism $\mathcal{U}_G \xrightarrow{\cong} \mathcal{U}'_G$ gives rise to a natural isomorphism $\pi_*^{G,\mathcal{U}_G} \cong \pi_*^{G,\mathcal{U}'_G}$, and two different isomorphisms of universes will in general give rise to two different isomorphisms of homotopy groups. This stands in contrast to the situation for orthogonal spectra. It means that any two notions of homotopy group are isomorphic, but not canonically so, and hence it is misleading to leave $\mathcal{U}_G$ out of the notation.

In particular, the homotopy group functor $\pi_*^{G,\mathcal{U}_G}$ has a natural action of the $G$-automorphisms of $\mathcal{U}_G$ (in fact, this action extends to an action of the monoid of $G$-self injections of $\mathcal{U}_G$, cf. [Hau15, Section 4]). Since no such action exists on the homotopy groups of orthogonal spectra, this is an indication that the homotopy groups of symmetric spectra defined in the way above are 'not the right thing'. This phenomenon is already present non-equivariantly (cf. [HSS00], [Sch08]). Globally it has another consequence: The $\mathrm{Aut}(\mathcal{U}_G)$-actions interact non-trivially with restrictions and transfers. In fact it turns out that the collection of equivariant homotopy groups of a symmetric spectrum does in general not form a $\mathcal{F}in$-global functor but inherits a more complicated structure (cf. [Hau15, Section 4]).

In summary, global equivalences need to be defined by other means. This is done similarly as for **I**-spaces by considering maps into global $\Omega$-spectra (the stable analog of static **I**-spaces) in a suitable level homotopy category. We do not need the general definition of global equivalence of symmetric spectra in this thesis, and refer to [Hau15, Section 2.2].

The reason why we do not need the general notion is the following: There is a class of symmetric spectra - called *globally semistable* - whose global homotopy theory behaves like that of orthogonal spectra. We first note that every orthogonal spectrum $Y$ has an underlying symmetric spectrum $U(Y)$ defined via $U(Y)(M) = Y(\mathbb{R}[M])$. The linearization $\mathbb{R}[\mathcal{U}_G]$ is a complete $G$-universe (in the sense of representations) and so it is not hard to see that there is a natural isomorphism

$$\pi_*^{G,\mathcal{U}_G}(U(Y)) \cong \pi_*^G(Y).$$

In particular, the homotopy groups of symmetric spectra which underlie an orthogonal spectrum do not depend on the $G$-set universe up to canonical isomorphism.

**Definition 3.5.4.** A symmetric spectrum is called *globally semistable* if there exists a zig-zag of global $\underline{\pi}_*$-isomorphisms to $U(Y)$ for some orthogonal spectrum $Y$.

Various other characterizations of global semistability are given in [Hau15, Proposition 4.13]. There it is also shown that globally semistable symmetric spectra have the following properties:

- A morphism between globally semistable symmetric spectra is a global equivalence if and only if it induces an isomorphism on equivariant homotopy groups.
- For every globally semistable $X$, the homotopy groups $\pi_*^{G,\mathcal{U}_G}(X)$ are independent of $\mathcal{U}_G$ up to canonical isomorphism and will hence be denoted $\pi_*^G(X)$.
- The equivariant homotopy groups of a globally semistable symmetric spectrum naturally form a $\mathcal{F}in$-global functor, which for restrictions of orthogonal spectra agrees with the one described in Section 3.1.

All symmetric spectra we encounter in this thesis are globally semistable, so we can treat them like orthogonal spectra.

Finally, we note:

**Theorem 3.5.5** ([Hau15, Theorems 2.18 and 5.3])**.** *The global equivalences are part of a model structure on the category of symmetric spectra, which is Quillen equivalent to orthogonal spectra with the $\mathcal{F}in$-global model structure of [Sch15].*

More precisely, the functor $U$ is the right adjoint of a Quillen equivalence.

## 3.6 Global algebraic $K$-theory

We recall the construction of the free global algebraic $K$-theory of a ring $R$, as introduced in [Sch16], which only forms a symmetric spectrum. More precisely, we describe a slight modification of Schwede's construction, as explained in [Hau15, Section 6.3].

Let $W$ be a free $R$-module and $A_+$ a finite pointed set. We define $kR(W, A_+)$ to be the nerve of the following category: Objects are tuples of the form $(W_a)_{a\in A}$ where the $W_a$ are finite rank free $R$-submodules of $W$ such that their sum $\sum_{a\in A} W_a$ in $W$ is direct and splits off from $W$ as a direct summand (but no such splitting is part of the data). Morphisms are tuples of abstract $R$-module isomorphisms, again indexed by $A$. This category can also be described as the quotient category

$$\coprod_{(n_a)_{a\in A}} \left( E(\mathrm{Emb}_R(\bigoplus_{a\in A} R^{n_a}, W)) / \prod_{a\in A} GL_{n_a}(R) \right),$$

where $E(-)$ of a set is the category with objects the set and exactly one morphism between any two objects and $\mathrm{Emb}(-,-)$ is the set of splittable $R$-module monomorphisms between two $R$-modules.

For fixed $W$, the assignment $A_+ \mapsto kR(W, A_+)$ possesses the structure of a $\Gamma$-space by forming the inner direct sum of objects and morphisms, similarly to the topological case.

For $W = R^\infty$, the realization $kR(R^\infty, \mathbb{S})$ - which even forms an orthogonal spectrum - is non-equivariantly a model for the free algebraic $K$-theory of $R$. To implement equivariance we again need to vary the free $R$-module functorially, and this forces us to use the more combinatorial category of symmetric spectra.

We obtain an $\mathbf{I}$-$\Gamma$-space via

$$(M, A_+) \mapsto kR(\mathrm{Sym}(R[M]), A_+)$$

where $\mathrm{Sym}(R[M]) = \mathbb{R} \otimes \mathrm{Sym}(\mathbb{Z}[M])$ denotes the polynomial ring with commuting variable set $M$. We note that this is well-defined even if $R$ is not commutative, even though the symmetric algebra on a general $R$-module does not make sense. The global algebraic $K$-theory spectrum $kR$ of $R$ is the realization of this $\mathbf{I}$-$\Gamma$-space, i.e.,

$$kR(M) = kR(\mathrm{Sym}(R[M]), S^M)$$

with diagonal $\Sigma_M$-action. The structure maps are defined analogously to those for the realization of an orthogonal $\Gamma$-space.

**Remark 3.6.1.** The levels $kR(M)$ can be interpreted similarly to the topological case. Each $kR(M)$ is the geometric realization of a simplicial space whose 0-simplices are labeled configurations $[(W_1, x_1), \ldots, (W_k, x_k)]$ of the following kind:

- The $x_i$ are points in the sphere $S^M$.
- The $W_i$ are finitely generated free submodules of the polynomial ring $\mathrm{Sym}(R[M])$ with variable set $M$, such that their sum is direct and the inclusion $W_1 \oplus \ldots \oplus W_k \hookrightarrow \mathrm{Sym}(R[M])$ allows an $R$-linear splitting.

These configurations are considered up to the equivalence relation that a labeled point $(W_i, x_i)$ can be left out if either $W_i$ is zero or $x_i$ the basepoint, and that if two $x_i$ are equal they can be replaced by a single one with label the sum of the previous labels. In other words, the 0-simplices are the direct algebraic analog of the spaces $ku(V)$. The difference is that while complex subspaces can vary continuously, there is in general no topology on the set of free submodules of an $R$-module which has the right properties. Instead one uses higher simplices to implement isomorphisms of free modules.

In [Sch16], Schwede shows the following:

- The symmetric spectrum $kR$ is globally semistable.
- Its $G$-fixed point spectrum (cf. [Sch16, Section 6]) represents the direct sum $K$-theory of $R[G]$-*lattices*, i.e., $R[G]$-modules that are finitely-generated free as $R$-modules. In particular, the equivariant homotopy groups $\pi_*^G(kR)$ are the $K$-groups of $R[G]$-lattices.

The second item implies that $\pi_0^G(kR)$ is isomorphic to the representation ring $\mathrm{Rep}_R(G)$, i.e., the group completion of the monoid of isomorphism classes of $R[G]$-lattices. Again we will need an explicit description of how to assign elements in $\pi_0^G(kR)$ to $R[G]$-lattices, which we now recall.

**Lemma 3.6.2.** *Let $M$ be a faithful non-empty $G$-set and $W$ an $R[G]$-lattice. Then there exists a $G$-embedding*

$$W \hookrightarrow \mathrm{Sym}(R[M]),$$

*which splits non-equivariantly.*

*Proof.* Since the canonical map from induction to coinduction is an isomorphism, we obtain a $G$-map

$$W \xrightarrow{\mathrm{coev}} \mathrm{map}(G, \mathrm{res}^G_{\{e\}} W) \xleftarrow{\cong} \bigoplus_G \mathrm{res}^G_{\{e\}} W$$

which is $R$-linearly (though not $R[G]$-linearly) split by the projection onto the component of the neutral element of $G$. In particular, $W$ allows a $G$-equivariant embedding into the permutation representation $\bigoplus_G \mathrm{res}^G_{\{e\}} W$. Hence it suffices to show that this permutation representation in turn sits inside $\mathrm{Sym}(R[M])$ as a direct summand. But this follows from the observation that any monomial $\prod_{m \in M} m^{i_m}$ with all $i_m$ pairwise different spans a free $G$-subset, since $M$ is faithful. This finishes the proof, since we assumed that $M$ is non-empty. $\square$

Hence, every $R[G]$-lattice $W$ gives rise to an element $[W] \in \pi_0^G(kR)$, the class of

$$\begin{array}{rcl} S^M & \to & kR(M) \\ x & \mapsto & (\varphi(W), x) \end{array}$$

for some choice of faithful finite $G$-set $M$ and splittable embedding $\varphi : W \hookrightarrow \mathrm{Sym}(R[M])$.

That this assignment does not depend on the choices follows similarly as for topological $K$-theory: Let $\mathbf{I}(R^n)$ be the $GL_n(R)$-$\mathbf{I}$-space

$$|E(\mathbf{I}(R^n, \mathrm{Sym}(R[-])))|.$$

The quotient $\mathbf{I}(R^n)/GL_n(R)$ is the $\mathbf{I}$-space which assigns to every finite set $M$ the nerve of the category of splittable rank $n$ free submodules of $\mathrm{Sym}(R[M])$ and isomorphisms between such. So one obtains a morphism

$$\alpha_n : \Sigma^\infty_+(\mathbf{I}(R^n)/GL_n(R)) \to kR$$

by sending $x \in S^M$ and $W \subseteq \mathrm{Sym}(R[M])$ to the configuration $(W, x)$.

**Lemma 3.6.3.** *Let $M$ be a faithful non-empty $G$-set. Then $\mathbf{I}(R^n)(M)$ is a universal space for the family of graph subgroups of $GL_n(R) \times G$. Moreover, the quotient $\mathbf{I}(R^n)/GL_n(R)$ is a global classifying space for $GL_n(R)$.*

*Proof.* Since $\mathbf{I}(R^n)(M)$ carries a free $GL_n(R)$-action, it follows that all fixed-points for non-graph subgroups are empty. Now let $\Gamma(\psi) \leq GL_n(R) \times G$ be a graph subgroup for a homomorphism $\psi : H \to GL_n(R)$. There is a homeomorphism

$$\mathbf{I}(R^n)(M)^{\Gamma(\psi)} = |E(\mathbf{I}(R^n, \mathrm{Sym}(R[-])))|^{\Gamma(\psi)} \cong |E(\mathbf{I}(R^n, \mathrm{Sym}(R[-]))^{\Gamma(\psi)})|,$$

so it suffices to show that the fixed points $\mathbf{I}(R^n, \mathrm{Sym}(R[-]))^{\Gamma(\psi)}$ are non-empty. Let $W$ denote the $R[H]$-lattice obtained by restricting the $R[GL_n(R)]$-lattice $R^n$ along $\psi$. By Lemma 3.6.2 above, if $M$ is non-empty, there exists a splittable $H$-embedding $W \hookrightarrow \mathrm{Sym}(R[M])$, and hence a fixed point for $\Gamma(\psi)$, proving that $\mathbf{I}(R^n)(M)$ is a universal space.

This implies that there is a zig-zag of positive levelwise weak $(GL_n(R) \times \Sigma_M)$-equivalences

$$U(\mathcal{E}GL_n(R)) \leftarrow U(\mathcal{E}GL_n(R)) \times \mathbf{I}(R^n) \rightarrow \mathbf{I}(R^n),$$

where $\mathcal{E}GL_n(R)$ is the orthogonal space model for the global universal space of $GL_n(R)$ introduced in Section 2.2. So it follows that on quotients we obtain a zig-zag of positive strong level equivalences of $\mathbf{I}$-spaces, and hence $\mathbf{I}(R^n)/GL_n(R)$ is globally equivalent to $U((\mathcal{E}GL_n(R))/GL_n(R))$, which is a global classifying space for $GL_n(R)$. $\square$

Hence we can again view the assignment $[-]$ as the composites

$$\pi_0^G(\mathbf{I}(R^n)/GL_n(R)) \rightarrow \pi_0^G(\Sigma_+^\infty(\mathbf{I}(R^n)/GL_n(R))) \xrightarrow{(\alpha_n)_*} \pi_0^G(kR).$$

If $R$ satisfies dimension invariance, i.e., if $R^n \cong R^m$ implies $n = m$, there is a natural map to the underlying symmetric spectrum of the infinite symmetric product $Sp^\infty$ given by mapping every free $R$-module to its rank.

# Part II

# Symmetric products and subgroup lattices

# Chapter 4

# Rational symmetric products of $G$-spheres

In this part we are concerned with the rational properties of the $\mathcal{F}in$-global symmetric products $Sp^n$, in particular a computation of their rational homotopy groups. We recall what we already discussed in Section 3.2. The first term $Sp^1$ is the global sphere spectrum $\mathbb{S}$ with 0-th homotopy group $\underline{\pi}_0(\mathbb{S})$ the Burnside ring $\mathcal{F}in$-global functor. By the tom Dieck splitting (cf. [tD75, Satz 2]) and the Adams isomorphism (cf. [Ada84, Theorem 5.4]), the higher equivariant homotopy groups decompose as

$$\pi_*^G(\mathbb{S}) \cong \bigoplus_{(H \leq G)} \pi_*(\Sigma_+^\infty BW_G(H)),$$

where the sum is taken over representatives of conjugacy classes of subgroups $H$ of $G$. Since all Weyl groups $W_G(H)$ are finite, it follows that the higher homotopy groups are rationally trivial. The last term $Sp^\infty$ is an Eilenberg-MacLane spectrum for the constant global functor $\underline{\mathbb{Z}}$, so its equivariant homotopy groups are constant $\mathbb{Z}$ in degree 0 and trivial in other degrees. Hence, their rationalization is a constant $\mathbb{Q}$ concentrated in degree 0.

As we recalled in Section 3.2.3, Schwede gave a formula for the 0-th homotopy groups $\underline{\pi}_0(Sp^n)$ and hence in particular for their rationalization. This leaves the question whether the rational homotopy of the $Sp^n$ is always concentrated in degree 0, as it is the case for $Sp^1$ and $Sp^\infty$. It turns out that the answer is a strong 'no': As $G$ and $n$ vary there are arbitrarily high non-trivial rational $\pi_k^G(Sp^n)$. Moreover, these groups are closely connected to the topology of the subgroup lattice of $G$.

In the next sections we give the statement of the theorem, relate it to Schwede's result and then discuss the $\mathcal{F}in$-global functoriality. Throughout Part II we will only be talking about finite groups, so for the sake of brevity we will write 'global' to mean '$\mathcal{F}in$-global' and always assume that $G$ is finite.

## 4.1 Subgroup lattices and their filtration

Let $L(G)$ denote the (nerve of the) subgroup lattice of $G$, and $L(G)_n \subseteq L(G)$ the sub-simplicial set of chains $H_0 \leq \ldots \leq H_k$ of total index $[H_k : H_0]$ at most $n$. The group $G$ acts on each $L(G)_n$ via conjugation. Furthermore, we let $H_*(-, \mathbb{Q})$ denote the ordinary singular homology groups of a space with coefficients in $\mathbb{Q}$. Then the main theorem of this part says:

**Theorem 4.1.1.** *For every $n \in \mathbb{N}$ there are isomorphisms*

$$\pi_*^G(Sp^n) \otimes \mathbb{Q} \cong H_*(L(G)_n, \mathbb{Q})/G.$$

We observe that this result matches the previously known values $\pi_*^G(\mathbb{S}) \otimes \mathbb{Q}$ and $\pi_*^G(Sp^\infty) \otimes \mathbb{Q}$, since $L(G)_1$ is the discrete set of subgroups of $G$ and $L(G)_\infty = L(G)$ is contractible for the lattice has a minimal and a maximal element.

## 4.2 Motivation and relation to Schwede's result

We give some motivation for the appearance of the subgroup lattice in Theorem 4.1.1. For this we recall from Section 3.2.3 that Schwede [Sch14] gave the following description of the 0-th global homotopy group $\underline{\pi}_0(Sp^n)$, where $\tau_n^\Sigma \in A(\Sigma_n)$ is the tautological $n$-element $\Sigma_n$-set:

(i) The map $A(-) \cong \underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(Sp^n)$ is surjective.

(ii) The kernel is generated as a global functor by the class $(\tau_n^\Sigma - n \cdot 1) \in A(\Sigma_n)$.

This result is purely about 0-th homotopy groups, but it can also be used as a starting point to construct elements in higher homotopy. We consider the example $G = \Sigma_3$, whose subgroup lattice modulo conjugation is depicted in Figure 4.1.

In $\pi_0^{\Sigma_3}(Sp^3)$, the classes $[\Sigma_3/\{e\}]$ and $6 \cdot 1$ become equal for two reasons: Their difference can on the one hand be written as

$$\mathrm{tr}_{\Sigma_2}^{\Sigma_3}(\tau_2 - 2 \cdot 1) + 2 \cdot (\tau_3 - 3 \cdot 1)$$

and on the other hand as

$$\mathrm{tr}_{A_3}^{\Sigma_3}(\mathrm{res}_{A_3}^{\Sigma_3}(\tau_3 - 3 \cdot 1)) + 3 \cdot \mathrm{sgn}^*(\tau_2 - 2 \cdot 1),$$

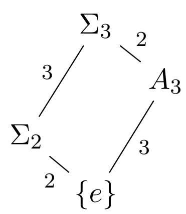



Figure 4.1: The subgroup lattice of $\Sigma_3$ modulo conjugation

where sgn : $\Sigma_3 \to \Sigma_3/A_3 \cong \Sigma_2$ denotes the sign homomorphism. These two reasons correspond to the two paths in the subgroup lattice from $\{e\}$ to $\Sigma_3$ and use that all of the edges in these paths are associated to subgroup inclusions of index at most 3. Moreover, Schwede's proof yields explicit null-homotopies of $(\tau_2 - 2 \cdot 1)$ and $(\tau_3 - 3 \cdot 1)$ in $Sp^3$, and so one obtains two explicit homotopies between $[\Sigma_3/\{e\}]$ and $6 \cdot 1$. These can be glued together to form a loop and hence give an element in $\pi_1^{\Sigma_3}(Sp^3)$. If we pass on to stage 6 of the filtration, there is another reason that $[\Sigma_3/\{e\}]$ and $6 \cdot 1$ become equal: Their difference is the restriction of $(\tau_6 - 6 \cdot 1)$ under the homomorphism $\Sigma_3 \to \Sigma_6$ corresponding to the free transitive $\Sigma_3$-set. We will see that this direct homotopy is itself homotopic to the two homotopies above in an explicit way and hence our constructed element becomes trivial in $\pi_1^{\Sigma_3}(Sp^6)$. In groups that allow longer chains of subgroups, one can now start gluing these explicit homotopies of homotopies together to obtain maps from the 2-sphere, and so on.

Theorem 4.1.1 can be loosely interpreted as saying that the above construction yields all elements of $\pi_*^G(Sp^n)$ up to torsion, and that the non-triviality of these classes (again up to torsion) can also be decided purely in terms of the subgroup lattice of $G$.

## 4.3 Global functor structure

As written in the introduction, we will prove a more highly structured version of Theorem 4.1.1. We start by describing the global functor structure on the side of lattices.

Since we aim to show that $H_*(L(G)_n, \mathbb{Q})/G$ is isomorphic to $\pi_*^G(Sp^n) \otimes \mathbb{Q}$ for every finite group $G$, the assignment

$$G \mapsto H_*(L(G)_n, \mathbb{Q})/G$$

must have the functoriality of a graded global functor. In fact, this structure already exists on rational chains, i.e., the assignment

$$G \mapsto \mathcal{C}_*(L(G)_n)/G$$

extends to a chain complex of rational global functors, where $\mathcal{C}_*$ denotes the reduced rational chain complex of a simplicial set. The transfers and restrictions work out as follows:

**Transfers**: The transfer along a subgroup inclusion $H \leq G$ sends a chain of subgroups of $H$ to $[G : H]$ times the same chain thought of as subgroups of $G$.

**Restriction maps**: The restriction along a group homomorphism $\psi : G \to K$ is a bit more complicated, it takes the class of a chain $H_0 \leq \ldots \leq H_m$ to

$$\sum_{[k] \in G_\psi \backslash K/H_0} \left( \frac{[G : \psi^{-1}(kH_0k^{-1})]}{[K : H_0]} \cdot [\psi^{-1}(kH_0k^{-1}) \leq \ldots \leq \psi^{-1}(kH_mk^{-1})] \right). \quad (4.3.1)$$

The sum is taken over a set of coset representatives of the $(G \times H_0^{op})$-action on $K$ given by $(g, h) \cdot k = \psi(g)kh$.

**Remark 4.3.1.** To explain the factor $[G:H]$ in the definition of the transfer and the fractions $\frac{[G:\varphi^{-1}(kH_0k^{-1})]}{[K:H_0]}$ in that of the restrictions, we consider the case $n=1$, where $L(G)_1$ is the discrete set of subgroups of $G$ and $\pi_0^G(Sp^1)\otimes\mathbb{Q}$ is the rationalized Burnside ring of $G$, a vector space with basis the isomorphism classes of transitive $G$-sets $G/H$. It is clear that $H_0(L(G)_1,\mathbb{Q})/G$ and $\pi_0^G(Sp^1)\otimes\mathbb{Q}$ are abstractly isomorphic, and one isomorphism would be given by simply mapping $H\in L(G)_1$ to $[G/H]\in\pi_0^G(Sp^1)\otimes\mathbb{Q}$. However, this isomorphism cannot be compatible with any choice of isomorphism

$$H_0(L(G)_\infty,\mathbb{Q})/G\cong\pi_0^G(Sp^\infty)\otimes\mathbb{Q},$$

which are both isomorphic to $\mathbb{Q}$. Any two subgroups of $G$ represent the same element in $H_0(L(G)_\infty,\mathbb{Q})$, since the lattice is connected. On the other hand, the augmentation $\pi_0^G(Sp^1)\to\pi_0^G(Sp^\infty)$ sends a finite $G$-set to its number of elements, so the orbits $[G/H]$ generally have different images. This can be corrected by sending $H\in L(G)_1$ to $\frac{1}{[G:H]}\cdot[G/H]$ - which has augmentation 1 - instead. So, in the case $n=1$, the formulas above express transfers and restrictions in the rationalized Burnside ring global functor written in terms of the basis $\{\frac{1}{[G:H]}\cdot[G/H]\}$ instead of the usual basis $\{[G/H]\}$, which leads to the appearance of the factors and fractions.

We claim that these transfers and restrictions turn $\mathbb{Q}[L(-)_n]/-$ into a simplicial global functor. The proof that each simplicial degree is a global functor is very similar to the Burnside ring global functor (the case $n=1$, cf. Remark 4.3.1 above), since the main role in the restrictions (4.3.1) is played by the smallest subgroup $H_0$ and the higher $H_i$ are carried along. The proof that the global structure maps commute with the simplicial operators is straightforward, except for the face $d_0$ and a restriction along a group homomorphism $\psi$. There we have to show that

$$\sum_{[k]\in G_\psi\backslash K/H_0}\left(\frac{[G:\psi^{-1}(kH_0k^{-1})]}{[K:H_0]}\cdot[\psi^{-1}(kH_1k^{-1})\leq\ldots\leq\psi^{-1}(kH_mk^{-1})]\right)\tag{4.3.2}$$

and

$$\sum_{[\widetilde{k}]\in G_\psi\backslash K/H_1}\left(\frac{[G:\psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})]}{[K:H_1]}\cdot[\psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})\leq\ldots\leq\psi^{-1}(\widetilde{k}H_m\widetilde{k}^{-1})]\right)\tag{4.3.3}$$

are the same. The class $[\psi^{-1}(kH_1k^{-1})\leq\ldots\leq\psi^{-1}(kH_mk^{-1})]$ only depends on the $(G\times H_1^{op})$-orbit of $k$, and so we can rewrite (4.3.2) as

$$\sum_{[\widetilde{k}]\in G_\psi\backslash K/H_1}\left(\left(\sum_{\substack{[k]\in G_\psi\backslash K/H_0\\ k\in G_\psi\widetilde{k}H_1}}\frac{[G:\psi^{-1}(kH_0k^{-1})]}{[K:H_0]}\right)\cdot[\psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})\leq\ldots\leq\psi^{-1}(\widetilde{k}H_m\widetilde{k}^{-1})]\right).$$

Hence, it suffices to show that

$$\sum_{\substack{[k]\in G_\psi\backslash K/H_0\\ k\in G_\psi\widetilde{k}H_1}}\frac{[G:\psi^{-1}(kH_0k^{-1})]}{[K:H_0]}=\frac{[G:\psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})]}{[K:H_1]}$$

for every $\widetilde{k} \in K$. This equality can be deduced from counting the number of elements in the $(G \times H_1^{op})$-orbit of $\widetilde{k} \in K$ in two different ways: The $(G \times H_1^{op})$-isotropy of $\widetilde{k}$ is given by the graph subgroup

$$\{(g, \widetilde{k}^{-1}\psi(g)^{-1}\widetilde{k}) \mid g \in \psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})\},$$

so the order of this orbit is

$$\frac{|G| \times |H_1|}{|\psi^{-1}(\widetilde{k}H_1\widetilde{k}^{-1})|}.$$

On the other hand, decomposing the orbit $G_\psi \widetilde{k} H_1$ into $(G \times H_0^{op})$-orbits yields the sum

$$\sum_{\substack{[k] \in G_\psi \backslash K / H_0 \\ k \in G_\psi \widetilde{k} H_1}} \frac{|G| \times |H_0|}{|\psi^{-1}(kH_1k^{-1})|},$$

and so dividing by $|K|$ gives the desired result.

Hence, the reduced chains of $\mathbb{Q}[L(-)_n]/-$ become a complex of global functors, which we denote by $\mathcal{C}L_n$. Then a more functorial version of Theorem 4.1.1 is the following:

**Theorem 4.3.2.** *There is an isomorphism of global functors*

$$\underline{\pi}_*(Sp^n) \otimes \mathbb{Q} \cong H_*(\mathcal{C}L_n).$$

This theorem will be implied by Theorem 5.2.2, a further strengthening, which is proved in Chapter 6.

# Chapter 5

# Geometric fixed points and rational global homotopy theory

In this chapter we describe how to reformulate Theorem 4.3.2 in terms of geometric fixed point homotopy groups, which turn out to be more directly accessible for the symmetric products $Sp^n$. Afterwards we explain a further strengthening of our main result, which allows to reconstruct the full global homotopy type of the rationalized $Sp^n_{\mathbb{Q}}$ from subgroup lattice data.

## 5.1 Geometric fixed points

Geometric fixed point homotopy groups are a different notion of homotopy groups for equivariant spectra. Rationally, they determine the usual (categorical) homotopy groups $\pi_*^G(-)$ and vice versa. We recall their definition:

**Definition 5.1.1** (Geometric fixed points)**.** The geometric fixed point homotopy groups of an orthogonal spectrum $X$ are defined as

$$\Phi_k^G(X) = \operatorname*{colim}_{n\in\mathbb{N}} [S^{k+n}, X(n\cdot\rho_G)^G]$$

for every finite group $G$ and $k \in \mathbb{Z}$, where the colimit is taken over the maps

$$[S^{k+n}, X(n\cdot\rho_G)^G] \xrightarrow{\wedge S^1} [S^{k+n+1}, X(n\cdot\rho_G)^G \wedge S^1] \xrightarrow{(\sigma^{\rho_G}_{n\cdot\rho_G})^G_*} [S^{k+n+1}, X((n+1)\cdot\rho_G)^G],$$

using that the $G$-fixed points of $S^{\rho_G}$ are homeomorphic to $S^1$.

The collection of geometric fixed point homotopy groups does not form a global functor, but it carries restrictions along surjective group homomorphisms ([Sch15, Construction III.7.8]). Again, inner conjugations act trivially. In other words, if we denote the category of finite groups and conjugacy classes of surjective group homomorphisms by Out, the collection $\Phi_*(X)$ forms a functor $\mathrm{Out}^{op} \to Ab$, an $\mathrm{Out}^{op}$-module. Moreover, one can show that a morphism of orthogonal spectra is a global equivalence if and only if it induces an isomorphism on all geometric fixed point homotopy groups (by applying [Sch15, Proposition III.7.20] for all finite groups $G$ and trivial $G$-action on the spectra).

There is a natural comparison map $\gamma_X : \pi_k^G(X) \to \Phi_k^G(X)$, which is easiest to describe if one defines $\pi_k^G(X)$ as the colimit over the $[S^{k+n\cdot\rho_G}, X(n\cdot\rho_G)]^G$, cf. Remark 3.1.3. Then $\gamma_X$ takes a $G$-map

$$S^{k+n\cdot\rho_G} \to X(n\cdot\rho_G)$$

to the induced map on fixed points

$$S^{k+n} \cong (S^{k+n\cdot\rho_G})^G \to X(n\cdot\rho_G)^G.$$

The map $\gamma_X$ has the following properties:

(i) It commutes with restrictions along surjective group homomorphisms.

(ii) It takes all elements of the form $\mathrm{tr}_H^G(x)$ for $H$ a proper subgroup of $G$ to 0.

Given a global functor $F$ and a finite group $G$ we let $\tau(F)(G)$ denote the quotient of $F(G)$ by all transfers from proper subgroups. Then the assignment $G \mapsto \tau(F)(G)$ no longer forms a global functor, but it inherits restrictions along surjective group homomorphisms, since these commute with transfers. In these terms, the two properties above mean that $\gamma_X$ factors through a morphism of $\mathrm{Out}^{op}$-modules $\widetilde{\gamma}_X : \tau(\underline{\pi}_*(X)) \to \Phi_*(X)$. It turns out that for rational global functors the construction $\tau$ can be reversed and that in this case $\widetilde{\gamma}_X$ is an isomorphism:

**Proposition 5.1.2** ([Sch15, Propositions IV.5.7 and IV.5.10])**.** *The functor*

$$\tau : (\mathbb{Q} - \text{global functors}) \ \to \ (\mathbb{Q}[\mathrm{Out}^{op}] - \text{mod})$$

*is an equivalence of categories. Moreover, for every orthogonal spectrum $X$ the map*

$$\widetilde{\gamma}_X : \tau(\underline{\pi}_*(X) \otimes \mathbb{Q}) \to \Phi_*(X) \otimes \mathbb{Q}$$

*is an isomorphism.*

**Remark 5.1.3.** When working over a fixed finite group $G$, the analogous statement is the equivalence of categories between rational $G$-Mackey functors and products of $\mathbb{Q}[W_G(H)]$-modules, where $H$ ranges through a set of conjugacy class representatives and $W_G(H) = N_G(H)/H$ denotes the Weyl group of $H$, cf. [GM95, Appendix A].

This means that instead of proving that $\underline{\pi}_*(Sp^n) \otimes \mathbb{Q}$ is isomorphic to $H_*(\mathcal{C}L_n)$ as global functors (Theorem 4.3.2), we can equivalently show:

**Theorem 5.1.4.** *There is an isomorphism of* $\mathrm{Out}^{op}$*-modules*

$$\Phi_*(Sp^n) \otimes \mathbb{Q} \cong \tau(H_*(\mathcal{C}L_n)).$$

The $\mathrm{Out}^{op}$-module $\tau(H_*(\mathcal{C}L_n))$ can be described as follows. Since $\tau$ is an equivalence, it is in particular exact. So there is a natural isomorphism

$$\tau(H_*(\mathcal{C}L_n)) \cong H_*(\tau(\mathcal{C}L_n)).$$

By the description given above, the transfers of $\mathcal{C}L_n(G)$ are generated by all chains of subgroups that end in a proper subgroup of $G$. This process can be carried out geometrically:

**Definition 5.1.5** (Reduced lattice)**.** We define the reduced subgroup lattice $\widetilde{L}(G)$ as the quotient of $L(G)$ by all chains that do not end in $G$, and similarly $\widetilde{L}(G)_n$.

So we find that there are isomorphisms

$$\tau(\mathcal{C}L_n)(G) \cong \mathcal{C}_*(\widetilde{L}(G)_n)/G \cong \mathcal{C}_*(\widetilde{L}(G)_n/G),$$

for every finite group $G$. Under this isomorphism, the restriction along a surjection $\varphi : G \twoheadrightarrow K$ sends a class $[H_0 \leq \ldots \leq H_m]$ to $[\varphi^*(H_0) \leq \ldots \leq \varphi^*(H_m)]$. In other words, the $\mathrm{Out}^{op}$-complex $\tau(\mathcal{C}L_n)$ arises by applying reduced rational chains to the simplicial set valued $\mathrm{Out}^{op}$-functor $G \mapsto \widetilde{L}(G)_n/G$. For this reason we from now on write $\mathcal{C}\widetilde{L}_n$ for $\tau(\mathcal{C}L_n)$.

## 5.2 Rational global homotopy theory and Wimmer's construction

We recall some rational global homotopy theory, again with respect to all finite groups. A morphism of orthogonal spectra is called a *rational global equivalence* if it induces an isomorphism on all $\underline{\pi}_*(-) \otimes \mathbb{Q}$. The localization of orthogonal spectra at rational global equivalences forms the *rational global homotopy category.* As in the non-equivariant case, the passage from the global homotopy category to the rational global homotopy category is a left Bousfield localization. It has a fully faithful right adjoint with essential image those orthogonal spectra whose categorical homotopy groups form $\mathbb{Q}$-vector spaces. Given an orthogonal spectrum $X$, its rationalization $X_{\mathbb{Q}}$ can be constructed as the homotopy colimit of the sequence

$$X \xrightarrow{\cdot 2} X \xrightarrow{\cdot 3} X \xrightarrow{\cdot 4} \ldots,$$

or as the smash product with the rational global sphere $\mathbb{S}_{\mathbb{Q}}$. Furthermore, the unit $X \to X_{\mathbb{Q}}$ induces an isomorphism $\underline{\pi}_*(X) \otimes \mathbb{Q} \cong \underline{\pi}_*(X_{\mathbb{Q}})$.

As an application of Morita theory for stable model categories (cf. [SS03]), the category of orthogonal spectra with rational global equivalences is Quillen equivalent to the derived category of rational global functors ([Sch15, Theorem IV.5.2]). The latter in turn – using Proposition 5.1.2 above – is equivalent to the derived category of rational $\mathrm{Out}^{op}$-modules. In his PhD thesis [Wim16], Christian Wimmer constructs an explicit equivalence $T$ between the rational global homotopy category and this derived category. We quickly recall his construction.

Let Epi denote the category of finite groups and surjective group homomorphisms. Then the equivalence $T$ is constructed as a chain of three functors

$$\text{orth. spectra} \xrightarrow{\Phi} (\text{orth. spectra})^{\mathrm{Epi}^{op}} \xrightarrow{c^{\mathrm{Epi}^{op}}} \mathrm{Ch}_{\mathbb{Q}}^{\mathrm{Epi}^{op}} \xrightarrow{q_!} \mathrm{Ch}_{\mathbb{Q}}^{\mathrm{Out}^{op}}.$$

We go through each one individually: The first functor $\Phi$ is the geometric fixed point spectrum functor. It sends a finite group $G$ and an orthogonal spectrum $X$ to the orthogonal spectrum $\Phi^G(X)$ with

$$\Phi^G(X)(V) = X(V \otimes \rho_G)^G$$

and structure maps

$$X(V \otimes \rho_G)^G \wedge S^{W-\varphi(V)} \to X(W \otimes \rho_G)^G$$

the induced map on $G$-fixed points of the structure map $(\varphi \otimes \rho_G)_*$ for $X$.

Given a surjection $\psi : G \twoheadrightarrow K$, the induced morphism $\Phi^\psi(X) : \Phi^K(X) \to \Phi^G(X)$ is defined in level $V$ by

$$X(V \otimes \rho_K)^K = X(V \otimes \psi^*(\rho_K))^G \xrightarrow{X(V \otimes i_\psi)^G} X(V \otimes \rho_G)^G.$$

Here, $\psi^*(\rho_K)$ denotes the restriction of $\rho_K$ to a $G$-representation along $\psi$, and $i_\psi : \psi^*(\rho_K) \hookrightarrow \rho_G$ is the $G$-equivariant linear isometry which sends a basis element $e_k$ to

$$\frac{1}{\sqrt{|\ker(\psi)|}} \sum_{g \in \psi^{-1}(k)} e_g.$$

To avoid confusion, one should no longer think of $\Phi^G(X)$ as a global spectrum, only the non-equivariant homotopy type is important. Almost by definition, there is a natural isomorphism between $\Phi_*^G(X)$ and the non-equivariant homotopy groups $\pi_*(\Phi^G(X))$. Moreover, the restriction maps on $\Phi(X)$ induce the restriction maps of $\Phi_*(X)$ under this isomorphism.

The second functor $c^{\mathrm{Epi}^{op}}$ is given by postcomposition with a certain functor

$$c : \text{orth. spectra} \to \mathrm{Ch}_{\mathbb{Q}}.$$

We do not need the definition of $c$, but only the following properties:

- There is a natural isomorphism between $\pi_*(X) \otimes \mathbb{Q}$ and $H_*(c(X))$. Hence, $c$ takes rational equivalences to quasi-isomorphisms.
- For all based spaces $A$ there is a natural quasi-isomorphism $\mathcal{C}_*(A) \simeq c(\Sigma^\infty A)$. In other words, for suspension spectra of spaces the associated rational chain complex $c(\Sigma^\infty A)$ is equivalent to the usual rational singular chains.

Finally, $q : \mathrm{Epi} \twoheadrightarrow \mathrm{Out}$ is the projection and we write $q_! : \mathrm{Ch}_{\mathbb{Q}}^{\mathrm{Epi}^{op}} \to \mathrm{Ch}_{\mathbb{Q}}^{\mathrm{Out}^{op}}$ for the left Kan extension along $q$. Concretely, $q_!$ divides out all inner conjugations. This process does not change the homology of complexes of the form $c(\Phi(X))$, since inner conjugations already act trivially on the homology of those.

Hence, the composite $T$ has the property that it turns rational geometric fixed point homotopy groups of an orthogonal spectrum into homology groups of the associated $\mathrm{Out}^{op}$-chain complex. In particular, it takes rational global equivalences to quasi-isomorphisms of $\mathrm{Out}^{op}$-chain complexes.

**Theorem 5.2.1** ([Wim16])**.** *The induced functor on homotopy categories*

$$T : \text{orth. spectra}[\mathbb{Q} - \text{global eq.}^{-1}] \to \mathcal{D}(\mathbb{Q}[\mathrm{Out}^{op}] - \mathrm{mod})$$

*is an equivalence.*

Recall that by $\mathcal{C}\widetilde{L}_n$ we denote the reduced rational chains on the $\mathrm{Out}^{op}$-functor $G \mapsto \widetilde{L}(G)_n/G$. Then the strongest version of the main result in Part II is the following:

**Theorem 5.2.2.** *There is a quasi-isomorphism of chain complexes of* $\mathbb{Q}[\mathrm{Out}^{op}]$*-modules*

$$T(Sp^n) \simeq \mathcal{C}\widetilde{L}_n.$$

This implies Theorem 5.1.4 (and hence also Theorem 4.3.2) by taking homology.

**Remark 5.2.3.** For a fixed finite group $G$, the category of rational $G$-spectra is Quillen-equivalent to the derived category of rational $G$-Mackey functors (which in turn is isomorphic to the product of the derived categories of rational $W_G(H)$-modules, cf. Remark 5.1.3). The category of rational $G$-Mackey functors is semisimple, hence a chain complex of such is determined up to quasi-isomorphism by its homology. Consequently, a rational $G$-spectrum is determined by its $G$-Mackey functor homotopy groups. Hence, for a fixed group $G$, the analogous statements of Theorem 5.1.4 and Theorem 5.2.2 are equivalent. However, since rational $\mathrm{Out}^{op}$-modules are not semisimple, these theorems are not equivalent globally. In Section 8 we use Theorem 5.2.2 to show that $Sp^n_{\mathbb{Q}}$ is not a product of global Eilenberg-MacLane spectra.

# Chapter 6

# Proof of the main theorem

Now we come to the proof of Theorem 5.2.2. Let $|-|$ denote the geometric realization of a simplicial set. We show that there exists a transformation of spectrum-level $\mathrm{Epi}^{op}$-functors

$$\widetilde{\alpha} : \Sigma^\infty|\widetilde{L}(-)_n| \to \Phi(Sp^n)$$

which induces an isomorphism of $\mathrm{Out}^{op}$-modules on rational homology modulo conjugation. Since rational homology is naturally isomorphic to rational stable homotopy, we see that this implies quasi-isomorphisms

$$\mathcal{C}\widetilde{L}_n = q_!(\mathcal{C}_*(\widetilde{L}(-)_n)) \simeq q_!(c(\Sigma^\infty_+|\widetilde{L}(-)_n|)) \simeq q_!(c(\Phi(Sp^n))) = T(Sp^n) \tag{6.0.1}$$

of $\mathrm{Out}^{op}$-complexes and hence yields Theorem 5.2.2. Here, the second and third quasi-isomorphisms make use of the properties of the chain functor $c$ described in the previous section.

More precisely, $\widetilde{\alpha}$ is a zig-zag, as we have to modify both $\widetilde{L}(G)_n$ and $\Phi(Sp^n)$ a little to be able to construct an honest map. First we note that by adjunction, spectrum maps

$$\Sigma^\infty|\widetilde{L}(G)_n| \to \Phi^G(Sp^n)$$

stand in bijection with maps of spaces

$$|\widetilde{L}(G)_n| \to (Sp^n(S^0))^G.$$

The target is just a discrete set of points, so there are no interesting maps on the point-set level. This can be resolved by stabilizing once, which we do via the following construction: The shift $\mathrm{sh}\, X$ of an orthogonal spectrum $X$ is defined via

$$(\mathrm{sh}\, X)(V) = X(\mathbb{R} \oplus V).$$

It allows a natural map $\lambda_X : S^1 \wedge X \to \mathrm{sh}\, X$ given in level $V$ by the composite

$$S^1 \wedge X(V) \cong X(V) \wedge S^1 \xrightarrow{\sigma^1_V} X(V \oplus \mathbb{R}) \xrightarrow{X(\tau_{\mathbb{R},V})} X(\mathbb{R} \oplus V).$$

Its adjoint is a morphism $\widetilde{\lambda}_X : X \to \Omega\, \mathrm{sh}\, X$. Both $\lambda_X$ and $\widetilde{\lambda}_X$ induce isomorphisms on homotopy groups (cf. [Sch15, Proposition III.2.25]). Hence, instead of $\Phi(Sp^n)$ we can equivalently consider the $\mathrm{Epi}^{op}$-diagram $\Omega\,\mathrm{sh}\,\Phi(Sp^n)$. Morphisms

$$\Sigma^\infty|\widetilde{L}(G)_n| \to \Omega\,\mathrm{sh}\,\Phi^G(Sp^n)$$

now correspond to maps of spaces

$$|\widetilde{L}(G)_n| \to \Omega(Sp^n(S^{\rho_G}))^G.$$

This one copy of the regular representation turns out to be enough to define $\widetilde{\alpha}$, though it only becomes a rational equivalence after further stabilization.

## 6.1 Geometric idea

We first focus on the case $n = |G|$ and start by describing a map

$$\overline{\alpha}_1 : |L(G)| \to \Omega(Sp^{|G|}(S^{\rho_G}))^G$$

from the non-reduced subgroup lattice. The regular representation $\rho_G$ decomposes as $\mathbb{R}\oplus\overline{\rho}_G$, where $\overline{\rho}_G$ is the reduced regular representation of tuples that add up to 0, and $\mathbb{R}$ denotes the trivial diagonal copy. To be explicit, we work with the splitting that sends an element $x = \sum x_g \cdot e_g$ to its trivial component

$$t(x) = (\frac{1}{|G|}\sum_{g\in G} x_g)\cdot(\sum_{g\in G} e_g)$$

and its reduced component $r(x) = x - t(x)$. This decomposition also induces a map

$$(\widetilde{\sigma}^1_{\overline{\rho}_G})^G : (Sp^{|G|}(S^{\overline{\rho}_G}))^G \to \Omega(Sp^{|G|}(S^{\rho_G}))^G,$$

the induced map on fixed points of the adjoint structure map $\widetilde{\sigma}^1_{\overline{\rho}_G}$ of the orthogonal spectrum $Sp^{|G|}$. Throughout this section we abbreviate $(\widetilde{\sigma}^1_{\overline{\rho}_G})^G$ by $\widetilde{\sigma}^G$. Our map $\overline{\alpha}_1$ is the composition of a map

$$\alpha_1 : |L(G)| \to (Sp^{|G|}(S^{\overline{\rho}_G}))^G$$

with this adjoint structure map. To define $\alpha_1$, we need one more piece of notation: Given a non-empty subset $M \subseteq G$, we denote by $e_M \in \rho_G$ the element $\frac{1}{\sqrt{|M|}}\sum_{g\in M} e_g$. Then $\alpha_1$ is defined by sending a chain of subgroups $H_0 \leq \ldots \leq H_k$ and $(t_0,\ldots,t_k)\in\Delta^k$ to the class

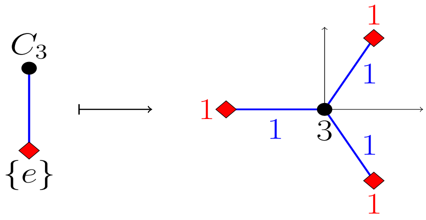
$$[(r(\sum_{i=0}^k t_i\cdot e_{gH_i}))_{g\in G}] \in (Sp^{|G|}(S^{\overline{\rho}_G}))^G.$$

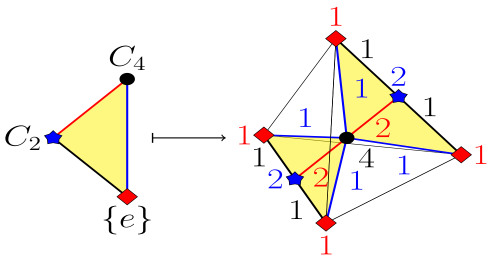
We shall explain this formula briefly: For each $g \in G$, the map

$$(\alpha_1)_g : (H_0 \leq \ldots \leq H_k; t_0,\ldots,t_k) \mapsto r(\sum_{i=0}^k t_i\cdot e_{gH_i})$$

defines an embedding of the subgroup lattice into $\overline{\rho}_G$. These different embeddings are permuted via the $G$-action, as an element $g'$ takes $(\alpha_1)_g$ to $(\alpha_1)_{g'g}$. So $\alpha_1$, the product of all $(\alpha_1)_g$, is $G$-fixed in $Sp^{|G|}(S^{\overline{\rho}_G})$.

**Example 6.1.** Elements in a symmetric product $Sp^n(X)$ of a space $X$ can be visualized as configurations in $X$ with labels in the natural numbers, cf. Remark 3.2.1. We use this to depict $\alpha_1$ in the case where $G$ is a cyclic group of order 3 or 4. For $G = C_3$ the reduced regular representation is isomorphic to $\mathbb{R}^2$ with rotation by 120 degrees. The image of the vertex $\{e\}$ in the subgroup lattice is the configuration of the three corners of an equilateral triangle with center 0, each equipped with the label 1. As one moves along the edge $\{e\} \leq C_3$, these points move straight towards the center at the same speed. Finally, the vertex $C_3$ is mapped to the zero vector with label 3.




The reduced regular representation of $C_4$ is three-dimensional and permutes the corners of a regular tetrahedron. The image of the vertex $\{e\}$ under $\alpha_1$ is the configuration of these corners with label 1, and the images of the other simplices are as depicted in the figure on the left. Denoting a generator of $C_4$ by $t$, the rightmost corner corresponds to $r(e_1)$, the left one to $r(e_t)$, the upper one to $r(e_{t^2})$ and the lower one to $r(e_{t^3})$.

This is the basic geometric idea, but some adjustments are necessary in order to make it have all the properties and compatibilities that we need. One problem is that $\alpha_1$ does not yet factor through the reduced lattice $\widetilde{L}(G)$, i.e., it does not send chains which end in a proper subgroup of $G$ to the basepoint. This problem can be resolved: Note that the full group $G$ is the only vertex that is sent to the 0-vector. So if we choose a ball around 0 of small enough radius and push everything that lies outside of it to $\infty$, the resulting map will send all proper subgroups $H$ of $G$ and the simplices connecting them to the basepoint.

In formulas, this is done as follows: Let $p : S^{\overline{\rho}_G} \to S^{\overline{\rho}_G}$ be a map of the form $p(v) = \mu(|v|) \cdot v$, where $\mu$ is a fixed continuous self-map of $[0, \infty]$ that restricts to an orientation-preserving homeomorphism $[0, \frac{1}{\sqrt{2}}] \cong [0, \infty]$ and sends $[\frac{1}{\sqrt{2}}, \infty]$ to $\infty$. In other words, $p$ collapses the hemisphere of vectors of length at least $\frac{1}{\sqrt{2}}$ to a point and identifies the resulting quotient with $S^{\overline{\rho}_G}$ again. Furthermore, we let

$$q : (\rho_G - \{0\}) \to S(\rho_G)$$

denote the projection to the unit sphere and

$$\overline{r} : (\rho_G - \{0\}) \to \overline{\rho}_G$$

the composite of $q$ and $r$. For every $g \in G$ we obtain a new map $(\alpha_2)_g : |L(G)| \to S^{\overline{\rho}_G}$ via the formula

$$(H_0 \leq \ldots \leq H_k; t_0, \ldots, t_k) \mapsto p(\overline{r}(\sum_{i=0}^{k} t_i \cdot e_{gH_i}))$$

and again let

$$\alpha_2 : |L(G)| \to (Sp^{|G|}(S^{\overline{\rho}_G}))^G$$

be the tuple of all $(\alpha_2)_g$ for $g \in G$. In words, we have made two changes: We project each of the lattices inside $\rho_G$ to the unit sphere before passing to the reduced $\overline{\rho}_G$, and in the end we quotient out all vectors of length at least $\frac{1}{\sqrt{2}}$. This has the desired effect:

**Lemma 6.1.1.** *The map $\alpha_2 : |L(G)| \to (Sp^{|G|}(S^{\overline{\rho}_G}))^G$ factors through the reduced lattice $|\widetilde{L}(G)|$.*

*Proof.* It suffices to see that the square of the norm of

$$q(\sum_{i=0}^{k}(t_i \cdot e_{gH_i})) + t \cdot e_G$$

is at least $1/2$ for any chain $H_0 \leq \ldots \leq H_k$, $(t_0, \ldots, t_k) \in \Delta^k$ and $t \in \mathbb{R}$, provided that $H_k$ is a proper subgroup of $G$. Dividing $\rho_G$ into the span of the basis elements of the form $e_{gh_k}$ with $h_k \in H_k$ and the span of the other basis elements, we see that this square is given by

$$|q(\sum_{i=0}^{k}(t_i \cdot e_{gH_i})) + t \cdot \sqrt{\frac{|H_k|}{|G|}} \cdot e_{gH_k}|^2 + |t \cdot \sqrt{\frac{|G| - |H_k|}{|G|}} \cdot e_{(G-gH_k)}|^2.$$

Using that $|e_{gH_k}| = 1 = |e_{(G-gH_k)}|$ and applying the triangle inequality yields that this square is at least as large as

$$\left(|q(\sum_{i=0}^{k}(t_i \cdot e_{gH_i}))| - |t| \cdot \sqrt{\frac{|H_k|}{|G|}}\right)^2 + \left(|t| \cdot \sqrt{\frac{|G| - |H_k|}{|G|}}\right)^2.$$

Since $q(-)$ by definition always has norm 1 and $|G| - |H_k|$ is at least $|H_k|$, we obtain the lower bound

$$\left(1 - |t| \cdot \sqrt{\frac{|H_k|}{|G|}}\right)^2 + \left(|t| \cdot \sqrt{\frac{|H_k|}{|G|}}\right)^2.$$

The minimum of this quadratic function equals $\frac{1}{2}$, which proves the claim. $\square$

**Example 6.1.2.** The effect of $\alpha_2$ is depicted in Figure 6.1 below for $G = C_3$. The first image illustrates the area of $\overline{\rho}_{C_3}$ that is quotiened out and the second the resulting map to $Sp^3(S^{\overline{\rho}_{C_3}})$.

However, there is a problem that is more complicated to resolve: We need the restriction to the subcomplex $\widetilde{L}(G)_n$ to take image in $(Sp^n(S^{\overline{\rho}_G}))^G$. This is simply not the case for $\alpha_2$, as one already sees in Figure 6.1: The image of the 0-chain with value $C_3$

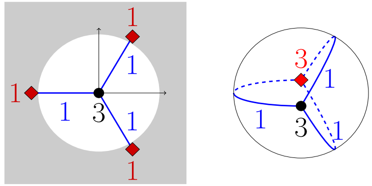



Figure 6.1: The map $\alpha_2$ for $G = C_3$

(which lies in $\widetilde{L}(G)_1$) is the tuple $[(0,0,0)]$. It has three non-basepoint components and hence does not lie in any smaller symmetric product. The idea to rectify this is to use that $[(0,0,0)]$ is stably the same as 'three times the element $[(0,*,*)]$', which does lie in the image of $Sp^1$.

We now make this precise and more generally let $H_0 \leq \ldots \leq H_k$ be a chain of total index $n$ and ending in $H_k = G$. We write

$$(\alpha_2)_g(\{H_i\}, -) : \Delta^k_+ \to S^{\bar{\rho}_G}$$

for the restriction of $(\alpha_2)_g$ to the $k$-simplex corresponding to this chain, and

$$\alpha_2(\{H_i\}, -) : \Delta^k_+ \to (Sp^{|G|}(S^{\bar{\rho}_G}))^G$$

for the analogous restriction of $\alpha_2$. These have the following properties:

- Each $(\alpha_2)_g(\{H_i\}, -)$ only depends on the coset $gH_0$, since multiplication with $h_0 \in H_0$ leaves all $e_{H_i}$ fixed. Hence there are only $n$ different components in $\alpha_2(\{H_i\}, -)$, each repeated $|H_0|$ times.

- Let $g_1, \ldots, g_n$ be a system of coset representatives of $G/H_0$. Then the tuple

$$(\alpha_2)_{G/H_0}(\{H_i\}, -) := [((\alpha_2)_{g_j}(\{H_i\}, -))_{j=1,\ldots,n}]$$

  defines a map $\Delta^k_+ \to (Sp^n(S^{\bar{\rho}_G}))^G$.

In other words, $\alpha_2(\{H_i\}, -)$ factors through the diagonal

$$(Sp^n(S^{\bar{\rho}_G}))^G \xrightarrow{\Delta} (Sp^{|G|}(S^{\bar{\rho}_G}))^G$$

that repeats each entry $|H_0|$ times, while we want it to factor through the standard inclusion $i_n^{|G|}$. In $(Sp^{|G|}(S^{\bar{\rho}_G}))^G$ there is no direct way to pass between $\Delta$ and $i_n^{|G|}$, but there is after stabilizing once, i.e., after postcomposing with

$$\widetilde{\sigma}^G : (Sp^{|G|}(S^{\bar{\rho}_G}))^G \to \Omega(Sp^{|G|}(S^{\rho_G}))^G.$$

To see this, we consider the following modified construction of the diagonal: We assume given $|H_0|$ closed subintervals $[a_i, b_i]$ of $[-\infty, \infty]$ and for each of these let $c(a_i, b_i)$ denote the self-map of $S^1$ which collapses everything outside $(a_i, b_i)$ to the basepoint and identifies $[a_i, b_i]$ with $[-\infty, \infty]$ in some fixed orientation-preserving way. To each such data one can associate a map

$$\Delta_{\{[a_i,b_i]\}} : \Omega(Sp^n(S^{\rho_G}))^G \to \Omega(Sp^{|G|}(S^{\rho_G}))^G$$

by sending $\varphi \in \Omega(Sp^n(S^{\rho_G}))^G$ to

$$[(\varphi \circ c(a_i, b_i))_{1 \leq i \leq |H_0|}].$$

So, instead of simply repeating it $|H_0|$ times, $\varphi$ is precomposed with every $c(a_i, b_i)$. If all of the $[a_i, b_i]$ are equal to $[-\infty, \infty]$, this construction gives back the usual diagonal $\Omega(\Delta)$. If on the other hand the interiors of the $[a_i, b_i]$ are pairwise disjoint, the map $\Delta_{\{[a_i,b_i]\}}$ factors through the standard inclusion $\Omega(i^{|G|}_{|G/H|})$. Indeed, in this case there is at most one $i$ such that $c(a_i, b_i)(t)$ is not the basepoint, for any fixed $t \in \mathbb{R}$. So each loop $\Delta_{\{[a_i,b_i]\}}(\varphi)$ has at most $n$ non-trivial components at every $t$. This means that it has at most $n$ non-trivial components globally and hence lies in the image of $\Omega(i^{|G|}_n)$, since we can always move the non-trivial components to the first $n$ entries. The self-map of $\Omega Sp^n(S^{\rho_G})$ obtained this way can also be described differently: It is given by precomposition with the self-map of $S^1$ that collapses everything outside the open intervals $(a_i, b_i)$ to the basepoint and identifies each $[a_i, b_i]/(a_i \sim b_i)$ with $S^1$. In particular, the homotopy class of $\Delta_{\{[a_i,b_i]\}}(\varphi)$ is the $|H_0|$-fold sum of $\varphi$ with itself.

Any choice of homotopies from the $c(a_i, b_i)$ to the identity of $S^1$ induces a homotopy between $\Omega(\Delta)$ and $\Delta_{\{[a_i,b_i]\}}$. So we see that, up to reparametrization of loops and in particular up to homotopy, $\widetilde{\sigma}^G \circ \alpha_2$ does map the simplex associated to $H_0 \leq \ldots \leq H_k$ to the image of $\Omega Sp^n(S^{\rho_G})$ under $\Omega(i^{|G|}_n)$. To turn this into honest maps from $\widetilde{L}(-)_n$ to $\Omega(Sp^n(S^{\rho_-}))^-$, we need to make choices of reparametrizations that are coherent for all chains $H_0 \leq \ldots \leq H_k$, all $n \in \mathbb{N}$ and all finite groups $G$. We deal with this by defining a modification of the subgroup lattice that contains a contractible choice of intervals as part of the data.

## 6.2 Fattening of the lattice

For this it turns out to be more convenient to work with subintervals of $[0, 1]$ instead of $[-\infty, \infty]$. Whenever we need to switch between the two, we use the homeomorphism that maps $t \in [0, 1]$ to $\frac{2t-1}{t(1-t)} \in [-\infty, \infty]$.

Let $J$ denote the space of closed subintervals $[a, b]$ of $[0, 1]$ (with $a < b$), topologized as a subspace of $[0, 1] \times [0, 1]$. We let $L_f(G)$ denote the following topological category: The object space is given by

$$\coprod_{H \leq G} Sp^{|H|}(J),$$

i.e., subgroups $H$ of $G$ together with an unordered $|H|$-tuple of subintervals of $[0, 1]$. The morphism space from a component $Sp^{|H|}(J)$ to $Sp^{|K|}(J)$ is empty if $H$ is not contained in $K$ and is otherwise given by $Sp^{|H|}(J)$ again. In this case the target map

$$Sp^{|H|}(J) \to Sp^{|K|}(J)$$

is the diagonal which repeats each subinterval $[K : H]$ times, and the source map is the identity. Hence, a $k$-simplex in the topological nerve of $L_f(G)$ - which we also denote by $L_f(G)$ - is given by a chain of subgroups $H_0 \leq \ldots \leq H_k$ together with $|H_0|$ many subintervals $[a_i, b_i]$ of $[0,1]$. We filter this nerve by saying that such a $k$-simplex lies in $L_f(G)_n$ if

(i) the total index $[H_k : H_0]$ is at most $n$, and

(ii) the intervals $\{[a_i, b_i]\}$ have at most $\frac{n}{[H_k:H_0]}$-fold intersections, i.e., every $t \in (0,1)$ lies in the interior of at most $\frac{n}{[H_k:H_0]}$-many $[a_i, b_i]$.

There is an obvious forgetful functor $\mu : L_f(G) \to L(G)$ to the usual subgroup lattice of $G$, whose nerve maps $L_f(G)_n$ into $L(G)_n$.

**Lemma 6.2.1.** *The maps $\mu : L_f(G)_n \to L(G)_n$ induce homotopy equivalences on geometric realizations.*

*Proof.* Given $l, m \in \mathbb{N}$, let $J^l_m$ denote the subspace of $Sp^l(J)$ of tuples of intervals with at most $m$-fold intersections. Then the space of $k$-simplices of $L_f(G)_n$ is given by the disjoint union

$$\coprod_{H_0 \leq \ldots \leq H_k; [H_k:H_0] \leq n} J^{|H_0|}_{\lfloor \frac{n}{[H_k:H_0]} \rfloor}, \tag{6.2.1}$$

where $\lfloor \frac{n}{[H_k:H_0]} \rfloor$ denotes the largest integer smaller than or equal to $\frac{n}{[H_k:H_0]}$. We first claim that each $J^l_m$ is contractible and hence $L_f(G)_n \to L(G)_n$ forms a degreewise homotopy equivalence. Since we have divided out the symmetric group action, every element of $J^l_m$ has a unique representative $([a_1, b_1], \ldots, [a_l, b_l])$ for which the $(a_i, b_i)$ are lexicographically ordered, i.e., $a_i \leq a_{i+1}$ and if $a_i = a_{i+1}$ then $b_i \leq b_{i+1}$. In fact, $J^l_m$ is homeomorphic to the space of such ordered tuples with at most $m$-fold intersections. But this space is star-shaped, it can be linearly contracted onto the tuple $([0, \frac{1}{l}], [\frac{1}{l}, \frac{2}{l}], \ldots, [\frac{l-1}{l}, 1])$. This proves the claim.

Hence it suffices to see that both $L_f(G)_n$ and $L(G)_n$ are Reedy cofibrant simplicial spaces (cf. [Hir03, Chapter 15]) with respect to the Strøm model structure on topological spaces ([Str72]). For $L(G)_n$ this is clear, since it is a discrete simplicial space. For $L_f(G)_n$, the $k$-th latching map is the inclusion of those components in the disjoint union (6.2.1) above which are associated to chains $H_0 \leq \ldots \leq H_k$ for which at least one containment is not proper. Every topological space is Strøm cofibrant, so this inclusion is a Strøm cofibration, which finishes the proof. □

Hence, the $L_f(G)_n$ indeed form a fattening of the $L(G)_n$, but we still need to explain their functoriality in surjective group homomorphisms. For this we let $\psi : G \twoheadrightarrow K$ be a surjection and denote by $k$ the order of the kernel. Then we define

$$\psi^* : L_f(K) \to L_f(G)$$

to send a subgroup $L$ of $K$ to $\psi^{-1}(L)$, and the associated collection of intervals $([a_i, b_i])$ to

$$(r^k_j([a_i, b_i]))_{i=1,\ldots,l; j=0,\ldots,k-1},$$

where $r^k_j : [0,1] \hookrightarrow [0,1]$ is the unique oriented affine embedding with image $[\frac{j}{k}, \frac{j+1}{k}]$. In other words, $\psi^*$ splits $[0,1]$ into $k$ parts of equal size and copies each $[a_i, b_i]$ into every one of them, yielding $|L| \cdot k = |\varphi^{-1}(L)|$ subintervals, as needed. This definition turns $L_f(-)$ into a functor from $\mathrm{Epi}^{op}$ to topological categories. After applying the nerve, it restricts to functors $L_f(-)_n$ from $\mathrm{Epi}^{op}$ to simplicial spaces and hence after geometric realization to functors $|L_f(-)_n| : \mathrm{Epi}^{op} \to \mathrm{Top}$. The forgetful functor $|L_f(-)_n| \to |L(-)_n|$ is natural for this $\mathrm{Epi}^{op}$-functoriality and a homotopy equivalence for all finite groups $G$.

Finally, we again define a reduced version $|\widetilde{L}_f(G)_n|$ by quotiening out all simplices associated to chains that do not end in the full group $G$. The $|\widetilde{L}_f(G)_n|$ again assemble to a functor $\mathrm{Epi}^{op} \to \mathrm{Top}$ and the forgetful map $|\widetilde{L}_f(G)_n| \to |\widetilde{L}(G)_n|$ defines a natural levelwise based homotopy equivalence.

## 6.3 Definition of $\widetilde{\alpha}$

Given an interval $[a, b]$ of $[0, 1]$, we from now on let $c(a, b)$ denote the self-map of $[0, 1]/\{0, 1\}$ (or $S^1 = \mathbb{R} \cup \{\infty\}$, using the fixed homeomorphism between the two) obtained by collapsing everything outside $(a, b)$ to a point and using the identification $[a, b] \cong [0, 1]$ that sends $x$ to $\frac{x-a}{b-a}$.

We are now ready to define the map

$$\alpha : |L_f(G)| \to \Omega(Sp^{|G|}(S^{\rho_G}))^G$$

by sending a simplex associated to a chain $H_0 \leq \ldots \leq H_k$ together with intervals $[a_1, b_1], \ldots, [a_{|H_0|}, b_{|H_0|}]$ to the composite

$$\Delta^k \xrightarrow{(\alpha_2)_{G/H_0}(\{H_i\},-)} (Sp^{|G/H_0|}(S^{\bar{\rho}_G}))^G \xrightarrow{\widetilde{\sigma}^G} \Omega(Sp^{|G/H_0|}(S^{\rho_G}))^G \xrightarrow{\Delta_{\{[a_i,b_i]\}}} \Omega(Sp^{|G|}(S^{\rho_G}))^G. \tag{6.3.1}$$

The maps $(\alpha_2)_{G/H_0}(\{H_i\}, -)$ and $\Delta_{\{[a_i,b_i]\}}$ are explained on page 69 (using the specific $c(a_i, b_i)$ defined above to construct the diagonal). If the $[a_j, b_j]$ are all equal to $[0, 1]$, we get back $\widetilde{\sigma}^G \circ \alpha_2(\{H_i\}, -)$.

In order for the maps (6.3.1) to glue to a map from the geometric realization, we need to check that they are still compatible with the simplicial structure maps. This is a consequence of the fact that the $(\alpha_2)_g(\{H_i\}, -)$ have this compatibility, except for the boundary $d_0$, since it changes the smallest subgroup $H_0$. We recall that $d_0^*$ of a tuple $(H_0 \leq \ldots \leq H_k, \{[a_i, b_i]\})$ is the chain $H_1 \leq \ldots \leq H_k$ together with the intervals $\Delta(\{[a_i, b_i]\})$, i.e., each interval repeated $|H_1/H_0|$ times. So the compatibility for $d_0$ follows from the commutativity of the diagram:

$$\begin{tikzcd}
\Delta^k \arrow[r, "(\alpha_2)_{G/H_0}"] & (Sp^{|G/H_0|}(S^{\bar{\rho}_G}))^G \arrow[r, "\widetilde{\sigma}^G"] & \Omega(Sp^{|G/H_0|}(S^{\rho_G}))^G \arrow[r, "\Delta_{\{[a_i,b_i]\}}"] & \Omega(Sp^{|G|}(S^{\rho_G}))^G \\
\Delta^{k-1} \arrow[u, "d_0"'] \arrow[r, "(\alpha_2)_{G/H_1}"] & (Sp^{|G/H_1|}(S^{\bar{\rho}_G}))^G \arrow[u, "\Delta"] \arrow[r, "\widetilde{\sigma}^G"] & \Omega(Sp^{|G/H_1|}(S^{\rho_G}))^G \arrow[u, "\Omega(\Delta)"] \arrow[r, "\Delta_{\Delta(\{[a_i,b_i]\})}"] & \Omega(Sp^{|G|}(S^{\rho_G}))^G \arrow[u, "="]
\end{tikzcd}$$

Since $(\alpha_2)_{G/H_0}(\{H_i\}, -)$ sends all chains that do not end in the full group $G$ to the basepoint (Lemma 6.1.1), it follows that $\alpha$ again factors through the reduced fat lattice, yielding a map $\widetilde{\alpha} : |\widetilde{L}_f(G)| \to \Omega(Sp^{|G|}(S^{\rho_G}))^G$. In addition, we now have:

**Proposition 6.3.1.** *The restriction of $\widetilde{\alpha}$ to $|\widetilde{L}_f(G)_n|$ factors through*

$$\Omega(i_n^{|G|}) : \Omega(Sp^n(S^{\rho_G}))^G \hookrightarrow \Omega(Sp^{|G|}(S^{\rho_G}))^G.$$

*Proof.* Let $H_0 \leq \cdots \leq H_k$ be a chain with $H_k = G$ and $[G : H_0] \leq n$, together with $|H_0|$ intervals $[a_i, b_i]$ with at most $\frac{n}{[G:H_0]}$-fold intersections. Then at any point $t \in (0,1)$, at most $\frac{n}{[G:H_0]}$-many values $c(a_i, b_i)(t)$ are not equal to the basepoint. Each one of them appears exactly $[G : H_0]$ times in the definition of $\Delta_{\{[a_i,b_i]\}}$, so it follows that the diagonal

$$\Omega(Sp^{|G/H_0|}(S^{\rho_G}))^G \xrightarrow{\Delta_{\{[a_i,b_i]\}}} \Omega(Sp^{|G|}(S^{\rho_G}))^G$$

factors through $\Omega(i_n^{|G|})$, which proves the claim. $\square$

## 6.4 Naturality

By adjunction, we obtain maps $\widetilde{\alpha} : \Sigma^\infty|\widetilde{L}_f(G)_n| \to \Omega\,\mathrm{sh}\,\Phi^G(Sp^n)$, compatible with the respective inclusions from $n$ to $n+1$. We now check their naturality with respect to surjective group homomorphisms $\psi : G \twoheadrightarrow K$.

First of all, we assume given a subset $M$ of $K$. Then the linear isometry $i_\psi : \psi^*(\rho_K) \hookrightarrow \rho_G$ (defined in Section 5.2 to describe the $\mathrm{Epi}^{op}$-functoriality of geometric fixed points) sends the element $e_M$ to $e_{\psi^{-1}(M)}$. This implies that for every chain of subgroups $H_0 \leq \ldots \leq H_k$ of $K$, the composite

$$\Delta^k \xrightarrow{(\alpha_2)_{K/H_0}(\{H_i\},-)} (Sp^{|K/H_0|}(S^{\bar{\rho}_K}))^K \xrightarrow{(i_\psi)_*} (Sp^{|G/\psi^{-1}(H_0)|}(S^{\bar{\rho}_G}))^G$$

equals the map $(\alpha_2)_{G/\psi^{-1}(H_0)}(\{\psi^{-1}(H_i)\},-)$. To compute the effect of $(\Omega\,\mathrm{sh}\,\Phi^\psi) \circ \widetilde{\alpha}$ on the tuple $(H_0 \leq \ldots \leq H_k, \{[a_i,b_i]\})$, we then have to postcompose with the diagonal $\Delta_{\{[a_i,b_i]\}}$. On the other hand, in order to compute the effect of $\widetilde{\alpha}$ on the tuple $\psi^*(H_0 \leq \ldots \leq H_k, \{[a_i,b_i]\})$ we have to postcompose with the diagonal $\Delta_{\{[r_k^l(a_i), r_k^l(b_i)]\}}$, where $k$ denotes the order of the kernel of $\psi$. The diagonal $\Delta_{\{[r_k^l(a_i), r_k^l(b_i)]\}}$ can be written as the composite

$$\Delta_{\{[\frac{l-1}{k}, \frac{l}{k}]\}} \circ \Delta_{\{[a_i,b_i]\}}.$$

A priori, this composite takes image in $\Omega(Sp^{|G|}(S^{\rho_G}))^G$, but since the intervals $(\frac{l-1}{k}, \frac{l}{k})$ are pairwise disjoint, the diagonal $\Delta_{\{[\frac{l-1}{k}, \frac{l}{k}]\}}$ factors as

$$\begin{tikzcd}
\Omega(Sp^{|K|}(S^{\rho_G}))^G \arrow[r, "\Delta_{\{[\frac{l-1}{k}, \frac{l}{k}]\}}"] \arrow[d, "l_k"'] & \Omega(Sp^{|G|}(S^{\rho_G}))^G \\
\Omega(Sp^{|K|}(S^{\rho_G}))^G \arrow[ur, "\Omega(i_{|K|}^{|G|})"'] &
\end{tikzcd}$$

where $l_k$ is the selfmap of $[0,1]/\{0,1\}$ which takes $x$ to $kx - \lfloor kx \rfloor$ (cf. page 69).

So, in summary, $\widetilde{\alpha}$ is natural for a different $\mathrm{Epi}^{op}$-functoriality on $\Omega\,\mathrm{sh}\,\Phi(Sp^n)$ that sends a surjection $\psi : G \twoheadrightarrow K$ to

$$\Omega\,\mathrm{sh}\,\Phi^K(Sp^n) \xrightarrow{\Omega\,\mathrm{sh}^\psi} \Omega\,\mathrm{sh}\,\Phi^G(Sp^n) \xrightarrow{l_k^*} \Omega\,\mathrm{sh}\,\Phi^G(Sp^n),$$

where $l_k^*$ is the self-map of $\Omega\,\mathrm{sh}\,\Phi^G(Sp^n)$ which precomposes each loop with $l_k$. This difference in the functoriality can be corrected: The self-maps

$$l_{|G|}^* : \Omega\,\mathrm{sh}\,\Phi^G(Sp^n) \to \Omega\,\mathrm{sh}\,\Phi^G(Sp^n)$$

assemble to a transformation from the usual $\mathrm{Epi}^{op}$-functor $\Omega\,\mathrm{sh}\,\Phi$ to the twisted one. Since $l_{|G|}$ induces multiplication with $|G|$ on homotopy, this transformation is a rational equivalence.

**Remark 6.4.1.** This 'defect' of $\widetilde{\alpha}$ can be explained: Recall from Remark 4.3.1 that on $\pi_0$ we should be sending a vertex $H$ of the subgroup lattice to the element $\frac{1}{[G:H]} \cdot [G/H]$ in the rationalized Burnside ring. But this is impossible, since $\alpha$ is geometrically defined to land in the not yet rationalized spectrum $\Omega\,\mathrm{sh}\,\Phi(Sp^n)$. Instead it sends $H$ to $|H| \cdot [G/H]$, which needs to be corrected by dividing by $|G|$ afterwards.

So we obtain a zig-zag of natural transformations of $\mathrm{Epi}^{op}$-functors

$$\Sigma^\infty|\widetilde{L}(-)_n| \xleftarrow{\simeq} \Sigma^\infty|\widetilde{L}_f(-)_n| \xrightarrow{\widetilde{\alpha}} (\Omega\,\mathrm{sh}\,\Phi(Sp^n))^{\text{twisted}} \xleftarrow{\simeq_{\mathbb{Q}}} \Omega\,\mathrm{sh}\,\Phi(Sp^n).$$

In order to prove Theorem 5.2.2 it now remains to show that for all $n$ and $G$ the map $\widetilde{\alpha}$ induces an isomorphism

$$H_*(|\widetilde{L}_f(G)_n|, \mathbb{Q})/G \xrightarrow{\cong} H_*(\Omega\,\mathrm{sh}\,\Phi^G(Sp^n), \mathbb{Q}).$$

Via an induction on $n$ and the five-lemma, this in turn can be reduced to showing that $\widetilde{\alpha}$ induces isomorphisms

$$H_*(|\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}|, \mathbb{Q})/G \xrightarrow{\cong} H_*(\Omega\,\mathrm{sh}\,\Phi^G(Sp^n/Sp^{n-1}), \mathbb{Q}), \tag{6.4.1}$$

and this is what we will do.

## 6.5 Rational splitting on subquotients

Our first aim is to show:

**Proposition 6.5.1.** *For all $n \in \mathbb{N}$ the map (6.4.1) above is split injective.*

We produce this splitting geometrically. Let $[(x_1, \ldots, x_n)] \in Sp^n(S^{k\cdot\rho_G})$ be a $G$-fixed point. Then the subset $\{x_1, \ldots, x_n\} \subseteq S^{k\cdot\rho_G}$ is closed under the $G$-action. Let $C^n$ be the subspectrum of $\Phi^G(Sp^n)$ consisting of those tuples for which this $G$-set is not transitive or contains less than $n$ elements. In particular, $C^n$ contains $\Phi(Sp^{n-1})$ and so we can consider the composite

$$\Sigma^\infty|\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}| \xrightarrow{\widetilde{\alpha}} \Omega\,\mathrm{sh}\,\Phi^G(Sp^n/Sp^{n-1}) \to \Omega\,\mathrm{sh}(\Phi^G(Sp^n)/C^n).$$

Our aim is to show that this composite induces an isomorphism

$$H_*(|\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}|, \mathbb{Q})/G \xrightarrow{\cong} H_*(\Omega \operatorname{sh}(\Phi^G(Sp^n)/C^n), \mathbb{Q}),$$

which proves Proposition 6.5.1.

Every non-basepoint element of $(\Phi^G(Sp^n)/C^n)_k$ is determined by any of its components $x_i \in S^{k\cdot\rho_G}$, and the isotropy of such a point is necessarily an index $n$ subgroup of $G$. Given an index $n$ subgroup $H$, we let $S(k, H)$ denote the space

$$(S^{k\cdot\rho_G})^H / \left(\operatorname*{colim}_{H \lneq K \leq G} (S^{k\cdot\rho_G})^K\right),$$

i.e., the $H$-fixed points of the $G$-space $S^{k\cdot\rho_G}$ modulo all fixed points of larger subgroups. The spaces $S(k, H)$ assemble to an orthogonal spectrum $S(H)$, with structure map sending $(x \wedge t) \in (S^{k\cdot\rho_G})^H \wedge S^1$ to the class of the element $(x + t \cdot e_G) \in S^{(k+1)\cdot\rho_G}$. There are morphisms

$$S(H) \to \Phi^G(Sp^n)/C^n$$

sending $x$ to $[(x, g_1 \cdot x, \ldots, g_{n-1} \cdot x)_{[g_i]\in G/H}]$. As we just argued, every element $x = [(x_1, \ldots, x_n)] \in S^{k\cdot\rho_G}$ lies in the image of one of these, by choosing a component $x_i$. Moreover, the choice of a different component amounts to multiplying with some element $g \in G$, since the $G$-set $\{x_1, \ldots, x_n\}$ is assumed to be transitive. So we find that there is an isomorphism of orthogonal spectra

$$\Phi^G(Sp^n)/C^n \cong \left(\bigvee_{H\leq G, [G:H]=n} S(H)\right)/G, \tag{6.5.1}$$

where the modded out $G$-action is given by translation, it sends $x \in (S^{k\cdot\rho_G})^H$ to $g \cdot x \in (S^{k\cdot\rho_G})^{gHg^{-1}}$.

**Remark 6.5.2.** This translation action should not be confused with the conjugation action on $\Phi^G(Sp^n)/C^n$ that comes from the $\mathrm{Epi}^{op}$-functoriality of geometric fixed points (as described in Section 5.2). The conjugation action sends an element $x \in (S^{k\cdot\rho_G})^H$ to $g \cdot x \cdot g^{-1} \in (S^{k\cdot\rho_G})^{gHg^{-1}}$, using that $\rho_G$ is both a left and a right module over $G$. Together the conjugation action and the translation action assemble to an action of the semi-direct product $G \ltimes G$ on $\bigvee S(H)$. Under the isomorphism (6.5.1) above, the conjugation action on $\Phi^G(Sp^n)/C^n$ is the induced one on translation orbits. To make clear which action we are talking about, we write $-/^t G$ for the quotient by the translation action and $-/^c G$ for the one by the conjugation action.

We again use the decomposition $\rho_G \cong \mathbb{R} \oplus \overline{\rho}_G$ to rewrite each $(S^{k\cdot\rho_G})^K$ as $S^k \wedge (S^{k\cdot\overline{\rho}_G})^K$. This induces a decomposition $S(k, H) \cong S^k \wedge \overline{S}(k, H)$ with $\overline{S}(k, H)$ defined similarly to $S(k, H)$, replacing each $\rho_G$ by $\overline{\rho}_G$. Through this identification, the structure map of $S(H)$ becomes the smash product of the associativity isomorphism $S^k \wedge S^1 \cong S^{k+1}$ and the closed inclusion $\overline{S}(k, H) \hookrightarrow \overline{S}(k+1, H)$. Hence, the inclusions $\overline{S}(k, H) \hookrightarrow$

$\overline{S}(\infty, H)$ assemble to a morphism of spectra

$$S(H) \to \Sigma^\infty \overline{S}(\infty, H). \tag{6.5.2}$$

By cofinality, this morphism induces an isomorphism on homotopy groups and is hence a stable equivalence.

We now turn to $\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}$, or rather its non-fat version $\widetilde{L}(G)_n/\widetilde{L}(G)_{n-1}$. Non-basepoint simplices in this quotient are given by chains which go from an index $n$ subgroup $H$ to $G$. So we find that there is an isomorphism

$$\widetilde{L}(G)_n/\widetilde{L}(G)_{n-1} \cong \bigvee_{\substack{H \le G \\ [G:H]=n}} \Big( L(G)^{[H,G]}/\{\text{non-max. chains}\} \Big), \tag{6.5.3}$$

where we write $L(G)^{[H,G]}$ for the poset of subgroups of $G$ which contain $H$ and the subcomplex {non-max. chains} is given by the chains that do not start in $H$ or do not end in $G$. The homotopy equivalence $L_f(G)_n^{[H,G]} \xrightarrow{\simeq} L(G)^{[H,G]}$ is split by the functor which sends a subgroup $K$ between $H$ and $G$ to itself together with the intervals $[0, \frac{1}{|H|}], [\frac{1}{|H|}, \frac{2}{|H|}], \ldots, [\frac{|H|-1}{|H|}, 1]$, each repeated $[K : H]$ times. Taking the wedge over these, we obtain a homotopy equivalence

$$|\widetilde{L}(G)_n/\widetilde{L}(G)_{n-1}| \cong \bigvee_{\substack{H \le G \\ [G:H]=n}} \Big( |L(G)^{[H,G]}|/\{\text{non-max. chains}\} \Big) \xrightarrow{\simeq} |\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}|.$$

Composing this equivalence with the morphism

$$\Sigma^\infty |\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}| \to \Omega\,\mathrm{sh}(\Phi(Sp^n)/C^n)$$

and using (6.5.1), (6.5.2) and (6.5.3), we obtain a morphism

$$\bigvee_{\substack{H \le G \\ [G:H]=n}} \Sigma^\infty \Big( L(G)^{[H,G]}/\{\text{non-max. chains}\} \Big) \to \Omega\,\mathrm{sh} \left( \left( \bigvee_{\substack{H \le G \\ [G:H]=n}} \Sigma^\infty \overline{S}(\infty, H) \right) /^t G \right). \tag{6.5.4}$$

It suffices to show that this morphism induces an isomorphism on rational homology, modulo conjugation in the domain. In formulas, (6.5.4) sends a chain $H_0 \le \ldots \le H_k$ (lieing between $H$ and $G$) together with coordinates $(t_0, \ldots, t_k) \in \Delta^k$ to the loop

$$[\widetilde{\sigma}^G((\alpha_2)_e(\{H_i\}, \{t_i\})) \circ l_{|H|}] \in \Omega(S^1 \wedge \overline{S}(\infty, H)). \tag{6.5.5}$$

Here, we have again used that the diagonal (in the sense of page 69) formed with respect to the intervals $[\frac{l-1}{|H|}, \frac{l}{|H|}]$ corresponds to precomposition with $l_{|H|}$, cf. Section 6.3. Multiplication by $l_{|H|}$ is a rational equivalence, and

$$\widetilde{\sigma}^G : \left(\Sigma^\infty \overline{S}(\infty, H)\right) /^t G \to \Omega\left(\left(\Sigma^\infty S^1 \wedge \overline{S}(\infty, H)\right) /^t G\right)$$

is a stable equivalence (as it agrees with the map $\widetilde{\lambda}_{(\Sigma^\infty \overline{S}(\infty,H))/^tG}$, cf. the beginning of Section 6). So we can leave both out of the formula. What remains is given by applying the suspension spectrum functor $\Sigma^\infty$ to a wedge of space level maps

$$\begin{aligned}\beta_H : L(G)^{[H,G]}/\{\text{non-max. chains}\} &\to \overline{S}(\infty, H)\\ (H_i, t_i) &\mapsto [(\alpha_2)_e(H_i, t_i)] = [p(\overline{r}(\sum_{i=0}^{k} t_i \cdot e_{H_i}))]\end{aligned}$$

followed by the projection to the translation $G$-orbits $\bigvee \overline{S}(\infty, H) \to \left(\bigvee \overline{S}(\infty, H)\right)/^tG$. We first show:

**Lemma 6.5.3.** *Each*

$$\beta_H : L(G)^{[H,G]}/\{\text{non-max. chains}\} \to \overline{S}(\infty, H)$$

*is a weak equivalence of spaces.*

*Proof.* If $H$ is equal to $G$ and hence $n = 1$, both sides consist of two points and $\beta_H$ is a bijection. So from now on we assume that $H$ is a proper subgroup of $G$ and consider the map

$$\begin{aligned}\overline{\beta}_H : L(G)^{[H,G]} &\to (S^{\infty\overline{\rho}_G})^H\\ (H_i, t_i) &\mapsto \overline{r}(\sum_{i=0}^{k} t_i \cdot e_{H_i}).\end{aligned} \tag{6.5.6}$$

The map $\beta_H$ in the statement of the lemma is obtained from $\overline{\beta}_H$ by quotiening out by all non-maximal chains in the domain and the subspace $A$ of the target given by all vectors which are either fixed by a larger subgroup than $H$ or have norm at least $\frac{1}{\sqrt{2}}$. Since both $L(G)^{[H,G]}$ and $(S^{\infty\overline{\rho}_G})^H$ are contractible and the inclusions of the respective subspaces are cofibrations, it suffices to show that $\overline{\beta}_H$ induces a weak equivalence between the complex of non-maximal chains and $A$. We note that the former is given by the pushout

$$L(G)^{(H,G]} \cup_{L(G)^{(H,G)}} L(G)^{[H,G)}$$

of half-closed respectively open subintervals of the subgroup lattice. The space $A$ can be expressed in a similar way: Let $A_1 \leq A$ be the subspace of vectors that have norm at most 1 and are fixed by a subgroup properly containing $H$, and $A_2 \leq A$ the subspace of all vectors of length at least $\frac{1}{\sqrt{2}}$. Together, the two cover $A$. Then $\overline{\beta}_H$ maps $L(G)^{(H,G]}$ into $A_1$ and, by the same proof as for Lemma 6.1.1, $L(G)^{[H,G)}$ into $A_2$. Since all of the spaces $L(G)^{(H,G]}, L(G)^{[H,G)}, A_1$ and $A_2$ are contractible (the latter two can be contracted onto 0 respectively $\infty$), we are left to show that $\overline{\beta}_H$ induces a weak equivalence from $L(G)^{(H,G)}$ to the intersection of $A_1$ and $A_2$. This intersection is the space of vectors of $(S^{\infty\overline{\rho}_G})^H$ that have norm in the interval $[\frac{1}{\sqrt{2}}, 1]$ and are fixed by a larger subgroup. It deformation retracts onto

$$S(G)^{(H,G)} := \operatorname*{colim}_{H \lneq K \lneq G} \left(S(\infty\overline{\rho}_G)^K\right).$$

We now consider the following commutative square:

$$\begin{tikzcd}
L(G)^{(H,G)} = \mathrm{colim}_{H\lneq K\lneq G}\left(L(G)^{[K,G)}\right) \arrow[r] & \mathrm{colim}_{H\lneq K\lneq G}\left(S(\infty\overline{\rho}_G)^K\right) = S(G)^{(H,G)} \\
\mathrm{hocolim}_{H\lneq K\lneq G}\left(L(G)^{[K,G)}\right) \arrow[r] \arrow[u] & \mathrm{hocolim}_{H\lneq K\lneq G}\left(S(\infty\overline{\rho}_G)^K\right) \arrow[u]
\end{tikzcd}$$

Both vertical maps are the respective canonical map from the homotopy colimit to the colimit, and the lower horizontal map is induced from the restriction of $\overline{\beta}_H$ to the intervals $[K,G)$. Since each $(S^{\infty\overline{\rho}_G})^K$ is contractible and so is $L(G)^{[K,G)}$, it follows by homotopy-invariance of homotopy colimits that the lower horizontal map is a weak equivalence. We claim that both vertical maps are also weak equivalences. For the left one, this can be seen by noting that it is split by the map

$$L(G)^{(H,G)} = \operatorname*{hocolim}_{H\lneq K\lneq G} * \to \operatorname*{hocolim}_{H\lneq K\lneq G}\left(L(G)^{[K,G)}\right)$$

induced from the inclusions $* \mapsto K \in (L(G)^{[K,G)})_0$. Again by homotopy-invariance of homotopy colimits, this splitting is a weak equivalence and hence so is the left vertical map. Finally, to derive that the right vertical map is a weak equivalence one can use a $G$-CW structure on $S^{\infty\overline{\rho}_G}$ to apply a general statement: For every $G$-cell complex $X$ the colimit over the $X^K$ with $K \in (H,G)$ is also a homotopy colimit. This can be seen by checking it for the orbits $G/L$ and using that cell attachments are both colimits and homotopy colimits. $\square$

Finally, we consider the following commutative diagram:

$$\begin{tikzcd}
H_*(\bigvee L(G)^{[H,G]}/\{\text{non-max. chains}\}, \mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & H_*(\bigvee \overline{S}(\infty,H), \mathbb{Q}) \arrow[d] \\
H_*(\bigvee L(G)^{[H,G]}/\{\text{non-max. chains}\}, \mathbb{Q})/^cG \arrow[r, "\cong"] \arrow[dr, "(*)"'] & H_*(\bigvee \overline{S}(\infty,H), \mathbb{Q})/^cG \arrow[d] \\
& H_*((\bigvee \overline{S}(\infty,H))/^tG, \mathbb{Q})
\end{tikzcd}$$

We want to show that the map $(*)$ is an isomorphism. The middle horizontal map is an isomorphism, since the wedge of the $\beta_H$ is equivariant for the conjugation actions. Furthermore, $\bigvee \overline{S}(\infty,H)$ is a cofibrant $G$-space under the translation action, so there is a natural isomorphism

$$H_*(\left(\bigvee \overline{S}(\infty,H)\right)/^tG, \mathbb{Q}) \cong H_*(\bigvee \overline{S}(\infty,H), \mathbb{Q})/^tG.$$

Hence it suffices to see that conjugation and translation induce the same action on rational homology. This comes out of the proof of Lemma 6.5.3 above: Up to equivalence we can replace $\bigvee \overline{S}(\infty,H)$ by the suspension of the unreduced suspension of

$$\bigvee_H \left(\operatorname*{hocolim}_{H\lneq K\lneq G}(S^{\infty\overline{\rho}_G})^K\right).$$

Collapsing all $(S^{\infty\overline{\rho}_G})^K$ to a point yields a weak equivalence to

$$\bigvee_H \left(\operatorname*{hocolim}_{H\lneq K\lneq G} *\right) = \bigvee_H L(G)^{(H,G)}.$$

This equivalence turns both the translation and the conjugation action on the $(S^{\infty\overline{\rho}_G})^K$ into the conjugation action on the wedge of subgroup intervals. So the two actions agree on homology, which finishes the proof of Proposition 6.5.1.

## 6.6 Abstract rational equivalence of the subquotients

We now show that there is an abstract rational equivalence

$$\Phi^G(Sp^n/Sp^{n-1}) \simeq_{\mathbb{Q}} \Sigma^\infty \left(|\widetilde{L}(G)_n/\widetilde{L}(G)_{n-1}|/G\right). \tag{6.6.1}$$

Since the latter is a finite complex, it has finite dimensional rational homology. So our map

$$H_*(|\widetilde{L}_f(G)_n/\widetilde{L}_f(G)_{n-1}|,\mathbb{Q})/G \xrightarrow{\widetilde{\alpha}_*} H_*(\Omega\,\mathrm{sh}\,\Phi^G(Sp^n/Sp^{n-1}),\mathbb{Q})$$

from the previous section must also be surjective, hence an isomorphism, proving Theorem 5.2.2.

To deduce the equivalence (6.6.1) we combine work of Arone [Aro15], Arone-Dwyer [AD01], Brantner [Bra16] and Schwede [Sch14]. In [Sch14, Proposition 1.11], Schwede showed that there is a global equivalence

$$Sp^n/Sp^{n-1} \simeq \Sigma^\infty (B_{gl}\mathcal{F}_n)^\diamond.$$

Here, $\mathcal{F}_n$ is the family of subgroups of $\Sigma_n$ that do not act transitively on $\underline{n} = \{1,\dots,n\}$ and $B_{gl}\mathcal{F}_n$ denotes a global classifying space in the sense of Section 2.2. Furthermore, the superscript $(-)^\diamond$ denotes the unreduced suspension of a space. We will recall Schwede's proof in Chapter 10.

The non-equivariant version of this statement was previously shown by Lesh in [Les00]. In [AD01, Section 7], also for the case $G = \{e\}$, Arone and Dwyer gave another description of this suspension spectrum, which we now mimic in the global equivariant context. For this we denote by $\Pi_n$ the $\Sigma_n$-poset of non-trivial proper partitions of the set $\underline{n}$, and by $E_{gl}\Sigma_n$ a global universal space for $\Sigma_n$. Our aim is to show:

**Theorem 6.6.1.** *The based orthogonal spaces $(B_{gl}\mathcal{F}_n)^\diamond$ and $(E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)$, i.e., the global homotopy orbits of $|\Pi_n|^\diamond \wedge S^n$, are globally equivalent after one suspension. Hence, there is a global equivalence*

$$Sp^n/Sp^{n-1} \simeq \Sigma^\infty \left((E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)\right).$$

Here, $\Sigma_n$ acts on $S^n$ by permuting the coordinates. This statement is even true in the full global category, i.e., with respect to all compact Lie groups.

The following pages are devoted to proving Theorem 6.6.1. The arguments are global equivariant adaptions of the arguments in [AD01, Section 7]. Let $\mathrm{Sing}(E_{gl}\mathcal{F}_n)$ denote the sub-orthogonal space of the global universal space $E_{gl}\mathcal{F}_n$ consisting of the points with non-trivial $\Sigma_n$-isotropy. We note that $\mathrm{Sing}(E_{gl}\mathcal{F}_n)$ is a global universal space for $\mathcal{F}_n^\circ$, the collection of non-transitive subgroups minus the trivial subgroup. Then there is a cofiber sequence

$$E_G\Sigma_{n+} \wedge (\mathrm{Sing}(E_G\mathcal{F}_n))^\diamond \to \mathrm{Sing}(E_G\mathcal{F}_n)^\diamond \to (E_G\Sigma_n * \mathrm{Sing}(E_G\mathcal{F}_n))^\diamond \simeq (E_G\mathcal{F}_n)^\diamond$$

of $\Sigma_n$-orthogonal spaces, where $*$ denotes the join. Smashing with the $\Sigma_n$-equivariant diagonal inclusion $i : S^1 \hookrightarrow S^n$ yields a commutative diagram

$$\begin{tikzcd}
E_{gl}\Sigma_{n+} \wedge \mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^1 \arrow[r] \arrow[d] & \mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^1 \arrow[r] \arrow[d] & (E_{gl}\mathcal{F}_n)^\diamond \wedge S^1 \arrow[d] \\
E_{gl}\Sigma_{n+} \wedge \mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^n \arrow[r] & \mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^n \arrow[r] & (E_{gl}\mathcal{F}_n)^\diamond \wedge S^n.
\end{tikzcd}$$

We have:

**Lemma 6.6.2.** *The map*

$$(E_{gl}\mathcal{F}_n)^\diamond \wedge i : (E_{gl}\mathcal{F}_n)^\diamond \wedge S^1 \to (E_{gl}\mathcal{F}_n)^\diamond \wedge S^n$$

*is a based* $(\Sigma_n \times G)$*-homotopy equivalence when evaluated on any complete* $G$*-universe* $\mathcal{U}_G$*.*

**Lemma 6.6.3.** *The quotient* $(\mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^n)/\Sigma_n$ *is based globally contractible.*

The proofs of these lemmas are given below. Hence, dividing out the $\Sigma_n$-action in the lower cofiber sequence yields a cofiber sequence of orthogonal spaces, which by Lemma 6.6.3 exhibits $((E_{gl}\mathcal{F}_n)^\diamond \wedge S^n)/\Sigma_n$ as the suspension of $(E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (\mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^n)$. By Lemma 6.6.2, the former is globally equivalent to $(B_{gl}\mathcal{F}_n)^\diamond \wedge S^1$, so we obtain:

**Corollary 6.6.4.** *There are global equivalences*

$$\begin{aligned}
(B_{gl}\mathcal{F}_n)^\diamond \wedge S^1 &\simeq ((E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (\mathrm{Sing}(E_{gl}\mathcal{F}_n)^\diamond \wedge S^n)) \wedge S^1 \\
&\simeq ((E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} ((E_{gl}\mathcal{F}_n^\circ)^\diamond \wedge S^n)) \wedge S^1.
\end{aligned}$$

*Proof of Lemma 6.6.2.* Since both sides are cofibrant based $(\Sigma_n \times G)$-spaces when evaluated on $\mathcal{U}_G$, it suffices to show that $((E_{gl}\mathcal{F}_n)^\diamond \wedge i)(\mathcal{U}_G)$ induces a weak equivalence on all fixed point spaces. Let $H$ be a subgroup of $\Sigma_n \times G$. If the intersection $H \cap (\Sigma_n \times 1)$ acts non-transitively on $\underline{n}$, the $H$-fixed points of $(E_{gl}\mathcal{F}_n)^\diamond(\mathcal{U}_G)$ are contractible and hence the map is necessarily a weak equivalence. If $H \cap (\Sigma_n \times 1)$ does act transitively, the inclusion $i^H : S^1 \to (S^n)^H$ is even a homeomorphism (in fact for this it would suffice that the projection of $H$ to $\Sigma_n$ acts transitively). So $((E_{gl}\mathcal{F}_n)^\diamond \wedge i)(\mathcal{U}_G)$ induces a weak equivalence on fixed points for all subgroups of $\Sigma_n \times G$, which finishes the proof. □

*Proof of Lemma 6.6.3.* For this we make use of the specific model for $E_{gl}\mathcal{F}_n$ that comes out of [Sch14, Proposition 1.11] (cf. Chapter 10). It is given by the $\Sigma_n$-orthogonal space $S(\overline{\mathbb{R}^n} \otimes -)$, the unit sphere in the tensor product with the reduced natural $\Sigma_n$-representation. What we need from this model is the property that all the $\Sigma_n$-isotropy lies in complete subgroups, i.e., subgroups of $\Sigma_n$ that are conjugate to one of the form $\Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$ with all $n_i > 0$, $\sum n_i = n$ and $k > 1$. Hence, the isotropy of $\mathrm{Sing}(S(\overline{\mathbb{R}^n} \otimes -))$ lies in non-trivial complete subgroups.

We now evaluate on a complete $G$-universe $\mathcal{U}_G$ and prove that more generally, the quotient $(X \wedge S^n)/\Sigma_n$ is based $G$-contractible for any based cofibrant $(\Sigma_n \times G)$-space with all $\Sigma_n$-isotropy non-trivial and complete, or possibly the whole group $\Sigma_n$ (which we have to include because of the cone points of $\mathrm{Sing}(E_{gl}\Pi_n)^\diamond$). Without loss of generality we can assume that $X$ is a $(\Sigma_n \times G)$-cell complex. Via induction over the cells and passing to the sequential colimit we can reduce to showing that

$$((\Sigma_n \times G)/H_+ \wedge A \wedge S^n)/\Sigma_n$$

is based $G$-weakly contractible for any space $A$ with trivial $(\Sigma_n \times G)$-action and $H \leq \Sigma_n \times G$ a subgroup for which $H \cap (\Sigma_n \times 1)$ is non-trivial and complete or equal to $\Sigma_n$. We denote this intersection by $H'$ and the image of $H$ under the projection to $G$ by $K$. For every $k \in K$ we choose an element $\psi(k) \in \Sigma_n$ such that $(\psi(k), k)$ lies in $H$. This property uniquely characterizes $\psi(k)$ up to multiplication with an element in $H'$, and every $\psi(k)$ automatically lies in the normalizer of $H'$. Altogether, $k \mapsto [\psi(k)]$ defines a homomorphism $\overline{\psi} : K \to W_{\Sigma_n} H'$ into the Weyl group. Then there is a $G$-homeomorphism

$$((\Sigma_n \times G)/H_+ \wedge A \wedge S^n)/\Sigma_n \cong G \ltimes_K (A \wedge (S^n/H')),$$

with $K$ acting on $S^n/H'$ via restriction along $\overline{\psi}$. So it suffices to see that $S^n/H'$ is $(W_{\Sigma_n} H')$-equivariantly contractible. Up to conjugacy, $H'$ is of the form $\Sigma_{n_1}^{\times i_1} \times \ldots \times \Sigma_{n_k}^{\times i_k}$ with all $n_j$ pairwise different and $\sum (i_j \cdot n_j) = n$. The Weyl group is given by $\Sigma_{i_1} \times \ldots \times \Sigma_{i_k}$. Then, $S^n/H'$ is homeomorphic to

$$(S^{n_1}/\Sigma_{i_1})^{\wedge i_1} \wedge \ldots \wedge (S^{n_k}/\Sigma_{n_k})^{\wedge i_k},$$

with the Weyl group permuting the smash factors in each $(S^{n_j}/\Sigma_{n_j})^{\wedge i_j}$. By [AD01, Lemma 7.10], $S^{n_j}/\Sigma_{n_j}$ is contractible whenever $n_j$ is greater than 1. Since $H'$ is non-trivial, this has to be the case for some $n_j$, which finishes the proof. □

**Remark 6.6.5.** A more conceptual way to phrase the first part of the above proof would be to say that the global universal space for the collection of complete subgroups is $\Sigma_n$-globally equivalent to the global universal space for the family of non-transitive subgroups $E_{gl}\mathcal{F}_n$. This follows directly from the fact that $S(\overline{\mathbb{R}^n} \otimes -)$ is a global universal space for both, but could also be proved along the lines of [AD01, Lemma 4.3].

In Corollary 6.6.4 one can further simplify $E_{gl}\mathcal{F}_n^\circ$. For this we think of $\mathcal{F}_n^\circ$ as a $\Sigma_n$-poset, ordered by inclusion, and let $|\mathcal{F}_n^\circ|$ denote its nerve (or the associated constant $\Sigma_n$-orthogonal space).

**Lemma 6.6.6.** *There is a zig-zag of morphisms of $\Sigma_n$-orthogonal spaces from $E_{gl}\mathcal{F}_n^\circ$*

*to* $|\mathcal{F}_n^\circ|$ *which induces a global equivalence*

$$E_{gl}\Sigma_n \times_{\Sigma_n} E_{gl}\mathcal{F}_n^\circ \xrightarrow{\simeq} E_{gl}\Sigma_n \times_{\Sigma_n} |\mathcal{F}_n^\circ|,$$

*and hence also a global equivalence*

$$(E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} ((E_{gl}\mathcal{F}_n^\circ)^\diamond \wedge S^n)) \xrightarrow{\simeq} (E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (|\mathcal{F}_n^\circ|^\diamond \wedge S^n)$$

*Proof.* This is a general fact about global universal spaces for collections of subgroups, as we discussed in Section 2.2.2. □

Finally we relate $\mathcal{F}_n^\circ$ to the partition poset $\Pi_n$. There is an equivariant map of posets $j : \Pi_n \to \mathcal{F}_n^\circ$ which sends a partition $\underline{n} = M_1 \sqcup \ldots \sqcup M_k$ to the associated non-transitive subgroup $\Sigma_{M_1} \times \ldots \times \Sigma_{M_k}$.

**Lemma 6.6.7.** *The map* $j : \Pi_n \to \mathcal{F}_n^\circ$ *induces a* $\Sigma_n$*-weak equivalence after applying the nerve.*

*Proof.* An equivariant left adjoint is given by the map of posets that associates to every non-transitive and non-trivial subgroup $H \leq \Sigma_n$ the partition of $\underline{n}$ into $H$-orbits. □

Combining these two lemmas with Corollary 6.6.4, we see that $(B_{gl}\mathcal{F}_n)^\diamond$ is globally equivalent to $(E_{gl}\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)$, after one suspension. This finishes the proof of Theorem 6.6.1.

In particular, the underlying $G$-homotopy type of $Sp^n/Sp^{n-1}$ is given by the suspension spectrum

$$\Sigma^\infty((E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)).$$

Since there are natural isomorphisms $\Phi^G(\Sigma^\infty X) \cong \Sigma^\infty X^G$ for based $G$-spaces $X$, Theorem 6.6.1 reduces the computation of $\Phi^G(Sp^n/Sp^{n-1})$ to the computation of the $G$-fixed points of the $G$-space $(E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)$. By Corollary 2.3.2, these are given by

$$((E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n))^G \cong \bigvee_{(\alpha:G\to\Sigma_n)} EC(\alpha)_+ \wedge_{C(\alpha)} \left((|\Pi_n|^\diamond \wedge S^n)^{\mathrm{im}(\alpha)}\right).$$

Each $\alpha : G \to \Sigma_n$ defines a $G$-set structure on $\underline{n}$. The centralizer $C(\alpha)$ is given by the automorphisms of that $G$-set, and the fixed points $(S^n)^{\mathrm{im}(\alpha)}$ are a sphere of dimension the number of its $G$-orbits. So, written in a more coordinate free way, we obtain a homeomorphism

$$((E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n))^G \cong \bigvee_{\substack{G\text{-sets } M\\ |M|=n}} E\mathrm{Aut}_G(M)_+ \wedge_{\mathrm{Aut}_G(M)} \left(|\Pi_M^G|^\diamond \wedge S^{M/G}\right),$$

where the wedge is taken over isomorphism classes of $G$-sets of order $n$. The fixed point sets $|\Pi_M^G|$ of partition posets that appear here have been studied by Arone [Aro15] and Brantner [Bra16]. We can make use of their results to see that a lot of the wedge summands are (rationally) contractible, simplifying the expression for $((E_G\Sigma_n)_+ \wedge_{\Sigma_n}$

$(|\Pi_n|^\diamond \wedge S^n))^G$. A finite $G$-set is called *isotypical* if the isotropy groups of all of its elements are conjugate.

**Proposition 6.6.8** (Fixed points of partition posets)**.** *We have:*

*(i) If $M$ is not isotypical, the fixed points $|\Pi_M^G|$ are contractible.*

*(ii) If $M$ is isotypical but not transitive, the space*

$$(E\mathrm{Aut}_G(M))_+ \wedge_{\mathrm{Aut}_G(M)} \left(|\Pi_M^G|^\diamond \wedge S^{M/G}\right)$$

*is rationally contractible.*

*(iii) If $M = G/H$ for a subgroup $H$ of $G$, then $\Pi_M^G$ is isomorphic to $L(G)^{(H,G)}$.*

*Proof.* $(i)$: This is [Aro15, Lemma 7.1].

$(ii)$: Let $M = \bigsqcup_m G/H$ with $m \geq 2$. Then the automorphism group of $M$ is given by the wreath product $\Sigma_m \wr W_G(H)$. By [Aro15, Proposition 9.1], there is a $W_G(H)^m$-equivariant map

$$W_G(H)^m \ltimes_{W_G(H)} (|\Pi_m| * |\Pi_{G/H}^G|^\diamond) \xrightarrow{\simeq} |\Pi_M^G| \tag{6.6.2}$$

that is a non-equivariant equivalence, where $W_G(H)$ sits inside $W_G(H)^m$ diagonally. Direct inspection of its definition shows that the adjoint $(|\Pi_m|*|\Pi_{G/H}^G|^\diamond) \to |\Pi_M^G|$ (basically given by the cartesian product of partitions) is not only equivariant over the Weyl group, but also over the symmetric group $\Sigma_m$, if one lets it act on $\Pi_M$ by permuting the $m$ copies of $G/H$. In other words, we can also think of (6.6.2) as a $(\Sigma_m \wr W_G(H))$-equivariant map

$$(\Sigma_m \wr W_G(H)) \ltimes_{\Sigma_m \times W_G(H)} (|\Pi_m| * |\Pi_{G/H}^G|^\diamond) \xrightarrow{\simeq} |\Pi_M^G|$$

that is a non-equivariant equivalence. Hence, we find that

$$E\,\mathrm{Aut}(M)_+ \wedge_{\mathrm{Aut}(M)} \left(|\Pi_M^G|^\diamond \wedge S^{M/G}\right)$$

is weakly equivalent to

$$((E\Sigma_m)_+ \wedge_{\Sigma_m} (|\Pi_m|^\diamond \wedge S^m)) \wedge \left(EW_G(H)_+ \wedge_{W_G(H)} |\Pi_{G/H}^G|^\diamond\right) \wedge S^1.$$

Here, we used that

$$|\Pi_m| * |\Pi_{G/H}^G|^\diamond \simeq |\Pi_m|^\diamond \wedge |\Pi_{G/H}^G|^\diamond,$$

as described in [Aro15, Lemma 2.5]. So it suffices to note that $(E\Sigma_m)_+ \wedge_{\Sigma_m} (|\Pi_m|^\diamond \wedge S^m)$ is rationally contractible for $m \geq 2$. For this in turn it is enough to see that the strict quotient $(|\Pi_m|^\diamond \wedge S^m)/\Sigma_m$ is contractible. Via an induction over the cells we can again reduce to showing that $S^m/K$ is contractible for any subgroup $K \leq \Sigma_m$ which appears as an isotropy group of $|\Pi_m|^\diamond$. The isotropy of the cone points are given by the full $\Sigma_m$ and we know that $S^m/\Sigma_m$ is contractible. All other isotropy groups are given by those of $|\Pi_m|$, so let $\mathcal{P} = \mathcal{P}_0 \subseteq \ldots \subseteq \mathcal{P}_k$ be a chain of non-trivial proper partitions of $\underline{m}$. Then the subgroup $K_0$ of $\Sigma_m$ of those elements which fix $\mathcal{P}_0$ strongly (i.e., which fix every set in the partition) is complete. The subgroup $K_0$ also fixes all larger partitions

$\mathcal{P}_i$ and is hence a subgroup of the isotropy group $\mathrm{Iso}(\mathcal{P})$ of the chain. Moreover, $\mathrm{Iso}(\mathcal{P})$ is contained in the normalizer of $K_0$, which is the isotropy of the 0-simplex $\mathcal{P}_0$. In the proof of Lemma 6.6.3 we saw that the quotient $S^m/K_0$ is equivariantly contractible over the Weyl group of $K_0$, hence also over the subgroup $\mathrm{Iso}(\mathcal{P})/K_0$. It follows that

$$S^m/\,\mathrm{Iso}(\mathcal{P}) \cong (S^m/K_0)/(\mathrm{Iso}(\mathcal{P})/K_0)$$

is contractible. which finishes the proof of item $(ii)$.

$(iii)$ : This follows from the fact that a $G$-fixed partition of $G/H$ is determined by its summand containing $H/H$ and that this summand has to be of the form $K/H$ for a subgroup $H \lneq K \lneq G$ (cf. [Aro15, Lemma 7.2]). $\square$

So we see that there is a rational equivalence

$$((E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n))^G \simeq_{\mathbb{Q}} \bigvee_{\substack{(H\leq G)\\ [G:H]=n}} \Big(EW_G(H)_+ \wedge_{W_G(H)} (|L(G)^{(H,G)}|^\diamond \wedge S^1)\Big)\,.$$

We claim that the right hand side is rationally equivalent to $|\widetilde{L}_n(G)/\widetilde{L}_{n-1}(G)|/G$. In fact, we already saw in the proof of Lemma 6.5.3 that $|\widetilde{L}_n(G)/\widetilde{L}_{n-1}(G)|$ is isomorphic to the wedge over all index $n$ subgroups $H$ of the spaces

$$|L(G)^{[H,G]}|/\left(|L(G)^{[H,G)}| \cup_{|L(G)^{(H,G)}|} |L(G)^{(H,G]}|\right) \simeq |L(G)^{(H,G)}|^\diamond \wedge S^1.$$

After taking $G$-orbits, we can equivalently form the wedge over representatives of conjugacy classes of such subgroups $H$, and quotient each summand by the Weyl-group action. Since orbits and homotopy orbits are rationally equivalent for finite groups, the claim follows.

So together with Theorem 6.6.1 and the fact that geometric fixed points commute with suspension spectra, this shows that there is a rational equivalence

$$\Phi^G(Sp^n/Sp^{n-1}) \simeq_{\mathbb{Q}} \Sigma^\infty(|\widetilde{L}(G)_n/\widetilde{L}(G)_{n-1}|/G),$$

which finishes the proof of Theorem 5.2.2.

### 6.6.1 The $\mathcal{F}in$-global Steenrod algebra

We digress a little to show another application of Theorem 6.6.1, since it follows almost directly from what we have seen. We define the $\mathcal{F}in$-global Steenrod algebra as the graded endomorphisms of the Eilenberg-MacLane spectrum $H\underline{\mathbb{Z}}$ (for the constant $\mathcal{F}in$-global functor $\underline{\mathbb{Z}}$) in the $\mathcal{F}in$-global homotopy category. As we recalled in Section 3.2.2, a model for $H\underline{\mathbb{Z}}$ is given by the infinite symmetric product $Sp^\infty$.

**Theorem 6.6.9.** *The $\mathcal{F}in$-global Steenrod algebra is a single copy of $\mathbb{Z}$ concentrated in degree 0.*

In other words, there are no global operations of positive degree on the cohomology theory represented by $H\underline{\mathbb{Z}}$ (and none of negative degree, but this is easier to see). The theorem also implies that more generally there are isomorphisms

$$[H\underline{A}, H\underline{B}]_*^{gl} \cong \mathrm{Ext}_{\mathbb{Z}}^{-*}(A, B)$$

for all finitely generated abelian groups $A$ and $B$. In particular, the mod-$p$ $\mathcal{F}in$-global Steenrod algebra is two-dimensional, generated by the identity and the Bockstein.

The $C_p$-equivariant mod-$p$ Steenrod algebra was studied by Caruso in [Car99]. He showed that the forgetful map to the non-equivariant Steenrod algebra only hits the identity and the Bockstein, but that there is a large kernel given by shifted copies of the cohomology of $BC_p$. Caruso's result already implies that none of the higher Steenrod squares can lift to the $\mathcal{F}in$-global category. By Theorem 6.6.9, the exotic $C_p$-equivariant operations which become trivial non-equivariantly are not part of a global family.

*Proof of Theorem 6.6.9.* We claim that the map $\mathbb{S} \to Sp^\infty$ induces an isomorphism

$$[Sp^\infty, Sp^\infty]_*^{\mathrm{gl}} \cong [\mathbb{S}, Sp^\infty]_*^{\mathrm{gl}},$$

which proves the theorem since the non-equivariant homotopy of $Sp^\infty$ are a copy of $\mathbb{Z}$ concentrated in degree 0. For this it suffices to see that the graded morphism groups $[Sp^n/Sp^{n-1}, Sp^\infty]_*^{\mathrm{gl}}$ are trivial for all $n \geq 2$. Using Theorem 6.6.1 and the adjunction mentioned in Section 3.1.4, we find that these agree with

$$[\Sigma^\infty((E_G\Sigma_n)_+ \wedge_{\Sigma_n} (|\Pi_n|^\diamond \wedge S^n)), Sp^\infty]_*^{\mathrm{gl}} \cong [\Sigma^\infty(|\Pi_n|^\diamond \wedge S^n), Sp^\infty_{\Sigma_n}]_*^{\Sigma_n}.$$

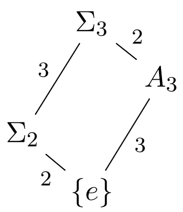
The latter term is naturally isomorphic to the Bredon cohomology of the $\Sigma_n$-space $|\Pi_n|^\diamond \wedge S^n$ with coefficients in the constant $\Sigma_n$-Mackey functor $\underline{\mathbb{Z}}$, or in other words the ordinary cohomology of the quotient $(|\Pi_n|^\diamond \wedge S^n)/\Sigma_n$. But, as we saw in the proof of item $(ii)$ of Proposition 6.6.8, this quotient is contractible for $n \geq 2$, which finishes the proof. $\square$

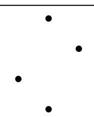
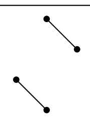
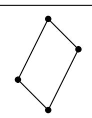
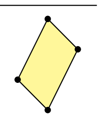
# Chapter 7

# Examples

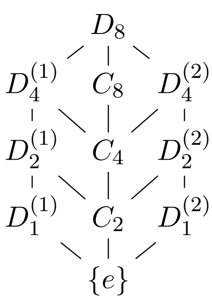
In this section we go through some examples, where we describe the respective filtrations on subgroup lattices and read off the rational equivariant homotopy groups of the symmetric products. In all the cases we discuss, the quotient $L(G)/G$ is isomorphic to the nerve of the poset of conjugacy classes of subgroups of $G$ (while in general there is only a natural surjection from the former to the latter).

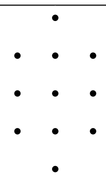
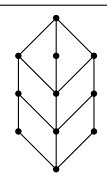
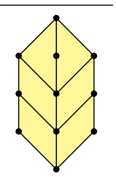
**Example 7.0.10** (Symmetric group $\Sigma_3$)**.** We start with the symmetric group on 3 letters. On the left we depict the subgroup lattice modulo conjugation, on the right the filtration by the $L(\Sigma_3)_n/\Sigma_3$ and the resulting $\Sigma_3$-homotopy groups of the symmetric products.




| $n$ | 1 | 2 | $3-5$ | $\geq 6$ |
|---|---|---|---|---|
| $L(\Sigma_3)_n/\Sigma_3$ |  |  |  |  |
| $\pi_1^{\Sigma_3}(Sp^n)\otimes\mathbb{Q}$ | 0 | 0 | $\mathbb{Q}$ | 0 |
| $\pi_0^{\Sigma_3}(Sp^n)\otimes\mathbb{Q}$ | $\mathbb{Q}^4$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

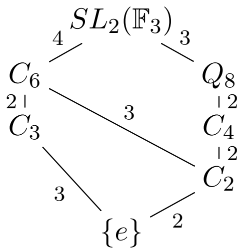
**Example 7.0.11** (Dihedral group $D_8$)**.** For the dihedral group with 16 elements the filtration stabilizes, up to homotopy, at $n = 4$. All minimal subgroup inclusions are of index 2.



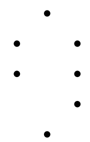
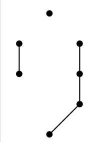
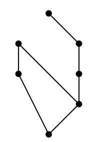
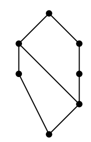
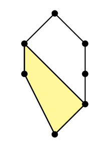
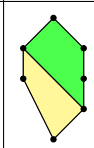

| $n$ | 1 | $2,3$ | $\geq 4$ |
|---|---|---|---|
| $L(D_8)_n/D_8$ |  |  |  |
| $\pi_1^{D_8}(Sp^n)\otimes\mathbb{Q}$ | 0 | $\mathbb{Q}^6$ | 0 |
| $\pi_0^{D_8}(Sp^n)\otimes\mathbb{Q}$ | $\mathbb{Q}^{11}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

**Example 7.0.12** ($SL_2(\mathbb{F}_3)$)**.** The figure below depicts the filtration for the special linear group $SL_2(\mathbb{F}_3)$, the semi-direct product of the quaternion group $Q_8$ with the cyclic

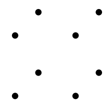
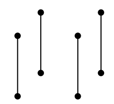
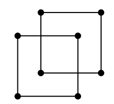
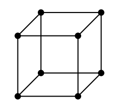
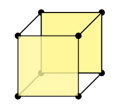
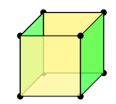
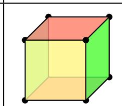
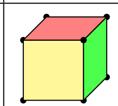
group $C_3$. The subgroup lattice modulo conjugation is given by



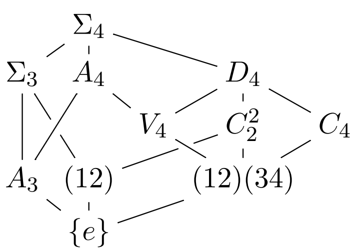

and the filtration works out as:

| $n$ | 1 | 2 | 3 | 4, 5 | 6 – 11 | ≥ 12 |
|---|---|---|---|---|---|---|
| $L(SL_2(\mathbb{F}_3))_n/SL_2(\mathbb{F}_3)$ |  |  |  |  |  |  |
| $\pi_1^{SL_2(\mathbb{F}_3)}(Sp^n)\otimes\mathbb{Q}$ | 0 | 0 | $\mathbb{Q}$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | 0 |
| $\pi_0^{SL_2(\mathbb{F}_3)}(Sp^n)\otimes\mathbb{Q}$ | $\mathbb{Q}^7$ | $\mathbb{Q}^3$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

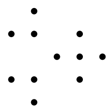
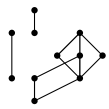
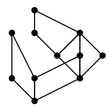
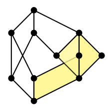
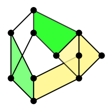
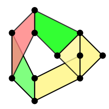
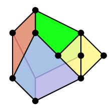
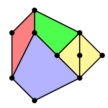
**Example 7.0.13** (Cyclic groups)**.** For cyclic groups $C_m$, the subgroups correspond to divisors of $m$ ordered by divisibility. If $m$ is the product of $k$ different primes, the subgroup lattice is a $k$-dimensional cube. In particular, the subcomplex $L(C_m)_{m-1}$ is the boundary of the cube and hence isomorphic to $S^{k-1}$. This shows that for every $k \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ and a finite group $G$ such that $\pi_k^G(Sp^n)\otimes\mathbb{Q}$ is non-trivial. Below is the 3-dimensional cube for the example $m = 30 = 2\cdot 3\cdot 5$, where the elements at the top are those divisible by 2, the ones on the right those divisible by 3 and the ones at the back those divisible by 5.

| 1 | 2 | 3, 4 | 5 | 6 – 9 | 10 – 14 | 15 – 29 | ≥ 30 |
|---|---|---|---|---|---|---|---|
|  |  |  |  |  |  |  |  |
| 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | 0 |
| 0 | 0 | $\mathbb{Q}^2$ | $\mathbb{Q}^5$ | $\mathbb{Q}^3$ | $\mathbb{Q}$ | 0 | 0 |
| $\mathbb{Q}^8$ | $\mathbb{Q}^4$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

**Example 7.0.14** (Symmetric group $\Sigma_4$)**.** The last example we draw explicitly is for the symmetric group on four letters. The subgroup lattice modulo conjugation is depicted below:



For example, one can imagine all vertices to lie in the plane, except for the one associated to a transposition (12), which is placed below the rest. The indices of the subgroup inclusions are left out for better readability. The filtration then works out as follows:

| 1 | 2 | 3 | $4,5$ | $6,7$ | $8-11$ | $12-23$ | $\geq 24$ |
|---|---|---|---|---|---|---|---|
|  |  |  |  |  |  |  |  |
| 0 | 0 | 0 | 0 | 0 | 0 | $\mathbb{Q}$ | 0 |
| 0 | $\mathbb{Q}^3$ | $\mathbb{Q}^5$ | $\mathbb{Q}^4$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | 0 | 0 |
| $\mathbb{Q}^{11}$ | $\mathbb{Q}^3$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $\mathbb{Q}$ |

We close this section with a few general remarks on the complexes of the form $L(G)_{|G|-1}$, the last non-trivial stage in the filtration. They can be identified with the unreduced suspension of the lattice $L(G)^{(1,G)}$ of proper non-trivial subgroups of $G$, about which there are various results in the literature. For example, in [KT85] it is shown that if $G$ is solvable, then $L(G)^{(1,G)}$ is homotopy-equivalent to a wedge of spheres of dimension two less than the chief length $c(G)$ of $G$. By [Thé85], the top homology

$$H_{c(G)-2}(L(G)^{(1,G)}, \mathbb{Z})$$

is a permutation representation under the conjugation $G$-action. The quotient

$$H_{c(G)-2}(L(G)^{(1,G)}, \mathbb{Z})/G$$

is still acted on by the outer automorphism group $\mathrm{Out}(G)$ of $G$. As we saw (Theorem 4.3.2), after tensoring with $\mathbb{Q}$ this corresponds to the action of $\mathrm{Out}(G)$ on

$$\pi^G_{c(G)-1}(Sp^{|G|-1}) \otimes \mathbb{Q}$$

which is part of the structure of a global functor. These actions can be interesting representation-theoretically:

**Example 7.0.15** ($(\mathbb{Z}/p)^n$ and the Steinberg module)**.** When $G = (\mathbb{Z}/p)^n$, the complex $L(G)^{(1,G)}$ is the Tits building for $\mathrm{Out}(G) = GL_n(\mathbb{F}_p)$. So, by a theorem of Solomon ([Sol69]), its homology

$$H_{n-2}(L(G)^{(1,G)}, \mathbb{Q})$$

(and hence also $\pi^G_{n-1}(Sp^{p^n-1}) \otimes \mathbb{Q}$) is isomorphic to the rational Steinberg module, a distinguished irreducible $GL_n(\mathbb{F}_p)$-representation of dimension $p^{\frac{n(n-1)}{2}}$. For example, when $p = n = 2$, the Steinberg module is the reduced natural representation of $\Sigma_3 \cong GL_2(\mathbb{F}_2)$.

A different relation between symmetric products of spheres and the Steinberg module - over $\mathbb{F}_p$ instead of $\mathbb{Q}$ - plays a major role in [AD01].

# Chapter 8

# Global properties of $Sp^n_{\mathbb{Q}}$

In this final section of Part II we describe homological properties of the $\mathrm{Out}^{op}$-complex models $\mathcal{C}\widetilde{L}_n$ for the rational symmetric products. We first show that they are degreewise projective and then use this to prove that for $1 < n < \infty$ they are not formal, i.e., not quasi-isomorphic to their homology with trivial differential. As a consequence, the rationalization $Sp^n_{\mathbb{Q}}$ is not a product of Eilenberg-MacLane spectra for any $n$ except 1 and $\infty$. This is a truly global phenomenon, since over a fixed finite group $G$ every rational $G$-spectrum is determined by its Mackey functor homotopy groups. Finally, we give a proof that the cyclic $p$-groups are the only groups for which $\pi_*^G(Sp^n)\otimes\mathbb{Q}$ is always concentrated in degree 0.

**Proposition 8.0.16.** *Each $\mathcal{C}\widetilde{L}_n$ is degreewise projective as a $\mathbb{Q}[\mathrm{Out}^{op}]$-module.*

*Proof.* We need the following notion: A chain of subgroup inclusions $H_0 \leq \ldots \leq H_k$ is called *simple* if $H_0$ does not contain a non-trivial normal subgroup of $H_k$, i.e., if this chain cannot be obtained via pull-back along a surjective group homomorphism that is not an isomorphism.

Now recall that the $k$-th level of $\mathcal{C}\widetilde{L}_n(G)$ is given by the $\mathbb{Q}$-linearization of the set of conjugacy classes of chains of proper subgroup inclusions which end in $G$, are of length $k$ and have total index at most $n$. The map

$$\left(\coprod_{H_0\lneq\ldots\lneq H_k \text{ simple},[H_k:H_0]\leq n} \mathrm{Epi}(G,H_k)\right)/\mathrm{iso} \to \{k\text{-chains of index} \leq n \text{ in } G\}/\mathrm{conj}$$
$$(\psi : G \twoheadrightarrow H_k) \mapsto [\psi^{-1}(H_0) \lneq \ldots \lneq \psi^{-1}(H_k) = G]$$

defines a natural bijection. On the left hand side, two pairs

$$(H_0 \lneq \ldots \lneq H_k; \psi : G \twoheadrightarrow H_k)$$

and

$$(H_0' \lneq \ldots \lneq H_k'; \psi' : G \twoheadrightarrow H_k')$$

are considered isomorphic if there exists an isomorphism $\varphi : H_k \xrightarrow{\cong} H_k'$ that takes the first chain to the second and which satisfies $\varphi\circ\psi = \psi'$. For the inverse map one associates to a $k$-chain $H_0 \lneq \ldots \lneq H_k = G$ the simple $k$-chain

$$H_0/H \lneq \ldots \lneq H_k/H = G/H$$

together with the projection $G \twoheadrightarrow G/H$, where $H$ is the intersection of all $G$-conjugates of $H_0$, or in other words the maximal subgroup of $H_0$ that is normal in $G$. Hence, we find that there is an isomorphism of $\mathrm{Out}^{op}$-modules

$$(\mathcal{C}\widetilde{L}_n)_k \cong \bigoplus_{[H_0 \lneq \ldots \lneq H_k \text{ simple}, [H_k : H_0] \leq n]} \mathbb{Q}[\mathrm{Out}(-, H_k)] / \mathrm{Out}(H_0 \lneq \ldots \lneq H_k), \qquad (8.0.1)$$

with $\mathrm{Out}(H_0 \lneq \ldots \lneq H_k)$ denoting the group of conjugacy classes of automorphisms of $H_k$ which map the chain $H_0 \lneq \ldots \lneq H_k$ to a conjugate of itself. The functors $\mathbb{Q}[\mathrm{Out}(-, H_k)]$ are by definition representable, hence projective. Furthermore, the orbits of a projective functor under any action of a finite group $K$ are again projective, since the projection is split by the map $[x] \mapsto \frac{1}{|K|} \sum_{k \in K} (k \cdot x)$. This finishes the proof. □

The proof also applies to $n = \infty$ and hence $\mathcal{C}\widetilde{L}$ itself, showing that it gives a projective resolution of the $\mathrm{Out}^{op}$-module that sends the trivial group to $\mathbb{Q}$ and all other finite groups to 0. Using once more that the functor $\tau$ of Section 5.1 is an equivalence, this shows that $\mathcal{C}L$ is a projective resolution of the constant global functor $\underline{\mathbb{Q}}$.

We can use our algebraic model to see:

**Theorem 8.0.17.** *The only $n$ for which the rationalization $Sp^n_{\mathbb{Q}}$ is a product of global Eilenberg-MacLane spectra are 1 and $\infty$.*

*Proof.* Using the equivalence between the rational global stable homotopy category and the derived category of rational $\mathrm{Out}^{op}$-modules (Theorem 5.2.1) together with Theorem 5.2.2, the statement is equivalent to the $\mathrm{Out}^{op}$-complex $\mathcal{C}\widetilde{L}_n$ not being quasi-isomorphic to its homology with trivial differential.

Each $\mathcal{C}\widetilde{L}_n$ with $n < \infty$ is concentrated in finitely many degrees

$$0, \ldots, \lfloor \log_2(n) \rfloor = a(n).$$

We show that if $n > 1$, the highest possible $k$-invariant is non-trivial, i.e., the map

$$\Sigma^{a_n} H_{a(n)}(\mathcal{C}\widetilde{L}_n) \to \mathcal{C}\widetilde{L}_n$$

does not have a section in the derived category. Since $\mathcal{C}\widetilde{L}_n$ is degreewise projective, this is equivalent to the inclusion

$$H_{a(n)}(\mathcal{C}\widetilde{L}_n) \hookrightarrow (\mathcal{C}\widetilde{L}_n)_{a(n)}$$

of $\mathrm{Out}^{op}$-modules not having a section. In fact we claim that any $\mathrm{Out}^{op}$-map

$$(\mathcal{C}\widetilde{L}_n)_{a(n)} \to H_{a(n)}(\mathcal{C}\widetilde{L}_n)$$

is necessarily zero on all abelian groups. To see this we use the decomposition (8.0.1) of

$(\mathcal{C}\widetilde{L}_n)_{a(n)}$ above. When restricted to abelian groups, only the summands associated to chains $K_0 \lneq \ldots \lneq K_{a(n)}$ for which $K_{a(n)}$ is abelian play a role, since there is no surjective map from an abelian group to a non-abelian one. In the abelian case, the simpleness implies that $K_0$ is the trivial group, and so the order of $K_{a(n)}$ is at most $n$. So, over abelian groups, $(\mathcal{C}\widetilde{L}_n)_{a(n)}$ is a quotient of a direct sum of representables for groups $K$ of order at most $n$. It now suffices to see that any map from these representables to $H_{a(n)}(\mathcal{C}\widetilde{L}_n)$ is trivial. By the Yoneda Lemma, such maps correspond to homology classes $H_{a(n)}(\mathcal{C}\widetilde{L}_n)(K)$. Since $\widetilde{L}(K)_n$ is contractible for groups $K$ of order $\leq n$ (unless $K$ is the trivial group, which can only appear if $a(n) = 0$ and hence $n = 1$), this homology is trivial. This proves the claim.

Hence it suffices to show that there exists an abelian group $G$ for which $H_{a(n)}(\mathcal{C}\widetilde{L}_n)(G)$ is non-trivial. By Example 7.0.15, such a $G$ is given by $(\mathbb{Z}/2)^{a(n)+1}$. □

Using similar arguments for other classes of groups, one can show the non-vanishing of many more $k$-invariants of $\mathcal{C}\widetilde{L}_n$. Finally, we have:

**Proposition 8.0.18.** *For a finite group $G$ the following are equivalent:*

*(i) For all $n \in \mathbb{N}$ the graded vector space $\pi_*^G(Sp^n) \otimes \mathbb{Q}$ is concentrated in degree 0.*

*(ii) $G \cong C_{p^n}$ for some prime $p$ and $n \in \mathbb{N}$.*

*Proof.* If $G \cong C_{p^n}$, the subgroup lattice of $G$ is linear, and it is not hard to see that all subcomplexes $L(G)_n$ are either discrete (for $n < p$) or contractible. So, by Theorem 4.3.2, $\pi_*^G(Sp^n) \otimes \mathbb{Q}$ is concentrated in degree 0.

For the other direction we fix a finite group $G$ that is not cyclic of prime power order. It suffices to show that some $\Phi_k^G(Sp^n) \otimes \mathbb{Q}$ with $k > 0$ is non-trivial, since it is a quotient of $\pi_k^G(Sp^n) \otimes \mathbb{Q}$. We now see that $k$ can in fact always be taken to be 1. For this we choose two non-conjugate maximal subgroups $H$ and $H'$ of $G$, which is possible since $G$ is not cyclic of prime power order. For example, their existence follows from the fact that the union of all conjugates of a proper subgroup can never be all of $G$. Let $n$ denote the index of $H$ in $G$, which we can without loss of generality assume to be at least as large as that of $H'$. Then the formal difference

$$[H \leq G] - [H' \leq G] \in \mathcal{C}\widetilde{L}_n(G)_1$$

is a non-trivial 1-cycle. Since any proper subgroup of $H$ has index larger than $n$ in $G$ and $G$ is the only subgroup containing $H$, there are no non-degenerate 2-simplices of $\widetilde{L}(G)_n/G$ that have $H \leq G$ or any of its conjugates as a face. Hence, $[H \leq G] - [H' \leq G]$ cannot be a boundary and thus defines a non-trivial element in

$$H_1(\mathcal{C}\widetilde{L}_n(G)) \cong \Phi_1^G(Sp^n) \otimes \mathbb{Q}.$$

This finishes the proof. □

# Part III

# Filtrations of global $K$-theory I: Subquotients

# Chapter 9

# Rank and complexity filtrations

In this part we give a definition of the modified rank and complexity filtration in the global context and then determine the global homotopy type of the filtration subquotients of these filtrations. The results are joint work with Dominik Ostermayr.

## 9.1 Arone and Lesh's construction

We recall Arone and Lesh's non-equivariant construction of the rank and complexity filtrations as well as some of their results.

In [AL07], the authors start with a permutative category $\mathcal{C}$ which is augmented over the permutative category $\mathbb{N}$, i.e., the category whose objects are the natural numbers, whose morphisms are only the identities and whose monoidal structure is addition. Their main examples for $\mathcal{C}$ are given by the (topological) category of finite dimensional real or complex vector spaces, the category of finitely generated free $R$-modules over a discrete ring $R$ satisfying dimension invariance and the category of finite sets.

Out of this data the authors construct a sequence of permutative categories

$$\mathcal{C} = \mathcal{K}_0\mathcal{C} \to \mathcal{K}_1\mathcal{C} \to \mathcal{K}_2\mathcal{C} \to \ldots \to \mathcal{K}_\infty\mathcal{C} \simeq \mathbb{N}$$

by inductively 'killing the bottom non-trivial component' via a homotopy pushout in permutative categories. Through Segal's delooping machine (i.e., first forming the $\Gamma$-space associated to a permutative category and then realizing to spectra), this yields a sequence of spectra

$$k\mathcal{C} = A_0^{\mathcal{C}} \to A_1^{\mathcal{C}} \to A_2^{\mathcal{C}} \to \ldots \to A_\infty^{\mathcal{C}} \simeq H\mathbb{Z}. \tag{9.1.1}$$

The authors did not attach a name to this construction. We call it the *complexity filtration for* $\mathcal{C}$, based on the usage of that term in [Les00]. For $\mathcal{C}$ the category of finite sets, the complexity filtration gives back the symmetric product filtration of Section 3.2.2 (as shown in [AL07, Corollary 8.4]). Hence, the authors suggest to view the filtrations (9.1.1) as $k\mathcal{C}$-analogs of symmetric products. They back this claim up by showing that many formal properties of symmetric products carry over to general complexity filtrations. In particular, the authors prove that the quotients are suspension spectra and determine their homotopy type.

In a later paper [AL10] Arone and Lesh show that there is a different - and perhaps simpler - route to obtain the spectra $A_n^{\mathcal{C}}$, which involves a filtration of the $K$-theory spectra $k\mathcal{C}$ themselves, the so-called *modified rank filtration.* It is constructed as follows: The evaluation of the $\Gamma$-space associated to $\mathbb{N}$ is isomorphic to $Sp^\infty$, because the infinite symmetric product of a based set/space can be identified with its reduced $\mathbb{N}$-linearization. Then the $n$-th level $k\mathcal{C}^n$ in the modified rank filtration is defined as the pullback

$$\begin{tikzcd} k\mathcal{C}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & k\mathcal{C} \arrow[d] \\ Sp^n \arrow[r] & Sp^\infty. \end{tikzcd} \tag{9.1.2}$$

In other words, the modified rank filtration is obtained by pulling back the symmetric product filtration along the augmentation. Here, it is important to note that one forms the point-set pullback and not a homotopy pullback.

**Remark 9.1.1.** The assignment

$$\underline{k}_+ \mapsto Sp^n(\underline{k}_+)$$

yields a $\Gamma$-space whose spectrum realization is isomorphic to $Sp^n$. The modified rank filtration can alternatively be defined by forming the (again strict) pullback as above in the category of $\Gamma$-spaces and then realizing to a spectrum. This yields the same result because at the level of $\Gamma$-spaces the pullback amounts to picking out certain components.

As Arone and Lesh show, one can get back the complexity filtration by forming the homotopy pushout in Diagram (9.1.2) above. For simplicity, let us assume that the preimage of $1 \in \mathbb{N}$ under the augmentation of $\mathcal{C}$ consists of a single isomorphism class. This is the case in all our examples. Then there exists an essentially unique functor of augmented permutative categories from the category of finite sets to $\mathcal{C}$. Hence, the complexity filtration of $\mathcal{C}$ receives a map from the complexity filtration for the $K$-theory of finite sets, which is equivalent to the symmetric product filtration by [AL07, Corollary 8.4]. Hence one can consider the diagram

$$\begin{tikzcd} k\mathcal{C}^n \arrow[r] \arrow[d] & k\mathcal{C} \arrow[d] \\ Sp^n \arrow[r] & A_n^{\mathcal{C}}. \end{tikzcd} \tag{9.1.3}$$

**Theorem 9.1.2** ([AL10, Theorem 4.4])**.** *The square* (9.1.3) *is homotopy cocartesian.*

Note that if Diagram (9.1.2) above was a homotopy pullback, $A_n^{\mathcal{C}}$ and $Sp^\infty$ would be equivalent by stability, which is usually not the case.

In the case of topological $K$-theory, Arone and Lesh also examine the modified rank filtration itself and show that the subquotients again turn out to be suspension spectra, related to the lattice of decompositions of finite dimensional complex vector spaces as an orthogonal sum of subspaces. We generalize this to the global equivariant context in Chapter 11.

**Remark 9.1.3** (Relation to Rognes' stable rank filtration)**.** In [Rog92], Rognes constructs a different spectrum level rank filtration of connective $K$-theory and uses it to prove that the $K$-group $K_4(\mathbb{Z})$ is trivial. Arone and Lesh show that there exists a natural map from their modified rank filtration to Rognes' stable rank filtration [AL10, Section 4.2]. For topological $K$-theory (complex and real) this map is an equivalence, and Arone and Lesh use this to prove a conjecture by Rognes on the connectivity of the subquotients in his rank filtration. For algebraic $K$-theory, the comparison map is in general not an equivalence and the two filtrations are different.

We do not know whether there exists a global version of Rognes' stable rank filtration.

## 9.2 Global versions

In this thesis we follow the route of the later paper [AL10] to construct global generalizations of the modified rank and complexity filtrations. For now we let $kX$ denote either one of the orthogonal spectra $ku, ko, k\mathcal{F}in$ of Sections 3.3 and 3.4 or the symmetric spectrum $kR$ for a ring satisfying dimension invariance of Section 3.6. We recall that each $kX$ comes with a natural morphism to $Sp^\infty$.

**Definition 9.2.1** (Global modified rank filtration)**.** The $n$-th level of the modified rank filtration for $kX$ is defined as the strict pullback

$$\begin{tikzcd} kX^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & kX \arrow[d] \\ Sp^n \arrow[r] & Sp^\infty. \end{tikzcd}$$

Again, the pullback could also be performed on the level of orthogonal $\Gamma$-spaces (respectively $\mathbf{I}$-$\Gamma$-spaces), where it picks out certain components. In the geometric picture of Remark 3.3.2, a labeled configuration $[(x_1, W_1), \ldots, (x_l, W_l)]$ - with all $x_i$ non-basepoint elements of some sphere $S^V$ - lies in $ku^n(V)$ if and only if the sum of the dimensions of the $W_i$ is at most $n$. Similar descriptions work for $ko^n$, $k\mathcal{F}in^n$ and $kR^n$.

Inspired by Theorem 9.1.2 above, we further define:

**Definition 9.2.2** (Global complexity filtration)**.** The $n$-th term $A_n^X$ of the complexity filtration for $kX$ is defined as the homotopy pushout

$$\begin{tikzcd} kX^n \arrow[r] \arrow[d] & kX \arrow[d] \\ Sp^n \arrow[r] \arrow[ur, phantom, "\ulcorner^h", very near end] & A_n^X. \end{tikzcd}$$

Concretely, we let $A_n^X$ be the spectrum

$$([0,1]_+ \wedge Sp^n) \vee_{kX^n} kX,$$

where the embedding $Sp^n \to [0,1]_+ \wedge Sp^n$ is via the endpoint 1. The inclusions $kX^{n-1} \hookrightarrow kX^n$ and $Sp^{n-1} \hookrightarrow Sp^n$ induce comparison morphisms $A^X_{n-1} \xrightarrow{p_{n-1}} A^X_n$ and give rise to the complexity filtration

$$kX \cong A^X_0 \xrightarrow{p_0} A^X_1 \xrightarrow{p_1} \ldots \to A^X_\infty \simeq Sp^\infty.$$

Since the morphism $kX^n \to kX$ is always a levelwise cofibration (cf. Appendix A.1), $A^X_n$ could also be defined as the strict pushout. We use the mapping cylinder construction to ensure that the induced maps $A^X_n \xrightarrow{p_n} A^X_{n+1}$ are level-cofibrations. This will make it easier to describe the filtration quotients.

# Chapter 10

# Rank filtration of $H\mathbb{Z}$/Symmetric product filtration

We now begin to study the subquotients in the modified rank and complexity filtrations. Our first example is the symmetric product filtration, or in other words the rank filtration for the $K$-theory of the permutative category $\mathbb{N}$. As Arone and Lesh show in [AL07], the symmetric product filtration is also equivalent to the complexity filtration for the $K$-theory of finite sets. This is also true globally, as follows from the results of Chapter 12.

The symmetric product filtration has been much studied non-equivariantly, in particular the homotopy type of the filtration quotients was determined by Lesh in [Les00]. As explained in Section 3.2.3, Schwede examined the behavior of the global $Sp^n$ on $\underline{\pi}_0$ in [Sch14], for which he also gave a description of the global filtration quotients $Sp^n/Sp^{n-1}$. This description already appeared in Section 6.6. In this section we recall his proof, because the symmetric products are the universal example of a rank filtration and because we need it later to describe subquotients in complexity filtrations.

Recall that the $V$-th level of the orthogonal spectrum $Sp^n$ is the $n$-th symmetric product $Sp^n(V)$ of the sphere $S^V$. So the quotient $Sp^n/Sp^{n-1}$ is given by

$$(Sp^n/Sp^{n-1})(V) = (S^V)^{\wedge n}/\Sigma_n,$$

or in other words the $\Sigma_n$-orbit space of the one-point compactification of the $(\Sigma_n \times O(V))$-representation $\mathbb{R}^n \otimes V$. The natural $\Sigma_n$-representation $\mathbb{R}^n$ decomposes as

$$\mathbb{R}^n = \overline{\mathbb{R}^n} \oplus \mathbb{R},$$

where $\overline{\mathbb{R}^n}$ is the reduced natural representation of vectors whose entries sum to zero and $\mathbb{R}$ the trivial representation which sits inside $\mathbb{R}^n$ via

$$t \mapsto t \cdot (\frac{1}{\sqrt{n}} \sum_{i=1}^{n} e_i).$$

Using this decomposition, we see that

$$(Sp^n/Sp^{n-1})(V) \cong \left(S^{\overline{\mathbb{R}^n}\otimes V}/\Sigma_n\right) \wedge S^V \cong \left(S(\overline{\mathbb{R}^n}\otimes V)^\diamond/\Sigma_n\right) \wedge S^V.$$

Here, $S(-)$ denotes the sphere of vectors of length one in an inner product space and the notation $X^\diamond$ stands for the unreduced suspension of a space $X$ equipped with the basepoint $X \times 1$. Moreover, under this homeomorphism the structure map

$$(Sp^n/Sp^{n-1})(V) \wedge S^{W-\varphi(V)} \to (Sp^n/Sp^{n-1})(W)$$

associated to a linear isometric embedding $\varphi : V \hookrightarrow W$ corresponds to the smash product of the inclusion

$$S(\overline{\mathbb{R}^n}\otimes V)^\diamond/\Sigma_n \xrightarrow{\varphi_*} S(\overline{\mathbb{R}^n}\otimes W)^\diamond/\Sigma_n$$

and the homeomorphism $S^V \wedge S^{W-\varphi(V)} \cong S^W$ induced by $\varphi$. In other words, $Sp^n/Sp^{n-1}$ is isomorphic (!) to the suspension spectrum of the based orthogonal space

$$V \mapsto S(\overline{\mathbb{R}^n}\otimes V)^\diamond/\Sigma_n.$$

Let $\mathcal{T}_n$ denote the collection of *non-transitive subgroups* of $\Sigma_n$, i.e., those subgroups whose tautological action on the set $\underline{n}$ is not transitive. Further we denote by $\mathcal{C}_n^\Sigma$ the collection of *complete subgroups* of $\Sigma_n$, i.e., those conjugate to one of the form $\Sigma_{n_1} \times \Sigma_{n_2} \times \ldots \times \Sigma_{n_k}$ with $n_1 + n_2 + \ldots + n_k = n$, all $n_i \geq 1$ and $k > 1$. We note that a subgroup of $\Sigma_n$ is non-transitive if and only if it is contained in a complete subgroup.

**Proposition 10.0.3.** *The $\Sigma_n$-orthogonal space $S(\overline{\mathbb{R}^n}\otimes -)$ is a global universal space for both $\mathcal{T}_n$ and $\mathcal{C}_n^\Sigma$.*

The notion of a global universal space for a collection is explained in Section 2.2.

*Proof.* Clearly, all structure maps $S(\overline{\mathbb{R}^n}\otimes V) \to S(\overline{\mathbb{R}^n}\otimes W)$ are closed inclusions. Now let $G$ be a compact Lie group and $\mathcal{U}_G$ a complete $G$-universe. As a consequence of a theorem of Illman (cf. [Ill83]), all $(\Sigma_n \times G)$-spheres $S(\overline{\mathbb{R}^n}\otimes V)$ are $(\Sigma_n \times G)$-cofibrant.

We are now going to show that all $\Sigma_n$-isotropy of $S(\overline{\mathbb{R}^n}\otimes \mathcal{U}_G)$ lies in complete subgroups and that the fixed points for a subgroup $H$ of $\Sigma_n \times G$ are contractible whenever the intersection $H \cap (\Sigma_n \times 1)$ is non-transitive, implying universality for both collections. An element of $\overline{\mathbb{R}^n}\otimes \mathcal{U}_G$ can be represented by an $n$-tuple $(v_1, \ldots, v_n)$ of vectors of $\mathcal{U}_G$ which sum up to zero, with $\Sigma_n$ acting by permuting the coordinates. Every such element defines a partition of the set $\underline{n}$ by the equivalence relation that $i \sim j$ if $v_i = v_j$. Let the equivalence classes be denoted by $A_1, \ldots, A_k$. Then a permutation in $\Sigma_n$ fixes the element $(v_1, \ldots, v_n)$ if and only if it maps each $A_i$ into itself, i.e., if and only if it lies in the subgroup $\Sigma(A_1) \times \ldots \times \Sigma(A_k)$. Since the $v_i$'s add up to zero and are of total length one, they cannot all be the same and hence $k$ is greater than 1 and the isotropy subgroup is complete.

Now let $H$ be an element of $\mathcal{C}_n^\Sigma\langle G\rangle$, i.e., a subgroup of $\Sigma_n \times G$ whose intersection

$K := H \cap (\Sigma_n \times 1)$ acts non-transitively. There is a short exact sequence of groups

$$1 \to K \to H \to \mathrm{pr}_G(H) \to 1,$$

where $\mathrm{pr}_G(H)$ denotes the image of $H$ under the projection to $G$. Hence, the $H$-fixed points of $S(\overline{\mathbb{R}^n} \otimes \mathcal{U}_G)$ equal the $\mathrm{pr}_G(H)$-fixed points of

$$S(\overline{\mathbb{R}^n} \otimes \mathcal{U}_G)^K = S((\overline{\mathbb{R}^n})^K \otimes \mathcal{U}_G) = S(\overline{\mathbb{R}^{n/K}} \otimes \mathcal{U}_G).$$

The $\mathrm{pr}_G(H)$-action on the latter representation is the tensor product of the action on $\overline{\mathbb{R}^{n/K}}$ induced from the short exact sequence above and the restricted action on $\mathcal{U}_G$. Since $\mathcal{U}_G$ is a complete $\mathrm{pr}_G(H)$-universe, it in particular contains an infinite direct sum of copies of the dual of $\overline{\mathbb{R}^{n/K}}$ (which is again isomorphic to $\overline{\mathbb{R}^{n/K}}$). The tensor product of any finite dimensional representation with its dual always contains a trivial representation, so it follows that the $\mathrm{pr}_H(G)$-fixed points of $S(\overline{\mathbb{R}^{n/K}} \otimes \mathcal{U}_G)$ are a unit sphere in an infinite dimensional vector space and hence contractible. So we are done. $\square$

Thus one obtains:

**Theorem 10.0.4** ([Sch14, Proposition 1.11])**.** *The quotient $Sp^n/Sp^{n-1}$ is globally equivalent (in fact, isomorphic) to the suspension spectrum of the unreduced suspension of a global classifying space for both the collection of non-transitive and complete subgroups of $\Sigma_n$, or in short:*

$$Sp^n/Sp^{n-1} \simeq \Sigma^\infty (B_{gl}\mathcal{T}_n)^\diamond \simeq \Sigma^\infty (B_{gl}\mathcal{C}_n^\Sigma)^\diamond$$

# Chapter 11

# Filtrations associated to global topological $K$-theory

In this section we describe the global homotopy type of the filtration subquotients of the modified rank and complexity filtration for connective global topological $K$-theory $ku$ and $ko$.

## 11.1 Quotients in the modified rank filtration

We recall from Section 3.3 that $ku^n$ is given by the realization of the orthogonal $\Gamma$-space

$$(V, A_+) \mapsto \coprod_{(n_a)_{a\in A}, \sum n_a \leq n} L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, \mathrm{Sym}(V_{\mathbb{C}}))/\prod_{a\in A} U(n_a),$$

or in short $(V, A_+) \mapsto ku^n(\mathrm{Sym}(V_{\mathbb{C}}), A)$.

In Appendix A.1, we argue that the maps $ku^{n-1} \to ku^n$ are levelwise equivariant cofibrations, so we can consider the strict quotient $ku^n/ku^{n-1}$ as a model for the homotopy cofiber. The first step consists of rewriting the quotient $\Gamma$-spaces in a slightly different way. For a complex vector space $W$ and a finite set $A$ the space $ku^n(W, A_+)/ku^{n-1}(W, A_+)$ is given by

$$\bigvee_{(n_a)_{a\in A}, \Sigma n_a = n} \left( L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, W)/\prod_{a\in A} U(n_a) \right)_+ .$$

Using that composition defines a homeomorphism

$$L_{\mathbb{C}}(\mathbb{C}^n, W)_+ \wedge_{U(n)} L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, \mathbb{C}^n)_+ \xrightarrow{\cong} L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, W)_+,$$

we obtain that this quotient is isomorphic to

$$L_{\mathbb{C}}(\mathbb{C}^n, W)_+ \wedge_{U(n)} \left( \bigvee_{(n_a)_{a\in A}, \Sigma n_a = n} \left( L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, \mathbb{C}^n)/\prod_{a\in A} U(n_a) \right)_+ \right).$$

Applying this to $W = \mathrm{Sym}(V_{\mathbb{C}})$, we see that we have rewritten the orthogonal $\Gamma$-space

$$ku^n(\mathrm{Sym}(-_{\mathbb{C}}), -)/ku^{n-1}(\mathrm{Sym}(-_{\mathbb{C}}), -)$$

as a balanced smash product of two parts. The first – $L_{\mathbb{C}}(\mathbb{C}^n, \mathrm{Sym}(-_{\mathbb{C}}))_+$– is constant in the $\Gamma$-space direction and equals the $U(n)$-orthogonal space $L(\mathbb{C}^n)$ that appeared in Section 3.3. As we argued there, it is a global universal space for $U(n)$.

The second smash factor is constant in the orthogonal space direction. It is given by the $U(n)$-$\Gamma$-space of decompositions of $\mathbb{C}^n$ into a direct sum of orthogonal sub-vector spaces. We give it the shorter notation

$$\mathcal{L}(n, A_+) = \bigvee_{(n_a)_{a\in A}, \Sigma n_a = n} \left( L_{\mathbb{C}}(\bigoplus_{a\in A} \mathbb{C}^{n_a}, \mathbb{C}^n)/\prod_{a\in A} U(n_a) \right)_+ .$$

We consider the evaluation of this $U(n)$-$\Gamma$-space on a representation sphere $S^V$ for some compact Lie group $G$ and $G$-representation $V$. Every element is represented by a tuple $(W_i, x_i)_{i\in I}$ for some finite indexing set $I$, where the $W_i$ form an orthogonal decomposition of $\mathbb{C}^n$ into complex subspaces and the $x_i$ are elements of $S^V$. (Again, this representative becomes unique up to a change of labels if we require all the $x_i$ to be distinct elements of $V$ and all the $W_i$ to be non-zero.) The action of $U(n)$ is through the partition, that of $G$ through the points $x_i$.

Sitting inside $\mathcal{L}(n, S^V)$ we have a copy of $S^V$ (with trivial $U(n)$-action) as elements of the form $(\mathbb{C}^n, x)$. In fact we can mimic the construction in Chapter 10 for the symmetric products to see that $S^V$ splits off $(U(n) \times G)$-equivariantly as a smash factor. The other factor is given by the subspace of $\mathcal{L}(n, S^V)$ consisting of elements represented by tuples $(W_i, x_i)_{i\in I}$ (with $x_i \in V$) that satisfy the relation

$$\sum_{i\in I} (\dim(W_i) \cdot x_i) = 0$$

as elements of $V$ (this property is independent of the representing tuple). We denote this 'reduced' subspace by $\overline{\mathcal{L}(n, S^V)}$. Then the $(U(n) \times G)$-homeomorphism

$$\mathcal{L}(n, S^V) \xrightarrow{\cong} \overline{\mathcal{L}(n, S^V)} \wedge S^V$$

is given by $[(W_i, x_i)_{i\in I}] \mapsto [(W_i, x_i - \overline{x})] \wedge \overline{x}$, where $\overline{x} = \frac{1}{n} \sum_{i\in I} (\dim(W_i) \cdot x_i)$.

Under this identification, for a linear isometric embedding $\varphi : V \hookrightarrow W$ the structure map

$$\overline{\mathcal{L}(n, S^V)} \wedge S^V \wedge S^{W-\varphi(V)} \to \overline{\mathcal{L}(n, S^W)} \wedge S^W$$

becomes the smash product of the inclusion

$$\overline{\mathcal{L}(n, S^V)} \xrightarrow{\varphi_*} \overline{\mathcal{L}(n, S^{V\oplus V'})}$$

with the homeomorphism $S^V \wedge S^{W-\varphi(V)} \cong S^W$ induced by $\varphi$. Thus we see that the orthogonal spectrum realization of $\mathcal{L}(n, -)$ (with induced $U(n)$-action) is isomorphic to the suspension spectrum of the based $U(n)$-orthogonal space sending $V$ to $\overline{\mathcal{L}(n, S^V)}$.

Finally, like in the symmetric product filtration, this based orthogonal space is itself the unreduced suspension of the subspace of 'norm 1 elements': Let $\overline{\mathcal{L}_{|.|=1}(n, S^V)}$ be the subspace of $\overline{\mathcal{L}(n, S^V)}$ consisting of those elements that are represented by a tuple $(W_i, x_i)_{i\in I}$ (with $x_i \in V$) satisfying the relation $\sum_{i=1}^m (\dim(W_i)|x_i|^2) = 1$. Then there is a $(U(n) \times G)$-homeomorphism

$$\begin{aligned} \overline{\mathcal{L}(n, S^V)} &\to \overline{\mathcal{L}_{|.|=1}(n, S^V)}^{\diamond} \\ [(W_i, x_i)_{i\in I}] &\mapsto ([(W_i, \frac{1}{|x|} \cdot x_i)_{i\in I}], \frac{|x|}{1+|x|}), \end{aligned}$$

where $|x| = \sqrt{\sum_{i=1}^m \dim(W_i)|x_i|^2}$ and the image of the basepoint is understood to be the endpoint at 1. Hence we obtain:

**Corollary 11.1.1.** *The quotient $ku^n/ku^{n-1}$ is isomorphic to the suspension spectrum of the based orthogonal space*

$$L(\mathbb{C}^n)_+ \wedge_{U(n)} (\overline{\mathcal{L}_{|.|=1}(n, S^-)}^{\diamond}).$$

It remains to determine the global homotopy type of the $U(n)$-orthogonal space $\overline{\mathcal{L}_{|.|=1}(n, S^-)}$, which we from now on abbreviate by $\overline{\mathcal{L}}_n$. We introduce two collections of subgroups of $U(n)$:

**Definition 11.1.2.** A subgroup of $U(n)$ is called

- *complete* if it is conjugate to one of the form $U(n_1) \times \ldots \times U(n_k)$ with each $n_i$ positive, $n_1 + \ldots + n_k = n$ and $k > 1$.
- *non-isotypical* if its tautological action on $\mathbb{C}^n$ is not isotypical, i.e., if $\mathbb{C}^n$ is not the direct sum of copies of one irreducible representation.

The collection of complete subgroups is denoted by $\mathcal{C}_n^u$, that of non-isotypical subgroups by $\mathcal{I}_n^u$.

We note that to every (unordered) decomposition $\mathbb{C}^n = \bigoplus_{i\in I} W_i$ into at least two pairwise orthogonal non-trivial subspaces we can associate a complete subgroup $\prod_{i\in I} U(W_i)$ of $U(n)$. This assignment is bijective, the inverse maps a complete subgroup to the decomposition of $\mathbb{C}^n$ into the isotypical components of its action. Note also that every complete subgroup is non-isotypical. Then we have:

**Proposition 11.1.3.** *The $U(n)$-orthogonal space $\overline{\mathcal{L}}_n$ is a global universal space for both $\mathcal{C}_n^u$ and $\mathcal{I}_n^u$.*

*Proof.* Let $G$ be a compact Lie group and $\mathcal{U}_G$ a complete $G$-universe. In Appendix A.2 it is proved that $\overline{\mathcal{L}}_n(\mathcal{U}_G)$ is $(U(n)\times G)$-cofibrant. We now show that all the $U(n)$-isotropy of $\overline{\mathcal{L}}_n(\mathcal{U}_G)$ lies in complete subgroups and that the $H$-fixed points are contractible whenever $H$ lies in $\mathcal{I}_n^u\langle G\rangle$. Since complete subgroups are non-isotypical, this implies universality for both collections. Any point $x$ in $\overline{\mathcal{L}}_n(\mathcal{U}_G)$ is represented by a tuple $(W_i, x_i)_{i\in I}$ satisfying the relations $\sum_{i\in I} \dim(W_i) \cdot x_i = 0$ and $\sum_{i\in I} \dim(W_i)|x_i|^2 = 1$. Without loss of generality we can assume that all the $x_i$ are distinct. Since an element $\varphi$ of $U(n)$ only acts through the $W_i$ and the presentation of $x$ as such a tuple is unique up to a permutation, $\varphi$ fixes $x$ if and only if it fixes each of the $W_i$. In other words, the isotropy group of $x$ is the product $\prod_{i \in I} U(W_i)$. The two relations force $|I|$ to be larger than 1 (the only element would have to be zero by the 'reduced' condition, contradicting that the tuple has norm 1) and hence this product is complete.

We move on to show that the relevant fixed point spaces are contractible. First let $K \subseteq U(n)$ be any subgroup and denote by $W_1, W_2, \ldots, W_k$ the isotypical components of its action on $\mathbb{C}^n$. Since every $K$-representation decomposes canonically into isotypical subrepresentations, we see that

$$\mathcal{L}(n, S^{\mathcal{U}_G})^K \cong \bigwedge_{i=1}^{k} \mathcal{L}(W_i, S^{\mathcal{U}_G})^K.$$

Here, the notation $\mathcal{L}(W_i, S^{\mathcal{U}_G})$ is used to denote the evaluation of the $\Gamma$-space of decompositions of the complex vector space $W_i$ on $\mathcal{U}_G$, i.e., $\mathcal{L}(n, S^{\mathcal{U}_G})$ with $\mathbb{C}^n$ replaced by $W_i$.

We can perform the manipulations of this section to each smash factor separately and obtain $k$ smash copies of $S^{\mathcal{U}_G}$, of which the diagonal corresponds to the one of $\mathcal{L}(\mathbb{C}^n, S^{\mathcal{U}_G})$ used as the suspension spectrum coordinate. Hence we have an isomorphism

$$\overline{\mathcal{L}(n, S^{\mathcal{U}_G})}^K \cong (S^{\overline{\mathbb{R}^k} \otimes \mathcal{U}_G}) \wedge \bigwedge_{i=1}^{k} \overline{\mathcal{L}(W_i, S^{\mathcal{U}_G})}^K.$$

Finally we make use of the fact that a smash product of unreduced suspensions is (based) homeomorphic to the unreduced suspension of the join (denoted by $- * -$) and obtain

$$\overline{\mathcal{L}}_n(\mathcal{U}_G)^K \cong S(\overline{\mathbb{R}^k} \otimes \mathcal{U}_G)^K * \overline{\mathcal{L}_{|.|=1}(W_1, S^{\mathcal{U}_G})}^K * \ldots * \overline{\mathcal{L}_{|.|=1}(W_K, S^{\mathcal{U}_G})}^K.$$

Now let $H$ be a subgroup of $U(n) \times G$ such that $K := H \cap (U(n) \times 1)$ acts non-isotypically. We have to show that the $H$-fixed points of $\overline{\mathcal{L}}_n(\mathcal{U}_G)$ are contractible. Again we make use of the short exact sequence

$$1 \to K \to H \to \mathrm{pr}_G(H) \to 1$$

to write these $H$-fixed points as the $\mathrm{pr}_G(H)$-fixed points of $\overline{\mathcal{L}}_n(\mathcal{U}_G)^K$. But by the homeomorphism above, these are given ($\mathrm{pr}_G(H)$-equivariantly) by the join of $S(\overline{\mathbb{R}^k} \otimes \mathcal{U}_G)$ with another space. Here, the $\mathrm{pr}_G(H)$-action on $S(\overline{\mathbb{R}^k} \otimes \mathcal{U}_G)$ comes from the fact that any element of $U(n)$ which normalizes $H$ permutes its isotypical components and hence acts on the set $\underline{k}$. But we have seen in the proof of Proposition 10.0.3 that the $\mathrm{pr}_G(H)$-fixed points of $S(\overline{\mathbb{R}^k} \otimes \mathcal{U}_G)$ under such an action are contractible if $k > 1$. Hence, so is the join and we are done. □

Putting everything together:

**Theorem 11.1.4.** *There are global equivalences*

$$\begin{aligned} ku^n/ku^{n-1} &\simeq \Sigma^\infty(E_{gl}U(n)_+ \wedge_{U(n)} (E_{gl}\mathcal{C}_n^u)^\diamond) \\ &\simeq \Sigma^\infty(E_{gl}U(n)_+ \wedge_{U(n)} (E_{gl}\mathcal{I}_n^u)^\diamond). \end{aligned}$$

The underlying non-equivariant statement of this theorem is due to Arone and Lesh [AL10, Section 2.2].

This can be simplified by relating $E_{gl}\mathcal{C}_n^u$ to the collection $\mathcal{C}_n^u$ viewed as a topological poset, as explained in Section 2.2. By Proposition 2.2.8 and the uniqueness of global universal spaces shown in Section 2.2.3, there is a zig-zag of $U(n)$-maps from $E_{gl}\mathcal{C}_n^u$ to the constant orthogonal space $|\mathcal{C}_n^u|$, which induces a global equivalence

$$E_{gl}U(n) \times_{U(n)} E_{gl}\mathcal{C}_n^u \xrightarrow{\simeq} E_{gl}U(n) \times_{U(n)} |\mathcal{C}_n^u|$$

to the global homotopy orbits of $|\mathcal{C}_n^u|$. The collection $\mathcal{C}_n^u$ is isomorphic to the lattice $\mathcal{L}_n$ of proper decompositions of $\mathbb{C}^n$ into an orthogonal sum of complex subspaces, ordered by refinement. So we find:

**Theorem 11.1.5.** *There is a global equivalence*

$$ku^n/ku^{n-1} \simeq \Sigma^\infty(E_{gl}U(n)_+ \wedge_{U(n)} |\mathcal{L}_n|^\diamond).$$

## 11.2 Quotients in the complexity filtration

Now we turn to the complexity filtration. We recall that its $n$-th level $A_n^u$ is defined as

$$([0,1]_+ \wedge Sp^n) \vee_{ku^n} ku.$$

Since we know the filtration quotients of the modified rank filtration and the symmetric product filtration, it is not difficult to obtain a description for the filtration quotients of the complexity filtration. By forming termwise quotients in the pushout diagram defining $A_n^u$, we see that the sequence

$$ku^n/ku^{n-1} \to Sp^n/Sp^{n-1} \to A_n^u/A_{n-1}^u \tag{11.2.1}$$

is a mapping cone sequence. Using the results and notation of the previous sections we can identify the first two terms with the suspension spectra of the orthogonal spaces $L(\mathbb{C}^n)_+ \wedge_{U(n)} \overline{\mathcal{L}}_n^\diamond$ respectively $S(\overline{\mathbb{R}^n} \otimes -)^\diamond/\Sigma_n$. Moreover, the map is induced from the map of orthogonal spaces which collapses $L(\mathbb{C}^n)$ to a point and sends an element in $\overline{\mathcal{L}}_n$ represented by a tuple $(W_i, x_i)_{i\in I}$ to the element of $S(\overline{\mathbb{R}^n} \otimes -)/\Sigma_n$ represented by

$$(\underbrace{x_{i_1}, \dots, x_{i_1}}_{\dim W_{i_1}}, \underbrace{x_{i_2}, \dots, x_{i_2}}_{\dim W_{i_2}}, \dots, \underbrace{x_{i_j}, \dots, x_{i_j}}_{\dim W_{i_j}})$$

for some enumeration $i_1, i_2, \ldots, i_j$ of $I$, on which it does not depend since the $\Sigma_n$-action is quotiened out. In fact, the map $\overline{\mathcal{L}}_n \to S(\overline{\mathbb{R}^n} \otimes -)/\Sigma_n$ induces an isomorphism

$$\overline{\mathcal{L}}_n/U(n) \xrightarrow{\cong} S(\overline{\mathbb{R}^n} \otimes -)/\Sigma_n$$

(and in particular, the global classifying space of complete subgroups of $U(n)$ is globally equivalent to the global classifying space of complete subgroups of $\Sigma_n$). In other words, the map $ku^n/ku^{n-1} \to Sp^n/Sp^{n-1}$ is induced – by forming $U(n)$-orbits and applying the suspension spectrum functor – from the map of $U(n)$-orthogonal spaces

$$L(\mathbb{C}^n)_+ \wedge \overline{\mathcal{L}}_n^{\diamond} \to \overline{\mathcal{L}}_n^{\diamond}$$

that collapses $L(\mathbb{C}^n)$ to a point. So we have:

**Corollary 11.2.1.** *The quotient $A_n^u/A_{n-1}^u$ is isomorphic to the suspension spectrum of the based orthogonal space $L(\mathbb{C}^n)^{\diamond} \wedge_{U(n)} \overline{\mathcal{L}}_n^{\diamond}$.*

Since the smash product of two unreduced suspensions is isomorphic to the unreduced suspension of the join $*$, this based orthogonal space can be rewritten as

$$(L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^{\diamond}/U(n).$$

From Section 11.1 we know that the first join factor is a global universal space for $U(n)$ and that the second is a global universal space for the collection of complete (or non-isotypical) subgroups of $U(n)$.

The global homotopy type of this join can then be determined by the following easy lemma:

**Lemma 11.2.2.** *Let $\mathcal{F}$ be any collection of subgroups of a Lie group $K$, $E_{gl}\mathcal{F}$ a global universal space for $\mathcal{F}$ and $E_{gl}K$ be a global universal space for $K$. Then the join $E_{gl}\mathcal{F} * E_{gl}K$ is a global universal space for the collection $\overline{\mathcal{F}}$, i.e., $\mathcal{F}$ with the trivial subgroup added.*

*Proof.* This follows directly from the fact that the join commutes with taking fixed points. □

Hence, denoting the collection of complete and trivial subgroups of $U(n)$ by $\overline{\mathcal{C}}_n^u$ and the collection of non-isotypical and trivial subgroups by $\overline{\mathcal{I}}_n^u$, we obtain:

**Theorem 11.2.3** (Subquotients in the complexity filtration)**.** *There are global equivalences*

$$A_n^u/A_{n-1}^u \simeq \Sigma^\infty(B_{gl}(\overline{\mathcal{C}}_n^u)^{\diamond}) \simeq \Sigma^\infty(B_{gl}(\overline{\mathcal{I}}_n^u)^{\diamond}).$$

There is a different description of these filtration quotients, which is specific to the case of topological $K$-theory. Also, it turns out that the map from the quotients of the symmetric product filtration to the quotients of the complexity filtrations is globally null-homotopic. Both of these statements are a straightforward equivariant generalization of [AL07, Section 9].

We consider the tautological complex $U(n)$-representation $\mathbb{C}^n$. By smashing the cofiber sequence

$$L(\mathbb{C}^n)_+ \wedge \overline{\mathcal{L}}_n^\diamond \to \overline{\mathcal{L}}_n^\diamond \to (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond$$

described above with the $U(n)$-representation sphere $S^{\mathbb{C}^n}$ we obtain a commutative diagram of based $U(n)$-orthogonal spaces

$$\begin{tikzcd}
L(\mathbb{C}^n)_+ \wedge \overline{\mathcal{L}}_n^\diamond \arrow[r] \arrow[d] & \overline{\mathcal{L}}_n^\diamond \arrow[r] \arrow[d] & (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond \arrow[d] \\
L(\mathbb{C}^n)_+ \wedge \overline{\mathcal{L}}_n^\diamond \wedge S^{\mathbb{C}^n} \arrow[r] & \overline{\mathcal{L}}_n^\diamond \wedge S^{\mathbb{C}^n} \arrow[r] & (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond \wedge S^{\mathbb{C}^n}
\end{tikzcd}$$

where the vertical arrows are induced by including $S^0$ as the fixed point sphere of $S^{\mathbb{C}^n}$. This diagram has the following two properties:

**Lemma 11.2.4.** *The $U(n)$-orbits of $\overline{\mathcal{L}}_n^\diamond \wedge S^{\mathbb{C}^n}$ are globally contractible.*

**Lemma 11.2.5.** *The map $(L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond \to (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond \wedge S^{\mathbb{C}^n}$ is a $(U(n) \times G)$-equivalence when evaluated on a complete $G$-universe, for any compact Lie group $G$. In particular, the map on $U(n)$-quotients is a global equivalence.*

So we see:

**Corollary 11.2.6.** *The map*

$$S(\overline{\mathbb{R}^n} \otimes -)/\Sigma_n \cong (\overline{\mathcal{L}}_n)^\diamond/U(n) \to (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond/U(n)$$

*is based globally null-homotopic and hence so is $Sp^n/Sp^{n-1} \to A_n^u/A_{n-1}^u$.*

**Corollary 11.2.7.** *There is a global equivalence*

$$S^1 \wedge (L(\mathbb{C}^n)_+ \wedge_{U(n)} (\overline{\mathcal{L}}_n)^\diamond \wedge S^{\mathbb{C}^n})) \simeq (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond/U(n).$$

*Putting in what we know about the global homotopy of these orthogonal spaces from Proposition 11.1.3, Theorem 11.2.3 and Proposition 2.2.8, we hence find that there are the following global equivalences of orthogonal spectra:*

$$\begin{aligned}
A_n^u/A_{n-1}^n &\simeq \Sigma^\infty(S^1 \wedge E_{gl}U(n)_+ \wedge_{U(n)} (E_{gl}(\mathcal{I}_n^u)^\diamond \wedge S^{\mathbb{C}^n})) \\
&\simeq \Sigma^\infty(S^1 \wedge E_{gl}U(n)_+ \wedge_{U(n)} (E_{gl}(\mathcal{C}_n^u)^\diamond \wedge S^{\mathbb{C}^n})) \\
&\simeq \Sigma^\infty(S^1 \wedge E_{gl}U(n)_+ \wedge_{U(n)} (|\mathcal{L}_n|^\diamond \wedge S^{\mathbb{C}^n})).
\end{aligned}$$

In words, $A_n^u/A_{n-1}^u$ is globally equivalent to the suspension spectrum of the global homotopy orbits of the based $U(n)$-space $S^1 \wedge |\mathcal{L}_n|^\diamond \wedge S^{\mathbb{C}^n}$.

*Proof of Lemma 11.2.4, cf. [AL07, Proposition 9.13].* Let $G$ be a compact Lie group and $\mathcal{U}_G$ a complete $G$-universe. The evaluation $\overline{\mathcal{L}}_n(\mathcal{U}_G)$ is a $(U(n) \times G)$-cell complex with all the $U(n)$-isotropy in complete subgroups (cf. Proposition 11.1.3). Hence, $\overline{\mathcal{L}}_n(\mathcal{U}_G)^\diamond$ can be built from two points by attaching cells of the form $D^k \times (U(n) \times G)/H$ with $H \cap (U(n) \times 1)$ complete. It follows that $\overline{\mathcal{L}}_n^\diamond(\mathcal{U}_G) \wedge_{U(n)} S^{\mathbb{C}^n}$ can be built from

$S^{\mathbb{C}^n}/U(n) \cong [0,1] \simeq *$ by attaching spaces of the form $(D^k \times (U(n)\times G)/H)_+ \wedge_{U(n)} S^{\mathbb{C}^n}$. We observe that for any space $A$ there is a $G$-homeomorphism

$$(A \times (U(n) \times G)/H)_+ \wedge_{U(n)} S^{\mathbb{C}^n} \cong (A \times G)_+ \wedge_{\mathrm{pr}_G(H)} S^{\mathbb{C}^n}/(H \cap (U(n) \times 1)),$$

where the $\mathrm{pr}_G(H)$-action on $S^{\mathbb{C}^n}/(H \cap (U(n) \times 1))$ is the restriction along the induced homomorphism $\mathrm{pr}_G(H) \to W_{U(n)}(H \cap (U(n) \times 1))$, cf. the proof of Lemma 6.6.3. Since the quotient of $S^{\mathbb{C}^n}$ by a complete subgroup is homeomorphic to the smash product of intervals, it is (Weyl-group equivariantly) contractible and hence so is this $G$-space. Thus it follows by induction on the cells of $\overline{\mathcal{L}}_n(\mathcal{U}_G)^\diamond$ that the quotient is contractible. □

*Proof of Lemma 11.2.5, cf. [AL07, Theorem 9.4].* Let $G$ be a compact Lie group and $\mathcal{U}_G$ a complete $G$-universe. We have to show that for every closed subgroup $H$ of $U(n) \times G$ the induced map on $H$-fixed points is a weak equivalence. Let $K$ be the intersection of $H$ with $U(n) \times 1$. If the action of $K$ on $\mathbb{C}^n$ has no non-trivial fixed points then neither has that of $H$ and so the map $(S^0)^H \to (S^{\mathbb{C}^n})^H$ is even a homeomorphism. If the action of $K$ on $\mathbb{C}^n$ does have non-trivial fixed points, $K$ is either the trivial subgroup of $U(n)$ or it acts non-isotypically. In both cases the space $(L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)^\diamond(\mathcal{U}_G)^H$ is contractible and hence the map on $H$-fixed points is necessarily a weak equivalence, so we are done. □

### 11.2.1 Real version

Let $\mathcal{C}_n^o$ denote the collection of complete subgroups of $O(n)$, $\mathcal{I}_n^o$ the collection of subgroups of $O(n)$ which act non-isotypically on $\mathbb{R}^n$ and $\mathcal{L}_n^o$ the topological lattice of proper decompositions of $\mathbb{R}^n$ as an orthogonal sum of subspaces. Then all proofs of the previous sections go through verbatim in the real case to give the following:

**Theorem 11.2.8** (Quotients in the modified rank filtration for $ko$)**.** *There are global equivalences*

$$\begin{aligned} ko^n/ko^{n-1} &\simeq \Sigma^\infty(E_{gl}O(n)_+ \wedge_{O(n)} (E_{gl}\mathcal{C}_n^o)^\diamond) \\ &\simeq \Sigma^\infty(E_{gl}O(n)_+ \wedge_{O(n)} (E_{gl}\mathcal{I}_n^o)^\diamond) \\ &\simeq \Sigma^\infty(E_{gl}O(n)_+ \wedge_{O(n)} |\mathcal{L}_n^o|^\diamond). \end{aligned}$$

Again, we let $\overline{\mathcal{C}}_n^u$ and $\overline{\mathcal{I}}_n^u$ be the collections obtained by adding the trivial group to $\mathcal{C}_n^u$ respectively $\mathcal{I}_n^u$ and denote by $\mathbb{R}^n$ the tautological $O(n)$-representation.

**Theorem 11.2.9** (Quotients in the complexity filtration for $ko$)**.** *There are global equivalences*

$$\begin{aligned} A_n^o/A_{n-1}^o &\simeq \Sigma^\infty(B_{gl}\mathcal{C}_n^o)^\diamond \\ &\simeq \Sigma^\infty(B_{gl}\mathcal{C}_n^o)^\diamond \\ &\simeq \Sigma^\infty(S^1 \wedge E_{gl}O(n)_+ \wedge_{O(n)} (|\mathcal{L}_n^o|^\diamond \wedge S^{\mathbb{R}^n})). \end{aligned}$$

*Moreover, the morphism $Sp^n/Sp^{n-1} \to A_n^o/A_{n-1}^o$ is globally null-homotopic.*

# Chapter 12

# The rank filtration for the category of finite sets and the global Barratt-Priddy-Quillen theorem

In this section we use the modified rank filtration to prove a global version of the Barratt-Priddy-Quillen theorem, which says that the global $K$-theory of finite sets is globally equivalent to the sphere spectrum.

The quotient $k\mathcal{F}in^n/k\mathcal{F}in^{n-1}$ is the realization of the orthogonal $\Gamma$-space which sends $(V, A_+)$ to the space

$$\bigvee_{(n_a\in\mathbb{N})_{a\in A}, \sum n_a = n} (L_{\mathbb{R}}(\bigoplus_{a\in A}\mathbb{R}^{n_a}, \mathrm{Sym}(V))/\prod_{a\in A}\Sigma_{n_a})_+.$$

By the same trick as in Section 11.1, this can be rewritten as

$$L_{\mathbb{R}}(\mathbb{R}^n, \mathrm{Sym}(V))_+ \wedge_{\Sigma_n} (\bigvee_{(n_a\in\mathbb{N})_{a\in A}, \sum n_a = n} \mathrm{Bij}(\bigsqcup_{a\in A} \underline{n_a}, n)_+).$$

The first smash factor $L(\mathbb{R}^n, \mathrm{Sym}(-))$ is untouched by the $\Gamma$-space structure. Since the permutation representation $\mathbb{R}^n$ of $\Sigma_n$ is faithful, this first factor is a global universal space for $\Sigma_n$. The second smash factor is the $\Sigma_n$-$\Gamma$-space of partitions of the set $\underline{n}$. But this $\Gamma$-space can be described in an easier way, it is isomorphic to the one that sends a finite pointed set $A_+$ to its $n$-fold smash product $(A_+)^{\wedge n}$. Hence its realization is given in level $V$ by $(S^V)^{\wedge n}$, the $n$-th quotient of the symmetric product filtration before taking orbits under the $\Sigma_n$-action. By Proposition 10.0.3, this is the suspension spectrum of the unreduced suspension of a global universal space for the collection of complete subgroups $\mathcal{C}_n^{\Sigma}$ of $\Sigma_n$. So we obtain:

**Proposition 12.0.10.** *The quotient $k\mathcal{F}in^n/k\mathcal{F}in^{n-1}$ is globally equivalent to*

$$\Sigma^\infty(E_{gl}\Sigma_{n_+} \wedge_{\Sigma_n} (E_{gl}\mathcal{C}_n^{\Sigma})^\diamond).$$

But for $n > 1$ the collection of complete subgroups of $\Sigma_n$ contains the trivial subgroup (unlike the collection of complete subgroups of $U(n)$ that appeared in the filtration quotients of $ku$). Hence the map from $E_{gl}\mathcal{C}_n^{\Sigma}$ to a point induces a global equivalence after taking $\Sigma_n$-homotopy orbits. In other words, all filtration quotients $k\mathcal{F}in^n/k\mathcal{F}in^{n-1}$ for $n > 1$ are globally trivial. Furthermore, the spectrum $k\mathcal{F}in^1$ is isomorphic to the suspension spectrum of the based orthogonal space $V \mapsto S(\mathrm{Sym}(V))_+$. Since the unit sphere in a complete $G$-universe is equivariantly contractible, we see that the unit map $S^0 \to S(\mathrm{Sym}(V))$ is a global equivalence. Hence we obtain:

**Corollary 12.0.11** (Global Barratt-Priddy-Quillen Theorem)**.** *The unit*

$$\mathbb{S} \to k\mathcal{F}in$$

*is a global equivalence.*

We note that this proof shows more generally that the maps $Sp^n \to A_n^{\mathcal{F}in}$ are global equivalences for all $n \geq 1$. Hence, the complexity filtration for the $K$-theory of finite sets is globally equivalent to the symmetric product filtration, generalizing [AL07, Corollary 8.4].

# Chapter 13

# Filtrations associated to global algebraic $K$-theory

In this chapter we describe the subquotients in the modified rank and complexity filtration for global algebraic $K$-theory of a ring $R$ satisfying the dimension invariance property. These, again, turn out to be suspension spectra, this time in the symmetric spectrum context. The way to see this is very similar to the topological case, but some differences arise in determining the global homotopy type of the relevant $\mathbf{I}$-spaces.

## 13.1 Quotients in the modified rank filtration

We begin with the modified rank filtration. The quotient $kR^n/kR^{n-1}$ is the realization of the $\mathbf{I}$-$\Gamma$-space which maps $(M, A_+)$ to the space

$$\bigvee_{(n_a)_{a\in A}, \sum n_a = n} \left( |E(\mathrm{Emb}_R(\bigoplus_{a\in A} R^{n_a}, \mathrm{Sym}(R[M])))| / \prod_{a\in A} GL_{n_a} R \right)_+ .$$

Here, $\mathrm{Emb}_R(-,-)$ denotes the set of $R$-linear split injections and $E(-)$ of a set is the category with objects the set and exactly one morphism between any two objects.

Once more, these spaces can be rewritten as

$$|E(\mathrm{Emb}_R(R^n, \mathrm{Sym}(R[M])))|_+ \wedge_{GL_n(R)} \left( \bigvee_{(n_a)_{a\in A}, \sum n_a = n} \mathrm{Iso}(\bigoplus_{a\in A} R^{n_a}, R^n)_+ \right).$$

The first factor $E(\mathrm{Emb}_R(R^n, \mathrm{Sym}(R[-])))$ is constant in the $\Gamma$-space direction and as we argued in Section 3.6 (where it is denoted $\mathbf{I}(R^n)$) a global universal space for $GL_n(R)$.

The second factor is constant in the $\mathbf{I}$-space direction and forms the $GL_n(R)$-$\Gamma$-space of partitions of $R^n$, we denote it by $\mathcal{P}^R(n, -)$. Its realization even forms a $GL_n(R)$-orthogonal spectrum, so we see:

**Corollary 13.1.1.** *Both the quotients $kR^n/kR^{n-1}$ and the $kR^n$ are globally semistable symmetric spectra.*

*Proof.* It follows from Section 3.6 that

$$\mathbf{I}(R^n)_+ \wedge_{GL_n(R)} U(\mathcal{P}^R(n,-)(\mathbb{S}))$$

is $\underline{\pi}_*$-isomorphic to

$$U(\mathcal{E}_{gl}GL_n(R))_+ \wedge_{GL_n(R)} U(\mathcal{P}^R(n,-)(\mathbb{S})),$$

which underlies an orthogonal spectrum.

The statement for the $kR^n$ then follows by induction on $n$. □

We proceed by examining $\mathcal{P}^R(n,-)$. A point in the $M$-th level of the realization of this $\Gamma$-space is represented by a tuple $(W_i, x_i)_{i\in I}$ where the $x_i$ are elements of $\mathbb{R}^M$ and the $W_i$ are free submodules of $R^n$ whose inner sum is direct and all of $R^n$. In other words, it is the direct algebraic analog of $\mathcal{L}(n, S^M)$ of Section 11.1. Many of the arguments we used there can also be applied here by formally replacing complex subspaces by free $R$-submodules. We define two subspaces (where $\mathrm{rk}(-)$ denotes the rank of a free $R$-module):

$$\overline{\mathcal{P}^R(n, S^V)} = \{[(W_i, x_i)_{i\in I}] \mid \sum \mathrm{rk}(W_i)\cdot x_i = 0\}$$

and

$$\overline{\mathcal{P}^R_{|.|=1}(n, S^V)} = \{[(W_i, x_i)_{i\in I}] \mid \sum \mathrm{rk}(W_i)\cdot x_i = 0,\ \sum \mathrm{rk}(W_i)|x_i|^2 = 1\}.$$

The same arguments as in Section 11.1 show that the realization of $\mathcal{P}^R(n,-)$ is $GL_n(R)$-isomorphic to the suspension spectrum of the unreduced suspension of the $GL_n(R)$-$\mathbf{I}$-space $\overline{\mathcal{P}^R_{|.|=1}(n, S^-)}$, which we abbreviate by $\overline{\mathcal{P}^R_n}$. Again we remark that this $GL_n(R)$-$\mathbf{I}$-space is in fact the restriction of a $GL_n(R)$-orthogonal space and hence can be examined by the means of Section 2.1. It turns out that not all descriptions from Section 11.1 for $ku$ can be carried over to this setting.

We now discuss in which cases $\overline{\mathcal{P}^R_n}$ is a universal space of a collection of subgroups, for which we have to make assumptions on the ring $R$. Let $\mathcal{C}^R_n$ denote the collection of complete subgroups of $GL_n(R)$, i.e., those that are conjugate to one of the form $GL_{n_1}(R)\times\ldots\times GL_{n_k}(R)$ with $n_1+\ldots+n_k = n$ and $k>1$. Then the following still holds for all rings $R$:

**Lemma 13.1.2.** *The $GL_n(R)$-$\mathbf{I}$-space $\overline{\mathcal{P}^R_n}$ is closed and has all isotropy in complete subgroups.*

*Proof (cf. Proposition 11.1.3).* An element of $GL_n(R)$ fixes each $W_i$ in a partition $R^n = \bigoplus W_i$ if and only if it lies in the product of the $GL(W_i)$, which is a complete subgroup. □

One might guess that $\overline{\mathcal{P}^R_n}$ is in fact a universal space for the collection of complete subgroups, but this is not true in general. The issue is the following: Assume given a complete subgroup $H = \prod GL(W_i)$ and a decomposition $R^n = \bigoplus W'_j$ that is (strongly) fixed by $H$. Over the complex numbers this implies that each $W_i$ must be contained in some $W'_j$, or in other words that the decomposition $\bigoplus W_i$ is a refinement of $\bigoplus W'_j$. This allowed an easy description of the $H$-fixed point space and led to the proof that it is contractible. However, for general $R$ this is not the case, as the following example shows:

**Example 13.1.3.** Let $R = \mathbb{F}_2$ be the field with two elements and consider the decomposition $\mathbb{F}_2^2 = \mathbb{F}_2 \oplus \mathbb{F}_2$. Then the associated complete subgroup of $GL_2(\mathbb{F}_2)$ is $GL_1(\mathbb{F}_2) \times GL_1(\mathbb{F}_2)$ and hence trivial. So it fixes all three decompositions of $\mathbb{F}_2^2$ as a sum of two 1-dimensional subspaces, not only the one it was associated to.

Arone and Lesh show that this phenomenon cannot occur under the following assumptions on $R$:

**Lemma 13.1.4** ([AL07, Lemma 8.8])**.** *Let $R$ be an integral domain with $2 \neq 0$, and further be given two proper decompositions $R^n = \bigoplus W_i$ and $R^n = \bigoplus W'_j$ into free submodules. Then the subgroup $\prod GL(W_i)$ fixes all of the $W'_j$ if and only if $\bigoplus W_i$ is a refinement of $\bigoplus W'_j$.*

Under these conditions we see:

**Proposition 13.1.5.** *Let $R$ be an integral domain with $2 \neq 0$. Then $\overline{\mathcal{P}_n^R}$ is a global universal space for the collection of complete subgroups of $GL_n(R)$.*

*Proof.* We have already seen that all the $GL_n(R)$-isotropy lies in complete subgroups. Now let $G$ be a finite group and $\mathcal{U}_G$ a complete $G$-set universe. In Appendix A.2 we show that $(\overline{\mathcal{P}_n^R})(\mathcal{U}_G)$ is a $(GL_n(R) \times G)$-cell complex. Let $H \leq GL_n(R) \times G$ be a subgroup whose intersection with $GL_n(R) \times 1$ (which we denote by $K$) is complete. Making use of Lemma 13.1.4, we can associate to $K$ the unique minimal partition $R^n = W_1 \oplus \ldots \oplus W_k$ that is fixed by it and it follows that $K = GL(W_1) \times \ldots \times GL(W_k)$. We have to show that the $H$-fixed points $(\overline{\mathcal{P}_n^R})(\mathcal{U}_G)^H$ are contractible. By the short exact sequence

$$1 \to K \to H \to \mathrm{pr}_G(H) \to 1$$

these $H$-fixed points are the $\mathrm{pr}_G(H)$-fixed points of the action on $(\overline{\mathcal{P}_n^R})(\mathcal{U}_G)^K$ which is induced from the associated group homomorphism $\mathrm{pr}_G H \to W_{GL_n(R)}K$. By Lemma 13.1.4, a partition is fixed by $K$ if and only if it refines $R^n = \bigoplus W_i$. Refinements of this partition stand in bijection to partitions of the set $\{1, \ldots, k\}$. Via this correspondence we see that the $K$-fixed points $(\overline{\mathcal{P}_n^R})(\mathcal{U}_G)^K$ are in fact homeomorphic to the $(\Sigma_{\dim W_1} \times \ldots \times \Sigma_{\dim W_k})$-fixed points of $S(\overline{\mathbb{R}^n} \otimes \mathbb{R}[\mathcal{U}_G])$. Moreover, the Weyl-groups $W_{GL_n(R)}K$ and $W_{\Sigma_n}(\Sigma_{\dim W_1} \times \ldots \times \Sigma_{\dim W_k})$ are canonically isomorphic and the homeomorphism is equivariant under this isomorphism. Hence the statement follows from the fact that $S(\overline{\mathbb{R}^n} \otimes \mathbb{R}[\mathcal{U}_G])$ is a universal space for the collection $\mathcal{C}_n^\Sigma\langle G\rangle$, which was proved in Proposition 10.0.3. $\square$

Let $\mathcal{P}_n^R$ denote the poset of proper decompositions of $R^n$ into direct sums of free $R$-submodules. Lemma 13.1.4 implies that if $R$ is an integral domain with $2 \neq 0$, the map of posets

$$\begin{aligned} \mathcal{P}_n^R &\to \mathcal{C}_n^R \\ (R^n = \bigoplus W_i) &\mapsto \prod GL(W_i) \end{aligned}$$

is an isomorphism. So, again using Proposition 2.2.8, we can also replace $E_{gl}\mathcal{C}_n^R$ by $\mathcal{P}_n^R$ and summarize:

**Theorem 13.1.6** (Quotients in the modified rank filtration)**.** *Let $R$ be an integral domain with $2 \neq 0$. Then there are global equivalences*

$$\begin{aligned} kR^n/kR^{n-1} &\simeq \Sigma^\infty(E_{gl}GL_n(R)_+ \wedge_{GL_n(R)} (E_{gl}\mathcal{C}_n^R)^\diamond) \\ &\simeq \Sigma^\infty(E_{gl}GL_n(R)_+ \wedge_{GL_n(R)} |\mathcal{P}_n^R|^\diamond). \end{aligned}$$

**Remark 13.1.7.** In fact, it turns out that the description in terms of the decomposition lattice $\mathcal{P}_n^R$ *always* holds, i.e., for every ring $R$ satisfying dimension invariance there is a global equivalence

$$kR^n/kR^{n-1} \simeq (E_{gl}GL_n(R)_+ \wedge_{GL_n(R)} |\mathcal{P}_n^R|^\diamond). \tag{13.1.1}$$

On the other hand, the description in terms of $E_{gl}\mathcal{C}_n^R$ does not hold in full generality, as one can see in the example $R = \mathbb{F}_2$ above. This is not a contradiction, because for general $R$ the posets $\mathcal{C}_n^R$ and $\mathcal{P}_n^R$ are not isomorphic. This means that one cannot use Proposition 2.2.8 to prove the global equivalence (13.1.1) and instead one has to relate the $GL_n(R)$-orthogonal space $\overline{\mathcal{P}_n^R}$ to $\mathcal{P}_n^R$ directly. Since the gain is relatively small compared to the length of the proof, we decided to leave it out of this thesis.

## 13.2 Quotients in the complexity filtration

There is a cofiber sequence

$$kR^n/kR^{n-1} \to Sp^n/Sp^{n-1} \to A_n^R/A_{n-1}^R$$

of symmetric spectra. As in the topological case, the first morphism arises by applying $GL_n(R)$-orbits and the suspension spectrum functor to the map

$$\mathbf{I}(R^n)_+ \wedge (\overline{\mathcal{P}_n^R})^\diamond \to (\overline{\mathcal{P}_n^R})^\diamond$$

which collapses $\mathbf{I}(R^n)$ to a point. So we find that there is an isomorphism

$$A_n^R/A_{n-1}^R \cong \Sigma^\infty((\mathbf{I}(R^n) * \overline{\mathcal{P}_n^R})/GL_n(R))^\diamond.$$

Under the same hypotheses as in the previous section we obtain the following:

**Theorem 13.2.1.** *Let $R$ be an integral domain with $2 \neq 0$. Then there is a global equivalence*

$$A_n^R/A_{n-1}^R \simeq \Sigma^\infty(B_{gl}(\overline{\mathcal{C}}_n^R)^\diamond),$$

*where $\overline{\mathcal{C}}_n^R$ denotes the collection of complete subgroups of $GL_n(R)$ plus the trivial subgroup.*

*Proof.* Using Lemma 11.2.2, this follows from the fact that $\mathbf{I}(R^n)$ is a global universal space for $GL_n(R)$ together with Proposition 13.1.5. $\square$

# Appendix A

## A.1 Cofibrancy properties of the rank filtration

In this appendix we show that the $V$-th level of the inclusions $ku^{n-1} \to ku^n$ is an $O(V)$-cofibration, guaranteeing that the quotient $ku^n/ku^{n-1}$ has the global homotopy type of the homotopy cofiber. For instance, this was used in the proof of Theorem 14.1.3. For finite subgroups of $O(V)$ (and hence for the $\mathcal{F}in$-global homotopy type of the quotient) this would follow quite directly from the results of [Ost14], but we need the general statement. In this and the next appendix we repeatedly make use of a theorem due to Illman (cf. [Ill83]), which says that every smooth manifold equipped with a smooth action by a compact Lie group allows the structure of an equivariant CW complex.

We recall from [Lyd99, Section 3] that the evaluation $X(A)$ of a $\Gamma$-space $X$ on a based space $A$ is naturally filtered by skeleta $sk_m(X(A))$. The $m$-skeleton is obtained from the $(m-1)$-st by forming a certain pushout ([Lyd99, Theorem 3.10]). Furthermore, given a map $i : X \to Y$ of $\Gamma$-spaces, one can define relative skeleta $sk_m[i](A)$ by

$$sk_m(Y(A)) \cup_{sk_m(X(A))} X(A)$$

and it follows that these are related by a similar pushout square. The colimit over the $sk_m[i](A)$ gives back $Y(A)$ and the map from $X(A) = sk_0[i](A)$ agrees with $i$. Now let $V$ be a finite dimensional real inner product space. We are interested in the case where $A$ is equal to $S^V$ and $i$ is the inclusion

$$ku^{n-1}(\mathrm{Sym}(V_{\mathbb{C}}), -) \hookrightarrow ku^n(\mathrm{Sym}(V_{\mathbb{C}}), -).$$

Here, the connecting pushout takes the form

$$\begin{tikzcd}
(\bigvee_{n_i}(L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, \mathrm{Sym}(V_{\mathbb{C}}))/\prod U(n_i))_+) \times_{\Sigma_m} F((S^V)^{\times m}) \arrow[r] \arrow[d] & sk_{m-1}[i](S^V) \arrow[d] \\
(\bigvee_{n_i}(L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, \mathrm{Sym}(V_{\mathbb{C}}))/\prod U(n_i))_+) \times_{\Sigma_m} (S^V)^{\times m} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & sk_m[i](S^V),
\end{tikzcd} \tag{A.1.1}$$

where the wedge is indexed over all $m$-tuples $(n_1, \ldots, n_m)$ which add up to $n$, with all $n_i$ larger than 0. The notation $F((S^V)^{\times m})$ stands for the subspace of $(S^V)^{\times m}$ of tuples which contain two equal entries or a basepoint. It suffices to show that

$$sk_{m-1}[i](S^V) \to sk_m[i](S^V)$$

is an $O(V)$-cofibration for all $m \in \mathbb{N}$, since the sequential colimit of $O(V)$-cofibrations is again an $O(V)$-cofibration. This follows from:

**Lemma A.1.1.** *The left hand vertical map in Diagram (A.1.1) is an $O(V)$-cofibration.*

*Proof.* We first argue that

$$\bigvee_{n_1+\ldots+n_m=n} (L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, \mathrm{Sym}(V_{\mathbb{C}}))/\prod U(n_i))_+$$

is $(\Sigma_m \times O(V))$-cofibrant. This would follow directly from Illman's theorem for

$$W_k = \bigoplus_{i=0,\ldots,k} \mathrm{Sym}^i(V_{\mathbb{C}})$$

instead of the full $\mathrm{Sym}(V_{\mathbb{C}})$, since each $L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, W_k)$ is a smooth manifold with a smooth action by $U(W_k) \times (N_{U(n)} \prod U(n_i))$. The subspace $L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, W_{k-1})$ is exactly the space of $U(\mathrm{Sym}^k(V_{\mathbb{C}}))$-fixed points under this action. Since $O(V)$ fixes $\mathrm{Sym}^k(V_{\mathbb{C}})$, it normalizes the subgroup $U(\mathrm{Sym}^k(V_{\mathbb{C}}))$. This implies that if we forget any $(U(W_k) \times (N_{U(n)} \prod U(n_i)))$-CW structure to an $(O(V) \times (N_{U(n)} \prod U(n_i)))$-cell structure, the space $L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i}, W_{k-1})$ is necessarily a subcomplex and hence the inclusion a cofibration. By dividing out the $\prod U(n_i)$-actions and passing to the colimit we see that the wedge is $(\Sigma_m \times O(V))$-cofibrant, as claimed.

Hence it suffices to show that

$$F((S^V)^{\times m}) \to (S^V)^{\times m}$$

is a $(\Sigma_m \times O(V))$-cofibration. By once more applying Illman's theorem we see that $S^V$ is an $O(V)$-CW complex, containing the basepoint $\infty$ as a 0-cell. This $O(V)$-CW structure induces a $(\Sigma_m \wr O(V))$-CW structure on the $m$-fold cartesian product $(S^V)^{\times m}$ and thus in particular a $(\Sigma_m \times O(V))$-cell structure by choosing $(\Sigma_m \times O(V))$-CW structures on the $(\Sigma_m \wr O(V))$-orbits. We now claim that $F((S^V)^{\times m})$ is a $(\Sigma_m \times O(V))$-subcomplex for this cell structure. By definition, $F((S^V)^{\times m})$ is the union of two subspaces: The space of tuples containing a basepoint and the space of tuples containing two equal entries. By definition, the former is even a $(\Sigma_m \wr O(V))$-CW subcomplex of $(S^V)^{\times m}$, since the basepoint is a 0-cell. But the latter is given precisely by those points that have non-trivial $\Sigma_m$-isotropy, hence it is an equivariant subcomplex for any $(\Sigma_m \times O(V))$-cell structure. This finishes the proof. □

## A.2 Equivariant CW structures

The content of this appendix is to show that the $U(n)$-orthogonal spaces $\overline{\mathcal{L}}_n$ that appeared in Section 11.1 give $(U(n) \times G)$-cell complexes when evaluated on any $G$-representation $V$ (at most countably infinite dimensional). This property was needed in

Proposition 11.1.3 for $\overline{\mathcal{L}}_n$ to be a global universal space for the family of complete/non-isotypical subgroups of $U(n)$. The same proof also shows that the spaces $\overline{\mathcal{P}_n^R}(M)$ arising in the filtrations of algebraic $K$-theory are $(GL_n(R)\times G)$-cell complexes.

The proof is similar to that of the previous section. This time we consider the (absolute) skeleta filtration for the $U(n)$-$\Gamma$-space $\mathcal{L}(n,-)$, where the relating pushouts take the form

$$\begin{tikzcd}
\bigvee_{n_1+\ldots+n_m=n}(L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i))_+ \times_{\Sigma_m} F((S^V)^{\times m}) \arrow[r] \arrow[d] & sk_{m-1}(\mathcal{L}(n,S^V)) \arrow[d] \\
\bigvee_{n_1+\ldots+n_m=n}(L_{\mathbb{C}}(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i))_+ \times_{\Sigma_m} (S^V)^{\times m} \arrow[r] & sk_m(\mathcal{L}(n,S^V)).
\end{tikzcd}$$

The wedge is taken over the same indexing system as in the previous section. We recall that the closed subspace $\overline{\mathcal{L}}_n(V)$ of $\mathcal{L}(n,S^V)$ was defined to contain those elements that can be represented by a tuple $(W_i,x_i)_{i\in I}$ with all $x_i$ non-equal to the basepoint and satisfying the equations $\sum \dim(W_i)\cdot x_i = 0$ and $\sum \dim(W_i)|x_i|^2 = 1$. Intersection with $sk_m(\mathcal{L}(n,S^V))$ gives subspaces $sk_m(\overline{\mathcal{L}}_n(V))$ whose colimit over $m$ is isomorphic to $\overline{\mathcal{L}}_n(V)$. Likewise, for fixed $n_1,\ldots,n_m$ we define closed subspaces

$$S_{\{n_i\}}((S^V)^{\times m}) \subseteq (S^V)^{\times m}$$

as those tuples satisfying $\sum n_i\cdot x_i = 0$ and $\sum n_i|x_i|^2 = 1$. With these definitions an element of

$$\Big(L(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i)\Big)\times (S^V)^{\times m}$$

is mapped to $sk_m(\overline{\mathcal{L}}_n(V))$ if and only if it lies in

$$\Big(L(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i)\Big)\times S_{\{n_i\}}((S^V)^{\times m}).$$

So we obtain a new pushout square

$$\begin{tikzcd}
(\bigsqcup_{n_1+\ldots+n_m=n}(L(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i))) \times_{\Sigma_m} F(S_{\{n_i\}}((S^V)^{\times m})) \arrow[r] \arrow[d] & sk_{m-1}(\overline{\mathcal{L}}_n(V)) \arrow[d] \\
(\bigsqcup_{n_1+\ldots+n_m=n}(L(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i))) \times_{\Sigma_m} S_{\{n_i\}}((S^V)^{\times m}) \arrow[r] & sk_m(\overline{\mathcal{L}}_n(V)).
\end{tikzcd} \tag{A.2.1}$$

Hence it suffices to show:

**Lemma A.2.1.** *The left hand vertical map Diagram* (A.2.1) *above is a* $(U(n)\times G)$-*cofibration.*

*Proof.* The proof is very similar to that of Lemma A.1.1. Again it suffices to see that

$$\bigsqcup_{\{n_i\neq 0\}_{0\leq i\leq m}\,\sum n_i=n}\Big(L(\bigoplus \mathbb{C}^{n_i},\mathbb{C}^n)/\prod U(n_i)\Big)$$

is a $(U(n) \times \Sigma_m)$-CW complex and that the map

$$F(S_{\{n_i\}}((S^V)^{\times m}))) \to S_{\{n_i\}}((S^V)^{\times m})$$

is a $(\Sigma_m \times G)$-cofibration. The former is easy to see, because each summand is $U(n)$-isomorphic to $U(n)/\prod U(n_i)$ and these summands are permuted by the $\Sigma_m$-action. For the latter we note that by a transformation of variables each $S_{\{n_i\}}((S^V)^{\times m})$ is homeomorphic to the usual unit sphere $S(V \otimes \mathbb{R}^m)$, which – by Illman's theorem for finite dimensional $V$ and the same trick as in Lemma A.1.1 for the infinite case – is a $(\Sigma_m \times G)$-CW complex. Since $F(S_{\{n_i\}}((S^V)^{\times m}))$ no longer contains any basepoints, it is exactly the subspace of elements with non-trivial $\Sigma_m$-isotropy, and hence always a $(\Sigma_m \times G)$-subcomplex. This finishes the proof. $\square$

# Part IV

# Filtrations of global $K$-theory II: Induced filtrations on representation rings

# Chapter 14

# Modified rank filtrations on 0-th homotopy

In this chapter we describe global formulas for the filtrations of representation rings that are induced from the modified rank filtration of global $K$-theory.

## 14.1 Statement of results

In this introduction we let $kX$ stand for the orthogonal spectra $ku$ or $ko$ of Section 3.3 or for the symmetric spectrum $kR$ for a discrete ring $R$ of Section 3.6. By $\mathrm{Rep}_X(G)$ we denote the associated representation ring. For now we neglect the difference between $\pi_0^G(kX)$ and the representation ring $\mathrm{Rep}_X(G)$ that exists for non-discrete compact Lie groups $G$ and $ku$ or $ko$. We will be precise in the following sections and discuss this issue in Remark 14.1.4.

Applying $\pi_0^G$ to the modified rank filtration yields a sequence of groups

$$\pi_0^G(kX^1) \to \pi_0^G(kX^2) \to \ldots \to \pi_0^G(kX) \cong \mathrm{Rep}_X(G).$$

By Theorems 11.1.4, 11.2.8 and 13.1.6, $kX^1$ is globally equivalent to the suspension spectrum of a global classifying space of $X^\times$. Here $X^\times$ is understood as $U(1)$ if $kX = ku$, as $O(1)$ if $kX = ko$ and as the group of units $R^\times$ if $kX = kR$. So, by Proposition 3.1.5, $\pi_0^G(kX^1)$ is a free abelian group with basis

$$\{\mathrm{tr}_H^G([\psi])\}_{(H,\psi)},$$

where $(H,\psi)$ ranges through representatives of conjugacy classes of subgroups $H$ of $G$ together with a character $\psi : H \to X^\times$. The map

$$\pi_0^G(kX^1) \to \pi_0^G(kX) \cong \mathrm{Rep}_X(G) \tag{14.1.1}$$

sends such a basis element $\mathrm{tr}_H^G([\psi])$ to $\mathrm{Ind}_H^G(W_\psi)$, where $W_\psi$ is the 1-dimensional representation associated to $\psi$.

**Example 14.1.1.** By Brauer's induction theorem (cf. [Ser77, Theorem 18, Section

10.2]), the complex representation ring is always generated by transfers of 1-dimensional representations, hence in that case the map (14.1.1) is surjective.

The 0-th homotopy group $\underline{\pi}_0^G(kX^1)$ can be interpreted as the 'free global functor on the 1-dimensional part of the representation ring global functor'. It is generated by 1-dimensional representations but does not yet see any representation-theoretic sum or induction, since these operations lead to representations of dimension at least 2. So, instead, sums and transfers are added on freely. As a consequence, 1-dimensional representations satisfy relations in the representation ring global functor that are not yet visible in $\pi_0^G(kX^1)$.

**Example 14.1.2.** We consider the case $G = C_p$ with $p$ a prime, $kX = ku$ and let $\eta_p$ be a primitive $p$-th root of unity in $\mathbb{C}$. Then $\pi_0^{C_p}(ku^1)$ is free with basis

$$\{[\eta_p^1], [\eta_p^2], \ldots, [\eta_p^p]\} \cup \{\mathrm{tr}_1^{C_p}[1]\}.$$

The elements $[\eta_p^i]$ also form a basis of the representation ring of $G$ and hence the only difference between $\pi_0^{C_p}(ku^1)$ and $\pi_0^{C_p}(ku)$ lies in the element $\mathrm{tr}_1^{C_p}[1]$, which is equal to the sum of the $[\eta_p^i]$ in $\pi_0^{C_p}(ku)$.

We show that something similar is true for higher $n$: $\underline{\pi}_0(kX^n)$ is the free global functor on the at most $n$-dimensional part of the representation ring global functor. It is generated by at most $n$-dimensional representations, and the homotopy-theoretic sum and transfers model the representation-theoretic sum and induction whenever the latter operation does not lead to a representation of dimension larger than $n$. Otherwise the sum and transfer are added on freely.

### 14.1.1 Global complex $K$-theory

We now make this precise and first concentrate on the case $kX = ku$. We recall from Section 3.3 the morphisms

$$\alpha_m : \Sigma_+^\infty(L(\mathbb{C}^m)/U(m)) \to ku$$

which we used to assign elements in $\pi_0^G(ku)$ to $G$-representations. If $m \leq n$, the morphism $\alpha_m$ takes image in $ku^n$. Hence every $m$-dimensional $G$-representation $W$ already defines an element $[W]$ in $\pi_0^G(ku^n)$ which only depends on its isomorphism type. We will see that $\pi_0^G(ku^n)$ is additively generated by transfers of these elements. To understand the relations, we observe: If $W$ is $n$-dimensional, the class $[W]$ does not make sense in $\pi_0^G(ku^{n-1})$ yet, but it might already secretly live there in the following sense:

- If $W = W_1 \oplus W_2$ is (non-trivially) decomposable, then the classes $[W_1], [W_2]$ already live in $\pi_0^G(ku^{n-1})$ and hence so does their homotopy theoretic sum $[W_1] + [W_2]$.
- If $W = \mathrm{Ind}_H^G W'$ is induced up from a proper finite index subgroup $H$, then $[W']$ is an element in $\pi_0^H(ku^{n-1})$ and hence one can form the homotopy theoretic transfer $\mathrm{tr}_H^G[W'] \in \pi_0^G(ku^{n-1})$.

We will see that both elements map to $[W]$ under $\pi_0^G(ku^{n-1}) \to \pi_0^G(ku^n)$ and that $[W]$ lies in the image if and only if one of those two conditions is satisfied. Furthermore, if $[W]$ does lie in the image, then it might be so for different reasons: It could both be decomposable and induced, or it could be induced up in different ways. It turns out that direct sums and induction are reflected in certain fixed points of the decomposition poset $\mathcal{L}_n$ and that it is exactly these different reasons that are identified via the boundary map

$$\partial : \pi_1^G(ku^n/ku^{n-1}) \to \pi_0^G(ku^{n-1}).$$

In global equivariant homotopy theory, all this can be phrased via universal examples: If an $n$-dimensional $G$-representation $W$ is the direct sum of two subrepresentations $W_1$ and $W_2$ of dimensions $k$ and $l$, then – up to conjugation – the associated homomorphism $\beta : G \to U(n)$ factors through the embedding $U(k) \times U(l) \hookrightarrow U(k + l = n)$. For $t \geq 1$ let $\tau_t^u$ denote the tautological complex $t$-dimensional representation of $U(t)$. Then the fact that $[W] = [W_1] + [W_2]$ in $RU(G)$ is the restriction along $\widetilde{\beta} : G \to U(k) \times U(l)$ of the relation

$$(\operatorname{res}^{U(k+l)}_{U(k)\times U(l)})^*(\tau^{\mathbb{C}}_{k+l}) = (p_1)^*(\tau_k^u) + (p_2)^*(\tau_l^u), \tag{14.1.2}$$

where $p_1$ and $p_2$ denote the projections from $U(k) \times U(l)$ to $U(k)$ respectively $U(l)$. We denote this relation by $a(k, l)$.

Likewise, if $W$ is the induction of a $j$-dimensional representation $W'$ of a subgroup $H$ of index $i$, the associated group homomorphism $\beta : G \to U(n)$ takes image in the wreath product $\Sigma_i \wr U(j)$, i.e., the semidirect product of $U(j)^{\times i}$ and $\Sigma_i$ via the permutation action on the product coordinates. Then the relation $[W] = \operatorname{tr}_H^G[W']$ in the representation ring global functor is the restriction along $\widetilde{\beta} : G \to \Sigma_i \wr U(j)$ of

$$\operatorname{res}^{U(i\cdot j)}_{\Sigma_i \wr U(j)}(\tau^u_{i\cdot j}) = \operatorname{tr}^{\Sigma_i \wr U(j)}_{U(j)\times \Sigma_{i-1}\wr U(j)}(p^*(\tau_j^u)). \tag{14.1.3}$$

Here, $p$ stands for the projection from $U(j) \times (\Sigma_{i-1} \wr U(j))$ to $U(j)$. Let $b(i, j)$ denote this relation.

In these terms the intermediate homotopy groups can be described as follows:

**Theorem 14.1.3.** *The global functor $\underline{\pi}_0(ku^n)$ is the free global functor generated by the elements*

$$\tau_1^u, \tau_2^u, \ldots, \tau_n^u$$

*modulo the relations $a(k, l)$ for all $k + l \leq n$ and $b(i, j)$ for all $i \cdot j \leq n$.*

We try to make clear what this means at a fixed group $G$ in a few examples of these filtrations in Chapter 16. However, the result at a specific group is often a lot more complicated than the global formula.

**Remark 14.1.4.** The formula in Theorem 14.1.3 is true as stated for all compact Lie groups, not only finite ones. In particular, we see that $\underline{\pi}_0(ku)$ is the free global functor on the elements $\tau_i^u$ modulo the relations $a(k, l)$ and $b(i, j)$, with no restrictions on $i, j, k$ or $l$. This shows that the only difference between $\underline{\pi}_0(ku)$ and the representation ring global functor are infinite index transfers, since these are not encoded in the $b(i, j)$.

We do not know whether Segal's smooth infinite index transfers can be modeled by universal relations.

### 14.1.2 Global real $K$-theory

The real case is completely analogous to the complex one. We denote by $\tau_n^o$ the class of the tautological $n$-dimensional real representation of $O(n)$ and define relations $a(k,l)$ and $b(i,j)$ by

$$\mathrm{res}^{O(k+l)}_{O(k)\times O(l)}(\tau_n^o) = p_1^*(\tau_k^o) + (p_2)^*(\tau_l^o),$$

respectively

$$\mathrm{res}^{O(i+j)}_{\Sigma_i \wr O(j)}(\tau_{i\cdot j}^o) = \mathrm{tr}^{\Sigma_i \wr O(j)}_{O(j)\times(\Sigma_{i-1}\wr O(j))}(p^*(\tau_j^o)).$$

Then we have:

**Theorem 14.1.5.** *The global functor $\underline{\pi}_0(ko^n)$ is the free global functor generated by the elements*

$$\tau_1^o, \tau_2^o, \ldots, \tau_n^o$$

*modulo the relations $a(k,l)$ for all $k+l \le n$ and $b(i,j)$ for all $i \cdot j \le n$.*

### 14.1.3 Global algebraic $K$-theory

We first treat the case where $R$ is a finite ring. Let $\tau_n^R$ be the tautological $R[GL_n(R)]$-lattice of rank $n$. Then analogously to above one defines universal relations $a(k,l)$ and $b(i,j)$ by

$$\mathrm{res}^{GL_{k+l}(R)}_{GL_k(R)\times GL_l(R)}(\tau_n^R) = p_1^*(\tau_k^R) + (p_2)^*(\tau_l^R),$$

respectively

$$\mathrm{res}^{GL_{i\cdot j}(R)}_{\Sigma_i \wr GL_j(R)}(\tau_{i\cdot j}^R) = \mathrm{tr}^{\Sigma_i \wr GL_j(R)}_{GL_j(R)\times(\Sigma_{i-1}\wr GL_j(R))}(p^*(\tau_j^R)).$$

And we obtain:

**Theorem 14.1.6** (Modified rank filtration on $\underline{\pi}_0$)**.** *Let $R$ be a finite ring. Then the $\mathcal{F}in$-global functor $\underline{\pi}_0(kR^n)$ is the free $\mathcal{F}in$-global functor generated by the elements*

$$\tau_1^R, \ldots, \tau_n^R$$

*modulo the relations $a(k,l)$ for all $k+l \le n$ and $b(i,j)$ for all $i \cdot j \le n$.*

The problem with infinite $R$ is that its general linear groups are not finite. Hence the universal elements above do not make sense, as the theory is 'not global enough' to include infinite discrete groups. One can still give the following concrete description:

**Proposition 14.1.7** (Description for arbitrary rings)**.** *Let $R$ be a ring satisfying dimension invariance. Then $\underline{\pi}_0(kR^n)$ is generated as a $\mathcal{F}in$-global functor by the elements $[W] \in \pi_0^G(kR)$, where $(G, W)$ runs through a system of representatives of isomorphism classes of a finite group $G$ together with an $R[G]$-lattice of rank $\le n$. The relations are generated by:*

- $[W \oplus W'] = [W] \oplus [W']$ *with* $\dim_R(W) + \dim_R(W') \le n$.

- $[\mathrm{Ind}_H^G(W)] = \mathrm{tr}_H^G[W]$ *with* $\dim_R(W) \cdot [G : H] \leq n$.

In this case both the generators and the relations are already closed under restrictions, so it suffices to apply transfers to obtain the concrete value at a given finite group. In this sense the global formula is no easier than the one for a specific group.

## 14.2 Proof

In this section we prove the results stated in the previous section. We concentrate on the proof for complex topological $K$-theory, as the arguments for orthogonal and algebraic $K$-theory are analogous. For infinite discrete rings $R$, one needs to replace the universal classes at each step by the set of all representations of the respective dimension, and similarly for the relations. This makes the notation and statements less clean, but otherwise the proofs can be carried out in the same way.

The proof proceeds by comparing the cofiber sequence

$$ku^{n-1} \to ku^n \to ku^n/ku^{n-1}$$

to another one with the same cofiber, namely

$$\begin{tikzcd}
\Sigma_+^\infty\left(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n\right) \arrow[r, "p"] \arrow[d, dashed, "\psi_n"'] & \Sigma_+^\infty\left(L(\mathbb{C}^n)/U(n)\right) \arrow[r] \arrow[d, "\alpha_n"] & \Sigma^\infty\left(L(\mathbb{C}^n)_+ \wedge_{U(n)} \overline{\mathcal{L}}_n^\diamond\right) \arrow[d, "\cong"] \\
ku^{n-1} \arrow[r, "i_n"'] & ku^n \arrow[r] & ku^n/ku^{n-1},
\end{tikzcd} \tag{14.2.1}$$

where the vertical isomorphism on the right is the one explained in Section 11.1. The map $\psi_n$ could be obtained (at least as a stable map) via the triangulated structure on the homotopy category, but we make it explicit below in order to understand its effect on $\underline{\pi}_0$.

The map of cofiber sequences exhibits the left square as a homotopy pushout, giving rise to a Mayer-Vietoris sequence on homotopy groups. In particular:

**Corollary 14.2.1.** *The sequence*

$$\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n)\times_{U(n)}\overline{\mathcal{L}}_n)) \xrightarrow{(p_*,-(\psi_n)_*)} \underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n)/U(n)))\oplus\underline{\pi}_0(ku^{n-1}) \xrightarrow{\binom{(\alpha_n)_*}{(i_n)_*}} \underline{\pi}_0(ku^n) \to 0$$

*is exact.*

We recall from Section 3.3 that $L(\mathbb{C}^n)$ is a global universal space for $U(n)$, and hence $L(\mathbb{C}^n)/U(n)$ is a global classifying space for $U(n)$. The image of the fundamental class under $\alpha_n$ is by definition $\tau_n^u$, so via induction one sees that $\underline{\pi}_0(ku^n)$ is globally generated by the $\tau_1^u, \ldots, \tau_n^u$. Hence, the main work goes into understanding the relations, which is divided into the following two parts:

1. A description of $\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n))$.

2. Constructing $\psi_n$ and determining its effect on $\underline{\pi}_0$.

We start with number (1). Applying Lemma 2.3.1 to $Y = (L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)(\mathcal{U}_G)$ and $K = U(n)$, we see that the $G$-fixed points of the quotient decompose as

$$\coprod_{\langle \alpha : G \to U(n) \rangle} EC(\alpha) \times_{C(\alpha)} ((\overline{\mathcal{L}}_n)(\mathcal{U}_G))^{\Gamma(\alpha)}.$$

A tuple $(W_i, x_i)_{i\in I} \in \overline{\mathcal{L}}_n(\mathcal{U}_G)$ (with pairwise different $x_i$) is $\Gamma(\alpha)$-fixed if and only if each pair $(\alpha(g)(W_i), g \cdot x_i)$ is equal to some $(W_j, x_j)$ in the tuple. Hence, any such fixed point in particular gives rise to a non-trivial decomposition $\mathbb{C}^n = \bigoplus_{i\in I} W_i$ that is weakly fixed by $\alpha(G)$. Here, *weakly fixed* means that not necessarily every $W_i$ is fixed itself, but they may be permuted in a way encoded by a $G$-action on the indexing set $I$. The path-component of $(W_i, x_i)_{i\in I}$ in the $\Gamma(\alpha)$-fixed points only depends on this associated decomposition and every weakly fixed decomposition is realized. Furthermore, if one decomposition refines another, the associated fixed points lie in the same path-component. Written in a more coordinate-free way we get:

**Proposition 14.2.2.** *The set $\pi_0^G((L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n))$ stands in natural bijection to the set of pairs*

$$\{(W, \oplus_{i\in I} W_i) \mid W \ \textit{n-dim G-rep.}, W = \bigoplus_{i\in I} W_i \ \textit{non-trivial and weakly G-fixed}\}$$

*modulo isomorphisms of representations and refinement of decompositions.*

*In this description, the induced map to*

$$\pi_0^G(L(\mathbb{C}^n)/U(n)) \cong \{\textit{isom. classes of n-dim G-rep.}\}$$

*is given by forgetting the decompositions.*

Let $W = \bigoplus_{i\in I} W_i$ be such a weakly $G$-fixed partition and denote by $A_1, \ldots, A_k$ the orbits of the induced $G$-action on $I$. Then the decomposition

$$W = \bigoplus_{j=1,\ldots k} (\bigoplus_{i\in A_j} W_i)$$

is strongly fixed. It is non-trivial if $I$ is not transitive, in which case it refines a strongly fixed decomposition with two summands. So we see:

**Corollary 14.2.3.** *Every point in $\pi_0^G(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$ is represented by a weakly $G$-fixed decomposition of at least one of the following two types:*

1. *$W = W_1 \oplus W_2$ and $W_1, W_2$ are $G$-subrepresentations.*
2. *$W = W_1 \oplus \ldots \oplus W_k$ and the $W_i$ are permuted transitively by the $G$-action.*

Decompositions of the second type can be interpreted in the following way: Let $H$ be the isotropy subgroup of $W_1$ under the $G$-action on the set $\{W_i\}_{i=1,\ldots,k}$. Then $H$ is a subgroup of index $k$ in $G$, $W_1$ is an $H$-representation and the map $\mathrm{Ind}_H^G(W_1) \to W$ adjoint to the inclusion gives an isomorphism of $G$-representations. Vice versa, every induced representation $\operatorname{Ind}_H^G W'$ (where $H$ has finite index in $G$) possesses the weakly $G$-fixed decomposition $\operatorname{Ind}_H^G W' = \bigoplus_{gH \in G/H} gW'$. Hence, a general $G$-fixed point of the decomposition poset can be interpreted as exhibiting $W$ as a combination of sums and inductions.

**Remark 14.2.4.** This also lets us determine $\pi_0^G$ of the cone $L(\mathbb{C}^n)_+ \wedge_{U(n)} \overline{\mathcal{L}}_n^\diamond$ (and hence of the quotient $ku^n/ku^{n-1}$ via Proposition 3.1.5) explicitly. It is given by isomorphism classes of irreducible $n$-dimensional $G$-representations which are not the induction of a representation from a proper finite index subgroup. The smallest finite group for which there exists such a representation of dimension greater than 1 is $SL_2(\mathbb{F}_3)$. The tautological $U(n)$-representation $\tau_n^u$ always has this property, so we see that the maps $\underline{\pi}_0(ku^{n-1}) \to \underline{\pi}_0(ku^n)$ are never globally surjective.

The decompositions of Corollary 14.2.3 have universal representatives: Given $k, l > 0$ with $k + l = n$, the $(U(k) \times U(l))$-representation obtained by restricting $\tau_n^u$ along the embedding $U(k) \times U(l) \hookrightarrow U(n)$ decomposes as $\tau_k^u \oplus \tau_l^u$. We denote by

$$\widetilde{\alpha}(k, l) \in \pi_0^{U(k)\times U(l)}(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$$

the element associated to this decomposition. Likewise, given $i, j \in \mathbb{N}$ with $i \cdot j = n$, the restriction of $\tau_n^u$ along $\Sigma_i \wr U(j) \hookrightarrow U(n)$ is the induction of $p^*(\tau_j^u)$, where $p$ is the projection to $U(j)$. Hence there is an associated weakly $G$-fixed decomposition of type (2) above, which we denote by $\widetilde{\beta}(i, j)$. We obtain:

**Corollary 14.2.5.** *The* Rep*-functor*

$$\underline{\pi}_0(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$$

*is generated by the elements* $\{\widetilde{\alpha}(k, l)\}_{k+l=n}$ *and* $\{\widetilde{\beta}(i, j)\}_{i \cdot j = n}$*. Hence, by Proposition 3.1.5, so is* $\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n))$ *as a global functor.*

So it remains to show that $\psi_n$ indeed maps the $\widetilde{\alpha}'$ and $\widetilde{\beta}'s$ to the right hand sides of Equations (14.1.2) and (14.1.3) respectively. For this we require an explicit construction of $\psi_n$, which we now explain.

We quickly recall the objects involved: An element of $L(\mathbb{C}^n)(V)$ is a linear isometric embedding $\mathbb{C}^n \hookrightarrow \operatorname{Sym}(V_\mathbb{C})$. Points in $\overline{\mathcal{L}}_n$ are represented by tuples $(W_i, x_i)_{i \in I}$ indexed on a finite set $I$, where the $x_i$ are elements of $V$ and the $W_i$ are pairwise orthogonal subspaces of $\mathbb{C}^n$ which add up to all of $\mathbb{C}^n$. Furthermore, these tuples are required to satisfy the two conditions $\sum \dim(W_i) \cdot x_i = 0$ and $\sum \dim(W_i)|x_i|^2 = 1$. Finally, elements of $ku^n(V)$ are also represented by tuples $(W_i, x_i)_{i \in I}$, but this time the $W_i$ are orthogonal subspaces of $\operatorname{Sym}(V_\mathbb{C})$ and the only requirement is that the sum of the dimensions is at most $n$.

Now we come to the construction of $\psi_n$. We would like to define each level

$$(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)(V)_+ \wedge S^V \to ku^{n-1}(V)$$

by sending $(\varphi, (W_i, x_i)_{i \in I}, v)$ to the tuple $(\varphi(W_i), x_i + v)_{i \in I}$. However, even though all the $W_i$ necessarily have smaller dimension than $n$, their sum is still $n$-dimensional. So for fixed $v$ this tuple does not represent an element in $ku^{n-1}$. The idea is to shrink the domain of each of the coordinate functions $v \mapsto (\varphi(W_i), x_i+v)$, so that they become equal to the basepoint outside a certain neighborhood of $-x_i$. For this let $s : [0,\infty] \to [0,\infty]$ be a map which induces a homeomorphism $[0, 1/(2n^2)] \xrightarrow{\cong} [0,\infty]$ and is constant $\infty$ on $[1/(2n^2), \infty]$. Furthermore, given a finite tuple $x = (x_i)_{i\in I}$ of vectors of a (finite dimensional) real inner product space $V$ we let $p_x : V \to \langle \{x_i\}_{i\in I}\rangle \subseteq V$ denote the linear map defined by

$$p_x(v) = \sum_{i\in I} \langle v, x_i\rangle \cdot x_i.$$

We need the following properties of this map:

**Lemma 14.2.6.** *The value $p_x(v)$ only depends on the orthogonal projection of $v$ onto the span of the $x_i$ and is an automorphism when restricted to this span. Furthermore, it satisfies the inequality*

$$|p_x(x_j)| \geq |x_j|^3$$

*for every $j$ in $I$.*

**Remark 14.2.7.** The reason for using $p_x$ instead of the orthogonal projection onto the span of the $x_i$ is that the latter is not continuous in the $x_i$. However, the linear homotopy from the identity to $p_x$ restricts to an isotopy on this span and hence for a fixed tuple $x$ there is essentially no difference.

*Proof.* If a vector is orthogonal to each of the $x_i$ it is sent to 0 under $p_x$ and hence the value only depends on the orthogonal projection onto the span. For the other two statements we note that the scalar product of $p_x(v)$ and $v$ is equal to the sum of the squares $\langle v, x_i\rangle^2$. Hence, if $v$ is a non-zero vector in the span of the $x_i$, this scalar product is non-zero and in particular $p_x(v)$ is non-zero, so the restriction of $p_x$ to the span is injective. Finally, the stated inequality follows from

$$|p_x(x_j)| \cdot |x_j| \geq \langle p_x(x_j), x_j\rangle = \sum_{i\in I} \langle x_j, x_i\rangle^2 \geq \langle x_j, x_j\rangle^2 = |x_j|^4,$$

where the first step is the Cauchy-Schwarz inequality. □

We use this to obtain a selfmap $s_x^V : S^V \to S^V$ via the formula

$$s_x^V(v) = (s(|p_x(v)|) - |p_x v|) \cdot p_x(v) + v.$$

This map sends every vector $v$ for which $p_x(v)$ has length larger than $1/(2n^2)$ to the basepoint and is the identity on the orthogonal complement of the span of the $x_i$. Finally, for an element $(\varphi, (W_i, x_i)_{i\in I}, v)$ of $(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)(V)$ we set

$$\psi_n(\varphi, (W_i, x_i)_{i\in I}, v) = (\varphi(W_i), s_x^V(x_i + v))_{i\in I}.$$

This gives a map of orthogonal spectra: It commutes with the action of elements $A$ of $O(V)$ because of the equality $A(\langle v, x_i\rangle \cdot x_i) = \langle Av, Ax_i\rangle \cdot Ax_i$. Furthermore, if $W$ is another vector space and $w$ an element, then

$$s_x^{V\oplus W}(x_i+v+w)=s_x^V(x_i+v)+w$$

since $w$ is orthogonal to all the $x_i$ and hence $\psi_n$ also commutes with the structure map. In fact, assuring this equality was the reason for introducing the projections into the formula.

Finally, we have to show that $\psi_n$ does indeed take image in $ku^{n-1}$. Each component function

$$(v\mapsto(\varphi(W_i),s_x^V(x_i+v))$$

is equal to the basepoint on all points $v$ for which $p_x(v)$ is more than $1/(2n^2)$ away from $-p_x(x_i)$. We claim that for every fixed $v$ there is at least one $i$ such that this is the case. If this was not true, it would imply that all the $p_x(x_i)$ are less than $1/n^2$ away from each other. Now the conditions for the $x_i$ come into play. Since $\sum\dim(W_i)|x_i|^2=1$, there is at least one $j$ such that $|x_j|^2\geq 1/n$ and hence, by Lemma 14.2.6, we have

$$|p_x(x_j)|\geq 1/n^{3/2}\geq 1/n^2.$$

The equality

$$\sum\dim(W_i)\cdot p_x(x_j-x_i)+\sum\dim(W_i)\cdot p_x(x_i)=n\cdot p_x(x_j)$$

implies that

$$|\sum\dim(W_i)\cdot p_x(x_i)|\geq n\cdot|p_x(x_j)|-\sum\dim(W_i)|p_x(x_i-x_j)|>1/n-1/n=0,$$

which contradicts the condition $\sum\dim(W_i)\cdot x_i=0$. Hence, for every $(\varphi;(W_i,x_i)_{i\in I})$ the tuple $(\varphi(W_i),s_x^V(x_i+v))_{i\in I}$ contains at least one basepoint and thus represents an element in $ku^{n-1}$, as the total dimension of the remaining $\varphi(W_i)$ is strictly less than $n$.

Some justification is also needed that $\psi_n$ indeed turns Diagram (14.2.1) into a map of cofiber sequences, but we outsource this to Appendix B.1.

Now we let $(\varphi,(W_i,x_i)_{i\in I})$ be a $G$-fixed point of $(L(\mathbb{C}^n)\times_{U(n)}\overline{\mathcal{L}}_n)(V)$, assume that the balls of radius $1/(2n^2)$ around the $p_x(x_i)$ are pairwise disjoint and denote the span of the $x_i$ by $V'$. As noted before, the set $\{x_i\}_{i\in I}$ is permuted by the $G$-action. Then the induced $G$-map $S^V\to ku^{n-1}(V)$ given by $v\mapsto\psi_n(\varphi,(W_i,x_i)_{i\in I},v)$ is equal to the composite

$$S^{V'}\wedge S^{V-V'}\xrightarrow{p_x\wedge id}S^{V'}\wedge S^{V-V'}\to(\{x_i\}_+\wedge S^{V'})\wedge S^{V-V'}\cong\bigvee_{i\in I}S^V\xrightarrow{\bigvee\varphi(W_i)}ku^{n-1}(V),$$

where the second map is the smash product of $S^{V-V'}$ with the pinch map which collapses everything outside the balls of radius $1/(2n^2)$ around the $x_i$ to a point, and $\varphi(W_i)$ maps $v$ to the configuration $(\varphi(W_i),v)$. Up to homotopy, the first map can be replaced by the identity.

This lets us prove:

**Proposition 14.2.8.** *If an element $y \in \pi_0^G(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$ is associated to $W = W_1 \oplus W_2$ and $W_1, W_2$ are $G$-fixed, then*

$$(\psi_n)_*(y) = [W_1] + [W_2] \in \pi_0^G(ku^{n-1}).$$

*If $y$ is associated to $W = W_1 \oplus \ldots \oplus W_k$ and the $W_i$ are permuted transitively by $G$, then*

$$(\psi_n)_*(y) = \mathrm{tr}_H^G[W_1] \in \pi_0^G(ku^{n-1}),$$

*where $H$ is the subgroup of elements fixing $W_1$. In particular,*

$$(\psi_n)_*(\widetilde{\alpha}(k,l)) = p_1^*(\tau_k^u) + p_2^*(\tau_l^u)$$

*and*

$$(\psi_n)_*(\widetilde{\beta}(i,j)) = \mathrm{tr}_{\Sigma_{i-1}\wr U(j)}^{\Sigma_i \wr U(j)}(p^*(\tau_j^u)).$$

*Proof.* Without loss of generality we can assume that $W$ is equal to $\mathbb{C}^n$ with some $G$-action. We start with the first case. Let

$$(\varphi : \mathbb{C}^n \hookrightarrow \mathrm{Sym}(V_{\mathbb{C}}), (W_1, x_1), (W_2, x_2)) \in (L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)(V)$$

be a fixed point giving rise to such a decomposition, for some finite dimensional $G$-representation $V$. Let $V'$ be the (one-dimensional) span of the $x_i$. It carries the trivial $G$-action, since the $x_i$ are fixed by assumption. Furthermore, as there are only two points it is automatic that the intervals of radius $1/(2n^2)$ around them (or their images under $p_x$) are disjoint and thus the description above shows that $(\psi_n)_*(y)$ is the class of the composition

$$S^{V'} \wedge S^{V-V'} \to (\{x_1, x_2\}_+ \wedge S^{V'}) \wedge S^{V-V'} \cong S^V \wedge S^V \xrightarrow{\varphi(W_1) \vee \varphi(W_2)} ku^{n-1}(V).$$

Since $G$ acts trivially on the set $\{x_1, x_2\}$, the first map is just the usual pinch map and the second one is the wedge of two $G$-equivariant maps. Hence, this composite represents their sum

$$[\varphi(W_1)] + [\varphi(W_2)] = [W_1] + [W_2] \in \pi_0^G(ku^{n-1}).$$

Now we let $y$ correspond to a decomposition of type two, i.e., $\mathbb{C}^n = W_1 \oplus \ldots \oplus W_k$ and the $W_i$ are permuted transitively by $G$. Let $(\varphi, (W_i, x_i)_{i=1,\ldots,k})$ be a representative for this fixed point, chosen in a way that the $x_i$ have distance larger than $1/(2n^2)$ from each other. We again denote by $V'$ their span. Then, as seen above, $(\psi_n)_*(y)$ is represented by the composite

$$S^V \wedge S^{V-V'} \to (\{x_1, \ldots, x_k\}_+ \wedge S^{V'}) \wedge S^{V-V'} \cong \bigvee_{i=1,\ldots,k} S^V \xrightarrow{\bigvee \varphi(W_i)} ku^{n-1}(V).$$

Using that the $G$-set $\{x_1, \ldots, x_k\}$ is isomorphic to $G/H$, we see that this is precisely the definition of the transfer of the class $[\varphi(W_1)] = [W_1] \in \pi_0^H(ku^{n-1})$ recalled in Section

3.1: The first map is the 'transfer pinch map' and each wedge summand of the second equals the composite

$$S^V \xrightarrow{g^{-1}} S^V \xrightarrow{\varphi(W_1)} ku^{n-1} \xrightarrow{g\cdot} ku^{n-1}.$$

This finishes the proof. □

Now we are ready for:

*Proof of Theorem 14.1.3.* We proceed by induction on $n$, the case $n = 0$ being clear. So now let $n$ be a positive natural number and assume the statement to be true for $n-1$. Together with the induction hypothesis and the fact that $\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n)/U(n)))$ is generated by the tautological $U(n)$-representation, the exact sequence

$$\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)) \to \underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n)/U(n))) \oplus \underline{\pi}_0(ku^{n-1}) \to \underline{\pi}_0(ku^n) \to 0$$

of Corollary 14.2.1 shows that the global functor $\underline{\pi}_0(ku^n)$ is generated by the elements $\tau_1^u, \dots, \tau_n^u$. It further shows that the relations are generated by the ones of $\underline{\pi}_0(ku^{n-1})$, which we know by induction, and the image of $\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n))$ in

$$\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n)/U(n))) \oplus \underline{\pi}_0(ku^{n-1}).$$

By Corollary 14.2.5, the global functor $\underline{\pi}_0(\Sigma_+^\infty(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n))$ is generated by the elements $\{\widetilde{\alpha}(k,l)\}_{k+l=n}$ and $\{\widetilde{\beta}(i,j)\}_{i\cdot j=n}$, which are sent to the relations $\alpha(k,l)$ and $\beta(i,j)$ by Proposition 14.2.8, so we are done. □

# Chapter 15

# Complexity filtrations on 0-th homotopy

In this chapter we deal with the effect of complexity filtrations on $\underline{\pi}_0$. The global formula turns out to be similar to the one for the symmetric product filtration, with the Burnside ring replaced by the representation ring.

The map

$$\mathrm{Rep}_X(G) \cong \pi_0^G(kX) \to \pi_0^G(A_\infty^X) \cong \mathbb{Z}$$

is the augmentation which sends a representation to its dimension respectively rank. This is again said with the usual caveat for $ku$ and $ko$ and non-discrete compact Lie groups $G$, where it restricts to the augmentation on the representation ring as a subgroup of $\pi_0^G(kX)$.

The complexity filtration induces a sequence of global functors

$$\underline{\pi}_0(kX) \to \underline{\pi}_0(A_1^X) \to \underline{\pi}_0(A_2^X) \to \ldots \to \underline{\pi}_0(A_\infty^X)$$

which factor this augmentation.

Again we begin by listing the results.

## 15.1 Statement of results

### 15.1.1 Global complex $K$-theory

We recall that $\tau_n^u$ denotes the tautological $n$-dimensional complex $U(n)$-representation. We show:

**Theorem 15.1.1** (Complexity filtration on $\underline{\pi}_0$)**.** *For all $n \in \mathbb{N}$ the map $q_n : ku \to A_n^u$ induces a surjection on $\underline{\pi}_0$ and the kernel is generated as a global functor by the single element*

$$\tau_n^u - n \cdot [1] \in \pi_0^{U(n)}(ku).$$

*In particular, there is an isomorphism of global functors*

$$\underline{\pi}_0(A_n^u) \cong \underline{\pi}_0(ku)/(\tau_n^u - n \cdot [1]).$$

It turns out that if $G$ is finite, this filtration already stabilizes at $\pi_0^G(A_1^u)$, for algebraic reasons closely related to Brauer's induction theorem, which says that the complex representation ring is generated by inductions of 1-dimensional representations. The fastest way to see that the filtration stabilizes is to make use of the fact that the representation rings form a global functor also for compact Lie groups, via Segal's smooth induction [Seg68b]. The smooth transfer takes on the following values:

$$\begin{aligned} \mathrm{Ind}_{U(1)\times U(n-1)}^{U(n)}(p^*(\tau_1^u)) &= \tau_n^u \\ \mathrm{Ind}_{U(1)\times U(n-1)}^{U(n)}([1]) &= n\cdot[1] \end{aligned}$$

These equalities follow from the character formula (cf. [Seg68b, page 119] and [Oli98, Proposition 2.3]). Hence, the class $\tau_n^u - n\cdot[1]$ can be obtained by applying restriction and induction to the class $\tau_1^u - [1]$ and thus lies in the global functor generated by it. Since $\pi_0^G(ku)$ agrees with the representation ring global functor on finite groups, this shows that all restrictions of $\tau_n^u - n\cdot[1]$ to finite groups already lie in the global functor generated by $\tau_1^u - [1]$. In addition, this shows that if $\pi_0^G(ku)$ was the representation ring also for infinite compact Lie groups, the filtration would stabilize at stage 1 globally. As it stands it does not, $\tau_n^u$ is identified with the trivial $n$-dimensional representation exactly in the $n$-th step, since $U(n)$ has no proper finite index subgroups.

### 15.1.2 Global real $K$-theory

We have the analogous result for real $K$-theory $ko$:

**Theorem 15.1.2** (Complexity filtration on $\underline{\pi}_0$)**.** *For all $n\in\mathbb{N}$ the map $q_n : ko\to A_n^o$ induces a surjection on $\underline{\pi}_0$ and the kernel is generated as a global functor by the single element*

$$\tau_n^o - n\cdot[1]\in\pi_0^{O(n)}(ko).$$

*In particular, there is an isomorphism of global functors*

$$\underline{\pi}_0(A_n^o)\cong\underline{\pi}_0(ko)/(\tau_n^o - n\cdot[1]).$$

This time the filtration does not stabilize at $A_1^o$ for all finite groups. One can see this by considering the cyclic group $C_3$, for which no subgroup has a non-trivial 1-dimensional real representation (cf. Example 16.0.10).

However, it stabilizes one step later. One can show the induction formulas

$$\begin{aligned} \mathrm{Ind}_{O(2)\times O(2k-2)}^{O(2k)}(p^*(\tau_2^o)) &= \tau_{2k}^o \\ \mathrm{Ind}_{O(2)\times O(2k-2)}^{O(2k)}([1]) &= k\cdot[1] \end{aligned}$$

for all $k\in\mathbb{N}$, again using the character formula. Hence, $\tau_n^o - n\cdot[1]$ lies in the global functor generated by $\tau_2^o - 2\cdot[1]$ for all even $n$ and thus for all $n$.

### 15.1.3 Global algebraic $K$-theory

We again start with the case of finite $R$, where the formula is analogous to the topological case.

**Theorem 15.1.3** (Complexity filtration on $\underline{\pi}_0$)**.** *Let $R$ be a finite ring and $n \in \mathbb{N}$. Then the map $(q_n)_* : \mathrm{Rep}_R(-) \cong \underline{\pi}_0(kR) \to \underline{\pi}_0(A_n^R)$ is surjective with kernel generated as a $\mathcal{F}in$-global functor by the element*

$$\tau_n^R - n \cdot [1] \in \pi_0^{GL_n(R)}(kR) \cong \mathrm{Rep}_R(GL_n(R)).$$

*In particular, there is an isomorphism of $\mathcal{F}in$-global functors*

$$\underline{\pi}_0(A_n^R) \cong \mathrm{Rep}_R(-)/(\tau_n^R - n \cdot [1]).$$

For general $R$ these universal elements are again not part of the theory and hence there is no such compact description. The result then reads as follows:

**Proposition 15.1.4** (Description for arbitrary rings)**.** *Let $R$ be a ring satisfying dimension invariance and $n \in \mathbb{N}$. Then the map $(q_n)_* : \mathrm{Rep}_R(-) \cong \underline{\pi}_0(kR) \to \underline{\pi}_0(A_n^R)$ is surjective with kernel generated as a $\mathcal{F}in$-global functor by the elements*

$$[W] - n \cdot [1] \in \pi_0^G(kR) \cong \mathrm{Rep}_R(G)$$

*where $(G, W)$ runs through isomorphism classes of pairs with $W$ an $n$-dimensional $G$-representation over $R$.*

In contrast to complex and real topological $K$-theory, the sequence of the $\pi_0^G(A_n^R)$ can take arbitrarily long to stabilize. For instance this is the case for $R = \mathbb{Q}$ or a finite field, as the examples in Chapter 16 show.

## 15.2 Proof

We give two proofs and again concentrate on the case of topological complex $K$-theory. The first proof makes use of the construction of a homotopy pushout square that describes the passage from $A_{n-1}^u$ to $A_n^u$. The second proof is an elementary computation in global functors which assumes Schwede's result for the symmetric product filtration as well as the description for the modified rank filtration of the previous chapter.

### 15.2.1 Geometric proof

The method is similar to that of the last section, but shorter as we can use the constructions we made there. The proof makes use of an exact sequence associated to a homotopy-cocartesian square, which this time takes the form

$$\begin{tikzcd} \Sigma_+^\infty((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n)) \arrow[r, "p"] \arrow[d, "\gamma_n"] & \mathbb{S} \arrow[d, "i"] \\ A_{n-1}^u \arrow[r, "p_n"] \arrow[ur, phantom, "h"] & A_n^u, \end{tikzcd} \qquad (15.2.1)$$

where $p$ is induced from the constant map $((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n) \to *$. This homotopy-cocartesian square is established by forming the homotopy-pushout of three homotopy-cocartesian squares:

$$\begin{tikzcd}
\Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)_+ \arrow[r] \arrow[d, "\psi_n"'] & \Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \arrow[d, "\alpha_n"'] \arrow[r, phantom, "\Longrightarrow"] & \Sigma^\infty_+(\overline{\mathcal{L}}_n/U(n)) \arrow[r] \arrow[d, "\psi_n"] & \mathbb{S} \arrow[d] \\
ku^{n-1} \arrow[r, "i_n"] \arrow[dr, phantom, "\Downarrow"] & ku^n & Sp^{n-1} \arrow[r, "i_n"] & Sp^n \\
\Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \arrow[r, "id"'] \arrow[d, "\alpha_n"'] & \Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \arrow[d, "\alpha_n"'] & & \\
ku \arrow[r, "id"] & ku & &
\end{tikzcd}$$

The fact that the first square is a homotopy-pushout (and in particular the construction of a homotopy between the two composites) is treated in Appendix B.1. The square on the right hand side can be dealt with by the same formulas, replacing complex subspaces by natural numbers, making the upper double arrow a homotopy-coherent natural transformation between the two squares. Finally, the lower square of course commutes on the nose and the vertical double arrow can be made a homotopy-coherent transformation by using the same homotopy as in the upper square. Hence we see that there exists a cocartesian square of the form (15.2.1) above.

A comparison of the associated long exact sequences shows:

**Corollary 15.2.1.** *There is an exact sequence of global functors*

$$\ker(p_*) \xrightarrow{(\gamma_n)_*} \underline{\pi}_0(A^u_{n-1}) \xrightarrow{(p_n)_*} \underline{\pi}_0(A^u_n) \to \underline{\pi}_0(A^u_n/A^u_{n-1}) \cong \operatorname{coker}(p_*) \to 0.$$

We now consider the inclusion

$$k_n : L(\mathbb{C}^n)/U(n) \hookrightarrow (L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n).$$

By the definition of $\gamma_n$ above, it fits into the following homotopy-commutative square:

$$\begin{tikzcd}
\Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \arrow[r, "\Sigma^\infty_+ k_n"] \arrow[d, "\alpha_n"'] & \Sigma^\infty_+((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n)) \arrow[d, "\gamma_n"] \\
ku \arrow[r, "q_{n-1}"'] & A_{n-1}
\end{tikzcd} \tag{15.2.2}$$

Furthermore, we have:

**Lemma 15.2.2.** *For every compact Lie group $G$ the induced map*

$$\pi_0^G(L(\mathbb{C}^n)/U(n)) \xrightarrow{(k_n)_*} \pi_0^G((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n))$$

*is surjective.*

*Proof.* Since the square

$$\begin{array}{ccc} \pi_0^G(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n) & \longrightarrow & \pi_0^G(L(\mathbb{C}^n)/U(n)) \\ \downarrow & & \downarrow \\ \pi_0^G(\overline{\mathcal{L}}_n/U(n)) & \longrightarrow & \pi_0^G((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n)) \end{array}$$

is a pushout of sets (it arises from applying $\pi_0$ to a homotopy pushout of spaces), it suffices to show that the projection

$$L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n \to \overline{\mathcal{L}}_n/U(n)$$

induces a surjection on $\pi_0^G$. An element of $\overline{\mathcal{L}}_n/U(n)$ is represented by a tuple $(W_i, x_i)_{i\in I}$, where the $x_i$ are elements of a complete $G$-universe $\mathcal{U}_G$ and the $W_i$ form an orthogonal decomposition of $\mathbb{C}^n$ (such that the equalities $\sum \dim W_i \cdot x_i = 0$ and $\sum \dim W_i |x_i|^2 = 1$ are satisfied). Since the $U(n)$-action is modded out, the represented element only depends on the $x_i$ and the dimensions of the $W_i$. That the tuple $(W_i, x_i)_{i\in I}$ is a $G$-fixed point means that every element of $G$ maps each $x_i$ to an element $x_{g(i)}$ such that $\dim W_i = \dim W_{g(i)}$. Now let $\alpha : G \to U(n)$ be any homomorphism such that $\alpha(G)\cdot W_i = W_{g(i)}$. For example one can choose orthonormal bases $\{a_{i,k}\}$ of the $W_i$ and define $\alpha(g)(a_{i,k}) = a_{g(i),k}$. Furthermore, let $\varphi : \mathbb{C}^n \hookrightarrow \mathcal{U}_G$ be a linear isometric embedding which is equivariant for the action on $\mathbb{C}^n$ induced by $\alpha$. Then the tuple $(\varphi, (W_i, x_i)_{i\in I}) \in (L(\mathbb{C}^n) \times \overline{\mathcal{L}}_n)(\mathcal{U}_G)$ is a fixed point for the graph $\Gamma(\alpha)$ and hence a $G$-fixed point of the quotient. Its projection to $\overline{\mathcal{L}}_n/U(n)$ gives back the tuple we started with, which hence lies in the image, and so we are done. □

**Remark 15.2.3.** Conceptually, the main input in the proof of the previous lemma is that for every complete subgroup $L$ of $U(n)$ the projection $N_{U(n)}L \to W_{U(n)}L$ splits.

**Corollary 15.2.4.** *The global functor*

$$\underline{\pi}_0(\Sigma_+^\infty((L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n)))$$

*is generated by the element $(k_n)_*(\tau_n^u)$. Hence, the kernel of $\gamma_n$ is generated as a global functor by the element $(k_n)_*(\tau_n^u - n \cdot [1])$.*

Now we are ready for:

*Proof of Theorem 15.1.1.* We make use of the exact sequence of Corollary 15.2.1. Since $(L(\mathbb{C}^n) * \overline{\mathcal{L}}_n)/U(n)$ is not the empty orthogonal space, the projection to a point splits and hence $p_*$ is surjective. It follows that $\underline{\pi}_0(A_n^u/A_{n-1}^u)$ is zero and we see that all the maps

$$(p_n)_* : \underline{\pi}_0(A_{n-1}^u) \to \underline{\pi}_0(A_n^u)$$

are surjective. Hence so is the composition $(q_n)_* : \underline{\pi}_0(ku) \to \underline{\pi}_0(A_n^u)$ and we have proved the first statement.

It remains to show that the kernel of $(q_n)_* : \underline{\pi}_0(ku) \to \underline{\pi}_0(A_n^u)$ is generated by the element $\tau_n^u - n \cdot [1] \in \pi_0^{U(n)}(ku)$. We proceed by induction on $n$. For $n = 0$ there is nothing to show. Now let $n \geq 1$ and assume the statement to be proved for $n-1$. By Corollary 15.2.4, the kernel of $(p_n)_* : \underline{\pi}_0(A^u_{n-1}) \to \underline{\pi}_0(A^u_n)$ is generated by the element

$$(\gamma_n)_*((k_n)_*(\tau^u_n - n \cdot [1]))$$

which by the commutativity of Square (15.2.2) above is equal to $(q_{n-1})_*(\tau^u_n - n \cdot [1])$. Thus, by induction hypothesis we see that the kernel of $(q_n)_*$ is generated as a global functor by the elements $(\tau^u_n - n \cdot [1])$ and $(\tau^u_{n-1} - (n-1) \cdot [1])$. But the latter is obtained from the former by restriction along the inclusion $U(n-1) \hookrightarrow U(n)$ and it follows that $(\tau^u_n - n \cdot [1])$ generates the whole kernel. This finishes the proof. □

### 15.2.2 Computation in global functors

For this proof we assume Schwede's result explained in Section 3.2.3 and recall the following homotopy-pushout defining $A^u_n$:

$$\begin{tikzcd} ku^n \arrow[r, "i^{(n)}"] \arrow[d, "q'_n"'] & ku \arrow[d, "q_n"] \\ Sp^n \arrow[r, "j^{(n)}"'] \arrow[ur, phantom, "\ulcorner\cdot\, h"] & A^u_n \end{tikzcd}$$

All the spectra involved are connective, so the associated Mayer-Vietoris sequence gives rise to a short exact sequence

$$\underline{\pi}_0(ku^n) \xrightarrow{(-(q'_n)_*, i^{(n)}_*)} \underline{\pi}_0(Sp^n) \oplus \underline{\pi}_0(ku) \xrightarrow{\binom{j^{(n)}_*}{(q_n)_*}} \underline{\pi}_0(A^u_n) \to 0. \tag{15.2.3}$$

By Theorem 3.2.2, we know that $\underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(Sp^n)$ is surjective. Since the map $\underline{\pi}_0(ku^1) \to \underline{\pi}_0(\mathbb{S})$ is also surjective (the unit map $\mathbb{S} \to ku^1$ induces a section), we find that so is $\underline{\pi}_0(ku^n) \to \underline{\pi}_0(Sp^n)$ and hence also $\underline{\pi}_0(ku) \to \underline{\pi}_0(A^u_n)$. So, by a diagram chase we see that the exact sequence (15.2.3) induces another short exact sequence

$$\ker((q'_n)_*) \xrightarrow{i^{(n)}_*} \underline{\pi}_0(ku) \xrightarrow{(q_n)_*} \underline{\pi}_0(A^u_n) \to 0. \tag{15.2.4}$$

Hence we need to study the map $(q'_n)_* : \underline{\pi}_0(ku^n) \to \underline{\pi}_0(Sp^n)$.

**Lemma 15.2.5.** *For $1 \leq i \leq n$ we have*

$$(q'_n)_*(\tau^u_i) = i \cdot [1] \in \pi_0^{U(n)}(Sp^n).$$

*Proof.* The class $\tau^u_i \in \pi_0(ku^n)$ is represented by the $U(i)$-map

$$\begin{aligned} S^{\mathbb{C}^i} &\to ku^n(\mathbb{C}^i) \\ v &\mapsto (v, \mathbb{C}^i), \end{aligned}$$

so we see that $(q'_n)_*(\tau_i^u)$ is represented by the $U(i)$-map

$$\begin{aligned} S^{\mathbb{C}^i} &\to Sp^n(S^{\mathbb{C}^i}) \\ v &\mapsto (v, i), \end{aligned}$$

or in other words, the $\mathbb{C}^i$-fold suspension of the map

$$\begin{aligned} S^0 &\to Sp^n(S^0) \cong \mathbb{N} \\ 0 &\mapsto i. \end{aligned} \tag{15.2.5}$$

In particular, $(q'_n)_*(\tau_i^u)$ is the restriction of the non-equivariant class associated to the same map (15.2.5). Hence it suffices to see that this class represents $i \cdot [1] \in \pi_0^{\{e\}}(Sp^n)$. Since the map $\pi_0^{\{e\}}(Sp^n) \to \pi_0^{\{e\}}(Sp^\infty)$ is an isomorphism, we can equivalently check this in $Sp^\infty$. But there it is clear, since $Sp^\infty$ is the realization of a special $\Gamma$-space and

$$\mathbb{N} \cong \pi_0^{\{e\}}(\mathbb{N}) \to \pi_0^{\{e\}}(Sp^\infty) \cong \mathbb{Z}$$

is the group completion map. □

So we see that $\tau_n^u - n \cdot [1]$ lies in the kernel of $(q'_n)_*$. Hence, by the short exact sequence (15.2.4), this relation also holds in $\pi_0^{U(n)}(A_n^u)$. We are left to show that there are no further relations, or in other words:

**Proposition 15.2.6.** *The kernel of $(q'_n)_*$ is generated as a global functor by the element*

$$\tau_n^u - n \cdot [1].$$

*Proof.* We claim that the induced map

$$\underline{\pi}_0(ku^n)/(\tau_n^u - n \cdot [1]) \to \underline{\pi}_0(Sp^n) \tag{15.2.6}$$

is an isomorphism, which proves the proposition.

The global functor $\underline{\pi}_0(\mathbb{S})$ is the free global functor on the class $[1] \in \pi_0^{\{e\}}(\mathbb{S})$ (cf. Example 3.1.6). So there is a unique map of global functors

$$\underline{\pi}_0(\mathbb{S}) \to \underline{\pi}_0(ku^n)/(\tau_n^u - n \cdot [1]) \tag{15.2.7}$$

sending $[1] \in \pi_0^{\{e\}}(\mathbb{S})$ to $[1] \in \pi_0^{\{e\}}(ku^n)/(\tau_n^u - n \cdot [1])$. It takes the element $\tau_n^\Sigma - n \cdot [1]$ to

$$\mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}([1]) - n \cdot [1]$$

which equals

$$\mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}[1] - n \cdot [1] \tag{15.2.8}$$

because of the relation $b(n, 1)$ in $\underline{\pi}_0(ku^n)$ (cf. Theorem 14.1.3). The $\Sigma_n$-representation $\mathrm{Ind}_{\Sigma_{n-1}}^{\Sigma_n}[1]$ is $n$-dimensional, so the class (15.2.8) can be obtained as a restriction of

$\tau_n^u - n \cdot [1]$. Hence, by Theorem 3.2.2, the map (15.2.7) factors as

$$\underline{\pi}_0(Sp^n) \to \underline{\pi}_0(ku^n)/(\tau_n^u - n \cdot [1]).$$

We claim that this map is an inverse to (15.2.6). It is clear that one composite is the identity of $\underline{\pi}_0(Sp^n)$, since it sends the generator [1] to [1].

The other composite takes the class of each $\tau_i^u$ to $i \cdot [1]$. The difference $\tau_i^u - i \cdot [1]$ equals the restriction of $\tau_n^u - n \cdot [1]$ along the embedding $U(i) \hookrightarrow U(n)$, making use of relation $a(i, n-i)$. Hence, each $\tau_i^u$ is mapped to itself. Since these classes generate $\underline{\pi}_0(ku^n)$ by Theorem 14.1.3, we are done. $\square$

This finishes the proof of Theorem 15.1.1.

# Chapter 16

# Examples

In this final chapter we give examples for the effect on $\pi_0^G$ of the modified rank and complexity filtrations for various finite groups $G$ and topological or discrete rings. The purpose is twofold: On the one hand we want to explain how one computes specific values $\pi_0^G(X)$ for the global spectra $X$ that appeared in this thesis, using the global formulas we gave in the previous sections. On the other hand we try to demonstrate that the behavior at a specific group is often quite complicated, while the global formula is not.

**Remark 16.0.7.** All the homotopy groups we compute turn out to be torsion-free. We do not know whether this is true in general. It does not hold for symmetric products of spheres, cf. [Sch14, Example 4.6].

**Example 16.0.8** (The symmetric group $\Sigma_3$, over $\mathbb{C}$)**.** We begin by going through the example $G = \Sigma_3$ in detail. In order to compute the values of the modified rank and complexity filtration for $\Sigma_3$, we need to know its subgroups, their complex representation rings (together with the conjugation action) and the induction maps between them. The conjugacy classes of subgroups are given by the trivial group $\{e\}$, the cyclic groups $C_2$ (represented by any transposition) and $C_3$ (the normal subgroup of 3-cycles), and the whole group $\Sigma_3$. Their representation rings are

$$\begin{aligned}
RU(\{e\}) &\cong \mathbb{Z}\{[1]\} \\
RU(C_2) &\cong \mathbb{Z}\{[1],[-1]\} \\
RU(C_3) &\cong \mathbb{Z}\{[1],[\eta_3],[\eta_3^2]\} \\
RU(\Sigma_3) &\cong \mathbb{Z}\{[1],[\mathrm{sgn}],[\nu_3]\},
\end{aligned}$$

where $\nu_3$ is the 2-dimensional reduced natural representation and $\eta_3$ is a primitive third root of unity. The $C_3$-representations $\eta_3$ and $\eta_3^2$ are conjugate under the Weyl-group action. Furthermore, we have the following formulas for induction:

$$\begin{aligned}
\mathrm{Ind}_{\{e\}}^{C_2}([1]) &= [1]+[-1] \\
\mathrm{Ind}_{\{e\}}^{C_3}([1]) &= [1]+[\eta_3]+[\eta_3^2] \\
\mathrm{Ind}_{\{e\}}^{\Sigma_3}([1]) &= [1]+[\mathrm{sgn}]+2\cdot[\nu_3]
\end{aligned}$$

$$\begin{aligned}
\mathrm{Ind}_{C_2}^{\Sigma_3}([1]) &= [1]+[\nu_3]\\
\mathrm{Ind}_{C_2}^{\Sigma_3}([-1]) &= [\mathrm{sgn}]+[\nu_3]\\
\mathrm{Ind}_{C_3}^{\Sigma_3}([1]) &= [1]+[\mathrm{sgn}]\\
\mathrm{Ind}_{C_3}^{\Sigma_3}([\eta_3]) &= [\nu_3]
\end{aligned}$$

To compute the first term $\pi_0^{\Sigma_3}(ku^1)$ we need to consider all transfers of 1-dimensional representations (modulo the respective Weyl group actions) so we see that it is given by the free abelian group

$$\mathbb{Z}\{[1],[\mathrm{sgn}],\mathrm{tr}_{C_3}^{\Sigma_3}([1]),\mathrm{tr}_{C_3}^{\Sigma_3}([\eta_3]),\mathrm{tr}_{C_2}^{\Sigma_3}([1]),\mathrm{tr}_{C_2}^{\Sigma_3}([-1]),\mathrm{tr}_{\{e\}}^{\Sigma_3}([1])\}.$$

For the second stage we add on everything that comes from a 2-dimensional irreducible representation (since, using the relation $a(1,1)$ of Theorem 14.1.3, we can replace a non-irreducible representation by the homotopy-theoretic sum of its summands). In this case there is only one 2-dimensional irreducible representation, the $\Sigma_3$-representation $\nu_3$. Taking into account the relation $b(2,1)$ we furthermore have to identify all representations that are at most 2-dimensional and an induction over a proper subgroup with the homotopy-theoretic transfer of that respective representation, transferred up to the whole group $\Sigma_3$ if necessary. Considering the formulas for induction above, this means that we have to identify the following:

$$\begin{aligned}
\mathrm{tr}_{\{e\}}^{\Sigma_3}([1]) = \mathrm{tr}_{C_2}^{\Sigma_3}(\mathrm{tr}_{\{e\}}^{C_2}([1])) &\sim \mathrm{tr}_{C_2}^{\Sigma_3}([1])+\mathrm{tr}_{C_2}^{\Sigma_3}([-1])\\
\mathrm{tr}_{C_3}^{\Sigma_3}([1]) &\sim [1]+[\mathrm{sgn}]\\
\mathrm{tr}_{C_2}^{\Sigma_3}([\eta_3]) &\sim [\nu_3]
\end{aligned}$$

So we see that $\pi_0^{\Sigma_3}(ku^2)$ is a free group with basis

$$\{[1],[\mathrm{sgn}],[\nu_3],\mathrm{tr}_{C_2}^{\Sigma_3}([1]),\mathrm{tr}_{C_2}^{\Sigma_3}([-1])\}.$$

Since there are no irreducible representations of dimension 3 or higher for any of the subgroups of $\Sigma_3$, we from now on do not add any new generators but only have to take into account new relations. In the third step the universal relation $b(3,1)$ shows that $\mathrm{tr}_{C_2}^{\Sigma_3}([1])$ is identified with $[1]+[\nu_3]$ and $\mathrm{tr}_{C_2}^{\Sigma_3}([-1])$ with $[\mathrm{sgn}]+[\nu_3]$. Hence, $\pi_0^{\Sigma_3}(ku^3)$ is isomorphic to $\mathbb{Z}\{[1],[\mathrm{sgn}],[\nu_3]\}$, which is the representation ring of $\Sigma_3$, and the rank filtration is constant from then on.

Hence, recording only the isomorphism types, we summarize:

| $n$ | 1 | 2 | $\geq 3$ |
|---|---|---|---|
| $\pi_0^{\Sigma_3}(ku^n)$ | $\mathbb{Z}^7$ | $\mathbb{Z}^5$ | $\mathbb{Z}^3$ |

For every finite group $G$ the complexity filtration over $\mathbb{C}$ stabilizes on $\pi_0^G(-)$ at stage 1, as noted in Section 15.1.1. For $\Sigma_3$ this can be seen concretely as follows: Recall that we start with $\pi_0^{\Sigma_3}(ku)$, the representation ring, a free group on the classes $[1],[\mathrm{sgn}]$ and $[\nu_3]$. As the sign representation is one-dimensional, it is identified with $[1]$ in $\pi_0^{\Sigma_3}(A_1^u)$.

Furthermore, $\nu_3$ is the induction of the 1-dimensional $C_3$-representation $\eta_3$. So, since $[\eta_3]$ is identified with the trivial representation, application of $\mathrm{Ind}_{C_3}^{\Sigma_3}(-)$ shows that $[\nu_3]$ is identified with $[1]+[\mathrm{sgn}]$. We already argued that $[\mathrm{sgn}]$ is identified with $[1]$, so this shows that $[\nu_3]$ becomes $2\cdot[1]$ in $\pi_0^{\Sigma_3}(A_1^u)$, which is hence isomorphic to $\mathbb{Z}$.

**Example 16.0.9** (The symmetric group $\Sigma_3$, over $\mathbb{R}$)**.** We again discuss the symmetric group on 3 letters, this time over $\mathbb{R}$. Even though every complex $\Sigma_3$-representation is induced up from a real one and hence the representation rings are isomorphic, the effect of the modified rank and complexity filtrations on $\pi_0^{\Sigma_3}$ differ. Again we have to start with the representation rings over all subgroups, and the only difference to the complex case is at the subgroup $C_3$, where $RO(C_3)$ only has rank 2 with basis $[1]$ and the reduced regular representation $[\overline{\rho}_{C_3}]$. Consequently, we find that $\pi_0^{C_3}(ko^1)$ has one basis element less, it is the free group on

$$\{[1],[\mathrm{sgn}],\mathrm{tr}_{C_3}^{\Sigma_3}([1]),\mathrm{tr}_{C_2}^{\Sigma_3}([1]),\mathrm{tr}_{C_2}^{\Sigma_3}([-1]),\mathrm{tr}_{\{e\}}^{\Sigma_3}([1])\}.$$

In the next step the irreducible representations $[\nu_3]$ and $\mathrm{tr}_{C_3}^{\Sigma_3}[\overline{\rho}_{C_3}]$ are added, and again $\mathrm{tr}_{\{e\}}^{\Sigma_3}([1])$ is identified with $\mathrm{tr}_{C_2}^{\Sigma_3}([1])+\mathrm{tr}_{C_2}^{\Sigma_3}([-1])$, as well as $\mathrm{tr}_{C_3}^{\Sigma_3}([1])$ with $[1]+[\mathrm{sgn}]$. This gives

$$\pi_0^{\Sigma_3}(ko^2)\cong\mathbb{Z}\{[1],[\mathrm{sgn}],[\nu_3],\mathrm{tr}_{C_3}^{\Sigma_3}[\overline{\rho}_{C_3}],\mathrm{tr}_{C_2}^{\Sigma_3}([1]),\mathrm{tr}_{C_2}^{\Sigma_3}([-1])\}.$$

In the third step the latter two classes are identified with $[1]+[\nu_3]$ and $[-1]+[\nu_3]$ respectively, hence they become algebraically dependent of the first three. Furthermore, applying $\mathrm{tr}_{C_3}^{\Sigma_3}(-)$ to the relation $\mathrm{tr}_{\{e\}}^{C_3}([1])=[1]+[\overline{\rho}_{C_3}]$ (plus using earlier relations) shows that $\mathrm{tr}_{C_3}^{\Sigma_3}([\overline{\rho}_3])$ represents the same class as $2\cdot[\nu_3]$, so $\pi_0^{\Sigma_3}(ko^3)$ is isomorphic to the representation ring $RO(\Sigma_3)$.

So we get:

| $n$ | 1 | 2 | $\geq 3$ |
|---|---|---|---|
| $\pi_0^{\Sigma_3}(ko^n)$ | $\mathbb{Z}^6$ | $\mathbb{Z}^6$ | $\mathbb{Z}^3$ |

We note, however, that the map $\pi_0^{\Sigma_3}(ko^1)\to\pi_0^{\Sigma_3}(ko^2)$ is not an isomorphism.

The complexity filtration is also different to the complex one: In the first step $[1]$ and $[\mathrm{sgn}]$ are identified, but this time there are no further relations. This is because applying $\mathrm{Ind}_{C_2}^{\Sigma_3}$ to the identification $[1]\sim[-1]$ contributes nothing new, and there is only one 1-dimensional representation of $C_3$ over $\mathbb{R}$. So

$$\pi_0^{\Sigma_3}(A_1^o)\cong\mathbb{Z}\{\overline{[1]},\overline{[\nu_3]}\}.$$

In the second step $[\nu]$ is identified with $2\cdot[1]$ and hence $\pi_0^{\Sigma_3}(A_2^o)\cong\mathbb{Z}$, which is true for all $\pi_0^G(A_2^o)$ with $G$ finite.

**Example 16.0.10** (Cyclic groups of prime order)**.** Having seen the general algorithm, we now go back to the easiest example and use it to illustrate the behavior over various rings. Let $p$ be a prime and $C_p$ the cyclic group with $p$ elements.

**Over** $\mathbb{C}$: The irreducible $\mathbb{C}[C_p]$-representations are given by $\eta_p^1,\eta_p^2,\ldots,\eta_p^p$, where $\eta_p$

is a primitive $p$-th root of unity. So we find that

$$\pi_0^{C_p}(ku^n) \cong \begin{cases} \mathbb{Z}\{[\eta_p^1], [\eta_p^2], \ldots, [\eta_p^p], \mathrm{tr}_1^{\mathbb{Z}/p}[1]\} & \text{for } 0 < n < p \\ \mathbb{Z}\{[\eta_p^1], [\eta_p^2], \ldots, [\eta_p^p]\} & \text{for } n \geq p. \end{cases}$$

As mentioned before, $\pi_0^G(A_n^u)$ is isomorphic to $\mathbb{Z}$ for all $n \geq 1$ and any finite group $G$.

**Over** $\mathbb{R}$: If $p$ is 2, all complex representations are already defined over the reals, so the filtrations are the same. If $p$ is odd, there are $(p-1)/2$ isomorphism classes of 2-dimensional indecomposable representations, which can be expressed as the underlying real representations of $\eta_p^1, \ldots, \eta_p^{(p-1)/2}$, plus the trivial 1-dimensional one. So we find that

$$\pi_0^{\mathbb{C}_p}(ko^n) \cong \begin{cases} \mathbb{Z}\{[1], \mathrm{tr}_{\{e\}}^{C_p}[1]\} & \text{for } n = 1 \\ \mathbb{Z}\{[1], \mathrm{res}_{\mathbb{R}}^{\mathbb{C}}(\eta_p^1), \ldots, \mathrm{res}_{\mathbb{R}}^{\mathbb{C}}(\eta_p^{(p-1)/2}), \mathrm{tr}_{\{e\}}^{C_p}[1]\} & \text{for } 1 < n < p \\ \mathbb{Z}\{[1], \mathrm{res}_{\mathbb{R}}^{\mathbb{C}}(\eta_p^1), \ldots, \mathrm{res}_{\mathbb{R}}^{\mathbb{C}}(\eta_p^{(p-1)/2})\} & \text{for } n \geq p. \end{cases}$$

Furthermore, $\pi_0^{\mathbb{C}_p}(A_1^o) \cong \pi_0^{\mathbb{C}_p}(ko) \cong RO(C_p)$ since there are no non-trivial 1-dimensional representations, and $\pi_0^{\mathbb{C}_p}(A_n^o) \cong \mathbb{Z}$ for all $n > 1$.

**Over** $\mathbb{Q}$: There are only two isomorphism classes of irreducible $C_p$-representations over $\mathbb{Q}$, the trivial 1-dimensional representation and the reduced regular representation $\overline{\rho}_{C_p}$ of dimension $p-1$. Hence we find that

$$\pi_0^{C_p}(k\mathbb{Q}^n) \cong \begin{cases} \mathbb{Z}\{[1], \mathrm{tr}_{\{e\}}^{C_p}[1]\} & \text{for } 0 < n < p-1 \\ \mathbb{Z}\{[1], [\overline{\rho}_{C_p}], \mathrm{tr}_{\{e\}}^{C_p}[1]\} & \text{for } n = p-1 \\ \mathbb{Z}\{[1], [\overline{\rho}_{C_p}]\} & \text{for } n \geq p. \end{cases}$$

Furthermore, we see that

$$\pi_0^{C_p}(A_n^{\mathbb{Q}}) \cong \begin{cases} \mathbb{Z}\{[1], [\overline{\rho}_{C_p}]\} & \text{for } 0 \leq n \leq p-1 \\ \mathbb{Z} & \text{for } n \geq p. \end{cases}$$

In particular, the complexity filtration over $\mathbb{Q}$ does not stabilize globally on $\underline{\pi}_0$.

**Over** $\mathbb{F}_p$: Unlike in characteristic 0 the group ring

$$\mathbb{F}_p[C_p] \cong \mathbb{F}_p[t]/(t^p - 1) \cong \mathbb{F}_p[t]/(t-1)^p \cong \mathbb{F}_p[t]/(t^p)$$

is no longer semisimple. Up to isomorphism, there is exactly one indecomposable representation $V_i$ in every dimension $1 \leq i \leq p$ (and none in higher dimensions) and every representation decomposes uniquely as a sum of these. So we see that

$$\pi_0^{C_p}((k\mathbb{F}_p)^n) \cong \begin{cases} \mathbb{Z}\{[V_1], \ldots, [V_n], \mathrm{tr}_{\{e\}}^{C_p}[1]\} & \text{for } n = 1, \ldots, p-1 \\ \mathbb{Z}\{[V_1], \ldots, [V_p]\} & \text{for } n \geq p, \end{cases}$$

where the cases $p-1$ and $p$ are only notationally different, since the map

$$\pi_0^{C_p}((k\mathbb{F}_p)^{p-1}) \to \pi_0^{C_p}((k\mathbb{F}_p)^p)$$

sends $\mathrm{tr}_{\{e\}}^{C_p}[1]$ to $[V_p]$. For the complexity filtration we obtain:

$$\pi_0^{C_p}(A_n^{\mathbb{F}_p}) \cong \begin{cases} \mathbb{Z}\{\overline{[V_{n+1}]}, \ldots, \overline{[V_p]}\} & n = 0, \ldots, p-1 \\ \mathbb{Z} & n \geq p \end{cases}$$

So, in summary, for $p$ odd the modified rank filtrations work out as:

| $n$ | $1$ | $2, \ldots, p-2$ | $p-1$ | $\geq p$ |
|---|---|---|---|---|
| $\pi_0^{C_p}(ku^n)$ | $\mathbb{Z}^{p+1}$ | $\mathbb{Z}^{p+1}$ | $\mathbb{Z}^{p+1}$ | $\mathbb{Z}^p$ |
| $\pi_0^{C_p}(ko^n)$ | $\mathbb{Z}^2$ | $\mathbb{Z}^{\frac{p+3}{2}}$ | $\mathbb{Z}^{\frac{p+3}{2}}$ | $\mathbb{Z}^{\frac{p+1}{2}}$ |
| $\pi_0^{C_p}(k\mathbb{Q}^n)$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}^3$ | $\mathbb{Z}^2$ |
| $\pi_0^{C_p}((k\mathbb{F}_p)^n)$ | $\mathbb{Z}^2$ | $\mathbb{Z}^{n+1}$ | $\mathbb{Z}^p$ | $\mathbb{Z}^p$ |

And the complexity filtrations as:

| $n$ | $0$ | $1$ | $2, \ldots, p-1$ | $\geq p$ |
|---|---|---|---|---|
| $\pi_0^{C_p}(A_n^u)$ | $\mathbb{Z}^p$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\pi_0^{C_p}(A_n^o)$ | $\mathbb{Z}^{\frac{p+1}{2}}$ | $\mathbb{Z}^{\frac{p+1}{2}}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\pi_0^{C_p}(A_n^{\mathbb{Q}})$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\pi_0^{C_p}(A_n^{\mathbb{F}_p})$ | $\mathbb{Z}^p$ | $\mathbb{Z}^p$ | $\mathbb{Z}^{p+1-n}$ | $\mathbb{Z}$ |

Finally we compute the complexity filtration of the alternating group $A_5$ over $\mathbb{Q}$. To achieve this we first need two preparatory examples:

**Example 16.0.11** (Complexity filtration of the alternating group $A_4$ over $\mathbb{Q}$)**.** The representation ring is given by

$$\mathrm{Rep}_{\mathbb{Q}}(A_4) \cong \mathbb{Z}\{[1], [\eta], [\nu_4]\},$$

where $\eta$ is of dimension 2 and $\nu_4$ is of dimension 3. There are two conjugacy classes of maximal subgroups, the alternating group $A_3$ and the Klein four-group $K$, with representation rings

$$\mathrm{Rep}_{\mathbb{Q}}(A_3) \cong \mathbb{Z}\{[1], \overline{[\rho_{A_3}]}\}$$

respectively

$$\mathrm{Rep}_{\mathbb{Q}}(K) \cong \mathbb{Z}\{[1], [\varphi_1], [\varphi_2], [\varphi_3]\}.$$

The $\varphi_i$ are all 1-dimensional and conjugate under the action of the Weyl group in $A_4$.

We have the inductions

$$\mathrm{Ind}_K^{A_4}([1]) = [1] + [\eta]$$
$$\mathrm{Ind}_K^{A_4}([\varphi_i]) = [\nu_4].$$

So $[\nu_4]$ is identified with $[1] + [\eta]$ in $\pi_0^{A_4}(A_1^{\mathbb{Q}})$ and this is the only relation (since $A_3$ has only one 1-dimensional representation). Hence

$$\pi_0^{A_4}(A_1^{\mathbb{Q}}) \cong \mathbb{Z}\{[1], [\eta], [\nu_4]\}/([1] + [\eta] - [\nu_4]) \cong \mathbb{Z}\{\overline{[1]}, \overline{[\eta]}\}.$$

In $\pi_0^{A_4}(A_2^{\mathbb{Q}})$ the representation $[\eta]$ is identified with $2 \cdot [1]$, so the filtration becomes constant $\mathbb{Z}$ from then on. This yields:

| $n$ | 0 | 1 | $\geq 2$ |
|---|---|---|---|
| $\pi_0^{A_4}(A_n^{\mathbb{Q}})$ | $\mathbb{Z}^3$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |

**Example 16.0.12** (Complexity filtration of the dihedral group $D_5$ over $\mathbb{Q}$)**.** The representation ring of $D_5$ is given by

$$\mathrm{Rep}_{\mathbb{Q}}(D_5) \cong \mathbb{Z}\{[1], [-1], [\psi], [(-1) \cdot \psi]\},$$

where $[-1]$ is restricted from the projection $D_5 \to D_5/C_5 \cong C_2$. The 4-dimensional irreducible representations $[\psi]$ and $[(-1) \cdot \psi]$ are characterized by

$$\mathrm{Ind}_{C_2}^{D_5}([1]) = [1] + [\psi] \qquad \text{and} \qquad \mathrm{Ind}_{C_2}^{D_5}([-1]) = [-1] + [(-1) \cdot \psi].$$

Hence we see that the kernel of

$$\mathrm{Rep}_{\mathbb{Q}}(D_5) \to \pi_0^{D_5}(A_1^{\mathbb{Q}})$$

is generated by $[1] - [-1]$ and $[1] + [\psi] - [-1] - [(-1) \cdot \psi]$, which can be simplified to $[1] - [-1]$ and $[\psi] - [(-1) \cdot \psi]$. So $\pi_0^{D_5}(A_1^{\mathbb{Q}})$ is free of rank 2 with basis the classes of $[1]$ and $[\psi]$. Since there are no 2- or 3-dimensional irreducible representations for any subgroup of $D_5$, this is also the case for $\pi_0^{D_5}(A_2^{\mathbb{Q}})$ and $\pi_0^{D_5}(A_3^{\mathbb{Q}})$. In $\pi_0^{D_5}(A_4^{\mathbb{Q}})$ we have the relation $[\psi] - 4 \cdot [1]$, so the filtration stabilizes. We obtain:

| $n$ | 0 | $1, 2, 3$ | $\geq 4$ |
|---|---|---|---|
| $\pi_0^{D_5}(A_n^{\mathbb{Q}})$ | $\mathbb{Z}^4$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |

**Example 16.0.13** (Complexity filtration of the alternating group $A_5$ over $\mathbb{Q}$)**.** The representation ring is given by

$$\mathrm{Rep}_{\mathbb{Q}}(A_5) \cong \mathbb{Z}\{[1], [\nu_5], [\psi], [\Lambda^2 \nu_5]\},$$

where $\nu_5$ is the restriction of the reduced natural $\Sigma_5$-representation, $\Lambda^2 \nu_5$ is its 6-dimensional exterior square and $[\psi]$ is 5-dimensional. There are 3 conjugacy classes of maximal subgroups given by $A_4$, $\Sigma_3$ (generated by (123) and (12)(45)) and $D_5$ (generated by (1234) and (13)). We note that the rational complexity filtration for $\Sigma_3$ is the same as the one over $\mathbb{R}$, since all real representations of its subgroups are already defined rationally. Using the notation from the previous examples, we have

$$\begin{aligned}
\mathrm{Ind}_{A_4}^{A_5}([1]) &= [1] + [\nu_5] \\
\mathrm{Ind}_{A_4}^{A_5}([\eta]) &= 2 \cdot [\psi] \\
\mathrm{Ind}_{A_4}^{A_5}(\nu_4) &= [\nu_5] + [\psi] + [\Lambda^2 \nu_5] \\
\mathrm{Ind}_{\Sigma_3}^{A_5}([1]) &= [1] + [\nu_5] + [\psi] \\
\mathrm{Ind}_{\Sigma_3}^{A_5}([\mathrm{sgn}]) &= [\nu_5] + [\Lambda^2 \nu_5] \\
\mathrm{Ind}_{\Sigma_3}^{A_5}([\nu_3]) &= [\nu_5] + 2 \cdot [\psi] + [\Lambda^2 \nu_5] \\
\mathrm{Ind}_{D_5}^{A_5}([1]) &= [1] + [\psi] \\
\mathrm{Ind}_{D_5}^{A_5}([-1]) &= [\Lambda^2 \nu_5] \\
\mathrm{Ind}_{D_5}^{A_5}([\psi]) &= 2 \cdot [\nu_5] + 2 \cdot [\psi] + [\Lambda^2 \nu_5] \\
\mathrm{Ind}_{D_5}^{A_5}([(-1) \cdot \psi]) &= 2 \cdot [\nu_5] + 2 \cdot [\psi] + [\Lambda^2 \nu_5].
\end{aligned}$$

From our previous calculations we know that the relations in $\pi_0^G(A_1^{\mathbb{Q}})$ are generated by $[1]+[\eta]-[\nu_4]$ for $G = A_4$, by $[1]-[\mathrm{sgn}]$ for $G = \Sigma_3$ and by $[1]-[-1]$ and $[\psi]-[(-1)\cdot\psi]$ for $G = D_5$. Applying inductions to these relations, we see that they only give the relation $([\psi] + [1] - [\Lambda^2 \nu_5])$ in $\pi_0^{A_5}(A_1^{\mathbb{Q}})$. From this we can read off that

$$\pi_0^{A_5}(A_1^{\mathbb{Q}}) \cong \mathrm{Rep}_{\mathbb{Q}}(A_5)/([\psi] + [1] - [\Lambda^2 \nu_5]),$$

hence it is free with basis $[1]$, $[\nu_5]$ and $[\psi]$. In step 2 we have to add the inductions of the relations $2 \cdot [1] - [\eta]$ for $A_4$ and $2 \cdot [1] - [\nu_3]$ for $\Sigma_3$. This yields the new relation $[1] + [\nu_5] - [\psi]$, so

$$\pi_0^{A_5}(A_2^{\mathbb{Q}}) \cong \mathrm{Rep}_{\mathbb{Q}}(A_5)/([\psi] + [1] - [\Lambda^2 \nu_5], [1] + [\nu_5] - [\psi])$$

is free with basis $[1]$ and $[\nu_5]$. In the third step nothing happens, because $A_5$ has no 3-dimensional irreducible representation and we have seen that there are no new relations for any of the maximal subgroups. At step 4 the element $[\nu_5]$ is identified with $4 \cdot [1]$, so $\pi_0^{A_5}(A_n^{\mathbb{Q}}) \cong \mathbb{Z}$ for all $n \geq 4$, and we find:

| $n$ | 0 | 1 | 2, 3 | $\geq 4$ |
|---|---|---|---|---|
| $\pi_0^{A_5}(A_n^{\mathbb{Q}})$ | $\mathbb{Z}^4$ | $\mathbb{Z}^3$ | $\mathbb{Z}^2$ | $\mathbb{Z}$ |

# Appendix B

## B.1 Verification of cofiber sequence

In this appendix we give the proof that the map

$$\psi_n : \Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n) \to ku^{n-1}$$

constructed in Section 14.2 makes the following diagram a morphism of triangles in the global homotopy category:

$$\begin{tikzcd}
\Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n) \arrow[r] \arrow[d, "\psi_n"'] & \Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \arrow[r] \arrow[d, "\alpha_n"] & \Sigma^\infty(L(\mathbb{C}^n)_+ \wedge_{U(n)} (\overline{\mathcal{L}}_n)^\diamond) \arrow[d, "\cong"] \\
ku^{n-1} \arrow[r, "i_{n-1}"] & ku^n \arrow[r] & ku^n/ku^{n-1}
\end{tikzcd} \tag{B.1.1}$$

In order to establish this we turn the upper sequence into a strict quotient sequence by replacing $L(\mathbb{C}^n)/U(n)$ with $L(\mathbb{C}^n) \times_{U(n)} C\overline{\mathcal{L}}_n$, where $C\overline{\mathcal{L}}_n$ denotes the cone on $\overline{\mathcal{L}}_n$. We construct a morphism

$$\overline{\psi}_n : \Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} C\overline{\mathcal{L}}_n) \to ku^n$$

with the following three properties:

1. The square

$$\begin{tikzcd}
\Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n) \arrow[r] \arrow[d, "\psi_n"'] & \Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} C\overline{\mathcal{L}}_n) \arrow[d, "\overline{\psi}_n"] \\
ku^{n-1} \arrow[r, "i_{n-1}"] & ku^n
\end{tikzcd}$$

   commutes.

2. The restriction of $\overline{\psi}_n$ to the copy of $\Sigma^\infty_+(L(\mathbb{C}^n)/U(n))$ at the cone point is equal to $\alpha_n$.

3. The induced map
$$\Sigma^\infty(L(\mathbb{C}^n)_+ \wedge (\overline{\mathcal{L}}_n)^\diamond) \to ku^n/ku^{n-1},$$
   obtained by quotiening out $\Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$ and $ku^{n-1}$, is homotopic to the isomorphism constructed in Section 11.1.

The first two properties show that $\overline{\psi}_n$ induces a homotopy between the two composites in the first square of Diagram B.1.1. The third property implies that there is a homotopy between the two composites in the square

$$\begin{tikzcd}
\Sigma^\infty(L(\mathbb{C}^n)_+ \wedge_{U(n)} (\overline{\mathcal{L}}_n)^\diamond) \arrow[r] \arrow[d, "\cong"'] & S^1 \wedge \Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n) \arrow[d, "S^1\wedge\psi_n"] \\
ku^n/ku^{n-1} \arrow[r] & S^1 \wedge (ku^n/ku^{n-1})
\end{tikzcd}$$

and so we are done. The map $\overline{\psi}_n$ is also used in Section 15.2 to determine the effect of the complexity filtration on $\underline{\pi}_0$.

In order to construct $\overline{\psi}_n$ we quickly recall the objects involved: An element of $L(\mathbb{C}^n)(V)$ is a linear isometry $\mathbb{C}^n \hookrightarrow \mathrm{Sym}(V_\mathbb{C})$. Points in $\overline{\mathcal{L}}_n(V)$ are represented by tuples $(W_i, x_i)_{i\in I}$ where the $x_i$ are elements of $V$ and the $W_i$ are pairwise orthogonal subspaces of $\mathbb{C}^n$ which add up to all of $\mathbb{C}^n$. Furthermore, these tuples have to be reduced and of norm 1 (cf. Section 11.1). Finally, elements of $ku^n(V)$ are also represented by tuples $(W_i, x_i)_{i\in I}$, but this time the $W_i$ are orthogonal subspaces of $\mathrm{Sym}(V_\mathbb{C})$ and the only requirement is that the sum of the dimensions is at most $n$.

We recall also that the definition of $\psi_n$ made use of a function $s : [0,\infty] \to [0,\infty]$ which maps the interval $[0, \frac{1}{2n^2}]$ homeomorphically onto $[0,\infty]$ and is constant on the rest. Finally, given a finite tuple of vectors $x = (x_i)_{i\in I}$ of a real inner product space $V$ we defined a map $p_x : V \to \langle \{x_i\}_{i\in I}\rangle \subseteq V$ by $p_x(v) = \sum_I \langle v, x_i\rangle \cdot x_i$.

Now let $H : [0,\infty]\times[0,1] \to [0,\infty]$ be a homotopy relative endpoints from the identity to $s$. Given a real inner product space $V$ with a finite tuple of vectors $x = (x_i)_{i\in I}$ as above, we define a map $H^V_x : S^V \times [0,1] \to S^V$ via

$$H^V_x(v,t) = (H(|p_x v|, t) - |p_x v|) \cdot p_x v + v.$$

This gives a homotopy from the identity to the map $s^V_x$ used in the definition of $\psi_n$.

Now we can define $\overline{\psi}_n$ by the formula

$$(\varphi, (W_i, x_i)_{i\in I}, t) \wedge v \mapsto \begin{cases} (\varphi(W_i), \frac{t}{1-t} \cdot x_i + v)_{i\in I} & \text{if } 0 \le t \le 1/2 \\ (\varphi(W_i), H^V_x(x_i + v, 2t-1))_{i\in I} & \text{if } 1/2 \le t \le 1 \end{cases}$$

where $x$ is short for the tuple of the $x_i$. Since $H_x(x_i + v, 0)$ is equal to $x_i + v$, these two definitions agree on the intersection and glue to a well-defined map. By definition, setting $t$ equal to 1 gives back $\psi_n$, thus property (1) is satisfied. Furthermore, the elements $(\varphi, (W_i, x_i)_{i\in I}, 0)$ are mapped to the tuple $(\varphi(W_i), v)_{i\in I}$, which is equal to $(\varphi(\mathbb{C}^n), v)$. Hence it is independent of the $W_i$ and $x_i$ and the induced map

$$\Sigma^\infty_+(L(\mathbb{C}^n)/U(n)) \to ku^n$$

equals $\alpha_n$, yielding property (2).

It remains to prove property (3), i.e., that the induced map

$$\overline{\psi}_n' : \Sigma^\infty(L(\mathbb{C}^n)_+ \wedge \overline{\mathcal{L}}_n^\diamond) \to ku^n/ku^{n-1}$$

obtained by quotiening out $\Sigma^\infty_+(L(\mathbb{C}^n) \times_{U(n)} \overline{\mathcal{L}}_n)$ and $ku^{n-1}$ is homotopic to the isomorphism from Section 11.1. For $t \leq 1/2$ the two maps are in fact equal and hence it suffices to construct a homotopy on the part where $t \geq 1/2$, relative to $t = 1/2$. This is achieved by the formula

$$(\varphi, (W_i, x_i)_{i\in I}, t, s) \wedge v \mapsto [(H_x^V((\frac{(1-s)t}{1-(1-s)t} + \frac{s-1}{s+1} + 1) \cdot x_i + v, s(2t-1)), \varphi(W_i))_{i\in I}]$$

for $s \in [0, 1]$. Continuity is only unclear at points for which $t = 1$ and $s = 0$, which are mapped to the basepoint. However, by the same estimate as in Section 14.2 one sees that the expression $(H_x^V((\frac{(1-s)t}{1-(1-s)t} + \frac{s-1}{s+1} + 1) \cdot x_i + v, s(2t-1)), \varphi(W_i))_{i\in I}$ lies in $ku^{n-1}$ already for all $t$ close enough to 1 and $s$ close enough to 0. So the homotopy is actually constant around $s = 0$ and $t = 1$, hence we are done.

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# Zusammenfassung

In dieser Dissertation beschäftigen wir uns mit verschiedenen Filtrierungen äquivarianter Spektren.

Zunächst betrachten wir die symmetrischen Produkte $Sp^n$ des $G$-Sphärenspektrums über einer endlichen Gruppe $G$, welche zwischen dem $G$-Sphärenspektrum und dem Eilenberg-MacLane Spektrum für den konstanten Mackey-Funktor $\underline{\mathbb{Z}}$ interpolieren. Die symmetrischen Produkte der Sphäre sind – vor allem unäquivariant – viel untersucht worden, beispielsweise haben sie Verbindungen zu Tits Gebäuden und dem Goodwillie Turm der Identität [AD01].

Wir zeigen, dass sich mit Hilfe von Untergruppenverbänden ein rationales Modell der symmetrischen Produkte konstruieren lässt. Der Untergruppenverband einer Gruppe $G$ ist der Komplex, dessen Eckpunkte die Untergruppen $H$ von $G$ sind und dessen höherdimensionale Simplizes zu Ketten von Untergruppen $H_0 \leq \ldots \leq H_k$ korrespondieren. Insbesondere impliziert unser Resultat, dass die rationalen $G$-äquivarianten Homotopiegruppen der $Sp^n$ natürlich isomorph zu den rationalen Homologiegruppen bestimmter Unterkomplexe des Untergruppenverbandes von $G$ sind, was in vielen Fällen eine konkrete Berechnung ermöglicht. Es lässt sich mithilfe des Resultats auch leicht einsehen, dass die symmetrischen Produkte beliebig hohe nicht-triviale rationale Homotopiegruppen besitzen (wenn man $n$ und $G$ variieren lässt). Dies steht im Kontrast zu den Grenzfällen $Sp^1$ und $Sp^\infty$, deren rationale Homotopiegruppen bereits bekannt waren und stets im Grad 0 konzentriert sind. Ein wichtiger Zwischenschritt im Beweis ist eine äquivariante Version eines Satzes von Arone und Dwyer [AD01], in welchem der Quotient $Sp^n/Sp^{n-1}$ mit dem Partitionskomplex der Menge $\{1, \ldots n\}$ in Verbindung gesetzt wird.

Wir zeigen zudem, dass die rationalisierten symmetrischen Produkte interessante Eigenschaften haben, wenn man sie in einem global äquivarianten Kontext, welcher von Schwede in seinem Buchprojekt [Sch15] eingeführt wurde, auffasst. Sofern $n$ nicht 1 oder $\infty$ ist, zerfällt $Sp^n_{\mathbb{Q}}$ nicht als Produkt globaler Eilenberg-MacLane Spektren, obwohl so eine Zerlegung für jede fixierte Gruppe $G$ existiert.

Im zweiten Projekt befassen wir uns mit einer äquivarianten Verallgemeinerung zweier Arten von Filtrierungen, welche von Arone und Lesh in den Artikeln [AL07] und [AL10] eingeführt wurden. Die erste, genannt Rang-Filtrierung, filtriert konnektive äquivariante topologische/algebraische $K$-Theoriespektren mittels der Dimension von Vektorräumen beziehungsweise dem Rang freier Moduln. Sie ist nahe verwandt mit der von Rognes [Rog92] konstruierten Rangfiltrierung, aber im allgemeinen nicht dazu äquivalent. Die zweite, welche wir als Komplexitätsfiltrierung bezeichnen, liefert eine Interpolation zwischen dem K-Theorie Spektrum und dem Eilenberg-MacLane Spektrum des konstanten Mackey-Funktors $\underline{\mathbb{Z}}$ und hat ähnliche Eigenschaften wie die symmetrischen Produkte der Sphäre.

Nach der Konstruktion der Filtrierungen bestimmen wir den äquivarianten Homotopietyp der Unterquotienten. Es stellt sich heraus, dass diese Einhängungsspektren äquivarianter Räume sind, welche in enger Beziehung zu der Kombinatorik von Zerlegungen endlich dimensionaler Vektoräume als orthogonale Summe von Unterräumen (beziehungsweise Zerlegungen freier Moduln im algebraischen Kontext) stehen. Dieser Teil ist aus gemeinsamer Arbeit mit Dominik Ostermayr enstanden und verallgemeinert Ergebnisse von Arone und Lesh.

Die Beschreibung der Subquotienten nutzen wir daraufhin für die Bestimmung algebraischer Filtrierungen, welche von den Rang- und Komplexitätsfiltrierungen durch Anwenden des Funktors $\pi_0^G$ induziert werden. Genauer gesagt liefert die Rangfiltrierung eine Filtrierung des Darstellungsrings von $G$ und die Komplexitätsfiltrierung eine Filtrierung des Augmentationsideals dieses Darstellungsrings. Wir zeigen, dass sich diese algebraischen Filtrierungen einfach beschreiben lassen, wenn man alle endlichen oder kompakten Lie Gruppen simultan betrachtet und die Einschränkungs- und Transferabbildungen verwendet, welche die verschiedenen Darstellungsringe miteinander verbinden. Mittels dieser Strukturen ist etwa jede Stufe der Filtrierung des Augmentationsideals durch ein einzelnes Element bestimmt, und auch die Filtrierung des Darstellungsrings lässt sich durch endlich viele universelle Elemente beschreiben. Um dies präzise zu machen, nutzen wir erneut den global äquivarianten Kontext. Diese Vorgehensweise ist durch einen Satz von Schwede [Sch14] motiviert, in welchem er analoge Formeln für die 0-te äquvariante Homotopiegruppe der symmetrischen Produkte von Sphären bewiesen hat.