KPZ Universality for Last Passage Percolation models

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A tutti coloro
che hanno avuto la pazienza
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Abstract

In this thesis we consider models of last passage percolation on $\mathbb{Z}^2$. These models belong to the Kardar–Parisi–Zhang universality class, a class of stochastic growth models that have been widely studied in the last 30 years.

Last passage percolation models provide a "physical" description of combinatoric problems, such as the Ulam's problem, in terms of zero temperature directed polymers, but also a geometrical interpretation of an interacting particle system, the totally asymmetric simple exclusion process (TASEP). Moreover, in the large time limit, they share statistical features with certain ensembles of random matrices.

We investigate the universality of the limit distributions of the last passage time for different settings. First, we study TASEP starting from a periodic configuration and show the universality of the GOE Tracy-Widom distribution for generic particle density. This result is proved in the last passage percolation framework and is obtained with soft probabilistic arguments, as the convergence of the last passage time to a variational formula involving the limit Airy_2 process.

Then, we analyze the correlations of two last passage times for different ending points in a neighbourhood of the characteristic. For the standard settings (step, flat and stationary), using similar techniques, we prove the converge of the covariance of the last passage times to the covariance of the limiting processes. For a more general class of random initial conditions, we prove the universality of the first order correction when the two observation times are close and provide a rigorous bound of the error term.

Finally, we consider a model of last passage percolation on half-space. We show that the stationary initial condition can be realized by adding weights on the axis and on the diagonal, and we obtain the distribution of the last passage time for this configuration. The limit distribution is analogous to the Baik–Rains distribution from the case of stationary full-space last passage percolation, but in our case, it depends on a parameter, the strength of the weights on the diagonal.

Contents

Introduction

The objects of study of this thesis are *last passage percolation* models. Last passage percolation (LPP) was introduced by Hammersley [Ham66] to answer the following question: given n points uniformly and independently distributed in $[0,1]^2$, what is the maximal number of points that can be collected by an up-right path? A solution to this inquiry provides an answer to an equivalent combinatorial problem, known as Ulam's problem [Ula61], of the longest increasing subsequence of a random permutation.

Last passage percolation has been extensively investigated, also unrelatedly to the combinatorics starting point. Indeed, last passage percolation is connected to a particle system on the integers line, the *totally asymmetric simple exclusion process* (TASEP). TASEP is a particular case of the asymmetric simple exclusion process (ASEP), introduced by Spitzer in [Spi70], a continuous-time Markov process on $\{0,1\}^{\mathbb{Z}}$, where the 1s represent particles and the 0s holes. In ASEP each particle waits an exponential time and then attempts a jump, to the neighbouring right site with probability p and to the neighbouring left site with probability $q = 1 - p$. The jump is performed only if there is no particle on the chosen site. In TASEP particles can jump only to the right (corresponding to the $p = 1$ case). (T)ASEP is exactly solvable: many exact results on ASEP can be achieved using techniques inspired by quantum mechanics, such as the Bethe Ansatz [Sch00]; for TASEP a Fredholm determinant formula for the joint distribution of particle positions have been obtained by Borodin and Ferrari [BF08]. Moreover, this model belongs to the so-called *Kardar–Parisi–Zhang* (KPZ) *universality class*.

The concept of KPZ universality class was defined after Kardar, Parisi and Zhang [KPZ86] proposed a non-linear stochastic partial differential equation as a model for the evolution of a growing interface. The *KPZ equation* is given by

$$\frac{\partial h}{\partial t} = \nu \Delta h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,t),$$

where $h(x,t) \in \mathbb{R}$ is the value of the height function describing the modeled interface at position x and at time t. On the right hand side of the equation, the first is a relaxation term with surface tension ν; the second is a non-linear term due to the later growth of the interface; the last term is a space-time white noise.

The KPZ universality class includes interface growth models in 1+1 dimension, which indicates one-space dimension and one-time dimension; in general, they are described in terms of a height function $h(x,t)$. Models in the KPZ class evolve according to a dynamics, which is characterized by locality of growth (which means that there are no long-range interactions entering the growth rules), a smoothing mechanism and lateral growth (this means that the speed of growth depends non-linearly on slope). The presence of the Laplacian ensures the existence of a law of large numbers for the height function and leads to a deterministic interface for large times, defined as the limit of the interface for a large parameter L going to infinity when space, time and height function are scaled by L,

$$h_{\text{macro}}(x,t) := \lim_{L\to\infty} \frac{h(Lt, Lt)}{L}. \tag{0.0.1}$$

Models in this class are expected to show large time universality under an appropriate scaling, which is determined by the order of fluctuations and spatial correlations: fluctuations around the limit shape scale as $t^{1/3}$, as spatial correlations occur on the scale $t^{2/3}$ (this means that two points are non-trivially correlated if they are at distance of order $t^{2/3}$). For $\epsilon > 0$, we define the rescaled height function

$$h_\epsilon(x,t) = \epsilon^\beta h(\epsilon^{-1}x, \epsilon^{-z}t) - C_\epsilon t, \tag{0.0.2}$$

where $\beta = 1/2$ is the fluctuation exponent imposed by the Brownian motion and $z = 3/2$ is the dynamic scaling exponent. These exponents correspond to scaling time : space : fluctuations as $3:2:1$ and define the so-called KPZ scaling. The universality conjecture states that, if we take the $\epsilon \to 0$ limit of the rescaled height function, we observe a universal field, known as the *KPZ fixed point* [MQR17], which does not depend on the model itself. This limit object is indeed universal, but whithin subclasses of initial/boundary conditions. For more on models in the KPZ universality class, see Chapter 1.

The Hammersley last passage percolation was one the first models for which the belonging to the KPZ class was proved [Joh00a, BDJ99a]. In the Hammersley LPP points are distributed according to a Poisson point process on the plane.

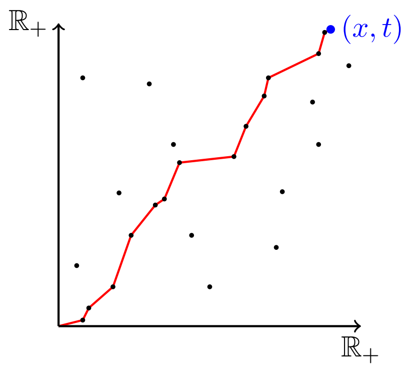

Figure 1: Hammersley last passage percolation on $\mathbb{R}^2_+$. The red path moves from the origin to the point (x,t) maximizing the number of Poisson points on the positive real plane.

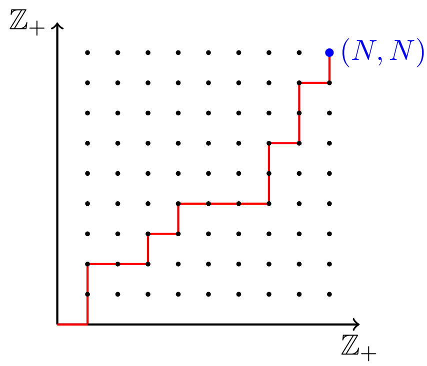

Figure 2: Last passage percolation on $\mathbb{Z}^2_+$. The red path moves from the origin to the point (N,N) maximizing the weights on the positive integer plane. The weights are given by independent exponential random variables.

In this work we consider a different version of LPP, defined on the lattice $\mathbb{Z}^2$, where on each site an independent exponentially distributed random variable is placed. The length of a path is defined as the sum of the random variables along the path and the last passage time is given by the length of the path maximizing the values of the random variables. The simplest version of LPP is given by paths between two fixed points, but its definition can be extended considering paths between sets of points in $\mathbb{Z}^2$. Beyond this setting, here we focus on LPP between a point and a set of points along a line, which is a “line-to-point” problem, considering deterministic, stationary and a general family of random initial conditions. For further details on last passage percolation models, see Chapter 2.

In this thesis all chapters are intented to be self-contained; each one containes a short review on the models and previous results that are used for the proofs.

In Chapter 1 we give a brief introduction to universality in random systems and present some integrable models in the KPZ universality class that are relevant for this thesis.

In Chapter 2 we describe in more details the last passage percolation models, which are the main objects studied here, and define their limit processes and distributions depending on the initial setting. Moreover, we include a short presentation of the results proved in the following chapters.

The three final chapters are based on the works [FO18], [FO19] and [BFO19].

In Chapter 3 we consider TASEP in continuous time with non-random initial conditions and arbitrary fixed density of particles $\rho \in (0, 1)$. We show that the one-point fluctuations of the associated height function are given by the GOE Tracy-Widom distribution function. This distribution was first observed in the context of random matrices as the limit distribution of the largest eigenvalue for a matrix in Gaussian orthogonal ensemble. In previous works [Joh05, BFP07] it was proved that the GOE Tracy-Widom distribution describes the fluctuations of the height function for specific values of the density, $\rho = 1/d$, $d = 2, 3, 4, \ldots$. Thus, showing that the same holds for $\rho \in (0, 1)$ provides a universality result for TASEP starting from flat initial conditions, i.e. with particles starting with a periodic configuration. The proof of the result is phrased in last passage percolation language and it shows the universality for a line-to-point problem where the line has an arbitrary slope.

In Chapter 4 we consider time correlations for KPZ growth in 1+1 dimensions in neighbourhoods of the characteristics. The latter are the characteristic curves of the associated large time limit partial differential equation (in this case, the Burgers equation), the lines along which a PDE becomes an ordinary differential equation. For last passage percolation the characteristic is given by a straight line connecting the starting and the ending point of the maximizing path.

Here we study the time-time covariance of LPP with droplet (point-to-point), flat (line-to-point), stationary and random non-stationary initial profiles. We prove the convergence of the covariance for the first three profiles. In particular, this provides a rigorous proof of the exact formula of the covariance for the stationary case obtained in [FS16]. Furthermore, we analyze the behaviour of the covariance when the two times are close to each other on a macroscopic scale. At first approximation, on a small scale, we observe the stationary state and recover a universal formula for the covariance up to the first order correction. This result holds also for random initial profiles which are not necessarily stationary.

In Chapter 5 we study last passage percolation with a different geometric setting, a LPP on the half-space quadrant of integers. This is equivalent to the full-space model with weights symmetric with respect to the diagonal. The half-space last passage percolation is the geometric counterpart of the Ulam's problem for random involution, i.e. for permutations consisting only of transpositions and fixed points. The permutation matrices of involutions are indeed symmetric.

Here we study stationary half-space last passage percolation with exponential weights. We use integrable probability (Pfaffian) techniques, analytic continuation, and asymptotic analysis to obtain the $N^{1/3}$ fluctuation limit for the last passage time. We observe that in the critical scaling regime the half-space LPP obeys a law analogous to the Baik–Rains distribution law from the case of stationary full space LPP [BR00]. This limit distribution is described in terms of Fredholm pfaffians, which are a generalization of Fredholm determinants for anti-symmetric matrix kernels. Unlike the Baik–Rains distribution, our distribution depends on a parameter giving the strength of the diagonal bounding the half-space.

Chapter 1

The KPZ universality class

1.1 Universality in random systems

The concept of universality arose in the context of statistical mechanics as a consequence of the study of critical phenomena. To understand what a critical phenomenon is, we consider the classical example of equilibrium statistical mechanics, the Ising model. This is a mathematical model of ferromagnetism that consists of discrete variables representing magnetic dipole moments of atomic spins (that can take only two values, ± 1) on a regular lattice. On the microscopic level, the evolution of the magnetic moments is the consequence of the interactions of many atoms. Macroscopically, one observes a change in the ferromagnet behaviour, which is dependant on external conditions, such as the temperature or the magnetic field. This happens suddenly in correspondence with *critical values* of the temperature or the magnetic field, which determine a phase transition from one state to another. Two types of transitions are possible. If the two states coexist at the critical point and the transition involves discontinuous behaviour of thermodynamic properties, we talk about "first-order" phase transition. If the transition is continuous, we talk about "second-order" phase transition. In this case, the two phases on either side of the transition must be identical at the critical point and the magnetization goes to zero.

The singular behaviour near the critical point is characterized by *critical exponents*. These exponents describe the non-analyticity of various thermodynamic observables. In the Ising model, we describe the magnetization with an order parameter M. As we heat the system, M decreases and eventually, at a certain critical temperature T_c , it reaches zero. The closer the parameter is to its critical value, the less sensitively the order parameter depends on the details of the system. The order parameter is well approximated by

$$M \sim |T - T_c|^{\alpha}, \tag{1.1.1}$$

where the exponent α is the critical exponent. Remarkably, it was found empirically that different systems such as the liquid/gas and ferromagnetic transition can be described by the same set of critical exponents. Such systems are said to belong to the same *universality class*.

In the last decade, the concept of universality class has been investigated and exploited, and now it plays a central role in probability and mathematical physics. In particular, mathematicians and physicists are interested in finding the "scaling exponents" of a system and the relations between them. Two scaling exponents are mostly considered: the *fluctuation exponent* χ, which quantifies the order of the fluctuations of the observable quantity from its typical value, and the *wandering exponent*, which quantifies the transversal fluctuations, the order of the maximal deviation. The Ising model is an example of critical phenomenon at equilibrium. But one might ask if there exist such a description for systems driven out of equilibrium.

1.1.1 Gaussian universality class

If we move away from statistical mechanics, we realize that the concept of universality class is more familiar than we thought. The simplest and historically first example of universality is provided by the *Central Limit Theorem.*

Consider a sample (i.e. a collection of data) obtained by a large number of randomly and independently generated observations, and compute the average of the observed values. Repeating the computation many times, one will notice that the computed averages are distributed according to a normal distribution: this is the content of the Central Limit Theorem.

Theorem 1.1.1. *Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed random variables with mean μ and variance $\sigma^2 < \infty$. Let $S_n := \frac{1}{n}\sum_{k=1}^{n} X_k$ be the sample average. Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - \mu n}{\sigma\sqrt{n}} \leq s\right) = \int_{-\infty}^{s} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx. \tag{1.1.2}$$

This means that, despite model dependent features, such as the mean and the variance, any sum of i. i. d. random variables with finite mean and variance will demonstrate the same limiting behaviour, described by the normal distribution, and fluctuations around the mean value of order $n^{1/2}$. Physical and mathematical systems accurately described in terms of Gaussian statistics are said to be in the *Gaussian universality class.*

1.1.2 Random and ballistic deposition

We present two models of randomly growing one-dimensional interface that show different limiting behaviours. The *random deposition model* consists of unit blocks falling independently and in parallel on $\mathbb{Z}$ after an exponentially distributed waiting time of parameter 1 (see Figure 1.1). Due to the memoryless property of the exponential distribution, this model is a Markov process and its evolution depends only on the present, not on the history. Each column evolves independently and is a sum of i. i. d. random variables. We indicate with $h(x, t)$ the *height function*, a function that gives the value of the height of the column on the site x at time t.

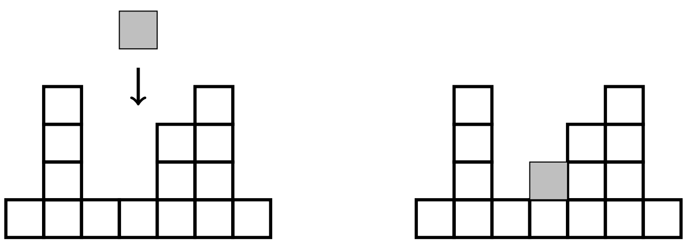

Figure 1.1: The random deposition model. Blocks fall from above each site with independent exponentially distributed waiting times and land at the top of each column.

By the Law of Large Numbers and the Central Limit Theorem, for any $x \in \mathbb{Z}$,

$$\begin{aligned} \lim_{t\to\infty} \frac{h(x,t)}{t} &= 1, \\ \frac{h(x,t)-t}{\sqrt{t}} &\Rightarrow \mathcal{N}(x), \end{aligned} \tag{1.1.3}$$

where the double arrow symbol indicates convergence in distribution and $\mathcal{N}$ the Gaussian distribution. The model shows linear growth speed and lack of spatial correlations, and the fluctuations belong to the Gaussian universality class, since they grow as $t^{1/2}$ and have Gaussian limit distribution.

If we modify the rules of the growth of the interface, we lose the Gaussian behaviour. Consider the same model, but now, instead of falling on the ground or the interface, a new block sticks to the first edge (see Figure 1.2): this is known as the *ballistic deposition model*, introduced by Vold in 1959.

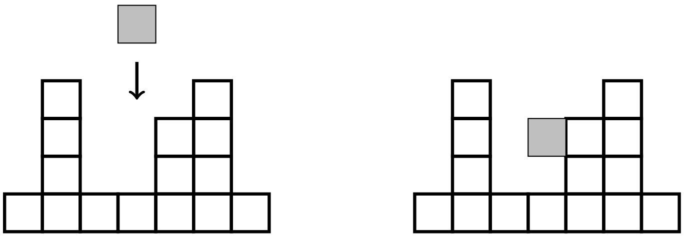

Figure 1.2: The ballistic deposition model. Blocks fall from above each site with independent exponentially distributed waiting times and stick to the first edge to which they become incident.

This change in the evolution rules turns into large time effects: the interface grows faster than in the random deposition model (the value of the velocity is still unknown). Simulations show that the height function has smaller fluctuations, on the scale $t^{1/3}$, and demonstrates non-trivial correlations on the transversal scale of $t^{2/3}$ (see Figure 1.5). Moreover, the rescaled height does not converge anymore to the Gaussian distribution.

1.2 KPZ Universality Class

The hypothesis of independence is essential for belonging to the Gaussian universality class. Nevertheless, we have seen that this class cannot exhaust all cases. There exists a group of systems that belongs to a different universality class, the so-called *Kardar–Parisi–Zhang* (KPZ) *universality class*. The features that a model is supposed to share to be a member of the KPZ class are:

- **Locality**: the dynamics depends only locally on the configuration of the interface.
- **Smoothing**: deep holes in the configuration are rapidly filled to smooth the interface; this ensures that there exists a macroscopic limit shape $h_{ma} = \lim_{t\to\infty} \frac{h(\xi t,t)}{t}$.
- **Rotationally invariant, slope dependent, growth speed**: vertical effective growth rate depends non-linearly on the local slope, which implies $v(\nabla h) \neq 0$.
- **Space-time uncorrelated noise**: growth is driven by noise which quickly decorrelates in space and time and does not show heavy tails.

The ballistic deposition model presents these properties, since the modification in the dynamics breaks the independence of the column heights and introduces spatial correlations.

1.2.1 The KPZ equation

The KPZ universality class was introduced in the context of studying the motion of growing interfaces. In 1986 Kardar, Parisi and Zhang [KPZ86] presented a model to describe the evolution of a continuum stochastically growing height function $h(x,t)$, given in terms of a stochastic PDE, known as the *KPZ equation*:

$$\frac{\partial h}{\partial t}(x,t) = \nu \frac{\partial^2 h}{\partial x^2}(x,t) + \frac{1}{2}\lambda \left(\frac{\partial h}{\partial x}(x,t)\right)^2 + \sqrt{D}\xi(x,t), \tag{1.2.1}$$

where $\lambda, \nu \in \mathbb{R}$, $D > 0$ are physical constant and $\xi(x,t)$ is a Gaussian space-time white noise, which is a distribution valued Gaussian field with correlation function

$$\mathbb{E}\left[\xi(x,t)\xi(y,s)\right] = \delta(t-s)\delta(x-y). \tag{1.2.2}$$

The equation contains, of course, the key features for KPZ membership, in fact the growth is local and the time derivative of the height function depends on three factors: the Laplacian, responsible for the smoothing mechanism, the square of the gradient, which leads to rotationally invariant, slope dependent, growth speed, and the white noise (space-time uncorrelated noise).

The presence of the non-linear term is justified by the fact that growth occurs locally along the direction normal to the interface. To understand this, we consider the *Eden model*, a model of cellular growth introduced by Eden [Ede61] in 1961. A finite connected subset of $\mathbb{Z}^2$ grows by adding sites in its exterior boundary at rate 1. Since the growth is normal to the surface h, when a particle is added, the projection of the increment on the h axis is

$$\begin{aligned} \delta h =& \sqrt{(v\delta t)^2 + (v\delta t \partial_x h)^2} \\ \approx& v\delta t + \frac{v}{2}\delta t(\partial_x h)^2 + \dots \end{aligned} \tag{1.2.3}$$

However, the KPZ equation has a problem of well-posedness. Because of the white noise term the solution should have the same regularity as Brownian notion, whose derivate has negative Hölder regularity, so the non-linearity has a priori no sense. The first question that arises naturally is: what does it mean to solve the KPZ equation? Later, we will see that the solutions can be written in terms of a classical well-posed stochastic partial differential equation. On the other hand, since the solution of the equation should be random, we are interested in the distribution of the solution and its dependence on the initial data. In other words, we ask if there is a limiting growth velocity

$$v = \lim_{t\to\infty} \frac{h(x,t)}{t}, \tag{1.2.4}$$

what is the scale χ of the fluctuations under which we observe a non-trivial distribution for large times

$$F(s) = \lim_{t\to\infty} \mathbb{P}\left(\frac{h(x,t) - vt}{t^\chi} \le s\right), \qquad s \in \mathbb{R}, \tag{1.2.5}$$

what is the order of transversal fluctuations t^ξ, and what is the exact form of this distribution. A conjecture on the scaling exponents χ and ξ is that, although they may depend on the dimension, they always satisfy the relation

$$2\xi = \chi + 1. \tag{1.2.6}$$

This was formulated for two-dimensional systems in [KPZ86], but it is believed to hold for any dimension [WK87, Kes93]. For a wider introduction to KPZ and exact solutions to the KPZ equation, we refer the interested reader to [Qua11] and [Cor18].

Rescaling the KPZ equation

We look for a scaling of the solution of (1.2.1) under which we expect to observe a non-trivial behaviour on large space and time scales. Without loss of generality we can consider the special case $\lambda = \nu = \frac{1}{2}$ and $D = 1$. We define the rescaled solution

$$h^\epsilon(x,t) = \epsilon^\beta h(\epsilon^{-1}x, \epsilon^{-z}t). \tag{1.2.7}$$

Under this scaling,

$$\begin{aligned} \partial_t h &= \epsilon^{z-\beta}\partial_t h^\epsilon, \\ \partial_x h &= \epsilon^{1-\beta}\partial_x h^\epsilon, \\ (\partial_x h)^2 &= \epsilon^{2(1-\beta)}(\partial_x h^\epsilon)^2, \\ \xi(x,t) &= \epsilon^{-(z+1/2)}\xi(\epsilon^{-1}x, \epsilon^{-2}t) \end{aligned} \tag{1.2.8}$$

in distribution. Plugging (1.2.7) in (1.2.1), we have

$$\partial_t h^\epsilon = \frac{1}{2}\epsilon^{2-z}\partial_x^2 h^\epsilon + \frac{1}{2}\epsilon^{2-z-\beta}(\partial_x h^\epsilon)^2 + \epsilon^{\beta-\frac{1}{2}z+\frac{1}{2}}\xi. \tag{1.2.9}$$

If we linearize the KPZ equation, we get

$$\partial_t h = \nu\partial_x^2 h + \sqrt{D}\xi, \tag{1.2.10}$$

which is solved by the Ornstein-Uhlenbeck process, a two-sided stationary Gaussian Markov process. A two-sided Brownian motion is invariant not only for this process, but also for (1.2.1), with the difference that, for the KPZ equation, it will be globally shifted in height as time proceeds. The invariance of Brownian motion imposes the scaling

$$\beta = 1/2, \tag{1.2.11}$$

and, to avoid divergence of the non-linear term, we have to take the dynamic scaling exponent

$$z = 3/2. \tag{1.2.12}$$

Thus, we observe non-trivial fluctuations at the scale

$$h^\epsilon(x,t) = \epsilon^{1/2}h(\epsilon^{-1}x, \epsilon^{-3/2}t). \tag{1.2.13}$$

The scaling $\beta = 1/2$ and $z = 3/2$ is called *KPZ scaling*.

A fundamental question regarding the KPZ class is whether there is an universal object such that, under the KPZ scaling, all models converge to it. It is also natural to ask if this object can be the KPZ equation. The answer to the second question is no. In fact, the KPZ equation is not invariant under the KPZ scaling and it was predicted [FNS77, BKS85, KPZ86] that, under the KPZ scaling, the KPZ equation should converge to some non-trivial process, called the *KPZ fixed point* [MQR17].

However, the KPZ equation is fixed by other scalings, defined *weak scalings*: in addition to scale space, time and fluctuation, a tuning parameter, depending on ϵ, is introduced. There are two weak scalings:

- Weak non-linearity scaling: take $\beta = 1/2$, $z = 2$ and multiply the non-linear term by $\lambda_\epsilon = \epsilon^{1/2}$.
- Weak noise scaling: take $\beta = 0$, $z = 2$ and multiply the noise term by $\beta_\epsilon = \epsilon^{1/2}$.

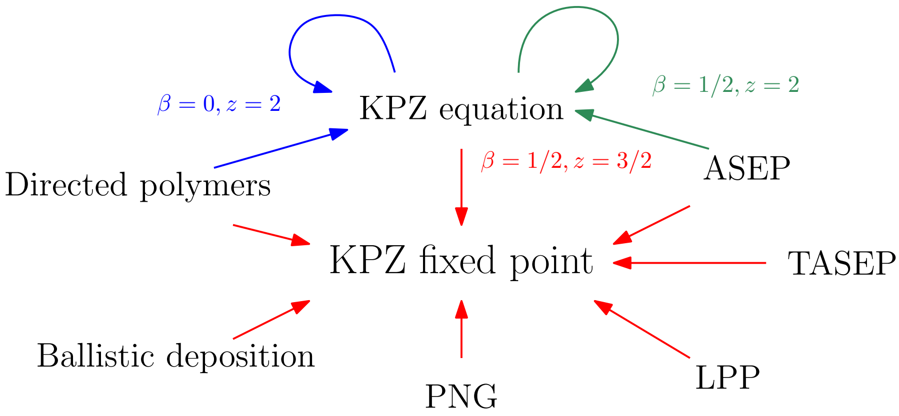

Figure 1.3: Scalings for the KPZ equation. The KPZ equation is invariant under weak noise (in blue) and weak non-linearity (in green) scaling, as it is supposed to converge (with other model within the class) to the KPZ fixed point under the KPZ scaling (in red).

Solution to the KPZ equation

We said that Brownian motion is an invariant measure for the KPZ equation; moreover, starting from any initial data, at any time, the solution will be locally Brownian. These are the reason why the non-linear term makes no sense and needs a sort of "infinite renormalization",

$$\partial_t h = -\left[\frac{1}{2}(\partial_x h)^2 - \infty\right] + \frac{1}{2}\partial_x^2 h + \xi. \tag{1.2.14}$$

An attempt to solve this problem was made in [Cha00], introducing a weak ordered version of the non linearity, $:(\partial_x h)^2:$, but it led to non-physical solutions.

In 1997 L. Bertini and G. Giacomin [BG97] provided a physically relevant solution (indeed, they proved that the equation is the scaling limit of a growth model), known as the *Cole-Hopf solution*. They proposed that the solution of the KPZ equation is

$$h(x,t) = -\log z(x,t), \tag{1.2.15}$$

where $z(x,t)$ is the solution of the *stochastic heat equation* (SHE)

$$\partial_t z(x,t) = \frac{1}{2}\partial_x^2 z(x,t) - z(x,t)\xi(x,t) \tag{1.2.16}$$

with initial data $z(x,0) = e^{-h(x,0)}$. This equation is well-posed in the Îto sense and its solution is strictly positive for $t > 0$, so taking the logarithm makes sense.

The Cole-Hopf transform of z is the solution to the equation (1.2.14), in the sense that z is the $\epsilon \to 0$ limit of a sequence of approximating solutions z_ϵ to the stochastic heat equation with smoothed noise

$$\partial_t z_\epsilon = \frac{1}{2}\partial_x^2 z_\epsilon - z_\epsilon \xi_\epsilon, \tag{1.2.17}$$

and $h_\epsilon = -\log z_\epsilon$ solves

$$\partial_t h_\epsilon = -\left[\frac{1}{2}(\partial_x h_\epsilon)^2 - C_\epsilon\right] + \frac{1}{2}\partial_x^2 h_\epsilon + \xi_\epsilon, \tag{1.2.18}$$

where C_ϵ is a divergent term.

In [BG97] the Cole-Hopf solution was obtained by approximating the KPZ equation by the height function of a weakly asymmetric exclusion process, but it can be also be obtained by the free energy of directed random polymers (see Section 1.2.2).

It was Hairer [Hai13] to prove that (1.2.14) is well-posed. Using the theory of rough paths [Gub04], he introduced a new concept of solution to the KPZ equation which extends the classical Cole-Hopf solution, obtaining new regularity results about the solution and its derivative with respect to the initial condition. Gonçalves and Jara [GJ14] introduced the notion of *energy solution* of the equation, based on a martingale problem approach to the stationary KPZ equation. Using a second-order Boltzmann-Gibbs principle (which allows to replace local functionals of a conservative, one-dimensional stochastic process by a function of the conserved quantity), they prove that the density fluctuations of a one-dimensional, stationary, weakly asymmetric, conservative particle system converge to the energy solutions of the stochastic Burgers equation and that the Cole-Hopf solution is the energy solutions of the KPZ equation. Later, the concept of energy solution was redefined by Gubinelli and Jara [GJ13], who gave a different definition of the generalized martingale problem with a forward-backward decomposition of the drift. Uniqueness of the energy solution was proved by Gubinelli and Perkowski [GP18]. More recently, they also provided an alternative point of view on Hairer's result, analyzing the KPZ equation in the language of paracontrolled distributions [GP17].

1.2.2 Directed polymers in random environment

Using Feynman-Kac formula, the solution to the SHE (1.2.16) with initial condition $z(x,0)$ can be written as

$$z(x,t) = \mathbb{E}_x\left[e^{\int_0^t \xi(b(s),s)ds} z(b(0),0)\right], \tag{1.2.19}$$

where $\mathbb{E}_x[\cdot]$ is the expectation w.r.t. a standard Brownian motion $b(s),\ 0 \le s \le t$, with $b(t) = x$. In the seminal paper [BG97], it was observed that, if we choose the initial condition $z(x,0) = \delta_x$[1], then $b(s),\ 0 \le s \le t$, can be considered a directed polymer starting at time 0 at 0 and ending at time t at x with (random) energy

$$\int_0^t \xi(b(s),s)ds, \tag{1.2.20}$$

given by the sum over the potential ξ along the path $b(s)$. Thus, it is possible to interpret the solution to the SHE as the partition function of what is called a *continuous directed random polymer* (CDRP). The free energy, i.e. the logarithm of the partition function, of the CDRP corresponds to the Cole-Hopf solution of the KPZ equation.

Continuous directed random polymers represent a one-dimensional prototype of a wider class of polymers, the *directed polymers in random media* (DPRM), introduced by Huse and Henley [HH85] in the study of an Ising ferromagnet with randomly placed impurities. The term "directed" is referred to the time direction, as the polymers are free to move in the other d spatial directions. Formally, they are defined as follows [Com05].

[1] Using the Feynman-Kac formula for the SHE with initial data $z(x,0)$, it is possible to find a polymer interpretation for other initial conditions beyond wedge ($z(x,0) = \delta_x$), corresponding to a point-to-point-polymer. For example, flat initial condition ($z(x,0) = 1$) corresponds to a point-to-line polymer, which starts from a fixed point and ends at a fixed time without a fixed endpoint; stationary initial condition ($z(x,0) = e^{B(x)}$ with $B(x)$ a two-sided Brownian motion) corresponds to a point-to-Brownian polymer, which starts at a fixed point and ends at the first point (y,s) where it meets a random walk started at t.

Consider a random environment $\omega = \{\omega(n,x), n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, a sequence of real-valued, non-constant, i. i. d. random variables defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ such that

$$\mathbb{P}\left(e^{\beta\omega(n,x)}\right) < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.2.21}$$

Let $(S = \{S_n\}_{n\geq 0}, P_x)$ be a simple random walk on $\mathbb{Z}^d$ starting from $x \in \mathbb{Z}^d$. For any $n \geq 0$, the polymer measure is defined as the probability measure $P_n^{\beta,\omega}$ on the path space $(\Omega_{\text{traj}}, \mathcal{F})$ by

$$P_n^{\beta,\omega}(d\mathbf{x}) = \frac{1}{Z_n(\omega,\beta)} e^{\beta H_n(\mathbf{x})} P(d\mathbf{x}), \tag{1.2.22}$$

where $\beta > 0$ is an inverse temperature and

$$H_n(\mathbf{x}) = H_n^{\omega}(\mathbf{x}) = \sum_{1\leq j\leq n} \omega(j, x_j) \tag{1.2.23}$$

is the energy of the path $\mathbf{x}$ in the random environment ω and

$$Z_n = Z_n(\omega,\beta) = P\left(\exp\left(\beta \sum_{1\leq j\leq n} \omega(j, S_j)\right)\right) \tag{1.2.24}$$

is the normalizing constant. So, $P_n^{\beta,\omega}$ is the Gibbs measure with weights $e^{\beta H_n}$ and partition function Z_n.

Directed polymers can be seen as a generalization of percolation models at positive temperature. For fixed n, if we take the zero-temperature limit $\beta \to \infty$, then, the free energy $\frac{1}{\beta}\ln Z_n(\omega,\beta)$ converges to $\max_{\mathbf{x}} H_n(\mathbf{x})$, called *last passage time*, and the polymer measure $P_n^{\beta,\omega}$ concentrates on the paths maximizing the energy. In this case, the term "directed" must be intended as oriented, site percolation. The polymer measure interpolates between the path measure for a standard simple random walk ($\beta = 0$) and geodesics for last passage percolation ($\beta = \infty$). For a more substantial overview on last passage percolation problems, see Chapter 2.

1.3 Integrable systems in the KPZ universality class

While the ballistic deposition and the Eden model cannot be investigated except with simulation, there is a rich class of systems that can be studied with analytic and algebraic methods. We refer to these models as "solvable" or *integrable*. For an integrable probabilistic system, it is possible to compute concise formulas for averages of a class of observables. In this sense, De Moivre (1738) and Laplace (1816) considered the first integrable system, studying the asymptotic distribution of a sum of i. i. d. random variables for Bernoulli trials. Furthermore, taking limits of the system, observables and formulas, it is possible to access detailed descriptions of universal classes. We will focus on a few examples in the Kardar-Parisi-Zhang class in the case of one space dimension.

1.3.1 The corner growth model

Consider an interface modeled by a height function $h(x,t)$, continuous, piecewise linear and made of line increments of length $\sqrt{2}$ and slope ± 1. The dynamics evolves as follows: each local minimum turns into a local maximum after an exponentially distributed waiting

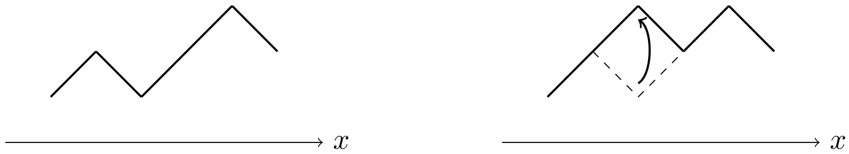

Figure 1.4: Evolution of the height function. A local minimum can grow into a local maximum after an exponential waiting time of parameter 1.

time of parameter 1, independently from each other (see Figure 1.4). This model is known as the *corner growth model.*

Particularly interesting are the cases of two initial configurations: *wedge* initial condition, which means that $h(x, 0) = |x|$, and *flat* initial condition, which means that $h(x, 0)$ is a sawtooth function between 0 an 1.
For wedge initial condition, if we rescale time and space by a large parameter L, the macroscopic limit shape is a parabola continued by two straight lines [Ros81].

$$h_{ma}(x,t) := \lim_{L\to\infty} \frac{h(Lx, Lt)}{L} = \begin{cases} t\frac{1+(x/t)^2}{2} & \text{for } |x| \le t, \\ |x| & \text{for } |x| > t. \end{cases}$$

Fluctuations of the height function around the limit shape are universal in the following sense. For $\epsilon > 0$, define the rescaled height function

$$h_\epsilon(x,t) := \epsilon^\beta h(\epsilon^{-1}x, \epsilon^{-z}t) - \frac{\epsilon^{-1}t}{2}, \tag{1.3.1}$$

where $z = 3/2$ is the dynamic scaling exponent and $b = 1/2$ is the fluctuation exponent. In [Joh00b] Johansson proved large time results for the rescaled height function: for fixed t, as $\epsilon \to 0$, $h_\epsilon(x, t)$ converges to a random variable, in particular,

$$\lim_{\epsilon\to\infty} \mathbb{P}(h_\epsilon(x, 1) \ge -2^{1/3}s) = F_{\rm GUE}(s), \tag{1.3.2}$$

where the function $F_{\rm GUE}$ is known as the GUE Tracy-Widom distribution, first discovered in random matrices [TW94] (see Section 1.4).
For flat initial condition, the macroscopic shape is constantly equal to 1/2. Sasamoto [Sas05] proved for this case a result analogous to (1.3.2) for the fluctuations around the limit shape:

$$\lim_{\epsilon\to 0} \mathbb{P}\left(h_\epsilon(x, 1) \ge -s\right) = F_{\rm GOE}(2s), \tag{1.3.3}$$

where $F_{\rm GOE}$ is known as GOE Tracy-Widom distribution, first observed in the random matrix context (see Section 1.4).

An interesting fact to observe is that, although the scaling exponent are invariant, the limit distribution depends on the initial condition.

1.3.2 Directed polymers and Last Passage Percolation

Consider the above introduced model of growing interface starting from wedge initial configuration. There is an alternative way to describe the evolution of the height function with model of a randomly growing cluster $\mathcal{B}(t)$ that over time invades the entire first quadrant of the plane $\mathbb{N}^2$, rotated by 45°. Each point $(i, j) \in \mathbb{N}^2$ is assigned a weight $\omega_{i,j}$, which is a non-negative random variable and represents the "waiting time" for the

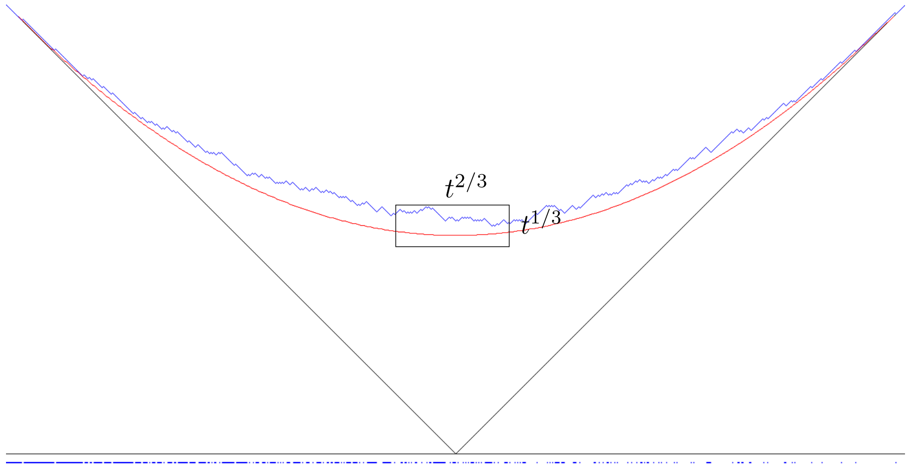

Figure 1.5: A simulation of the height function fluctuations starting from wedge initial condition [Fer]. The curve in red represents the limit shape (a parabola) while the piecewise linear line represents the height function. Fluctuations live on the $t^{1/3}$ scale and are correlated spatially in the $t^{2/3}$ scale.

local valley at (i, j) to become a local maximum, in other words, it is the time it takes to occupy point (i, j), but only after its two neighbours to the left and below are either occupied or lie outside $\mathbb{N}^2$. Once occupied, a point remains occupied: the growing cluster can only add points. Such a model is called *totally asymmetric*.

The evolution is described in terms of the times when points join the interface. Let $L_{i,j}$ for $i, j \in \mathbb{N}$ denote the time when the point (i, j) becomes occupied. Since a point can be added only once the points $(i - 1, j)$ and $(i, j - 1)$ are in the cluster, $L_{i,j}$ must satisfy the recursive relation

$$L_{i,j} = L_{i-1,j} \vee L_{i,j-1} + \omega_{i,j}, \quad \text{for } (i, j) \in \mathbb{N}^2. \tag{1.3.4}$$

Iterating (1.3.4) backwards until we reach the corner $(1, 1)$, we get the formula

$$L_{i,j} = \max_{\pi:(1,1)\to(i,j)} \sum_{(k,l)\in\pi} \omega_{k,l}, \tag{1.3.5}$$

where π are all steps made of consecutive steps of $(1, 0)$ or $(0, 1)$. This model is called *directed last-passage percolation* model: "directed" because of the restrictions on admissible paths, "last-passage" because the occupation time $L_{i,j}$ is determined by the slowest path to (i, j).

In terms of the last passage times, the growing cluster at time t is given by

$$\mathcal{B}(t) = \{(i, j) \in \mathbb{N}^2 : L_{i,j} \le t\}. \tag{1.3.6}$$

When the weights $\omega_{i,j}$ have exponential or geometric distribution, $\mathcal{B}(t)$ becomes a Markov chain in the state space of possible finite clusters in $\mathbb{N}^2$. In the latter case, the model degenerates in the following problem: we consider a (homogeneous) Poisson point process with density 1 in $\mathbb{R}^2_+$ and look for the directed path that maximizes the number of Poisson points. Baik, Deift and Johansson [BDJ99b] showed that the behaviour of these models is typical of the KPZ class, using the connection of the corner growth model to a combinatorial one, known as *Ulam's problem* [Ula61]. Let $\mathcal{S}_N$ be the permutation group of $\{1, \ldots, N\}$. For $\sigma \in \mathcal{S}_N$, we say that the sequence $(\sigma(1), \ldots, \sigma(N))$ has an increasing subsequence of length k, $(n_1, \ldots, n_k) \subset (\sigma(1), \ldots, \sigma(k))$, if $1 \leq n_1 < n_2 < \cdots < n_k \leq N$. We denote with $\ell_N(\sigma)$ the length of the longest increasing subsequence for the permutation σ. The Ulam's problem studies the asymptotic law of L_N for uniform distributions on $\mathcal{S}_N$. In 1961 it was conjectured that $\mathbb{E}[\ell_N] \approx c\sqrt{N}$. Hammersley [Ham66] proved the existence of $c = \lim_{N\to\infty} \mathbb{E}[\ell_N]/N$, but only in 1968 the fact that the right value was $c = 2$ was pointed out via numeric simulations [BB68]. The final proof that $c = 2$ was completed in 1977 by Vershik and Kerov [VK77]. In [BDJ99b] it was proved that the fluctuations of the length of the longest increasing subsequence of a uniformly distributed permutation σ of $\{1, \ldots, n\}$ are GUE Tracy-Widom distributed, i.e.

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell_n(\sigma) - 2\sqrt{n}}{n^{1/6}} \leq s\right) = F_{\mathrm{GUE}}(s). \tag{1.3.7}$$

They studied a Poissonized version of the problem: instead of fixing the length of the permutations to N, they assigned the probability $e^{-N}k^N/N!$ that a sequence is of length k. The Poissonized Ulam's problem is related to directed polymers on Poisson points.

For a more detailed and exhaustive description of last passage percolation models we refer the reader to Chapter 2.

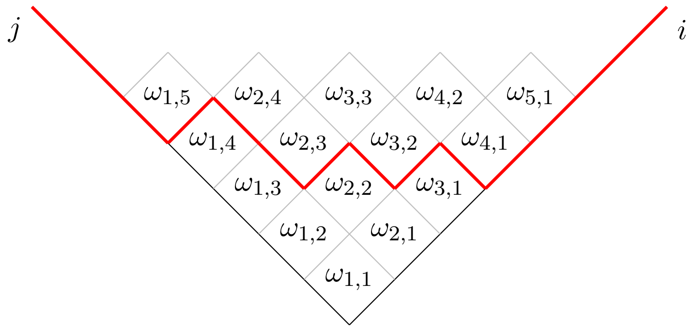

Figure 1.6: An example of last passage percolation with exponential weights.

1.3.3 Interacting particle systems: the Totally Asymmetric Simple Exclusion Process

The last passage model can be related to a particle process on $\mathbb{Z}$ in discrete time, in the case of geometric weights, or in continuos time, in the exponential case. This process is called *totally asymmetric simple exclusion process* (TASEP).

TASEP is a Markov process that describes the motion of particles on the integer lattice $\mathbb{Z}$ with the constraint that two particles cannot occupy the same site at the same time (exclusion rule). We can label the particles from right to left and denote with $x_i(t), i \in I \subset \mathbb{Z}$ the position of the i-th particle at time t. Because of the constraint, the ordering of particles is preserved: if initially

$$\ldots < x_2(0) < x_1(0) < 0 \leq x_0(0) < x_{-1}(0) < \cdots ,$$

then, for all times $t \geq 0$, also $x_{n+1}(t) < x_n(t)$, $n \in \mathbb{Z}$.

In discrete time particles evolve with the following rules. Let $\{x_i(0)\}_{i\in I}$ be the initial particles configuration. For each $t \in \mathbb{N}$,

- if site $x_i(t) + 1$ is occupied at time t, then $x_i(t + 1) = x_i(t)$;

- if site $x_i(t)+1$ is vacant at time t, then

$$x_i(t+1) = \begin{cases} x_i(t)+1 & \text{with probability } p, \\ x_i(t) & \text{with probability } 1-p. \end{cases} \tag{1.3.8}$$

This defines a Markov process in discrete time, since particle positions at time t depend only on the configuration at time $t-1$ and all jumps occur independently.

To run the process in continuous time, we equip each particle with a "Poisson clock", a Poisson process with rate 1 on the positive real line: whenever the Poisson clock of a particle "rings", the particle attempts to jump to the right, compatibly with the exclusion rule.

Now, we consider TASEP with "step" initial condition: at time 0 all particles are on the left of the origin, occupying all negative integer sites, $x_i(0) = -i$. In this case, the particles determine a growing cluster in $\mathbb{N}^2$,

$$\mathcal{A}(t) = \{(i,j) \in \mathbb{N}^2 : 1 \le j \le x_i(t) + i\}, \tag{1.3.9}$$

such that the height of the column above i is the number of steps taken by the particle i up to time t, $x_i(t) - x_i(0)$. Then, the processes $\mathcal{A}(t)$ and $\mathcal{B}(t)$, $t \ge 0$ are equal in distribution. This is true for both discrete time/geometric weights and continuous time/exponential weights. For last passage percolation with geometric and exponential weights, $\omega_{i,j}$ can be interpreted as the waiting time of particle j to jump from site $i-j-1$ to site $i-j$

1.3.4 The polynuclear growth model

The *discrete polynuclear growth model* (PNG) is a growth model with discrete space, $x \in \mathbb{Z}$, and discrete time, $t \in \mathbb{Z}_+$. The height function is integer-valued, $h(x,t) \in \mathbb{Z}$ for any $x \in \mathbb{Z}$, $t \in \mathbb{Z}_+$.

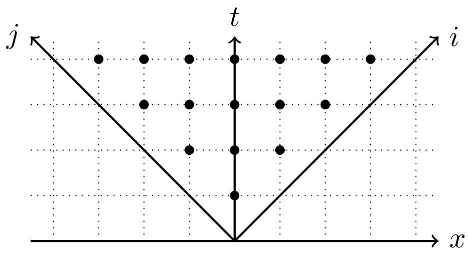

Figure 1.7: Setting for the PNG model. Representation of the weights: a black dot is drawn on sites where $\widetilde{\omega}(x,t) \neq 0$.

We consider the PNG model in the so-called "droplet" geometry, corresponding to the wedge initial configuration for the corner growth model introduced in Section 1.3.1. Fix the initial condition $h(x,0) = 0$, for any $x \in \mathbb{Z}$, and define the dynamics as

$$h(x,t+1) = \max\{h(x-1,t), h(x,t), h(x+1,t)\} + \widetilde{\omega}(x,t+1), \tag{1.3.10}$$

where $\widetilde{\omega}(x,t)$ are independent random variables such that $\widetilde{\omega}(x,t) = 0$ if $x-t$ is odd or $|x| > t$. Then, $\widetilde{\omega}(x,t) \neq 0$ in the discrete lattice rotated by $\pi/4$ (see Figure 1.7). We indicate the vertices with $i = (x+t)/2$ and $j = (t-x)/2$ and denote $\omega(i,j) = \widetilde{\omega}(i-j, i+j-1)$. In Figure 1.8 an example of the evolution of the PNG model is

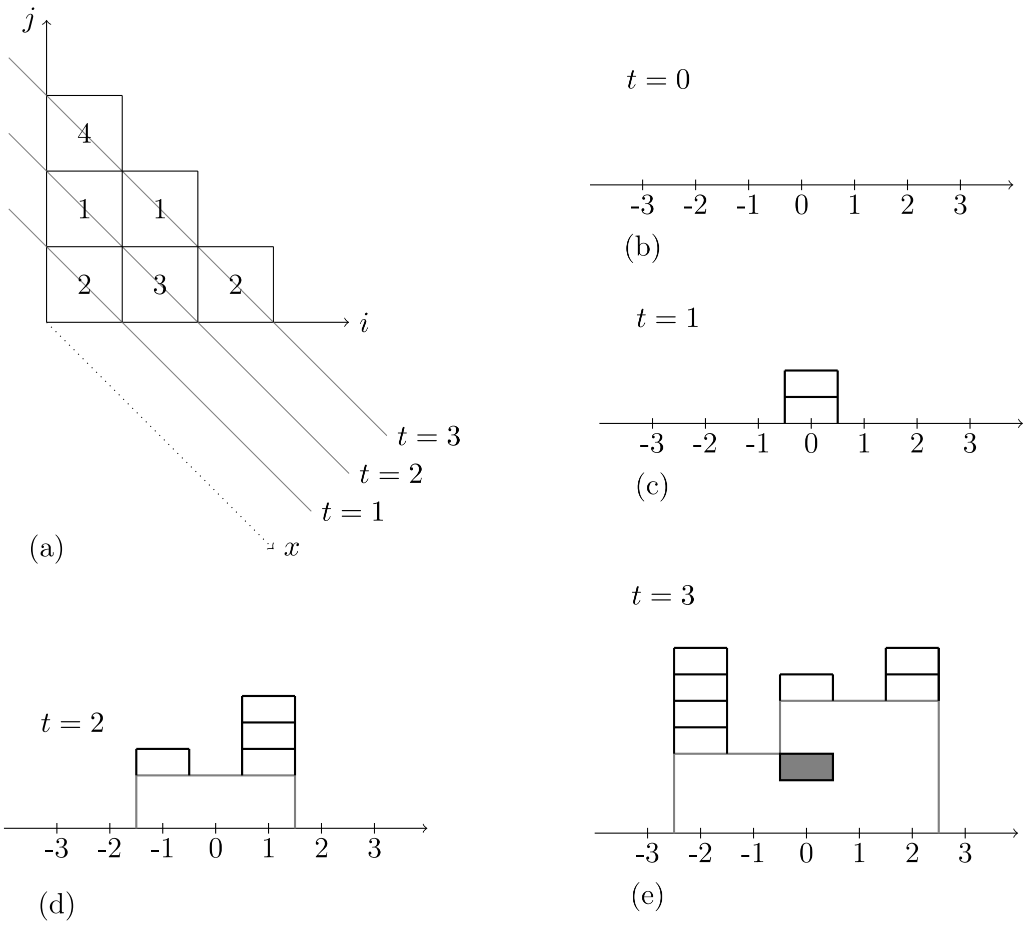

Figure 1.8: Evolution of the PNG model. The height function is represented with a continuous line, but it is defined only on $\mathbb{Z} + \frac{1}{2}$. In figure (a) the values of the random variables ω are represented on the plane (i, j). From (b) to (e) the first three iterations of the height function: we start from $h(x, 0) = 0$, then we add two "boxes" on the site 0, since $\omega(1, 1)$=2; at the next step we move all the points of the previous configuration of one step to the left and to the right, corresponding to taking the maximum in (1.3.10), and we add a number of boxes on the sites 1,-1 indicated by the values of $\omega(1, 2)$, $\omega(2, 1)$, and so on. At the third step two "islands" meet and we observe the formation of an overlap.

represented up to the first three steps[2]. In the case when $\omega(i, j)$ are geometric random variables, i.e. $\mathbb{P}(\omega(i, j) = k) = (1-a_i b_j)(a_i b_j)^k, k \geq 0$, with $a_i, b_j \in (0, 1)$, $\forall i, j \in \mathbb{Z}_+$, the

[2] In Figure 1.8 we can observe that until time $t = 2$ we can recover from the graph of the height function the values of the $\omega(i, j)$'s, but at time $t = 3$ this is no longer possible due to the overlap, that makes lose the initial information. So, given $\{h(x, t), |x| \leq t\}$, we cannot recover the values $\{\omega(i, j)\}$: even a nice measure on the $\omega(i, j)$ will not translate to a simple measure on $\{h(x, t), |x| \leq t\}$. The answer to the question about how to recover this information is to extend this model to a set of height functions which are bijective with the $\omega(i, j)$'s. To solve this problem Imamura and Sasamoto [IS05] introduced the *multilayer PNG model*, which defines a 1:1 map between $\{h(x, t), |x| \leq t\}$ and $\{\omega(x, t), |x| \leq t\}$, by keeping track of the overlaps during the evolution. This is defined as follows: let $h_1(x, t) := h(x, t)$ and introduce a set of height functions $\{h_\ell(x, t), x \in \mathbb{Z}, t \in \mathbb{Z}_+, \ell \geq 1\}$ with initial condition $h_\ell(x, 0) = -\ell + 1$, for any $x \in \mathbb{Z}, \ell \geq 1$. The dynamics of $h_1(x, t)$ is the same as in the discrete PNG, as for $\ell \geq 2$, $h_\ell(x, t+1) = \max\{h_\ell(x-1, t), h_\ell(x, t), h_\ell(x+1, t)\} + w_\ell(x, t+1)$, where $w_\ell(x, t+1)$ is given by the overlap on the site x at level $\ell - 1$. The lines $\{h_\ell(x, t), x \in \mathbb{Z}, t \in \mathbb{Z}_+, \ell \geq 1\}$ are non-intersecting by construction. They can be seen as the trajectories of fermions, particles which can not occupy simultaneously the same position (state).

model is integrable and is a generalization of a last passage percolation, which is recovered for $a_i = b_i = \sqrt{q}$, $q \in (0,1)$ with the relation

$$h(x,t) = L_{\frac{x+t}{2},\frac{t-x}{2}}, \tag{1.3.11}$$

where L is defined in (1.3.5). In [Joh03] Johansson proved that the fluctuations of the height function are governed by the GUE Tracy-Widom distribution: for $\tau \in [0,1)$, we define $\gamma = (1-\tau)/(1+\tau)$ and

$$\begin{aligned} \mu(\gamma,q) &= \frac{1}{1+\gamma}\left(\frac{(1-\sqrt{q\gamma})^2}{1-q} - 1\right), \\ \sigma(\gamma,q) &= \frac{q^{1/6}\gamma^{-1/6}}{(1-q)(1+\gamma)^{1/3}}\left((\sqrt{\gamma}+\sqrt{q})^{2/3}(1+\sqrt{q})^{2/3}\right). \end{aligned} \tag{1.3.12}$$

Then,

$$\lim_{t\to\infty} \mathbb{P}\left(h(\tau t,t) \leq \mu(\gamma,q)t + \sigma(\gamma,q)t^{1/3}s\right) = F_{\mathrm{GUE}}(s). \tag{1.3.13}$$

We can obtain a continuous version of the PNG taking the limit $q \to 0$ with lattice spacing $\sqrt{q}$. The dynamics is constructed from a space-time Poisson process of intensity 2 of nucleation events. At points of the Poisson process the height is increased by 1, creating an adjacent pair of up-step and down-step. Up-steps move to the left with velocity -1, down-steps move to the right with velocity $+1$ and steps disappear upon coalescence. As for the discrete case, it was proved [PS00] that the fluctuations of the height function are distrubuted according to the GUE Tracy-Widom distribution.

The polynuclear growth model on a flat substrate, i.e. $h(x,0) = 0$, was firstly considered in [BR01c] and in [PS00], where it was shown that the one-point distribution is the GOE Tracy-Widom distribution (see Section 1.4.2). In the attempt to study the joint distribution of the full process with respect to x, Ferrari [Fer04] extended the connection between the GOE Tracy-Widom distribution and the PNG model: he considered the multilayer PNG, which is a stack of non-intersecting lines, the top one being the PNG height, and proved that the edge statistics of this point process is described by F_{GOE} for large times.

1.4 Random matrices

In Section 1.3.1 we introduced the distributions F_{GUE} and F_{GOE} and mentioned that they were first observed in the context of random matrices. In this section we explain in which framework they appear, introducing the classical Gaussian ensemble of random matrices. The interested reader can refer to [AGZ10] or [Meh91] for standard books on random matrices.

1.4.1 Gaussian Unitary Ensemble

The Gaussian Unitary Ensemble (GUE) of random matrices consists of Hermitian matrices H of size $n \times n$ distributed according to the probability measure

$$p^{\mathrm{GUE}}(H)\mathrm{d}H = \frac{1}{Z_n}\exp\left(-\frac{1}{2n}\operatorname{Tr}(H^2)\right)\mathrm{d}H, \tag{1.4.1}$$

where $\mathrm{d}H = \prod_{i=1}^n \mathrm{d}H_{i,i} \prod_{1\leq i<j\leq n} \mathrm{dRe}(H_{i,j})\mathrm{dIm}(H_{i,j})$ is the reference measure and Z_n the normalization constant. Or, equivalently, it consists of matrices with entries distributed

according to complex gaussians, i.e. $H_{i,j} \sim \mathcal{N}(0, \frac{n}{2}) + \mathrm{i}\mathcal{N}(0, \frac{n}{2})$, $H_{i,i} \sim \mathcal{N}(0, n)$. Denote by $\lambda_{n,\max}^{\mathrm{GUE}}$ the largest eigenvalue of a $n \times n$ GUE matrix. Tracy and Widom [TW94] proved that the asymptotic distribution of the (properly rescaled) largest eigenvalue is F_{GUE} (see Figure 1.9):

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\lambda_{n,\max}^{\mathrm{GUE}} - 2n}{n^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s). \tag{1.4.2}$$

The GUE Tracy-Widom distribution is given by

$$F_{\mathrm{GUE}}(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)dx\right), \tag{1.4.3}$$

where q is the unique solution of the Painlevé II equation $q'' = sq + 2q^3$ satisfying the asymptotic condition $q(s) \sim \mathrm{Ai}(s)$ for $s \to \infty$, with $\mathrm{Ai}(s)$ the Airy function

$$\mathrm{Ai}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{\frac{\mathrm{i}(z-\mathrm{i}w)^3}{3} - \mathrm{i}s(z-\mathrm{i}w)}, \tag{1.4.4}$$

where $w > 0$ is arbitrary. F_{GUE} can also be written as Fredholm determinant of the Airy operator (see Section 1.5.2). Consider the *Airy kernel*

$$A(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}, \tag{1.4.5}$$

as an integral kernel on $L^2[s,\infty)$. Then,

$$F_{\mathrm{GUE}}(s) = \det(\mathbb{1} - A)|_{L^2[s,\infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(A(x_i,x_j))_{i,j=1}^k d^k x. \tag{1.4.6}$$

GUE is invariant under unitary conjugation. If $C \in \mathbb{C}^{n\times n}$ is unitary (i.e. $CC^\star = I$), then $C^\star HC$ has the same distribution as H. If we translate the invariance of the matrices into space-time symmetries for physical system, this describes a system which is not invariant with respect to time-inversion, e.g. in the presence of an external magnetic field.

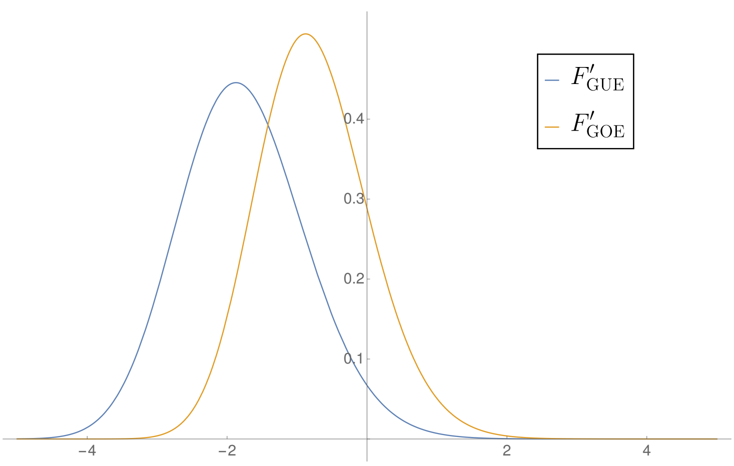

Figure 1.9: The densities of the GUE Tracy–Widom distribution (in blue) and the GOE Tracy–Widom distribution (in yellow).

1.4.2 Gaussian Orthogonal Ensemble.

The Gaussian Orthogonal Ensemble (GOE) of random matrices consists of symmetric matrices H of size $n \times n$ distributed according to the probability measure

$$p^{\mathrm{GOE}}(H)dH = \frac{1}{Z_n} \exp\left(-\frac{1}{4n} Tr(H^2)\right) dH, \tag{1.4.7}$$

where $dH = \prod_{1 \le i < j \le n} dH_{i,j}$ is the reference measure and Z_n the normalization constant. Or, equivalently, of matrices with entries $H_{i,j} \sim \mathcal{N}(0, n)$ and $H_{i,i} \sim \mathcal{N}(0, 2n)$. Denote by $\lambda_{n,\max}^{\mathrm{GOE}}$ the largest eigenvalue of a $n \times n$ GOE matrix. The asymptotic distribution of the (properly rescaled) largest eigenvalue is F_{GOE} [TW96] (see Figure 1.9):

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\lambda_{n,\max}^{\mathrm{GOE}} - 2n}{n^{1/3}} \le s\right) = F_{\mathrm{GOE}}(s). \tag{1.4.8}$$

The GOE Tracy-Widom distribution is defined as

$$F_{\mathrm{GOE}}(s) = \exp\left(-\frac{1}{2}\int_s^\infty q(x)dx\right) F_{\mathrm{GUE}}(s)^{1/2}, \tag{1.4.9}$$

with q as above. GOE is invariant under orthogonal conjugation. If $C \in \mathbb{R}^{n\times n}$ is orthogonal (i.e. $CC^T = I$), then $C^T HC$ has the same distribution as H. This ensemble describes systems which are invariant with respect to time-inversion and with integer total angular momentum.

In [FS05] it was proved that the GOE Tracy-Widom distribution can be expressed with a determinantal formula involving the Airy function. Let B_s be the operator with kernel

$$B_s(x, y) = \mathrm{Ai}(x + y + s). \tag{1.4.10}$$

Then,

$$F_{\mathrm{GOE}}(s) = \det(\mathbb{1} - B_s)_{L^2(\mathbb{R}_+)}. \tag{1.4.11}$$

F_{GOE} can be also written as a Fredholm pfaffian [BBCS18] (see Section 1.5.3 for the definition of Fredholm pfaffian)

$$F_{\mathrm{GOE}}(s) = \mathrm{pf}(J - K^{\mathrm{GOE}})_{L^2(s,\infty)}, \tag{1.4.12}$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ is the symplectic matrix and $\mathcal{C}_a^\phi$ is the union of two semi-infinite rays departing from $a \in \mathbb{C}$ with angles ϕ and $-\phi$, oriented from $a + e^{-i\phi}$ to $a + e^{+i\phi}$.
K^{GOE} is the 2×2 matrix valued kernel defined by

$$\begin{aligned}
K_{11}^{\mathrm{GOE}}(x, y) =& \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1^{\pi/3}} dz \int_{\mathcal{C}_1^{\pi/3}} dw \frac{z - w}{z + w} e^{z^3/3 + w^3/3 - xz - yw}, \\
K_{12}^{\mathrm{GOE}}(x, y) =& -K_{21}^{\mathrm{GOE}}(x, y) = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1^{\pi/3}} dz \int_{\mathcal{C}_{-1/2}^{\pi/3}} dw \frac{w - z}{2w(z + w)} e^{z^3/3 + w^3/3 - xz - yw}, \\
K_{22}^{\mathrm{GOE}}(x, y) =& \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1^{\pi/3}} dz \int_{\mathcal{C}_1^{\pi/3}} dw \frac{z - w}{4zw(z + w)} e^{z^3/3 + w^3/3 - xz - yw} \\
&+ \frac{1}{(2\pi i)^2} \int_{\mathcal{C}_1^{\pi/3}} \frac{dz}{4z} e^{z^3/3 - xz} - \int_{\mathcal{C}_1^{\pi/3}} \frac{dz}{4z} e^{z^3/3 - yz} - \frac{\mathrm{sgn}(x - y)}{4}.
\end{aligned} \tag{1.4.13}$$

The joint distributions of the eigenvalues of these matrix ensembles are contained in the following one-parameter family of densities:

$$f(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_\beta} \prod_{i<j} |\lambda_j - \lambda_i|^\beta e^{-\frac{\beta}{4n} \sum_{i=1}^n \lambda_i^2}. \tag{1.4.14}$$

For a given $\beta > 0$, the resulting distribution (on ordered n-tuples in $\mathbb{R}$) is called *Dyson's β-ensemble.* For $\beta = 1$ one gets the eigenvalue density of GOE, for $\beta = 2$ the GUE. The $\beta = 4$ case is also special: it is related another classical random matrix model, the Gaussian Symplectic Ensemble (GSE), which can be defined using quaternions. This is invariant under symplectic unitary transformation: if $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ be the symplectic matrix, then, $JHJ^{-1} = H$. This ensemble describes physical systems which are invariant with respect to time-inversion and have half-integer total angular momentum.

The limit distribution of the largest eigenvalue of a $n \times n$ GSE matrix is also a Fredholm pfaffian and is known as the GSE Tracy-Widom distribution F_{GSE}.

1.5 Determinantal and Pfaffian processes

In the previous section we gave a description of the limit distributions of the rescaled eigenvalues of Gaussian ensembles of random matrices in terms of Fredholm determinants and pfaffians. Here we introduce these concepts to the reader, giving a brief overview on determinantal and Pfaffian point processes.

1.5.1 On determinants and point processes

Consider a complete separable metric space Λ equipped with its Borel σ-algebra $\mathcal{B}(\Lambda)$. Let Ω be the set of *locally finite* particle configurations: a point configuration $x = (x_i)_{i \in I}$, $x_i \in \Lambda$, $I \subset \mathbb{N}$ is locally finite if, for every bounded set $B \subset \Lambda$, $\xi(B) = \#\{x_i \in B\} < \infty$. Let $\mathcal{F}$ be the σ-algebra generated by the cylinder sets $C_n^B = \{\xi \in \Omega \,|\, \xi(B) = n\}$, $n \geq 0$, $B \subset \Lambda$ bounded, and let $\mathbb{P}$ be a probability measure on (Ω, Λ).

A *point measure* on Λ is a positive measure ν on $(\Lambda, \mathcal{B}(\Lambda))$, which is a locally finite sum of Dirac measures, i.e. $\nu = \sum_{i \in I} \delta_{x_i}$, $I \subset \mathbb{N}$, and for every bounded set $B \subset \Lambda$, $\nu(\mathbb{1}_B) = \#\{x_i \in B\} < \infty$. Denote with $M_p(\Lambda)$ the set of all positive measures on Λ and with $\mathcal{M}_p(\Lambda)$ the σ-algebra generated by the mappings from $M_p(\Lambda)$ to $\mathbb{N} \cup \{\infty\}$, $\nu \mapsto \nu(f)$, when f spans $\mathcal{B}(\Lambda)$.

Definition 1.5.1. *A* point process η *in* Λ *is a measurable mapping from* $(\Omega, \mathcal{F}, \mathbb{P})$ *to* $(M_p(\Lambda), \mathcal{M}_p(\Lambda))$. *The law of* η *is* $\mathbb{P} \circ \eta^{-1}$.

We say that η is a *simple point process* if $\mathbb{P}(\eta(\{x\}) \leq 1, \forall x \in \Lambda) = 1$. Given a set $A \subset \Lambda$, we denote the measure of the set as $\eta(A) := \eta(\mathbb{1}_A)$. A point process is usually described in terms of *correlation functions.* For any $A_1, \dots, A_n$ disjoint Borel sets of Λ, define

$$M_n(A_1, \dots, A_n) = \mathbb{E}\left[\prod_{i=1}^n \eta(\mathbb{1}_{A_i})\right]. \tag{1.5.1}$$

For generic $A_1, \dots, A_n \in \mathcal{B}(\Lambda)$,

$$M_n(A_1, \dots, A_n) = \mathbb{E}\left[\sum_{\substack{(x_1,\dots,x_n) \in \mathrm{supp}(\eta) \\ x_i \neq x_j, \forall i,j}} \prod_{i=1}^n \mathbb{1}_{A_i}(x_i)\right]. \tag{1.5.2}$$

If M_n is absolutely continuous with respect to a reference measure $\mu^{\otimes n}$ on Λ, i.e. if for any $A_1, \dots, A_n \in \mathcal{B}(\Lambda)$, there exists a function of the configuration $\rho^{(n)}$ such that

$$M_n(A_1, \dots, A_n) = \int_{A_1 \times \cdots \times A_n} d\mu(x_1) \cdots d\mu(x_n) \rho^{(n)}(x_1, \dots, x_n), \tag{1.5.3}$$

then $\rho^{(n)}$ is called the *n-point correlation function*. It is immediate to see that $\rho^{(n)}$ is symmetric in its arguments.

We often consider random point processes with fixed number of particles N, described by a joint probability distribution $P_N(dx_1, \dots, dx_N)$ symmetric with respect to permutations of the x_i's, In this case, the correlation measures assume a simplified form.

Proposition 1.5.2. *Let P_N be a symmetric density on Λ^N w.r.t. $\mu^{\otimes N}$, i.e. the probability measure on Λ^N is given by $P_N(x_1, \dots, x_N) d\mu(x_1) \dots d\mu(x_N)$. Then,*

$$\rho^{(n)}(x_1, \dots, x_n) d\mu(x_1) \cdots d\mu(x_n) = \begin{cases} \frac{N!}{(N-n)!} \int_{\Lambda^N} P_N(x_1, \dots, x_N) d\mu(x_{n+1}) \dots d\mu(x_N), & n \leq N \\ 0, & n > N. \end{cases} \tag{1.5.4}$$

Correlation functions are particularly useful to characterize the measures of "empty sets". The distribution of a height function is often formulated in terms of a *gap probability*, $\mathbb{P}(h_0 \leq a) = \mathbb{P}(\eta(\mathbb{1}_{(a,\infty)}) = 0)$, which is the probability that there are no particles above a level a.

Proposition 1.5.3. *Let $B \in \mathcal{B}(\Lambda)$. Then*

$$\mathbb{P}(\eta(\mathbb{1}_B) = 0) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{B^n} d\mu(x_1) \cdots d\mu(x_n) \rho^{(n)}(x_1, \dots, x_n). \tag{1.5.5}$$

An important class of simple point processes are the *determinantal point processes.*

Definition 1.5.4. *A point process in Λ is said to be* determinantal *if there exists a function $K(x, y)$ on $\Lambda \times \Lambda$ such that the correlation functions (with respect to some reference measure) are given by the determinantal formula*

$$\rho^{(n)}(x_1, \dots, x_n) = \det \left[K(x_i, x_j) \right]_{1 \leq i,j \leq n} \tag{1.5.6}$$

for $n = 1, 2, \dots$. The function K is called correlation kernel *and is the kernel of an integral operator $\mathcal{K} : L^2(\Lambda, d\mu) \to L^2(\Lambda, d\mu)$.*

This means that

$$\begin{aligned} \rho^{(1)}(x) &= K(x, x), \\ \rho^{(2)}(x_1, x_2) &= \det \begin{bmatrix} K(x_1, x_1) & K(x_1, x_2) \\ K(x_2, x_1) & K(x_2, x_2) \end{bmatrix}, \\ &\cdots \end{aligned}$$

The correlation kernel is a single function of two variables while the correlation functions form an infinite sequence of functions of growing number of variables. Thus, if a point process happens to be determinantal, it can be described by a substantially reduced amount of data. As a consequence of Proposition 1.5.3, we can express the gap probability of a set as follows.

Proposition 1.5.5. *Let $B \in \mathcal{B}(\Lambda)$. For a determinantal point process with correlation kernel $K(x, y)$,*

$$\begin{aligned}\mathbb{P}(\eta(\mathbb{1}_B) = 0) &= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_{B^n} d\mu(x_1)\cdots d\mu(x_n) \det\left[K(x_i, x_j)\right]_{1\leq i,j\leq n} \\ &=: \det\left[\mathbb{1} - K\right]_{L^2(B,d\mu)} .\end{aligned} \tag{1.5.7}$$

The determinant defined in (1.5.7) is called *Fredholm determinant* of the operator K on the space $L^2(B, d\mu)$.
Notice that the correlation kernel is not unique, indeed the process is invariant for gauge transformations of the kernel of the form

$$K(x, y) \mapsto \frac{f(x)}{f(y)} K(x, y) \tag{1.5.8}$$

for non-vanishing $f : \Lambda \to \mathbb{C}$.

1.5.2 Fredholm determinants

The theory of Fredholm determinants started with the study of the integral equation

$$(\mathbb{1} + K)u = f, \tag{1.5.9}$$

where K is the integral operator $Kf(x) = \int_Y K(x, y)f(y)dy$ mapping functions on Y to functions on X, with X, Y compact metric spaces. If $K : [0, 1]^2 \to \mathbb{C}$ is continuous and $f \in C[0, 1]$, by Riesz' theorem, the equation

$$u(x) + \int_0^1 K(x, y)u(y)dy = f(x) \tag{1.5.10}$$

admits solutions on $C[0, 1]$ iff the integral operator K is injective on the space. Fredholm's idea to solve the equation was to take advantage of linear algebra discretizing the integral equation, and then take the limit. This introduced the object known as Fredholm determinant, which determines whether the given integral equation is solvable. Here we discuss Fredholm determinants for operators on Hilbert spaces and state the fundamental properties. For a more detailed discussion we refer to [Sim00].

Let H be a separable Hilbert space on $\mathbb{C}$. Let T be a compact operator on H and $T^\star$ its adjoint. Then, $T^\star T$ is a non-negative self-adjoint operator and both $T^\star T$ and $A = \sqrt{T^\star T}$ are compact operators with non-negative eigenvalues. The singular values of T are the positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ of A, counting with multiplicity. We say that T is a trace class operator if the sum of its singular values is finite (counting with multiplicity). In this case, the trace norm of T is defined as the sum of its singular values,

$$\|T\|_{\text{tr}} = \sum_i \lambda_i. \tag{1.5.11}$$

For a trace class operator T the following properties hold:

i. Let $T^\star$ denote the adjoint of T. Then, $\|T\|_{\text{tr}} = \|T^\star\|_{\text{tr}}$;

ii. For any bounded operator B on H, BT and TB are trace class. Moreover

$$\begin{aligned}\|TB\|_{\text{tr}} &\leq \|T\|_{\text{HS}}\|B\|_{\text{HS}}, \\ \|TB\|_{\text{tr}} &\leq \|T\|_{\text{tr}}\|B\|_{\text{op}},\end{aligned} \tag{1.5.12}$$

where $\|\cdot\|_{\text{op}}$ is the operator norm and $\|\cdot\|_{\text{HS}}$ is the Hilbert-Schimdt norm.

iii. $\|T_1 + T_2\|_{\text{tr}} \leq \|T_1\|_{\text{tr}} + \|T_2\|_{\text{tr}}$.

Moreover, one can prove the following equivalent characterization of the trace norm.

Proposition 1.5.6. *For any trace class operator T,*

$$\|T\|_{\text{tr}} = \sup_{f_n, e_n} \sum_n |\langle Tf_n, e_n\rangle|, \tag{1.5.13}$$

where $\{f_n\}$ and $\{e_n\}$ are orthonormal bases of H.

Given a trace class operator, one can define the linear functional

$$\text{Tr}(T) = \sum_n \langle Tf_n, f_n\rangle, \tag{1.5.14}$$

where $\{f_n\}$ is an orthonormal basis of H. This is called the trace of T. Lidskii's trace formula says that, if T is a trace class operator on a separable Hilbert space, then

$$\text{Tr}(T) = \sum_i \lambda_i, \tag{1.5.15}$$

where λ_i are the eigenvalues of T. Since T is compact, it has a countable set of nonzero eigenvalues.

Now we define an inner product on H^k by

$$\langle (w_1, \dots, w_k), (v_1, \dots, v_k)\rangle = \det(\langle w_i, v_j\rangle)_{i,j}. \tag{1.5.16}$$

Then, T extends to a trace class operator T_k on H^k, defined by $T_k(w_1, \dots, w_k) = (Tw_1, \dots, Tw_k)$. Thus, we can define

$$\det(\mathbb{1} + T) = \sum_{k\geq 0} \text{Tr}(T_k). \tag{1.5.17}$$

This is known as the *Fredholm determinant* of T. It can be shown that

$$\det(\mathbb{1} + T) = \prod_i (1 + \lambda_i), \tag{1.5.18}$$

where λ_i are the eigenvalues of T. Moreover, it satisfies the following properties:

i. $|\det(\mathbb{1} + T_1) - \det(\mathbb{1} + T_2)| \leq \|T_1 - T_2\|_{\text{tr}} \exp(\|T_1\|_{\text{tr}} + \|T_2\|_{\text{tr}} + 1)$;

ii. $\det(\mathbb{1} + T_1 + T_2 + T_1T_2) = \det(\mathbb{1} + T_1)\det(\mathbb{1} + T_2)$;

iii. $\det(\mathbb{1} + T_1T_2) = \det(\mathbb{1} + T_2T_1)$,

for any T_1, T_2 trace class operators. In this context, we deal with integral operators. We say that T is an *integral operator* on the space $L^2(X)$, if there exists a function $K : X \times X \to \mathbb{R}$ such that

$$(Tf)(x) = \int_X K(x, y)f(y)dy. \tag{1.5.19}$$

We call K the integral kernel. The Hilbert-Schmidt norm of an integral operator is defined as

$$\|T\|^2_{\text{HS}} = \int_{X\times X} |K(x, y)|^2 dxdy. \tag{1.5.20}$$

The Fredholm determinant of T is given by

$$\det(\mathbb{1} + T)_{L^2(X)} = \sum_{n\geq 0} \frac{1}{n!} \int_{X^n} dx_1 \dots dx_n \det_{1\leq i,j\leq n}[K(x_i, x_j)]. \tag{1.5.21}$$

1.5.3 On Pfaffian processes

Given an anti-symmetric $2n \times 2n$ matrix $(a_{i,j})$, its pfaffian is defined as:

$$\operatorname*{pf}_{i<j} a_{i,j} = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) a_{\sigma(1),\sigma(2)} a_{\sigma(3),\sigma(4)} \cdots a_{\sigma(2n-1),\sigma(2n)} \tag{1.5.22}$$

where $\mathcal{S}_{2n}$ is the permutation group on $2n$ letters and we write $\mathrm{pf}_{i<j}$ to emphasize the fact that our matrices are skew-symmetric (and thus determined entirely by the upper triangular part). One can show that

$$\left(\operatorname*{pf}_{i<j} a_{i,j} \right)^2 = \det_{i,j} a_{i,j}. \tag{1.5.23}$$

Suppose that one has a 2×2 anti-symmetric matrix kernel $K(x,y)$, i.e. K is a 2×2 matrix function of (x,y) satisfying $K(x,y) = -K^t(y,x)$ with t denoting transposition. Notice the interchange of x and y. Given such a kernel and points $x_1, \dots, x_n$, one can define a $2n \times 2n$ anti-symmetric matrix $K^{(n)}$ block-wise as follows: its 2×2 block (i,j) for $1 \le i,j \le n$ is given by the matrix $K(x_i, x_j)$. Because $K(x,y) = -K^t(y,x)$, $K^{(n)}$ thus defined is even-dimensional anti-symmetric and its pfaffian well-defined.

A point process (measure)[3] on a configuration space X is called *Pfaffian with* 2×2 *matrix correlation kernel* K if there exists a 2×2 matrix K satisfying $K(x,y) = -K^t(y,x)$ such that, for all $n \ge 1$, the n-point correlation functions $\rho^{(n)}(x_1, x_2, \dots, x_n) := \mathbb{P}(S : x_1 \in S, \dots, x_n \in S)$ of the process are pfaffians of the associated $2n \times 2n$ matrix $K^{(n)}$:

$$\rho^{(n)}(x_1, x_2, \dots, x_n) = \operatorname*{pf}_{i<j} K^{(n)}(x_i, x_j). \tag{1.5.24}$$

A simple observation says that if this is the case, the one-point function is the K_{12} entry: $\mathbb{P}(S : x \in S) = \rho_1(x) = K_{12}(x,x)$.

Given a 2×2 anti-symmetric matrix kernel K defined on a configuration space X equipped with a measure dx, the *Fredholm pfaffian* of K restricted to the subspace $Y \subset X$ is defined as

$$\mathrm{pf}(J + \lambda K)_{L^2(Y)} := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{Y^n} \operatorname*{pf}_{i<j} K^{(n)}(x_i, x_j) \prod_{i=1}^{n} dx_i. \tag{1.5.25}$$

Here J is the anti-symmetric matrix kernel $J(x,y) = \delta_{x,y} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, but as is oftentimes the case in the literature, this technicality is overlooked and we think of J just as the corresponding 2×2 matrix.

Technically speaking, the Fredholm pfaffian $\mathrm{pf}(J + \lambda K)_{L^2(Y)}$ is finite whenever K (or rather its entries) is a (are) trace-class operator(s) on $L^2(Y)$. Moreover, Fredholm pfaffians are defined up to conjugation, in the following sense. Suppose $\tilde{K}$ is the anti-symmetric matrix kernel

$$\tilde{K}(x,y) := \begin{pmatrix} e^{f(x)} & 0 \\ 0 & e^{-f(x)} \end{pmatrix} K(x,y) \begin{pmatrix} e^{f(y)} & 0 \\ 0 & e^{-f(y)} \end{pmatrix} \tag{1.5.26}$$

for a dx-measurable function f. Then it is not hard to check that $\mathrm{pf}_{i<j} K^{(n)}(x_i, x_j) = \mathrm{pf}_{i<j} \tilde{K}^{(n)}(x_i, x_j)$ and so $\mathrm{pf}(J + \lambda K)_{L^2(Y)} = \mathrm{pf}(J + \lambda \tilde{K})_{L^2(Y)}$ whenever both are defined. Importantly, we can use this to define $\mathrm{pf}(J + \lambda K)_{L^2(Y)}$ even if K is not trace-class provided we find an appropriate f which makes $\tilde{K}$ trace-class.

[3]For more on point processes in general and determinantal ones in particular, see e.g. [Joh06] or [BOO00]

Owing to identity (1.5.23), we have the following relation between Fredholm pfaffians with 2×2 matrix kernels K and block Fredholm determinants with related kernel $J^{-1}K$:

$$\mathrm{pf}(J + \lambda K)^2_{L^2(Y)} = \det(\mathbb{1} + \lambda J^{-1} K)_{L^2(Y)} \tag{1.5.27}$$

The Fredholm determinant on the right hand side is defined in (1.5.21).

Finally, for any two bounded operators $A : L^2(Y) \to L^2(Z), B : L^2(Z) \to L^2(Y)$ we have

$$\mathrm{pf}(J + \lambda JAB)_{L^2(Z)} = \mathrm{pf}(J + \lambda JBA)_{L^2(Y)}, \tag{1.5.28}$$

whenever both sides are defined. We note that oftentimes AB is infinite dimensional, while BA is finite dimensional. This fact is immediately implied by the corresponding Fredholm determinant identity $\det(\mathbb{1} + \lambda AB)_{L^2(Z)} = \det(\mathbb{1} + \lambda BA)_{L^2(Y)}$ and (1.5.27).

For more on pfaffians, the reader is referred to the the Appendix of [OQR17] for the analytic side and to [Ste90] for the algebraic and combinatorial ones.

Chapter 2

Last Passage Percolation models

In 1957 [BH57] Broadbent and Hammersley gave a mathematical formulation of percolation theory, analyzing the influence of the random properties of a medium on the percolation of a fluid passing through it. Their work led to the study of *first passage percolation* problems. First passage percolation is now one of the most classical areas in probability theory, but it was introduced in 1965 by Morgan and Welsh, who considered a two-dimensional Poisson growth process. The following year Hammersley [Ham66] generalized their setting, considering a generic distribution function.

First passage percolation theory considers the following problem. Let $\mathcal{G}$ be a connected graph. Let $\{t(e)\}_{e\in\mathcal{G}}$ be non-negative independent, identically distributed random variables, placed at each nearest-neighbour edge e of the graph $\mathcal{G}$. The random variable $t(e)$ is called passage time of the edge e and is interpreted as the time or the cost needed to traverse the edge e. Since each edge in first passage percolation has an independent weight, we can write the total time of a path γ on $\mathcal{G}$ as the summation of weights of each edge in the path,

$$T(\gamma) = \sum_{e\in\gamma} t(e). \tag{2.0.1}$$

Given two vertices $x, y \in \mathcal{G}$, the first passage time is defined as the shortest time of travel between x and y,

$$T(x,y) = \min_{\substack{\gamma\in\mathcal{G} \\ \gamma: x\to y}} T(\gamma). \tag{2.0.2}$$

It was conjectured that the fluctuation exponent χ (which gives the order of the length fluctuations of the first passage time around its mean) and the wandering exponent ξ (which quantifies the magnitude of the maximal deviation of the minimizing path between two vertices from the straight line connecting the vertices) satisfy the KPZ relation $2\xi = \chi + 1$. This result was rigorously proved by Chatterjee [Cha13] for a general version of this model on $\mathbb{Z}^d$ under assumptions on the weights and the existence of the exponents.

It was Rost [Ros81] to connect for the first time a corner growth model to a simple exclusion jump process on $\mathbb{Z}$ studying its limit density profile. Even if he did not use this formulation to prove his results, he was the first one to give a geometric interpretation of the profile as the asymptotic shape of a subset of $\mathbb{R}^2_+$. This opened the way to an extensive study of *last passage percolation* problems.

Last passage percolation is the counterpart of the first passage percolation, where we take the max instead of the min and it is defined as follows. Let $\{\omega_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ be non-negative, independent random variables. Let $\mathcal{L}, \mathcal{E}$ be disjoint subsets of $\mathbb{Z}^2$. An *up-right path* $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ on $\mathbb{Z}^2$ between $\mathcal{L}$ and $\mathcal{E}$ is a sequence of points in $\mathbb{Z}^2$ with

$\pi(k+1) - \pi(k) \in \{(0,1),(1,0)\}$ and $\pi(0) \in \mathcal{L}$ and $\pi(n) \in \mathcal{E}$. The number of points n is called the length $\ell(\pi)$ of π. We define the last passage time from $\mathcal{L}$ to $\mathcal{E}$ as

$$L_{\mathcal{L}\to\mathcal{E}} = \max_{\substack{\pi: A\to E\\ A\in\mathcal{L}, E\in\mathcal{E}}} \sum_{1\le k\le \ell(\pi)} \omega_{\pi(k)}. \tag{2.0.3}$$

We denote by $\pi^{\max}_{\mathcal{L}\to\mathcal{E}}$ any maximizer of the last passage time $L_{\mathcal{L}\to\mathcal{E}}$. For continuous random variables, the maximizer is a.s. unique.

Here we presented the most general case, where $\mathcal{L}$ and $\mathcal{E}$ are two sets of points. If $\mathcal{L}$ and $\mathcal{E}$ are both lines, we speak of "line-to-line" problem; if $\mathcal{L}$ and $\mathcal{E}$ are points, we speak of "point-to-point" problem; if $\mathcal{L}$ is line and $\mathcal{E}$ a point (or viceversa), we speak of "line-to-point" (or point-to-line) problem (see Figure 2.1). We denote with $L_{(k,\ell)\to(m,n)}$ the point-to-point last passage time between (k,ℓ) and (m,n), but for the special case $(k,\ell) = (1,1)$, we use the notation $L_{(m,n)}$.

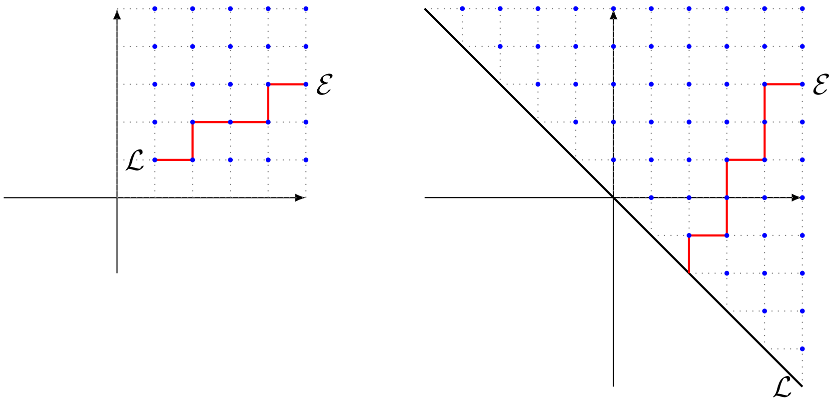

Figure 2.1: An example of point-to-point and line-to-point last passage percolation.

2.1 LPP as a queuing model

Consider a series of m single-server queues and n customers visiting the servers in order and then leaving the system. The queues follow the first-in-first-out (FIFO) discipline: at each server one customer at a time is served in the order of arrival and, if there are no customers waiting to be served, the server rests.

Let $X(n,k)$ be the service time of customer n at queue k and let $G(n,k)$ be the time at which customer n finishes service at queue k. Before customer n starts the service at queue k, customer n must complete the service at queue $k-1$ and customer $n-1$ must complete service at queue k; thus, the time at which customer n is served at queue k is $\max\{G(n,k-1), G(n-1,k)\}$ and

$$G(n,k) = X(n,k) + \max\{G(n,k-1), G(n-1,k)\}. \tag{2.1.1}$$

If we start from the situation where all the customers are waiting at queue 0, and at time 0 customer 0 moves to queue 1, then, using induction on (2.1.1), we have that $G(n,k) = L_{(n,k)}$. Indeed, a directed path to (n,k) is the union of a directed path to $(n,k-1)$ or to $(n-1,k)$. This give a correspondence between queues in tandem and last passage percolation, first observed by [Mut79].

If $X(n,k)$ are i. i. d. exponential random variables, then the queueing system evolves as a Markov process. This is known in queueing theory as $M/M/1$ queues, where the two M's describes the Markov (or memoryless) property of the distribution of durations between each arrival to the queue and of the distribution of service times for jobs, and 1 is the number of servers at the node. In this case, there is a correspondence with a particle exclusion process, the *totally asymmetric simple exclusion process* (TASEP) in continuous time. In TASEP, particles on $\mathbb{Z}$ try to jump to the neighbouring right site after an exponential waiting time of rate 1; a jump succeeds only if the right site is empty. If we label the particles from right to left and the holes from left to right, particles play the role of servers, the holes between particle k and $k-1$ the role of customers in the queue of server k, and $G(n,k)$ is the time at which the particle n occupies the hole k.

2.2 LPP as an interacting particle system: TASEP

Here we explain the connection between LPP and TASEP, mentioned in Section 1.3.3. TASEP is an interacting particle system on $\mathbb{Z}$ with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. For a configuration $\eta \in \Omega$, $\eta = \{\eta_j, j \in \mathbb{Z}\}$, η_j is the occupation variable at site j, and $\eta_j = 1$ if and only if j is occupied by a particle. TASEP is a Markov process with generator L given by [Lig99]

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1})\big(f(\eta^{j,j+1}) - f(\eta)\big), \tag{2.2.1}$$

where f are local functions (depending only on finitely many sites) and $\eta^{j,j+1}$ denotes the configuration η with the occupations at sites j and $j+1$ interchanged,

$$\eta^{j,j+1}(z) = \begin{cases} \eta(z) & \text{if } z \neq j, j+1, \\ \eta(j+1) & \text{if } z = j, \\ \eta(j) & \text{if } z = j+1. \end{cases} \tag{2.2.2}$$

TASEP can be also described as a growth process by introducing the height function $h(j,t)$ as

$$h(j,t) = \begin{cases} 2J(t) + \sum_{i=1}^{j}(1-2\eta_i(t)) & \text{for } j \geq 1, \\ 2J(t) & \text{for } j = 0, \\ 2J(t) - \sum_{i=j+1}^{0}(1-2\eta_i(t)) & \text{for } j \leq -1, \end{cases} \tag{2.2.3}$$

for $j \in \mathbb{Z}$, $t \geq 0$, where $J(t)$ is the current on the bond 0-1, the number of jumps from site 0 to site 1 during the time-span $[0,t]$.

Denote with $x_k(t)$ the position of the k-th particle at time t, with particles labeled from right to left (if there is a left- or rightmost particles, k runs over an interval of $\mathbb{Z}$). Consider TASEP in continuous time with initial configuration $\{x_k(0), k \in \mathbb{Z}\}$. The j-th particle attempts to jump after an exponential waiting time of parameter μ_j. Now, consider a line-to-point last passage percolation from

$$\mathcal{L} = \{(k + x_k(0), k), k \in \mathbb{Z}\} \tag{2.2.4}$$

to (m,n), and choose weights

$$\omega_{i,j} \sim \exp(\mu_j). \tag{2.2.5}$$

Then, the following relation between TASEP and LPP holds:

$$\mathbb{P}(L_{\mathcal{L}\to(m,n)} \leq t) = \mathbb{P}(x_n(t) \geq m-n). \tag{2.2.6}$$

This correspondence can be expressed also with the height function, as follows.

$$\mathbb{P}(L_{\mathcal{L}\to(m,n)} \leq t) = \mathbb{P}(h(m-n,t) \geq m+n). \tag{2.2.7}$$

Throughout this work, we consider always the case $\mu_j = 1$.

2.3 Point-to-point LPP

Now we consider point-to-point last passage percolation with i. i. d. weights. As we explained in the previous chapter, it is one of the possible description of growth models, like the corner growth model, introduced in Section 1.3.1, or the polynuclear growth model in the droplet geometry, introduced in Section 1.3.4. It is also related to the interacting particle system $\{\eta_x\}_{x\in\mathbb{Z}}$ on $\mathbb{Z}$ called TASEP (see Sections 1.3.3 and 2.2) starting from step initial condition, when all the particles are on the left of the origin at time 0, i.e. $\eta_x(0) = 1$ for $x \leq 0$ and $\eta_x(0) = 0$ for $x > 0$; in terms of height function, this translate to $h(x, 0) = |x|$ (see Figure 2.2).

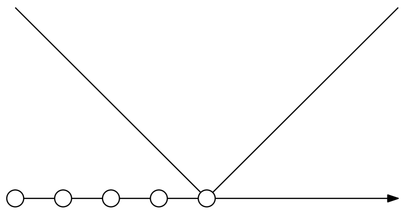

Figure 2.2: Height function for TASEP with step initial condition.

We are interested in understanding the deterministic limit on large scales for this model. Thus, we analyze the behaviour of the random variables $L_{(m,n)}$ for large values of m and n. The first result is a law of large number.

Theorem 2.3.1 (Theorem 2.1 in [Sep09]). *Consider the point-to-point LPP model with i. i. d. weights $\{\omega_{i,j}, i, j \geq 1\}$. Then, there exists a deterministic function $\Psi : (0,\infty)^2 \to [0,\infty]$ such that, for all $(x, y) \in (0,\infty)^2$,*

$$\Psi(x, y) = \lim_{N\to\infty} \frac{1}{N} L_{(\lfloor Nx\rfloor,\lfloor Ny\rfloor)} \quad a.s. \tag{2.3.1}$$

Either $\Psi = \infty$ or $\Psi < \infty$ on all $(0,\infty)^2$. In the latter case, Ψ is continuous, superadditive, i.e. for $(x_1, y_1), (x_2, y_2) \in (0,\infty)^2$,

$$\Psi(x_1, y_1) + \Psi(x_2, y_2) \leq \Psi(x_1 + x_2, y_1 + y_2), \tag{2.3.2}$$

concave, i.e. for $s \in (0, 1)$

$$s\Psi(x_1, y_1) + (1-s)\Psi(x_2, y_2) \leq \Psi(s(x_1, y_1) + (1-s)(x_2, y_2)), \tag{2.3.3}$$

homogeneous, i.e. for $c > 0$, $\Psi(cx, cy) = c\Psi(x, y)$, and symmetric, i.e. $\Psi(x, y) = \Psi(y, x)$. Moreover, Ψ in non-decreasing in both arguments and $\Psi(x + h, y) \geq \Psi(x, y) + h\mathbb{E}[\omega_{1,1}]$.

Using the subadditivity of the last passage time

$$L_{(nx,ny)} \geq L_{(mx,my)} + L_{(mx,my)\to(nx,ny)}, \quad m < n, \tag{2.3.4}$$

the existence of the limit shape is ensured by Liggett's subadditive ergodic theorem [Lig8511] (in a modified version).

This theorem says that asymptotically $L_{(\lfloor nx\rfloor,\lfloor ny\rfloor)}$ grows linearly in n. However, we are interested in computing the limit shape explicitly. This turns out to be possible only for special cases, when the weights ω are geometric or exponential random variables.

In the former case, given a parameter $p \in (0,1)$, the weights have probability distribution given by

$$\mathbb{P}(\omega_{1,1} = k) = p(1-p)^k, k \in \mathbb{Z}_+. \tag{2.3.5}$$

The law of large number was proved by Johansson [Joh00a].

Theorem 2.3.2. *Consider the point-to-point last passage percolation L^{geom} with weights geometrically distributed of parameter p. Then, for $(x,y) \in (0,\infty)^2$,*

$$\Psi(x,y) = p^{-1}\left((1-p)x + (1-p)y + \sqrt{(1-p)xy}\right). \tag{2.3.6}$$

This says that the boundary curve of the limit shape is an arc of a circonference tangent to the x- and y-axis at the points $(1-p,0)$ and $(0,1-p)$. The limit for the geometric case satisfies

$$\Psi(x,y) = \mu(x+y) + 2\sqrt{\sigma^2 xy}, \tag{2.3.7}$$

where $\mu = \mathbb{E}[\omega_{1,1}]$ and $\sigma^2 = \text{Var}[\omega_{1,1}]$ are the mean and the variance of the weights distribution. Johansson considered a slightly different version of the model, defining the random variable

$$L^\star_{(m,n)} = \max_{\pi:(1,1)\to(m,n)} \sum_{(i,j)\in\pi} \omega^\star_{i,j}, \tag{2.3.8}$$

where $\omega^\star_{i,j} = \omega_{i,j} + 1$ so that $\mathbb{P}(\omega^\star_{i,j} = k) = p(1-p)^{k-1}$, $k \geq 1$. It holds

$$L^\star_{(m,n)} = L^{\text{geom}}_{(m,n)} + m + n - 1, \tag{2.3.9}$$

since all paths have the same length. Using this random variable, he could define, for each $t \geq 0$, a random subset of the first quadrant by

$$A(t) = \{(m,n) \in \mathbb{Z}^2_+ : L^\star_{(m,n)} \leq t\} + [-1,0]^2. \tag{2.3.10}$$

Then, he proved that $\frac{A(t)}{t}$ has an asymptotic shape A_0 as $t \to \infty$, in the sense that, for any $\epsilon > 0$,

$$(1-\epsilon)A_0 \subseteq \frac{1}{t}A(t) \subseteq (1+\epsilon)A_0 \tag{2.3.11}$$

for all sufficiently large t.

For the latter case, Rost [Ros81] proved an analogous statement.

Theorem 2.3.3. *Consider the point-to-point last passage percolation $L^{\exp}$ with weights exponentially distributed of parameter 1. Then, for $(x,y) \in (0,\infty)^2$,*

$$\Psi(x,y) = \left(\sqrt{x} + \sqrt{y}\right)^2. \tag{2.3.12}$$

In [Joh00a] Johansson proved also a result about the fluctuations of $A(t)$ around its asymptotic shape A_0, i.e. the fluctuations of $L^{\text{geom}}_{(\lfloor nx\rfloor,\lfloor ny\rfloor)}$ around $n\Psi(x,y)$. Fix a parameter $0 < p < 1$ and $q = 1-p$. Theorem 2.3.2 gives an explicit formula for the limit shape of $L^{\text{geom}}_{(\lfloor nw\rfloor,n)}$, given by

$$\Psi(w,1) = p^{-1}(qw + q + 2\sqrt{qw}). \tag{2.3.13}$$

Then, set

$$\sigma = p^{-1}q^{1/6}w^{-1/6}(\sqrt{w} + \sqrt{q})^{2/3}(\sqrt{wq} + 1)^{2/3}. \tag{2.3.14}$$

Theorem 2.3.4 (Theorem 1.2 of [Joh00a]). *For $w \geq 1$ and $s \in \mathbb{R}$*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L^{\text{geom}}_{(\lfloor nw \rfloor, n)} - n\Psi(w,1)}{\sigma n^{1/3}} \leq s\right) = F_{\text{GUE}}(s). \tag{2.3.15}$$

F_{GUE} is the GUE Tracy-Widom distribution [TW94], the distribution function of the appropriately scaled largest eigenvalue of an $n \times n$ random matrix from the Gaussian Unitary Ensemble (GUE) in the limit $n \to \infty$ (see Section 1.4.1).

Johansson also pointed out the connection of $A(t)$ with a particle system $\{x_k(t), k \in \mathbb{Z}\}$, the totally asymmetric simple exclusion process with step initial condition, i.e. with starting configuration $x_k(0) = \mathbb{1}_{(-\infty,0]}(0)$. Here $x_k(t) = 1$ means that there is a particle at k, as $x_k(t) = 0$ means that there is no particle at k. The stochastic growth of $A(t)$ corresponds to the following stochastic dynamics of the particle system. At time t each particle independently moves to the right-neighbouring site with probability p, provided there is no particle, otherwise it does not move. In this particle model $L^\star_{(m,n)} = k$ means that the particle initially at position $-(n-1)$ has moved m steps at time k. This simple exclusion process is exactly the one considered by Rost [Ros81]. So, taking the limit $p \to 1$, it is possible to obtain the fluctuations of the last passage time with exponential weights[1].

Theorem 2.3.5 (Theorem 1.6 of [Joh00a]). *Consider the point-to-point LPP with weights exponentially distributed of parameter* 1. *Assume $a_n = \mathcal{O}(n^{1/3})$ as $n \to \infty$ and choose d_n so that $d_n - (1 + 1/\sqrt{w})a_n = o(n^{1/3})$ as $n \to \infty$. Then, for each $w \geq 1$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{L^{\text{exp}}_{(wn+a_n, n)} - (1+\sqrt{w})^2 n}{w^{-1/6}(1+\sqrt{w})^{4/3} n^{1/3}} \leq s\right) = F_{\text{GUE}}(s). \tag{2.3.16}$$

From now on, we will consider only the last passage percolation with exponential weights and we will omit the apex "exp".

Given the law of large numbers for the LPP from the origin to (wn, n), as in Theorem 2.3.3, we can define a rescaled last passage time under the KPZ scaling, considering ending points at distance of order $\mathcal{O}(n^{2/3})$ from the characteristic line, given by the line with direction $(w, 1)$,

$$L^{\text{resc}}_n(u) = \frac{L_{(wn+\beta_1 u n^{2/3}, n)} - n\left(1 + \sqrt{w + \beta_1 u n^{-1/3}}\right)^2}{\beta_2 n^{1/3}}. \tag{2.3.17}$$

Thus, we have a process $u \mapsto L^{\text{resc}}_n(u)$. With the choice $\beta_1 = 2(1+\sqrt{w})^{2/3} w^{2/3}$, we obtain the convergence in distribution to a GUE Tracy-Widom random variable, as in Theorem 2.3.5. If we choose $\beta_2 = (1+\sqrt{w})^{4/3} w^{-1/6}$, then,

$$\lim_{n\to\infty} L^{\text{resc}}_n(u) = \mathcal{A}_2(u) - u^2, \tag{2.3.18}$$

in the sense of finite-dimensional distribution. The process $\mathcal{A}_2$ is the *Airy$_2$ process* (see Section 2.3.1).

Last passage percolation is linked to the ensemble of complex Gaussian sample covariance matrices [BBP06]. Given $\pi_1, \dots, \pi_n$ positive numbers, if the weights $\omega_{i,j}$ are

[1] If X_ℓ is a random variable geometrically distributed with parameter $1 - 1/\ell$, then, as $\ell \to \infty$, X_ℓ/ℓ converges in distribution to an exponential random variable with parameter 1.

exponential random variables of mean $\frac{1}{\pi_i m}$, then $L_{(m,n)}$ has the same distribution of the complex Gaussian sample covariance matrix of m sample vectors of n variables, where the π_i's are the inverse of the eigenvalues of the covariance matrix of the Gaussian sample vectors. Using this connection and estimates on the kernel of the Fredholm determinant of the distribution of the matrix ensemble (Proposition 3.1 of [BBP06]), one can prove that the upper tail of the distribution of the rescaled LPP has an exponential decay, more precisely, there exist constants s_0, n_0 and C, c uniform in n such that

$$\mathbb{P}(L_n^{\text{resc}} \geq s) \leq Ce^{-cs}, \tag{2.3.19}$$

for all $n \geq n_0$ and $s \geq s_0$. An estimate on the lower tail was found in [BFP14]; in particular, there exist constants s_0, n_0 and C, c uniform in n such that

$$\mathbb{P}(L_n^{\text{resc}} < s) \leq Ce^{-c|s|^{3/2}}, \tag{2.3.20}$$

for all $n \geq n_0$ and $s \leq -s_0$.

Beyond the one-point distribution, it is object of wide interest the correlation in time,

$$\text{Cov}(L_{(n,n)}, L_{(\tau n,\tau n)}) = \mathbb{E}[L_{(n,n)} L_{(\tau n,\tau n)}] - \mathbb{E}[L_{(n,n)}]\mathbb{E}[L_{(\tau n,\tau n)}], \tag{2.3.21}$$

for $0 \leq \tau \leq 1$, but less results are available on this. Johansson [Joh16] obtained the long time asymptotics for the joint distribution of point-to-point semi-discrete directed polymers. In [Tak12, Tak13, TS10], the authors measured the temporal correlation function in a turbulent liquid crystal (here expressed in terms of height function) and determined that, for large n and $\tau \ll 1$,

$$\lim_{n\to\infty} \text{Cov}(L_{(n,n)}, L_{(\tau n,\tau n)}) = \Theta(\tau^{2/3}). \tag{2.3.22}$$

A conjecture on the behaviour of the correlation for large time has also been made by Ferrari and Spohn [FS16] on the basis of heuristic arguments and numerical simulations. They obtained the power laws of the covariance for short and large time differences: for $\tau \to 0$, consistently with Takeuchi's experiments, $\lim_{n\to\infty} \text{Cov}(L_{(n,n)}, L_{(\tau n,\tau n)}) = \Theta(\tau^{2/3})$, as for $\tau \to 1$, they observed at first approximation the variance of the stationary process with a correction of order $\mathcal{O}(1-\tau)$. The strategy is the following. They decompose the LPP from the origin to (n, n) in the sum of two independent LPPs, the first ending in $I(u) = (\tau n, \tau n) + u(2n)^{2/3}u(1, -1)$, the second starting from $I(u)$ and ending in (n, n). We know that, as $n \to \infty$,

$$\begin{aligned}
\frac{L_{(0,0)\to(\tau n,\tau n)} - 4\tau n}{2^{4/3}n^{1/3}} &\approx \tau^{1/3}\mathcal{A}_2(0), \\
\frac{L_{(0,0)\to I(u)} - 4\tau n}{2^{4/3}n^{1/3}} &\approx \tau^{1/3}[\mathcal{A}_2(u) - u^2], \\
\frac{L_{I(u)\to(n,n)} - 4(1-\tau)n}{2^{4/3}n^{1/3}} &\approx (1-\tau)^{1/3}\left[\widetilde{\mathcal{A}}_2\left(u\left(\tfrac{\tau}{1-\tau}\right)^{2/3}\right) - \left(u\left(\tfrac{\tau}{1-\tau}\right)^{2/3}\right)^2\right],
\end{aligned} \tag{2.3.23}$$

where $\mathcal{A}_2$ and $\widetilde{\mathcal{A}}_2$ are independent Airy$_2$ processes. Then, using the relation

$$L_{(0,0)\to(n,n)} = \max_{u\in\mathbb{R}}\{L_{(0,0)\to I(u)} + L_{I(u)\to(n,n)}\}, \tag{2.3.24}$$

one has

$$\lim_{n\to\infty} L_n^{\text{resc}}(0) = \max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u) - u^2 - \left(\tfrac{1-\tau}{\tau}\right)^{1/3}\widetilde{\mathcal{A}}_2\left(u\left(\tfrac{\tau}{1-\tau}\right)^{2/3}\right)\right\}. \tag{2.3.25}$$

To give an estimate on the behaviour of the covariance, they use the fact that the Airy$_2$ process converges to Brownian motion, in the sense that, as $\tau \to 0$,

$$\left(\tfrac{1-\tau}{\tau}\right)^{1/3}\left[\mathcal{A}_{\rm stat}\left(\left(\tfrac{\tau}{1-\tau}\right)^{2/3}u\right)-\mathcal{A}_{\rm stat}(0)\right]\approx\sqrt{2}B(u), \tag{2.3.26}$$

where B is a standard Brownian motion. Therefore, as $\tau \to 0$, the covariance of the limit processes becomes

$$\begin{aligned}&\tau^{2/3}\,{\rm Cov}\left(\mathcal{A}_2(0),\max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u)-u^2+\sqrt{2}B(u)\right\}+\left(\tfrac{1-\tau}{\tau}\right)^{1/3}\widetilde{\mathcal{A}}_2(0)\right)\\ =&\tau^{2/3}\,{\rm Cov}\left(\mathcal{A}_2(0),\max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u)-u^2+\sqrt{2}B(u)\right\}\right),\end{aligned} \tag{2.3.27}$$

since $\mathcal{A}_2$ and $\widetilde{\mathcal{A}}_2$ are independent. Finally, they rewrite the covariance as the expectation of the conditional expectation with respect to the Brownian motion B and they conclude the $\Theta(\tau^{2/3})$ behaviour, since for typical realizations of B, the maximum is attained for $u=\Theta(1)$.

When $\tau \to 1$, by the symmetry of the point-to-point LPP, the maximum is attained for $u=\Theta((1-\tau)^{2/3})$. Therefore, they set $v=\left(\frac{\tau}{1-\tau}\right)^{2/3}u$ so that

$$\lim_{n\to\infty}L^{\rm resc}_{(0,0)\to(n,n)}=(1-\tau)^{1/3}\max_{v\in\mathbb{R}}\left\{\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\mathcal{A}_2\left(v\left(\tfrac{1-\tau}{\tau}\right)^{2/3}\right)-v^2\tfrac{1-\tau}{\tau}+\widetilde{\mathcal{A}}_2(v)-v^2\right\}. \tag{2.3.28}$$

Using the elementary identity

$${\rm Cov}\,(X_1,X_2)=\tfrac{1}{2}\,{\rm Var}\,(X_1)+\tfrac{1}{2}\,{\rm Var}\,(X_2)-\tfrac{1}{2}\,{\rm Var}(X_2-X_1), \tag{2.3.29}$$

for any two random variables X_1, X_2, they reduce the problem to an estimate on the variance of the difference of the limit processes

$$\begin{aligned}&\lim_{n\to\infty}(L^{\rm resc}_{(0,0)\to(n,n)}-L^{\rm resc}_{(0,0)\to(\tau n,\tau n)})\\ =&(1-\tau)^{1/3}\max_{v\in\mathbb{R}}\left\{\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\left[\mathcal{A}_2\left(v\left(\tfrac{1-\tau}{\tau}\right)^{2/3}\right)-\mathcal{A}_2(0)+\widetilde{\mathcal{A}}_2(v)-v^2\tau^{-1}\right]\right\}.\end{aligned} \tag{2.3.30}$$

The first term in (2.3.30), given by the increment of the Airy$_2$ process, converges to $\sqrt{2}B(v)$ as $\tau\to 1$ and, since the maximum is reached for $v=\Theta(1)$, (2.3.30) will give the variance of the process $\max_{v\in\mathbb{R}}\left\{\mathcal{A}_2(v)-v^2+\sqrt{2}B(v)\right\}$ (which is the stationary process) plus a correction of order $\mathcal{O}(1-\tau)$.

To turn this reasoning into rigorous statements, one should have control on the convergence of the Airy process to the Brownian motion and on the convergence of the covariance of the last passage times. This argument has been formalized and generalized in [FO19], where it is proved the convergence of the covariance of the LPP with ending points out the diagonal to the covariance of the respective limit processes (Theorem 4.2.2) and precise bounds are made on the error term for $\tau \to 1$.

Theorem 2.3.6 (see Theorem 4.2.5)**.** *Consider the last percolation from the origin to* $E_\tau=(\tau n,\tau n)+(2n)^{2/3}w_\tau(1,-1)$, *for* $0<\tau\leq 1$. *Let us scale* $w_1=\tilde{w}_1(1-\tau)^{2/3}$ *and* $w_\tau=\tilde{w}_\tau(1-\tau)^{2/3}$. *Then, as* $\tau\to 1$ *we have*

$$\begin{aligned}\lim_{n\to\infty}{\rm Cov}\left(L^{\rm resc}_{(0,0)\to E_1},L^{\rm resc}_{(0,0)\to E_\tau}\right)=&\frac{1}{2}\,{\rm Var}\,(\xi(w_1))+\frac{\tau^{2/3}}{2}\,{\rm Var}\left(\xi(w_\tau\tau^{-2/3})\right)\\ &-\frac{(1-\tau)^{2/3}}{2}\,{\rm Var}\,(\xi_{{\rm stat},\tilde{w}_1-\tilde{w}_\tau})+\mathcal{O}(1-\tau)^{1-\delta},\end{aligned} \tag{2.3.31}$$

for any $\delta > 0$. Here $\xi(w) + w^2$ is distributed according to a GUE Tracy-Widom law and $\xi_{\text{stat},w}$ has distribution given by

$$\mathbb{P}(\xi_{\text{stat},w} \leq s) = \mathbb{P}\Big(\max_{v \in \mathbb{R}} \{\sqrt{2}B(v) + \mathcal{A}_2(v) - (v-w)^2\} \leq s \Big). \tag{2.3.32}$$

2.3.1 The Airy$_2$ process

The Airy$_2$ process was introduced by Prähofer and Spohn [PS02b] as the limit of the top layer in the polynuclear growth model (see Section 1.3.4). In the same work, they also proved that the process is almost surely continuous, stationary and invariant under time-reversal. Its one-point distribution is the GUE Tracy-Widom distribution and has super-exponential decay,

$$\begin{aligned} \mathbb{P}(\mathcal{A}_2(u) > s) &\approx e^{-\frac{4}{3}s^{3/2}} \quad \text{as } s \to \infty, \\ \mathbb{P}(\mathcal{A}_2(u) < s) &\approx e^{-|s|^3/12} \quad \text{as } s \to -\infty. \end{aligned} \tag{2.3.33}$$

The correlations of the process $\mathbb{E}[\mathcal{A}_2(u)\mathcal{A}_2(0)] - \mathbb{E}[\mathcal{A}_2(u)]\mathbb{E}[\mathcal{A}_2(0)]$ decay as u^{-2} and, in particular,

$$\mathbb{E}[\mathcal{A}_2(u)\mathcal{A}_2(0)] - \mathbb{E}[\mathcal{A}_2(u)]\mathbb{E}[\mathcal{A}_2(0)] = \begin{cases} c(\infty) - u + \mathcal{O}(u^2) & \text{for } |u| \text{ small}, \\ u^{-2} + \mathcal{O}(u^{-4}) & \text{for } |u| \text{ large}, \end{cases} \tag{2.3.34}$$

with $c(\infty) = \mathrm{Var}(\mathcal{A}_2(0)) = 1.6264\ldots$.

The Airy$_2$ process is defined by its finite-dimensional distributions [Joh05]. It is the process with n-point joint distribution at $u_1 < u_2 < \cdots < u_n$ given by the Fredholm determinant

$$\mathbb{P}\left(\bigcap_{k=1}^{n} \{\mathcal{A}_2(u_k) \leq s_k\} \right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_2} \chi_s)_{L^2(\{u_1,\ldots,u_n\}\times\mathbb{R})}, \tag{2.3.35}$$

where $\chi_s(u_k, x) = \mathbb{1}_{x > u_k}$. The correlation kernel $K_{\mathcal{A}_2}$ is given by

$$K_{\mathcal{A}_2}(s_1, u_1; s_2, u_2) = -V_{u_1,u_2}(s_1, s_2)\mathbb{1}_{u_1 < u_2} + K_{u_1,u_2}(s_1, s_2), \tag{2.3.36}$$

with

$$\begin{aligned} V_{u_1,u_2}(s_1,s_2) &= \frac{e^{\frac{2}{3}u_2^3 + u_2 s_2}}{e^{\frac{2}{3}u_1^3 + u_1 s_1}} \int_{\mathbb{R}} dx e^{-x(u_1-u_2)} \mathrm{Ai}(u_1^2 + s_1 + x)\mathrm{Ai}(u_2^2 + s_2 + x), \\ K_{u_1,u_2}(s_1,s_2) &= \begin{cases} \int_0^\infty dx\, e^{-x(u_1-u_2)} \mathrm{Ai}(s_1+x)\mathrm{Ai}(s_2+x) & \text{for } u_1 \geq u_2, \\ -\int_{-\infty}^0 dx\, e^{-x(u_1-u_2)} \mathrm{Ai}(s_1+x)\mathrm{Ai}(s_2+x) & \text{for } u_1 < u_2, \end{cases} \end{aligned} \tag{2.3.37}$$

Using the Fredholm determinant description of the Airy process, Hägg [Häg08] proved that the Airy$_2$ process behaves locally like a Brownian motion, in the sense of finite-dimensional distributions, i. e.

$$\lim_{\epsilon \to 0} \epsilon^{-1/2}(\mathcal{A}_2(\epsilon u) - \mathcal{A}_2(0)) \stackrel{\text{dist.}}{=} \sqrt{2}\mathcal{B}(u). \tag{2.3.38}$$

Corwin and Hammond [CH13] proved that its sample paths are locally absolutely continuous with respect to Brownian motion. Using the fact that the Airy line process, of which the top line corresponds to the Airy$_2$ process, can be seen as a limit of non-intersecting

Brownian bridges, they showed that the local Brownian behaviour of the Airy_2 process holds in a stronger functional sense: the convergence (2.3.38) holds in the sense of weak convergence of probability measures in the space of continuous functions. The same result was proved by Cator and Pimentel [CP15], but in a different setting. They used the fact that the Airy process is a limiting process for the Hammersley last-passage percolation model (see Section 2.3.2), and showed that local fluctuations can be controlled by equilibrium versions of this model, which are simply Poisson processes.

We said before that the last passage time and the largest eigenvalue of a random matrix in the Gaussian unitary ensemble have the same limit distribution, namely the GUE Tracy-Widom distribution. The analogy between point-to-point last passage percolation and random matrices extends to the process defined in (2.3.17) and the point process associated to the ordered eigenvalue of a GUE matrix.

Consider a $n \times n$ GUE matrix $H(t)$ evolving according to the stationary Ornstein–Uhlenbeck process

$$dH(t) = -\gamma H(t)dt + B(t), \tag{2.3.39}$$

where $\gamma > 0$ and $B(t)$ is a Brownian motion on the space of hermitian matrices. Then, the eigenvalues of $H(t)$ evolve according to a Dyson's Brownian, motion [Dys62]

$$d\lambda_j(t) = \left(-\gamma\lambda_j(t) + \frac{\beta}{2} \sum_{i \neq j} \frac{1}{\lambda_j(t) - \lambda_i(t)} \right) dt + db_j(t), \quad j = 1, \dots, n, \tag{2.3.40}$$

where $b_j(t)$ are independent standard Brownian motions. Dyson's Brownian motion describes the diffusion of n particles with positions $\lambda_j(t)$, $j = 1, \dots, n$, at time t on the real line in a harmonic potential. For $\beta = 2$ we can associate a (determinantal) point process to the eigenvalues $\lambda_j(t)$ of $H(t)$. Under the scaling

$$\lambda_j^n(u) = \sqrt{2\gamma} n^{1/6} \left(\lambda_j(u/(\gamma n^{1/3})) - \sqrt{2n/\gamma} \right), \tag{2.3.41}$$

as $n \to \infty$, the kernel converges to the Airy kernel (2.3.36). This means that the rescaled largest eigenvalue $\lambda_n^n(u)$ converges to the Airy_2 process,

$$\lim_{n \to \infty} \lambda_n^n(u) = \mathcal{A}_2(u), \tag{2.3.42}$$

in the sense of finite-dimensional distributions [Joh03].

2.3.2 Hammersley LPP

The Hammersley model [AD95] is a model of last passage percolation on $\mathbb{R}^2$ introduced by Aldous and Diaconis to investigate the problem of finding the length ℓ_n of longest increasing subsequence in a random n-permutation [Ham72]. By studying the hydrodynamic limit for the Hammersley's process, they showed that $n^{-1/2}\mathbb{E}[\ell_n] \to 2$, as $n \to \infty$, considerably simplifying the approach of [LS77, VK77], which relied on hard analysis of combinatorial asymptotics.

The Hammersley last passage percolation model is the continuous space analogue of the LPP on $\mathbb{Z}^2$ and is constructed from a two-dimensional homogeneous Poisson point process of intensity 1. The presentation of the model follows [AD95]. Consider n points (x_i, t_i) in the rectangle $[0, x] \times [0, t]$ with distinct coordinates. The set of points specifies a permutation σ: the point with i-th smallest t-coordinate has the $\sigma(i)$-th smallest x-coordinate. The length $\ell_n(\sigma)$ of the longest increasing subsequence of σ is the maximal length of a sequence (i_k) such that

$$x_{i_1} < x_{i_2} < \cdots < x_{i_{\ell_n}}, \quad t_{i_1} < t_{i_2} < \cdots < t_{i_{\ell_n}}. \tag{2.3.43}$$

This defines an up-right path π from $(0,0)$ to (x,t). If we take a Poisson process of rate 1 on $\mathbb{R}^2$ and, for $x, t \geq 0$, we define $L_{(x,t)}$ as the maximal number of points on π, then, the number of points $M(x,t)$ in the rectangle $[0,x] \times [0,t]$ has Poisson distribution and

$$L_{(x,t)} \stackrel{\text{dist}}{=} \ell_{M(x,t)}. \tag{2.3.44}$$

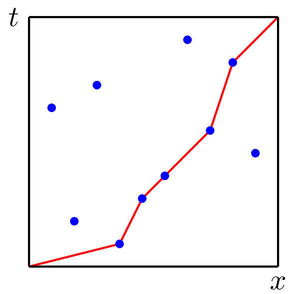

Figure 2.3: Hammersley's last passage percolation associated to the permutation $\sigma = (7\ 2\ 8\ 1\ 3\ 4\ 10\ 6\ 9\ 5)$. The longest increasing subsequence is $(1\ 3\ 4\ 6\ 9)$ with length $\ell(\sigma) = 5$.

In [BDJ99b], Baik, Deift and Johansson showed that the fluctuations of the length of a maximal path from (0,0) to (n,n) are of order n^χ with $\chi = 1/3$. In [Joh00c] Johansson proved that the transversal fluctuations of a maximizing path are of order n^ξ with $\xi = 2/3$. By transversal fluctuations we mean the typical deviations of a maximal path from the diagonal $\overline{(0,0)(n,n)}$. These scaling exponents are consistent with the conjectured relation between the fluctuation exponent χ and the wandering exponent ξ,

$$\chi = 2\xi - 1. \tag{2.3.45}$$

We present the heuristic argument of [Joh00c]. We know from [AD95] that the length of the maximal path from the origin to (x,y) is $\sim \sqrt{xy}$. If we consider the maximizing path from the origin to (N,N) that passes through $(N(t-\delta), N(t+\delta))$, where $0 < t < 1$ and δ is small, it will be shorter than the maximizer from the origin to (N,N) without the constraint by the amount

$$2N\sqrt{(t-\delta)(t+\delta)} + 2N\sqrt{(1-t+\delta)(1-t-\delta)} - 2N. \tag{2.3.46}$$

This should have the same order as the length fluctuations, which are $\mathcal{O}(N^\chi)$. This gives $\delta^2 = \mathcal{O}(N^{\chi-1})$ and, thus, $N^\xi \sim N\delta \sim N^{\frac{1}{2}(\chi+1)}$, leading to $2\xi = \chi + 1$.

2.4 Point-to-line LPP

A point-to-line last passage percolation is defined as in (2.0.3) with $\mathcal{L} = \{(i,j) \in \mathbb{Z}^2 \mid i+j = 0\}$. This is equivalent to the polynuclear growth model with flat geometry, a continuous version of the growth model $h(x,t)$ presented in Section 1.3.4 with weights distributed

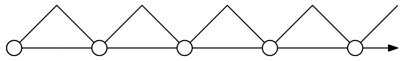

Figure 2.4: Height function for TASEP with flat initial condition.

according to a Poisson process of intensity $\rho = 2$ in $\mathbb{R} \times \mathbb{R}_+$ and starting from $h(x, 0) = 0$. In terms of particle systems, it represents the geometric interpretation of TASEP starting from flat initial condition, i.e. $\eta_x(0) = 1$ for x even and $\eta_x(0) = 0$ for x odd; the height function at time 0 is saw-tooth function between 0 and 1 (see Figure 2.4).
More generally, we can consider *k-periodic initial data* for TASEP, for which particles have positions

$$x_n(0) = -kn, \quad n \in \mathbb{Z}, k \in \{2, 3, \dots\}. \tag{2.4.1}$$

The density of particle is $\rho = 1/k$. In this case, we can still provide a last passage percolation picture by choosing the starting line $\mathcal{L}_\rho^{\text{flat}} = \{\left(\lfloor \frac{\rho-1}{\rho} n \rfloor, n\right), n \in \mathbb{Z}\}$. By universality, fluctuations of particle positions/limit shape should be governed by the GOE Tracy-Widom distribution, as proved in [FS05, Sas05] for the 2-periodic case and in [BFP07] for $k > 2$. More recently, we proved the same result for generic density $\rho \in (0, 1)$.

Theorem 2.4.1 (see Theorem 3.2.1)**.** *Let $\rho \in (0, 1)$ and consider the last passage percolation from $\mathcal{L}_\rho^{\text{flat}}$ to (n, n). Set $a_0 = 1/(\rho(1-\rho))$ and $a_1 = 1/(\rho(1-\rho))^{2/3}$. Then, for any $s \in \mathbb{R}$,*

$$\lim_{N \to \infty} \mathbb{P}\left(L_{\mathcal{L}_\rho^{\text{flat}} \to (n,n)} \leq a_0 n + a_1 s n^{1/3}\right) = F_{\text{GOE}}(2^{2/3} s), \tag{2.4.2}$$

where F_{GOE} is the GOE Tracy-Widom distribution function.

To prove this result, we used the variational formula [Joh05]

$$F_{\text{GOE}}(2^{2/3} s) = \mathbb{P}\Big(\max_{v \in \mathbb{R}} \{\mathcal{A}_2(v) - v^2\} \leq s\Big), \tag{2.4.3}$$

where $\mathcal{A}_2$ is the Airy$_2$ process [PS02b, Joh03] (see Section 2.3.1), and showed convergence of the rescaled last passage time to the variational problem. First, we proved convergence of a restricted LPP to the formula (2.4.3) with $|v| \leq M$ for a positive M. Then, we obtained bounds on the probability that the maximizing path is not localized in a region of order $MN^{2/3}$. The quadratic term provides localization for the position of the maximum of $\mathcal{A}_2(v) - v^2$: bounds can be found in [CH13, QR15].

As for the point-to-point LPP, we can define a process $u \mapsto L_{\mathcal{L}^{\text{flat}} \to E}(u)$ choosing the ending point $E = (n, n) + 2^{2/3} u(1, -1)$. The rescaled process

$$L_n^{\text{resc}}(u) = \frac{L_{\mathcal{L}^{\text{flat}} \to E} - 4n}{2^{4/3} n^{1/3}} \tag{2.4.4}$$

converges, as $n \to \infty$, to the *Airy$_1$ process* (see Section 2.4.1)

$$\lim_{n \to \infty} L_n^{\text{resc}}(u) = 2^{1/3} \mathcal{A}_1(2^{-2/3} u), \tag{2.4.5}$$

in the sense of finite-dimensional distribution. The distribution of the Airy$_1$ process is expressed in terms of Fredholm determinants. The result in (2.4.5) was proved in terms of particle positions for TASEP. In [BFP07] Borodin, Ferrari and Prähofer showed convergence of Fredholm determinants of the kernels for discrete time TASEP with k-periodic initial condition. Later, they proved the result for continuous time TASEP in the 2-periodic case [BFPS07].

Also for the point-to-line LPP, the tails of the distribution have exponential decay. An estimate on the lower tail can be immediately obtained from (2.3.20). Since $\mathbb{P}(L_{\mathcal{L}^{\text{flat}}\to(n,n)} \le s) \le \mathbb{P}(L_{(0,0)\to(n,n)})$, we have

$$\mathbb{P}(L_n^{\text{resc}}(0) \le s) \lesssim e^{-|s|^{3/2}}, \quad s < 0 \tag{2.4.6}$$

For the upper tail we have

$$\mathbb{P}(L_n^{\text{resc}}(0) \ge s) \lesssim e^{-s}, \quad s > 0. \tag{2.4.7}$$

In [FS16] heuristic arguments similar to the ones for the point-to-point case are used to study the time covariance for flat initial conditions: in this case, the absence of the quadratic term, that provided the localization of the maximizer, is responsible for a different behaviour of the covariance of $L_{(n,n)}$ and $L_{(\tau n,\tau n)}$ for small τ: as $\tau \to 0$,

$$\lim_{n\to\infty} \mathrm{Cov}\left(L_{(n,n)}, L_{(\tau n,\tau n)}\right) = \Theta(\tau^{4/3}). \tag{2.4.8}$$

Instead, for $\tau \to 1$, we showed a universal behaviour of the covariance, confirming Ferrari and Spohn's conjecture.

Theorem 2.4.2 (see Theorem 4.2.5)**.** *Consider the last percolation from $\mathcal{L}^{\text{flat}}$ to $E_\tau = (\tau n, \tau n) + (2n)^{2/3} w_\tau (1,-1)$, for $0 < \tau \le 1$. Let us scale $w_1 = \tilde{w}_1 (1-\tau)^{2/3}$ and $w_\tau = \tilde{w}_\tau (1-\tau)^{2/3}$. Then, as $\tau \to 1$ we have*

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Cov}\left(L^{\text{resc}}_{\mathcal{L}^{\text{flat}}\to(n,n)}, L^{\text{resc}}_{\mathcal{L}^{\text{flat}}\to(\tau n,\tau n)}\right) &= \frac{1}{2}\mathrm{Var}\left(\xi(w_1)\right) + \frac{\tau^{2/3}}{2}\mathrm{Var}\left(\xi(w_\tau \tau^{-2/3})\right) \\ &\quad - \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{\text{stat},\tilde{w}_1-\tilde{w}_\tau}\right) + \mathcal{O}(1-\tau)^{1-\delta},\end{aligned} \tag{2.4.9}$$

for any $\delta > 0$. Here $2^{2/3}\xi(w)$ is distributed according to a GOE Tracy-Widom law and $\xi_{\text{stat},w}$ is given in (2.3.32).

2.4.1 The Airy$_1$ process

The Airy$_1$ process was discovered by Sasamoto [Sas05] as limit process for TASEP with flat initial condition. It is the analogue of the Airy$_2$ process for flat growth and also this process is stationary and has sample paths with locally Brownian fluctuations, as proved in [QR13]. In [BFS08] it was proved that the Airy$_1$ process is also the limit process for the PNG model with flat initial conditions. It was conjectured in [Sas05] and proved in [FS05] that its one-point distribution is the GOE Tracy-Widom distribution. It was shown in [BBD08] that the lower tail decays as

$$\mathbb{P}(\mathcal{A}_1(0) < s) \approx e^{-|s|^3/24} \quad \text{as } s \to -\infty. \tag{2.4.10}$$

As shown in Figure 2.5, the correlation of the process decay much faster than the correlation for the Airy$_2$: in [BFP08] it was shown that $\mathrm{Cov}(\mathcal{A}_1(0), \mathcal{A}_1(u)) \approx c(\infty) - u + \mathcal{O}(u^2)$ for u small, where $c(\infty) = \mathrm{Var}(\mathcal{A}_1(0)) = 0.402\ldots$, as it has super-exponential decay for u large.

The Airy$_1$ process is defined in terms of its finite-dimensional distribution. It is the process with n-point distribution at $u_1 < u_2 < \cdots < u_n$ given by the Fredholm determinant

$$\mathbb{P}\left(\bigcap_{k=1}^{n} \{\mathcal{A}_1(u_k) \le s_k\}\right) = \det(\mathbb{1} - \chi_s K_{\mathcal{A}_1} \chi_s)_{L^2(\{u_1,\ldots,u_n\}\times\mathbb{R})}, \tag{2.4.11}$$

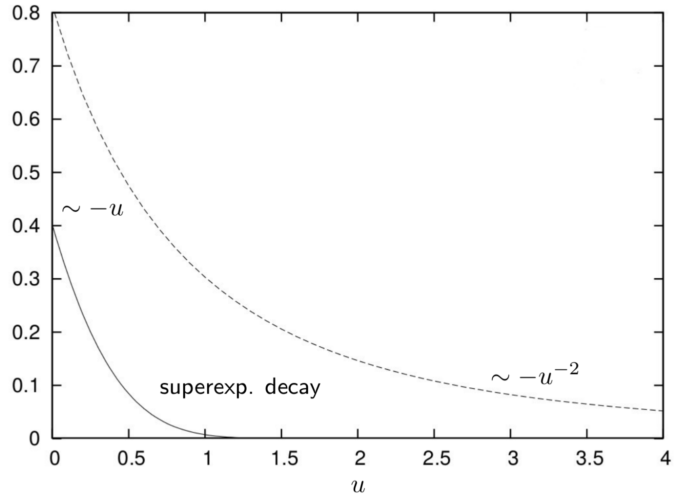

Figure 2.5: With dashed line $\mathrm{Cov}(\mathcal{A}_2(0), \mathcal{A}_2(u))$ and with solid line $\mathrm{Cov}(\mathcal{A}_1(0), \mathcal{A}_1(u))$ obtained in [BFP08] by numerical evaluation of Fredholm determinants.

where $\chi_s(u_k, x) = \mathbb{1}_{x>u_k}$. The correlation kernel $K_{\mathcal{A}_1}$ is

$$\begin{aligned} K_{\mathcal{A}_1}(s_1, u_1; s_2, u_2) = &- V_{u_1,u_2}(s_1, s_2)\mathbb{1}_{u_1<u_2} \\ &+ \mathrm{Ai}(s_1 + s_2 + (u_2 - u_1)^2)e^{(u_2-u_1)(s_1+s_2)+\frac{2}{3}(u_2-u_1)^3}, \end{aligned} \tag{2.4.12}$$

where $V_{u_1,u_2}(s_1, s_2)$ was defined in (2.3.37).

As for the Airy$_2$ process, it seems natural to expect that the Airy$_1$ process is also the limit of the largest eigenvalue in GOE matrix diffusion. This was conjectured in [BFPS07], but later refuted by [BFP08]: they compared with numerical evaluation the joint distribution functions for the Airy$_1$ processes (given in terms of Fredholm determinants of integral operators) and the correlation function for GOE matrix diffusion and showed that they differ in the limit of large matrices.

2.5 Stationary LPP

The stationary last passage percolation can be realized choosing the following setting: the starting line $\mathcal{L} = \{(i, j) \in \mathbb{Z}^2 | i + j = 0\}$ and a set of random variables h^0 given by

$$h^0(x, -x) = \begin{cases} \sum_{k=1}^{x}(X_k - Y_k), & \text{for } x \geq 1, \\ 0, & \text{for } x = 0, \\ -\sum_{k=x+1}^{0}(X_k - Y_k), & \text{for } x \leq -1, \end{cases} \tag{2.5.1}$$

where $\{X_k\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$ are independent random variables with $X_k \sim \mathrm{Exp}(1 - \rho)$ and $Y_k \sim \mathrm{Exp}(\rho)$.

The statistics of growth models in the equilibrium situation was first observed by Baik and Rains [BR00] as a result of the study of a polynuclear growth model with two external sources, considered by Prähofer and Spohn in [PS00]. They constructed a set of points in $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ in the following way. Given three parameters $t > 0$ and $\alpha_\pm \geq 0$, they selected points in $(0, 1) \times (0, 1)$ from a Poisson point process $P(t^2)$ of density t^2, on the open bottom edge $(0, 1) \times \{0\}$ from $P(\alpha_- t)$ and on the open left edge $\{0\} \times (0, 1)$ from $P(\alpha_+ t)$. They studied the statistics of the length $L(t)$ of the longest up-right path in this random configuration of points. This process can be alternatively seen as a Poisson process of intensity 1 in $\mathbb{R}_+^2$, together with a Poisson process of intensity α_+ on the open half-line $\mathbb{R}_+ \times \{0\}$ and a Poisson process of intensity α_- on the open half-line $\{0\} \times \mathbb{R}_+$. Then, $L(t)$ is equal to the length of the longest up-right path from $(0,0)$ to (t,t).

When $\alpha_\pm = 0$, the limiting fluctuations of $L(t)$ are described by the GUE Tracy-Widom distribution. When $\alpha_\pm > 0$, before entering the bulk, the longest path follows one of the edges, the time spent being proportional to $\alpha_\pm$: if $\alpha_\pm$ is small, one expects GUE fluctuations, similar to the $\alpha_\pm = 0$ case; if $\alpha_\pm$ is large, then most of the time is spent on one of the edges, giving Gaussian fluctuations. They distinguished four cases:

- If $\alpha_\pm < 1$, we observe F_{GUE} fluctuations;
- If $\alpha_+ > 1$ or $\alpha_- > 1$, we observe Gaussian fluctuations;
- If $\alpha_+ = 1$ and $\alpha_- < 1$, or viceversa, we observe F^2_{GOE} fluctuations. The F^2_{GOE} describes the limiting fluctuations of the largest of the superimposition of the eigenvalues of two random matrices in the Gaussian orthogonal ensemble;
- If $\alpha_\pm = 1$, we are in the critical case and observe fluctuations distributed according to the distribution F_{BR} that we will define later.

Summarizing, we have the following results.

Theorem 2.5.1 (Theorem 1.1 of [BR00]). *For each fixed $\alpha_\pm$, as $t \to \infty$, we have:*

i. When $0 \le \alpha_\pm \le 1$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t)-2t}{t^{1/3}} \le x\right) = \begin{cases} F_{\text{GUE}}(x), & 0 \le \alpha_\pm < 1, \\ F_{\text{GOE}}(x)^2, & \alpha_+ = 1, 0 \le \alpha_- < 1 \\ & \quad \textit{or } \alpha_- = 1, 0 \le \alpha_+ < 1 \\ F_{\text{BR}}(x), & \alpha_\pm = 1. \end{cases} \tag{2.5.2}$$

ii. When at least of the $\alpha_\pm$ is greater than 1, setting $\alpha = \max\{\alpha_+, \alpha_-\}$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L(t)-(\alpha+\alpha^{-1})t}{\sqrt{\alpha+\alpha^{-1}}t^{1/2}} \le x\right) = \begin{cases} \operatorname{erf}(x), & \alpha_+ \ne \alpha_-, \\ \operatorname{erf}(x)^2, & \alpha_+ = \alpha_-. \end{cases} \tag{2.5.3}$$

The limiting distribution F_{BR} of the critical case $\alpha_\pm = 1$ is defined as

$$F_{\text{BR}}(x) = \{1 - (x + 2u'(x) + 2u(x)^2)v(x)\} \left(e^{\frac{1}{2}\int_x^\infty u(s)ds}\right)^4 F_{\text{GUE}}(x), \tag{2.5.4}$$

where u is the solution of the Painlevé II equation,

$$u_{xx} = 2u^3 + xu, \tag{2.5.5}$$

with the boundary condition $u(x) \sim -\text{Ai}(x)$ as $x \to +\infty$. The distribution was renamed after the authors as *Baik–Rains distribution*.

In [FS06] Ferrari and Spohn obtained analogous results for TASEP starting from Bernoulli-ρ measure, $0 < \rho < 1$, which is the stationary measure. Here they used a mapping to a directed polymer problem. In the initial configuration of TASEP, let $\zeta_+ + 1$ be the location of the first particle to the right of 1 and let $-\zeta_-$ be the location of the first hole to the left of 0. The random variables ζ_+, ζ_- are independent and geometrically distributed, $\mathbb{P}(\zeta_+ = k) = \rho(1-\rho)^k$, $\mathbb{P}(\zeta_- = k) = \rho^k(1-\rho)$. Consider a family of independent random variables $w_{i,j}$, $i,j \geq 0$,

$$\begin{aligned} &w_{i,0} \sim \mathrm{Exp}(1-\rho), && i \geq 1,\\ &w_{0,j} \sim \mathrm{Exp}(\rho), && j \geq 1,\\ &w_{i,j} \sim \mathrm{Exp}(1), && i,j \geq 1,\\ &w_{0,0} = 0. \end{aligned} \tag{2.5.6}$$

The random variable $w_{i,j}$ represents the j-th waiting time of the i-th particle; in particular, $w_{\zeta_+ + k,0}$ is the k-th waiting time of the first particle to the right of 0 and $w_{0,\zeta_- + k}$ is the k-th waiting time of the first hole to the left of 0. If we consider the last passage percolation with weights

$$\omega_{i,j} = \begin{cases} 0 & 1 \leq i \leq \zeta_+, j = 0,\\ 0 & i = 0, 1 \leq j \leq \zeta_-,\\ w_{i,j} & \text{otherwise}, \end{cases} \tag{2.5.7}$$

then, (2.2.7) holds.

By means of the same translation in terms of LPP, in [BFP10] it was determined the limiting multi-point distribution of the current fluctuations of TASEP. In analogy with the other two cases, the limit process was renamed *Airy*$_{\text{stat}}$ *process* (see Section 2.5.1).

However, the above described setting looks quite unnatural if one wants to work in a last passage percolation picture. In [BCS06], Balász, Cator and Seppäläinen constructed a last passage growth model with exponential weights with boundary conditions that represent the equilibrium exclusion process as seen from a particle right after its jump. The equilibrium distribution of a particle system as seen from a "typical" particle is the Palm distribution. For TASEP the Palm distribution is the Bernoulli-ρ equilibrium conditioned on having a particle in 0. The stationarity of the initial measure is attained by a theorem from queuing theory.

Theorem 2.5.2 (Burke's Theorem, Theorem 3.1 of [BCS06])**.** *Consider TASEP started from the Palm distribution (i.e. a particle at the origin, Bernoulli measure elsewhere). Then, the position of the particle started at the origin is marginally a Poisson process with jump rate* $1-\rho$.

The last passage percolation with weights given by (2.5.6) corresponds to TASEP started from Bernoulli-ρ measure, conditioned on having a hole at the origin and a particle at site one. Despite this small alteration of the Palm distribution, a version of Burke's Theorem still holds.

Proposition 2.5.3 (Corollary 3.2 of [BCS06])**.** *Let $P_j(t)$ be the position of the j-th particle at time t and let $H_i(t)$ be the position of the i-th hole at time t. Marginally, $P_0(t)-1$ and $-H_0(t)$ are two independent Poisson processes with respective jump rates $1-\rho$ and ρ.*

TASEP can be interpreted as a queuing system, if we represents the particles as servers and the holes between the j-th and the $(j-1)$-th particles as customers waiting at server j. In the LPP, the occupation of the point (i,j) corresponds to the customer i being served by server j. This can formulated in the following way. Let $I_{i,j}$ be the time it takes for the j-th particle to jump to site $i-j$ and let $J_{i,j}$ be the time it takes for the i-th hole

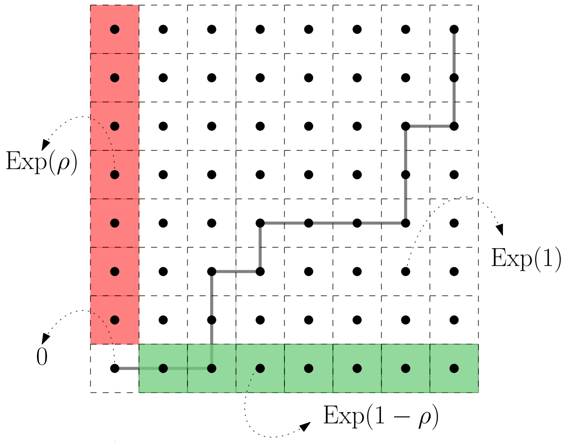

Figure 2.6: Stationary last passage percolation with boundary conditions (2.5.6). The random variables are exponentially distributed of parameter 1 in the bulk (white squares), of parameter ρ on the horizontal axis (green squares) and of parameter $1-\rho$ on the vertical axis (red squares). The weight in the origin is identically zero.

to jump to site $i-j+1$. Then

$$\begin{aligned} I_{i,j} =& L_{(i,j)} - L_{(i-1,j)} \\ J_{i,j} =& L_{(i,j)} - L_{(i,j-1)} \end{aligned} \tag{2.5.8}$$

The connection with LPP with boundary condition (2.5.6) is established, if we show that Burke's Theorem holds for every hole and particle in the last-passage picture. Let Σ be the set of down-right paths in the first quadrant and define the interior of the set enclosed by $\sigma \in \Sigma$ as

$$\mathcal{I}(\sigma) = \{(i,j) : 0 \le i < p_k, 0 \le j < q_k \text{ for some } (p_k, q_k) \in \sigma\}. \tag{2.5.9}$$

Proposition 2.5.4 (Lemma 4.2 of [BCS06]). *For any $\sigma \in \Sigma$, the collections of random variables $\{I_{i,j}, (i,j) \in \mathcal{I}(\sigma)\}$ and $\{J_{i,j}, (i,j) \in \mathcal{I}(\sigma)\}$ are mutually independent exponentials with parameters $1-\rho$ and ρ respectively.*

This means that the increments along a down-right path are sums of independent random variables, $\mathrm{Exp}(1-\rho)$ for horizontal steps and $\mathrm{Exp}(\rho)$ for vertical steps.

By Proposition 2.5.4, the stationary situation can be equivalently realized considering a point-to-line LPP with $\mathcal{L} = \{(i,j) \in \mathbb{Z}^2 \,|\, i+j=0\}$ and randomness h^0 on the boundary given by a two-sided random walk

$$h^0(x,-x) = \begin{cases} \sum_{k=1}^{x}(X_k - Y_k), & \text{for } x \ge 1, \\ 0, & \text{for } x = 0, \\ -\sum_{k=x+1}^{0}(X_k - Y_k), & \text{for } x \le -1, \end{cases} \tag{2.5.10}$$

where $\{X_k\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$ are independent random variables with $X_k \sim \mathrm{Exp}(1-\rho)$ and $Y_k \sim \mathrm{Exp}(\rho)$.

At page 32 we explained the link between LPP and the ensemble of complex Gaussian sample covariance matrices. The stationary LPP can be expressed as the maximum of two LPPs

$$L_{\mathcal{L}^{\text{stat}} \to (n,n)} = \max\{L^{|}_{(0,0)\to(n,n)}, L^{-}_{(0,0)\to(n,n)}\}, \tag{2.5.11}$$

where $L^{|}_{(0,0)\to(n,n)}$ and $L^{-}_{(0,0)\to(n,n)}$ are rank-one perturbations of the point-to-point LPP with weights only on the boundary $i=0$ and $j=0$ respectively. Using this fact, we obtain the exponential decay of the upper tail of the distribution of the rescaled stationary LPP. An estimate like (2.3.20) for the lower tail can be immediately obtained, since $\mathbb{P}(L_{\mathcal{L}^{\rm stat}\to(n,n)} \leq s) \leq \mathbb{P}(L_{(0,0)\to(n,n)} \leq s)$.

With an argument similar to the one made for the point-to-point LPP, in [FS16] it was made a prediction on the behaviour of the covariance for the stationary case,

$$\mathrm{Cov}(L_{\mathcal{L}^{\rm stat}\to(n,n)}, L_{\mathcal{L}^{\rm stat}\to(\tau n,\tau n)}), \tag{2.5.12}$$

for $0 \leq \tau \leq 1$. Close to $\tau = 0$, in the limit $n \to \infty$, they predicted the same $\Theta(\tau^{2/3})$ behaviour of the step case. But, in this case, it is due to the randomness of the initial conditions, not to the correlations generated at small times. For the stationary case, it is possible to obtain an exact expression for the covariance for τ in the entire interval [0,1]. This is due to the fact that the increments of the limit $\mathrm{Airy}_{\rm stat}$ process are not only locally Brownian, but exactly Brownian, namely

$$\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\left[\mathcal{A}_{\rm stat}\left(\tfrac{1-\tau}{\tau}v\right)^{2/3} - \mathcal{A}_{\rm stat}(0)\right] \stackrel{\rm dist}{=} \sqrt{2}B(v), \tag{2.5.13}$$

where B is a standard Brownian motion. In [FS16] this was obtained in the special case of ending points on the diagonal. For the general case, we consider the LPP with ending point $E_\tau = (\tau n, \tau n) + (2n)^{2/3}w_\tau(1,-1)$, for $0 \leq \tau \leq 1$ and $w_\tau \in \mathbb{R}$.

Theorem 2.5.5 (see Corollary 4.2.4)**.** *For the stationary LPP, the covariance of the limiting height function for all $\tau \in (0,1)$ can be expressed as*

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Cov}\left(L^{\rm resc}_{\mathcal{L}^{\rm stat}\to(n,n)}, L^{\rm resc}_{\mathcal{L}^{\rm stat}\to(\tau n,\tau n)}\right) &= \frac{\tau^{2/3}}{2}\mathrm{Var}\left(\xi_{{\rm stat},\tau^{-2/3}w_\tau}\right) + \frac{1}{2}\mathrm{Var}\left(\xi_{{\rm stat},w_1}\right) \\ &- \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{{\rm stat},(1-\tau)^{-2/3}(w_1-w_\tau)}\right).\end{aligned} \tag{2.5.14}$$

Here $\xi_{{\rm stat},w}$ has distribution given by (2.3.32)*.*

2.5.1 The $\mathrm{Airy}_{\rm stat}$ process

The $\mathrm{Airy}_{\rm stat}$ process arises as limit process for models started from stationary initial condition. Unlike the Airy_1 and Airy_2 processes, this process is *not* stationary, in spite of its name. Studying the limiting distribution functions for a polynuclear growth model with two external sources, previously considered by Prähofer and Spohn [PS00], Baik and Rains identified the one-point distribution of the $\mathrm{Airy}_{\rm stat}$ process, subsequently baptized Baik–Rains distribution. Baik, Ferrari and Péché [BFP14] obtained the multi-point distribution for TASEP with Bernoulli measure as initial condition.

The increments of the $\mathrm{Airy}_{\rm stat}$ process have exactly the statistics of a Brownian motion. This follows from the model directly, since the stationary initial condition is realized as a random walk, which converges to a Brownian motion. A proof of this property can be found in Section 8 of [FSW15a]. Moreover, the Baik–Rains distribution has zero mean (Proposition 2.1 of [BR00]). This fact was suggested by numerical simulations in [PS00], where also a justification is provided: with an indirect argument for the PNG with $\alpha_\pm = 1$, they show that the average of the length $L(t)$ of the longest up-right path in a random configuration of Poisson points is $2t$, which implies the zero mean property of $F_{\rm BR}$. To see this formally, we write (2.5.4) as

$$F_{\rm BR}(s) = \left\{1 - E(s)^{-4}\int_{\infty}^{s} E(x)^4 dx\right\} E(s)^4 F_{\rm GUE}(s), \tag{2.5.15}$$

where $E(s) = e^{\frac{1}{2}\int_s^\infty u(x)dx}$. The identity is verified, since $y(s) := s + 2u' + 2u^2$ satisfies

$$y'(s) = 1 + 2u(s)y(s), \qquad y(s) = \frac{1}{\sqrt{-2s}}(1+o(1)), \qquad s \to -\infty. \tag{2.5.16}$$

Then, we have

$$F_{\rm BR}(s) = \frac{d}{ds}\left\{F_{\rm GUE}(s)\int_{-\infty}^{s} E(t)^4 dt\right\}, \tag{2.5.17}$$

and integrating,

$$\begin{aligned}\int_{-\infty}^{s} F_{\rm BR}(x)dx &= F_{\rm GUE}(s)\int_{-\infty}^{s} E(t)^4 dt \\ &= F_{\rm GUE}(s)E(s)^4(s + 2u'(s) + 2u(s)^2).\end{aligned} \tag{2.5.18}$$

Now, the mean is given by

$$\int_{-\infty}^{\infty} x\frac{d}{dx}F_{\rm BR}(x)dx = \lim_{s\to\infty}\left[sF_{\rm BR}(s) - \int_{-\infty}^{s} F_{\rm BR}(y)dy\right]. \tag{2.5.19}$$

Subtracting (2.5.18) from $sF_{\rm BR}(s)$ and performing the $s \to \infty$ limit, we obtain

$$\int_{-\infty}^{\infty} xF_{\rm BR}(x)dx = 0. \tag{2.5.20}$$

In [BFP14] the $\text{Airy}_{\rm stat}$ process is defined in terms of its finite-dimensional distributions given by Fredholm determinants. For real numbers $u_1 < u_2 < \cdots < u_n$ and $s_1, \ldots, s_n$, define

$$\begin{aligned}\mathcal{R} =& s_1 + e^{-\frac{2}{3}u_1^3}\int_{s_1}^{\infty} dx \int_0^\infty dy \mathrm{Ai}(x+y+u_1^2)e^{-u_1(x+y)}, \\ \Psi_j(y) =& e^{\frac{2}{3}u_j^3 + u_j y} - \int_0^\infty dx \mathrm{Ai}(x+y+u_j^2)e^{-u_j x}, \\ \Phi_i(x) =& e^{-\frac{2}{3}u_1^3}\int_0^\infty d\lambda \int_{s_1}^\infty dy e^{-\lambda(u_1-u_i)}e^{-u_1 y}\mathrm{Ai}(x+u_i^2+\lambda)\mathrm{Ai}(y+u_1^2+\lambda) \\ &+ \mathbb{1}_{[i\geq 2]}\frac{e^{-\frac{2}{3}u_i^3 - u_i x}}{\sqrt{4\pi(u_i-u_1)}}\int_{-\infty}^{s_1-x} dy e^{-\frac{y^2}{4(u_i-u_1)}} - \int_0^\infty dy \mathrm{Ai}(x+y+u_i^2)e^{\tau_i y}.\end{aligned} \tag{2.5.21}$$

Then, the n-point distribution at $u_1 < u_2 < \cdots < u_n$ is given by

$$\mathbb{P}\left(\bigcap_{k=1}^{n}\{\mathcal{A}_{\rm stat}(u_k) \leq s_k\}\right) = \sum_{k=1}^{n}\frac{\partial}{\partial s_k}\left(g_n(\vec{u},\vec{s})\det\left(\mathbb{1} - \chi_s \hat{K}_{\rm Ai}\chi_s\right)_{L^2(\{u_1,\ldots,u_n\}\times\mathbb{R})}\right), \tag{2.5.22}$$

where $\chi_s(u_k, x) = \mathbb{1}_{x > u_k}$ and the function $g_n(\vec{u},\vec{s})$ is defined by

$$\begin{aligned}g_n(\vec{u},\vec{s}) &= \mathcal{R} - \langle \rho\chi_s\Phi, \chi_s\Psi\rangle \\ &= \mathcal{R} - \sum_{i,j=1}^{n}\int_{s_i}^{\infty} dx \int_{s_j}^{\infty} dy \Psi_j(y)\rho_{j,i}(y,x)\Phi_i(x),\end{aligned} \tag{2.5.23}$$

with

$$\rho := (\mathbb{1} - \chi_s \hat{K}_{\mathrm{Ai}} \chi_s)^{-1}, \quad \rho_{j,i}(y,x) := \rho((j,y),(i,x)), \\ \Phi((i,x)) := \Phi_i(x), \qquad \Psi((j,y)) := \Psi_j(y). \tag{2.5.24}$$

The correlation kernel $\hat{K}_{\mathrm{Ai}}$ is the so-called *extended Airy kernel* with entries defined by

$$\begin{aligned} &\hat{K}_{\mathrm{Ai}}((i,x),(j,y)) := [\hat{K}_{\mathrm{Ai}}]_{i,j}(x,y) \\ &= \begin{cases} \int_0^\infty d\lambda \mathrm{Ai}(x+\lambda+u_i^2)\mathrm{Ai}(y+\lambda+u_j^2)e^{-\lambda(u_j-u_i)} & \text{if } u_i \le u_j \\ -\int_{-\infty}^0 d\lambda \mathrm{Ai}(x+\lambda+u_i^2)\mathrm{Ai}(y+\lambda+u_j^2)e^{-\lambda(u_j-u_i)} & \text{if } u_i > u_j. \end{cases} \end{aligned} \tag{2.5.25}$$

2.6 Half-space LPP

So far we presented models of last passage percolation defined on the positive quadrant of $\mathbb{Z}^2$ or $\mathbb{R}^2$ or on full spaces. However, beyond these cases, it is possible to define models with different geometries, like the last passage percolation in the half-quadrant of integers. The so-called *half-space last passage percolation* is a variant of Johansson's full-quadrant corner growth model and, as for the full-space case, it is integrable when the weights are geometric or exponential (or come from a Poisson process of constant intensity). Consider

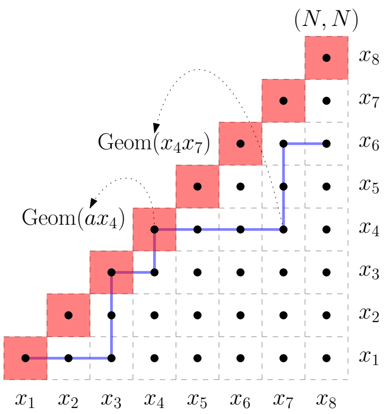

Figure 2.7: A possible LPP path (polymer) starting at the origin to the point $(N, N-n)$, for $n \ge 0$. The types of geometric random variables are assigned according to the row and column x parameters, and the diagonal has an extra parameter a.

a sequence of independent geometric random variables

$$w_{n,m} \sim \begin{cases} \mathrm{Geom}(x_n x_m), & n \ge m+1, \\ \mathrm{Geom}(a x_n), & n = m, \end{cases} \tag{2.6.1}$$

where $a, x_1, \ldots, x_N$ are real parameters satisfying

$$0 \le a < \min_i \tfrac{1}{x_i}, \qquad 0 < x_1, \ldots, x_N < 1 \tag{2.6.2}$$

and a random variable X is geometric $\mathrm{Geom}(q)$ if $\mathbb{P}(X = k) = (1-q)q^k, \forall k \in \mathbb{N}$. We depict this in Figure 2.7. It is helpful to visualize the x's as indexing the rows and columns of the half-space. Consider up-right paths π from $(0,0)$ to (n,m) with $n \ge m$. The last passage time on the half-quadrant, denoted by $H^{\mathrm{geom}}(n,m)$, is defined as

$$H^{\mathrm{geom}}(n,m) = \max_{\pi:(0,0)\to(n,m)} \sum_{(i,j)\in\pi} w_{i,j} \tag{2.6.3}$$

and satisfies the recurrence relation

$$H^{\text{geom}}(n,m) = w_{n,m} + \begin{cases} \max\{H^{\text{geom}}(n-1,m), H^{\text{geom}}(n,m-1)\}, & \text{if } n \geq m+1, \\ H^{\text{geom}}(n,m-1), & \text{if } n = m. \end{cases} \tag{2.6.4}$$

In the exponential case, the definition is identical with weights given by

$$\omega_{n,m} \sim \begin{cases} \text{Exp}(1), & n \geq m+1, \\ \text{Exp}(\alpha), & n = m, \end{cases} \tag{2.6.5}$$

for $\alpha > 0$. This model is also known as *symmetric last passage percolation*, since it is equivalent to a model of LPP in the full-quadrant where the weights are symmetric with respect to the diagonal, i.e. $\omega_{i,j} = \omega_{j,i}$.

As the full-space last passage percolation was introduced to study the asymptotic distribution of the size of the longest increasing subsequence in a uniformly random permutation [BDJ99b], the half-space LPP is related to the problem of the longest increasing subsequence in a random involution. A random involution $\sigma \in \mathcal{S}_n$ is a random permutation that does not contain any permutation cycle of length greater than 2, so it consists only of fixed points and transpositions. The description with the half-space model is justified by the fact that the permutation matrix of an involution is symmetric and its graph $i \mapsto \sigma(i)$ is symmetric with respect to the diagonal.

The problem of the longest increasing subsequence in a random involution was initially studied in [BR01a, BR01c, BR01b] together with the symmetrized last passage percolation with geometric weights. In [Rai00] it was proved that the half-space LPP is a Pfaffian point process (see Section 1.5.3) and in [BR01b] the asymptotic of the last passage time $H(n,m)$ was obtained as $n,m \to \infty$. In particular, it was shown that the limiting fluctuations depend on the end point (n,m) and on choice of the parameters x_n, a. These results have been proved many times in different contexts and with different techniques [Rai00, BR01b, IS04, BR06, FR07, BBNV18]. Here we present them as in [BBNV18] for an end point on the diagonal and for $x_n = \sqrt{q}$, $q \in (0,1)$.

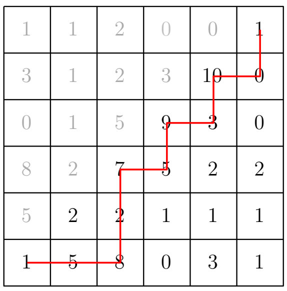

Figure 2.8: Last passage percolation for a 6×6 symmetric matrix.

Theorem 2.6.1 (Special case of Theorem 2.7 of [BBNV18]). *The distribution of the last passage time $H^{\text{geom}}(n,n)$ is a pfaffian*

$$\mathbb{P}(H^{\text{geom}}(n,n) < s) = \text{pf}(J-K)_{\ell^2\{s+\frac{1}{2}, s+\frac{3}{2}, s+\frac{5}{2}, \ldots\}} \tag{2.6.6}$$

with 2×2 *matrix correlation kernel* $K : \left(\mathbb{Z} + \frac{1}{2}\right)^2 \to \mathrm{Mat}_2(\mathbb{R})$ *given by*

$$\begin{aligned}
K_{11}(k,\ell) &= \frac{1}{(2\pi i)^2} \oint \frac{dz}{z^{k+1}} \oint \frac{dw}{w^{\ell+1}} F(z)F(w) \frac{\sqrt{zw}(z-w)(z-a)(w-a)}{(z^2-1)(w^2-1)(zw-1)},\\
K_{12}(k,\ell) &= -K_{21}(\ell,k) = \frac{1}{(2\pi i)^2} \oint \frac{dz}{z^{k+1}} \oint \frac{dw}{w^{-\ell+1}} \frac{F(z)}{F(w)} \frac{\sqrt{zw}(zw-1)(z-a)}{(z-w)(z^2-1)(w-a)},\\
K_{22}(k,\ell) &= \frac{1}{(2\pi i)^2} \oint \frac{dz}{z^{-k+1}} \oint \frac{dw}{w^{-\ell+1}} \frac{1}{F(z)F(w)} \frac{\sqrt{zw}(z-w)}{(zw-1)(z-a)(w-a)},
\end{aligned} \tag{2.6.7}$$

where

$$F(z) := \prod_{i=1}^{N} \frac{1 - x_k/z}{1 - x_k z} \tag{2.6.8}$$

and where the contours are positively oriented circles centered around the origin satisfying the following conditions:

- *for* K_{11}, $1 < |z|, |w| < \min_i \frac{1}{x_i}$;
- *for* K_{12}, $\max\{\max_i x_i, a\} < |w| < |z|$ *and* $1 < |z| < \min_i \frac{1}{x_i}$;
- *for* K_{22}, $\max\{\max_i x_i, a\} < |w|, |z|$ *and* $1 < |zw|$.

With a suitable choice of a and of the end point we obtain a limiting crossover regime.

Theorem 2.6.2 (Theorem 2.8 of [BBNV18])**.** *Consider the geometric half-space LPP with* $x_j = \sqrt{q}$, $q \in (0,1)$, $j \geq 1$ *and* $a = 1 - 2c_q n^{-1/3}$, *where* $c_q = \frac{1-\sqrt{q}}{q^{1/6}(1+\sqrt{q})^{1/3}}$, $v \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\left(H^{\mathrm{geom}}(n - \lfloor un^{2/3} \rfloor, n) \leq \frac{2\sqrt{q}n}{1-\sqrt{q}} - u\frac{\sqrt{q}n^{2/3}}{1-\sqrt{q}} + c_q^{-1} s n^{-1/3} \right) = F_{u,v}(s), \tag{2.6.9}$$

where $F_{u,v}(s)$ *performs a crossover between the classical distributions from random matrix theory. One has*

$$\begin{aligned}
F_{0,0}(s) &= F_{\mathrm{GOE}}(s),\\
\lim_{v\to\infty} F_{0,v}(s) &= F_{\mathrm{GSE}}(s),\\
\lim_{u\to\infty} F_{u,v}(s - u^2 d_q^2) &= F_{\mathrm{GUE}}(s),
\end{aligned}$$

where $d_q = \frac{q^{1/6}}{2(1+\sqrt{q})^{2/3}}$.

An analogue theorem for the exponential last passage time was stated and proved in [BBCS18]. They show that the limiting fluctuations depend on the strength of the weights on the diagonal and that a transition in the behaviour of the fluctuations occurs at $\alpha = 1/2$.

Theorem 2.6.3 (Theorem 1.4 of [BBCS18])**.** *Consider the half-space last passage time on the diagonal* $H(n,n)$ *with exponential weights.*

a) For $\alpha > 1/2$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H^{\exp}(n,n) - 4n}{2^{4/3} n^{1/3}} < s \right) = F_{\mathrm{GSE}}(s).$$

b) For $\alpha = 1/2$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H^{\exp}(n,n) - 4n}{2^{4/3}n^{1/3}} < s\right) = F_{\mathrm{GOE}}(s).$$

c) For $\alpha < 1/2$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H^{\exp}(n,n) - \frac{n}{\alpha(1-\alpha)}}{\sigma n^{1/2}} < s\right) = G(s),$$

where $G(s)$ *is the standard Gaussian distribution function and* $\sigma = \frac{(1-2\alpha)^{1/2}}{\alpha(1-\alpha)}$.

Moreover, if we consider end points far away from the diagonal, the last passage time $H(n,m)$ satisfies the same limit theorem of the full-space model.

Theorem 2.6.4 (Theorem 1.5 of [BBCS18])**.** *For any* $\kappa \in (0,1)$ *and* $\alpha > \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H^{\exp}(n,\kappa n) - (1+\sqrt{\kappa})^2 n}{\sigma n^{1/3}} < s\right) = F_{\mathrm{GUE}}(s),$$

where $\sigma = \frac{(1+\sqrt{\kappa})^{4/3}}{\kappa^{1/6}}$.

We can give an heuristic explanation of the phase transition of the limit fluctuations for α varying in $(0,\infty)$. As α goes to infinity, the weights on the diagonal go to 0, so the maximizing path to (n,n) will tend to avoid the diagonal. Then, the last passage time to (n,n) is the last passage time from (1,0) to $(n,n-1)$ and this is equal in distribution to $H(n-1,n-1)$ with $\alpha = 1$. This implies that the fluctuations behave in the same way for $\alpha \in (1,\infty)$. On the contrary, if α is very small, the weights on the diagonal will be much more relevant than the ones in the bulk and the path will stick to the diagonal; thus, the last passage time will be a sum of $\mathcal{O}(n)$ i. i. d. random variables and we expect fluctuations of order $n^{1/2}$. The critical value $\alpha = 1/2$ is due to symmetries in the underlying Pfaffian process.

In [BFO19] we studied the stationary version of the half-space LPP with exponential weights. Here stationarity has to be interpreted in the sense of [BCS06], as it refers to the stationarity of the increments of the last passage time along the vertical and the horizontal directions. It is realized choosing the weights as follows:

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right), & i=j>1, \\ \mathrm{Exp}\left(\frac{1}{2}-\alpha\right), & j=1, i>1, \\ 0, & \text{if } i=j=1, \\ \mathrm{Exp}(1), & \text{otherwise} \end{cases} \tag{2.6.10}$$

where $\alpha \in (-1/2, 1/2)$.

We considered the last passage percolation from the origin to a point $(N, N-n)$ not necessarily on the characteristic and obtained an exact formula for the distribution of the last passage time, denoted $L_{N,N-n}$ for simplicity, together with asymptotic results for the distribution under the KPZ scaling. The method to approach the stationary half-space LPP was inspired by the works of Baik–Rains [BR00] in the study of the PNG with external sources, that lead to the definition of the distribution of the stationary LPP in the full-space, and of Baik–Ferrari–Péché [BFP10], that generalized the previous approach to obtain the multi-point distribution of stationary TASEP. Since no useful formulas are available to study the statistics of $L_{N,N-n}$, first we consider a slightly different version of the model, which turns out to be more manageable: an half-space LPP with weights

$$\tilde{\omega}_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right), & i=j>1, \\ \mathrm{Exp}\left(\frac{1}{2}+\beta\right), & j=1, i>1, \\ \mathrm{Exp}\left(\alpha+\beta\right), & i=j=1, \\ \mathrm{Exp}(1), & \text{otherwise}, \end{cases} \tag{2.6.11}$$

where $\alpha \in (-1/2, 1/2), \beta \in (-1/2, 1/2)$ are parameters satisfying $\alpha+\beta > 0$. As a corollary of [Rai00] and Theorem 2.6.1, the distribution of the last passage time is integrable and is given by a Fredholm Pfaffian. For this reason, we denote the last passage time with $L^{\mathrm{pf}}_{N,N-n}$.

Theorem 2.6.5. *Let $\beta \in (0, 1/2)$ and $\alpha \in (-1/2, 1/2)$. Then, for $s \in \mathbb{R}_+$,*

$$\mathbb{P}(L^{\mathrm{pf}}_{N,N-n} \le s) = \mathrm{pf}(J-K)_{L^2(s,\infty)} \tag{2.6.12}$$

where $K = K(x,y)$ is defined in Theorem 5.3.1.

To recover the desired distribution, we need to find an expression that links the distribution of $L_{N,N-n}$ and $L^{\mathrm{pf}}_{N,N-n}$. The difference between the two models is given by the random variable $\tilde{\omega}_{1,1}$ and the parameter β. We can remove $\tilde{\omega}_{1,1}$ using a standard shift argument.

Lemma 2.6.6. *Let $\alpha, \beta \in (-1/2, 1/2)$ with $\alpha+\beta > 0$. Then*

$$\left(1 + \frac{1}{\alpha+\beta}\partial_s\right)\mathbb{P}(L^{\mathrm{pf}}_{N,N-n} \le s) = \mathbb{P}(\widetilde{L}_{N,N-n} \le s). \tag{2.6.13}$$

Then, we take the $\beta \to -\alpha$ limit. To do this, we need to find a decomposition of $\frac{1}{\alpha+\beta}\mathbb{P}(L^{\mathrm{pf}}_{N,N-n} \le s)$ which is well-defined when we take the limit as $\alpha+\beta \to 0$. Using the residue theorem for the integral kernels, we obtain the following decomposition.

Proposition 2.6.7. *Let $\alpha \in (-1/2, 1/2)$, $\beta > 0$. Then the kernel K splits as*

$$K = \overline{K} + (\alpha+\beta)R \tag{2.6.14}$$

where

$$\begin{aligned} \overline{K}_{11}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi \mathrm{i}} \oint_{\Gamma_{-1/2}} \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[(\tfrac{1}{2}-z)(\tfrac{1}{2}+w)\right]^n \frac{(z+\beta)(w-\beta)}{(z-\beta)(w+\beta)} \cdot \\ & \cdot \frac{(z+\alpha)(w-\alpha)(z+w)}{4zw(z-w)}, \\ \overline{K}_{12}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi \mathrm{i}} \oint_{\Gamma_{-1/2,-\alpha,-\beta}} \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[\frac{(\frac{1}{2}-z)}{(\frac{1}{2}-w)}\right]^n \frac{z+\alpha}{w+\alpha}\frac{z+\beta}{z-\beta}\frac{w-\beta}{w+\beta}\frac{z+w}{2z(z-w)} \\ =& -\overline{K}_{21}(y,x), \\ \overline{K}_{22}(x,y) =& \varepsilon(x,y) + \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n} \frac{1}{(z-\alpha)(w+\alpha)} \cdot \\ & \cdot \frac{z+\beta}{z-\beta}\frac{w-\beta}{w+\beta}\frac{z+w}{z-w} \end{aligned} \tag{2.6.15}$$

and where the integration contours for $\overline{K}_{22}$ *are, for* (z,w)*, the union of* $\Gamma_{1/2,\alpha,\beta}\times\Gamma_{-1/2}$*,* $\Gamma_{1/2,\beta}\times\Gamma_{-\alpha}$*, and* $\Gamma_{1/2,\alpha}\times\Gamma_{-\beta}$*.*

The operator R *is of rank two and given by*

$$R=\begin{pmatrix} g_1(x)f_+^\beta(y)-f_+^\beta(x)g_1(y) & f_+^\beta(x)\tilde{g}_2(y) \\ -\tilde{g}_2(x)f_+^\beta(y) & 0 \end{pmatrix}. \tag{2.6.16}$$

With Γ_I we indicate a counter-clockwise contour around a set of points I. The functions $g_1,\tilde{g}_2,f_+^\beta$ are defined in (5.3.18) and (5.3.19).

Once the analyticity of $\frac{1}{\alpha+\beta}\mathbb{P}(L_{N,N-n}^{\mathrm{pf}}\le s)$ is determined (details are carried out in Section 5.3.3), taking the $\alpha+\beta\to 0$ limit, we recover the distribution of the stationary model.

Theorem 2.6.8. *Let* $\alpha\in(-1/2,1/2)$ *be a real number and* $1\le N$*,* $0\le n\le N-1$ *be positive integers. Let* $L_{N,N-n}$ *be the stationary LPP time from* $(1,1)$ *to* $(N,N-n)$ *in the model of weights given by* (2.6.10)*. Then*

$$\mathbb{P}(L_{N,N-n}\le s)=\partial_s\left\{\mathrm{pf}(J-\overline{\mathsf{K}})\cdot\left[e^\alpha(s)-\left\langle -g_1\quad \tilde{g}_2\middle|(\mathbb{1}-J^{-1}\overline{\mathsf{K}})^{-1}\begin{pmatrix}-h_1\\ h_2\end{pmatrix}\right\rangle\right]\right\} \tag{2.6.17}$$

where the Fredholm pfaffian is taken over $L^2(s,\infty)$*.*

The functions e^α, h_1 and h_2 are defined in (5.2.12) and (5.2.16).

To obtain the limit distribution, we recover the appropriate scaling by computing the limit shape for the last passage time. Since this can be separated into two contributions as $L_{N,N-\eta N}=L_{N,1}+(L_{N,N-\eta N}-L_{N,1})$, where each of the terms is a sum of independent random variables, we get

$$\begin{aligned} N^{-1}L_{N,N-\eta N} &\simeq \frac{1}{\frac12-\alpha}+\frac{(1-\eta)}{\frac12+\alpha}=\frac{1}{\frac14-\alpha^2}-\frac{\eta}{\frac12+\alpha} \\ &\simeq 4N-2u2^{5/3}N^{2/3}+\delta(2u+\delta)2^{4/3}N^{1/3}, \end{aligned} \tag{2.6.18}$$

after setting $\alpha=\delta 2^{-4/3}N^{-1/3}$ and $\eta N=n=u2^{5/3}N^{2/3}$. Thus, we consider the scaling

$$(s,x,y)=4N-2u2^{5/3}N^{2/3}+(S,X,Y)\,2^{4/3}N^{1/3}. \tag{2.6.19}$$

Here we got rid of the quadratic term in δ, since it effects the limit distribution only cosmetically, but it plays a role in the analysis of the kernels asymptotics for $\delta\to\pm\infty$. Accordingly, we scale the variables $z=\zeta/(2^{4/3}N^{1/3})$ and $w=\omega/(2^{4/3}N^{1/3})$. Under this scaling,

$$\lim_{N\to\infty}\overline{\mathsf{K}}_{ij}^{\mathrm{resc}}(X,Y)=\overline{\mathcal{A}}_{ij}(X,Y),\quad i,j\in\{1,2\}, \tag{2.6.20}$$

where the limit kernels are

$$
\begin{aligned}
\overline{\mathcal{A}}_{11}(X,Y) &= -\int_{0\langle} \frac{d\zeta}{2\pi i} \int_{\rangle 0,\zeta} \frac{d\omega}{2\pi i} \frac{e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}+\omega^2 u-\omega Y}} (\zeta-\delta)(\omega+\delta)\frac{\zeta+\omega}{4\zeta\omega(\zeta-\omega)},\\
\overline{\mathcal{A}}_{12}(X,Y) &= -\int_{0\langle} \frac{d\zeta}{2\pi i} \int_{\delta\rangle \zeta} \frac{d\omega}{2\pi i} \frac{e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{\zeta-\delta}{\omega-\delta}\frac{\zeta+\omega}{2\zeta(\zeta-\omega)}\\
&= -\overline{\mathcal{A}}_{21}(Y,X),\\
\overline{\mathcal{A}}_{22}(X,Y) &= \mathcal{E}(X,Y) + \int \frac{d\zeta}{2\pi i} \int \frac{d\omega}{2\pi \mathrm{i}} \frac{e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{1}{\zeta-\omega}\left(\frac{1}{\zeta+\delta}+\frac{1}{\omega-\delta}\right).
\end{aligned}
\tag{2.6.21}
$$

For two sets of points I, J, we indicated with $_{I}\langle_{J}$ and $_{I}\rangle_{J}$ the typical Airy contours, as in Figure 5.3. In $\overline{\mathcal{A}}_{22}$ the integration contours, for (ζ,ω), are $\langle_{-\delta} \times \rangle_{\zeta}$ for the term $1/(\zeta+\delta)$, and $\langle \times {}_{\delta}\rangle_{\zeta}$ for the term $1/(\omega-\delta)$. We have denoted $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$ with

$$
\begin{aligned}
\mathcal{E}_0(X,Y) &= -\operatorname{sgn}(X-Y)e^{\delta|X-Y|+2\delta^2 u},\\
\mathcal{E}_1(X,Y) &= -\operatorname{sgn}(X-Y)\int_{\pm\delta\langle} \frac{d\zeta}{2\pi \mathrm{i}} e^{-\zeta|X-Y|+2\zeta^2 u}\frac{2\zeta}{\zeta^2-\delta^2}.
\end{aligned}
\tag{2.6.22}
$$

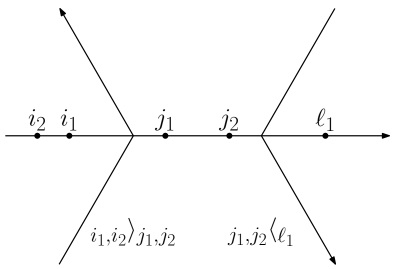

Figure 2.9: The two Airy integration contours $\rangle,\langle$ with acute angles of $\pi/3$ with the horizontal axis. Note they have opposite orientations.

Denoting with $\mathscr{e}^{\delta,u}\mathscr{h}_1^{\delta,u}, \mathscr{h}_1^{\delta,u}, \mathscr{g}_1^{\delta,u}, \widetilde{\mathscr{g}}_2^{\delta,u}$ the limits of the rescaled functions appearing in Theorem 2.6.8, we obtain a formula for the limit distribution of $L_{N,N-n}$ when $N\to\infty$.

Theorem 2.6.9. *Let $\delta\in\mathbb{R}$, $u>0$. Consider the stationary LPP time $L_{N,N-n}$ from $(1,1)$ to $(N,N-n)$ and the scaling*

$$
n = u2^{5/3}N^{2/3}, \qquad \alpha = \delta 2^{-4/3}N^{-1/3}. \tag{2.6.23}
$$

Then,

$$
\lim_{N\to\infty}\mathbb{P}\left(\frac{L_{N,N-n}-4N+4u(2N)^{2/3}}{2^{4/3}N^{1/3}} \leq S\right) = F^{(\delta,u)}_{0,\,\mathrm{half}}(S) \tag{2.6.24}
$$

with

$$
F^{(\delta,u)}_{0,\,\mathrm{half}}(S) = \partial_S\left\{\mathrm{pf}(J-\overline{\mathcal{A}})\cdot\left[\mathscr{e}^{\delta,u}(S) - \left\langle -\mathscr{g}_1^{\delta,u} \quad \widetilde{\mathscr{g}}_2^{\delta,u} \middle| (\mathbb{1}-J^{-1}\overline{\mathcal{A}})^{-1}\begin{pmatrix}-\mathscr{h}_1^{\delta,u}\\ \mathscr{h}_2^{\delta,u}\end{pmatrix}\right\rangle\right]\right\}
\tag{2.6.25}
$$

where the Fredholm pfaffian is taken over $L^2(S,\infty)$.

The distribution $F^{(\delta,u)}_{0,\,\mathrm{half}}(S)$ depends on the parameter δ, which gives the limiting strength of the weights on the boundaries and it interpolates between two Gaussian distribution, obtained in the limiting cases $\delta \to \pm\infty$. To see this, we need to take into account also the $\mathcal{O}(N^{1/3})$ term in the limit shape approximation, but, in the limit kernels, this results only in a linear shift of $\delta(2u+\delta)$ in the variables S, X, Y. Moreover, one can guess that for large values of the parameter u, it will be possible to recover the distribution of the full-space case, i.e. the Baik–Rains distribution (see (2.5.22) for $n = 1$ with $u_1 = \tau$), since (as in the point-to-point problem) the path will not be influenced too much by the simmetry of the system when the ending point is far enough from the diagonal. This is indeed proved in Theorem 5.2.10

2.7 Slow decorrelation

We have seen that the exact form of the statistics for LPP and, in general, for models in the KPZ class depends on the initial geometry of the process. These results have been proved only for a group of solvable models such as LPP, TASEP, PNG, that we presented in Chapter 1. Moreover, most of the results deal only with the spatial process at a fixed time and not with the time evolution of the process. Here we discuss a property of KPZ models that allows us to obtain the scale of time correlations. It was shown in [Fer08a] that non-trivial correlations survive on the macroscopic time scale, if one considers space-time points along special directions. This phenomenon is called "slow decorrelation". As a consequence, the space-time is non-trivially fibred, but there exist directions along which the decorrelation exponent is 1 and not 2/3. These directions are given by the characteristic lines of the PDE associated to the macroscopic evolution of the height function.

The slow decorrelation phenomenon was first shown and proved for the PNG model [Fer08a] and for TASEP with stationary initial distribution [BFP10]. This was proved in [CFP12] for a generalized LPP model that includes many KPZ models; moreover, sufficient conditions under which such LPP models display slow decorrelation have been given. They considered a growth model in $\mathbb{R}^{d+1}$, for $d \geq 1$, on a regular lattice or on a Poisson point process. A directed LPP model is defined as an almost surely sigma-finite random non-negative measure $\mu \in \mathbb{R}^{d+1}$. They studied the directed half-line to point last passage time, where the half-line is given by

$$\mathrm{HL} = \{p : p_1 = \cdots = p_{d+1} \leq 0\}. \tag{2.7.1}$$

A directed path is a curve $\pi \in \mathbb{R}^{d+1}$ such that $R\pi$ is a 1-Lipschitz function of t, where R is the rotation matrix that takes HL to the real half-line $\mathbb{R}_{\leq 0}$. The passage time of a directed path is defined as the measure of the curve π,

$$T(\pi) = \mu(\pi). \tag{2.7.2}$$

The last passage time from HL to a point p is

$$L_{\mathrm{HL}}(p) = \sup_{\pi:\mathrm{HL}\to p} T(\pi). \tag{2.7.3}$$

The following theorem states that slow decorrelation can be observed for any model that can be phrased in terms of this LPP model under certain conditions on the height function.

Theorem 2.7.1 (Theorem 2.1 of [BFP10]). *Consider a last passage model in dimension $d+1$ with $d \geq 1$ with a specific distribution of the random variables in the environment. Consider a point $p \in \mathbb{R}^{d+1}$ and a time-like direction $u \in \mathbb{R}_+^{d+1}$. If there exist constants (depending on p, u and the model) $\ell_{\mathrm{HL}}, \ell_{\mathrm{PP}} \geq 0$, $\gamma_{\mathrm{HL}}, \gamma_{\mathrm{PP}} \in (0,1)$, $\nu \in (0, \gamma_{\mathrm{HL}}/\gamma_{\mathrm{PP}})$, distributions D, D', and scaling constants $c_{\mathrm{HL}}, c_{\mathrm{PP}}$ such that*

$$\begin{aligned} &\frac{L_{\mathrm{HL}}(tp) - t\ell_{\mathrm{HL}}}{c_{\mathrm{HL}} t^{\gamma_{\mathrm{HL}}}} \stackrel{(d)}{\to} D, \\ &\frac{L_{\mathrm{HL}}(tp + t^\nu u) - t^\nu \ell_{\mathrm{HL}} - t^\nu \ell_{\mathrm{PP}}}{c_{\mathrm{HL}} t^{\gamma_{\mathrm{HL}}}} \stackrel{(d)}{\to} D \quad t \to \infty, \\ &\frac{L_{\mathrm{PP}}(tp + t^\nu u) - t^\nu \ell_{\mathrm{PP}}}{c_{\mathrm{PP}} t^{\gamma_{\mathrm{PP}}}} \stackrel{(d)}{\to} D', \end{aligned} \tag{2.7.4}$$

then we have slow decorrelation of the half-line to point last passage time at tp in the direction u with scaling exponents ν, which means that for all $\varepsilon > 0$,

$$\lim_{t \to \infty} \mathbb{P}(|L_{\mathrm{HL}}(tp + t^\nu u) - L_{\mathrm{HL}}(tp) - t^\nu \ell_{\mathrm{PP}}| \geq \varepsilon t^{\gamma_{\mathrm{HL}}}) = 0. \tag{2.7.5}$$

This theorem says that, given the existence of a law of large numbers and a central limit theorem, the height function at time $tp + t^\nu u$, scaled by $t^{1/3}$ and in a spatial scale of $t^{2/3}$, will be asymptotically the same as at time tp.

Slow decorrelation has been widely used, since it represents a technical tool to translate results on the limit processes from one observable to another and to extend known results on limit processes for more general initial condition, as we did in the following chapters.

Here we state the one-point slow decorrelation theorem in the setting of point-to-point LPP with homogeneous waiting times, since it is what we employ in our proofs. The application to finitely many points is straightforward using union bound and it was already used for instance in [CFP10, BFP10].

Theorem 2.7.2 (One-point slow decorrelation). *Let $p \in \mathbb{R}_+^2$ be a direction. Assume that there exist constants $\mu = \mu(p)$, $\alpha \in (0,1)$ and $\nu \in (0,1)$, and a distribution D, such that*

$$\frac{L_{(0,0)\to[p\ell]} - \mu\ell}{\ell^\alpha} \Rightarrow D, \text{ as } t \text{ goes to infinity.} \tag{2.7.6}$$

Then, for any $\varepsilon > 0$,

$$\lim_{\ell \to \infty} \mathbb{P}\left(|L_{(0,0)\to[p(\ell+\ell^\nu)]} - L_{(0,0)\to[p\ell]} - \mu\ell^\nu| \geq \varepsilon \ell^\alpha\right) = 0. \tag{2.7.7}$$

This pointwise property can be extended to a finite interval $[-M, M]$ and it is possible to formulate a uniform slow decorrelation property [CLW16], as stated in Theorem 3.4.1.

Chapter 3

Universality of the GOE Tracy-Widom distribution for TASEP with arbitrary particle density

This chapter is based on [FO18]. In this work we consider TASEP in continuous time with non-random initial conditions and arbitrary fixed density of particles $\rho \in (0, 1)$. We show GOE Tracy-Widom universality of the one-point fluctuations of the associated height function. The result phrased in last passage percolation language is the universality for the point-to-line problem where the line has an arbitrary slope.

3.1 Introduction

We consider the totally asymmetric simple exclusion process (TASEP) in continuous time on $\mathbb{Z}$. It is an interacting particle system with the constraint that there is at most one particle per site. Particles jump to their right-neighbouring site with rate 1, provided the arrival site is empty. A very natural and important observable is the integrated current at (for example) the origin, that is,

$$J(t) = \#\text{ particles which jumped from site 0 to site 1 during time } [0, t]. \tag{3.1.1}$$

TASEP is a model in the Kardar-Parisi-Zhang (KPZ) universality class and thus one expects that for some model-dependent constants, c_1, c_2,

$$t \mapsto \frac{J(t) - c_1 t}{c_2 t^{1/3}} \tag{3.1.2}$$

has in the $t \to \infty$ limit a non-trivial distribution function, say D. It is well-known that for KPZ models the distribution D depends on classes of initial conditions [BR01c, BR01b, PS00] (see also the reviews [Fer08b, Cor12]). In particular, consider the case of non-random initial condition with density $\rho = 1/2$, realized by placing at time 0 particles on every even sites. The joint distribution of the current at different points has been studied [Sas05, BFPS07]. As a particular case, the one-point distribution is given by the Fredholm determinant, which is shown to be equal to the GOE Tracy-Widom distribution in [FS05],

$$\lim_{t\to\infty} \mathbb{P}\left(J(t) \geq \tfrac{1}{4}t - s2^{-2/3}t^{1/3}\right) = F_{\rm GOE}(2^{2/3}s), \tag{3.1.3}$$

where F_{GOE} denotes the GOE Tracy-Widom distribution function discovered first in random matrix theory [TW96]. The analogue result was previously known for discrete time TASEP with parallel update and for a combinatorial model of longest increasing subsequences with involutions [BR01c, BR01b]. This latter model was brought in connection to the KPZ world in [PS00], where it was reinterpreted as a stochastic growth model (the so-called polynuclear growth model).

From [Joh05] we also have the variational formula

$$F_{\mathrm{GOE}}(2^{2/3}s) = \mathbb{P}\Big(\max_{v\in\mathbb{R}} \{\mathcal{A}_2(v) - v^2\} \leq s \Big), \tag{3.1.4}$$

where $\mathcal{A}_2$ is called the Airy$_2$ process [PS02b, Joh03]. There are many more variational formulas related with the Airy$_2$ process, see e.g. [BL13] and the review [QR14].

By universality one expects that the GOE Tracy-Widom distribution describes the fluctuations of $J(t)$ in the large time limit for any non-random initial condition with density $\rho \in (0,1)$. Beyond the case of $\rho = 1/2$, this was proven for densities $\rho = 1/d$, $d = 2,3,4,\ldots$ in [BFP07], and for the low-density limit of reflecting Brownian motions in [FSW15b] (in these works also the joint distribution of the current have been analyzed). In these papers, the results are achieved by exact formulas for a correlation kernel which describes the system. However, beyond the $d = 2$ case, the asymptotic analysis in these special cases turned out to be quite involved. An exact formula has very recently been derived for arbitrary initial condition as well [MQR17]. Formulas for the system with periodic boundary condition are also know only for densities $1/2, 1/3, \ldots$ [BL16, BL17].

In this paper we prove that for any $\rho \in (0,1)$,

$$\lim_{t\to\infty} \mathbb{P}\left(J(t) \geq \rho(1-\rho)t - s(\rho(1-\rho))^{2/3} t^{1/3} \right) = F_{\mathrm{GOE}}(2^{2/3}s); \tag{3.1.5}$$

compare this with Corollary 3.2.8. The proof of our result is in his core probabilistic, where the only input from exactly solvable cases is the convergence to the Airy$_2$ process for the so-called step initial condition and bounds on the tails of its one-point distribution. We prove the convergence to the variational problem (3.1.4), which does not depend on ρ. For $\rho = 1/2$ the limiting distribution function was already known to be given by F_{GOE}. The method allows for more general, including random initial conditions, we first prove convergence to a more generic variational process in Theorem 3.2.7.

To show the convergence to the variational problem, we work in the last passage percolation (LPP) framework (see Section 3.2.1 for definitions and details). In that language we need to study a "line-to-point" problem with the line having arbitrary slopes. Using a tightness result for the "point-to-point" problem (see Theorem 3.2.3) and a slow decorrelation result (see Theorem 2.7.2) (which is then extended to a functional slow decorrelation theorem (see Theorem 3.4.1)) we can show, analogously to [CLW16], the convergence of a restricted "line-to-point" LPP problem to the variational problem (3.1.4) with $|u| \leq M$. The second step of the proof consists in showing that the original LPP is localized, which is obtained by obtaining a bound on the probability that the maximizer of the LPP is not localized on a $\mathcal{O}(Mt^{2/3})$ region. In particular, for the flat initial condition case, we obtain a Gaussian bound in M, see Lemma 3.4.3 (for an analogue bound on the limit process, see Proposition 4.4 of [CH13]).

The strategy to prove the convergence for the restricted was first developed by Corwin, Liu and Wang in [CLW16]. In that paper, for generic initial conditions (possibly random) they obtained universal results showing that the distribution converges to a variational problem (which depends on how the initial condition scales under diffusive scaling), for cases which are macroscopically at density 1/2. In the continuous time setting, this was studied in [CFS18]. In particular, if the initial condition "scales subdiffusively", then for $\rho = 1/2$ one still sees F_{GOE} fluctuations. This fact was predicted in the context of the KPZ equation in [QR19].

The main technical novelty of our proof concerns the localization. In particular, unlike in [CLW16, CFS18], we do not require any extra input from solvable models beyond the ones which are used to prove convergence in the restricted LPP problem. All we need is a good control on the point-to-point process along a horizontal line. The key idea is to bound the increment of the process by the ones of two stationary initial conditions, with densities slightly higher/lower than ρ, which are chosen such that the inequality holds on a set of high probability. This probability is given in terms of some exit point probabilities. This comparison was used first by Cator and Pimentel in [CP15] (see also [Pim17b]) to show tightness for the Hammersley process and the point-to-point LPP along a characteristic direction with "speed" 0. In Lemma 3.2.5 we obtain much stronger exit point probabilities than in [Pim17b]. More importantly, we use the inequality in two ways: (a) to extend the tightness result to any characteristic direction (which is needed to the analysis any density ρ), and (b) to control the fluctuations of the process over large distances (of order $Mt^{2/3}$).

The control of the fluctuations over large distances is indeed a key ingredient to obtain the localization bound. This reduces the input from exactly solvable models with respect to [CLW16, CFS18]. In [CLW16] they introduced a non-intersecting line ensemble and the bound followed using its Gibbs-Brownian property in a smart way. In [CFS18] the bound was obtained using an explicit correlation kernel for the so-called "half-flat" initial condition. This approach allowed to simplify [CLW16], but it has the drawback that it is restricted to the case $\rho = 1/2$.

The main problem in analyzing directly $\rho \notin \{1/2, 1/3, 1/4, \ldots\}$ was that an explicit expression for the correlation kernel was not known. In the recent paper on KPZ fixed point by Matetski, Quastel and Remenik [MQR17] they found an explicit representation of it which could be used to obtain our result (and also the convergence to the Airy_1 process). However, the analysis has been made only for $\rho = 1/2$, since it was enough for answering the question on the KPZ fixed-point considered in the paper.

Although the method in this paper allows to get convergence only for the one-point distribution, its strategy could be used also for other models in the KPZ universality class. For instance, for the partially asymmetric simple exclusion process (PASEP), where an analogue of the work [MQR17] seems out of reach (an exact formula allowing the asymptotic analysis for PASEP even with $\rho = 1/2$ is not known, although heavy efforts have been made in particular by Ortmann, Quastel and Remenik [OQR16, OQR17]). On the other hand, ingredients like slow decorrelation hold also for PASEP using basic coupling [CFP12]. Furthermore, as shown in [Fer18], the mapping to LPP is actually not needed to analyze TASEP. This observation is relevant since for PASEP this mapping does not exist anymore. The main missing ingredient for an extension to PASEP is the convergence to the Airy_2 process for step initial condition. This is an open problem, but it looks easier than the analysis of PASEP with general densities ρ through exact formulas (compare with the formulas for $\rho = 1/2$ of [OQR16, OQR17]).

Outline. In Section 3.2 we define TASEP, LPP and present the main results. Section 3.3 contains the proof of tightness and the derivation of a bound needed to control localization as well. Finally, we prove the main theorem for LPP and TASEP in Section 3.4.

3.2 Main results

3.2.1 LPP and TASEP

A last passage percolation (LPP) model on $\mathbb{Z}^2$ with independent random variables $\omega_{i,j}$, $i,j \in \mathbb{Z}$ is the following. An *up-right path* π on $\mathbb{Z}^2$ from a point A to a point E is a sequence of points $(\pi(0), \pi(1), \ldots, \pi(n))$ in $\mathbb{Z}^2$ with $\pi(k+1) - \pi(k) \in \{(0,1),(1,0)\}$, with $\pi(0) = A$ and $\pi(n) = E$, and where n is called the length $\ell(\pi)$ of π. Now, given a set of points S_A and E, one defines the last passage time $L_{S_A \to E}$ as

$$L_{S_A\to E} = \max_{\substack{\pi:A\to E\\ A\in S_A}} \sum_{1\le k\le \ell(\pi)} \omega_{\pi(k)}. \tag{3.2.1}$$

Finally, we denote by $\pi^{\max}_{S_A\to E}$ any maximizer of the last passage time $L_{S_A\to E}$. For continuous random variables, the maximizer is a.s. unique.

TASEP is an interacting particle system on $\mathbb{Z}$ with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. For a configuration $\eta \in \Omega$, $\eta = (\eta_j, j\in\mathbb{Z})$, η_j is the occupation variable at site j, which is 1 if and only if j is occupied by a particle. TASEP has generator L given by [Lig99]

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1})\big(f(\eta^{j,j+1}) - f(\eta)\big), \tag{3.2.2}$$

where f are local functions (depending only on finitely many sites) and $\eta^{j,j+1}$ denotes the configuration η with the occupations at sites j and $j+1$ interchanged. Notice that for the TASEP the ordering of particles is preserved. That is, if initially one orders from right to left as

$$\ldots < x_2(0) < x_1(0) < 0 \le x_0(0) < x_{-1}(0) < \cdots,$$

then for all times $t \ge 0$ also $x_{n+1}(t) < x_n(t)$, $n\in\mathbb{Z}$.

TASEP can be also though as a growth process by introducing the height function $h(j,t)$ as

$$h(j,t) = \begin{cases} 2J(t) + \sum_{i=1}^{j}(1-2\eta_i(t)) & \text{for } j\ge 1,\\ 2J(t) & \text{for } j = 0,\\ 2J(t) - \sum_{i=j+1}^{0}(1-2\eta_i(t)) & \text{for } j \le -1,\end{cases} \tag{3.2.3}$$

for $j\in\mathbb{Z}$, $t\ge 0$, where $J(t)$ counts the number of jumps from site 0 to site 1 during the time-span $[0,t]$.

The connection between TASEP and LPP is as follows. Take $\omega_{i,j}$ to be the waiting time of particle j to jump from site $i-j-1$ to site $i-j$. Then $\omega_{i,j}$ are $\mathrm{Exp}(1)$ i. i. d. random variables. Further, setting the set $S_A = \{(u,k)\in\mathbb{Z}^2 : u = k + x_k(0), k\in\mathbb{Z}\}$, we have that

$$\mathbb{P}\left(L_{S_A\to(m,n)} \le t\right) = \mathbb{P}\left(x_n(t) \ge m-n\right) = \mathbb{P}\left(h(m-n,t)\ge m+n\right). \tag{3.2.4}$$

3.2.2 Universality for LPP

For any fixed $\rho\in(0,1)$, we consider the LPP model with S_A corresponding to TASEP with initial condition $x_k^{\rm flat}(0) = -\lfloor k/\rho\rfloor$, $k\in\mathbb{Z}$. We denote this initial set by

$$\mathcal{L}_\rho^{\rm flat} = \left\{\left(\lfloor \tfrac{\rho-1}{\rho}k\rfloor, k\right), k\in\mathbb{Z}\right\} \tag{3.2.5}$$

and we are interested in the LPP from $\mathcal{L}_\rho^{\rm flat}$ to $E_N(w)$ in the limit $N\to\infty$ illustrated in Figure 3.1. However, the approach used in the proof allows to consider more general (also random) initial conditions. Thus we consider TASEP with initial condition close to the flat initial condition with density ρ as well. Denote by

$$u_k = x_k(0) - x_k^{\rm flat}(0) \tag{3.2.6}$$

the deviation of the particle position with respect to the flat initial condition with density ρ. In this setting, in the LPP setting, we need to consider the initial set

$$\mathcal{L}_\rho = \left\{ \left(\lfloor \tfrac{\rho-1}{\rho} k \rfloor + u_k, k \right), k \in \mathbb{Z} \right\}. \tag{3.2.7}$$

We also denote

$$\chi = \rho(1-\rho). \tag{3.2.8}$$

Let

$$A^{\rm flat}(v) = \left(-2(1-\rho)\chi^{-1/3} v N^{2/3}, 2\rho\chi^{-1/3} v N^{2/3} \right) \tag{3.2.9}$$

and define by $A(v)$ the closest point on $\mathcal{L}_\rho$ to the characteristic line with direction $\mathbf{e}_\rho = ((1-\rho)^2, \rho^2)$ passing by $A^{\rm flat}(v)$. Then define $\lambda(v)$ by

$$A(v) = A^{\rm flat}(v) + \lambda(v)\mathbf{e}_\rho \tag{3.2.10}$$

To avoid that the randomness in the initial condition dominates the bulk ones, we assume **Assumption A:**

$$\lim_{N\to\infty} \frac{\lambda(v)}{\chi^{-2/3} N^{1/3}} = \mathcal{R}(v) = \sqrt{2}\sigma\mathcal{B}(v), \tag{3.2.11}$$

weakly on the space of continuous functions on bounded sets, where $\mathcal{B}$ is a two-sided Brownian motion and $\sigma \geq 0$ a coefficient. The stationary initial condition is $\sigma = 1$, while the flat initial condition is $\sigma = 0$.

Furthermore, we assume that globally the starting height function (or particle positions) are not deviating too much from the flat case, so that the maximization problem is non-trivially correlated only with the randomness in a $N^{2/3}$-neighbourhood of the origin. **Assumption B:** For any given $\delta > 0$ and $M > 0$, there exists a N_0 such that for all $N \geq N_0$,

$$\mathbb{P}(\lambda(v) \geq -\delta v^2 N^{1/3} \text{ for all } |v| \geq M) \geq 1 - Q(M), \quad \lim_{M\to\infty} Q(M) = 0, \tag{3.2.12}$$

where v are restricted to those such that $A(v)$ is connected to the end-point of the LPP by an up-right path.

These assumptions clearly holds for LPP corresponding to flat initial conditions, but also to the case where the deviation of the initial height function scales diffusively like in the stationary initial conditions. Under these assumptions we show the following universality result.

Theorem 3.2.1. *Let $\rho \in (0,1)$, $\chi = \rho(1-\rho)$. Set the end-point of the LPP as $E_N(w) = (m_N(w), n_N(w))$ with*

$$\begin{aligned} m_N(w) &= \tfrac{1-\rho}{\rho} N - 2w(1-\rho)\chi^{-1/3} N^{2/3}, \\ n_N(w) &= \tfrac{\rho}{1-\rho} N + 2w\rho\chi^{-1/3} N^{2/3}, \end{aligned} \tag{3.2.13}$$

Under Assumptions A and B, for any $s \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\left(L_{\mathcal{L}_\rho \to E_N(w)} \leq \frac{N}{\chi} + \frac{sN^{1/3}}{\chi^{2/3}} \right) = \mathbb{P}\left(\max_{v\in\mathbb{R}} \{\mathcal{A}_2(v) - (v-w)^2 + \mathcal{R}(v)\} \leq s \right). \tag{3.2.14}$$

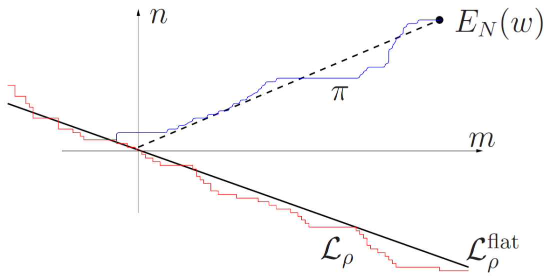

Figure 3.1: The last passage percolation setting considered in Theorem 3.2.1. The maximizer π from $\mathcal{L}_\rho$ (red) to $E_N(w)$ starts in a $\mathcal{O}(N^{2/3})$-neighbourhood of the origin. The straight thick line represents $\mathcal{L}_\rho^{\rm flat}$.

where $\mathcal{A}_2$ is the Airy$_2$ process [PS02b]. *In particular, for LPP from $\mathcal{L}_\rho^{\rm flat}$, for which $\mathcal{R}=0$, we have*

$$\lim_{N\to\infty} \mathbb{P}\left(L_{\mathcal{L}_\rho^{\rm flat}\to E_N(w)} \leq N/\chi + sN^{1/3}/\chi^{2/3}\right) = F_{\rm GOE}(2^{2/3}s), \tag{3.2.15}$$

where $F_{\rm GOE}$ is the GOE Tracy-Widom distribution function [TW96].

In [BLS12] the distribution of the position where the maximum of $\mathcal{A}_2(v)-v^2$ is attained has been derived. Due to the quadratic term it is localized and bounds can be found in [CH13, QR15]. These bounds can be compared with our Lemma 3.4.3, where we obtain a Gaussian bound in M of the probability that the maximizers is not in a main region of order $\mathcal{O}(MN^{2/3})$ (uniformly for all N large enough).

Remark 3.2.2. From the work on KPZ equation of Remenik and Quastel [QR19] it is conjectured that for KPZ growth models, if the initial configuration is flat with subdiffusive scaling, then the limiting distribution is the same as for the flat case (see Theorem 1.5 and subsequent remarks in [QR19]). In the LPP framework this corresponds to have $\mathcal{L}_\rho$ replaced by a (possibly random) down-right line, which at distance X from the origin has fluctuations at most $\mathcal{O}(|X|^\delta)$ for some $\delta<1/2$. Theorem 3.2.1 confirms it for general densities (since in that case $\mathcal{R}=0$); compare with [CFS18, CLW16] for the analogue result at $\rho=1/2$.

The proof of the main theorem (Theorem 3.2.1) is in his core probabilistic and it is based on the comparison of the LPP problem from a horizontal line to $E_N(w)$, where the line is around the region where the LPP from $\mathcal{L}_\rho$ to $E_N(w)$ is achieved. If we look the maximizers from the $E_N(w)$ position backwards, this is equivalent to consider the LPP from $(0,0)$ to a horizontal line crossing $(\gamma^2 n, n)$ for some $\gamma\in(0,\infty)$ with n proportional to N. Therefore consider the following LPP setting: for $i,j\geq 1$, let $\omega_{i,j}$ be i. i. d. Exp(1) random variables, $\omega_{i,j}=0$ for $i\leq 0$ or $j\leq 0$.

The estimate from law of large numbers for the LPP from the origin to (M,N) is given by $(\sqrt{M}+\sqrt{N})^2$ (as shown by Rost [Ros81] in the TASEP setting). Due to KPZ scaling we define the rescaled last passage time[1]

$$L_n^{\rm resc,h}(u) := \frac{L_{(0,0)\to(\gamma^2 n+\beta_1 u n^{2/3},n)} - n(1+\sqrt{\gamma^2+\beta_1 u n^{-1/3}})^2}{\beta_2 n^{1/3}}, \tag{3.2.16}$$

[1]Here and below we will not write the integer parts explicitly in the entries of the LPP.

where we set $\beta_1 = 2(1+\gamma)^{2/3}\gamma^{4/3}$ and $\beta_2 = (1+\gamma)^{4/3}\gamma^{-1/3}$. The coefficient β_2 is chosen to have the one-point distribution given by the GUE Tracy-Widom distribution [TW94], as shown by Johansson in Theorem 1.6 of [Joh00b]. The coefficient β_1 is chosen such that the limit process converges to the Airy$_2$ process [PS02b], $\mathcal{A}_2$. The finite-dimensional convergence to the Airy$_2$ process is a special case of [BF08, BP08, IS07]. Note that since

$$n(1+\sqrt{\gamma^2+\beta_1 u n^{-1/3}})^2 = (1+\gamma)^2 n + 2u(1+\gamma)^{5/3}\gamma^{1/3}n^{2/3} - \beta_2 u^2 n^{1/3} + \mathcal{O}(1) \quad (3.2.17)$$

we can replace in (3.2.16) also the approximation of the LLN until the order $n^{1/3}$ only without any relevant changes.

Theorem 3.2.3. *Fix any $M \in (0,\infty)$. Then, $u \mapsto L_n^{\rm resc}(u)$ is tight in the space of continuous functions on $[-M, M]$, $\mathcal{C}([-M, M])$.*

As a direct consequence of the convergence of finite-dimensional distributions and tightness we have:

Corollary 3.2.4. *For any given finite $M > 0$, $u \mapsto L_n^{\rm resc}(u)$ converges weakly to an Airy$_2$ process $u \mapsto \mathcal{A}_2(u)$ in $\mathcal{C}([-M, M])$.*

The next result which is in itself interesting is a bound of the exit point probability for the stationary situation, which can be achieved (see more details in Section 3.3.1) if we consider the LPP as before but with extra random variables if $i = 0$ or $j = 0$, namely with

$$\omega_{i,j} = \begin{cases} 0 & i=0, j=0, \\ \mathrm{Exp}(1-\rho) & i\geq 1, j=0, \\ \mathrm{Exp}(\rho) & i=0, j\geq 1, \\ \mathrm{Exp}(1) & i\geq 1, j\geq 1. \end{cases} \quad (3.2.18)$$

Here $\mathrm{Exp}(a)$ denotes exponential random variables with parameter a (thus average $1/a$). For the LPP with boundary conditions (3.2.18) we define the *exit point* as the last point of a path $\pi_{(0,0)\to(m,n)}$ on the x-axis or the y-axis. Since we need to distinguish whether the exit point is on the x- or on the y-axis, we introduce a random variable $Z^\rho(m,n) \in \mathbb{Z}$ such that, if $Z^\rho(m,n) > 0$, then the exit point is $(Z^\rho(m,n), 0)$, as if $Z^\rho(m,n) < 0$, then the exit point is $(0, -Z^\rho(m,n))$.

Lemma 3.2.5 (Exit point probability)**.** *Let $\kappa > 0$ be given and set*

$$\rho_\pm = \rho_0 \pm \kappa n^{-1/3} \text{ with } \rho_0 = \frac{1}{\gamma+1}. \quad (3.2.19)$$

Then there exists a n_0 such that for all $n \geq n_0$,

$$\begin{aligned} &\mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) > 0) \geq 1 - C\exp(-c\kappa^2), \\ &\mathbb{P}(Z^{\rho_-}(\gamma^2 n, n) < 0) \geq 1 - C\exp(-c\kappa^2), \end{aligned} \quad (3.2.20)$$

for some constants C, c independent of κ (and which can be taken uniform for γ in a bounded set).

A simple change of variables gives the following result.

Corollary 3.2.6. *In the settings of Lemma 3.2.5, for any given $M > 0$ and κ satisfying*

$$\tilde\kappa = \kappa - M\gamma^{1/3}(1+\gamma)^{-4/3} > 0 \quad (3.2.21)$$

it holds

$$\begin{aligned} &\mathbb{P}(Z^{\rho_+}(\gamma^2 n - \beta_1 M n^{2/3}, n) \geq 0) \geq 1 - C\exp(-c\tilde\kappa^2), \\ &\mathbb{P}(Z^{\rho_-}(\gamma^2 n + \beta_1 M n^{2/3}, n) \geq 0) \geq 1 - C\exp(-c\tilde\kappa^2). \end{aligned} \quad (3.2.22)$$

3.2.3 Universality for TASEP

The LPP with $\mathcal{L}_\rho^{\rm flat}$ as initial set corresponds to TASEP in continuous time with initial condition $x_k(0) = -\lfloor k/\rho \rfloor$, $k \in \mathbb{Z}$. We have the following universality result for the one-point fluctuations for TASEP with flat initial conditions for any density $\rho \in (0,1)$. For the more general initial condition, in terms of height Assumptions A and B rewrite as follows.

Assumption A:

$$\lim_{L\to\infty} \frac{h(2v\chi^{1/3}L^{2/3},0) - 2v(1-2\rho)\chi^{1/3}L^{2/3}}{2\chi^{2/3}L^{1/3}} = \mathcal{R}(v) = \sqrt{2}\sigma\mathcal{B}(v), \tag{3.2.23}$$

weakly on the space of continuous functions on bounded sets, where $\mathcal{B}$ is a two-sided Brownian motion and $\sigma \geq 0$ a coefficient. The stationary initial condition is $\sigma = 1$, while the flat initial condition is $\sigma = 0$.

Assumption B: For any given $\delta > 0$ and $M > 0$, there exists a L_0 such that for all $L \geq L_0$,

$$\mathbb{P}(h(2v\chi^{1/3}L^{2/3},0) - 2v(1-2\rho)\chi^{1/3}L^{2/3} \geq -\delta v^2 L^{1/3} \text{ for all } |v| \geq M) \geq 1 - Q(M), \tag{3.2.24}$$

with Q independent on L and $\lim_{M\to\infty} Q(M) = 0$.

Theorem 3.2.7. *Let $\rho \in (0,1)$ and set $\chi = \rho(1-\rho)$. Then, for any $s \in \mathbb{R}$,*

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{P}\left(h((1-2\rho)t + 2w\chi^{1/3}t^{2/3}, t) \geq (1-2\chi)t + 2w(1-2\rho)\chi^{1/3}t^{2/3} - 2s\chi^{2/3}t^{1/3}\right)\\ &= \mathbb{P}\left(\max_{v\in\mathbb{R}}\{\mathcal{A}_2(v) - (v-w)^2 + \mathcal{R}(v)\} \leq s\right).\end{aligned} \tag{3.2.25}$$

Proof. The first equality follows from (3.2.3). The rest is a direct consequence of Theorem 3.2.1 and the relation (3.2.4). □

The flat TASEP is the special case $\mathcal{R} = 0$ and the result is independent of w since the Airy$_2$ process is stationary. Thus we have proven the following result, which motivated the study of this paper.

Corollary 3.2.8. *Consider TASEP with flat initial condition and density $\rho \in (0,1)$, and set $\chi = \rho(1-\rho)$. Then, for any $s \in \mathbb{R}$,*

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}\left(J(t) \geq \chi t - s\chi^{2/3}t^{1/3}\right) &= \lim_{t\to\infty} \mathbb{P}\left(h((1-2\rho)t, t) \geq (1-2\chi)t - 2s\chi^{2/3}t^{1/3}\right)\\ &= \mathbb{P}\left(\max_{v\in\mathbb{R}}\{\mathcal{A}_2(v) - v^2\} \leq s\right) = F_{\rm GOE}(2^{2/3}s).\end{aligned} \tag{3.2.26}$$

3.3 Comparison with stationary LPP and proof of Theorem 3.2.3

In this section we will prove tightness of the process $L_n^{\rm resc,h}$. This mainly follows the approach of Cator and Pimentel [CP15]. The key observation in [CP15] is that the increments of the LPP with end-points on a horizontal line can be bounded by the increments of the LPP for the stationary case on the set of events where the "exit point" is on the right or the left of the origin. Then the idea is to consider stationary LPP with slightly higher/lower density so that the given exit point events are highly probable and at the same time the increments of the LPP are controlled by the ones in the stationary LPPs. In [CP15] the case of the Hammersley process was studied in details and it was stated the result for the exponential random variable along the diagonal only, i.e. $\gamma = 1$. The proof of the latter is left to the reader as it was mentioned that it is similar to the case of the Hammersley.

We have a few reasons to present the details for the result with generic densities:

(a) here we consider the space of continuous functions instead of the càdlàg functions and there are some minor twists which have to be taken into account for generic density $\rho \neq 1/2$;

(b) we get a much stronger bound for the exit point distributions with respect to [CP15] (see Lemma 3.2.5);

(c) we derive an estimate on the increments, which is not needed for proving tightness, but it is the key for the control of the probability that the maximizer of the LPP from $\mathcal{L}_\rho$ to $E_N(w)$ is localized: the derivation of this result is noticeably simplified with respect to the previous papers [CLW16] (they made use of a Brownian-Gibbs property) and [CFS18] (an ad-hoc comparison with half-line problem with slope -1 was used).

3.3.1 Stationary LPP and exit points

Let us now explain what we mean with stationary LPP with density $\rho \in (0,1)$ and report a result of Balázs, Cator and Seppäläinen [BCS06]. Consider the LPP as given by (3.2.18). We denote by $L^\rho(m,n)$ the last passage percolation from $(0,0)$ to (m,n) in this setting, while we use $L(m,n)$ for the last passage percolation from $(0,0)$ to (m,n) if we set $\omega_{i,0} = \omega_{0,j} = 0$.

The boundary conditions (3.2.18) correspond to a TASEP starting from the stationary Bernoulli(ρ) measure, conditioned on $\eta_0(0) = 0$ and $\eta_1(0) = 1$. Let $P_0(t)$ be the position at time t of the particle which started in 1 at time 0, and $H_0(t)$ be the position at time t of the hole which started in 0 at time 0. It was shown in Corollary 3.2 of [BCS06] (as a corollary of Burke's theorem [Bur56]) that $P_0(t) - 1$ and $-H_0(t)$ are two independent Poisson processes with jump rates $1-\rho$ and ρ. They extended the result to get independent increments also in the bulk of the system. The result we will use is the following:

Lemma 3.3.1 (Special case of Lemma 4.2 of [BCS06])**.** *Fix any $n \geq 1$. Then the increments*

$$\{L^\rho(m+1,n) - L^\rho(m,n), m \geq 1\} \tag{3.3.1}$$

are i. i. d. exponential random variables with parameter $1-\rho$.

With this definition we have the following lower and upper bounds in the increments of the process $m \mapsto L(m,n)$ that we want to study:

Lemma 3.3.2 (Lemma 1 of [CP15])**.** *Let $0 \leq m_1 \leq m_2$. Then if $Z^\rho(m_1,n) \geq 0$, it holds*

$$L(m_2,n) - L(m_1,n) \leq L^\rho(m_2,n) - L^\rho(m_1,n), \tag{3.3.2}$$

while, if $Z^\rho(m_2,n) \leq 0$, then we have

$$L(m_2,n) - L(m_1,n) \geq L^\rho(m_2,n) - L^\rho(m_1,n). \tag{3.3.3}$$

From the law of large numbers results one easily obtains that $Z^\rho(\gamma^2 n, n)$ is typically around 0 (it will fluctuates over a $n^{2/3}$ scale), if one chooses $\rho = 1/(\gamma+1)$. Therefore we set

$$\rho_\pm = \rho_0 \pm \kappa n^{-1/3} \text{ with } \rho_0 = \frac{1}{\gamma+1}. \tag{3.3.4}$$

The choice of $n^{-1/3}$ is due to the fact that the increments of the scaled process are just increased/decreased by a finite amount (proportional to κ), but on the other hand $\mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) > 0)$ and $\mathbb{P}(Z^{\rho_-}(\gamma^2 n, n) < 0)$ goes to 1 as $\kappa \to \infty$. The first step is to get an estimate on these probabilities.

3.3.2 Bounds on exit points

Now we want to derive a bound on $\mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) > 0)$ and on $\mathbb{P}(Z^{\rho_-}(\gamma^2 n, n) < 0)$. The last passage time L^ρ is the maximum between the last passage time from $(0,1)$ and the one from $(1,0)$, since any up-right path from $(0,0)$ has to go through one of these points. These LPP are denoted by

$$L^\rho_-(m,n) = L_{(0,0)\to(1,0)\to(m,n)}, \quad L^\rho_|(m,n) = L_{(0,0)\to(0,1)\to(m,n)}. \tag{3.3.5}$$

In terms of these two random variables, we have

$$\begin{aligned} \mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) > 0) &= \mathbb{P}\left(L^{\rho_+}_-(\gamma^2 n, n) > L^{\rho_+}_|(\gamma^2 n, n)\right), \\ \mathbb{P}(Z^{\rho_-}(\gamma^2 n, n) < 0) &= \mathbb{P}\left(L^{\rho_-}_|(\gamma^2 n, n) > L^{\rho_-}_-(\gamma^2 n, n)\right). \end{aligned} \tag{3.3.6}$$

Now we are ready to prove Lemma 3.2.5 and Corollary 3.2.6.

Proof of Lemma 3.2.5. By symmetry of the problem under the exchanges $\gamma \to 1/\gamma$ and $\rho \to 1-\rho$ it is enough to deal with the first estimate. We are going to prove that $\mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) < 0) \leq C\exp(-c\kappa^2)$.

First notice that for any $x \in \mathbb{R}$ we have

$$\begin{aligned} \mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) < 0) &= \mathbb{P}\left(L^{\rho_+}_-(\gamma^2 n, n) < L^{\rho_+}_|(\gamma^2 n, n)\right) \\ &\leq \mathbb{P}\left(L^{\rho_+}_-(\gamma^2 n, n) \leq x\right) + \mathbb{P}\left(L^{\rho_+}_|(\gamma^2 n, n) > x\right). \end{aligned} \tag{3.3.7}$$

Further, since for $\kappa > 0$ we have $\rho_+ > \rho_0$, and thus $\mathbb{E}(\omega_{0,i}) = 1/\rho_+ < 1/\rho_0$, implying

$$\mathbb{P}\left(L^{\rho_+}_|(\gamma^2 n, n) > x\right) \leq \mathbb{P}\left(L^{\rho_0}_|(\gamma^2 n, n) > x\right). \tag{3.3.8}$$

The bounds of Lemma 3.3.3 below with $x = (1+\gamma)^2 n + a\kappa^2\beta_2 n^{1/3}$ (where we can choose any value $a \in (0, (1+\gamma)^{8/3}\gamma^{-2/3})$) together with (3.3.7) and (3.3.8) give the desired result. □

Proof of Corollary 3.2.6. Setting $\tilde\gamma^2 n = \gamma^2 n \pm \beta_1 M n^{2/3}$ and $\frac{1}{1+\gamma} \pm \kappa n^{-1/3} = \frac{1}{1+\tilde\gamma} \pm \tilde\kappa n^{-1/3}$ we find the value of $\tilde\kappa$. Then the bound follows by Lemma 3.2.5. □

Lemma 3.3.3. *Let $x = (1+\gamma)^2 n + a\kappa^2\beta_2 n^{1/3}$ with $a \in (0, (1+\gamma)^{8/3}\gamma^{-2/3})$. Then, uniformly for n large enough, we have*

$$\begin{aligned} \mathbb{P}\left(L^{\rho_0}_|(\gamma^2 n, n) > x\right) &\leq Ce^{-c\kappa^2}, \\ \mathbb{P}\left(L^{\rho_+}_-(\gamma^2 n, n) \leq x\right) &\leq Ce^{-c\kappa^3}, \end{aligned} \tag{3.3.9}$$

for some κ-independent constants $C, c \in (0,\infty)$ (c is depending on a).

Proof. Denoting $L^{\rho_0,\text{resc}} := \frac{L_{|}^{\rho_0}(\gamma^2 n,n)-(1+\gamma)^2 n}{\beta_2 n^{1/3}}$, the first inequality becomes an estimate on $1-\mathbb{P}(L^{\rho_0,\text{resc}} \leq a\kappa^2)$. The distribution of $L^{\rho_0,\text{resc}}$ has been studied in [BBP06] in the framework of sample covariance matrices. One can use the connection of this LPP to a rank-one problem in sample covariance matrices (see Section 6 of [BBP06]) to recover the result. Let us explain how it goes.

From (62) of [BBP06] we have that

$$\mathbb{P}(L^{\rho_0,\text{resc}} \leq \xi) = \det\left(\mathbb{1} - K_n\right)_{L^2(\mathbb{R}_+)} \tag{3.3.10}$$

where K_n is a trace-class operator acting on $L^2(\mathbb{R}_+)$. The integral kernel of K_n can be expressed as

$$K_n(u,v) = \int_{\mathbb{R}_+} d\lambda H_n(u,\lambda) J_n(\lambda, v), \tag{3.3.11}$$

where $H_n(u,v) = \mathcal{H}(\xi+u+v)$ and $J_n(u,v) = \mathcal{J}(\xi+u+v)$ with $\mathcal{H}$, $\mathcal{J}$ given in (93)-(96) of [BBP06]. Using the triangular inequality and a standard inequality on Fredholm determinants (see e.g. Theorem 3.4 of [Sim00]) we have

$$\begin{aligned} |1-\det(\mathbb{1}-K_n)| &\leq |1-\det(1-K_\infty)| + |\det(1-K_\infty)-\det(1-K_n)| \\ &\leq (\|K_\infty\|_1 + \|K_\infty - K_n\|_1)\exp(\|K_\infty\|_1 + \|K_n\|_1 + 1). \end{aligned} \tag{3.3.12}$$

The limits of $\mathcal{H}$ and $\mathcal{J}$ are denoted by $\mathcal{H}_\infty$ and $\mathcal{J}_\infty$ and they are given in (120) and (122) of [BBP06]. For $k=1$ $\mathcal{H}_\infty(u) = e^{-\varepsilon u}\int_{\mathbb{R}_+}\text{Ai}(\xi+\lambda+u)d\lambda$ and $\mathcal{J}_\infty(u) = e^{\varepsilon u}\text{Ai}'(\xi+u)$ with $\varepsilon > 0$ being any small constant. Using triangular inequalities and the identity $\|AB\|_1 \leq \|A\|_{\text{HS}}\|B\|_{\text{HS}}$ (see e.g. Theorem VI.22 of [RS78]) we can bound each of the norms in (3.3.12) by a finite sum of product of two of the following Hilbert-Schmidt norms,

$$\|H_\infty\|_{\text{HS}}, \quad \|J_\infty\|_{\text{HS}}, \quad \|H_\infty - H_n\|_{\text{HS}}, \quad \|J_\infty - J_n\|_{\text{HS}}, \tag{3.3.13}$$

As a function of ξ, the latter two have exponential bounds (see Proposition 3.1 of [BBP06]) uniformly for n large enough, while the first two have (super-)exponential decay from the known asymptotics of the Airy functions (e.g., $|\text{Ai}(x)| \leq e^{-x}$ and $|\text{Ai}'(x)| \leq e^{-x}$, for all $x \in \mathbb{R}$).

To prove the second inequality, it is enough to have a bound on the probability for a lower bound for $L_{-}^{\rho_+}$. For any choice of $\xi_0 > 0$, we have

$$\begin{aligned} L_{-}^{\rho_+}(\gamma^2 n, n) &\geq L^{\rho_+}(\xi_0 n^{2/3},0) + L^{\rho_+}_{(\xi_0 n^{2/3},0)\to(\gamma^2 n,n)} \\ &\geq L^{\rho_+}(\xi_0 n^{2/3},0) + L_{(\xi_0 n^{2/3},0)\to(\gamma^2 n,n)}, \end{aligned} \tag{3.3.14}$$

where the L without ρ_+ means the LPP with all ω's to be Exp(1). Then

$$\mathbb{P}\left(L_{-}^{\rho_+}(\gamma^2 n,n) \leq x\right) \leq \mathbb{P}\left(L^{\rho_+}(\xi_0 n^{2/3},0) + L_{(\xi_0 n^{2/3},0)\to(\gamma^2 n,n)} \leq x\right). \tag{3.3.15}$$

Let us see what is a good choice for ξ_0. The estimate from the law of large numbers gives

$$L^{\rho_+}(\xi_0 n^{2/3},0) \simeq \xi_0 n^{2/3}/(1-\rho_+) = \tfrac{1+\gamma}{\gamma}\xi_0 n^{2/3} + \tfrac{(1+\gamma)^2}{\gamma^2}\xi_0\kappa n^{1/3} + O(1) \tag{3.3.16}$$

and

$$L_{(\xi_0 n^{2/3},0)\to(\gamma^2 n,n)} \simeq \left(\sqrt{n} + \sqrt{\gamma^2 n - \xi_0 n^{2/3}}\right)^2 = (1+\gamma)^2 n - \tfrac{1+\gamma}{\gamma}\xi_0 n^{2/3} - \frac{\xi_0^2}{4\gamma^3} n^{1/3} + O(1). \tag{3.3.17}$$

The sum of (3.3.16) and (3.3.17) (up to $O(n^{1/3})$) is maximal for $\xi_0 = 2\gamma(1+\gamma)^2\kappa$, which is the value that we choose. Let us define the rescaled LPP by

$$\begin{aligned} L_-^{\rm resc} &= \frac{L^{\rho_+}(\xi_0 n^{2/3},0) - \left(\frac{1+\gamma}{\gamma}\xi_0 n^{2/3} + \frac{(1+\gamma)^2}{\gamma^2}\xi_0\kappa n^{1/3}\right)}{n^{1/3}}, \\ L_{\rm bulk}^{\rm resc} &= \frac{L_{(\xi_0 n^{2/3},0)\to(\gamma^2 n,n)} - \left((1+\gamma)^2 n - \frac{1+\gamma}{\gamma}\xi_0 n^{2/3} - \frac{\xi_0^2}{4\gamma^3} n^{1/3}\right)}{n^{1/3}} \end{aligned} \tag{3.3.18}$$

Since $x = (1+\gamma)^2 n + a\kappa^2\beta_2 n^{1/3}$, we have that

$$(3.3.15) \leq \mathbb{P}\left(L_-^{\rm resc} + L_{\rm bulk}^{\rm resc} \leq -\tilde{s}\right) \leq \mathbb{P}\left(L_-^{\rm resc} \leq -\tilde{s}/2\right) + \mathbb{P}\left(L_{\rm bulk}^{\rm resc} \leq -\tilde{s}/2\right) \tag{3.3.19}$$

with $\tilde{s} = \left((1+\gamma)^4/\gamma - a\beta_2\right)\kappa^2$.

For any $a \in (0, (1+\gamma)^{8/3}\gamma^{-2/3})$ we have $\tilde{s} > 0$. Then, uniformly for n large enough, by Proposition A.1.1(c) we have[2]

$$\mathbb{P}\left(L_{\rm bulk}^{\rm resc} \leq -\tilde{s}/2\right) \leq C e^{-c\tilde{s}^{3/2}} = C e^{-\tilde{c}\kappa^3} \tag{3.3.20}$$

for some constants $C, c, \tilde{c} \in (0,\infty)$.

To bound the distribution of $L_-^{\rm resc}$, note that $L^{\rho_+}(\xi_0 n^{2/3}, 0)$ is a sum of $\lfloor \xi_0 n^{2/3} \rfloor$ i. i. d. random variables $\mathrm{Exp}(1-\rho_+)$. Let X_i i. i. d. $\mathrm{Exp}(1-\rho_+)$ random variables. Consider the centered random variables $Y_i = 1/(1-\rho_+) - X_i$. Set $\hat{s} = \tilde{s} n^{1/3}/2$ and $N = \lfloor \xi_0 n^{2/3} \rfloor$. Then by the exponential Tchebishev inequality,

$$\mathbb{P}\left(L_-^{\rm resc} \leq -\tilde{s}/2\right) = \mathbb{P}\left(\sum_{i=1}^N Y_i \geq \hat{s}\right) \leq \inf_{t\geq 0} e^{-\hat{s}t}\left(\mathbb{E}\left(e^{tY_1}\right)\right)^N. \tag{3.3.21}$$

We have $\mathbb{E}\left(e^{tY_1}\right) = e^{t/(1-\rho_+)}/(1 + t/(1-\rho_+))$ and thus (3.3.21) $\leq \exp(\inf_{t\geq 0} I(t))$ with $I(t) = Nt/(1-\rho_+) + N\ln((1-\rho_+)/(t+1-\rho_+)) - \hat{s}t$. A simple computation gives

$$\begin{aligned} \inf_{t\geq 0} I(t) &= \hat{s}(1-\rho_+) + N\ln(1 - \hat{s}(1-\rho_+)/N) \\ &= -\frac{\tilde{s}^2\gamma^2}{8\xi_0(1+\gamma)^2} + \mathcal{O}(n^{-1/3}) \leq -\hat{c}\kappa^3, \end{aligned} \tag{3.3.22}$$

for some constant $\hat{c}$ (which can be taken independent on $n \geq n_0$, n_0 large enough), since $\xi_0 \sim \kappa$ and $\tilde{s} \sim \kappa^2$ as well. □

3.3.3 Tightness

Now we prove tightness of the rescaled process $L_n^{\rm resc,h}$ (see (3.2.16)). Following the ideas in [CP15] we prove it using the bounds of Lemma 3.3.2 together with the estimates of Lemma 3.2.5 and of the fluctuations of sums of i. i. d. random variables.

First let us see what Lemma 3.3.2 becomes for the rescaled processes. This bounds will be used to show tightness, but also to control the fluctuations beyond the central region of the maximisation problem (see Lemma 3.4.3). Let us shortly recall the scaling (3.2.16)

[2] The constant c is not the same as in Proposition A.1.1(c), due to the 1/2 term and the fact that $L_{\rm bulk}^{\rm resc}$ converges to a GUE Tracy-Widom distribution once divided by β_2.

under which $L_n^{\rm resc,h}$ converges in the sense of *finite-dimensional distributions* [BF08, BP08, IS07] to the Airy$_2$ process, $\mathcal{A}_2$,

$$L_n^{\rm resc,h}(u) := \frac{L_{(0,0)\to(\gamma^2 n+\beta_1 u n^{2/3},n)} - \big((1+\gamma)^2 n + 2u(1+\gamma)^{5/3}\gamma^{1/3}n^{2/3} - \beta_2 u^2 n^{1/3}\big)}{\beta_2 n^{1/3}}, \tag{3.3.23}$$

with $\beta_1 = 2(1+\gamma)^{2/3}\gamma^{4/3}$ and $\beta_2 = (1+\gamma)^{4/3}\gamma^{-1/3}$.

Lemma 3.3.4. *Let us define*

$$B_n^{\rho_\pm}(u) := \frac{L^{\rho_\pm}(\gamma^2 n+\beta_1 u n^{2/3},n) - (L^{\rho_\pm}(\gamma^2 n,n) + \frac{1}{1-\rho_\pm}\beta_1 u n^{2/3})}{\beta_2 n^{1/3}}. \tag{3.3.24}$$

For any fixed constants M_1, M_2, consider any two points satisfying $-M_1 \le v \le u \le M_2$. Then we have:
(a) If $Z^{\rho_+}(\gamma^2 n - \beta_1 M_1 n^{2/3}, n) \ge 0$, then

$$L_n^{\rm resc,h}(u) - L_n^{\rm resc,h}(v) \le B_n^{\rho_+}(u) - B_n^{\rho_+}(v) + (u^2-v^2) + 2\beta_2\kappa(u-v) + \mathcal{O}(n^{-1/3}). \tag{3.3.25}$$

(b) If $Z^{\rho_-}(\gamma^2 n + \beta_1 M_2 n^{2/3}, n) \le 0$, then

$$L_n^{\rm resc,h}(u) - L_n^{\rm resc,h}(v) \ge B_n^{\rho_-}(u) - B_n^{\rho_-}(v) + (u^2-v^2) - 2\beta_2\kappa(u-v) + \mathcal{O}(n^{-1/3}). \tag{3.3.26}$$

Here $\mathcal{O}(n^{-1/3})$ is uniformly for κ and γ in bounded sets of $(0,\infty)$.

Proof. We wrote the conditions on the left-most and right-most point, since by monotonicity they imply the conditions needed to apply Lemma 3.3.2 for the full interval $[-M_1, M_2]$. By Lemma 3.3.2 and the definition of the scalings (3.3.23) and (3.3.24) we have

$$\begin{aligned} L_n^{\rm resc,h}(u) - L_n^{\rm resc,h}(v) &\le B_n^{\rho_+}(u) - B_n^{\rho_+}(v) + (u^2-v^2) \\ &+ \left(\frac{\beta_1}{1-\rho_+} - 2(1+\gamma)^{5/3}\gamma^{1/3}\right)\frac{(u-v)}{\beta_2}n^{1/3}. \end{aligned} \tag{3.3.27}$$

Using the explicit expressions for β_1, β_2, and ρ_+ we get (3.3.25).

Similarly, we have

$$\begin{aligned} L_n^{\rm resc,h}(u) - L_n^{\rm resc,h}(v) &\ge B_n^{\rho_-}(u) - B_n^{\rho_-}(v) + (u^2-v^2) \\ &+ \left(\frac{\beta_1}{1-\rho_-} - 2(1+\gamma)^{5/3}\gamma^{1/3}\right)\frac{(u-v)}{\beta_2}n^{1/3}, \end{aligned} \tag{3.3.28}$$

giving (3.3.26). □

Let us denote the modulus of continuity for the rescaled process $L_n^{\rm resc,h}$ in the interval $[-M, M]$ by $\varpi_n(\delta)$:

$$\varpi_n(\delta) = \sup_{\substack{|u|,|v|\le M\\ |u-v|\le\delta}} |L_n^{\rm resc,h}(u) - L_n^{\rm resc,h}(v)|. \tag{3.3.29}$$

Proof of Theorem 3.2.3. First of all, notice that the random variable $L_n^{\rm resc,h}(0)$ is tight, see the upper and lower tail estimates in Proposition A.1.1. Thus to show tightness it remains to control the modulus of continuity, namely we need to prove that for any $\varepsilon, \tilde{\varepsilon} > 0$, there exists a $\delta > 0$ and a n_0 such that

$$\mathbb{P}(\varpi_n(\delta) \geq \varepsilon) \leq \tilde{\varepsilon}, \tag{3.3.30}$$

for all $n \geq n_0$.

For any $\varepsilon > 0$, for n large enough, by Lemma 3.2.5 it holds

$$\mathbb{P}(\varpi_n(\delta) \geq \varepsilon) \leq 2Ce^{-c\kappa^2} + \mathbb{P}(\{\varpi_n(\delta) \geq \varepsilon\} \cap \{Z_M^{\rho_+} > 0\} \cap \{Z_M^{\rho_-} < 0\}), \tag{3.3.31}$$

where we shorten $Z_M^{\rho_+} = Z^{\rho_+}(\gamma^2 n - \beta_1 M n^{2/3}, n)$ and $Z_M^{\rho_-} = Z^{\rho_-}(\gamma^2 n + \beta_1 M n^{2/3}, n)$. From Lemma 3.3.4, for $|u|, |v| \leq M$ and $|u - v| \leq \delta$, if we choose n large enough so that the $\mathcal{O}(n^{-1/3})$ are smaller than δ, then on the set $\{Z_M^{\rho_+} > 0\} \cap \{Z_M^{\rho_-} < 0\}$ we have

$$|L_n^{\text{resc}}(u) - L_n^{\text{resc}}(v)| \leq |B_n^{\rho_+}(u) - B_n^{\rho_+}(v)| + |B_n^{\rho_-}(u) - B_n^{\rho_-}(v)| + K(\delta, M, \kappa) \tag{3.3.32}$$

with $K(\delta, M, \kappa) = (2M+1+2\beta_2\kappa)\delta$. Now choose δ small enough so that $K(\delta, M, \kappa) < \varepsilon/2$. Then, for all n large enough,

$$\begin{aligned} &\mathbb{P}(\{\varpi_n(\delta) \geq \varepsilon\} \cap \{Z_M^{\rho_+} > 0\} \cap \{Z_M^{\rho_-} < 0\}) \\ &\leq \mathbb{P}\Big(\sup_{\substack{|u|,|v| \leq M \\ |u-v| \leq \delta}} |B_n^{\rho_+}(u) - B_n^{\rho_+}(v)| \geq \varepsilon/4 \Big) \\ &+ \mathbb{P}\Big(\sup_{\substack{|u|,|v| \leq M \\ |u-v| \leq \delta}} |B_n^{\rho_-}(u) - B_n^{\rho_-}(v)| \geq \varepsilon/4 \Big). \end{aligned} \tag{3.3.33}$$

Dividing the interval $[-M, M]$ into pieces of length δ and using stationarity of the increments of $B^{\rho_\pm}$ (and $B^{\rho_\pm}(0) = 0$) we readily have

$$\mathbb{P}\Big(\sup_{\substack{|u|,|v| \leq M \\ |u-v| \leq \delta}} |B_n^{\rho_\pm}(u) - B_n^{\rho_\pm}(v)| \geq \varepsilon/4 \Big) \leq \frac{2M}{\delta} \mathbb{P}\Big(\sup_{0 \leq u \leq \delta} |B_n^{\rho_\pm}(u)| \geq \varepsilon/12 \Big), \tag{3.3.34}$$

compare e.g. with sentence around (5.60) in [Joh03]. A short computation and the use of Donsker's invariance principle theorem imply that the processes $u \mapsto B_n^{\rho_\pm}(u)$ converges weakly in $\mathcal{C}([-M, M])$ to $u \mapsto \sigma\mathcal{B}(u)$, where $\mathcal{B}$ is a standard Brownian motion and $\sigma = \sigma(\gamma) = \sqrt{2\gamma/(1+\gamma)}$. This implies that for n large enough,

$$\text{r.h.s. of (3.3.34)} \leq \frac{8M}{\delta} \mathbb{P}\Big(\sup_{0 \leq u \leq \delta} |\mathcal{B}(u)| \geq \varepsilon/12 \Big) \leq \frac{8M}{\delta} \exp\left(-\frac{\varepsilon^2}{288\,\delta\sigma^2} \right), \tag{3.3.35}$$

where we use the bound $\mathbb{P}\left(\sup_{t \in [0,T]} |\mathcal{B}(t)| > \lambda \right) \leq e^{-\lambda^2/2T}$.

To resume, we have obtained that for any $\varepsilon > 0$ and n large enough, it holds for $\tilde{\kappa} = \kappa - M\gamma^{1/3}(1+\gamma)^{-4/3} > 0$,

$$\mathbb{P}(\varpi_n(\delta) \geq \varepsilon) \leq 2Ce^{-c\tilde{\kappa}^2} + \frac{8M}{\delta} \exp\left(-\frac{\varepsilon^2}{288\,\delta\sigma^2} \right). \tag{3.3.36}$$

For any fixed $\tilde{\varepsilon} > 0$, we choose κ large enough such that $2Ce^{-c\tilde{\kappa}^2} \leq \tilde{\varepsilon}/2$ and then δ small enough such that $\frac{8M}{\delta} \exp(-\varepsilon^2/(288\,\delta\sigma^2)) \leq \tilde{\varepsilon}/2$ for any n large enough. This proves (3.3.30). $\square$

3.4 Proof of Theorem 3.2.1

In this section we prove the main theorem of LPP. The proof consists in showing that the LPP converges to a variational process. One essentially shows that (a) the LPP from $\mathcal{L}_\rho$ to $E_N(w)$ is with high probability the same as the LPP from a subset of $\mathcal{L}_\rho$ of size $\mathcal{O}(MN^{2/3})$, and (b) that in that region the LPP converges to the variational process of the theorem restricted to $|u| \leq M$. The most important novelty of our proof, with respect to the works in [CLW16, CFS18], is part (a). In [CLW16] they first needed to prove a Brownian-Gibbs property for an associated non-intersecting line ensemble. In [CFS18] one bounded a Fredholm determinant of a half-line problem corresponding to density $\rho = 1/2$ for TASEP (and this approach can not be extended to the generic $\rho \in (0, 1)$ case).

Proof of Theorem 3.2.1. Let us recall that we study the LPP from $\mathcal{L}_\rho$ and $\mathcal{L}_\rho^{\text{flat}}$ to $E_N(w)$. From the law of large numbers of the point-to-point LPP, see Proposition A.1.1(a), by optimizing over the positions on $\mathcal{L}_\rho^{\text{flat}}$ we obtain that the maximizer starts around 0 (in a $\mathcal{O}(N^{2/3})$ neighbourhood). Remember the definition of the points $A^{\text{flat}}(v)$ and $A(v)$ given in (3.2.9) and (3.2.10). For a fixed $M > 0$, define the following LPP problems:

$$L_M = \max_{|v| \leq M} L_{A(v) \to E_N(w)} \quad \text{and} \quad L_{M^c} = \max_{|v| > M} L_{A(v) \to E_N(w)}. \tag{3.4.1}$$

According to (3.2.14) we need to determine the $N \to \infty$ limit of

$$\mathbb{P}\left(\max\{L_M, L_{M^c}\} \leq S(s)\right), \quad S(s) = N/\chi + s\chi^{-2/3}N^{1/3}. \tag{3.4.2}$$

For large M (as we will show) one expects that $L_M > L_{M^c}$ with high probability. Thus we define the events

$$R_M = \{L_{M^c} > S(s)\}, \quad G_M = \{L_M \leq S(s)\}. \tag{3.4.3}$$

With these definitions we have

$$(3.4.2) = \mathbb{P}\left(R_M^c \cap G_M\right) = \mathbb{P}\left(G_M\right) - \mathbb{P}\left(R_M \cap G_M\right). \tag{3.4.4}$$

In Lemma 3.4.3 we show that, $\mathbb{P}(R_M \cap G_M) \leq Ce^{-cM^2} + Q(M)$ uniformly in N, where the function Q is the one in Assumption B. This implies that

$$\lim_{M\to\infty} \lim_{N\to\infty} \mathbb{P}\left(R_M \cap G_M\right) = 0. \tag{3.4.5}$$

Thus it remains to determine $\lim_{M\to\infty} \lim_{N\to\infty} \mathbb{P}(G_M)$.

The limit is obtained by first considering the last passage percolation problem from points on the horizontal line crossing $(0, 0)$, see Figure 3.2, for which the finite-dimensional distribution is known, and then using the functional slow decorrelation result of Theorem 3.4.1 we transport the fluctuations to the line $\mathcal{L}_\rho$. We define

$$\widetilde{A}(v) = (-\alpha_1 v N^{2/3}, 0), \quad \alpha_1 = 2\frac{(1-\rho)^{2/3}}{\rho^{4/3}}, \tag{3.4.6}$$

and

$$\widetilde{G}_M = \Big\{ \max_{|v|\leq M} L_{\widetilde{A}(v) \to E_N(w)} - \alpha_2 v N^{2/3} \leq S(s) \Big\}, \quad \alpha_2 = \frac{2}{\rho^{4/3}(1-\rho)^{1/3}}. \tag{3.4.7}$$

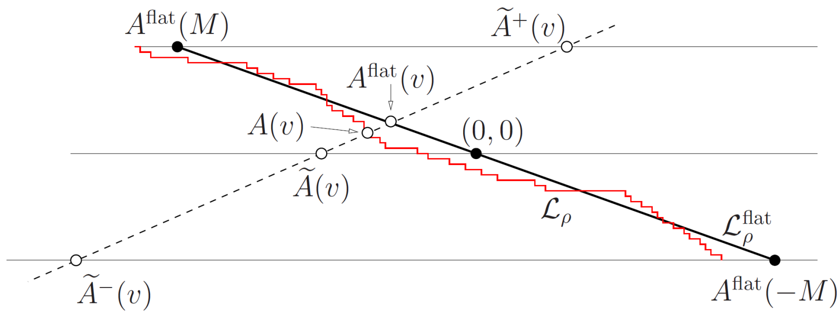

Figure 3.2: Zoom of the LPP around the line relevant region of $\mathcal{L}_\rho$ (red line) where the maximizers starts. For a given v, $\widetilde{A}^{\pm}(v)$, $\widetilde{A}(v)$, and $A(v)$ are on the same line, the line parallel to $\overline{(0,0),E_N(w)}$.

In [BP08] it is shown[3] the *convergence of finite dimensional distributions* of the rescaled process:

$$\widetilde{L}_N^{\mathrm{resc}}(v) := \frac{L_{\widetilde{A}(v)\to E_N(w)} - (N/\chi + \alpha_2 v N^{2/3})}{\chi^{-2/3}N^{1/3}} \to \mathcal{A}_2(v) - (v-w)^2 \tag{3.4.8}$$

as $N \to \infty$, with $\mathcal{A}_2$ an Airy$_2$ process. In Theorem 3.2.3 we show that as a process $v \mapsto \widetilde{L}_N^{\mathrm{resc}}(v)$ is *tight* in the set of continuous functions with supremum norm, $\mathcal{C}([-M,M])$, extending the sense of convergence to the weak*-convergence.

The rescaled process we want to study is

$$L_N^{\mathrm{resc}}(v) := \frac{L_{A(v)\to E_N(w)} - N/\chi}{\chi^{-2/3}N^{1/3}}. \tag{3.4.9}$$

In terms of the rescaled process, we indeed have

$$\mathbb{P}(G_M) = \mathbb{P}\Big(\max_{|v|\leq M} L_N^{\mathrm{resc}}(v) \leq s\Big). \tag{3.4.10}$$

For any realization of initial condition, the random line $\mathcal{L}_\rho$ passes in a neighbourhood of the origin. Restricted to a $MN^{2/3}$-neighbourood of the origin, by Assumption A we have that the points on $\mathcal{L}_\rho$ are given by

$$A(v) = A^{\mathrm{flat}}(v) + \lambda(v)\mathbf{e}_\rho, \text{ with } \lambda(v) \simeq \chi^{-2/3}N^{1/3}\mathcal{R}(v) \tag{3.4.11}$$

as $N \to \infty$. Define the set

$$F_\varepsilon = \Big\{\max_{|v|\leq M} |L_N^{\mathrm{resc}}(v) - \widetilde{L}_N^{\mathrm{resc}}(v)| \leq \varepsilon\Big\}. \tag{3.4.12}$$

By Theorem 3.4.1, for any $\varepsilon > 0$, $\lim_{N\to\infty}\mathbb{P}(F_\varepsilon) = 1$. Thus, for any $\varepsilon > 0$,

$$\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{P}(G_M) = \lim_{M\to\infty}\lim_{N\to\infty}\mathbb{P}(G_M \cap F_\varepsilon). \tag{3.4.13}$$

[3]The convergence of finite dimensional distributions can be also obtained from the finite-dimensional distributions along other lines using slow decorrelation [CFP12, Fer08b]. For instance it can be obtained starting from the analogue result for the joint distributions of TASEP particle positions [BF08]; see [BFP10] for an application of this technique.

The centerings in $L_N^{\rm resc}(v)$ and $\widetilde{L}_N^{\rm resc}(v)$ are the law of large number approximation from $A^{\rm flat}(v)$ and $\widetilde{A}(v)$ respectively. Define $\mu(m,n) = (\sqrt{m}+\sqrt{n})^2$ (see Proposition A.1.1), then we define

$$\Delta_N(v) := \frac{\mu(E_N(w)-A(v)) - \mu(E_N(w)-A^{\rm flat}(v))}{\chi^{-2/3}N^{1/3}}. \tag{3.4.14}$$

Then

$$\mathbb{P}(G_M \cap F_\varepsilon) \leq \mathbb{P}\Big(\Big\{\max_{|v|\leq M}[\widetilde{L}_N^{\rm resc}(v) + \Delta_N(v)] \leq s+\varepsilon\Big\} \cap F_\varepsilon\Big). \tag{3.4.15}$$

A lower bound on $\mathbb{P}(G_M \cap F_\varepsilon)$ is obtained with $-\varepsilon$ instead of ε.

By Assumption A, $\lim_{N\to\infty}\Delta_N(v) = \mathcal{R}(v) = \sqrt{2}\sigma\mathcal{B}(v)$ weakly. Together with the weak convergence of (3.4.6), we obtain

$$\begin{aligned}\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{P}(G_M\cap F_\varepsilon) &\leq \lim_{M\to\infty}\mathbb{P}\Big(\max_{|v|\leq M}[\mathcal{A}_2(v)-(v-w)^2+\mathcal{R}(v)] \leq s+\varepsilon\Big)\\ &= \mathbb{P}\Big(\max_{v\in\mathbb{R}}[\mathcal{A}_2(v)-(v-w)^2+\mathcal{R}(v)] \leq s+\varepsilon\Big).\end{aligned} \tag{3.4.16}$$

The last inequality holds since both the maximum of the Airy$_2$ minus a parabola and of $\mathcal{R}(v)$ minus a parabola are tight. For the special case of flat initial condition, i.e., when $\mathcal{R}=0$,

$$\mathbb{P}\big(\max_{v\in\mathbb{R}}[\mathcal{A}_2(v)-(v-w)^2] \leq s\big) \stackrel{(d)}{=} \mathbb{P}\big(\max_{v\in\mathbb{R}}[\mathcal{A}_2(v)-v^2]\leq s\big) = F_{\rm GOE}(2^{2/3}s), \tag{3.4.17}$$

where we used the fact that the Airy$_2$ process is stationary, and the last equality was proven in [Joh03]. This ends the proof of Theorem 3.2.1. □

Theorem 3.4.1 (Functional slow decorrelation)**.** *Consider any down-right path $\mathcal{L}$ passing a.s. at a finite-distance from the origin. Let $\widetilde{A}(v)$ be as in (3.4.6) and let $B(v)$ be the closest point on $\mathcal{L}$ to the line from $\widetilde{A}(v)$ to $E_N(w)$. Consider the rescaled processes (defined for any $v\in\mathbb{R}$ through linear interpolation)*

$$L_N^{\rm resc,B}(v) := \frac{L_{B(v)\to E_N(w)} - \mu(E_N(w)-B(v))}{\chi^{-2/3}N^{1/3}}, \quad \mu(m,n) = (\sqrt{m}+\sqrt{n})^2 \tag{3.4.18}$$

as well as $\widetilde{L}_N^{\rm resc}$ given in (3.4.8). Then $L_N^{\rm resc,B} - \widetilde{L}_N^{\rm resc}$ converges in probability to 0 in $\mathcal{C}([-M,M])$ as $N\to\infty$. More precisely, for any $\varepsilon,\tilde\varepsilon>0$ there is a N_0 such that for all $N\geq N_0$,

$$\mathbb{P}\left(\max_{|v|\leq M}|L_N^{\rm resc,B}(v) - \widetilde{L}_N^{\rm resc}(v)| \geq \varepsilon\right) \leq \tilde\varepsilon. \tag{3.4.19}$$

Proof. The proof is almost identical to the one of Theorem 2.10 in [CFS18], see also Theorem 2.15 of [CLW16] (which is two pages long) and therefore we do not repeat it. Let us just mention the strategy and on the way the inputs which are needed. Using Theorem 3.2.3 one knows that the processes along the horizontal lines $\mathcal{L}^\pm$ crossing $A(\pm M)$ are tight. One defines the rescaled processes $\widetilde{L}_N^{\rm resc,\pm}(v)$ to be the analogues of $\widetilde{L}_N^{\rm resc}(v)$ but with starting points on $\mathcal{L}^\pm$, which we call $\widetilde{A}^\pm(v)$, see Figure 3.2. Using *tightness* of $\widetilde{L}_N^{\rm resc}$ (see Theorem 3.2.3) and *one-point slow decorrelation* (see Theorem 2.7.2) one bounds $\max_{|v|\leq M}|\widetilde{L}_N^{\rm resc,\pm}(v) - \widetilde{L}_N^{\rm resc}(v)|$. Finally one needs to control for example the increments of $\widetilde{L}_N^{\rm resc,+}(v) - L_N^{\rm resc}(v)$. For this one employs use of the subadditivity property of LPP, $L_{\widetilde{A}^+(v)\to E_N(w)} \geq L_{\widetilde{A}^+(v)\to A(v)} + L_{A(v)\to E_N(w)}$, and *the bound on the left tail* of $L_{\widetilde{A}^+(v)\to A(v)}$ provided in Proposition A.1.1. □

A direct consequence of tightness of $\widetilde{L}_N^{\rm resc}$ and the functional slow decorrelation result (Theorem 3.4.1) is the following.

Corollary 3.4.2. *Fix any $M \in (0,\infty)$. Then the rescaled LPP process from $\mathcal{L}_\rho$ to $E_N(w)$, $v \mapsto L_N^{\rm resc}(v)$ defined in (3.4.9), is tight in the space of continuous functions on $[-M,M]$, $\mathcal{C}([-M,M])$. It converges weakly to an Airy$_2$ process $u \mapsto \mathcal{A}_2(u)$.*

Lemma 3.4.3. *Define $G_M = \{\max_{|v|\leq M} L_{A(v)\to E_N(w)} \leq a_0 N + a_1 s N^{1/3}\}$ and $R_M = \{\max_{|v|>M} L_{A(v)\to E_N(w)} > a_0 N + a_1 s N^{1/3}\}$, with $a_0 = 1/\chi$ and $a_1 = 1/\chi^{2/3}$. Under Assumption B, there exists a finite M_0 such that for any given $M \geq M_0$,*

$$\mathbb{P}\left(G_M \cap R_M\right) \leq Ce^{-cM^2} + Q(M) \tag{3.4.20}$$

for some constants $C, c > 0$ which are uniform in N. In particular, for flat initial conditions (where $Q = 0$),

$$\mathbb{P}(\textit{the LPP maximizer starts from } A^{\rm flat}(v) \textit{ with } |v| \leq M) \geq 1 - 2Ce^{-cM^2}. \tag{3.4.21}$$

Proof. For $s \leq -\frac{1}{4}M^2$, we have

$$\begin{aligned}\mathbb{P}(G_M \cap R_M) &\leq \mathbb{P}(G_M) \leq \mathbb{P}(L_{(0,0)\to E_N(w)} \leq a_0 N + a_1 s N^{1/3}) \\ &\leq Ce^{-c|s|^{3/2}} \leq Ce^{-cM^2/8},\end{aligned} \tag{3.4.22}$$

where we used the lower tail estimate of the point-to-point LPP from Proposition A.1.1.

Thus we consider below any $s \geq -\frac{1}{4}M^2$. Let us define a set of points $\widehat{L}$ and we say that $\widehat{L} \prec \mathcal{L}_\rho$ if each point in $\mathcal{L}_\rho \cap \{A(v), |v| > M\}$ can be reached by an up-right paths from a point in $\widehat{L}$. Then

$$\begin{aligned}\mathbb{P}(G_M \cap R_M) &\leq \mathbb{P}(R_M) \leq \mathbb{P}\big(\max_{|v|>M} L_{A(v)\to E_N(w)} > a_0 N - \tfrac{1}{4} a_1 M^2 N^{1/3}\big) \\ &\leq \mathbb{P}(L_{\widehat{L}\to E_N(w)} > a_0 N - \tfrac{1}{4} a_1 M^2 N^{1/3}) + \mathbb{P}(\widehat{L} \not\prec \mathcal{L}_\rho).\end{aligned} \tag{3.4.23}$$

Our choice for $\widehat{L}$ will be such that $\mathbb{P}(\widehat{L} \not\prec \mathcal{L}_\rho) \leq Q(M)$ for all N large enough. To realize it, it is enough to take any $\widehat{L}$ such that it stays to the left of a parabola close enough to $\mathcal{L}_\rho^{\rm flat}$. In Figure 3.3 we illustrate $\widehat{L}$. For a $\delta > 0$, we define the points

$$\widehat{A}(v) = A^{\rm flat}(v) - \delta v^2 N^{1/3} \mathbf{e}_\rho, \quad \mathbf{e}_\rho = ((1-\rho)^2, \rho^2), \tag{3.4.24}$$

the segments $\mathcal{D}_k = \overline{\widehat{A}(kM)\widehat{A}((k+1)M)}$ and $\widetilde{\mathcal{D}}_\ell = \overline{\widehat{A}(-\ell M)\widehat{A}(-(\ell+1)M)}$, and the points $C_+ = (-(1+\frac{1-\rho}{16}), \frac{\rho}{1-\rho}(1-\frac{\rho}{16}))N$ and $C_- = (\frac{1-\rho}{\rho}(1-\frac{1-\rho}{16}), -(1-\frac{\rho}{16}))N$. Then, we define

$$\widehat{L} = C_+ \cup C_- \bigcup_{|v|\geq N^{\nu/3}} \widehat{A}(v) \bigcup_{k=1}^{N^{\nu/3}} \mathcal{D}_k \bigcup_{\ell=1}^{N^{\nu/3}} \widetilde{\mathcal{D}}_\ell, \tag{3.4.25}$$

with $\nu \in (0, 1/2)$ ($\nu < 1/2$ is needed only in the last estimate of this lemma), and the union $A(v)$ is for v up to the v such that $A(v)$ is reachable by an up-right path from C_+ or C_- (there are $\mathcal{O}(N^{1/3})$ of such v). The constant δ is now chosen small enough such that taking $v_+ = \frac{\chi^{1/3}}{2(1-\rho)} N^{1/3}$, which corresponds to $A^{\rm flat}(v_+) = (-N, \frac{\rho}{1-\rho}N)$, then $C_+ \prec \widehat{A}(v_+)$, and similarly for side close to C_-.

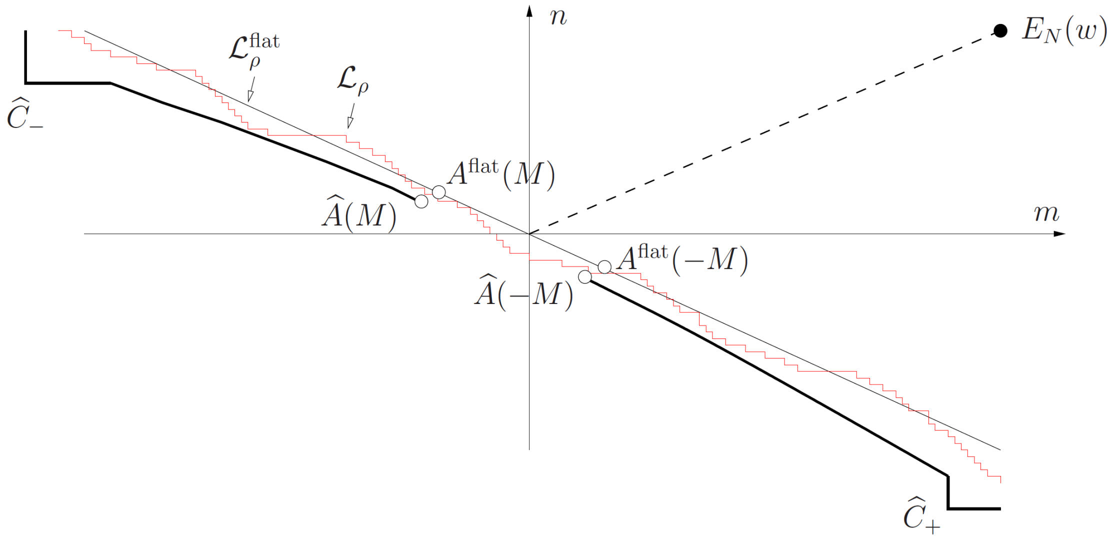

Figure 3.3: The setting used to control the LPP outside the central part. The thick black line is $\widehat{L}$.

With the $\widehat{L}$ defined as above, we can apply Assumption B to bound $\mathbb{P}(\widehat{L} \not\prec \mathcal{L}_\rho)$. It thus remains to get a bound for $\mathbb{P}(L_{\widehat{L}\to E_N(w)} > a_0 N - \frac{1}{4} a_1 M^2 N^{1/3})$. This can be bounded by

$$\begin{aligned}&\mathbb{P}(L_{C_+\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}) + \sum_{k=1}^{N^{\nu/3}} \mathbb{P}(L_{\mathcal{D}_k\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3})\\ &+ \mathbb{P}(L_{C_-\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}) + \sum_{\ell=1}^{N^{\nu/3}} \mathbb{P}(L_{\widetilde{\mathcal{D}}_\ell\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3})\\ &+ \sum_{N^{\nu/3}\leq |v|\leq \mathcal{O}(N^{1/3})} \mathbb{P}(L_{\widehat{A}(v)\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}).\end{aligned} \tag{3.4.26}$$

For the point-to-point estimates we can use the bounds of Proposition A.1.1, which are uniform for the slopes η in a bounded set of $(0,\infty)$. To avoid slopes which are close to 0 or ∞, we need to restrict the use of the point-to-point estimates for the LPP from $\widehat{A}(v)$ and add the LPP from the starting points $C_\pm$ as well.

1st bound. The points $C_\pm$ are chosen such that from the law of large numbers approximation of $L_{C_\pm\to E_N(w)}$ is less then $a_0 N - N/2$ for any $\rho \in (0,1)$. This means that a deviation of $-\frac{a_1 M^2}{4} N^{1/3}$ from $a_0 N$ of $L_{C_+\to E_N(w)}$ corresponds to look at the right tail at a value at least $N/2 - \mathcal{O}(M^2 N^{1/3})$. Thus for any given M, for all N large enough, Proposition A.1.1 implies

$$\mathbb{P}(L_{C_+\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}) \leq C e^{-cN^{2/3}} \tag{3.4.27}$$

for some constants C, c which depend only on ρ. Similarly one has the estimate for $\mathbb{P}(L_{C_-\to E_N(w)} > a_0 N - \frac{1}{4} a_1 M^2 N^{1/3})$.

2nd bound. In a similar way, using the bound of Proposition A.1.1, for any N large enough,

$$\mathbb{P}(L_{\widehat{A}(v)\to E_N(w)} > a_0 N - \tfrac{1}{4} a_1 M^2 N^{1/3}) \leq C e^{-cN^{2\nu/3}} \tag{3.4.28}$$

for any $v \in [N^{\nu/3}, \mathcal{O}(N^{1/3})]$, and thus

$$\sum_{N^{\nu/3} \leq |v| \leq \mathcal{O}(N^{1/3})} \mathbb{P}(L_{\widehat{A}(v) \to E_N(w)} > a_0 N - \tfrac{1}{4} a_1 M^2 N^{1/3}) \leq C N^{1/3} e^{-cN^{2\nu/3}} \leq C e^{-\frac{1}{2} c N^{2\nu/3}} \tag{3.4.29}$$

for $N \gg 1$.

3rd bound. Finally we need a bound for $\mathbb{P}(L_{\mathcal{D}_k \to E_N(w)} > a_0 N - \frac{1}{4} a_1 M^2 N^{1/3})$ uniform in N, which is summable in k and such that its sum is going to zero as $M \to \infty$. The bound for $\mathbb{P}(L_{\widetilde{\mathcal{D}}_\ell \to E_N(w)} > a_0 N - \frac{1}{4} a_1 M^2 N^{1/3})$ is completely analogue and thus we present in details only the first one.

For a given v, we define the point $\widehat{D}(v)$ such that its second coordinate equals the one of $\widehat{A}(kM)$ and the segment $\overline{\widehat{D}(v), \widehat{A}(v)}$ has direction $\mathbf{e}_\rho$. We have

$$\widehat{D}(v) = A^{\text{flat}}(v) - \theta \mathbf{e}_\rho, \quad \theta = \delta (kM)^2 N^{1/3} + \frac{2(v - kM) N^{2/3}}{\rho \chi^{1/3}}. \tag{3.4.30}$$

Then, for any $k \geq 1$ and M,

$$\begin{aligned} &\mathbb{P}\left(L_{\mathcal{D}_k \to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}\right) \leq \mathbb{P}\left(L_{\widehat{A}(kM) \to E_N(w)} > a_0 N - \tfrac{3 a_1 k^2 M^2}{4} N^{1/3}\right) \\ &+ \mathbb{P}\left(\max_{kM \leq v \leq (k+1)M} \{L_{\widehat{A}(v) \to E_N(w)} - L_{\widehat{D}(v) \to E_N(w)} + \beta N^{2/3}\} \geq \tfrac{a_1 k^2 M^2}{4} N^{1/3}\right) \\ &+ \mathbb{P}\left(\max_{kM \leq v \leq (k+1)M} \{L_{\widehat{D}(v) \to E_N(w)} - L_{\widehat{A}(kM) \to E_N(w)} - \beta N^{2/3}\} \geq \tfrac{a_1 k^2 M^2}{4} N^{1/3}\right), \end{aligned} \tag{3.4.31}$$

where $\beta = \frac{2(v-kM)}{\rho \chi^{1/3}} - \delta(v^2 - (kM)^2) N^{-1/3}$ (which is positive for all N large enough, since $v \in [kM, (k+1)M]$ with $k \in [1, \mathcal{O}(N^{\nu/3})]$).

Bound on first term of (3.4.31). The law of large numbers estimate of $L_{\widehat{A}(kM) \to E_N(w)}$ is $a_0 N + N^{1/3}(\delta (kM)^2 - a_1 (kM - w)^2)$. Thus for any $\delta < \chi^{2/3}/8$ and M large enough, we can use again the point-to-point estimate and obtain

$$\mathbb{P}\left(L_{\widehat{A}(kM) \to E_N(w)} > a_0 N - \tfrac{3}{4} a_1 k^2 M^2 N^{1/3}\right) \leq C e^{-ck^2 M^2/8}. \tag{3.4.32}$$

Bound on second term of (3.4.31). Using $L_{\widehat{D}(v) \to E_N(w)} \geq L_{\widehat{D}(v) \to \widehat{A}(v)} + L_{\widehat{A}(v) \to E_N(w)}$ we have

$$\begin{aligned} &\mathbb{P}\left(\max_{kM \leq v \leq (k+1)M} \{L_{\widehat{A}(v) \to E_N(w)} - L_{\widehat{D}(v) \to E_N(w)} + \beta N^{2/3}\} \geq \tfrac{a_1 k^2 M^2}{4} N^{1/3}\right) \\ &\leq \sum_{kM \leq v \leq (k+1)M} \mathbb{P}\left(L_{\widehat{A}(v) \to E_N(w)} - L_{\widehat{D}(v) \to E_N(w)} + \beta N^{2/3} \geq \tfrac{a_1 k^2 M^2}{4} N^{1/3}\right) \\ &\leq \sum_{kM \leq v \leq (k+1)M} \mathbb{P}\left(L_{\widehat{D}(v) \to \widehat{A}(v)} - \beta N^{2/3} \leq -\tfrac{a_1 k^2 M^2}{4} N^{1/3}\right). \end{aligned} \tag{3.4.33}$$

Since $L_{\widehat{D}(v) \to \widehat{A}(v)}$ centered by $\beta N^{2/3}$ and scaled by $\mathcal{O}(N^{2/9})$ converges to a F_{GUE} distributed random variable, by the lower tail estimate of Proposition A.1.1 we get

$$\mathbb{P}(L_{\widehat{D}(v) \to \widehat{A}(v)} - \beta N^{2/3} \leq -a_1 k M N^{1/3}) \leq C e^{-ck^2 M^2 N^{1/9}} \tag{3.4.34}$$

for some constants C, c which can be taken independent of $v \in [kM, (k+1)M]$. Since the sum in (3.4.33) is over a number of terms $\mathcal{O}(N^{2/3})$ we get

$$(3.4.33) \leq Ce^{-\frac{1}{2}ck^2M^2N^{1/9}} \tag{3.4.35}$$

for all N large enough.

Bound on third term of (3.4.31). For this bound we will employ, between other results, Lemma 3.3.4. Let us first reformulate what we need to prove in terms of $L_n^{\mathrm{resc,h}}$. One looks the picture from the point $E_N(w)$, which becomes the origin. The point $\widehat{A}(kM)$ as seen from $E_N(w)$ becomes the point $(\gamma^2 n, n)$ and the point $\widehat{D}(v)$ is $(\gamma^2 n + \beta_1 u(v) n^{2/3}, n)$. This means that we need to take

$$\begin{aligned}
n &= \frac{\rho}{1-\rho}N - \frac{2\rho(kM-w)}{\chi^{1/3}}N^{2/3} + \delta\rho^2(kM)^2N^{1/3},\\
\gamma &= \frac{1-\rho}{\rho}\Big(1 + \frac{kM-w}{\chi^{1/3}}N^{-1/3} + \frac{(kM-w)^2(3-4\rho)}{2\chi^{2/3}}N^{-2/3} + \mathcal{O}(N^{-1})\Big),\\
u(v) &= (v-kM)(1+\mathcal{O}(N^{-2/3})).
\end{aligned} \tag{3.4.36}$$

We have, in distribution,

$$L_{\widehat{D}(v)\to E_N(w)} \stackrel{d}{=} L_{(0,0)\to(\gamma^2 n + \beta_1 u(v) n^{2/3}, n)}. \tag{3.4.37}$$

Recall that $\widehat{D}(kM) = \widehat{A}(kM)$. Furthermore, the difference between the laws of large numbers of $L_{\widehat{D}(v)\to E_N(w)}$ and $L_{\widehat{A}(kM)\to E_N(w)}$ is given by

$$\begin{aligned}
&\beta N^{2/3} - \chi^{-2/3}N^{1/3}\Big[(v-kM)^2(1+\delta\chi^{2/3}) + (v-kM)(2w + 2kM(1+\delta\chi^{2/3}))\Big]\\
\leq&\beta N^{2/3} - \chi^{-2/3}N^{1/3}u(v)^2(1+\mathcal{O}(N^{-2/3})),
\end{aligned} \tag{3.4.38}$$

for all M large enough.

As a consequence, the third term of (3.4.31) can be rewritten as

$$\mathbb{P}\left(\max_{kM\leq v\leq (k+1)M}\{L_n^{\mathrm{resc,h}}(u(v)) - L_n^{\mathrm{resc,h}}(0) - u(v)^2 + \mathcal{O}(n^{-2/3})\} \geq \tfrac{1}{4}k^2M^2\right). \tag{3.4.39}$$

Applying the upper bound of Lemma 3.3.4 we obtain

$$\begin{aligned}
(3.4.39) \leq{}& \mathbb{P}(Z^{\rho+}(\gamma^2 n, n) < 0)\\
&+ \mathbb{P}\left(\max_{u\in I_M}\{B_n^{\rho+}(u(v)) + 2\beta_2\kappa u(v) + \mathcal{O}(n^{-2/3})\} \geq \tfrac{1}{4}k^2M^2\right),
\end{aligned} \tag{3.4.40}$$

where $I_M = [0, M(1+\mathcal{O}(n^{-2/3}))]$. With the choice $\kappa = \varepsilon_0 kM$ and, taking M large enough so that we get to use Lemma 3.2.5, we have

$$(3.4.40) = Ce^{-c\varepsilon_0^2k^2M^2} + \mathbb{P}\left(\max_{u\in I_M}\{B_n^{\rho+}(u) + 2\beta_2\kappa u + \mathcal{O}(n^{-2/3})\} \geq \tfrac{1}{4}k^2M^2\right) \tag{3.4.41}$$

We choose ε_0 small enough such that for any $M, k \geq 1$, $\max_{u\in I_M} 2\beta_2\kappa u + \mathcal{O}(n^{-2/3})$ is bounded by $\frac{1}{8}k^2M^2$ (uniformly for large n). Then

$$(3.4.41) \leq Ce^{-c\varepsilon_0^2k^2M^2} + \mathbb{P}\left(\max_{u\in I_M} B_n^{\rho+}(u) \geq \tfrac{1}{8}k^2M^2\right). \tag{3.4.42}$$

In the stationary setting, recall that we defined $\rho_0 = \rho_0(\gamma) := 1/(1+\gamma)$. By stationarity

$$B_n^{\rho_+}(u) = \frac{1}{\beta_2 n^{1/3}} \sum_{m=1}^{\beta_1 u n^{2/3}} (X_m - (1-\rho_+)^{-1}), \tag{3.4.43}$$

where $X_1, X_2, \ldots$ are i. i. d. random variables $\mathrm{Exp}(1-\rho_+)$ with $\rho_+ = \rho_0 + \varepsilon_0 k M n^{-1/3}$. Denote by $Y_m = X_m - (1-\rho_+)^{-1}$. Then $T \mapsto Z_T = \sum_{m=1}^{T} Y_m$ is a martingale. Using the generic maximal inequality for martingale $\mathbb{P}(\max_{1\leq t\leq T} Z_t \geq S) \leq \frac{\mathbb{E}(f(Z_T))}{f(S))}$ with $f(x) = e^{\lambda x}$, $\lambda > 0$, we have

$$\mathbb{P}\left(\max_{u\in I_M} B_n^{\rho_+}(u) \geq \tfrac{1}{8}k^2M^2\right) \leq \min_{\lambda>0} \frac{(\mathbb{E}(e^{\lambda Y_1}))^T}{e^{\lambda S}} = e^{-S(1-\rho_+)+T\ln[1+(1-\rho_+)S/T]}, \tag{3.4.44}$$

with $S = \frac{1}{8}k^2M^2\beta_2 n^{1/3}$ and $T = \beta_1 u(M) n^{2/3} = 2M\beta_1 n^{2/3}(1+\mathcal{O}(n^{-1/3}))$. A computation then leads to

$$(3.4.44) = \exp\left(-\frac{k^4M^3}{512}(1+\mathcal{O}(k^2n^{-1/3}))\right). \tag{3.4.45}$$

Remember that the range of k is from 1 to $\mathcal{O}(n^{\nu/3})$. Thus the error term is in the worst case $\mathcal{O}(n^{(2\nu-1)/3})$. Therefore we can now set the value of ν to be any number in $(0, 1/2)$, e.g., $\nu = 1/3$. With this choice, for n large enough, the error term is not larger than 1 and thus for any k, M,

$$(3.4.45) \leq \exp(-ck^2M^2). \tag{3.4.46}$$

Summing up the estimates we have

$$\sum_{k\geq 1} \mathbb{P}\left(L_{\mathcal{D}_k\to E_N(w)} > a_0 N - \tfrac{a_1 M^2}{4} N^{1/3}\right) \leq \sum_{k\geq 1}\Big((3.4.32) + (3.4.35) + (3.4.46)\Big) \leq Ce^{-cM^2} \tag{3.4.47}$$

for all N large enough. Here the constants C, c are uniform in N and M.

Finally we need to prove (3.4.21). Notice that for flat initial condition we have $Q = 0$ and thus

$$\begin{aligned} &\mathbb{P}(\text{the LPP maximizer starts from } A^{\text{flat}}(v) \text{ with } |v| \leq M) \\ =&\mathbb{P}\Big(\max_{|v|\leq M} L_{A(v)\to E_N(w)} > \max_{|v|>M} L_{A(v)\to E_N(w)}\Big) \geq \mathbb{P}(G_M^c \cap R_M^c) \\ \geq& 1 - \mathbb{P}(G_M) - \mathbb{P}(R_M), \end{aligned} \tag{3.4.48}$$

for any choice of s. With the choice $s = -M^2/4$, the bounds obtained above lead to the claimed result. □

Chapter 4

Time-time covariance for last passage percolation with generic initial profile

This chapter is based on [FO19]. In this work we consider time correlation for KPZ growth in 1+1 dimensions in a neighborhood of a characteristic. We prove convergence of the covariance with droplet, flat and stationary initial profile. In particular, this provides a rigorous proof of the exact formula of the covariance for the stationary case obtained in [FS16]. Furthermore, we prove the universality of the first order correction when the two observation times are close and provide a rigorous bound of the error term. This result holds also for random initial profiles which are not necessarily stationary.

4.1 Introduction

Stochastic growth models in the Kardar-Parisi-Zhang (KPZ) universality class [KPZ86] on a one-dimensional substrate are described by a height function $h(x,t)$ with x denoting space and t time. The height function evolves microscopically according to a random and local dynamics, while on a macroscopic scale the evolution is a deterministic PDE and the limit shape is non-random. In particular, if the speed of growth as a function of the gradient of the interface is a strictly convex or concave function, then the model is in the KPZ universality class. One expects large time universality under an appropriate scaling limit.

By studying special models in the KPZ class, the law of the one-point fluctuations and of the spatial statistics are well-known. In particular, the fluctuations scales as $t^{1/3}$ and the correlation length as $t^{2/3}$ (see surveys and lecture notes [FS11, Cor12, QS15, BG12, Qua11, Fer10, Tak16])[1]. Furthermore, it is known that non-trivial correlations survive on the macroscopic time scale if one considers space-time points along characteristic lines of the PDE for the macroscopic evolution [Fer08a, CFP12]. This phenomenon is called slow-decorrelation and it indicates that non-trivial processes in a spatial $t^{2/3}$-neighbourhood of a characteristic and for macroscopic temporal scale is to be expected. The limit process depends on the initial condition, since this is already the case for the processes at a fixed time.

[1] This holds true around point with smooth limit shape. Around shocks there are some differences, see e.g. [FN15, FN17, FF94].

The study of the time-time process started much more recently. On the experimental and numerical simulation side observables like the persistence probability or the covariance of an appropriately rescaled height function have been studied [TS12, Tak13, TA16, Tak12]. On the analytic and rigorous side, the two-time joint distribution of the height function is known for special initial conditions: Johansson analyzed a model on full space [Joh16, Joh18], while Baik and Liu considered a model on a torus [BL16, BL19]. There are also non-rigorous works on the time-time covariance and on the upper tail of distributions using replica approach [ND17, ND18, NDT17]. For general (random) initial conditions exact formulas on the joint distributions are not yet available. Also, the analysis of the covariance starting from the available formulas [Joh18, BL19] seems to be a difficult task.

In [FS16] Ferrari and Spohn made some predictions for the behavior of the two-time covariance for three typical initial conditions based on a last passage percolation (LPP) picture. In particular, for the stationary case, an exact formula for the covariance of two points along a characteristic has been derived. Furthermore, the behavior when the macroscopic times were either close or far from each others were provided. However, the work is not mathematically rigorous since the exchange of the large time limit and maximum over sums of Airy processes as well as justification for convergence of the covariances are not provided. The work by Corwin, Liu and Wang [CLW16] showed the way to obtain a rigorous convergence of distribution in terms of the variational process used in [FS16], by lifting the finite-dimensional slow-decorrelation result of [Fer08a, CFP12] to a functional slow-decorrelation statement.

In this paper we consider a last passage percolation model, which can be also seen as a (version of the) polynuclear growth model. As initial condition we consider the three standard cases (called droplet, flat and stationary) as in [FS16], but we extend the study to random but not stationary initial profiles (see [CFS18] for a related model). In the first three cases by the method of [CLW16] (simplified in some aspects in [CFS18, FO18]) one knows that the limiting distributions of (rescaled) LPP times can be expressed as a variational problem in terms of some Airy processes. The first result proven in this paper is the convergence of the covariance of the LPP time to the covariance of the limiting processes, see Theorem 4.2.2. As a corollary, this provides a proof for the exact formula of the covariance for the stationary case of [FS16]. We actually extend the result by taking points not exactly on the characteristics, but in a $t^{2/3}$-neighbourhood of it.

Our second result concerns the behavior of the covariance when the two times are close to each other on a macroscopic scale. Physically we expect to see the signature of the stationary state as first approximation. This was noticed also in numerical experiments [Tak13]. This is proven in Theorem 4.2.5 for all the initial conditions considered. We also provide a rigorous error term, which is compatible with the experiments[2]. To obtain the result, we need to control the spatial process at fixed time on small scales. This is achieved by comparing with stationary cases on sets of high probability. The idea goes back to Cator and Pimentel [CP15] for the droplet case (extended to general case in [Pim17b]). The control on the high probability sets requires bounds on exit point probabilities, which has to be obtained for each initial profile. In particular, to achieve a good control in the error term, one can not use soft bounds as in [BCS06, Pim17b]. Finally, for droplet initial condition we derive a result also when times are far apart, see

[2]The next order correction is sensitive to the scaling used to define the process. For the scaling used in this paper the error term seems to be optimal. However, if one scales the random variables to have the same one-point distribution function, then experimentally the error term is smaller: instead of an error term with exponent 1^-, one gets an exponent $\min\{5/3, 2/3+\alpha\}$, where α is the exponent controlling the convergence of the variance of the height difference to that of the Baik-Rains distribution [Tak18].

Theorem 4.2.6.

Outline. In Section 4.2 we introduce the model, state some known limiting results necessary for the rest of the paper and provide the main results. In Section 4.3 we recall the stationary LPP and the comparison lemmas. In Section 4.4 we prove Theorem 4.2.2 on the convergence of the covariance. In Section 4.5 we prove Theorem 4.2.5 on the close time behaviour, while in Section 4.6 we sketch the proof of Theorem 4.2.6. The appendix contains several bounds on the one-point distribution or on increments, which are used in the proofs.

4.2 Model and results

4.2.1 LPP and polynuclear growth

Consider a collection of i.i.d. random variables $\omega_{i,j}, i, j \in \mathbb{Z}$ with exponential distribution of parameter one. An *up-right path* $\pi = (\pi(0), \pi(1), \ldots, \pi(n))$ on $\mathbb{Z}^2$ from a point A to a point E is a sequence of points in $\mathbb{Z}^2$ with $\pi(k+1) - \pi(k) \in \{(0,1),(1,0)\}$, with $\pi(0) = A$ and $\pi(n) = E$, and n is called the length $\ell(\pi)$ of π. Given a set of points S_A with some random variables (not necessarily independent) h^0 on S_A, but independent of the ω's, and given a point E, one defines the last passage time $L_{S_A \to E}$ as

$$L_{S_A \to E} = \max_{\substack{\pi: A \to E \\ A \in S_A}} \left(h^0(\pi(0)) + \sum_{1 \le k \le n} \omega_{\pi(k)} \right). \tag{4.2.1}$$

Also, for two points P, Q which are not on the initial set S_A, we define $L_{P \to Q}$ as above but without the term $h^0(\pi(0))$. $\pi^{max}_{S_A \to E}$ indicates the maximizer of the last passage time. For continuous random variables, the maximizer is a.s. unique[3].

LPP can be though as a stochastic growth model, a version of the polynuclear growth model, as follows. Let $S_A = \mathcal{L} := \{(i,j) \in \mathbb{Z}^2 \,|\, i + j = 0\}$ and let h^0 represents a height function at time $t = 0$. Then one defines the height function at time t by the relation

$$h(x,t) = L_{\mathcal{L} \to ((x+t)/2,(t-x)/2)} \tag{4.2.2}$$

for all $x - t$ being even numbers (and set $h(x,t) = L_{\mathcal{L} \to ((x+t-1)/2,(t-x-1)/2)}$ for $x - t$ odd). The dynamics of the height function is

$$h(x,t) = \max\{h(x-1,t-1), h(x,t-1), h(x+1,t-1)\} + \omega_{(x+t)/2,(t-x)/2} \tag{4.2.3}$$

with initial conditions $h(x,0) = h^0(x/2, -x/2)$ (here $\omega_{(x+t)/2,(t-x)/2} = 0$ if $x - t$ is odd).

We are interested in the scaling limit of the height function

$$(w,\tau) \mapsto \lim_{t \to \infty} \frac{h(w 2^{1/3} t^{2/3}, \tau t) - \tau t}{2^{2/3} t^{1/3}} \tag{4.2.4}$$

or, equivalently, setting $E = (\tau N, \tau N) + w(2N)^{2/3}(1,-1)$,

$$(w,\tau) \mapsto \lim_{N \to \infty} \frac{L_{S_A \to E} - 4\tau N}{2^{4/3} N^{1/3}}, \tag{4.2.5}$$

[3]The only exception will be if h^0 is not random, since then the maximizer is unique up to the initial point, which has weight 0 and thus it is irrelevant.

for different initial conditions[4]

1. *Droplet case.* In this case one sets $h^0 = 0$ and further set $\omega(i,j) = 0$ whenever $(i,j) \notin \mathbb{Z}_+^2$. In terms of LPP this is equivalent to take $S_A = (0,0)$ and $h^0 = 0$.
2. *Flat with zero-slope.* This means that we take $h^0 = 0$.
3. *Stationary with zero-slope.* Let $\{X_k, Y_k\}_{k\in\mathbb{Z}}$ be i.i.d. random variable Exp(1/2)-distributed. Then define

$$h^0(x,-x) = \begin{cases} \sum_{k=1}^{x}(X_k - Y_k), & \text{for } x \geq 1, \\ 0, & \text{for } x = 0, \\ -\sum_{k=x+1}^{0}(X_k - Y_k), & \text{for } x \leq -1. \end{cases} \tag{4.2.6}$$

4. *A family of random initial conditions.* We consider the case where for a given $\sigma \geq 0$, h^0 is given by (4.2.6) multiplied by σ. Clearly, the cases $\sigma = 0$ and $\sigma = 1$ correspond to the flat and to the stationary cases.

Remark 4.2.1. In the setting of TASEP, a random initial condition maps to a LPP starting from a random line. Due to functional slow-decorrelation, the weight h^0 should be taken to reflect the first order LPP from a point on the line to its projection onto the antidiagonal. Thus a-priori one could try to start with the random line used in [CFS18, FO18], but since in the scaling limit the result is identical to the one of our choice, we did not attempt to use this precise mapping.

Limiting variational formulas

For $0 < \tau \leq 1$, we set[5] $E_\tau = (\tau N, \tau N) + (2N)^{2/3} w_\tau (1,-1)$ and define the LPP and its limit as

$$L_N^\star(\tau) = \frac{L^\star_{S_A \to E_\tau} - 4\tau N}{2^{4/3} N^{1/3}}, \quad \chi^\star(\tau) := \lim_{N\to\infty} L_N^\star(\tau), \tag{4.2.7}$$

where the superscript $\star$ denotes the different configurations, point-to-point ($\bullet$), point-to-line ($\diagdown$), stationary ($\mathcal{B}$) and random (σ).

The convergence in distribution of the random variables $L_N^\star(\tau)$ are well-known. Recall that for LPP we have the identity

$$L^\star_{S_A\to E_1} = \max_{u\in\mathbb{R}} \{ L^\star_{S_A \to I(u)} + L_{I(u)\to E_1} \} \tag{4.2.8}$$

with

$$I(u) = (\tau N, \tau N) + u(2N)^{2/3}(1,-1). \tag{4.2.9}$$

Provided that the limit $N \to \infty$ and $\max_{u\in\mathbb{R}}$ can be exchanged (which is the case in all the cases considered here, see [CLW16, FO18, CFS18] for related works), the limiting processes can be written in terms of Airy processes as follows.

1. *Droplet case.* Let $\mathcal{A}_2$ and $\tilde{\mathcal{A}}_2$ be two independent Airy$_2$ processes. Then

$$\begin{aligned} \chi^\bullet(\tau) &= \tau^{1/3}\left[\widetilde{\mathcal{A}}_2(\tfrac{w_\tau}{\tau^{2/3}}) - \tfrac{w_\tau^2}{\tau^{4/3}}\right], \\ \chi^\bullet(1) &= \max_{u\in\mathbb{R}}\left\{\tau^{1/3}\left[\widetilde{\mathcal{A}}_2(\tfrac{u}{\tau^{2/3}}) - \tfrac{u^2}{\tau^{4/3}}\right] + (1-\tau)^{1/3}\left[\mathcal{A}_2(\tfrac{u-w_1}{(1-\tau)^{2/3}}) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\right]\right\}, \end{aligned} \tag{4.2.10}$$

[4]The choice of zero-slope is just for convenience as it avoids to introduce a further parameter in the scaling. However, the inputs used in the proofs are available for non-zero slopes as well.

[5]Throughout the paper we do not write explicitly integer parts.

The Airy$_2$ process has been discovered in a related polynuclear growth model setting [PS02b] (see [Joh03] for the case of geometric random variables, or [BP08] for a two-parameter generalization). Tightness in this setting was shown in [FO18], building on the approach of [CP15] (while for the geometric case tightness was shown already in [Joh03]).

2. *Flat case.* Let $\mathcal{A}_1$ be an Airy$_1$ process and $\mathcal{A}_2$ an Airy$_2$ process, independent of each other. Then

$$\begin{aligned}
&\chi^{\diagdown}(\tau) = (2\tau)^{1/3}\mathcal{A}_1(\tfrac{w_\tau}{(2\tau)^{2/3}}),\\
&\chi^{\diagdown}(1) = \max_{u\in\mathbb{R}}\left\{(2\tau)^{1/3}\mathcal{A}_1(\tfrac{u}{(2\tau)^{2/3}}) + (1-\tau)^{1/3}\left[\mathcal{A}_2(\tfrac{u-w_1}{(1-\tau)^{2/3}}) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\right]\right\}.
\end{aligned} \tag{4.2.11}$$

The Airy$_1$ process has been discovered in the framework of the totally asymmetric simple exclusion process [Sas05, BFPS07], equivalent to the LPP through slow-decorrelation [CFP10, CFP12, Fer08a].

3. *Stationary case.* Let $\mathcal{A}_2$ be an Airy$_2$ process and $\mathcal{A}_{\rm stat}$ an Airy$_{\rm stat}$ process, independent of each other. Then

$$\begin{aligned}
&\chi^{\mathcal{B}}(\tau) = \tau^{1/3}\mathcal{A}_{\rm stat}(\tfrac{w_\tau}{\tau^{2/3}}),\\
&\chi^{\mathcal{B}}(1) = \max_{u\in\mathbb{R}}\left\{\tau^{1/3}\mathcal{A}_{\rm stat}(\tfrac{u}{\tau^{2/3}}) + (1-\tau)^{1/3}\left[\mathcal{A}_2(\tfrac{u-w_1}{(1-\tau)^{2/3}}) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\right]\right\}.
\end{aligned} \tag{4.2.12}$$

The limit process Airy$_{\rm stat}$ (which, in spite of the name, is not stationary) was obtained in [BFP10].

4. *Random initial conditions.* For this case, the one-point distribution is given by the following expression[6]

$$\mathbb{P}(\chi^\sigma(1) \le s) = \mathbb{P}\left(\max_{u\in\mathbb{R}}\{\mathcal{A}_2(u) - u^2 + \sqrt{2}\sigma B(u)\} \le s\right), \tag{4.2.13}$$

where the Airy$_2$ process and the two-sided standard Brownian motion B are independent of each other. Furthermore, we could write formulas similar to the one of the first three cases in terms of *an* Airy sheet [MQR17]. However uniqueness in law of Airy sheet is so-far not proven [MQR17, Pim17a]. Therefore we state the convergence of the covariance to the covariance of its limit process only for the other cases. However, the proof could be adapted to the general σ as well, once uniqueness of the limit is established.

4.2.2 Main results

Convergence of the covariance

As our first result we give a rigorous proof of the convergence of the covariances.

Theorem 4.2.2. *We have*

$$\lim_{N\to\infty} \mathrm{Cov}\left(L_N^\star(\tau), L_N^\star(1)\right) = \mathrm{Cov}\left(\chi^\star(\tau), \chi^\star(1)\right), \tag{4.2.14}$$

for $\star \in \{\bullet, \diagdown, \mathcal{B}\}$.

[6] This was actually proven for the LPP model where instead of the random function on the antidiagonal one has a random line in [CFS18], see also [FO18] for general slope. These works were based on the approach in the geometric random variables case of [CLW16]. Adapting the proof of [FO18] to this setting to get the variational formula is straightforward (it is actually even slightly simpler).

Remark 4.2.3. The motivation of this paper is the study of the covariance. However, by inspecting the proof, one sees that one can generalize the proof to get convergence of any joint moments of $L_N^\star(\tau)$ and $L_N^\star(1)$ without the need to new ideas and bounds.

For the stationary process $\mathcal{A}_{\rm stat}(w) \stackrel{(d)}{=} \max_{v\in\mathbb{R}}\{\sqrt{2}B(v)+\mathcal{A}_2(v)-(v-w)^2\}$ where the Airy$_2$ process, $\mathcal{A}_2$, and the two-sided standard Brownian motion, $B(v)$, are independent [QR14]. We denote

$$F_w(s)=\mathbb{P}\Big(\max_{v\in\mathbb{R}}\{\sqrt{2}B(v)+\mathcal{A}_2(v)-(v-w)^2\}\leq s\Big) \tag{4.2.15}$$

and use the notation $\xi_{{\rm stat},w}$ for a random variable distributed according to F_w. Due to stationarity one has the property [PS02a, BR00] $\mathbb{E}(\mathcal{A}_{\rm stat}(w))=0$, which implies

$$\mathrm{Var}(\xi_{{\rm stat},\tilde w})=\mathbb{E}\Big(\max_{v\in\mathbb{R}}\{\sqrt{2}B(v)+\mathcal{A}_2(v)-(v-w)^2\}\leq s\Big)^2. \tag{4.2.16}$$

For the stationary case, an exact expression for the covariance has been obtained in [FS16] for τ in the entire interval [0,1], in the special case $w_\tau=w_1=0$. For general values of w_τ and w_1, we obtain

Corollary 4.2.4. *For the stationary LPP, the covariance of the limiting height function for all $\tau\in(0,1)$ can be expressed as*

$$\begin{aligned}\mathrm{Cov}\left(\chi^{\mathcal{B}}(\tau),\chi^{\mathcal{B}}(1)\right)=&\frac{\tau^{2/3}}{2}\mathrm{Var}\left(\xi_{{\rm stat},\tau^{-2/3}w_\tau}\right)+\frac{1}{2}\mathrm{Var}\left(\xi_{{\rm stat},w_1}\right)\\&-\frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{{\rm stat},(1-\tau)^{-2/3}(w_1-w_\tau)}\right).\end{aligned} \tag{4.2.17}$$

Universal behavior for $\tau\to 1$

In [FS16] there is a conjecture on the behaviour of the covariance of the limit process for $\tau\to1$ for the other initial profiles as well. Our second goal is to provide a proof of such statements together with a rigorous error bound. We also extend the result to all initial conditions 1-4. Recall that for any random variables X_1, X_2 it holds

$$\mathrm{Cov}\left(X_1,X_2\right)=\tfrac12\mathrm{Var}\left(X_1\right)+\tfrac12\mathrm{Var}\left(X_2\right)-\tfrac12\mathrm{Var}(X_2-X_1). \tag{4.2.18}$$

Theorem 4.2.5. *Let us scale $w_1=\tilde w_1(1-\tau)^{2/3}$ and $w_\tau=\tilde w_\tau(1-\tau)^{2/3}$. Then as $\tau\to1$ we have*[7]

$$\mathrm{Var}\left(\chi^\star(\tau)-\chi^\star(1)\right)=(1-\tau)^{2/3}\mathrm{Var}\left(\xi_{{\rm stat},\tilde w_1-\tilde w_\tau}\right)+\mathcal{O}(1-\tau)^{1-\delta}, \tag{4.2.19}$$

for any $\delta>0$. In particular, by (4.2.18), for $\star=\{\bullet,\diagdown,\mathcal{B}\}$, we can rewrite

$$\begin{aligned}\mathrm{Cov}\left(\chi^\star(\tau),\chi^\star(1)\right)=&\frac12\mathrm{Var}\left(\xi^\star(w_1)\right)+\frac{\tau^{2/3}}{2}\mathrm{Var}\left(\xi^\star(w_\tau\tau^{-2/3})\right)\\&-\frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{{\rm stat},\tilde w_1-\tilde w_\tau}\right)+\mathcal{O}(1-\tau)^{1-\delta}.\end{aligned} \tag{4.2.20}$$

Here $\xi^\bullet(w)+w^2$ (resp. $2^{2/3}\xi^\diagdown(w)$) is distributed according to a GUE (resp. GOE) Tracy-Widom law and $\xi^{\mathcal{B}}(w)=\xi_{{\rm stat},w}$.

[7] One could also reformulate the result by saying that the error term is $\mathcal{O}((1-\tau)/\ln(1-\tau))$.

Small τ behavior for droplet initial conditions

Theorem 4.2.6. *For point-to-point LPP, let $w_\tau = \hat{w}_\tau \tau^{2/3}$. Then the covariance of the limiting height function for $\tau \to 0$ can be expressed as*

$$\mathrm{Cov}\left(\chi^\bullet(\tau), \chi^\bullet(1)\right) = \tau^{2/3}\mathbb{E}(\mathcal{A}_2(\hat{w}_\tau) \max_{u\in\mathbb{R}}\{\mathcal{A}_2(u) - u^2 + \sqrt{2}B(u)\}) + \mathcal{O}(\tau^{1-\delta}). \quad (4.2.21)$$

4.3 The stationary LPP and its comparison lemmas

As shown in [BCS06] the stationary situation can be realized in different ways. For the purpose of this paper, we will consider the following situations

- On $\mathbb{Z}_+^2$: consider the LPP from $S_A = \{(0,0)\}$ with

$$\omega_{i,j} = \begin{cases} 0 & \text{for } i=0, j=0, \\ \mathrm{Exp}(1-\rho) & \text{for } i\geq 1, j=0, \\ \mathrm{Exp}(\rho) & \text{for } i=0, j\geq 1, \\ \mathrm{Exp}(1) & \text{for } i\geq 1, j\geq 1. \end{cases} \quad (4.3.1)$$

  This is called stationary LPP with density ρ since the increments of the LPP along horizontal lines are still sums of iid. $\mathrm{Exp}(1-\rho)$ random variables, as a special case of Lemma 4.2 of [BCS06]. More generically, the increments along a down-right path are sums of independent random variables, $\mathrm{Exp}(1-\rho)$ for horizontal steps, and $-\mathrm{Exp}(\rho)$ for vertical steps.

- Consider $S_A = \mathcal{L} = \{(i,j) \in \mathbb{Z}^2 \,|\, i+j=0\}$ and with boundary terms

$$h^0(x,-x) = \begin{cases} \sum_{k=1}^{x}(X_k - Y_k), & \text{for } x\geq 1, \\ 0, & \text{for } x=0, \\ -\sum_{k=x+1}^{0}(X_k - Y_k), & \text{for } x\leq -1, \end{cases} \quad (4.3.2)$$

  where $\{X_k\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$ are independent random variables with $X_k \sim \mathrm{Exp}(1-\rho)$ and $Y_k \sim \mathrm{Exp}(\rho)$. Then by Lemma 4.2 of [BCS06] the increments of the LPP in this model are as in the first case.

We will call a stationary LPP model either of this two settings, depending on the cases. When we consider the point-to-point problem, we will refer to the stationary case as the first setting, while, when considering the other initial conditions, the stationary LPP will be the second setting.

To prove Theorem 4.2.5 we are going to use a comparison with the stationary model of density slightly higher or lower than 1/2. The comparison idea was first used in [CP15] and then generalized in [Pim17b], with applications in [Pim17a, FGN17, FO18, Nej18]. For that purpose, we need to introduce the notion of exit point, which is the location where the maximizer of the LPP exits its boundary terms. Let us define it for both stationary settings.

Definition 4.3.1.
- *The exit point for the stationary LPP to (m,n) with boundary* (4.3.1) *is the last point on the x-axis or the y-axis of the maximizer ending at (m,n). We introduce the random variable $Z^\rho(m,n) \in \mathbb{Z}$ such that, if $Z^\rho(m,n) > 0$, then the exit point is $(Z^\rho(m,n), 0)$, as if $Z^\rho(m,n) < 0$, then the exit point is $(0, -Z^\rho(m,n))$.*

- *The exit point for the stationary LPP to* (m,n) *with boundary* (4.3.2) *is the starting point of the maximizer ending at* (m,n)*. We use the notation* $\widetilde{Z}^\rho(m,n)\in\mathbb{Z}$ *such that the exit point is* $(\widetilde{Z}^\rho(m,n),-\widetilde{Z}^\rho(m,n))$*.*
- *The exit point for the LPP from* $\mathcal{L}$ *with initial condition* h^0 *is the starting point of the maximizer ending at* (m,n)*. We use the notation* $Z_{h^0}(m,n)\in\mathbb{Z}$ *such that the exit point is* $(Z_{h^0}(m,n),-Z_{h^0}(m,n))$*. For the random initial condition with parameter* σ*, we denote* $Z_{h^0}=Z_\sigma$*, and for flat initial condition* $Z_{h^0}=Z^{\diagdown}$*.*

Now we state the two comparison lemmas which we are going to use in the proof of Theorem 4.2.5.

Lemma 4.3.2. *Denote by* L^ρ *the LPP* (4.3.1) *and* $L^\bullet$ *the LPP in the droplet case. Let* $0\leq m_1\leq m_2$ *and* $n_1\geq n_2\geq 0$*. Then if* $Z^\rho(m_1,n_1)\geq 0$*, it holds*

$$L^\bullet(m_2,n_2)-L^\bullet(m_1,n_1)\leq L^\rho(m_2,n_2)-L^\rho(m_1,n_1), \tag{4.3.3}$$

while, if $Z^\rho(m_2,n_2)\leq 0$*, then we have*

$$L^\bullet(m_2,n_2)-L^\bullet(m_1,n_1)\geq L^\rho(m_2,n_2)-L^\rho(m_1,n_1). \tag{4.3.4}$$

Lemma 4.3.3. *Denote by* L^ρ *the LPP* (4.3.2) *and* $L^\star$ *be the LPP from* $\mathcal{L}$ *with boundary term* h^0*. Let* $0\leq m_1\leq m_2$ *and* $n_1\geq n_2\geq 0$*. Then if* $\widetilde{Z}^\rho(m_1,n_1)\geq \widetilde{Z}_{h^0}(m_2,n_2)$*, it holds*

$$L^\star(m_2,n_2)-L^\star(m_1,n_1)\leq L^\rho(m_2,n_2)-L^\rho(m_1,n_1), \tag{4.3.5}$$

while, if $\widetilde{Z}^\rho(m_2,n_2)\leq \widetilde{Z}_{h^0}(m_1,n_1)$*, then we have*

$$L^\star(m_2,n_2)-L^\star(m_1,n_1)\geq L^\rho(m_2,n_2)-L^\rho(m_1,n_1). \tag{4.3.6}$$

For $n_1=n_2$, Lemma 4.3.2 is in Lemma 1 of [CP15], while Lemma 4.3.3 is Lemma 2.1 of [Pim17b]. The generalization to points on a down-right path is straightforward. It was made for instance in the LPP setting (4.3.2) in Lemma 3.5 of [FGN17].

4.4 Convergence of the covariance

4.4.1 Preliminaries and notations

A law of large number for point-to-point LPP was proven in [Ros81], namely, for large (m,n), $L_{(0,0)\to(m,n)}\approx(\sqrt{m}+\sqrt{n})^2$. From this we can estimate

$$\begin{aligned}
L^\star_{(0,0)\to E_\tau}&\approx 4\tau N-w_\tau^2\tau^{-1}2^{4/3}N^{1/3},\\
L^\star_{(0,0)\to I(u)}&\approx 4\tau N-u^2\tau^{-1}2^{4/3}(\tau N)^{1/3},\\
L_{I(u)\to E_1}&\approx 4(1-\tau)N-\frac{(u-w_1)^2}{1-\tau}2^{4/3}N^{1/3}.
\end{aligned} \tag{4.4.1}$$

Denote the rescaled LPP by

$$L^\star_N(u,\tau):=\frac{L^\star_{S_A\to I(u)}-4(1-\tau)N}{2^{4/3}N^{1/3}}, \tag{4.4.2}$$

with $I(u)=(\tau N,\tau N)+u(2N)^{2/3}(1,-1)$ and

$$L^{\rm pp}_N(u,\tau):=\frac{L_{I(u)\to E_1}-4(1-\tau)N}{2^{4/3}N^{1/3}}, \tag{4.4.3}$$

where we recall that $E_1 = (N,N) + w_1(2N)^{2/3}(1,-1)$. Then, (4.2.7) and (4.2.8) become

$$\begin{aligned} L_N^\star(\tau) &\equiv L_N^\star(w_\tau,\tau),\\ L_N^\star(1) &\equiv L_N^\star(w_1,1) = \max_{u\in\mathbb{R}}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\}. \end{aligned} \tag{4.4.4}$$

Furthermore,

$$\lim_{N\to\infty} L_N^{\rm pp}(u,\tau) = (1-\tau)^{1/3}\left[\mathcal{A}_2\big(\tfrac{u-w_1}{(1-\tau)^{2/3}}\big) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\right], \tag{4.4.5}$$

and

$$\lim_{N\to\infty} L_N^\star(u,\tau) = \tau^{1/3}\mathcal{A}^\star\big(\tfrac{u}{\tau^{2/3}}\big), \quad \star\in\{\bullet,\diagdown,\mathcal{B}\}, \tag{4.4.6}$$

where

$$\mathcal{A}^\bullet(u) = \tilde{\mathcal{A}}_2(u) - u^2, \quad \mathcal{A}^\diagdown(u) = 2^{1/3}\mathcal{A}_1(u2^{-2/3}), \quad \mathcal{A}^\mathcal{B}(u) = \mathcal{A}_{\rm stat}(u). \tag{4.4.7}$$

4.4.2 Localization of the maximizer at time τN

The maximizer of the process $L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)$ is confined in the region with $|u|\leq M$ if the following event holds

$$\Omega_M^G = \Big\{\max_{|u|\leq M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\} > \max_{|u|>M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\}\Big\}. \tag{4.4.8}$$

Thus we need to estimate $\mathbb{P}(\Omega_M^G)$. For any choice of $s\in\mathbb{R}$ we can write

$$\begin{aligned} \mathbb{P}(\Omega_M^G) &\geq \mathbb{P}\left(\max_{|u|\leq M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\} > s > \max_{|u|>M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\}\right)\\ &\geq 1-\mathbb{P}(G_M)-\mathbb{P}(B_M), \end{aligned} \tag{4.4.9}$$

where we defined

$$\begin{aligned} G_M &= \{\max_{|u|\leq M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\}\leq s\},\\ B_M &= \{\max_{|u|>M}\{L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau)\}> s\}. \end{aligned} \tag{4.4.10}$$

The right hand side of (4.4.9) is estimated using the following lemma.

Lemma 4.4.1. *Let $s=-M^2\tilde{c}$ with $\tilde{c}=1/(16(1-\tau))$. Then, there exists a finite M_0 such that for any $M\geq M_0$*

$$\begin{aligned} \mathbb{P}(G_M) &\leq Ce^{-cM^2}\\ \mathbb{P}(B_M) &\leq Ce^{-cM^2} \end{aligned} \tag{4.4.11}$$

for some constants $C,c>0$ uniform in N.

As a direct consequence we have the following localization result.

Corollary 4.4.2. *For any $M\geq M_0$,*

$$\mathbb{P}\big(\textit{the maximizer of } L_N^\star(u,\tau)+L_N^{\rm pp}(u,\tau) \textit{ passes by } I(u) \textit{ with } |u|>M\big)\leq 2Ce^{-cM^2} \tag{4.4.12}$$

uniformly in N.

Denote by

$$\chi^\star_M(1) = \max_{|u|\leq M} \left\{\tau^{1/3}\mathcal{A}^\star(\tau^{-2/3}u) + (1-\tau)^{1/3}\big[\mathcal{A}_2\big(\tfrac{u-w_1}{(1-\tau)^{2/3}}\big) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\big]\right\} \tag{4.4.13}$$

and recall

$$\chi^\star(\tau) = \tau^{1/3}\mathcal{A}^\star(\tau^{-2/3}w_\tau). \tag{4.4.14}$$

Lemma 4.4.3. *We have the convergence of joint distributions*

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}&\left(\max_{|u|\leq M}\{L^\star_N(u,\tau)+L^{\rm pp}_N(u,\tau)\}\leq s_1; L^\star_N(w_\tau,\tau)\leq s_2\right)\\ =&\mathbb{P}\left(\chi^\star_M(1)\leq s_1;\chi^\star(\tau)\leq s_2\right).\end{aligned} \tag{4.4.15}$$

Proof. It is enough to have weak convergence of the two rescaled process to the terms in the rhs. As mentioned above, the point-wise convergence have been already proven. So we need tightness in the space of continuous functions of $[-M,M]$. Tightness $L^{\rm pp}_N(u,\tau)$ and $L^\bullet_N(u,\tau)$ can be found in Corollary 4.2 of [FO18], for $L^{\mathcal{B}}_N(u,\tau)$ it is a direct a direct consequence of Lemma 4.2 of [BCS06] and the standard Donsker's theorem. Finally, tightness for $L^{\diagdown}_N(u,\tau)$ has been established in [Pim17b]. □

Localization of the process

Let us prove Lemma 4.4.1 and Corollary 4.4.2.

Proof of Lemma 4.4.1. Recall that to prove this lemma, we take $s=s_0$, with the choice $s_0=-M^2\tilde{c}=-M^2/(16(1-\tau))$.

(1) Bound on $\mathbb{P}(G_M)$.
We have

$$\begin{aligned}\mathbb{P}(G_M)=&\mathbb{P}\Big(\max_{|u|\leq M}\{L^\star_N(u,\tau)+L^{\rm pp}_N(u,\tau)\}\leq s_0\Big)\\ \leq&\mathbb{P}\left(L^\bullet_N(0,\tau)+L^{\rm pp}_N(0,\tau)\leq s_0\right)\\ \leq&\mathbb{P}\left(L^\bullet_N(0,\tau)\leq s_0/2\right)+\mathbb{P}\left(L^{\rm pp}_N(0,\tau)\leq s_0/2\right).\end{aligned} \tag{4.4.16}$$

Now we can use standard estimates on the lower tail of the point-to-point LPP (see Prop. A.1.1 in Appendix A.1) to obtain that (4.4.16) is bounded by Ce^{-cM^3} uniformly in N, for some constants C,c.

(2) Bound on $\mathbb{P}(B_M)$. *Since similar estimates will be used to derive another result, we add an extra variable $\hat{s}\geq 0$ in the following computations. The case $\hat{s}=0$ is the one relevant for the present proof.*
We have

$$\begin{aligned}\mathbb{P}(B_M)&=\mathbb{P}\Big(\max_{|u|>M}\left\{L^\star_N(u,\tau)+L^{\rm pp}_N(u,\tau)\right\}>s_0+\hat{s}\Big)\\ &\leq\mathbb{P}\Big(\max_{|u|>M}\left\{L^\star_N(u,\tau)-\frac{u^2}{2(1-\tau)}\right\}>\frac{s_0+\hat{s}}{2}\Big)\\ &+\mathbb{P}\Big(\max_{|u|>M}\left\{L^{\rm pp}_N(u,\tau)+\frac{u^2}{2(1-\tau)}\right\}>\frac{s_0+\hat{s}}{2}\Big).\end{aligned} \tag{4.4.17}$$

We study separately the two terms of (4.4.17) and rename them $\mathbb{P}\left(B^1_M\right)$ and $\mathbb{P}\left(B^2_M\right)$ respectively. Remark that the maximum over u is actually a maximum over $M<|u|\leq$

$\mathcal{O}(N^{1/3})$, since $I(u)$ need to stay in the backward light cone of the end-point E_1. We will not write this explicitly all the time.
(a) Estimate of $\mathbb{P}\left(B_M^2\right)$:

$$\mathbb{P}\left(B_M^2\right) = \mathbb{P}\left(\max_{|u|>M}\left\{L_N^{\mathrm{pp}}(u,\tau) + \frac{u^2}{2(1-\tau)}\right\} > \frac{s_0+\hat{s}}{2}\right) \tag{4.4.18}$$

The bound can be obtained through the decay of the kernel for half-flat initial condition. The bound on the Fredholm determinant and the kernel are given as in Theorem 2.6 and Lemma 2.7 of [CFS18] to get

$$\mathbb{P}\left(B_M^2\right) \leq Ce^{-cM^2(1-\tau)^{-4/3}}e^{-\tilde{c}\hat{s}}. \tag{4.4.19}$$

Alternatively, one could adapt the proof of Lemma 4.3 of [FO18] to get the same result.
(b) Estimate of $\mathbb{P}\left(B_M^1\right)$:

$$\mathbb{P}\left(B_M^1\right) = \mathbb{P}\Big(\max_{|u|>M}\left\{L_N^{\star}(u,\tau) - \frac{u^2}{2(1-\tau)}\right\} > \frac{s_0+\hat{s}}{2}\Big). \tag{4.4.20}$$

- Droplet initial condition: for this case, one can estimate it like we made for (4.4.18) (with minor changes in the terms depending on τ). However, since $L_N^{\bullet}(u,\tau) \leq L_N^{\diagdown}(u,\tau)$, the droplet upper tail is simply bounded by the upper tail of the flat initial condition case.

- Flat initial condition: the bound is obtained in Lemma 4.4.4 below.

- Stationary initial condition: the bounds for the maximum over $u > M$ and for $u < -M$ are similar and thus we present the details only for the first one.

$$\begin{aligned}
&\mathbb{P}\Big(\max_{u>M}\left\{L_N^{\star}(u,\tau) - \frac{u^2}{2(1-\tau)}\right\} > \frac{s_0+\hat{s}}{2}\Big)\\
\leq&\mathbb{P}\Big(\max_{u>M}\left\{L_N^{\star}(u,\tau) - L_N^{\star}(M,\tau) - \frac{u^2}{2(1-\tau)}\right\} > s_0 + \frac{\hat{s}}{4}\Big)\\
&+\mathbb{P}\Big(L_N^{\star}(0,\tau) > -\frac{s_0}{4}+\frac{\hat{s}}{8}\Big) + \mathbb{P}\Big(L_N^{\star}(M,\tau) - L_N^{\star}(0,\tau) > -\frac{s_0}{4}+\frac{\hat{s}}{8}\Big).
\end{aligned} \tag{4.4.21}$$

We study separately the three terms of the last line of (4.4.21). The first term is bounded using (A.1.20), the second with (A.1.16) and the third one with (A.1.19), with the final result

$$\mathbb{P}(B_M^1) \leq Ce^{-cM^2-\tilde{c}\hat{s}} \tag{4.4.22}$$

for some $c, \tilde{c}$ depending on τ, but uniform for all N large enough.

□

Proof of Corollary 4.4.2. By (4.4.16), (4.4.22), (4.4.19) we can conclude that

$$\mathbb{P}\left(\max_{|u|\leq M}\left\{L_N^{\star}(u,\tau) + L_N^{\mathrm{pp}}(u,\tau)\right\} > \max_{|u|>M}\left\{L_N^{\star}(u,\tau) + L_N^{\mathrm{pp}}(u,\tau)\right\}\right) \geq 1 - 2Ce^{-cM^2}, \tag{4.4.23}$$

which implies that the probability that the maximizer of $L_N^{\star}(u,\tau) + L_N^{\mathrm{pp}}(u,\tau)$ passes through $I(u)$ with $|u| > M$ goes to zero as $2Ce^{-cM^2}$, for some constants $C, c > 0$. □

Here for simplicity of notation we rename $\hat{s}$ as s.

Lemma 4.4.4. *For flat initial condition, there exist N_0, M_0 large enough such that for all $N \geq N_0$ and $M \geq M_0$ it holds*

$$\mathbb{P}\Big(\max_{M<|u|<\mathcal{O}(N^{1/3})} \Big\{L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)}\Big\} > \frac{s_0+s}{2}\Big) \leq Ce^{-cs}e^{-\tilde{c}M^2}, \tag{4.4.24}$$

for some constants $C, c, \tilde{c}$ independent of N and M.

Proof. By symmetry we can consider only the case $u > M$, since the bounds for $u < -M$ are similar. Fix an $\varepsilon \in (0, 1/6)$. Then,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{M<u<\mathcal{O}(N^{1/3})} \Big\{L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)}\Big\} > \frac{s_0+s}{2}\Big) \\
&\leq \sum_{\ell=1}^{N^\varepsilon} \mathbb{P}\Big(\max_{u\in[\ell M,(\ell+1)M]} \Big\{L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)}\Big\} > \frac{s_0+s}{2}\Big) \\
&+ \sum_{u\in[N^\varepsilon,\mathcal{O}(N^{2/3})]} \mathbb{P}\Big(L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)} > \frac{s_0+s}{2}\Big).
\end{aligned} \tag{4.4.25}$$

Notice that $v \mapsto L_N^{\searrow}(u+v,\tau)$ and $v \mapsto L_N^{\searrow}(v,\tau)$ have the same law for any u. Thus, we can simply bound (using also (A.1.7))

$$\begin{aligned}
\mathbb{P}\Big(L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)} > \frac{s_0+s}{2}\Big) &\leq \mathbb{P}\Big(L_N^{\searrow}(0,\tau) > \frac{s}{2} - \frac{M^2}{32(1-\tau)} + \frac{N^{2\varepsilon}}{2(1-\tau)}\Big) \\
&\leq Ce^{-cs/2+cM^2/(32(1-\tau))-cN^{2\varepsilon}/(2(1-\tau))} \\
&\leq Ce^{-\tilde{c}s-\hat{c}M^2}e^{-cN^{2\varepsilon}/(4(1-\tau))},
\end{aligned} \tag{4.4.26}$$

for some constants $C, c, \tilde{c}, \hat{c}$, where the last inequality holds for all $N \geq N_0(M)$. From this it immediately follows that

$$\sum_{u\in[N^\varepsilon,\mathcal{O}(N^{2/3})]} \mathbb{P}\Big(L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)} > \frac{s_0+s}{2}\Big) \leq Ce^{-\tilde{c}s-\hat{c}M^2}. \tag{4.4.27}$$

Now we evaluate the first term in (4.4.25). Using translation-invariance in u we get

$$\begin{aligned}
&\mathbb{P}\Big(\max_{u\in[\ell M,(\ell+1)M]} \Big\{L_N^{\searrow}(u,\tau) - \frac{u^2}{2(1-\tau)}\Big\} > \frac{s_0+s}{2}\Big) \\
&\leq \mathbb{P}\Big(\max_{u\in[0,M]} L_N^{\searrow}(u,\tau) > \frac{s}{2} - \frac{M^2}{32(1-\tau)} + \frac{\ell^2M^2}{2(1-\tau)}\Big) \\
&\leq \mathbb{P}\Big(\max_{u\in[0,M]} L_N^{\searrow}(u,\tau) > \frac{s}{2} + \frac{\ell^2M^2}{4(1-\tau)}\Big)
\end{aligned} \tag{4.4.28}$$

(a) The first case is $s \geq N^{2\varepsilon}$. We can still just use the union bound and the exponential decay to get

$$\begin{aligned}
(4.4.28) &\leq N^{2/3}\exp\Big(-c\frac{\ell^2M^2}{4(1-\tau)} - c\frac{s}{2}\Big) \leq N^{2/3}\exp\Big(-c\frac{\ell^2M^2}{4(1-\tau)} - c\frac{s}{4} - c\frac{N^{2\varepsilon}}{4}\Big) \\
&\leq \exp(-c_1\ell^2M^2 - c_2 s),
\end{aligned} \tag{4.4.29}$$

for some constants $c_1, c_2 > 0$ and all N large enough.

(b) The second case is $s \in (0, N^{2\varepsilon})$. Since also $\ell \le N^{\varepsilon}$ we have that $x := \frac{s}{2} + \frac{\ell^2 M^2}{8(1-\tau)} = \mathcal{O}(N^{2\varepsilon}) \ll N^{1/3}$. The idea is to bound the process in terms of the increments of a stationary case. For that reason we first need to get a formula including the increments of the rescaled LPP, namely we get

$$\begin{aligned}(4.4.50) \le \mathbb{P}\Big(L_N^{\searrow}(0,\tau) > \frac{s}{4} + \frac{\ell^2 M^2}{8(1-\tau)}\Big) \\ + \mathbb{P}\Big(\max_{u\in[0,M]} \Big\{L_N^{\searrow}(u,\tau) - L_N^{\searrow}(0,\tau)\Big\} > \frac{s}{2} + \frac{\ell^2 M^2}{8(1-\tau)}\Big).\end{aligned} \tag{4.4.30}$$

For the first term, we just use (A.1.7) and obtain

$$\mathbb{P}\Big(L_N^{\searrow}(0,\tau) > \frac{s}{4} + \frac{\ell^2 M^2}{8(1-\tau)}\Big) \le C e^{-cs/4 - c\ell^2 M^2/(8(1-\tau))}. \tag{4.4.31}$$

The sum of this bound over $\ell \ge 1$ leads to a bound $\tilde{C} e^{-cs/4 - cM^2/(8(1-\tau))}$. For the second term in (4.4.30), define $\rho_+ = \frac{1}{2} + \kappa N^{-1/3}$ and the event

$$\Omega_{N,\kappa} = \{\widetilde{Z}^{\rho_+}(I(0)) > \widetilde{Z}^{\searrow}(I(u)), \text{ for all } u \in [0,M]\}. \tag{4.4.32}$$

On this event, by Lemma 4.3.3, we have

$$L_N^{\searrow}(u,\tau) - L_N^{\searrow}(0,\tau) \le L_N^{\rho_+}(u,\tau) - L_N^{\rho_+}(0,\tau), \tag{4.4.33}$$

which in turns gives

$$\begin{aligned}&\mathbb{P}\Big(\max_{u\in[0,M]} \Big\{L_N^{\searrow}(u,\tau) - L_N^{\searrow}(0,\tau)\Big\} > \frac{s}{2} + \frac{\ell^2 M^2}{8(1-\tau)}\Big) \\ &\le \mathbb{P}\Big(\max_{u\in[0,M]} \Big\{L_N^{\rho_+}(u,\tau) - L_N^{\rho_+}(0,\tau)\Big\} > \frac{s}{2} + \frac{\ell^2 M^2}{8(1-\tau)}\Big) + \mathbb{P}(\Omega_{N,\kappa}^c).\end{aligned} \tag{4.4.34}$$

By stationarity of the increments we have

$$L_N^{\rho_+}(u,\tau) - L_N^{\rho_+}(0,\tau) = \frac{1}{2^{4/3} N^{1/3}} \sum_{i=1}^{\lfloor uN^{2/3} \rfloor} Z_i, \tag{4.4.35}$$

where $Z_i = X_i - Y_i$ with X_i's i.i.d. $\mathrm{Exp}(1-\rho_+)$ and Y_i's i.i.d. $\mathrm{Exp}(\rho_+)$ random variables. Since $\mathcal{M}_u = \sum_{i=1}^{\lfloor uN^{2/3} \rfloor} Z_i$ is a submartingale, so it is $\exp(t\mathcal{M}_u)$ for $t > 0$ (at least for t small enough) and we can use Doob's inequality for submartingales,

$$\mathbb{P}\left(\max_{u\in[0,M]} \mathcal{M}_u \ge x\right) \le \inf_{t\ge 0} \frac{\mathbb{E}\big[e^{t\mathcal{M}_M}\big]}{e^{tx}} = \inf_{t\ge 0} \frac{\mathbb{E}\big[e^{tZ_1}\big]^{\lfloor MN^{2/3} \rfloor}}{e^{tx}}. \tag{4.4.36}$$

For and $\rho_+ = \frac{1}{2} + \kappa N^{-1/3}$. An explicit computation gives, for $\kappa \in (0, x/(2^{5/3} M))$,

$$(4.4.36) \le \exp\left(-\frac{(2^{1/3} x - 4M\kappa)^2}{4M} + \mathcal{O}(x^4 N^{-2/3}; \kappa^4 N^{-2/3})\right). \tag{4.4.37}$$

Thus, with the choice $\kappa = x/(2^{8/3}M)$, we find

$$\begin{aligned}\mathbb{P}\Big(\max_{u\in[0,M]}\Big\{L_N^{\rho_+}(u,\tau)-L_N^{\rho_+}(0,\tau)\Big\} > \frac{s}{2}+\frac{\ell^2M^2}{8(1-\tau)}\Big)\\ \leq \exp\left(-\frac{x^2}{16M}\right)\leq \exp\left(-c_1s^2-c_2\ell^4M^3\right),\end{aligned} \tag{4.4.38}$$

for some constants $c_1, c_2 > 0$ and all N large enough. Summing this bound over $\ell \geq 1$ we get $Ce^{-c_1s^2-c_2M^3}$ for some constants C, completing the proof of (4.4.24). □

Lemma 4.4.5. *Let $\rho_\pm = \frac{1}{2} \pm \kappa N^{-1/3}$. Define the event*

$$\begin{aligned}\Omega_{N,\kappa} = \Big\{\widetilde{Z}^{\rho_+}(I(0)) \geq \widetilde{Z}^{\searrow}(I(u)),\forall u\in[0,M]\Big\}\cap\\ \Big\{\widetilde{Z}^{\rho_-}(I(0)) \leq \widetilde{Z}^{\searrow}(I(u)),\forall u\in[-M,0]\Big\},\end{aligned} \tag{4.4.39}$$

where the exit points are as in Definition 4.3.1. Then, for all N large enough and all $\kappa > 0$ with $\kappa = o(N^{1/3})$,

$$\mathbb{P}(\Omega^c_{N,\kappa}) \leq Ce^{-c\kappa^2}. \tag{4.4.40}$$

Proof. We need to estimate the complement of the probabilities of the two terms in (4.4.39), for instance

$$\mathbb{P}(\widetilde{Z}^{\searrow}(I(u)) > \widetilde{Z}^{\rho_+}(I(0))),\ \text{for some } u\in[0,M]). \tag{4.4.41}$$

The estimates are completely analogous, thus we provide the details only for the first one.

Since $\widetilde{Z}^{\searrow}(I(u)) \leq \widetilde{Z}^{\searrow}(I(M))$ for all $u \in [0,M]$, we have

$$\begin{aligned}(4.4.41) &\leq \mathbb{P}(\widetilde{Z}^{\searrow}(I(M)) > \widetilde{Z}^{\rho_+}(I(0)))\\ &\leq \mathbb{P}(\widetilde{Z}^{\searrow}(I(M)) > \alpha(2N)^{2/3}) + \mathbb{P}(\widetilde{Z}^{\rho_+}(I(0))) < \alpha(2N)^{2/3}).\end{aligned} \tag{4.4.42}$$

By Lemma 4.3 of [FO18], we have that $\mathbb{P}(\widetilde{Z}^{\searrow}(I(M)) > \alpha(2N)^{2/3}) \leq Ce^{-c\alpha^2}$, for some constants $C, c \in (0,\infty)$. Using stationarity of the increments along the antidiagonal, we have

$$\widetilde{Z}^{\rho_+}(n-k,n+k) \overset{d}{=} \widetilde{Z}^{\rho_+}(n,n) - k. \tag{4.4.43}$$

Thus,

$$\begin{aligned}\mathbb{P}(\widetilde{Z}^{\rho_+}(I(0)) < \alpha(2N)^{2/3}) &= \mathbb{P}(\widetilde{Z}^{\rho_+}(I(0)) - \alpha(2N)^{2/3} \leq 0)\\ &= \mathbb{P}(\widetilde{Z}^{\rho_+}(I(-\alpha)) < 0) = \mathbb{P}(Z^{\rho_+}(I(-\alpha)) < 0).\end{aligned} \tag{4.4.44}$$

The last equality follows from the fact that we can construct the two models on the same randomness (define the random variables in the model (4.3.1) as image of the ones of (4.3.2) by [BCS06]), for which $\widetilde{Z}^{\rho_+}(m,n) < 0$ iff $Z^{\rho_+}(m,n) < 0$ by simple geometric considerations.

Setting $(\gamma^2 n, n) = I(-\alpha)$, and writing $\rho_+ = 1/(1+\gamma) + \tilde{\kappa} n^{-1/3}$, we deduce that $\tilde{\kappa} = \tau^{2/3}\kappa - 2^{-4/3}\alpha\tau^{-1/3} + \mathcal{O}(\kappa N^{-1/3})$. Lemma 2.5 of [FO18] states[8] that if $\tilde{\kappa} > 0$, then $\mathbb{P}(Z^{\rho_+}(\gamma^2 n, n) < 0) \leq Ce^{-c\tilde{\kappa}^2}$ for some constants $C, c > 0$. We choose $\alpha = 2^{1/3}\tau\kappa$, which gives $\tilde{\kappa} = \frac{1}{2}\kappa\tau^{2/3}(1 + \mathcal{O}(\kappa/N^{1/3}))$. Then, for all N large enough, we obtain

$$(4.4.44) \leq Ce^{-\tilde{c}\kappa^2} \tag{4.4.45}$$

for some constants $C, c > 0$. □

[8] By inspecting the proof of Lemma 2.5 of [FO18], one sees that it actually holds true not only for any given κ, but also for all $\kappa \in [0, o(n^{1/3})]$.

Convergence of the covariance

To prove Theorem 4.2.2, first we show that the $N \to \infty$ limit of the covariance of $L_N^\star(\tau)$ and $\max_{|u|\leq M} \left\{L_N^\star(u,\tau) + L_N^{\rm pp}(u,\tau)\right\}$ is the covariance of $\chi^\star(\tau)$ and $\chi_M^\star(1)$, for fixed $M > 0$. Now that we have proved the localization of the process, we need to show that the covariance of $\chi^\star(\tau)$ and $\chi^\star(1)$ is the $M \to \infty$ limit of the covariance of the LPP restricted to the region $|u| \leq M$.

Proposition 4.4.6. *For any fixed $M > 0$,*

$$\lim_{N\to\infty} \mathrm{Cov}\left(L_N^\star(w_\tau,\tau), \max_{|u|\leq M}\{L_N^\star(u,\tau) + L_N^{\rm pp}(u,\tau)\}\right) = \mathrm{Cov}\left(\chi^\star(\tau), \chi_M^\star(1)\right). \quad (4.4.46)$$

Proof. Let us denote

$$L_{N;M}^\star(1) = \max_{|u|\leq M}\{L_N^\star(u,\tau) + L_N^{\rm pp}(u,\tau)\}. \quad (4.4.47)$$

By Lemma 4.4.15 we already have the convergence of joint distributions of $L_{N;M}^\star(1)$ and $L_N^\star(\tau) \equiv L_N^\star(w_\tau,\tau)$ to $\chi_M^\star(1)$ and $\xi^\star(\tau)$. By Cauchy-Schwarz it is enough to show the convergence of the second moments of $L_{N;M}^\star(1)$ and $L_N^\star(\tau)$.

For a random variable X_N with distribution function $F_N(s) = \mathbb{P}(X_N \leq s)$, we can write

$$\mathbb{E}(X_N^2) = 2\int_{\mathbb{R}_+} s(1-F_N(s))ds - 2\int_{\mathbb{R}_-} sF_N(s)ds. \quad (4.4.48)$$

If we know that $X_N \to X$ in distribution, to show convergence of the second moment we need only to find $g(s)$ independent of N such that $1-F_N(s) \leq g(s)$ for $s \geq 0$, $F_N(s) \leq g(s)$ for $s < 0$ and that $g \in L^1(\mathbb{R})$. Then dominated convergence allows to take the limit in the integrals and obtain $\mathbb{E}(X_N^2) \to \mathbb{E}(X^2)$. Thus our task is to find such bounds. Since $F_N(s) \in [0,1]$, it is enough to get bounds for the tails, i.e., a bound for $1 - F_N(s)$ for $s \geq s_0$ and for $F_N(s)$ for $s \leq -s_0$ for some s_0.

(1) $\lim_{N\to\infty} \mathbb{E}[(L_N^\star(\tau))^2] = \mathbb{E}[(\chi^\star(\tau))^2]$.

- bound on lower tails: due to $L_N^\star(\tau) \geq L_N^\bullet(\tau)$, we can use for all cases the lower bound for the droplet initial condition, which is in Proposition A.1.1 (by appropriate change of variables).

- bound on upper tails: (a) for the droplet initial condition, this is in Proposition A.1.1, (b) for the flat initial condition, this is given in Proposition A.1.2, (c) for the stationary initial condition[9], we have

$$\mathbb{P}(L_N^{\mathcal{B}}(\tau) \leq s) \leq \mathbb{P}(L_N^{\mathcal{B}}(0) \leq s/2) + \mathbb{P}(L_N^{\mathcal{B}}(\tau) - L_N^{\mathcal{B}}(0) \leq s/2). \quad (4.4.49)$$

  The first term is bounded using Proposition A.1.4, while the second using Proposition A.1.5.

In all cases we have at least exponential decay of the both the upper and lower tails. This implies the convergence of the second moment as well.

(2) $\lim_{N\to\infty} \mathbb{E}[(L_{N;M}^\star(1))^2] = \mathbb{E}[(\chi_M^\star(1))^2]$.

[9] For stationary initial condition, the convergence of all moments was already proven in [BFP14].

- bound on lower tails: we have

$$\begin{aligned}\mathbb{P}(L^\star_{N;M}(1)\le s) &\le \mathbb{P}(L^\star_N(0,\tau)\le s/2)+\mathbb{P}(L^{\rm pp}_N(0,\tau)\le s/2)\\ &\le \mathbb{P}(L^\bullet_N(0,\tau)\le s/2)+\mathbb{P}(L^{\rm pp}_N(0,\tau)\le s/2)\le Ce^{-c|s|^{3/2}}\end{aligned} \tag{4.4.50}$$

  by Proposition A.1.1.

- bound on upper tails: we have $L^\star_{N;M}(1)\le L^\star_N(1)$ and thus by the estimates used already in part (1) we have

$$\mathbb{P}(L^\star_N(1)\ge s)\le Ce^{-cs}. \tag{4.4.51}$$

These bounds implies convergence of the second moment as well. □

What remains to prove Theorem 4.2.2 is a control on the contribution to the covariance from the events when the maximizer passed by $I(u)$ for some $|u|>M$. We have the decomposition

$$\begin{aligned}&\mathrm{Cov}\,(L^\star_N(\tau),L^\star_N(1))\\ &=\mathrm{Cov}\left(L^\star_N(\tau),L^\star_{N;M}(1)\right)+\mathrm{Cov}\left(L^\star_N(\tau),L^\star_N(1)-L^\star_{N;M}(1)\right).\end{aligned} \tag{4.4.52}$$

Given the convergence of the second moments for fixed M by Proposition 4.4.6, there is one term left to study:

$$\begin{aligned}&|\,\mathrm{Cov}\left(L^\star_N(\tau),L^\star_N(1)-L^\star_{N;M}(1)\right)|\\ &=|\mathbb{E}\left[L^\star_N(\tau)\left(L^\star_N(1)-L^\star_{N;M}(1)\right)\right]-\mathbb{E}\left[L^\star_N(\tau)\right]\mathbb{E}\left[L^\star_N(1)-L^\star_{N;M}(1)\right]|\\ &\le 2\left(\mathbb{E}\left[(L^\star_N(\tau))^2\right]\mathbb{E}\left[(L^\star_N(1)-L^\star_{N;M}(1))^2\right]\right)^{1/2},\end{aligned} \tag{4.4.53}$$

where we used Cauchy-Schwarz to control the second term.

Lemma 4.4.7. *For any $M>0$,*

$$\lim_{N\to\infty}\mathbb{E}\left[(L^\star_N(1)-L^\star_{N;M}(1))^2\right]\le Ce^{-cM^2} \tag{4.4.54}$$

as well as

$$\mathbb{E}[(\chi^\star(1)-\chi^\star_M(1))^2]\le Ce^{-cM^2}. \tag{4.4.55}$$

where $C,c>0$ are positive constants, uniformly in N.

Proof. Let us denote

$$L^\star_{N;M^c}(1)=\max_{|u|>M}\{L^\star_N(u,\tau)+L^{\rm pp}_N(u,\tau)\}. \tag{4.4.56}$$

Since $L^\star_N(1)=\max\{L^\star_{N;M}(1),L^\star_{N;M^c}(1)\}$, we can write

$$L^\star_N(1)-L^\star_{N;M}(1)=\max\{0,L^\star_{N;M^c}(1)-L^\star_{N;M}(1)\}. \tag{4.4.57}$$

Integrating by parts, we obtain

$$\mathbb{E}\left[\left(L^\star_N(1)-L^\star_{N;M}(1)\right)^2\right]=2\int_{\mathbb{R}_+}s\mathbb{P}\left(L^\star_{N;M^c}(1)-L^\star_{N;M}(1)>s\right). \tag{4.4.58}$$

The probability in the r.h.s. of (4.4.58) can be bounded as

$$\mathbb{P}\left(L^{\star}_{N;M^c}(1)-L^{\star}_{N;M}(1)>s\right)\leq \mathbb{P}\left(L^{\star}_{N;M^c}(1)>\frac{s+\alpha}{2}\right)+\mathbb{P}\left(L^{\star}_{N;M}(1)\leq\frac{\alpha-s}{2}\right) \tag{4.4.59}$$

for any choice of α.

(a) We use the first inequality in (4.4.28) and obtain

$$\mathbb{P}\left(L^{\star}_{N;M}(1)<\frac{\alpha-s}{2}\right)\leq \mathbb{P}\left(L^{\bullet}_{N}(0,\tau)<\frac{\alpha-s}{4}\right)+\mathbb{P}\left(L^{\rm pp}(0,\tau)<\frac{\alpha-s}{4}\right). \tag{4.4.60}$$

For any $\alpha<0$ and $s\geq 0$, by Proposition A.1.1 we get

$$\mathbb{P}\left(L^{\star}_{N;M}(1)<\frac{\alpha-s}{2}\right)\leq Ce^{-c(s-\alpha)^{3/2}} \tag{4.4.61}$$

for some constants C,c. Thus it is enough to choose $\alpha=-\gamma M^2$ for some $\gamma>0$.

(b) Next we bound $\mathbb{P}\left(L^{\star}_{N;M^c}(1)>\frac{s+\alpha}{2}\right)$. Choosing $\alpha=-\frac{M^2}{16(1-\tau)}$ and using the bounds for $\mathbb{P}(B_M)$ in the proof of Lemma 4.4.1, we obtain

$$\mathbb{P}\left(L^{\star}_{N;M^c}(1)>\frac{s+\alpha}{2}\right)\leq Ce^{-cM^2}e^{-\tilde{c}s}. \tag{4.4.62}$$

Plugging the bounds (4.4.61) and (4.4.62) into (4.4.58) leads (4.4.54).

Finally, (4.4.55) is proven as follows. By dominated convergence we have that

$$\begin{aligned}\mathbb{E}[(\chi^{\star}(1)-\chi^{\star}_M(1))^2]&=2\int_{\mathbb{R}_+}s\mathbb{P}\left(\chi^{\star}_{M^c}(1)-\chi^{\star}_M(1)>s\right)\\&=\lim_{N\to\infty}2\int_{\mathbb{R}_+}s\mathbb{P}\left(L^{\star}_{N;M^c}(1)-L^{\star}_{N;M}(1)>s\right)\leq Ce^{-cM^2}\end{aligned} \tag{4.4.63}$$

where the last inequality follows from (4.4.54). □

Proof of Theorem 4.2.2. We have

$$\begin{aligned}&\lim_{N\to\infty}\mathrm{Cov}\left(L^{\star}_N(\tau),L^{\star}_N(1)\right)\\&=\lim_{M\to\infty}\lim_{N\to\infty}\mathrm{Cov}\left(L^{\star}_N(\tau),L^{\star}_{N,M}(1)\right)+\lim_{M\to\infty}\lim_{N\to\infty}\mathrm{Cov}\left(L^{\star}_N(\tau),L^{\star}_N(1)-L^{\star}_{N,M}(1)\right).\end{aligned} \tag{4.4.64}$$

By Proposition 4.4.6, the first term equals $\mathrm{Cov}(\chi^{\star}(\tau),\chi^{\star}_M(1))$. By Lemma 4.4.7, the second term is 0. Thus what remains is to show that

$$\mathrm{Cov}(\chi^{\star}(\tau),\chi^{\star}(1))=\lim_{M\to\infty}\mathrm{Cov}(\chi^{\star}(\tau),\chi^{\star}_M(1)). \tag{4.4.65}$$

This is obtained once we prove that

$$\lim_{M\to\infty}\mathbb{E}[(\chi^{\star}(1)-\chi^{\star}_M(1))^2]=0, \tag{4.4.66}$$

which is also part of Lemma 4.4.7. □

4.4.3 Formula for the stationary case

Now we prove the claimed formula for the stationary case. It follows by a simple computation using the result of Theorem 4.2.2 for the stationary case and the identity (4.2.18).

Proof of Corollary 4.2.4. Setting $X_1 = \chi^{\mathcal{B}}(\tau)$ and $X_2 = \chi^{\mathcal{B}}(1)$ in (4.2.18) we get

$$\mathrm{Cov}\big(\chi^{\mathcal{B}}(\tau),\chi^{\mathcal{B}}(1)\big) = \tfrac{1}{2}\mathrm{Var}(\chi^{\mathcal{B}}(\tau)) + \tfrac{1}{2}\mathrm{Var}(\chi^{\mathcal{B}}(1)) - \tfrac{1}{2}\mathrm{Var}(\chi^{\mathcal{B}}(1)-\chi^{\mathcal{B}}(\tau)). \quad (4.4.67)$$

The first two terms in (4.2.17) are an immediate consequence of the convergence of moments, see proof of Proposition 4.4.6. For the last term, we have

$$\begin{aligned}\chi^{\mathcal{B}}(1)-\chi^{\mathcal{B}}(\tau) = \max_{u\in\mathbb{R}}\Big\{&(1-\tau)^{1/3}\Big[\mathcal{A}_2\big(\tfrac{u-w_1}{(1-\tau)^{2/3}}\big) - \tfrac{(u-w_1)^2}{(1-\tau)^{4/3}}\Big]\\ &+\tau^{1/3}[\mathcal{A}_{\rm stat}(\tau^{-2/3}u)-\mathcal{A}_{\rm stat}(\tau^{-2/3}w_\tau)]\Big\}.\end{aligned} \quad (4.4.68)$$

Changing the variable $u = w_\tau + z(1-\tau)^{2/3}$, and calling $\xi = \frac{w_1-w_\tau}{(1-\tau)^{2/3}}$, it gives

$$\begin{aligned}\chi^{\mathcal{B}}(1)-\chi^{\mathcal{B}}(\tau) =&(1-\tau)^{1/3}\max_{z\in\mathbb{R}}\big\{\mathcal{A}_2\big(z-\xi\big)-(z-\xi)^2\\ &+\frac{\tau^{1/3}}{(1-\tau)^{1/3}}\Big[\mathcal{A}_{\rm stat}(\tau^{-2/3}(w_\tau+z(1-\tau)^{2/3}))-\mathcal{A}_{\rm stat}(\tau^{-2/3}w_\tau)\Big]\Big\}.\end{aligned} \quad (4.4.69)$$

Next we use the facts: (a) $\mathcal{A}_2(z-\xi)\stackrel{(d)}{=}\mathcal{A}_2(z)$, (b) $\mathcal{A}_{\rm stat}(a+x)-\mathcal{A}_{\rm stat}(a)\stackrel{(d)}{=}\sqrt{2}B(x)$ with B a standard Brownian motion, and (c) the scaling of Brownian motion, to get

$$\chi^{\mathcal{B}}(1)-\chi^{\mathcal{B}}(\tau)\stackrel{(d)}{=}(1-\tau)^{1/3}\max_{z\in\mathbb{R}}\Big\{\sqrt{2}B(z)+\mathcal{A}_2(z)-(z-\xi)^2\Big\}\stackrel{(d)}{=}(1-\tau)^{1/3}\mathcal{A}_{\rm stat}(\xi). \quad (4.4.70)$$

□

4.5 Behavior around $\tau = 1$

What we have to prove is

$$\mathrm{Var}\left[\chi^\star(1)-\chi^\star(\tau)\right] = (1-\tau)^{2/3}\,\mathrm{Var}\left(\xi_{{\rm stat},\tilde w}\right)+\mathcal{O}(1-\tau)^{1-\delta}, \quad (4.5.1)$$

as $\tau\to 1$ for all the initial conditions. Clearly the flat and stationary are special case of the more generic random initial conditions. Define

$$\begin{aligned}\chi^\star_M &= \lim_{N\to\infty}\max_{|u|\leq(1-\tau)^{2/3}M}(L^\star_N(u,\tau)+L^{\rm pp}_N(u,\tau))\\ &= \lim_{N\to\infty}\max_{|v|\leq M}(L^\star_N((1-\tau)^{2/3}v,\tau)+L^{\rm pp}_N((1-\tau)^{2/3}v,\tau)).\end{aligned} \quad (4.5.2)$$

In particular, for droplet, flat, stationary initial conditions, we have

$$\chi^\star_M = (1-\tau)^{1/3}\max_{|v|\leq M}\left(\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\mathcal{A}^\star\left(v\left(\tfrac{1-\tau}{\tau}\right)^{2/3}\right)+\mathcal{A}_2(v-\tilde w_1)-(v-\tilde w_1)^2\right), \quad (4.5.3)$$

with $\mathcal{A}^\star$ being the Airy$_2$, Airy$_1$ or Airy$_{\rm stat}$ process for $\star = \bullet, \diagdown, \mathcal{B}$ respectively. Also, recall the notation

$$\chi^\star(\tau) = \lim_{N\to\infty}L^\star_N((1-\tau)^{2/3}\tilde w_\tau,\tau) = (1-\tau)^{1/3}\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\mathcal{A}^\star\left(\tilde v(\tau)\left(\tfrac{1-\tau}{\tau}\right)^{2/3}\right). \quad (4.5.4)$$

On short scales, $\mathcal{A}^\star$ is expected to behave similar to the stationary state, which is a two-sided Brownian motion with diffusion coefficient 2. Since the Airy$_2$ process is stationary, for $\tau \to 1$, $\chi^\star_M - \chi^\star(\tau)$ should be close to the following expression

$$\xi_{M,\tilde{w}_\tau,\tilde{w}_1} := (1-\tau)^{1/3} \max_{|v|\leq M} \left\{ \sqrt{2}B(v-\tilde{w}_\tau) + \mathcal{A}_2(v) - (v-\tilde{w}_1)^2 \right\}. \tag{4.5.5}$$

In this proof we set

$$\tilde{w} = \tilde{w}_1 - \tilde{w}_\tau. \tag{4.5.6}$$

For $M = \infty$, replacing $v - \tilde{w}_\tau \to \tilde{v}$ and using the stationarity of $\mathcal{A}_2$ we obtain

$$\xi_{\infty,\tilde{w}_\tau,\tilde{w}_1} \stackrel{(d)}{=} (1-\tau)^{1/3} \max_{\tilde{v}\in\mathbb{R}} \{\sqrt{2}B(\tilde{v}) + \mathcal{A}_2(\tilde{v}) - (\tilde{v}-\tilde{w})^2\} = (1-\tau)^{1/3}\xi_{\mathrm{stat},\tilde{w}}. \tag{4.5.7}$$

Note that in distribution

$$(1-\tau)^{1/3}(\xi_{\mathrm{GUE}} - \tilde{w}^2) \leq \xi_{M,\tilde{w}_\tau,\tilde{w}_1} \leq (1-\tau)^{1/3}\xi_{\mathrm{stat},\tilde{w}} \tag{4.5.8}$$

and therefore we know that the mth moment of $\xi_{M,\tilde{w}_\tau,\tilde{w}_1}$ is finite and of order $(1-\tau)^{m/3}$.

To control the error term, the idea is to take M depending on τ such that $M \to \infty$ as τ goes to 1. Then the task is to prove that the difference between the second moment of $\chi^\star_M(1) - \chi^\star(\tau)$ and the second moment of $\xi_{M,\tilde{w}_\tau,\tilde{w}_1}$ goes to zero as $\tau \to 1$.

Lemma 4.5.1. *Let $M = \frac{1}{(1-\tau)^{\delta/2}}$ with $\delta > 0$. Then*

$$\left| \mathbb{E}\left[(\chi^\star_M(1) - \chi^\star(\tau))^2\right] - \mathbb{E}\left[\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}\right] \right| = \mathcal{O}(1-\tau)^{1-\delta}. \tag{4.5.9}$$

We need to control how close the increments of the process over distances of order $(1-\tau)^{2/3}$ at time τ are with respect to the increments of Brownian motion.

We present a short technical lemma that will be used in the proof of Lemma 4.5.1. Recall Definition 4.3.1 of the exit point for a LPP with boundary conditions (4.3.1) (for the droplet case) or (4.3.2) (for the random case) and define $\rho^\pm = \frac{1}{2} \pm \kappa N^{-1/3}$. Denote by $L^{\rho^\pm}$ its associated LPP.

Lemma 4.5.2. *There is an event Ω_κ with $\mathbb{P}(\Omega_\kappa) \geq 1 - C\exp\left(-c\tilde{\kappa}^2\right)$, with*
(a) for droplet initial condition, $\tilde{\kappa} = \kappa - \frac{M(1-\tau)^{2/3}}{2^{4/3}\tau}$,
(b) for random initial condition, $\tilde{\kappa} = \kappa - 2\frac{M(1-\tau)^{2/3}}{2^{4/3}\tau}$,
and constants $C, c, M_0 \in (0,\infty)$, such that on Ω_κ the inequalities

$$\xi_{M,\tilde{w}_\tau,\tilde{w}_1} - \varepsilon_0 \leq \chi^\star_M(1) - \chi^\star(\tau) \leq \xi_{M,\tilde{w}_\tau,\tilde{w}_1} + \varepsilon_0, \tag{4.5.10}$$

hold in distribution, for all $M \geq M_0$ with $\varepsilon_0 = \mathcal{O}(\kappa M(1-\tau)^{2/3})$, under the condition $\tilde{\kappa} > 0$.

Proof. Let us define

$$\Delta^\bullet_N(u) = \frac{L_{(0,0)\to I(u)} - L_{(0,0)\to E_\tau}}{2^{4/3}N^{1/3}}, \quad \Delta^\sigma_N(u) = \frac{L^\sigma_{\mathcal{L}\to I(u)} - L^\sigma_{\mathcal{L}\to E_\tau}}{2^{4/3}N^{1/3}}, \tag{4.5.11}$$

and recall the definitions

$$\begin{aligned} L^\bullet_N(u,\tau) &= \frac{L_{(0,0)\to I(u)} - 4\tau N}{2^{4/3}N^{1/3}}, \quad L^\sigma_N(u,\tau) = \frac{L^\sigma_{\mathcal{L}\to I(u)} - 4\tau N}{2^{4/3}N^{1/3}}, \\ L^{\mathrm{pp}}_N(u,\tau) &= \frac{L_{I(u)\to E_1} - 4(1-\tau)N}{2^{4/3}N^{1/3}}. \end{aligned} \tag{4.5.12}$$

Then we have

$$L_N^\star(u,\tau) = L_N^\star(0,\tau) + \Delta_N^\star(u). \tag{4.5.13}$$

Also, recall that we will use the notation $u = v(1-\tau)^{2/3}$ and $\tilde{M} = M(1-\tau)^{2/3}$.

Define the event

$$\Omega_{N,\kappa} = \begin{cases} \{Z^{\rho_+}(I(-\tilde{M})) > 0, Z^{\rho_-}(I(\tilde{M})) < 0\}, & \text{for droplet IC,} \\ \{\widetilde{Z}^{\rho_+}(I(-\tilde{M})) > Z_\sigma(I(\tilde{M})), \widetilde{Z}^{\rho_-}(I(\tilde{M})) < Z_\sigma(I(-\tilde{M}))\}, & \text{for random IC.} \end{cases} \tag{4.5.14}$$

Then, on the event $\Omega_{N,\kappa}$ we can bound $\Delta_N^\star(u)$ with the increments of the stationary LPP with density $\rho_\pm$, defined as

$$B^\pm(u) = \frac{L^{\rho_\pm}(I(u)) - L^{\rho_\pm}(I(0)) - m_{\rho_\pm} u(2N)^{2/3}}{2^{4/3}N^{1/3}}, \tag{4.5.15}$$

where $m_{\rho^\pm} = \frac{1}{1-\rho^\pm} - \frac{1}{\rho^\pm} = 8\kappa N^{-1/3} + \mathcal{O}(N^{-1})$. Indeed a minimal modification of Lemma 4.3.2 implies, for $-\tilde{M} \leq w < u \leq \tilde{M}$,

$$\begin{aligned} \big[B^-(u) - B^-(w)\big] - 4(u-w)\kappa &\leq L_N^\star(u,\tau) - L_N^\star(w,\tau) \\ &\leq \big[B^+(u) - B^+(w)\big] + 4(u-w)\kappa \end{aligned} \tag{4.5.16}$$

for N large enough. Furthermore, $\mathrm{Var}(B^\pm(u)) = u2^{1/2}(1+\mathcal{O}(N^{-2/3}))$ and $B^\pm(0) = 0$. Thus by Donsker's theorem, $\lim_{N\to\infty} B^\pm(u) = \sqrt{2}B(u)$, with $B(u)$ a standard two-sided Brownian motion in the space of continuous functions on bounded sets.

Recall that

$$\chi_M^\star(1) - \chi^\star(\tau) = \lim_{N\to\infty} \max_{|v|\leq M} \Big\{ L_N^\star\big(v(1-\tau)^{2/3},\tau\big) - L_N^\star(\tilde{w}_\tau(1-\tau)^{2/3},\tau) + L_N^{\rm pp}(v(1-\tau)^{2/3},\tau) \Big\} \tag{4.5.17}$$

and also that $v \mapsto L_N^{\rm pp}(v(1-\tau)^{2/3},\tau)$ converges weakly to $(1-\tau)^{1/3}[\mathcal{A}_2(v) - (v-\tilde{w}_1)^2]$. Thus, taking the $N\to\infty$ limit and using the inequalities (4.5.16) we obtain

$$\begin{aligned} &\mathbb{P}\left(\chi_M^\star(1) - \chi^\star(\tau) \leq (1-\tau)^{1/3}s\right) \\ &\leq \mathbb{P}\left(\max_{|v|\leq M}\left\{\sqrt{2}(B(v) - B(\tilde{w}_\tau) + \mathcal{A}_2(v) - (v-\tilde{w}_1)^2 - 4\kappa(v-\tilde{w}_\tau)(1-\tau)^{1/3}\right\} \leq s\right). \end{aligned} \tag{4.5.18}$$

Denoting $\varepsilon = \max_{|v|\leq M} |4\kappa(v-w)(1-\tau)^{1/3}| = 6\kappa M(1-\tau)^{1/3}$ we obtain

$$\begin{aligned} &\mathbb{P}\left(\chi_M^\star(1) - \chi^\star(\tau) \leq (1-\tau)^{1/3}s\right) \\ &\leq \mathbb{P}\left(\max_{|v|\leq M}\left\{\sqrt{2}B(v-\tilde{w}_\tau) + \mathcal{A}_2(v) - (v-\tilde{w}_1)^2\right\} \leq s+\varepsilon\right) \\ &= \mathbb{P}\left(\xi_{M,\tilde{w}_\tau,\tilde{w}_1} \leq (1-\tau)^{1/3}s + \varepsilon_0\right) \end{aligned} \tag{4.5.19}$$

with $\varepsilon_0 = (1-\tau)^{1/3}\varepsilon$.

Similarly for the lower bound we get

$$\mathbb{P}\left(\chi_M^\star(1) - \chi^\star(\tau) > (1-\tau)^{1/3}s\right) \leq \mathbb{P}\left(\xi_{M,\tilde{w}_\tau,\tilde{w}_1} > (1-\tau)^{1/3}s + \varepsilon_0\right). \tag{4.5.20}$$

To conclude the proof, we need to estimate $\mathbb{P}(\Omega_{N,\kappa})$.

(a) Droplet initial condition: For this case, we apply Lemma 2.5 of [FO18]. To estimate $\mathbb{P}(Z^{\rho_\pm}(I(\mp\tilde{M})) > 0)$, we need to set $I(\mp\tilde{M}) = (\gamma^2 n, n)$. This gives

$$\rho_\pm = \frac{1}{2} \pm \frac{\tilde{M}}{2^{4/3}\tau N^{1/3}} \pm \frac{\tilde{\kappa}}{\tau^{2/3}N^{1/3}}. \tag{4.5.21}$$

Then, Lemma 2.5 of [FO18] gives

$$\mathbb{P}(Z^{\rho_\pm}(I(\mp\tilde{M})) > 0) \geq 1 - Ce^{-c\tilde{\kappa}^2} = 1 - Ce^{-c(\tau^{2/3}\kappa - 2^{-4/3}\tilde{M}\tau^{-1/3})^2}. \tag{4.5.22}$$

The estimates are uniform for all N large enough. Renaming $c\tau^{4/3}$ as a new constant c, and $2C$ by C, we get

$$\mathbb{P}(\Omega_{N,\kappa}) \geq 1 - C\exp\left(-c\Big(\kappa - \frac{M(1-\tau)^{2/3}}{2^{4/3}\tau}\Big)^2\right). \tag{4.5.23}$$

(b) Random initial condition: We derive a bound only for $\mathbb{P}(\widetilde{Z}^{\rho_+}(I(-\tilde{M})) < Z_\sigma(I(\tilde{M})))$, since bounding $\mathbb{P}(\widetilde{Z}^{\rho_-}(I(\tilde{M})) < Z_\sigma(I(-\tilde{M})))$ is completely analogue.

The probability we want to bound is smaller than

$$\mathbb{P}(\widetilde{Z}^{\rho_+}(I(-\tilde{M})) \leq \alpha(2N)^{2/3}) + \mathbb{P}(Z_\sigma(I(\tilde{M})) > \alpha(2N)^{2/3}), \tag{4.5.24}$$

and we choose $\alpha = 2^{1/3}\tau\kappa$. Exactly as in (4.4.44), we have

$$\mathbb{P}(\widetilde{Z}^{\rho_+}(I(-\tilde{M})) \leq \alpha(2N)^{2/3}) = \mathbb{P}(Z^{\rho_+}(I(-\tilde{M}-\alpha)) < 0) \leq Ce^{-c\tilde{\kappa}^2} \tag{4.5.25}$$

with $\tilde{\kappa} = 2\tau^{2/3}\left(\kappa - \frac{(1-\tau)^{2/3}M}{2^{1/3}\tau}\right)$, provided $\tilde{\kappa} > 0$.

Now we bound $\mathbb{P}(Z_\sigma(I(\tilde{M})) > \alpha(2N)^{2/3})$. Let $J(v) = v(2N)^{2/3}(1,-1)$, define the scaled variables

$$L_N(v) = \frac{L_{J(v)\to I(\tilde{M})} - 4(1-\tau)N}{2^{4/3}N^{1/3}}, \quad \mathcal{W}_N(v) = \frac{h^0(J(v))}{2^{4/3}N^{1/3}}. \tag{4.5.26}$$

Then,

$$\begin{aligned}\mathbb{P}(Z_\sigma(I(\tilde{M})) > \alpha(2N)^{2/3}) \leq{}& \mathbb{P}\left(\max_{v\leq\alpha}(L_N(v) + \mathcal{W}_N(v)) \leq -s\right) \\ &+ \mathbb{P}\left(\max_{v>\alpha}(L_N(v) + \mathcal{W}_N(v)) \geq -s\right).\end{aligned} \tag{4.5.27}$$

Since $L_N(v) \sim -(v-\tilde{M})^2/(\tau)$, we choose $s = (\alpha - \tilde{M})^2/4$.

The first term in (4.5.27) is bounded by

$$\mathbb{P}\left(L_N(\alpha) + \mathcal{W}_N(\alpha) \leq -s\right) \leq \mathbb{P}\left(L_N(\alpha) \leq -\tfrac{3}{2}s\right) + \mathbb{P}\left(\mathcal{W}_N(\alpha) \leq \tfrac{1}{2}s\right). \tag{4.5.28}$$

The first term bounded by $C_1e^{-c_2(\alpha-\tilde{M})^3}$ by (A.1.4). Since $\mathcal{W}_N$ is a (rescaled) sum of iid. random variables, we can use the the exponential Chebyshev inequality (see e.g. the proof of (A.1.19)) and obtain a bound $C_2e^{-c_2(\alpha-\tilde{M})^4/\alpha}$.

The second term in (4.5.27) is bounded by

$$\mathbb{P}\left(\max_{v\geq\alpha}\left(L_N(v) + \tfrac{1}{2}(v-\tilde{M})^2\right) \geq -\tfrac{1}{2}s\right) + \mathbb{P}\left(\max_{v\geq\alpha}\left(\mathcal{W}_N(v) - \tfrac{1}{2}(v-\tilde{M})^2\right) \geq -\tfrac{1}{2}s\right). \tag{4.5.29}$$

The first term is estimated similarly to (4.4.18) and leads to a bound $C_3 e^{-c_3(\alpha-\tilde{M})^2}$. The second term is bounded using Doob's maximal inequality (see e.g. the proof of (A.1.20) leading to a bound $C_4 e^{-c_4(\alpha-\tilde{M})^4/\alpha}$).

Combining these bounds we get $\mathbb{P}(Z_\sigma(I(\tilde{M})) > \alpha(2N)^{2/3}) \leq Ce^{-c\tilde{\kappa}^2}$, provided $\tilde{\kappa} > 0$, for some constants $C, c \in (0,\infty)$ uniformly for all τ in a compact subset of $(0,1]$. Up to renaming $c\tau^{4/3}$ to c and the constant $2C$ to C we get the claimed result. □

Now we can prove Lemma 4.5.1.

Proof of Lemma 4.5.1. By Lemma 4.5.2 we have, on a event Ω_κ with $\mathbb{P}(\Omega_\kappa^c) \leq Ce^{-c\tilde{\kappa}^2}$, with

$$\tilde{\kappa} = \kappa - \frac{M(1-\tau)^{2/3}}{2^{4/3}\tau}. \tag{4.5.30}$$

the inequality

$$(\chi_M^\star(1) - \chi^\star(\tau))\mathbb{1}_{\Omega_\kappa} = \xi_{M,\tilde{w}_\tau,\tilde{w}_1}\mathbb{1}_{\Omega_\kappa} + \zeta\mathbb{1}_{\Omega_\kappa}, \tag{4.5.31}$$

for some random variables ζ with $|\zeta| \leq \varepsilon_0$. Thus

$$\mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2] = \mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Omega_\kappa}] + \mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Omega_\kappa^c}]. \tag{4.5.32}$$

Using (4.5.31) we get

$$\mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Omega_\kappa}] = \mathbb{E}(\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}) - \mathbb{E}(\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}\mathbb{1}_{\Omega_\kappa^c}) + 2\mathbb{E}(\zeta\xi_{M,\tilde{w}_\tau,\tilde{w}_1}\mathbb{1}_{\Omega_\kappa}) + \mathbb{E}(\zeta^2\mathbb{1}_{\Omega_\kappa}). \tag{4.5.33}$$

Using Cauchy-Schwarz and the fact that $|\zeta| \leq \varepsilon_0$, we get the bounds

$$\begin{aligned} \mathbb{E}(\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}\mathbb{1}_{\Omega_\kappa^c}) &\leq \sqrt{\mathbb{E}(\xi^4_{M,\tilde{w}_\tau,\tilde{w}_1})\mathbb{P}(\Omega_\kappa^c)} \leq C_1(1-\tau)^{2/3}e^{-c\tilde{\kappa}^2/2}, \\ |\mathbb{E}(\zeta\xi_{M,\tilde{w}_\tau,\tilde{w}_1}\mathbb{1}_{\Omega_\kappa})| &\leq \varepsilon_0\sqrt{\mathbb{E}(\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1})} \leq C_2(1-\tau)^{1/3}\varepsilon_0, \\ \mathbb{E}(\zeta^2\mathbb{1}_{\Omega_\kappa}) &\leq \varepsilon_0^2, \end{aligned} \tag{4.5.34}$$

for some constants C_1, C_2 (since, as already mentioned, the mth moment of $\xi_{M,\tilde{w}_\tau,\tilde{w}_1}$ is of order $(1-\tau)^{m/3}$).

It remains to bound the last term of (4.5.32). Let $\Lambda = \{|\chi_M^\star(1) - \chi^\star(\tau)| \leq \lambda\}$ and decompose $(\chi_M^\star(1) - \chi^\star(\tau))\mathbb{1}_{\Omega_\kappa^c}$ as $(\chi_M^\star(1) - \chi^\star(\tau))\mathbb{1}_{\Omega_\kappa^c}(\mathbb{1}_\Lambda + \mathbb{1}_{\Lambda^c})$. Then,

$$\mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Omega_\kappa^c}] \leq \mathbb{E}[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Lambda^c}] + \lambda^2\mathbb{P}(\Omega_\kappa^c). \tag{4.5.35}$$

Integration by parts gives

$$\begin{aligned} &\mathbb{E}\left[(\chi_M^\star(1) - \chi^\star(\tau))^2\mathbb{1}_{\Lambda^c}\right] = \lambda^2\mathbb{P}(|\chi_M^\star(1) - \chi^\star(\tau)| > \lambda) \\ &+ 2\int_\lambda^\infty s\mathbb{P}(\chi_M^\star(1) - \chi^\star(\tau) > s)ds - 2\int_{-\infty}^{-\lambda} s\mathbb{P}(\chi_M^\star(1) - \chi^\star(\tau) \leq s)ds. \end{aligned} \tag{4.5.36}$$

Now, for $s > 0$,

$$\mathbb{P}(\chi_M^\star(1) - \chi^\star(\tau) > s) \leq \mathbb{P}(\chi_M^\star(1) \geq s/2) + \mathbb{P}(\chi^\star(\tau) \leq -s/2), \tag{4.5.37}$$

and for $s < 0$,

$$\mathbb{P}(\chi_M^\star(1) - \chi^\star(\tau) \leq s) \leq \mathbb{P}(\chi_M^\star(1) \leq s/2) + \mathbb{P}(\chi^\star(\tau) \geq -s/2). \tag{4.5.38}$$

Recall that

$$\begin{aligned} &\mathbb{P}(\chi^\star_M(1) > s) \leq \mathbb{P}(\chi^\star(1) > s), \\ &\mathbb{P}(\chi^\star_M(1)) \leq s) \leq \mathbb{P}((1-\tau)^{1/3}\widetilde{\mathcal{A}}_2(0) \leq s) = F_{\rm GUE}(s/(1-\tau)^{1/3}). \end{aligned} \tag{4.5.39}$$

Since both tails of $\chi^\star(1)$ and of the GUE Tracy-Widom distributions have (at least) exponential decay (see Prop. A.1.3 and A.1.4 in Appendix A.1), it then follows that

$$\mathbb{E}\left[(\chi^\star_M(1) - \chi^\star(\tau))^2 \mathbb{1}_{\Lambda^c}\right] \leq C\lambda^2 e^{-c\lambda/(1-\tau)^{1/3}} \tag{4.5.40}$$

for some constants C, c.

Summing up we have obtained

$$\begin{aligned} &\mathbb{E}[(\chi^\star_M(1) - \chi^\star(\tau))^2] - \mathbb{E}(\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}) \\ &\qquad = \mathcal{O}\Big((1-\tau)^{2/3}e^{-c\tilde{\kappa}^2/2}; (1-\tau)^{1/3}\varepsilon_0; \varepsilon_0^2; \lambda^2 e^{-c\tilde{\kappa}^2}; \lambda^2 e^{-c\lambda/(1-\tau)^{1/3}}\Big), \end{aligned} \tag{4.5.41}$$

with $\varepsilon_0 = \mathcal{O}(\kappa M(1-\tau)^{2/3})$. Now we choose M, κ, λ. Let $\delta \in (0, 1/3)$ be any fixed number and choose

$$M = \frac{1}{(1-\tau)^{\delta/2}}, \quad \kappa = \frac{1}{(1-\tau)^{\delta/2}}, \quad \lambda = 1. \tag{4.5.42}$$

Then, the error term in (4.5.41) is just of order $\mathcal{O}((1-\tau)^{1-\delta})$. □

Now we are ready to prove Theorem 4.2.5.

Proof of Theorem 4.2.5. We have

$$\begin{aligned} \mathbb{E}\left[(\chi^\star(1) - \chi^\star(\tau))^2\right] &= \mathbb{E}\left[(\chi^\star(1) - \chi^\star_M(1))^2\right] + \mathbb{E}\left[(\chi^\star_M(1) - \chi^\star(\tau))^2\right] \\ &\quad + 2\mathbb{E}\left[(\chi^\star(1) - \chi^\star_M(1))(\chi^\star_M(1) - \chi^\star(\tau))\right] \end{aligned} \tag{4.5.43}$$

With the choice $M = (1-\tau)^{-\delta/2}$, by Lemma 4.5.1 we have

$$\mathbb{E}\left[(\chi^\star_M(1) - \chi^\star(\tau))^2\right] = \mathbb{E}[\xi^2_{M,\tilde{w}_\tau,\tilde{w}_1}] + \mathcal{O}(1-\tau)^{1-\delta}. \tag{4.5.44}$$

By Lemma 4.4.7, the Cauchy-Schwarz inequality, and $\xi_{\infty,\tilde{w}} = (1-\tau)^{1/3}\xi_{{\rm stat},\tilde{w}}$, we obtain

$$\mathbb{E}\left[(\chi^\star(1) - \chi^\star(\tau))^2\right] = (1-\tau)^{2/3}\mathbb{E}[\xi_{{\rm stat},\tilde{w}}]^2 + \mathcal{O}((1-\tau)^{1-\delta}). \tag{4.5.45}$$

Since $\mathbb{E}[\xi_{{\rm stat},\tilde{w}}] = 0$ the claimed result is proven. □

4.6 Behavior around $\tau = 0$ for droplet initial conditions

Let us finally explain the asymptotic for $\tau \to 0$. The details are simple modifications of what we made for the case $\tau \to 1$. By Theorem 1 of [Pim17b], we have local weak convergence of the Airy$_2$ process to a Brownian motion for $\tau \to 0$,

$$\lim_{\tau\to 0} \left(\tfrac{\tau}{1-\tau}\right)^{-1/3} \left(\tilde{\mathcal{A}}_2\left(\left(\tfrac{\tau}{1-\tau}\right)^{2/3} v\right) - \tilde{\mathcal{A}}_2(0)\right) = \sqrt{2}B(v). \tag{4.6.1}$$

Lemma 4.5.1 and Lemma 4.5.2 can be easily readjusted for this case. Let us call $w_\tau = \tau^{2/3}\hat{w}_\tau$. Then, by Theorem 4.2.2, renaming $u = z\tau^{2/3}$,

$$\begin{aligned}
&\mathrm{Cov}\left(\chi^\bullet(\tau), \chi^\bullet(1)\right) \\
=&\mathrm{Cov}\left(\tau^{1/3}[\widetilde{\mathcal{A}}_2(\hat{w}_\tau) - \hat{w}_\tau^2], \tau^{1/3}\max_{u\in\mathbb{R}}\left\{\widetilde{\mathcal{A}}_2(z) - z^2 + \left(\tfrac{1-\tau}{\tau}\right)^{1/3}\mathcal{A}_2(z\tfrac{\tau^{2/3}}{(1-\tau)^{2/3}}) - z^2\tfrac{\tau}{1-\tau}\right\}\right) \\
=&\tau^{2/3}\,\mathrm{Cov}\left(\widetilde{\mathcal{A}}_2(\hat{w}_\tau), \max_{z\in\mathbb{R}}\left\{\widetilde{\mathcal{A}}_2(z) - z^2 + \sqrt{2}B(z)\right\} + \left(\tfrac{1-\tau}{\tau}\right)^{1/3}\mathcal{A}_2(0)\right) + \mathcal{O}(\tau^{1-\delta}) \\
=&\tau^{2/3}\,\mathrm{Cov}\left(\widetilde{\mathcal{A}}_2(\hat{w}_\tau), \max_{z\in\mathbb{R}}\left\{\widetilde{\mathcal{A}}_2(z) - z^2 + \sqrt{2}B(z)\right\}\right) + \mathcal{O}(\tau^{1-\delta}),
\end{aligned} \tag{4.6.2}$$

for any $\delta > 0$, where the covariance of $\widetilde{\mathcal{A}}_2(\hat{w}_\tau)$ and $\mathcal{A}_2(0)$ is zero, since they are independent processes. The second term in the covariance has the same distribution as $\xi_{\mathrm{stat},0}$, which is has expected value 0. This leads to the claimed result of Theorem 4.2.6.

Chapter 5

Stationary half-space last passage percolation

This chapter is based on [BFO19]. In this paper we study stationary last passage percolation (LPP) in half-space geometry. We determine the limiting distribution of the last passage time in a critical window close to the origin. The result is a new two-parameter family of distributions: one parameter for the strength of the diagonal bounding the half-space (strength of the source at the origin in the equivalent TASEP language) and the other for the distance of the point of observation from the origin. It should be compared with the one-parameter family giving the Baik–Rains distributions for full-space geometry. We finally show that far enough away from the characteristic line, our distributions indeed converge to the Baik–Rains family. We derive our results using a related integrable model having Pfaffian structure together with careful analytic continuation and steepest descent analysis.

5.1 Introduction

Background and motivation. A stochastic growth model in the one-dimensional Kardar–Parisi–Zhang (KPZ) universality class [KPZ86] describes the evolution of a height function $h(x,t)$ at position x and time t subject to a stochastic and local microscopic evolution. On a macroscopic scale, that is with space of order t, the evolution of the height function evolves according to a certain PDE and one has a non-random limit shape.

The following, among others, belong to the the KPZ class: the KPZ equation; directed random polymer models (where the free energy plays the role of the height function); their zero-temperature limits known as last passage percolation; and interacting particle systems like the exclusion process. Some of these models have been analyzed in the last two decades for many classes of initial conditions and/or boundary conditions. The fluctuations of the height function are of order $t^{1/3}$ and the correlation length scales as $t^{2/3}$, as conjectured in [FNS77, BKS85][1].

In particular, it is known that the limiting processes depend on subclasses of initial conditions. In full-space, that is for $x \in \mathbb{R}$ or $x \in \mathbb{Z}$, one sees the Airy$_2$ process around curved limit shape points [PS02b, Joh03, BF08], with one-point distribution the GUE Tracy–Widom distribution [TW94], discovered in [BDJ99b] and shown later to hold for a variety of models in the KPZ class [Joh00b, SS10b, ACQ11, SS10a, FV15, BCF14, Bar14].

[1] This holds true around points with smooth limit shape. Around shocks there are some differences, see e.g. [FN15, FN17, FF94, FGN17, Nej18].

For flat limit shapes and non-random initial conditions, the limit process is known as the Airy_1 process [Sas05, BFPS07] with the GOE Tracy–Widom as one-point distribution [TW96]—see [BR01c, BR01b, PS00]. Finally, stationary initial conditions also lead to flat limit shapes and the $\mathrm{Airy}_{\mathrm{stat}}$ process [BFP10], having the Baik–Rains distribution as the one-point distribution [BR00, FS06, BCFV15, IS13, Agg16, IS19, IS17]. The stationary model is obtained as a limit of some specific two-sided random initial condition[2].

For further details and recent developments around the KPZ universality class, see also the following surveys and lecture notes: [FS11, Cor12, QS15, BG12, Qua11, Fer10, Tak16].

In this paper we consider a stationary model in half-space, where the latter means having a height function $h(x,t)$ defined only on $x \in \mathbb{N}$ (or $x \in \mathbb{R}_+$). Our model, called stationary half-space last passage percolation (LPP), is defined in Section 5.2.1. In this geometry there are considerably fewer results compared to the case of full-space geometry. Of course, one has to prescribe the dynamics at site $x = 0$. If the influence on the height function of the growth mechanism at $x = 0$ is very strong, then close to the origin one will essentially see fluctuations induced by it, and since the dynamics in KPZ models has to be local (in space but also in time), one will observe Gaussian fluctuations. If the influence of the origin is small, then it will not be seen in the asymptotic behaviour. Between the two situations there is typically a critical value where a third different distribution function is observed. Furthermore, under a critical scaling, one obtains a family of distributions interpolating between the two extremes. For some versions of half-space LPP and related stochastic growth models (with non-random initial conditions) this has indeed been proven: one has a transition of the one-point distribution from Gaussian to GOE Tracy–Widom at the critical value, and GSE Tracy–Widom distribution [BR01b, SI04, BBC18, KD19][3] Furthermore, the limit process under critical scaling around the origin is also analyzed and the transition processes have been characterized [SI04, BBCS18, BBNV18, BBCS17].

However, the limiting distribution of the stationary LPP in half-space remained unresolved. In this paper we close this gap: in Theorem 5.2.3 we determine the distribution function of the stationary LPP for the finite size system and in Theorem 5.2.6 we determine the large time limiting distribution under critical scaling. A second reason for the study of the stationary case is the following. In the full-space case, it was shown in [FO19] that the first order correction of the time-time covariance for times close to each others on a macroscopic scale is governed by the variance of the Baik–Rains distribution. The reason is that the system locally converges to equilibrium. Thus, for any comparable study in half-space geometry, the knowledge of the stationary limiting distribution and/or process is necessary.

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The physical difference with the full-space problem is the source at the origin, and this makes the half-space problem richer. The influence of the boundary process is still visible in the limiting distribution: we have a new two-parameter family of distribution functions, one parameter for the strength of the source at the origin, the second for the distance from the diagonal. As a comparison, in the full-space analogue it was a one-parameter

[2]The random initial condition on the two sides is recovered using boundary sources, by the use of some Burke-type property [Bur56, DMO05], as shown for the exclusion process in [PS04].

[3]The results are obtained through a description in terms of Pfaffian structures [BR01a, BR06, Rai00, FR07, Gho17, BBCW18, BBNV18].

family of distributions with the parameter describing the distance from the characteristic line.

The limiting distribution we observe should be universal within the KPZ universality class to the same extent that the Baik–Rains distribution is for the case of full-space. By choosing the observation position far from the diagonal and setting the strength of the source to stay around a characteristic line from the origin, we furthermore recover, in the limit, the aforementioned Baik–Rains distribution from full-space. See Theorem 5.2.10.

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Concerning the methods used in this paper, there are some similarities but also important differences with respect to the full-space situation studied in [FS06, BFP10]. To identify the stationary model, it has been useful to start from the exclusion process analogy, for which the stationary measures in half-space were obtained in [Lig77]. Next we observe that, as in full-space, the desired distribution function can be obtained in a two-step procedure. First we study a two-parameter integrable model which has a Pfaffian structure. By a so-called shift argument, we can write the two-parameter distribution function in terms of the distribution function of the Pfaffian model. Finally we need to perform a limit when the sum of the two parameters goes to zero. This is achieved through analytic continuation. This last step turned out to be considerably more complicated that in the full-space case [FS06] as some exact cancellations of diverging terms happened only using the 2×2 structure of the Pfaffian kernel. Such issues did not show up in the full-space analysis.

Outline. In Section 5.2 we define the model and state the two main results of this paper. The finite-time formula for the stationary LPP in half-space, Theorem 5.2.3, is derived in Section 5.3, while in Section 5.4 we prove the asymptotic result of Theorem 5.2.6.

Notations. Throughout this work we handle numerous complex integrals. To simplify matters, we choose the following special notation for types of contours we will often encounter. First, Γ_I will indicate any simple counter-clockwise contour around the set of points I. We remark that sometimes such a contour will just be a disjoint union of simple counter-clockwise contours each encircling one of the points in I. In the large time asymptotics sections we use the following notation for the typical Airy contours, denoting with ${}_{I}\langle_{J}$ a down-oriented contour coming in a straight line from $\exp(\pi i/3)\infty$ to a point on the real line to the right of I and to the left of J, and continuing in a straight line to $\exp(5\pi i/3)\infty$, and with ${}_{I}\rangle_{J}$ an up-oriented contour from $\exp(4\pi i/3)\infty$ to $\exp(2\pi i/3)\infty$. Examples are depicted in Figure 5.3.

For two functions f, g we use (the usual) bra-ket notation as follows: the scalar product on $L^2(s,\infty)$ (or (S,∞) depending on the section) is denoted by

$$\langle f \,| g\rangle = \int_s^\infty f(x)g(x)dx \tag{5.1.1}$$

while by $|f\rangle\langle g|$ we denote the outer product kernel

$$|f\rangle\langle g|\,(x,y) = f(x)g(y). \tag{5.1.2}$$

5.2 Model and main results

5.2.1 Last passage percolation

Before going to the specific model studied in this paper, let us recall the more generic last passage percolation (LPP) model on $\mathbb{Z}^2$ and explain where the denomination half-space comes from.

Consider independent random variables $\{\omega_{i,j},\, i,j \in \mathbb{Z}\}$. An *up-right path* π on $\mathbb{Z}^2$ from a point A to a point E is a sequence of points $(\pi(0), \pi(1), \ldots, \pi(n))$ in $\mathbb{Z}^2$ such that $\pi(k+1) - \pi(k) \in \{(0,1),(1,0)\}$, with $\pi(0) = A$ and $\pi(n) = E$, and where n is called the length $\ell(\pi)$ of π. Now, given a set of points S_A and E, one defines the *last passage time* $L_{S_A \to E}$ as

$$L_{S_A\to E} = \max_{\substack{\pi:A\to E\\ A\in S_A}} \sum_{1\le k\le \ell(\pi)} \omega_{\pi(k)}. \tag{5.2.1}$$

Finally, we denote by $\pi^{\max}_{S_A\to E}$ any maximizer of the last passage time $L_{S_A\to E}$. For continuous random variables, the maximizer is a.s. unique. In this paper we consider exponentially distributed random variables, which give the well-known connection with the totally asymmetric simple exclusion process (TASEP).

TASEP is an interacting particle system on $\mathbb{Z}$ with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. For a configuration $\eta \in \Omega$, $\eta = (\eta_j, j \in \mathbb{Z})$, η_j is the occupation variable at site j, which is 1 if and only if j is occupied by a particle. TASEP has generator L given by [Lig99]

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1})\big(f(\eta^{j,j+1}) - f(\eta)\big) \tag{5.2.2}$$

where f are local functions (depending only on finitely many sites) and $\eta^{j,j+1}$ denotes the configuration η with occupations at sites j and $j+1$ interchanged. Notice that for TASEP the ordering of particles is preserved. That is, if initially one orders particles from right to left as

$$\ldots < x_2(0) < x_1(0) < 0 \le x_0(0) < x_{-1}(0) < \cdots \tag{5.2.3}$$

then for all times $t \ge 0$ also $x_{n+1}(t) < x_n(t)$, $n \in \mathbb{Z}$.

The connection between TASEP and LPP is as follows. Take $\omega_{i,j}$ to be the waiting time of particle j to jump from site $i-j-1$ to site $i-j$. Then $\omega_{i,j}$ are exponential random variables. Further, choosing the set $S_A = \{(u,k) \in \mathbb{Z}^2 : u = k + x_k(0), k \in \mathbb{Z}\}$, we have that

$$\mathbb{P}\left(L_{S_A\to(m,n)} \le t\right) = \mathbb{P}\left(x_n(t) \ge m-n\right) = \mathbb{P}\left(h(m-n,t) \ge m+n\right). \tag{5.2.4}$$

The denomination full-space (respectively half-space) LPP comes from the fact that the height function and particles live on $\mathbb{Z}$ (respectively $\mathbb{N}$). The relation (5.2.4) implies that the random variables in LPP are restricted to $\{(m,n)|m \ge n\}$ (equivalently, we can think that the other random variables are set to be 0).

In the framework of some interacting particle systems, with TASEP being the simplest case, Liggett studied the invariant measures for the full-space geometry—see Theorem 1.1 of [Lig77]. To achieve his result, he first considered a finite system from which the half-space model is a simple limiting case. In particular, for TASEP defined on $\mathbb{N}$ with particles entering at the origin with a given rate $\lambda \in [0,1]$, i.e. the origin playing the role of a reservoir of particles, he showed that the stationary measure with particle density $\rho = \lambda$ on $\mathbb{N}$ is a product measure. For this reason, the LPP analogue is obtained by considering weights on the diagonal as being exponentially distributed of parameter ρ (below set $\rho = \frac{1}{2}+\alpha$), while the random initial condition in $\mathbb{N}$ can be replaced by Burke's theorem [Bur56] with a first row in the LPP geometry having exponentially distributed random variables of parameter $1-\rho$.

It is worth mentioning that for half-space TASEP with input rate higher than 1, there are stationary measures different from blocking measures, and which are not product measures. A representation using matrix product ansatz is given in [?Theorem3.2, Gro04]. The mapping between LPP and TASEP would imply that the ω_{ij} of the corresponding LPP are not independent random variables anymore. Our techniques do not apply however in such cases.

5.2.2 The stationary half-space model

Let us now focus on the half-space LPP model. On the set $\mathcal{D} = \{(i,j) \in \mathbb{Z}^2 | 1 \le j \le i\}$ we consider independent non-negative random variables $\{\omega_{i,j}\}_{(i,j)\in\mathcal{D}}$. Then, the half-space LPP time to the point (n,m) (for $m \le n$), denoted $L_{n,m}$, is given by

$$L_{n,m} = \max_{\pi:(1,1)\to(n,m)} \sum_{(i,j)\in\pi} \omega_{i,j} \tag{5.2.5}$$

where the maximum is over up-right paths in $\mathcal{D}$ from $(1,1)$ to (m,n), i.e. paths with increments in $\{(0,1),(1,0)\}$.

We are interested in the stationary version of this model, which as we will see, can be obtained as follows. Let us write $\mathrm{Exp}(a)$ for an exponential random variable with parameter $a > 0$. Then, the stationary version is given by setting

$$\omega_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2}+\alpha\right), & i = j > 1, \\ \mathrm{Exp}\left(\frac{1}{2}-\alpha\right), & j = 1, i > 1, \\ 0, & \text{if } i = j = 1, \\ \mathrm{Exp}(1), & \text{otherwise} \end{cases} \tag{5.2.6}$$

where $\alpha \in (-1/2, 1/2)$ is a fixed parameter. A schematic depiction is drawn in Figure 5.1.

This model is stationary in the sense of [BCS06], i.e. it has stationary increments as stated in the following lemma.

Lemma 5.2.1. *(Half-space version of Lemma 4.2 of* [BCS06]*) For any $j \ge 1$, the increments along the horizontal direction*

$$\{L_{i+1,j} - L_{i,j}, i \ge 1\} \tag{5.2.7}$$

are i.i.d. $\mathrm{Exp}\left(\frac{1}{2}-\alpha\right)$ *random variables; those along the vertical direction*

$$\{L_{j,i+1} - L_{j,i}, i \ge 1\} \tag{5.2.8}$$

are i.i.d. $\mathrm{Exp}\left(\frac{1}{2}+\alpha\right)$ *random variables; finally, the increments along the anti-diagonal direction*

$$\{L_{i,j} - L_{i+1,j-1}, i,j \ge 1\} \tag{5.2.9}$$

are i.i.d. $\mathrm{Exp}(1)$ *random variables.*

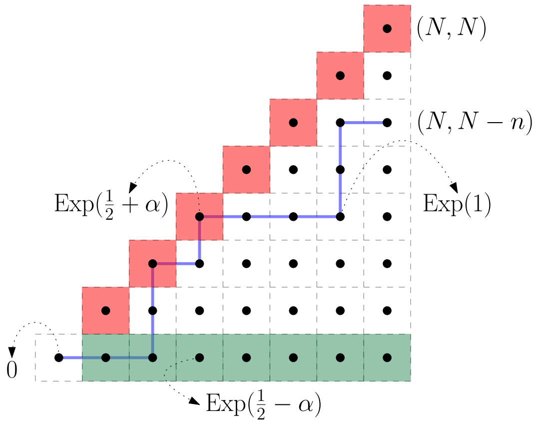

Figure 5.1: A possible LPP path (polymer) starting at the origin to the point $(N, N-n)$. The dots are independent random variables: deterministically 0 at the origin, $\mathrm{Exp}(\frac{1}{2}+\alpha)$ (respectively $\mathrm{Exp}(\frac{1}{2}-\alpha)$) on the rest of the diagonal (respectively the bottom line), and Exp(1) everywhere else in the bulk.

Proof. The result for the increments along the vertical and the horizontal directions is an immediate consequence of Lemma 4.1 of [BCS06] for the full-space model. For the increments along the anti-diagonal, we just have to observe that, in distribution, $L_{i,j} - L_{i+1,j-1} = X - Y$, with $X \sim \mathrm{Exp}(\frac{1}{2}+\alpha)$ and $Y \sim \mathrm{Exp}(\frac{1}{2}-\alpha)$. □

Remark 5.2.2. For the stationary model, we do not have a good formula to study the statistics of $L_{N,N-n}$. However, as it was already the case for the full-space stationary LPP, there is a way to recover it in a two-step procedure. First we consider a related LPP model, with two parameters, α, β, for which we have an explicit formula of the distribution in terms of a Fredholm pfaffian. The distribution of the original model is recovered by a standard shift argument and an analytic continuation. This last step turns out to be far from trivial and different from both the one for the full-space stationary model [FS06] and its multi-point extension [BFP10].

5.2.3 Finite time distribution for the stationary LPP

The first result of this paper is a formula for the distribution of the stationary LPP time $L = L_{N,N-n}$. In order to state the result, we need to introduce a few functions and kernels, which are going to be in the final expression. Let us define the following f functions

$$f_+^{-\alpha}(x) = \Phi(x,-\alpha)\left(\tfrac{1}{2}+\alpha\right)^n, \qquad f_-^{-\alpha}(x) = \frac{\Phi(x,-\alpha)}{\left(\frac{1}{2}-\alpha\right)^n} \tag{5.2.10}$$

where $\Phi(x,z) = \left[\frac{\frac{1}{2}+z}{\frac{1}{2}-z}\right]^{N-1} e^{-xz}$, the following g functions

$$\begin{aligned} g_1(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}}\Phi(x,z)\left(\tfrac{1}{2}-z\right)^n \frac{z+\alpha}{2z}, \quad \tilde{g}_2(x) = \oint_{\Gamma_{1/2,\alpha}} \frac{dz}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\left(\frac{1}{2}+z\right)^n}\frac{1}{z-\alpha}, \\ \mathrm{g}_3(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}}\Phi(x,z)\left(\tfrac{1}{2}-z\right)^n \frac{1}{z+\alpha}, \quad \tilde{\mathrm{g}}_4(x) = \oint_{\Gamma_{1/2,\pm\alpha}} \frac{dz}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\left(\frac{1}{2}+z\right)^n}\frac{2z}{(z-\alpha)(z+\alpha)^2} \end{aligned} \tag{5.2.11}$$

and finally let

$$e^{\alpha}(s) = -\oint_{\Gamma_{1/2,\alpha}} \frac{dz}{2\pi\mathrm{i}} \frac{\Phi(s,z)}{\Phi(s,\alpha)} \frac{(\frac{1}{2}+\alpha)^n}{(\frac{1}{2}+z)^n} \frac{1}{(z-\alpha)^2}. \tag{5.2.12}$$

Also, define the following *anti-symmetric* kernel $\overline{\mathsf{K}}$:

$$\begin{aligned}
\overline{\mathsf{K}}_{11}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[(\tfrac{1}{2}-z)(\tfrac{1}{2}+w)\right]^n \frac{(z-\alpha)(w+\alpha)(z+w)}{4zw(z-w)},\\
\overline{\mathsf{K}}_{12}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2,\alpha}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[\frac{\frac{1}{2}-z}{\frac{1}{2}-w}\right]^n \frac{z-\alpha}{w-\alpha} \frac{z+w}{2z(z-w)}\\
=& -\overline{\mathsf{K}}_{21}(y,x),\\
\overline{\mathsf{K}}_{22}(x,y) =& \varepsilon(x,y) + \oint \frac{dz}{2\pi\mathrm{i}} \oint \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n} \frac{1}{z-w} \left(\frac{1}{z+\alpha}+\frac{1}{w-\alpha}\right)
\end{aligned} \tag{5.2.13}$$

where the integration contours for $\overline{\mathsf{K}}_{22}$ are $\Gamma_{1/2,-\alpha} \times \Gamma_{-1/2}$ for the term with $1/(z+\alpha)$ and $\Gamma_{1/2} \times \Gamma_{-1/2,\alpha}$ for the term with $1/(w-\alpha)$, and where $\varepsilon = \varepsilon_0 + \varepsilon_1$ with

$$\varepsilon_0(x,y) = -\operatorname{sgn}(x-y) \frac{e^{-\alpha|x-y|}}{\left(\frac{1}{4}-\alpha^2\right)^n}, \qquad \varepsilon_1(x,y) = -\operatorname{sgn}(x-y) \oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}} \frac{2ze^{-z|x-y|}}{(z^2-\alpha^2)\left(\frac{1}{4}-z^2\right)^n}. \tag{5.2.14}$$

Finally, define

$$\begin{aligned}
\widetilde{\mathsf{K}}_{12}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[\frac{(\frac{1}{2}-z)}{(\frac{1}{2}-w)}\right]^n \frac{z-\alpha}{w-\alpha} \frac{z+w}{2z(z-w)},\\
\widetilde{\mathsf{K}}_{22}(x,y) =& \oint_{\Gamma_{1/2,-\alpha}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n} \frac{1}{(z+\alpha)(w-\alpha)} \frac{z+w}{z-w}
\end{aligned} \tag{5.2.15}$$

and

$$\begin{aligned}
h_1 &= \widetilde{\mathsf{K}}_{22} f_+^{-\alpha} + \varepsilon_1 f_+^{-\alpha} - \tilde{\mathsf{g}}_4 + j^{\alpha}(s,\cdot),\\
h_2 &= \widetilde{\mathsf{K}}_{12} f_+^{-\alpha} + \mathsf{g}_3
\end{aligned} \tag{5.2.16}$$

with

$$j^{\alpha}(s,y) = \left(\frac{\sinh \alpha(s-y)}{\alpha} + (s-y)e^{\alpha(s-y)}\right) f_-^{-\alpha}(s). \tag{5.2.17}$$

Our first main theorem, a finite size result, is as follows.

Theorem 5.2.3. *Let $\alpha \in (-1/2, 1/2)$ be a real number and $1 \le N$, $0 \le n \le N-1$ be positive integers. Let $L_{N,N-n}$ be the stationary LPP time from $(1,1)$ to $(N, N-n)$ in the model of weights given by* (5.2.6)*. Then*

$$\mathbb{P}(L_{N,N-n} \le s) = \partial_s \left\{ \operatorname{pf}(J-\overline{\mathsf{K}}) \cdot \left[e^{\alpha}(s) - \left\langle -g_1 \quad \tilde{g}_2 \middle| (\mathbb{1} - J^{-1}\overline{\mathsf{K}})^{-1} \begin{pmatrix} -h_1 \\ h_2 \end{pmatrix} \right\rangle \right] \right\} \tag{5.2.18}$$

where the Fredholm pfaffian is taken over $L^2(s,\infty)$ and where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

The proof of Theorem 5.2.3 is carried out in Section 5.3.

Remark 5.2.4. We remark that our kernel $\overline{\mathsf{K}}$ with parameter α and size N is identical to the integrable kernel K of Section 5.3.1 with parameters $-\alpha$ and $\beta = 1/2$ and size $N-1$. The latter corresponds to a model that has been studied in [BBCS18]. As a consequence, the pfaffian $\mathrm{pf}(J - \overline{\mathsf{K}})$ on $L^2(s,\infty)$ is the distribution of the corresponding LPP and is in $(0,1)$ for any fixed s.

5.2.4 Asymptotic result

In order to discuss the scaling limit, we first have to determine: (a) the limit shape approximation; and (b) the position of the end-point $(N, N-n)$ which is connected with the origin by the characteristic line. The reason is that if one does not take a point in a $N^{2/3}$ neighborhood of the characteristic leaving from the origin, then one will not see the $N^{1/3}$ fluctuations typical for KPZ models; rather one will only see Gaussian fluctuations whose origin is in the boundary terms—the polymer spend a time much larger that $N^{2/3}$ either on the diagonal or in the first row.

Concerning the limit shape, notice that $L_{N,N-\eta N} = L_{N,1} + (L_{N,N-\eta N} - L_{N,1})$. The two terms are not independent, but individually are sums of independent random variables (see Lemma 5.2.1). Thus, one expects that for $N \gg 1$,

$$L_{N,N-\eta N} \simeq \frac{N}{\frac{1}{2}-\alpha} + \frac{(1-\eta)N}{\frac{1}{2}+\alpha} = \frac{N}{\frac{1}{4}-\alpha^2} - \frac{\eta N}{\frac{1}{2}+\alpha}. \tag{5.2.19}$$

For a stationary situation in TASEP with particle density ρ, the characteristic line has speed $1-2\rho$. In terms of last passage percolation, the density ρ becomes a parameter $\alpha = \rho - 1/2$ (see e.g. [PS02a]) and the speed $1-2\rho$ becomes a slope in the LPP geometry given by $y/x = \rho^2/(1-\rho)^2$ (see e.g. [BFP10]). Thus, with $x = N$ and $y = N - \eta N$ we have

$$n = \eta N = -\frac{2\alpha}{(\frac{1}{2}-\alpha)^2} N. \tag{5.2.20}$$

Therefore, to obtain a non-trivial scaling limit, given the value of α, one needs to choose n in a $\mathcal{O}(N^{2/3})$-neighborhood of (5.2.20) and to consider fluctuations on a $N^{1/3}$-scale around (5.2.19). Choosing an end-point order N away from (5.2.20) leads to Gaussian behaviour in the $N^{1/2}$ scale, for the maximizer will spend $\mathcal{O}(N)$ of its time either in the first row or on the diagonal in that case. Recalling that $n \geq 0$, (5.2.20) cannot hold for $\alpha > 0$. In that case, the polymer will spend a macroscopic portion of its time on the diagonal and will have Gaussian fluctuations.

In this paper we consider the critical scaling where α is close to 0, namely we set

$$\alpha = \delta 2^{-4/3} N^{-1/3}, \qquad \eta N = n = u 2^{5/3} N^{2/3}. \tag{5.2.21}$$

With this choice we have

$$L_{N,N-\eta N} \simeq 4N - 2u 2^{5/3} N^{2/3} + \delta(2u+\delta) 2^{4/3} N^{1/3} \tag{5.2.22}$$

and

$$-\frac{2\alpha}{(\frac{1}{2}-\alpha)^2} N = -\delta 2^{5/3} N^{2/3} + \mathcal{O}(N^{1/3}). \tag{5.2.23}$$

Remark 5.2.5. We decided not to include the $\mathcal{O}(N^{1/3})$ term $\delta(2u+\delta)2^{4/3}N^{1/3}$ of the limit shape approximation in our calculations below, since many formulas are more compact without it. That is, we consider the scaling

$$s = 4N - 2u2^{5/3}N^{2/3} + S\,2^{4/3}N^{1/3}. \tag{5.2.24}$$

However, it has to be taken into account if one wants to determine various limits, e.g. $u \to \infty$ and/or $\delta \to \pm\infty$. In these limits, we first have to replace S by $S + \delta(2u+\delta)$. Also, when taking $u \to \infty$, by (5.2.23), we will have to set $u = -\delta + \tau$ and take $\delta \to -\infty$ to get a non-Gaussian limit. This is performed in Section 5.2.5, where we recover the Baik–Rains distribution.

As for the finite N case, the main theorem in the $N \to \infty$ limit requires definitions of various objects. Let us define the functions

$$\begin{aligned}
f^{-\delta,u}(X) &= e^{-\frac{\delta^3}{3}-\delta^2 u+\delta X},\\
e^{\delta,u}(S) &= -\int_{\circlearrowleft_{\delta}} \frac{d\zeta}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta S}}{e^{\frac{\delta^3}{3}+\delta^2 u-\delta S}} \frac{1}{(\zeta-\delta)^2},\\
j^{\delta,u}(S,X) &= \left[\frac{\sinh\delta(X-S)}{\delta} + (X-S)e^{\delta(X-S)}\right] f^{-\delta,-u}(S)
\end{aligned} \tag{5.2.25}$$

as well as

$$\begin{aligned}
&\mathscr{g}_1^{\delta,u}(X) = \int_{{}_0\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X} \frac{\zeta+\delta}{2\zeta}, \quad \tilde{\mathscr{g}}_2^{\delta,u}(X) = \int_{\circlearrowleft_{\delta}} \frac{d\zeta}{2\pi\mathrm{i}} e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X} \frac{1}{\zeta-\delta},\\
&\mathscr{g}_3^{\delta,u}(X) = \int_{{}_{-\delta}\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X} \frac{1}{\zeta+\delta}, \quad \tilde{\mathscr{g}}_4^{\delta,u}(X) = \int_{\circlearrowleft_{\pm\delta}} \frac{d\zeta}{2\pi\mathrm{i}} e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X} \frac{2\zeta}{(\zeta-\delta)(\zeta+\delta)^2}.
\end{aligned} \tag{5.2.26}$$

The limit of $\overline{\mathsf{K}}$ is the following *anti-symmetric* kernel $\overline{\mathcal{A}}$:

$$\begin{aligned}
\overline{\mathcal{A}}_{11}(X,Y) &= -\int_{{}_0\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} \int_{\circlearrowright_{0,\zeta}} \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}+\omega^2 u-\omega Y}} (\zeta-\delta)(\omega+\delta) \frac{\zeta+\omega}{4\zeta\omega(\zeta-\omega)},\\
\overline{\mathcal{A}}_{12}(X,Y) &= -\int_{{}_0\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} \int_{{}_{\delta}\circlearrowright_{\zeta}} \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{\zeta-\delta}{\omega-\delta} \frac{\zeta+\omega}{2\zeta(\zeta-\omega)}\\
&= -\overline{\mathcal{A}}_{21}(Y,X),\\
\overline{\mathcal{A}}_{22}(X,Y) &= \mathcal{E}(X,Y) + \int \frac{d\zeta}{2\pi\mathrm{i}} \int \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{1}{\zeta-\omega} \left(\frac{1}{\zeta+\delta} + \frac{1}{\omega-\delta}\right)
\end{aligned} \tag{5.2.27}$$

where in $\overline{\mathcal{A}}_{22}$ the integration contours, for (ζ,ω), are $\circlearrowleft_{-\delta} \times \circlearrowright_{\zeta}$ for the term $1/(\zeta+\delta)$, and $\circlearrowleft \times {}_{\delta}\circlearrowright_{\zeta}$ for the term $1/(\omega-\delta)$. We have denoted $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1$ with

$$\begin{aligned}
\mathcal{E}_0(X,Y) &= -\operatorname{sgn}(X-Y)e^{\delta|X-Y|+2\delta^2 u},\\
\mathcal{E}_1(X,Y) &= -\operatorname{sgn}(X-Y)\int_{{}_{\pm\delta}\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} e^{-\zeta|X-Y|+2\zeta^2 u} \frac{2\zeta}{\zeta^2-\delta^2}.
\end{aligned} \tag{5.2.28}$$

Finally, we also set

$$\begin{aligned}\widetilde{\mathcal{A}}_{12}(X,Y) &= -\int_{0\circlearrowleft} \frac{d\zeta}{2\pi\mathrm{i}} \int_{\circlearrowright \delta,\zeta} \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}-\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{\zeta-\delta}{\omega-\delta} \frac{\zeta+\omega}{2\zeta(\zeta-\omega)}, \\ \widetilde{\mathcal{A}}_{22}(X,Y) &= \int_{\circlearrowleft -\delta} \frac{d\zeta}{2\pi\mathrm{i}} \int_{\circlearrowright \delta,\zeta} \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{1}{(\zeta+\delta)(\omega-\delta)} \frac{\zeta+\omega}{\zeta-\omega}\end{aligned} \tag{5.2.29}$$

and

$$\begin{aligned}\mathscr{h}_1^{\delta,u}(Y) &= \int_S^\infty dV \widetilde{\mathcal{A}}_{22}(Y,V) f^{-\delta,u}(V) + \int_S^\infty dV \mathcal{E}_1(Y,V) f^{-\delta,u}(V) - \tilde{\mathscr{g}}_4^{\delta,u}(Y) + j^{\delta,u}(S,Y), \\ \mathscr{h}_2^{\delta,u}(Y) &= \int_S^\infty dV \widetilde{\mathcal{A}}_{12}(Y,V) f^{-\delta,u}(V) + \mathscr{g}_3^{\delta,u}(Y).\end{aligned} \tag{5.2.30}$$

Then, the limiting distribution of the rescaled last passage percolation in the stationary case is given as follows.

Theorem 5.2.6. *Let $\delta \in \mathbb{R}$, $u > 0$ be parameters. Consider the stationary LPP time $L_{N,N-n}$ from $(1,1)$ to $(N, N-n)$ and the scaling*

$$n = u2^{5/3}N^{2/3}, \qquad \alpha = \delta 2^{-4/3}N^{-1/3}. \tag{5.2.31}$$

We have that

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N-n} - 4N + 4u(2N)^{2/3}}{2^{4/3}N^{1/3}} \le S \right) = F_{0,\,\mathrm{half}}^{(\delta,u)}(S) \tag{5.2.32}$$

with

$$F_{0,\,\mathrm{half}}^{(\delta,u)}(S) = \partial_S \left\{ \mathrm{pf}(J-\overline{\mathcal{A}}) \cdot \left[e^{\delta,u}(S) - \left\langle -\mathscr{g}_1^{\delta,u} \quad \mathscr{g}_2^{\delta,u} \middle| (\mathbb{1} - J^{-1}\overline{\mathcal{A}})^{-1} \begin{pmatrix} -\mathscr{h}_1^{\delta,u} \\ \mathscr{h}_2^{\delta,u} \end{pmatrix} \right\rangle \right] \right\} \tag{5.2.33}$$

where the Fredholm pfaffian is taken over $L^2(S,\infty)$ and where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark 5.2.7. The origin of the terms $\widetilde{\mathcal{A}}_{12}$ and $\widetilde{\mathcal{A}}_{22}$ in lieu of $\overline{\mathcal{A}}_{12}$ and $\overline{\mathcal{A}}_{22}$ stems from the fact that for $\delta \ge 0$, the product of the latter with $f^{-\delta,u}$ is not well-defined. However, for $\delta < 0$ this is not the case and so using the tilde kernels is not necessary. For that reason and for $\delta < 0$, we can thus simplify the expression of the $\mathscr{h}_k^{\delta,u}$ entering equation (5.2.33) as follows.

Lemma 5.2.8. *For $\delta < 0$, the formula* (5.2.33) *holds also for $\mathscr{h}_k^{\delta,u}$ replaced by $\tilde{\mathscr{h}}_k^{\delta,u}$, $k = 1, 2$, with the latter defined by*

$$\begin{aligned}\tilde{\mathscr{h}}_1^{\delta,u}(Y) &= \int_S^\infty dV \overline{\mathcal{A}}_{22}(Y,V) f^{-\delta,u}(V) - \tilde{\mathscr{g}}_4^{\delta,u}(Y), \\ \tilde{\mathscr{h}}_2^{\delta,u}(Y) &= \int_S^\infty dV \overline{\mathcal{A}}_{12}(Y,V) f^{-\delta,u}(V) + \mathscr{g}_3^{\delta,u}(Y).\end{aligned} \tag{5.2.34}$$

Remark 5.2.9. The kernel $\overline{\mathcal{A}}$ with parameter δ corresponds to the limiting crossover kernel from [BBCS18, BBNV18, SI04] after matching the δ parameter with notations therein. This arises when considering LPP in half-space with boundary term only along the diagonal. The respective distribution, for $u = 0$, is the interpolating GOE to GSE distribution of Baik and Rains [BR01b]. Similar to Remark 5.2.4, the Pfaffian $\mathrm{pf}(J - \overline{\mathcal{A}})$ on $L^2(S,\infty)$ is in $(0, 1)$ for any fixed S.

5.2.5 Limit transition to the Baik–Rains distribution

One might wonder if the Baik–Rains distribution with parameter τ arises in some appropriate limit[4]. The answer is affirmative. Heuristically, when u is increasing (the end-point of the LPP is moving away from the diagonal), the path maximizing the polymer will visit the diagonal a distance of $\mathcal{O}(N^{2/3})$ away from the origin less and less frequently. Thus, in the $u \to \infty$ limit, the geometry is similar to the full-space problem and one might expect to recover the Baik–Rains distribution. This is the result presented in this section.

To state the theorem, we need to introduce a few functions and a limiting kernel. Let us define[5]

$$
\begin{aligned}
\mathcal{R}_\tau(s) &= -e^{-\frac{2}{3}\tau^3 - s\tau} \int_{\swarrow_{-\tau}} \frac{dz}{2\pi \mathrm{i}} e^{\frac{z^3}{3} - z(s+\tau^2)} \frac{1}{(z+\tau)^2}, \\
\Psi_\tau(x) &= \int_{\swarrow_{-\tau}} \frac{dz}{2\pi \mathrm{i}} e^{\frac{z^3}{3} - z(x+\tau^2)} \frac{1}{z+\tau}, \\
\Phi_\tau(y) &= e^{-\frac{2}{3}\tau^3 - s\tau} \int_{\swarrow} \frac{dz}{2\pi \mathrm{i}} \int_{\nearrow_{\tau, z}} \frac{dw}{2\pi \mathrm{i}} \frac{e^{\frac{z^3}{3} - z(y+\tau^2)}}{e^{\frac{w^3}{3} - w(s+\tau^2)}} \frac{1}{(z-w)(w-\tau)}
\end{aligned}
\tag{5.2.35}
$$

and the shifted Airy kernel[6]

$$
\mathcal{K}_{\mathrm{Ai},\tau}(x,y) = -\int_{\swarrow} \frac{dz}{2\pi \mathrm{i}} \int_{\nearrow_{z}} \frac{dw}{2\pi \mathrm{i}} \frac{e^{\frac{z^3}{3} - z(x+\tau^2)}}{e^{\frac{w^3}{3} - w(y+\tau^2)}} \frac{1}{(z-w)}. \tag{5.2.36}
$$

Theorem 5.2.10. *Let $S = s + \delta(2u+\delta)$ and $u + \delta = \tau$ fixed. Then we have:*

$$
\lim_{u\to\infty} F^{(\delta,u)}_{0,\,\mathrm{half}}(S) = F_{\mathrm{BR},\tau}(s) \tag{5.2.37}
$$

where $F_{\mathrm{BR},\tau}(s)$ is the extended Baik–Rains distribution, defined by

$$
F_{\mathrm{BR},\tau}(s) = \partial_s \left[F_{\mathrm{GUE}}(s+\tau^2) \cdot \left(\mathcal{R}_\tau - \left\langle \Psi_\tau \left| (\mathbb{1} - \mathcal{K}_{\mathrm{Ai},\tau})^{-1} \Phi_\tau \right\rangle \right) \right] \tag{5.2.38}
$$

with the operators in the scalar product being on $L^2(s,\infty)$ and with F_{GUE} the Tracy–Widom distribution.

5.3 Finite system stationary model: proof of Theorem 5.2.3

5.3.1 The integrable model

In this section we consider the slightly modified LPP model with weights

$$
\tilde{\omega}_{i,j} = \begin{cases} \mathrm{Exp}\left(\frac{1}{2} + \alpha\right), & i = j > 1, \\ \mathrm{Exp}\left(\frac{1}{2} + \beta\right), & j = 1, i > 1, \\ \mathrm{Exp}\left(\alpha + \beta\right), & i = j = 1, \\ \mathrm{Exp}(1), & \text{otherwise} \end{cases} \tag{5.3.1}
$$

[4]The Baik–Rains distribution away from the characteristic line is given in [BR00], formula (3.35).

[5]Here we use notation as in [FS06, BFP10] with one exception: for the functions involved, we keep their contour integral representations instead of rewriting them in terms of (integrals of) Airy and exponential functions. The original expression in [BR00] looks quite different, but as is shown in [?AppendixA, FS05a], gives the same distribution.

[6]The reason for the minus sign in front is that our z contour $\swarrow$ is oriented downwards.

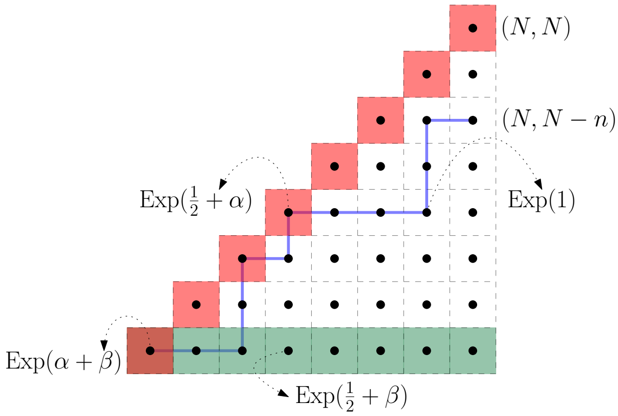

Figure 5.2: A possible LPP path (polymer) starting at the origin to the point $(N, N-n)$. The dots are independent exponential random variables: $\mathrm{Exp}(\alpha+\beta)$ at the origin, $\mathrm{Exp}(\frac{1}{2}+\alpha)$ (respectively $\mathrm{Exp}(\frac{1}{2}+\beta)$) on rest of the diagonal (respectively the bottom line), and Exp(1) everywhere else in the bulk.

where $\alpha \in (-1/2, 1/2), \beta \in (-1/2, 1/2)$ are parameters satisfying $\alpha + \beta > 0$—see Figure 5.2 for an illustration. We denote by $L^{\mathrm{pf}}_{N,N-n}$ the LPP with weights $\tilde{\omega}$ to the point $(N, N-n)$.

For the case of $\beta > 0$, it has been shown that the distribution of $L^{\mathrm{pf}}_{N,N-n}$ is given by a Fredholm pfaffian. The next theorem, which is the starting point of our analysis, is a simple corollary of the work of Baik–Barraquand–Corwin–Suidan [BBCS18] and Betea–Bouttier–Nejjar–Vuletić [BBNV18].

Theorem 5.3.1. *Let $\beta \in (0, 1/2)$ and $\alpha \in (-1/2, 1/2)$. Then, for $s \in \mathbb{R}_+$,*

$$\mathbb{P}(L^{\mathrm{pf}}_{N,N-n} \leq s) = \mathrm{pf}(J - K)_{L^2(s,\infty)}, \tag{5.3.2}$$

where $K = K(x, y)$ is the following 2×2 matrix kernel:

$$\begin{aligned} K_{11}(x,y) &= -\oint_{\Gamma_{1/2,\beta}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2,-\beta}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[(\tfrac{1}{2}-z)(\tfrac{1}{2}+w)\right]^n \frac{\theta(z)}{\theta(w)} \frac{(z+\alpha)(w-\alpha)(z+w)}{4zw(z-w)}, \\ K_{12}(x,y) &= -\oint_{\Gamma_{1/2,\beta}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2,-\alpha,-\beta}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[\frac{\frac{1}{2}-z}{\frac{1}{2}-w}\right]^n \frac{\theta(z)}{\theta(w)} \frac{z+\alpha}{w+\alpha} \frac{z+w}{2z(z-w)} \\ &= -K_{21}(y,x), \\ K_{22}(x,y) &= \oint_{\Gamma_{1/2,\alpha,\beta}} \frac{dz}{2\pi\mathrm{i}} \oint_{\Gamma_{-1/2,-\alpha,-\beta}} \frac{dw}{2\pi\mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n} \frac{\theta(z)}{\theta(w)} \frac{(z+w)(z-w)^{-1}}{(z-\alpha)(w+\alpha)} \\ &\quad + \tilde{\varepsilon}(x,y). \end{aligned} \tag{5.3.3}$$

Here we have denoted

$$\begin{aligned} \theta(z) &= \frac{z+\beta}{z-\beta}, \\ \Phi(x,z) &= e^{-xz}\phi(z), \qquad \textit{with} \quad \phi(z) = \left(\frac{\frac{1}{2}+z}{\frac{1}{2}-z}\right)^{N-1} \end{aligned} \tag{5.3.4}$$

and $\tilde{\varepsilon} = \varepsilon_1 + \varepsilon_2$ *with*

$$\varepsilon_1(x,y) = -\operatorname{sgn}(x-y) \oint_{\Gamma_{1/2}} \frac{dz}{2\pi \mathrm{i}} \frac{2ze^{-z|x-y|}}{(z^2-\alpha^2)\left(\frac{1}{4}-z^2\right)^n}, \quad \varepsilon_2(x,y) = -\operatorname{sgn}(x-y) \frac{e^{-\alpha|x-y|}}{\left(\frac{1}{4}-\alpha^2\right)^n}. \tag{5.3.5}$$

Proof. We now explain how we obtained the kernel representation above. We start from the kernel of Theorem 2.6.1 for the analogous model with geometric weights and take the geometric to exponential limit. More precisely, in the statement of Theorem 2.6.1 we take $x_1 = b$, $x_2 = \cdots = x_N = \sqrt{q}$ for parameters $b, q < 1$ and then take the limit $\epsilon \to 0$ while scaling the parameters as

$$(a,b) = (1-\epsilon\alpha, 1-\epsilon\beta), \qquad q = 1-\epsilon, \qquad (k,\ell) = \epsilon^{-1}(x,y). \tag{5.3.6}$$

The resulting kernel is the following:

$$\begin{aligned} K_{11}(x,y) &= \frac{1}{(2\pi \mathrm{i})^2} \int_{0\langle\beta} dz \int_{0\rangle\beta} dw F(z)F(w) \frac{z-w}{4zw(z+w)}, \\ K_{12}(x,y) &= -K_{21}(y,x) = \frac{1}{(2\pi \mathrm{i})^2} \int_{0,w\langle\beta} dz \int_{-\alpha,-\beta\rangle z} dw \frac{F(z)}{F(w)} \frac{z+w}{2z(z-w)}, \\ K_{22}(x,y) &= \frac{1}{(2\pi \mathrm{i})^2} \int_{0\langle} dz \int_{0\rangle} dw \frac{1}{F(z)F(w)} \frac{z-w}{z+w}, \end{aligned} \tag{5.3.7}$$

where

$$F(z) = e^{-xz}\phi_n(z) \frac{(\alpha+z)(\beta+z)}{\beta-z}, \qquad \phi_n(z) = \frac{\left(\frac{1}{2}+z\right)^{N-1}}{\left(\frac{1}{2}-z\right)^{N-1-n}} = \phi(z)\left(\tfrac{1}{2}-z\right)^n. \tag{5.3.8}$$

The contours become bottom-to-top oriented vertical lines parallel to the imaginary axis. In addition, they need to satisfy the following conditions:

- for K_{11}, both the z and the w contours need to satisfy

$$0 < \mathrm{Re} z, \mathrm{Re} w < \min\{\beta, 1/2\} = \beta; \tag{5.3.9}$$

- for K_{12},

$$\begin{aligned} &\max\{-1/2, -\alpha, -\beta\} = \max\{-\alpha, -\beta\} < \mathrm{Re} w < \mathrm{Re} z, \text{ for } w, \\ &0 < \mathrm{Re} z < \min\{\beta, 1/2\} = \beta, \text{ for } z; \end{aligned} \tag{5.3.10}$$

- for K_{22}, we need

$$\mathrm{Re} z + \mathrm{Re} w > 0 \text{ and } \mathrm{Re} z, \mathrm{Re} w > \max\{-1/2, -\alpha, -\beta\} = \max\{-\alpha, -\beta\}. \tag{5.3.11}$$

Recalling that $\alpha \in (-1/2, 1/2), \beta \in (0, 1/2)$, our choices satisfy these conditions.

We next do the change of variables $w \to -w$ in K_{11} from (5.3.7) and $z \to -z$ in K_{22}. This leads to

$$\begin{aligned}
K_{11}(x,y) &= \int \frac{dz}{2\pi\mathrm{i}} \int \frac{dw}{2\pi\mathrm{i}} \frac{\phi(z)}{\phi(w)} \left[\left(\tfrac{1}{2}-z\right)\left(\tfrac{1}{2}+w\right)\right]^n \frac{\theta(z)}{\theta(w)} (z+\alpha)(w-\alpha) \frac{z+w}{4zw(z-w)} \frac{e^{yw}}{e^{xz}}, \\
K_{12}(x,y) &= \int \frac{dz}{2\pi\mathrm{i}} \int \frac{dw}{2\pi\mathrm{i}} \frac{\phi(z)}{\phi(w)} \left[\frac{\frac{1}{2}-z}{\frac{1}{2}+w}\right]^n \frac{\theta(z)}{\theta(w)} \frac{z+\alpha}{w+\alpha} \frac{z+w}{2z(z-w)} \frac{e^{yw}}{e^{xz}}, \\
K_{22}(x,y) &= -\int \frac{dz}{2\pi\mathrm{i}} \int \frac{dw}{2\pi\mathrm{i}} \frac{\phi(z)}{\phi(w)} \frac{1}{\left[\left(\tfrac{1}{2}+z\right)\left(\tfrac{1}{2}-w\right)\right]^n} \frac{\theta(z)}{\theta(w)} \frac{1}{(z-\alpha)(w+\alpha)} \frac{z+w}{z-w} \frac{e^{yw}}{e^{xz}} \\
&= -\int \frac{dz}{2\pi\mathrm{i}} \int \frac{dw}{2\pi\mathrm{i}} \frac{\phi(z)}{\phi(w)} \frac{1}{\left[\left(\tfrac{1}{2}+z\right)\left(\tfrac{1}{2}-w\right)\right]^n} \frac{\theta(z)}{\theta(w)} \frac{1}{(z-\alpha)(w+\alpha)} \frac{z+w}{z-w} \frac{e^{yw}}{e^{xz}} \\
&\quad + \tilde{\varepsilon}(x,y),
\end{aligned} \tag{5.3.12}$$

with appropriate integration contours, i.e. the images of the ones in (5.3.7). The last line of K_{22} comes from switching the two contours and picking up the residue at $w \to z$ which is the $\varepsilon(x,y)$ term.

Finally, since the LPP time is non-negative, we only need to consider $x, y > 0$. The exponential term e^{-xz} in $\phi(z)$ (respectively e^{yw} in $1/\phi(w)$) allows us to close the z contours to the right of $1/2$ (respectively the w contour to the left of $-1/2$). Closing them as indicated and reversing the direction of z to make it counter-clockwise, we arrive at (5.3.3). □

Remark 5.3.2. We remark the following trivial but useful identities

$$\phi(-z) = \phi(z)^{-1}, \quad \Phi(x,-z) = \Phi(x,z)^{-1}, \tag{5.3.13}$$

which we shall use throughout many times without explicit reference. Further, note that for $n = 0$, $\tilde{\varepsilon}$ simplifies to $\tilde{\varepsilon}(x,y) = -\operatorname{sgn}(x-y)e^{-\alpha|x-y|}$ since the pole at $z = 1/2$ vanishes.

5.3.2 From integrable to stationary

Shift argument

To recover the desired distribution, we need to remove $\tilde{\omega}_{1,1}$ and then take the $\beta \to -\alpha$ limit. The former is achieved by a standard shift argument, used already in the full-space stationary LPP problem [BR00, FS06, BFP10, IS04][7]. We present the short proof for completeness.

We recall that $L^{\mathrm{pf}}_{N,N-n}$ denotes the LPP time for the random variables $\tilde{\omega}_{i,j}$ of (5.3.1). Denote by $\widetilde{L}_{N,N-n} = L^{\mathrm{pf}}_{N,N-n} - \tilde{\omega}_{1,1}$ and recall that $L_{N,N-n}$ is the $\beta \to -\alpha$ limit of $\widetilde{L}_{N,N-n}$. The shift argument captured in the following lemma.

Lemma 5.3.3. *Let $\alpha, \beta \in (-1/2, 1/2)$ with $\alpha + \beta > 0$. Then*

$$\left(1 + \frac{1}{\alpha+\beta}\partial_s\right) \mathbb{P}(L^{\mathrm{pf}}_{N,N-n} \le s) = \mathbb{P}(\widetilde{L}_{N,N-n} \le s). \tag{5.3.14}$$

[7] Baik–Rains [BR00] treat the Poisson case instead of the exponential one but the shift argument is similar.

Proof. Due to the independence of $\widetilde{L}_{N,N-n}$ and $\tilde{\omega}_{1,1}$, we have

$$\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s) = \mathbb{P}(\tilde{\omega}_{1,1} + \widetilde{L}_{N,N-n} \leq s) = (\alpha+\beta)\int_0^\infty d\lambda e^{-\lambda(\alpha+\beta)}\mathbb{P}(\widetilde{L}_{N,N-n} \leq s-\lambda). \tag{5.3.15}$$

Performing the Laplace transform and the change of variable $s-\lambda = u$, we obtain

$$\begin{aligned}\int_0^\infty dse^{-ts}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s) &= \int_0^\infty du \int_0^\infty d\lambda(\alpha+\beta)e^{-ts}e^{-\lambda(\alpha+\beta)}\mathbb{P}(\widetilde{L}_{N,N-n} \leq s-\lambda)\\ &= (\alpha+\beta)\int_0^\infty d\lambda e^{-\lambda(\alpha+\beta+t)}\int_0^\infty due^{-tu}\mathbb{P}(L \leq u).\end{aligned} \tag{5.3.16}$$

Computing the first integral on the right-hand side of (5.3.16), and then integrating by parts, we obtain

$$\begin{aligned}\int_0^\infty dse^{-ts}\mathbb{P}(\widetilde{L}_{N,N-n} \leq s) &= \left(1+\frac{t}{\alpha+\beta}\right)\int_0^\infty dse^{-ts}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s)\\ &= \int_0^\infty dse^{-ts}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s) + \frac{1}{\alpha+\beta}e^{-ts}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s)\big|_0^\infty\\ &\quad + \frac{1}{\alpha+\beta}\int_0^\infty dse^{-ts}\frac{d}{ds}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s),\end{aligned} \tag{5.3.17}$$

which gives (5.3.14) since the first term is 0. □

Kernel decomposition

From Lemma 5.3.3, we need to find a decomposition of $\frac{1}{\alpha+\beta}\mathbb{P}(L^{\rm pf}_{N,N-n} \leq s)$ which has a well-defined limit as $\alpha+\beta \to 0$. For that purpose, we first decompose the kernel by separating the contributions of the different poles in a way that will be convenient for future computations. The result will be given in terms of the following functions:

$$f^\beta_+(x) = \Phi(x,\beta)\left(\tfrac12-\beta\right)^n, \qquad f^\beta_-(x) = \frac{\Phi(x,\beta)}{\left(\tfrac12+\beta\right)^n}, \tag{5.3.18}$$

and

$$\begin{aligned}g_1(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi{\rm i}}\Phi(x,z)\left(\tfrac12-z\right)^n\frac{z+\alpha}{2z},\\ \tilde{g}_2(x) &= \oint_{\Gamma_{1/2,\alpha}} \frac{dz}{2\pi{\rm i}}\frac{\Phi(x,z)}{\left(\tfrac12+z\right)^n}\frac{1}{z-\alpha},\\ g_3(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi{\rm i}}\Phi(x,z)\left(\tfrac12-z\right)^n\frac{1}{z-\beta},\\ \tilde{g}_4(x) &= \oint_{\Gamma_{1/2,\pm\alpha,\beta}} \frac{dz}{2\pi{\rm i}}\frac{\Phi(x,z)}{\left(\tfrac12+z\right)^n}\frac{2z}{(z-\alpha)(z+\alpha)(z-\beta)},\\ g_5(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi{\rm i}}\Phi(x,z)\left(\tfrac12-z\right)^n\frac{(z-\alpha)(z+\beta)}{2z(z-\beta)},\\ g_6(x) &= \oint_{\Gamma_{1/2}} \frac{dz}{2\pi{\rm i}}\frac{\Phi(x,z)}{\left(\tfrac12+z\right)^n}\frac{z+\beta}{(z+\alpha)(z-\beta)}.\end{aligned} \tag{5.3.19}$$

We use the letter g for functions where the integration contour encloses only $1/2$, and symbol $\tilde{g}$ for those whose integration contour encloses $1/2$ and some other poles.

With these notations we can now write the kernel decomposition used later.

Proposition 5.3.4. *Let $\alpha \in (-1/2, 1/2)$, $\beta > 0$. Then the kernel K splits as*

$$K = \overline{K} + (\alpha+\beta) R \tag{5.3.20}$$

where

$$\begin{aligned}
\overline{K}_{11}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi \mathrm{i}} \oint_{\Gamma_{-1/2}} \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[(\tfrac{1}{2}-z)(\tfrac{1}{2}+w)\right]^n \frac{\theta(z)}{\theta(w)} \frac{(z+\alpha)(w-\alpha)(z+w)}{4zw(z-w)},\\
\overline{K}_{12}(x,y) =& -\oint_{\Gamma_{1/2}} \frac{dz}{2\pi \mathrm{i}} \oint_{\Gamma_{-1/2,-\alpha,-\beta}} \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \left[\frac{(\frac{1}{2}-z)}{(\frac{1}{2}-w)}\right]^n \frac{\theta(z)}{\theta(w)} \frac{z+\alpha}{w+\alpha} \frac{z+w}{2z(z-w)}\\
=& -\overline{K}_{21}(y,x),\\
\overline{K}_{22}(x,y) =& \tilde{\varepsilon}(x,y) + \oint \frac{dz}{2\pi \mathrm{i}} \oint \frac{dw}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{\Phi(y,w)} \frac{1}{\left[(\frac{1}{2}+z)(\frac{1}{2}-w)\right]^n} \frac{\theta(z)}{\theta(w)} \frac{1}{(z-\alpha)(w+\alpha)} \frac{z+w}{z-w}
\end{aligned} \tag{5.3.21}$$

and where the integration contours for $\overline{K}_{22}$ are, for (z,w), the union of $\Gamma_{1/2,\alpha,\beta} \times \Gamma_{-1/2}$, $\Gamma_{1/2,\beta} \times \Gamma_{-\alpha}$, and $\Gamma_{1/2,\alpha} \times \Gamma_{-\beta}$.

The operator R is of rank two and given by

$$R = \begin{pmatrix} |g_1\rangle \left\langle f_+^\beta\right| - \left|f_+^\beta\right\rangle \langle g_1| & \left|f_+^\beta\right\rangle \langle \tilde{g}_2| \\ -|\tilde{g}_2\rangle \left\langle f_+^\beta\right| & 0 \end{pmatrix}. \tag{5.3.22}$$

Proof. The proof amounts to residue computations. In K_{11} the terms coming into R are the residue at $(z=\beta, w=-1/2)$ and at $(z=1/2, w=-\beta)$. Furthermore, the residue at $(z=\beta, w=-\beta)$ is identically zero. For K_{12}, the residue at $(z=\beta, w=-\beta)$ is zero as well, and the terms in R are the residues from $(z=\beta, w=-1/2)$ and $(z=\beta, w=-\alpha)$. Finally, for K_{22}, there is no contribution to R and the residues at $(z=\beta, w=-\beta)$ and $(z=\alpha, w=-\alpha)$ are also zero. ☐

Decomposition of the rank-two perturbation R

Since we are going to work with Fredholm determinants instead of pfaffians for a while, we define

$$G = J^{-1}K, \qquad \overline{G} = J^{-1}\overline{K}, \qquad T = J^{-1}R \tag{5.3.23}$$

where $J(x,y) = \delta_{x,y} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ so that, for instance, the matrix kernel G is given by

$$G(x,y) = \begin{pmatrix} -K_{21}(x,y) & -K_{22}(x,y) \\ K_{11}(x,y) & K_{12}(x,y) \end{pmatrix} \tag{5.3.24}$$

while T is given by

$$T = |X_1\rangle \langle Y_1| + |X_2\rangle \langle Y_2| \tag{5.3.25}$$

with

$$X_1 = \left| \begin{matrix} \tilde{g}_2 \\ g_1 \end{matrix} \right\rangle, \qquad X_2 = \left| \begin{matrix} 0 \\ f_+^\beta \end{matrix} \right\rangle, \tag{5.3.26}$$

$$Y_1 = \left\langle f_+^\beta \quad 0 \right|, \qquad Y_2 = \langle -g_1 \quad \tilde{g}_2 | \,. \tag{5.3.27}$$

Since all Fredholm determinants/pfaffians as well as (scalar) products will be on $L^2(s,\infty)$, we consider all operators to be in $L^2(s,\infty)$ and omit the latter from the notation for brevity. Recall also that $\mathrm{pf}(J-K) = \sqrt{\det(\mathbb{1}-G)}$.

From the shift argument Lemma 5.3.3, we need to determine the $\alpha+\beta \to 0$ limit of

$$\frac{1}{\alpha+\beta}\,\mathrm{pf}(J-G) = \frac{1}{(\alpha+\beta)}\sqrt{\det(\mathbb{1}-G)} \tag{5.3.28}$$

where

$$\det(\mathbb{1}-G) = \det(\mathbb{1}-\overline{G})\cdot\det(\mathbb{1}-(\alpha+\beta)(\mathbb{1}-\overline{G})^{-1}T). \tag{5.3.29}$$

Setting

$$Z_i = (\mathbb{1}-\overline{G})^{-1}X_i, \qquad i=1,2, \tag{5.3.30}$$

we get

$$\begin{aligned}(\mathbb{1}-(\alpha+\beta)(\mathbb{1}-\overline{G})^{-1}S) &= \det\left(\mathbb{1}-(\alpha+\beta)\begin{pmatrix}\langle Y_1\,|Z_1\rangle & \langle Y_2\,|Z_1\rangle \\ \langle Y_1\,|Z_2\rangle & \langle Y_2\,|Z_2\rangle\end{pmatrix}\right) \\ &= \det(\mathbb{1}-\overline{G})(1-(\alpha+\beta)\,\langle Y_2\,|Z_2\rangle\,)^2.\end{aligned} \tag{5.3.31}$$

In (5.3.31), the first equality is just a rewriting and holds in any inner product space and for any vectors $Y_i, Z_i, i=1,2$—see for instance [TW96] for more details—and for the second we used the equalities

$$\langle Y_1\,|Z_2\rangle = \langle Y_2\,|Z_1\rangle = 0, \qquad \langle Y_1\,|Z_1\rangle = \langle Y_2\,|Z_2\rangle \tag{5.3.32}$$

proven in Section 5.6.

Summarizing, we need to determine the $\beta\to-\alpha$ limit of

$$\begin{aligned}\frac{1}{\alpha+\beta}\,\mathrm{pf}(J-K) &= \mathrm{pf}(J-\overline{K})\left(\frac{1}{\alpha+\beta}-\langle Y_2\,|Z_2\rangle\right) \\ &= \mathrm{pf}(J-\overline{K})\left(\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle-\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle\right).\end{aligned} \tag{5.3.33}$$

5.3.3 Analytic continuation

Recall that we started with our kernels defined for $\beta>0$ only. Now we have to deal with the analyticity of the right-hand side of (5.3.33) and determine the desired limit.

Throughout, when we say that a function is analytic in $\alpha,\beta\in(-1/2,1/2)$, we mean that for any $0<\epsilon\ll 1$, the function is analytic in $\alpha,\beta\in[-1/2+\epsilon,1/2-\epsilon]$.

Remark 5.3.5. Hereinafter we will denote, in up-right sans-serif font, the limits as $\beta\to -\alpha$ of the various kernels and functions we use and which depend explicitly on β. The ones that are independent of β we leave unchanged. Thus by definition:

$$(\mathsf{g}_3,\tilde{\mathsf{g}}_4,\mathsf{g}_5,\mathsf{g}_6) = \lim_{\beta\to-\alpha}(g_3,\tilde{g}_4,g_5,g_6), \qquad (\overline{\mathsf{K}},\widetilde{\mathsf{K}},\widehat{\mathsf{G}}) = \lim_{\beta\to-\alpha}(\overline{K},\widetilde{K},\widehat{G}) \tag{5.3.34}$$

where the g's are defined in (5.3.19), $\overline{K}$ in (5.3.21), and $\widetilde{K},\widehat{G}$ will be defined below.

Analyticity of the Fredholm pfaffian

Lemma 5.3.6. *The kernel $\overline{K}$ is analytic for $\alpha,\beta\in(-1/2,1/2)$. The limit kernel $\overline{\mathsf{K}}=\lim_{\beta\to-\alpha}\overline{K}$ has the following entries:*

$$\begin{aligned}\overline{\mathsf{K}}_{11}(x,y)=&-\oint_{\Gamma_{1/2}}\frac{dz}{2\pi\mathrm{i}}\oint_{\Gamma_{-1/2}}\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\left[(\tfrac12-z)(\tfrac12+w)\right]^n\frac{(z-\alpha)(w+\alpha)(z+w)}{4zw(z-w)},\\ \overline{\mathsf{K}}_{12}(x,y)=&-\oint_{\Gamma_{1/2}}\frac{dz}{2\pi\mathrm{i}}\oint_{\Gamma_{-1/2,\alpha}}\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\left[\frac{\frac12-z}{\frac12-w}\right]^n\frac{z-\alpha}{w-\alpha}\frac{z+w}{2z(z-w)}\\ =&-\overline{\mathsf{K}}_{21}(y,x),\\ \overline{\mathsf{K}}_{22}(x,y)=&\varepsilon(x,y)+\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\frac{1}{\left[(\frac12+z)(\frac12-w)\right]^n}\frac{1}{z-w}\left(\frac{1}{z+\alpha}+\frac{1}{w-\alpha}\right)\end{aligned}\tag{5.3.35}$$

where the integration contours for $\overline{\mathsf{K}}_{22}$ are $\Gamma_{1/2,-\alpha}\times\Gamma_{-1/2}$ for the term with $1/(z+\alpha)$ and $\Gamma_{1/2}\times\Gamma_{-1/2,\alpha}$ for the term with $1/(w-\alpha)$.

Proof. Analyticity for $\overline{K}_{11}$ (respectively $\overline{K}_{12}$) is obvious since we can take the integration contour for z as close to $1/2$ as desired and the contour for w to include $-1/2$ (respectively $-1/2,-\alpha,-\beta$) without crossing z. For $\overline{K}_{22}$, we can decompose the kernel using the identities

$$\frac{\theta(z)}{\theta(w)}\frac{1}{z-w}=\frac{(z+\beta)(w-\beta)}{(z-\beta)(w+\beta)(z-w)}=\frac{1}{z-w}+\frac{2\beta}{(w+\beta)(z-\beta)},\tag{5.3.36}$$

$$\frac{z+w}{(z-\alpha)(w+\alpha)}=\frac{1}{z-\alpha}+\frac{1}{w+\alpha}.\tag{5.3.37}$$

We get that the double integral part $\overline{K}_{22}-\varepsilon$ becomes the sum of

$$\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\frac{1}{\left[(\frac12+z)(\frac12-w)\right]^n}\frac{1}{z-w}\left(\frac{1}{z-\alpha}+\frac{1}{w+\alpha}\right)\tag{5.3.38}$$

and

$$\oint\frac{dz}{2\pi\mathrm{i}}\oint\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\frac{1}{\left[(\frac12+z)(\frac12-w)\right]^n}\frac{2\beta}{(w+\beta)(z-\beta)}\left(\frac{1}{z-\alpha}+\frac{1}{w+\alpha}\right).\tag{5.3.39}$$

The integration contour for the term in (5.3.38) with $1/(z-\alpha)$ is $\Gamma_{1/2,\alpha}\times\Gamma_{-1/2}$, while the one for the term with $1/(w+\alpha)$ is $\Gamma_{1/2}\times\Gamma_{-1/2,-\alpha}$. These can be chosen non-intersecting for all α in any subset of $[-1/2+\epsilon,1/2-\epsilon]$ for $0<\epsilon\ll1$. The contours for the term in (5.3.39) with $1/(z-\alpha)$ are $\Gamma_{1/2,\alpha,\beta}\times\Gamma_{-1/2,-\beta}$, while the one with $1/(w+\alpha)$ are $\Gamma_{1/2,\beta}\times\Gamma_{-1/2,-\alpha,-\beta}$. Notice that since the term $1/(z-w)$ is absent, the contours can cross without problems. Thus this term is also clearly analytic.

Comparing (5.3.35) with (5.3.21) one notices the change of $\tilde{\varepsilon}$ into ε, which corresponds to replacing ε_2 with ε_0. Let us start with (5.3.21). The contribution of the poles at $(z,w)=(1/2,-\alpha)$ and from $(z,w)=(\alpha,-1/2)$ vanishes as $\beta\to-\alpha$. The contributions from $(z,w)=(1/2,-1/2)$, $(z,w)=(1/2,-\beta)$ and $(z,w)=(\beta,-1/2)$ give, in the limit $\beta\to-\alpha$, the double integral in (5.3.35). Finally, the contributions from $(z,w)=(\alpha,-\beta)$ and $(z,w)=(\beta,-\alpha)$ become, in the $\beta\to-\alpha$ limit, as follows:

$$\frac{e^{-\alpha(x-y)}-e^{\alpha(x-y)}}{(\frac14-\alpha^2)^n}=\operatorname{sgn}(x-y)\frac{e^{-\alpha|x-y|}-e^{\alpha|x-y|}}{(\frac14-\alpha^2)^n}.\tag{5.3.40}$$

Summing this term with ε_2 gives ε_0. □

Proposition 5.3.7. $\mathrm{pf}(J-\overline{K})$ *is analytic in* $\alpha,\beta\in(-1/2,1/2)$ *and with a well-defined limit*

$$\lim_{\beta\to-\alpha}\mathrm{pf}(J-\overline{K})=\mathrm{pf}(J-\mathsf{K}). \tag{5.3.41}$$

Proof. Fix a $0<\epsilon\ll 1$ and consider $\alpha,\beta\in[-1/2+\epsilon,1/2-\epsilon]$. Take $\mu=1/2-3\epsilon/4$. Then, for some constant C independent of x,y, we have the bounds

$$\begin{aligned}
|\overline{K}_{11}(x,y)|&\leq Ce^{-(1/2-\epsilon/2)x}e^{-(1/2-\epsilon/2)y}=Ce^{-\mu(x+y)}e^{-\epsilon(x+y)/4},\\
|\overline{K}_{12}(x,y)|&\leq Ce^{-(1/2-\epsilon/2)x}e^{(1/2-\epsilon)y}=Ce^{-\mu(x-y)}e^{-\epsilon(x+y)/4},\\
|\overline{K}_{21}(x,y)|&\leq Ce^{(1/2-\epsilon)x}e^{-(1/2-\epsilon/2)y}=Ce^{\mu(x-y)}e^{-\epsilon(x+y)/4},\\
|\overline{K}_{22}(x,y)|&\leq Ce^{(1/2-\epsilon)x}e^{(1/2-\epsilon)y}=Ce^{\mu(x+y)}e^{-\epsilon(x+y)/4}.
\end{aligned} \tag{5.3.42}$$

This is achieved as follows: for $\overline{K}_{11}$, choose the contours as $|z-1/2|=\epsilon/2$ and $|w+1/2|=\epsilon/2$; for $\overline{K}_{12}$, take $|z-1/2|=\epsilon/2$ and the poles at $w=-\alpha,-\beta$ gives the leading asymptotic behaviour in y, namely $e^{-\min\{\alpha,\beta\}y}$. This is controlled by the $e^{-\mu y}$ from the conjugation. For $\overline{K}_{22}$ it is similar. In this case the leading behaviour is given by the residues a $\pm\alpha$ and $\pm\beta$.

Then,

$$\begin{aligned}
\mathrm{pf}(J-\overline{K})&=\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_s^\infty dx_1\cdots\int_s^\infty dx_n\,\mathrm{pf}[\overline{K}(x_i,x_j)]_{1\leq i,j\leq n}\\
&=\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_s^\infty dx_1\cdots\int_s^\infty dx_n\left(\det[\overline{G}(x_i,x_j)]_{1\leq i,j\leq n}\right)^{1/2}.
\end{aligned} \tag{5.3.43}$$

Using the standard Hadamard bound on the $2n\times 2n$ determinant together with the estimates (5.3.42), we have that the Fredholm expansion of $\mathrm{pf}(J-\overline{K})$ is absolutely convergent. Furthermore, each entry of the series is analytic in the claimed domain, so is $\mathrm{pf}(J-\overline{K})$. □

Analyticity of the term $\frac{1}{\alpha+\beta}-\langle Y_2|X_2\rangle$

The analyticity of the term $\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle$ is relatively easy.

Lemma 5.3.8. *The term* $\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle$ *is analytic for* $\alpha,\beta\in(-1/2,1/2)$*, with*

$$\lim_{\beta\to-\alpha}\left(\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle\right)=-\oint_{\Gamma_{1/2,\alpha}}\frac{dz}{2\pi\mathrm{i}}\frac{\Phi(s,z)}{\Phi(s,\alpha)}\frac{(\frac12+\alpha)^n}{(\frac12+z)^n}\frac{1}{(z-\alpha)^2}. \tag{5.3.44}$$

Proof. We have

$$\begin{aligned}
\langle Y_2\,|X_2\rangle=\left\langle\tilde g_2\,\middle|\,f_+^\beta\right\rangle&=\oint_{\Gamma_{1/2,\alpha}}\frac{dz}{2\pi\mathrm{i}}\frac{\Phi(s,z)}{\Phi(s,-\beta)}\frac{(\frac12-\beta)^n}{(\frac12+z)^n}\frac{1}{(z-\alpha)(z+\beta)}\\
&=\oint_{\Gamma_{1/2,\alpha,\beta}}\frac{dz}{2\pi\mathrm{i}}\frac{\Phi(s,z)}{\Phi(s,-\beta)}\frac{(\frac12-\beta)^n}{(\frac12+z)^n}\frac{1}{(z-\alpha)(z+\beta)}+\frac{1}{\alpha+\beta}
\end{aligned} \tag{5.3.45}$$

where the last term is the residue at $z=-\beta$ chosen so that the first term is analytic. The residue exactly cancels the $1/(\alpha+\beta)$ in $\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle$. The latter is therefore analytic with limit

$$\lim_{\beta\to-\alpha}\frac{1}{\alpha+\beta}-\langle Y_2\,|X_2\rangle=-\oint_{\Gamma_{1/2,\alpha}}\frac{dz}{2\pi\mathrm{i}}\frac{\Phi(s,z)}{\Phi(s,\alpha)}\frac{(\frac{1}{2}+\alpha)^n}{(\frac{1}{2}+z)^n}\frac{1}{(z-\alpha)^2}. \tag{5.3.46}$$

□

Analyticity of the $\langle Y_2|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\rangle$

Finding an analytic decomposition of $\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle$ turns out to be more intricate than in the full-space stationary case. Let us explain first where the problems are. We have

$$\overline{G}\,|X_2\rangle=\left|\begin{array}{c}-\overline{K}_{22}f_+^\beta\\ \overline{K}_{12}f_+^\beta\end{array}\right\rangle. \tag{5.3.47}$$

The issues are the following:

(a) The pole at $w=-\beta$ of $\overline{K}_{22}$ leads to a term of the form $|a\rangle\left\langle f_-^\beta\right|$ for some explicit function a, and similarly for $\overline{K}_{12}$. When these terms are multiplied by $\left|f_+^\beta\right\rangle$, they give terms proportional to $\int_s^\infty dye^{-2\beta y}<\infty$ iff $\beta>0$ whereas the model is defined for any α,β with $\alpha+\beta>0$.

(b) The pole at $w=-\alpha$ of $\overline{K}_{22}$ leads to terms of the form $|a\rangle\left\langle f_-^\alpha\right|$, and similarly for $\overline{K}_{12}$. When multiplied with $\left|f_+^\beta\right\rangle$, and taking into account the prefactors, one gets a term proportional to $1/(\beta-\alpha)$. This is well-defined in the $\beta\to-\alpha$ limit, except when $\alpha=0$. Also, the single terms in $1/(\beta-\alpha)$ are not analytic at $\alpha=\beta$. This would not be a serious problem if we did not want to consider also the $\alpha=\beta=0$ case, which we of course do.

Thus what we have to prove is that the terms in (a) give a zero contribution, within the product $\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle$ for any $\beta>0$; we also need to rewrite (b) such that we do not have divergent terms for $\beta=\alpha$. These issues did not occur in the full-space stationary problem, but can be put under control using the 2×2 structure of our kernels.

The idea to overcome this issue is the following. We decompose

$$\overline{G}=\widehat{G}+O,\quad\text{with}\quad O=\left|\begin{array}{c}\tilde{g}_2\\ g_1\end{array}\right\rangle\langle a\quad b| \tag{5.3.48}$$

for some functions a,b to be written down in the sequel. The following Lemma tells us that the matrix kernel O is irrelevant in $\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle$.

Lemma 5.3.9. *We have:*

$$\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle=\left\langle Y_2\,\middle|(\mathbb{1}-\overline{G})^{-1}\widehat{G}X_2\right\rangle. \tag{5.3.49}$$

Proof. First of all, notice that

$$(\mathbb{1}-\overline{G})^{-1}\overline{G}-(\mathbb{1}-\overline{G})^{-1}\widehat{G}=(\mathbb{1}-\overline{G})^{-1}O. \tag{5.3.50}$$

Therefore

$$\left\langle Y_2 \,\middle|(\mathbb{1}-\overline{G})^{-1}\overline{G}X_2\right\rangle - \left\langle Y_2 \,\middle|(\mathbb{1}-\overline{G})^{-1}\widehat{G}X_2\right\rangle = \mathrm{const}\left\langle -g_1 \quad \tilde{g}_2 \,\middle|(\mathbb{1}-\overline{G})^{-1}\begin{pmatrix}\tilde{g}_2\\ g_1\end{pmatrix}\right\rangle \tag{5.3.51}$$

with const $= \langle a \quad b\,|X_2\rangle$. Multiplying the 2×2 matrices we have that the scalar product without the constant is given by

$$-\left\langle g_1 \,|(\mathbb{1}-\overline{G})^{-1}_{11}\tilde{g}_2\right\rangle + \left\langle \tilde{g}_2 \,|(\mathbb{1}-\overline{G})^{-1}_{22}g_1\right\rangle - \left\langle g_1 \,|(\mathbb{1}-\overline{G})^{-1}_{12}g_1\right\rangle + \left\langle \tilde{g}_2 \,|(\mathbb{1}-\overline{G})^{-1}_{21}\tilde{g}_2\right\rangle . \tag{5.3.52}$$

The property $\overline{K}_{12}(x,y) = -\overline{K}_{21}(y,x)$ translates into $\overline{G}_{11}(x,y) = \overline{G}_{22}(y,x)$ and so into $(\mathbb{1}-\overline{G})^{-1}_{11}(x,y) = (\mathbb{1}-\overline{G})^{-1}_{22}(y,x)$—see Proposition 5.6.1 for details—implying the first two terms cancels each other. The anti-symmetry of $\overline{K}_{11}$ and $\overline{K}_{22}$ implies the anti-symmetry of $\overline{G}_{21}$ and $\overline{G}_{12}$ which in turn implies the same for the respective $(\mathbb{1}-\overline{G})^{-1}$ entries. Thus the last two terms are each equal to zero and this finishes the proof. □

We now state the announced further decomposition of $\overline{K}$.

Proposition 5.3.10. *Let $\alpha \in (-1/2, 1/2)$, $\beta > 0$. Then the kernel $\overline{K}$ splits as*

$$\overline{K} = \widetilde{K} + \begin{pmatrix}0 & 0\\ 0 & \tilde{\varepsilon}\end{pmatrix} + \widetilde{O} + \widetilde{P} \tag{5.3.53}$$

where

$$\begin{aligned}
\widetilde{K}_{11} &= \overline{K}_{11}, \qquad \widetilde{K}_{21} = \overline{K}_{21},\\
\widetilde{K}_{12}(x,y) &= -\oint_{\Gamma_{1/2}}\frac{dz}{2\pi\mathrm{i}}\oint_{\Gamma_{-1/2}}\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\left[\frac{(\frac12 - z)}{(\frac12 - w)}\right]^n\frac{\theta(z)}{\theta(w)}\frac{z+\alpha}{w+\alpha}\frac{z+w}{2z(z-w)},\\
\widetilde{K}_{22}(x,y) &= \oint_{\Gamma_{1/2,\alpha,\beta}}\frac{dz}{2\pi\mathrm{i}}\oint_{\Gamma_{-1/2}}\frac{dw}{2\pi\mathrm{i}}\frac{\Phi(x,z)}{\Phi(y,w)}\frac{1}{\left[(\frac12 + z)(\frac12 - w)\right]^n}\frac{\theta(z)}{\theta(w)}\frac{1}{(z-\alpha)(w+\alpha)}\frac{z+w}{z-w}
\end{aligned} \tag{5.3.54}$$

and

$$\begin{aligned}
\widetilde{O} &= \left|\begin{matrix}-g_1\\ \tilde{g}_2\end{matrix}\right\rangle\left\langle 0 \quad \tfrac{2\beta}{\beta-\alpha}f_-^{\beta} - \tfrac{\alpha+\beta}{\beta-\alpha}f_-^{\alpha}\right|,\\
\widetilde{P} &= \left|\begin{matrix}(\alpha+\beta)g_3\\ -(\alpha+\beta)\tilde{g}_4 - f_-^{-\alpha}\end{matrix}\right\rangle\left\langle 0 \quad f_-^{\alpha}\right| .
\end{aligned} \tag{5.3.55}$$

Proof. We first have to compute the (12) and (22) components of $\widetilde{O} + \widetilde{P}$ and then divide up accordingly. We have

$$\widetilde{O}_{12} + \widetilde{P}_{12} = \oint_{\Gamma_{1/2}}\oint_{\Gamma_{-\alpha,-\beta}}\cdots . \tag{5.3.56}$$

The residue computations at $w = -\alpha$ and $w = -\beta$ lead to

$$\begin{aligned}
\widetilde{O}_{12} + \widetilde{P}_{12} &= -\frac{2\beta}{\beta-\alpha}\left|g_1\right\rangle\left\langle f_-^{\beta}\right| + \frac{\alpha+\beta}{\beta-\alpha}\left|g_5\right\rangle\left\langle f_-^{\alpha}\right|\\
&= -\frac{2\beta}{\beta-\alpha}\left|g_1\right\rangle\left\langle f_-^{\beta}\right| + \frac{\alpha+\beta}{\beta-\alpha}\left|g_1\right\rangle\left\langle f_-^{\alpha}\right| + (\alpha+\beta)\left|g_3\right\rangle\left\langle f_-^{\alpha}\right|
\end{aligned} \tag{5.3.57}$$

where in the second equality we used Lemma 5.3.11. Similarly,

$$\widetilde{O}_{22} + \widetilde{P}_{22} = \oint_{\Gamma_{1/2,\alpha}} \oint_{\Gamma_{-\beta}} \cdots + \oint_{\Gamma_{1/2,\beta}} \oint_{\Gamma_{-\alpha}} \cdots . \tag{5.3.58}$$

Computing the residues at $w = -\alpha$ and $w = -\beta$ we get

$$\begin{aligned}\widetilde{O}_{22} + \widetilde{P}_{22} &= \frac{2\beta}{\beta-\alpha} |\tilde{g}_2\rangle \left\langle f_-^\beta \right| - \frac{2\beta}{\beta-\alpha} \left| f_-^\beta \right\rangle \langle f_-^\alpha| - \frac{\alpha+\beta}{\beta-\alpha} |g_6\rangle \langle f_-^\alpha| \\ &= \frac{2\beta}{\beta-\alpha} |\tilde{g}_2\rangle \left\langle f_-^\beta \right| - \frac{\alpha+\beta}{\beta-\alpha} |\tilde{g}_2\rangle \langle f_-^\alpha| - (\alpha+\beta) |\tilde{g}_4\rangle \langle f_-^\alpha| - \left| f_-^{-\alpha} \right\rangle \langle f_-^\alpha| .\end{aligned} \tag{5.3.59}$$

Recombining the terms leads to the claimed result, provided we prove the two equalities used in equations (5.3.57) and (5.3.59). We do this in the next lemma. □

Now we prove the two identities used in the proof of Proposition 5.3.10.

Lemma 5.3.11. *We have the following identities:*

$$\begin{aligned} g_5(x) &= g_1(x) + (\beta-\alpha) g_3(x), \\ g_6(x) + \frac{2\beta}{\alpha+\beta} f_-^\beta(x) &= \tilde{g}_2(x) + (\beta-\alpha)\left(\tilde{g}_4(x) + \frac{1}{\alpha+\beta} f_-^{-\alpha}(x)\right). \end{aligned} \tag{5.3.60}$$

Proof. The first identity follows directly from the relation $(z-\alpha)(z+\beta)/(z-\beta) - (z+\alpha) = (\beta-\alpha)/(z-\beta)$. To prove the second, first rewrite

$$g_6(x) + \frac{2\beta}{\alpha+\beta} f_-^\beta(x) = \oint_{\Gamma_{1/2,\beta}} \frac{dz}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{(\frac{1}{2}+z)^n} \frac{z+\beta}{(z+\alpha)(z-\beta)} \tag{5.3.61}$$

and recall

$$\tilde{g}_2(x) = \oint_{\Gamma_{1/2,\alpha}} \frac{dz}{2\pi \mathrm{i}} \frac{\Phi(x,z)}{(\frac{1}{2}+z)^n} \frac{1}{z-\alpha}. \tag{5.3.62}$$

Taking the same contours (i.e. including $1/2, \alpha, \beta$ inside both) and then computing the difference (5.3.61)−(5.3.62) leads to $(\beta-\alpha)\tilde{g}_4(x)$ minus the pole coming from $z = -\alpha$ in $\tilde{g}_4(x)$. The latter is $-\frac{1}{\alpha+\beta} f_-^{-\alpha}(x)$ and this finishes the proof. □

Coming back to the decomposition (5.3.48), namely

$$\overline{G} = \widehat{G} + O \tag{5.3.63}$$

with $O = J^{-1}\widetilde{O}$, we notice the latter has exactly the form to apply Lemma 5.3.9. We also explicitly have

$$\widehat{G} = \begin{pmatrix} -\widetilde{K}_{21} & -\widetilde{K}_{22} - \tilde{\varepsilon} \\ \widetilde{K}_{11} & \widetilde{K}_{12} \end{pmatrix} + \begin{pmatrix} 0 & \left|(\alpha+\beta)\tilde{g}_4 + f_-^{-\alpha}\right\rangle \left\langle f_-^\alpha \right| \\ 0 & (\alpha+\beta) |g_3\rangle \left\langle f_-^\alpha \right| \end{pmatrix}. \tag{5.3.64}$$

What remains to be done is to show that $\left\langle Y_2 \middle| (\mathbb{1} - \overline{G})^{-1} \widehat{G} X_2 \right\rangle$ is analytic for $\alpha, \beta \in (-1/2, 1/2)$ and determine its $\beta \to -\alpha$ limit. This will be accomplished in Proposition 5.3.15, itself following from the following three lemmas.

Lemma 5.3.12. *The vector* $Y_2(x) = (-g_1(x) \quad \tilde{g}_2(x))$ *is independent of* β*. Furthermore, for any* $\alpha \in [-1/2+\epsilon, 1/2-\epsilon]$*,*

$$\begin{aligned} |g_1(x)| &\leq Ce^{-x(1/2-\epsilon/2)}, \\ |\tilde{g}_2(x)| &\leq Ce^{(1/2-\epsilon)x}, \end{aligned} \tag{5.3.65}$$

for some constants C *uniform in* x*.*

Proof. The bound on g_1 is simply obtained by taking the integration contour $|z-1/2| = \epsilon/2$, while for $\tilde{g}_2$, the leading asymptotics comes from the pole at $z=\alpha$. □

Lemma 5.3.13. $\widehat{G}X_2$ *analytic in* $\alpha, \beta \in (-1/2, 1/2)$ *and, for any* $\alpha, \beta \in [-1/2+\epsilon, 1/2-\epsilon]$*, we have the following bounds:*

$$\begin{aligned} \left|(\widehat{G}X_2)_1(y)\right| &\leq Ce^{y(1/2-\epsilon)}, \\ \left|(\widehat{G}X_2)_2(y)\right| &\leq Ce^{-y(1/2-\epsilon/2)} \end{aligned} \tag{5.3.66}$$

for some constant C *independent of* y*. Moreover we have*

$$\lim_{\beta \to -\alpha} \widehat{G}\,|X_2\rangle = \left| \begin{matrix} -h_1 \\ h_2 \end{matrix} \right\rangle \tag{5.3.67}$$

where

$$\begin{aligned} h_1 &= \widetilde{\mathsf{K}}_{22} f_+^{-\alpha} + \varepsilon_1 f_+^{-\alpha} - \tilde{\mathsf{g}}_4 + j^\alpha(s,\cdot), \\ h_2 &= \widetilde{\mathsf{K}}_{12} f_+^{-\alpha} + \mathsf{g}_3 \end{aligned} \tag{5.3.68}$$

with

$$j^\alpha(s,y) = f_-^{-\alpha}(s)\left[\frac{\sinh(\alpha(y-s))}{\alpha} + (y-s)e^{\alpha(y-s)}\right]. \tag{5.3.69}$$

Proof. We start with

$$\widehat{G}\,|X_2\rangle = \begin{pmatrix} * & a \\ * & b \end{pmatrix} \left| \begin{matrix} 0 \\ f_+^\beta \end{matrix} \right\rangle = \left| \begin{matrix} af_+^\beta \\ bf_+^\beta \end{matrix} \right\rangle, \tag{5.3.70}$$

where the kernels a, b are read from the decomposition (5.3.64), namely

$$a = -\widetilde{K}_{22} - \varepsilon_1 - \varepsilon_2 + (\alpha+\beta)\,|\tilde{g}_4\rangle\,\langle f_-^\alpha| + |f_-^{-\alpha}\rangle\,\langle f_-^\alpha|, \qquad b = \widetilde{K}_{12} + (\alpha+\beta)\,|g_3\rangle\,\langle f_-^\alpha|. \tag{5.3.71}$$

The term which in the y-variable have a decay like $e^{-y(1-\varepsilon)/2}$, i.e., the ones for which in the integral representation we integrate only around the pole at $w=-1/2$, are clearly analytic when multiplied by f_+^β and its limits are straightforward, namely

$$\begin{aligned} \lim_{\beta \to -\alpha} \widetilde{K}_{12} f_+^\beta &= \widetilde{\mathsf{K}}_{12} f_+^{-\alpha}, \\ \lim_{\beta \to -\alpha} -(\widetilde{K}_{22} f_+^\beta + \varepsilon_1 f_+^\beta) &= -(\widetilde{\mathsf{K}}_{22} f_+^{-\alpha} + \varepsilon_1 f_+^{-\alpha}). \end{aligned} \tag{5.3.72}$$

Next we compute $\left\langle f_-^\alpha \middle| f_+^\beta \right\rangle$ with the result

$$\begin{aligned} \left\langle f_-^\alpha \middle| f_+^\beta \right\rangle &= \phi(\alpha)\phi(\beta)\left[\frac{\frac{1}{2}-\beta}{\frac{1}{2}+\alpha}\right]^n \int_s^\infty e^{-(\alpha+\beta)x}dx \\ &= \phi(\alpha)\phi(\beta)\left[\frac{\frac{1}{2}-\beta}{\frac{1}{2}+\alpha}\right]^n \frac{e^{-(\alpha+\beta)s}}{\alpha+\beta} = \frac{f_-^\alpha(s) f_+^\beta(s)}{\alpha+\beta}. \end{aligned} \tag{5.3.73}$$

Since $f_-^\alpha, f_+^\beta, g_3, \tilde{g}_4$ are clearly analytic—see their representations in (5.3.18) and (5.3.19), then also the two terms involving g_3 and $\tilde{g}_4$ are analytic as the $\alpha+\beta$ prefactor cancels with the one in (5.3.73). Their limits are given by

$$\begin{aligned}\lim_{\beta\to-\alpha}(\alpha+\beta)\tilde{g}_4\left|f_-^\alpha\right\rangle\left\langle f_+^\beta\right| &= \tilde{\mathrm{g}}_4,\\ \lim_{\beta\to-\alpha}(\alpha+\beta)g_3\left|f_-^\alpha\right\rangle\left\langle f_+^\beta\right| &= \tilde{\mathrm{g}}_3.\end{aligned} \tag{5.3.74}$$

Finally, it remains to analyze the term $-\varepsilon_2+\left|f_-^{-\alpha}\right\rangle\left\langle f_-^\alpha\right|$ applied to f_+^β. We have

$$\begin{aligned}\left\langle f_-^\alpha\middle|f_+^\beta\right\rangle f_-^{-\alpha}(y)-\varepsilon_2 f_+^\beta(y) &= \frac{\phi(\beta)(\frac12-\beta)^n}{(\frac14-\alpha^2)^n}\left[\frac{e^{-(\alpha+\beta)s+\alpha y}}{\alpha+\beta}+\int_s^\infty \operatorname{sgn}(y-u)e^{-\alpha|y-u|-\beta u}du\right]\\ &= \frac{\phi(\beta)(\frac12-\beta)^n}{(\frac14-\alpha^2)^n}\left[\frac{e^{-(\alpha+\beta)s+\alpha y}}{\alpha+\beta}+\frac{e^{-\beta y}-e^{(\alpha-\beta)s-\alpha y}}{\alpha-\beta}-\frac{e^{-\beta y}}{\alpha+\beta}\right]\\ &= \frac{\phi(\beta)(\frac12-\beta)^n e^{-\beta s}}{(\frac14-\alpha^2)^n}\left[\frac{e^{\beta(s-y)}-e^{\alpha(s-y)}}{\alpha-\beta}+\frac{e^{\alpha(y-s)}-e^{-\beta(y-s)}}{\alpha+\beta}\right]\end{aligned} \tag{5.3.75}$$

where in the second line inside the brackets the second and third terms come from explicitly integrating $\int_s^y du\cdots$ and $\int_y^\infty du\cdots$ respectively. The $\beta\to-\alpha$ of this last term is

$$\lim_{\beta\to-\alpha}\left\langle f_-^\alpha\middle|f_+^\beta\right\rangle f_-^{-\alpha}(y)-\varepsilon_2 f_+^\beta(y)=\frac{\phi(-\alpha)e^{\alpha s}}{(\frac12-\alpha)^n}\left[\frac{\sinh(\alpha(y-s))}{\alpha}+(y-s)e^{\alpha(y-s)}\right]. \tag{5.3.76}$$

It remains to discuss the decay properties. For bound in the first component follows directly by the fact that in the representation of $\widetilde{K}-12$ and g_3 we can take the contour as $|z-1/2|=1/2-\epsilon/2$. This decay is also correct for the term involving ε_1. Furthermore, the asymptotic behaviour of $\widetilde{K}_{22}$ is coming from the poles at $z=\alpha,\beta$, thus $e^{-\min\{\alpha,\beta\}y}$. Similarly, for $\tilde{g}_4$ the behaviour is $e^{-\min\{\alpha,\pm\beta\}y}$. Finally, the behaviour of (5.3.75) is clear from its final form. Thus, by choosing $\alpha,\beta\in[-1/2+\epsilon,1/2-\epsilon]$ the claimed bounds holds. ☐

Lemma 5.3.14. *The kernel $\overline{G}$ is analytic for $\alpha,\beta\in(-1/2,1/2)$. Moreover, for any $\alpha,\beta\in[-1/2+\epsilon,1/2-\epsilon]$ we have the following bounds:*

$$\begin{aligned}|\overline{G}_{11}(x,y)| &\le Ce^{(1/2-\epsilon)x}e^{-(1/2-\epsilon/2)y},\\ |\overline{G}_{12}(x,y)| &\le Ce^{(1/2-\epsilon)x}e^{(1/2-\epsilon)y},\\ |\overline{G}_{21}(x,y)| &\le Ce^{-(1/2-\epsilon/2)x}e^{-(1/2-\epsilon/2)y},\\ |\overline{G}_{22}(x,y)| &\le Ce^{-(1/2-\epsilon/2)x}e^{(1/2-\epsilon)y},\end{aligned} \tag{5.3.77}$$

for some constant C independent of x,y.

Proof. It is a rewriting of the results of Lemma 5.3.6 and Proposition 5.3.7. ☐

Proposition 5.3.15. *The term $\left\langle Y_2\middle|(\mathbb{1}-\overline{G})^{-1}\widehat{G}X_2\right\rangle$ is analytic for $\alpha,\beta\in(-1/2,1/2)$. Its $\beta\to-\alpha$ limit is given by*

$$\lim_{\beta\to-\alpha}\left\langle Y_2\middle|(\mathbb{1}-\overline{G})^{-1}\widehat{G}X_2\right\rangle=\left\langle -g_1\quad \tilde{g}_2\middle|\begin{pmatrix}\mathsf{R}_{11} & \mathsf{R}_{12}\\ \mathsf{R}_{21} & \mathsf{R}_{22}\end{pmatrix}\begin{pmatrix}-h_1\\ h_2\end{pmatrix}\right\rangle \tag{5.3.78}$$

where for brevity we denoted $\mathsf{R} = (\mathbb{1} - \overline{\mathsf{G}})^{-1}$ *and where* h_1, h_2 *are as in* (5.3.68)*.*

Proof. Due to the bounds of Lemma 5.3.12, 5.3.13, and 5.3.14, when multiplying the different terms, in each integral we have a integrals on (s, ∞) of integrands bounded for instance by $e^{-\epsilon x/2}$, thus the product is well-defined. Analyticity follows from the analyticity of the different entries of the scalar product. □

Remark 5.3.16. The formula we obtained might not look very practical to get numerical results, due to the $(\mathbb{1} - \overline{G})^{-1} = (\mathbb{1} - J^{-1}\overline{K})^{-1}$ term. However, we can always write $\mathrm{pf}(\mathbb{1} - \overline{K}) \left\langle Y_2 \middle| (\mathbb{1} - J^{-1}\overline{K})^{-1}\widehat{G}X_2 \right\rangle$ as a difference of two Fredholm pfaffians, see Lemma 5.3.17. This kind of property has been noticed already in Imamura–Sasamoto paper [IS13] in the context of the stationary KPZ. Thus one does not strictly speaking never needs to verify that the inverse is well-defined, and the formulation with the inverse can be though as well as a compact notation for (5.3.79).

Lemma 5.3.17. *Let* K *be a* 2×2 *anti-symmetric kernel and* $J(x, y) = \delta_{x,y} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$*. Let* a, b, c, d *be functions such that the scalar products in the following formulas are well-defined. Then,*

$$\begin{aligned} &\mathrm{pf}\,(J - K) \langle c \quad d| \,(\mathbb{1} - J^{-1}K)^{-1} \left| \begin{matrix} a \\ b \end{matrix} \right\rangle \\ &\qquad = \mathrm{pf}\,(J - K) - \mathrm{pf}\left(J - K - \left| \begin{matrix} a \\ b \end{matrix} \right\rangle \langle c \quad d| - \left| \begin{matrix} -d \\ c \end{matrix} \right\rangle \langle -b \quad a| \right). \end{aligned} \tag{5.3.79}$$

Proof. The proof consists in computations as in the derivation of (5.3.31). □

Proof of Theorem 5.2.3. The shift argument, Lemma 5.3.3, together with Theorem 5.3.1, gives a formula for the distribution for α, β with $\beta > 0$. The analytic continuation and their limits provided in Proposition 5.3.7, Lemma 5.3.8, and Proposition 5.3.15 imply to the claimed result. □

5.4 Large time asymptotics: proof of Theorem 5.2.6

In this section we prove our main asymptotic result. Let us recall the scaling (5.2.24), namely

$$s = 4N - 2u2^{5/3}N^{2/3} + S\,2^{4/3}N^{1/3}. \tag{5.4.1}$$

Accordingly, in the functions and/or kernels, we need to scale x, y in the same way, i.e.

$$(x, y) = 4N - 2u2^{5/3}N^{2/3} + (X, Y)2^{4/3}N^{1/3}. \tag{5.4.2}$$

Also, in the integrals we will consider the change of variables

$$z = \zeta/(2^{4/3}N^{1/3}), \qquad w = \omega/(2^{4/3}N^{1/3}). \tag{5.4.3}$$

Furthermore, the m-point correlation function has to be multiplied by the volume element $(2^{4/3}N^{1/3})^m$ in order to make sense. As our Pfaffian kernel has a 2×2 structure, not all the kernel elements have to be multiplied by the same volume element. Indeed, in our case, the rescaled and conjugated kernel elements are as follows:

$$\begin{aligned} \mathcal{K}_{11}^{\mathrm{resc}}(X, Y) &= (2^{4/3}N^{1/3})^2 2^{2n}\mathcal{K}_{11}(x, y), \\ \mathcal{K}_{12}^{\mathrm{resc}}(X, Y) &= 2^{4/3}N^{1/3}\mathcal{K}_{12}(x, y), \\ \mathcal{K}_{22}^{\mathrm{resc}}(X, Y) &= 2^{-2n}\mathcal{K}_{22}(x, y) \end{aligned} \tag{5.4.4}$$

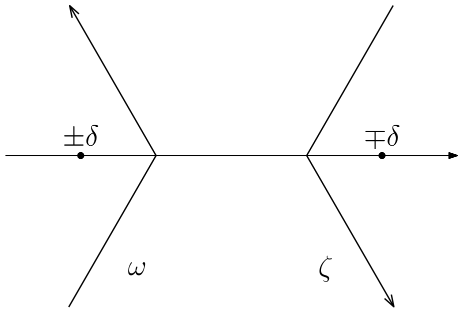

Figure 5.3: The two Airy integration contours $\rangle, \langle$ with acute angles of $\pi/3$ with the horizontal axis. Note they have opposite orientations.

where $\mathcal{K}$ can stand for $\overline{\mathsf{K}}$ or for $\widetilde{K}$. We also set $\mathcal{E}_k^{\mathrm{resc}}(X,Y) = 2^{-2n}\mathcal{E}_k(x,y)$, $k = 1,2$ or empty. Similarly, we rescale the functions

$$f_+^{-\delta,\mathrm{resc}}(X) = 2^n f_+^{-\alpha}(x), \quad h_1^{\mathrm{resc}}(X) = 2^{-4/3}N^{-1/3}2^{-n}h_1(x), \quad h_2^{\mathrm{resc}}(X) = 2^n h_2(x) \tag{5.4.5}$$

as well as

$$\begin{aligned} e^{\delta,\mathrm{resc}}(S) &= 2^{-4/3}N^{-1/3}e^{\alpha}(s), & j^{\delta,\mathrm{resc}}(S,X) &= 2^{-4/3}N^{-1/3}2^{-n}j^{\delta}(s,x), \\ g_1^{\mathrm{resc}}(X) &= 2^{4/3}N^{1/3}2^{-n}g_1(x) & \tilde{g}_2^{\mathrm{resc}}(X) &= 2^{-n}\tilde{g}_2(x), \\ \mathsf{g}_3^{\mathrm{resc}}(X) &= 2^{4/3}N^{1/3}2^{-n}\mathsf{g}_3(x), & \tilde{\mathsf{g}}_4^{\mathrm{resc}}(X) &= 2^{-n}\tilde{\mathsf{g}}_4(x). \end{aligned} \tag{5.4.6}$$

Both the functions and the kernels have a similar structure to the ones of the full-space stationary case, analyzed in great detail in [BFP10]. The integrals have terms of the form $e^{Nf_0(z)+N^{2/3}f_1(z)+N^{1/3}f_2(z)}$—and similarly for w in the case of double integrals, and a finite product of terms independent of N. Since the function f_0 is the same as the one in [BFP10], the steepest descent paths used for the asymptotics and for the uniform bounds are the same. The only minor differences are in the functions f_1 and f_2 and in the N-independent terms, but these do not generate any issues in the asymptotic analysis. For this reason we are not going to repeat all the details of the asymptotic analysis, but rather indicate which Lemmas in [BFP10] we are using analogously. A detailed description on the general approach in the asymptotic of single integrals having Airy-type behaviours can be found for instance in Section 6.1 of [BF14]. This follows the scheme introduced in our field by Gravner–Tracy–Widom in [GTW01].

The limits of the functions entering in the statement of Theorem 5.2.3 are the following.

Lemma 5.4.1. *For any given $L > 0$, the following limits holds uniformly for $X \in [-L, L]$:*

$$\lim_{N\to\infty} f_+^{-\delta,\mathrm{resc}}(X) = f^{-\delta,u}(X), \quad \lim_{N\to\infty} e^{\delta,\mathrm{resc}}(S) = e^{\delta,u}(X), \quad \lim_{N\to\infty} j^{\delta,\mathrm{resc}}(S,X) = j^{\delta,u}(S,X) \tag{5.4.7}$$

as well as

$$\begin{aligned} \lim_{N\to\infty} g_1^{\mathrm{resc}}(X) &= g_1^{\delta,u}(X), & \lim_{N\to\infty} \tilde{g}_2^{\mathrm{resc}}(X) &= \tilde{g}_2^{\delta,u}(X), \\ \lim_{N\to\infty} \mathsf{g}_3^{\mathrm{resc}}(X) &= g_3^{\delta,u}(X), & \lim_{N\to\infty} \mathsf{g}_4^{\mathrm{resc}}(X) &= \tilde{g}_4^{\delta,u}(X). \end{aligned} \tag{5.4.8}$$

Furthermore, for any $X \geq -L$, we have the following bounds which holds uniformly in N:

$$|f_+^{-\delta,\mathrm{resc}}(X)| \leq Ce^{\delta X}, \quad |j^{\delta,u}(S,X)| \leq C|X|e^{|\delta X|} \tag{5.4.9}$$

for some constant C. For any $\kappa > 0$ we have

$$\begin{aligned} &|g_1^{\rm resc}(X)| \leq Ce^{-\kappa X}, \quad |\tilde{g}_2^{\rm resc}(X)| \leq C(e^{-\delta X} + e^{-\kappa X}), \\ &|\mathsf{g}_3^{\rm resc}(X)| \leq Ce^{-\kappa X}, \quad |\mathsf{g}_4^{\rm resc}(X)| \leq C(|X|e^{|\delta X|} + e^{-\kappa X}). \end{aligned} \tag{5.4.10}$$

Proof. Inserting the new variables, we have $f_+^{-\delta,\rm resc}(X) = e^{\delta X} e^{Q_N}$ with Q_N independent of X and with $Q_N \to -\delta^3/3 - \delta^2 u$ as $N \to \infty$. The limit of $e^{\delta,\rm resc}(S)$ follows the patterns of Lemma 4.6 of [BFP10]. For $j^{\delta,\rm resc}(S,X)$ we have

$$j^{\delta,\rm resc}(S,X) = \left[\frac{\sinh \delta(X-S)}{\delta} + (X-S)e^{\delta(X-S)}\right] 2^{-n} f_-^{-\alpha}(s) \tag{5.4.11}$$

and the last term is analyzed as $f_+^{-\delta,\rm resc}$.

The limits of the g functions and their bounds are obtained as in Lemma 4.7 of [BFP10]. The terms $Ce^{-\kappa X}$ comes from the integrals with the contours to the right of the poles $\pm\alpha$ (if present), since the real decay is Airy-like, i.e. $e^{-cX^{3/2}}$. The contributions of the poles at α are bounded by $Ce^{-\alpha X}$, while the pole of order 2 in $-\alpha$ is bounded by $C|X|e^{\alpha X}$. □

The limits of the kernels are the following.

Lemma 5.4.2. *For any given $L > 0$, the following limits hold uniformly for $X, Y \in [-L, L]$:*

$$\lim_{N\to\infty} \overline{\mathsf{K}}_{ij}^{\rm resc}(X,Y) = \overline{\mathcal{A}}_{ij}(X,Y), \quad i,j \in \{1,2\}. \tag{5.4.12}$$

Furthermore, for any $X, Y \geq -L$ and $\kappa > 0$, we have the following bounds which hold uniformly in N:

$$\begin{aligned} &|\overline{\mathsf{K}}_{11}^{\rm resc}(X,Y)| \leq Ce^{-\kappa(X+Y)}, && |\overline{\mathsf{K}}_{12}^{\rm resc}(X,Y)| \leq C(e^{-\kappa(X+Y)} + e^{-\kappa X}e^{\delta Y}), \\ &|\overline{\mathsf{K}}_{21}^{\rm resc}(X,Y)| \leq C(e^{-\kappa(X+Y)} + e^{\delta X}e^{-\kappa Y}), && |\overline{\mathsf{K}}_{22}^{\rm resc}(X,Y)| \leq |\mathcal{E}^{\rm resc}(X,Y)| + C(e^{-\kappa X}e^{\delta Y} + e^{\delta X}e^{-\kappa Y}), \\ &|\mathcal{E}_1^{\rm resc}(X,Y)| \leq Ce^{-(|\delta|+\kappa)|X-Y|}, && |\mathcal{E}_2^{\rm resc}(X,Y)| \leq Ce^{-\delta|X-Y|} \end{aligned} \tag{5.4.13}$$

for some constant C.

Proof. The asymptotics of the double integrals is as in Lemma 4.4 of [BFP10] and the uniform bounds as in Lemma 4.5 of [BFP10]. To get the bounds, we first compute explicitly the poles at $\pm\alpha$—if they are inside the integration contours—while the rest has an Airy-like decay in both variables, from which we have the terms $e^{-\kappa X}$ and $e^{-\kappa Y}$. For $\mathcal{E}_1^{\rm resc}(X,Y)$, we take a contour passing on the right of $|\alpha|$ by an amount $\kappa 2^{-4/3} N^{-1/3}$, which can be deformed to become vertical, as the convergence comes from the quadratic term in Z. □

Finally, in order to define the limits of $h_1^{\rm resc}$ and $h_2^{\rm resc}$ we need the limits of $\widetilde{K}_{12}^{\rm resc}$ and $\widetilde{K}_{22}^{\rm resc}$, which are as follows.

Lemma 5.4.3. *For any given $L > 0$, the following limits hold uniformly for $X, Y \in [-L, L]$:*

$$\lim_{N\to\infty} \widetilde{\mathsf{K}}_{12}^{\rm resc}(X,Y) = \widetilde{\mathcal{A}}_{12}(X,Y), \quad \lim_{N\to\infty} \widetilde{\mathsf{K}}_{22}^{\rm resc}(X,Y) = \widetilde{\mathcal{A}}_{22}(X,Y). \tag{5.4.14}$$

Furthermore, for any $X, Y \geq -L$ and $\kappa > 0$, we have the following bounds which hold uniformly in N:

$$|\widetilde{\mathsf{K}}_{12}^{\rm resc}(X,Y)| \leq Ce^{-\kappa(X+Y)}, \quad |\widetilde{\mathsf{K}}_{22}^{\rm resc}(X,Y)| \leq C(e^{-\kappa X} + e^{\delta X})e^{-\kappa Y}, \tag{5.4.15}$$

for some constant C.

Proof. The proof is similar to that of Lemma 5.4.2, with the only difference being that some poles are not present anymore. □

Corollary 5.4.4. *For any given $S \in \mathbb{R}$, we have*

$$\lim_{N\to\infty} \mathrm{pf}(J - \overline{\mathsf{K}}^{\mathrm{resc}})_{L^2(S,\infty)} = \mathrm{pf}(J - \overline{\mathcal{A}})_{L^2(S,\infty)}. \tag{5.4.16}$$

Proof. We write the Fredholm expansion as in the proof of Proposition 5.3.7. Then, taking $\kappa > |\delta|$, the bounds of Lemma 5.4.2 allow us to exchange the $N \to \infty$ limit with the sums/integrals by dominated convergence. The result follows. □

Corollary 5.4.5. *For any given $L > 0$, the following limits hold uniformly for $Y \in [-L, L]$:*

$$\lim_{N\to\infty} h_1^{\mathrm{resc}}(Y) = \hbar_1^{\delta,u}(Y), \qquad \lim_{N\to\infty} h_2^{\mathrm{resc}}(Y) = \hbar_2^{\delta,u}(Y). \tag{5.4.17}$$

Furthermore, for any $Y \geq -L$ and $\kappa > 0$, we have the following bounds which hold uniformly in N:

$$|h_1^{\mathrm{resc}}(Y)| \leq C|Y|e^{|\delta Y|}, \qquad |h_2^{\mathrm{resc}}(Y)| \leq Ce^{-\kappa Y} \tag{5.4.18}$$

for some constant C.

Proof. The bounds of Lemmas 5.4.1, 5.4.2, and 5.4.3 imply that, taking $\kappa > |\delta|$, we can take $N \to \infty$ inside $\int_S^\infty dV \widetilde{K}_{22}^{\mathrm{resc}}(Y,V) f_+^{-\delta,\mathrm{resc}}(V)$ and inside $\int_S^\infty dV \mathcal{E}_1^{\mathrm{resc}}(Y,V) f_+^{-\delta,\mathrm{resc}}(V)$. Together with the bounds on the remaining terms we get the stated result. □

With the above results, we are now ready to finish the proof of Theorem 5.2.6.

Proof of Theorem 5.2.6. The result is now a direct consequence of Corollary 5.4.4; the bounds and their limits on the rescaled kernel of Lemma 5.4.2; of Lemma 5.4.1 for the functions $g_1^{\mathrm{resc}}(X)$ and $\tilde{g}_2^{\mathrm{resc}}(X)$ entering in left hand side of the scalar product; and of Corollary 5.4.5 for the functions $h_1^{\mathrm{resc}}(Y)$ and $h_2^{\mathrm{resc}}(Y)$ entering in the right hand side of the scalar product. The bounds indeed imply that we can take the $N \to \infty$ inside the integrals which appear when writing the scalar product explicitly by use of dominated convergence. This leads to the claimed result. We remark that it is not really needed to worry about the inverse operator, since when multiplied by the Fredholm pfaffian in front, it can be rewritten as linear combination of two Fredholm pfaffians—see Remark 5.3.16. □

5.5 Limit to the Baik–Rains distribution: proof of Theorem 5.2.10

In the $u \to \infty$ limit, we want to take $\delta = -u + \tau$ with τ fixed. Thus for u large enough we also have $\delta < 0$. In this case, i.e. for $u > 0$ and $\delta < 0$, there are some simplifications in the expression of the distribution.

Lemma 5.5.1. *Consider $u > 0$ and $\delta < 0$. Then the following equality holds:*

$$\overline{\mathcal{A}}_{22}(X,Y) = -\int_{-\mu+\mathrm{i}\mathbb{R}} \frac{d\zeta}{2\pi\mathrm{i}} \int_{\mu+\mathrm{i}\mathbb{R}} \frac{d\omega}{2\pi\mathrm{i}} \frac{e^{\frac{\zeta^3}{3}+\zeta^2 u-\zeta X}}{e^{\frac{\omega^3}{3}-\omega^2 u-\omega Y}} \frac{1}{\zeta-\omega}\left(\frac{1}{\zeta+\delta}+\frac{1}{\omega-\delta}\right) \tag{5.5.1}$$

for any choice of $0 < \mu < \min\{-\delta, u\}$ (the contours for ζ, ω are oriented with increasing imaginary parts).

Furthermore we have:

$$\begin{aligned}\widetilde{\mathcal{A}}_{22}(X,Y)+\mathcal{E}_1(X,Y)=&\overline{\mathcal{A}}_{22}(X,Y)-\tilde{g}_2^{\delta,u}(X)e^{-\frac{\delta^3}{3}+\delta^2u+\delta Y}\\ &+\mathbb{1}_{X>Y}e^{2\delta^2u}(e^{\delta(X-Y)}+e^{-\delta(X-Y)}).\end{aligned} \tag{5.5.2}$$

As a consequence of this representation we have:

$$\int_S^\infty dV\,\mathbb{1}_{Y>V}e^{2\delta^2u}(e^{\delta(Y-V)}+e^{-\delta(Y-V)})f^{-\delta,u}(V)=j^{\delta,u}(S,Y). \tag{5.5.3}$$

Finally we also have:

$$\widetilde{\mathcal{A}}_{12}(X,Y)=\overline{\mathcal{A}}_{12}(X,Y)+g_1^{\delta,u}(X)e^{-\frac{\delta^3}{3}+\delta^2u+\delta Y}. \tag{5.5.4}$$

Proof. First notice that for $\delta<0$, the contours in the double integral of (5.2.27) can be chosen to be the same for the two cases, with $\delta<\mathrm{Re}(\omega)<\mathrm{Re}(\zeta)<-\delta$. Next, notice that we can deform the contours to be vertical provided $\mathrm{Re}(\zeta)>-u$ and $\mathrm{Re}(\omega)<u$. Finally, we exchange the positions of ζ and ω, so now $\mathrm{Re}(\zeta)<\mathrm{Re}(\omega)$, which is the formula (5.5.1), minus the pole at $\omega=\zeta$. This pole gives as residue

$$-\int_{\mathrm{i}\mathbb{R}}\frac{d\zeta}{2\pi\mathrm{i}}e^{2\zeta^2u-\zeta(X-Y)}\frac{2\zeta}{(\zeta+\delta)(\zeta-\delta)} \tag{5.5.5}$$

which is equal to $-\mathcal{E}(X,Y)$. To verify this identity, it is enough by anti-symmetry to consider $X>Y$. Extracting the pole at $\zeta=-\delta$ leads to $-\mathcal{E}_0(X,Y)$, while the remaining integral is $-\mathcal{E}_1(X,Y)$. Finally, (5.5.3) is an elementary computation and (5.5.4) follows by taking the residue at $\omega=\delta$. □

With the above decomposition we can prove Lemma 5.2.8.

Proof of Lemma 5.2.8. Using the representations (5.5.2)—(5.5.4), our claim holds if we can show that

$$\left\langle -g_1^{\delta,u}\quad \tilde{g}_2^{\delta,u}\middle|(\mathbb{1}-J^{-1}\overline{\mathcal{A}})^{-1}\begin{pmatrix}\tilde{g}_2^{\delta,u}\left\langle f^{-\delta,-u}\middle|f^{-\delta,u}\right\rangle\\ g_1^{\delta,u}\left\langle f^{-\delta,-u}\middle|f^{-\delta,u}\right\rangle\end{pmatrix}\right\rangle=0. \tag{5.5.6}$$

The proof of this is the same as proving that (5.3.51) = 0 in Lemma 5.3.9. Notice that for $\delta<0$ (but not for $\delta\geq 0$) the scalar product $\left\langle f^{-\delta,-u}\middle|f^{-\delta,u}\right\rangle$ is well-defined. □

In order to analyze the $u\to\infty$ limit with $u+\delta=\tau$ constant, we need to consider a conjugation in the kernel entries, but also to shift the positions by $\delta(2u+\delta)$ as discussed above in Remark 5.2.5. Finally, to clearly see the limit $u\to\infty$, we shift the ζ,ω integration variables to remove the ζ^2,ω^2 terms in the exponential.

Lemma 5.5.2. *Let us consider $u>0$, $\delta<0$ and $u+\delta=\tau$. Shifting the positions as $X=x+\delta(2u+\delta)$ and $Y=y+\delta(2u+\delta)$, we have:*

$$\begin{aligned}\frac{e^{\frac{2}{3}u^3+uX}}{e^{-\frac{2}{3}u^3-uY}}\overline{\mathcal{A}}_{11}(X,Y)&=-\int_{-u\langle}\frac{dz}{2\pi\mathrm{i}}\int_{\rangle u,z+2u}\frac{dw}{2\pi\mathrm{i}}\frac{e^{\frac{z^3}{3}-z(x+\tau^2)}}{e^{\frac{w^3}{3}-w(y+\tau^2)}}\frac{(w+z)(w+\tau-2u)(z-\tau+2u)}{4(w-u)(z+u)(z-w+2u)},\\ \frac{e^{\frac{2}{3}u^3+uX}}{e^{\frac{2}{3}u^3+uY}}\overline{\mathcal{A}}_{12}(X,Y)&=-\int_{-u\langle}\frac{dz}{2\pi\mathrm{i}}\int_{\tau-2u\rangle z}\frac{dw}{2\pi\mathrm{i}}\frac{e^{\frac{z^3}{3}-z(x+\tau^2)}}{e^{\frac{w^3}{3}-w(y+\tau^2)}}\frac{(w+z+2u)(z-\tau+2u)}{2(z+u)(z-w)(w-\tau+2u)},\end{aligned} \tag{5.5.7}$$

$$\frac{e^{-\frac{2}{3}u^3-uX}}{e^{\frac{2}{3}u^3+uY}}\overline{\mathcal{A}}_{22}(X,Y) = -\int_{r+\mathrm{i}\mathbb{R}} \frac{dz}{2\pi\mathrm{i}} \int_{r+\mathrm{i}\mathbb{R}} \frac{dw}{2\pi\mathrm{i}} \frac{e^{\frac{z^3}{3}-z(x+\tau^2)}}{e^{\frac{w^3}{3}-w(y+\tau^2)}} \frac{z+w}{(w-\tau+2u)(z-w-2u)(z+\tau-2u)}$$

where for $\overline{\mathcal{A}}_{22}$ *the integration contours for* z, w *are oriented with increasing imaginary part and* $0 < r < \min\{u, 2u-\tau\}$.

Proof. The first equality is obtained by the change of variables $\zeta = z+u$ and $\omega = w-u$; the second is obtained by the change of variables $\zeta = z+u$ and $\omega = w+u$; finally, the third comes from substituting $\zeta = z-u$ and $\omega = w+u$ in the representation (5.5.1). □

We have a similar result for $e_1^{\delta,u}$, $g_1^{\delta,u}$, $\tilde{g}_2^{\delta,u}$, $\tilde{h}_1^{\delta,u}$, and $\tilde{h}_2^{\delta,u}$.

Lemma 5.5.3. *Let us consider* $u > 0$, $\delta < 0$ *and* $u + \delta = \tau$. *Shifting the positions as* $X = x + \delta(2u+\delta)$, $Y = y + \delta(2u+\delta)$, *and* $S = s + \delta(2u+\delta)$, *we have:*

$$\begin{aligned} e^{\delta,u}(S) &= \mathcal{R}_{-\tau}(s), \\ e^{\frac{2}{3}u^3+uX} g_1^{\delta,u}(X) &= \int_{-u\langle} \frac{dz}{2\pi\mathrm{i}} e^{\frac{z^3}{3}-z(x+\tau^2)} \frac{z+\tau}{2(z+u)}, \\ e^{-\frac{2}{3}u^3-uX} \tilde{g}_2^{\delta,u}(X) &= \int_{\langle\tau} \frac{dz}{2\pi\mathrm{i}} e^{\frac{z^3}{3}-z(x+\tau^2)} \frac{1}{z-\tau} \end{aligned} \tag{5.5.8}$$

as well as

$$\begin{aligned} e^{-\frac{2}{3}u^3-uY} \tilde{h}_1^{\delta,u}(Y) &= e^{\frac{2}{3}\tau^3+s\tau} \int_{r+\mathrm{i}\mathbb{R}} \frac{dz}{2\pi\mathrm{i}} \int_{-r+\mathrm{i}\mathbb{R}} \frac{dw}{2\pi\mathrm{i}} \frac{e^{\frac{z^3}{3}-z(y+\tau^2)}}{e^{\frac{w^3}{3}-w(s+\tau^2)}} \frac{1}{(w-\tau+2u)(z+\tau-2u)} \\ &\qquad \times \frac{(z+w)}{(z-w-2u)(w+\tau)} \\ &\quad - \int_{\langle\tau,2u-\tau} \frac{dz}{2\pi\mathrm{i}} e^{\frac{z^3}{3}-z(y+\tau^2)} \frac{2(z-u)}{(z-\tau)(z-2u+\tau)}, \\ e^{\frac{2}{3}u^3+uY} \tilde{h}_2^{\delta,u}(Y) &= e^{\frac{2}{3}\tau^3+s\tau} \int_{-u\langle} \frac{dz}{2\pi\mathrm{i}} \int_{\tau-2u\rangle z,-\tau} \frac{dw}{2\pi\mathrm{i}} \frac{e^{\frac{z^3}{3}-z(y+\tau^2)}}{e^{\frac{w^3}{3}-w(s+\tau^2)}} \frac{(w+z+2u)}{2(z+u)} \\ &\qquad \times \frac{(z-\tau+2u)}{(z-w)(w-\tau+2u)(w+\tau)} \\ &\quad + \int_{-\tau\langle} \frac{dz}{2\pi\mathrm{i}} e^{\frac{z^3}{3}-z(y+\tau^2)} \frac{1}{z+\tau} \end{aligned} \tag{5.5.9}$$

where $0 < r < \min\{u, 2u-\tau\}$.

Proof. The proofs for $e^{\delta,u}$, $g_1^{\delta,u}$ and $\tilde{g}_2^{\delta,u}$ consist, as above, respectively in the following changes of variables: $\zeta = z-u$, $\zeta = z+u$ and $\zeta = z-u$. The last two also work for $g_3^{\delta,u}$

and $\tilde{\mathscr{g}}_4^{\delta,u}$ respectively, as summands for $\tilde{\hbar}_2^{\delta,u}$ and $\tilde{\hbar}_1^{\delta,u}$:

$$\begin{aligned} e^{\frac{2}{3}u^3+uY}\mathscr{g}_3^{\delta,u}(Y) &= \int_{-\tau\langle} \frac{dz}{2\pi \mathrm{i}} e^{\frac{z^3}{3}-z(y+\tau^2)}\frac{1}{z+\tau}, \\ e^{-\frac{2}{3}u^3-uY}\tilde{\mathscr{g}}_4^{\delta,u}(Y) &= \int_{\langle\tau,2u-\tau} \frac{dz}{2\pi \mathrm{i}} e^{\frac{z^3}{3}-z(y+\tau^2)}\frac{2(z-u)}{(z-\tau)(z-2u+\tau)}. \end{aligned} \tag{5.5.10}$$

For the term $\overline{\mathcal{A}}_{12}f^{-\delta,u}$ we have:

$$\begin{aligned} &e^{\frac{2}{3}u^3+uY}\left(\overline{\mathcal{A}}_{12}f^{-\delta,u}\right)(Y) = \\ &= e^{\frac{2}{3}\tau^3+s\tau}\int_{-u\langle}\frac{dz}{2\pi\mathrm{i}}\int_{\tau-2u\rangle z,-\tau}\frac{dw}{2\pi\mathrm{i}}\frac{e^{\frac{z^3}{3}-z(y+\tau^2)}}{e^{\frac{w^3}{3}-w(s+\tau^2)}}\frac{(w+z+2u)(z-\tau+2u)}{2(z+u)(z-w)(w-\tau+2u)(w+\tau)} \end{aligned} \tag{5.5.11}$$

which can be obtained by explicitly performing the integration $\int_S^\infty dV e^{(\delta+\omega)V}$ in the product and then changing variables $(\zeta,\omega)=(z+u,w+u)$. The computation for $e^{-\frac{2}{3}u^3-uY}\overline{\mathcal{A}}_{22}f^{-\delta,u}$ is similar. Combining them with the $\mathscr{g}_3^{\delta,u}$ term, respectively the $\tilde{\mathscr{g}}_4^{\delta,u}$ term, we obtain the result for $\tilde{\hbar}_2^{\delta,u}$ and respectively $\tilde{\hbar}_1^{\delta,u}$. □

Proof of Theorem 5.2.10. The finiteness of the Fredholm Pfaffian and of the scalar products depends on the behavior in x,y in the above expressions. The u dependence is only marginal and, with the chosen conjugation, all the terms remain bounded as $u\to\infty$. By dominated convergence we can take the $u\to\infty$ limit inside both the Fredholm Pfaffians and the scalar product. We have the following limits:

$$\lim_{u\to\infty}\frac{e^{\frac{2}{3}u^3+uX}}{e^{-\frac{2}{3}u^3-uY}}\overline{\mathcal{A}}_{11}(X,Y)=\lim_{u\to\infty}\frac{e^{-\frac{2}{3}u^3-uX}}{e^{\frac{2}{3}u^3+uY}}\overline{\mathcal{A}}_{22}(X,Y)=0 \tag{5.5.12}$$

and

$$\lim_{u\to\infty}\frac{e^{\frac{2}{3}u^3+uX}}{e^{\frac{2}{3}u^3+uY}}\overline{\mathcal{A}}_{12}(X,Y)=\mathcal{K}_{\mathrm{Ai},\tau}(x,y). \tag{5.5.13}$$

Dominated convergence then implies that

$$\lim_{u\to\infty}\mathrm{pf}(J-\overline{\mathcal{A}})_{L^2(S,\infty)}=\det(\mathbb{1}-\mathcal{K}_{\mathrm{Ai},\tau})_{L^2(s,\infty)}=F_{\mathrm{GUE}}(s+\tau^2) \tag{5.5.14}$$

and that

$$\lim_{u\to\infty}J^{-1}\overline{\mathcal{A}}(X,Y)=\begin{pmatrix}\mathcal{K}_{\mathrm{Ai},\tau}(x,y) & 0\\ 0 & \mathcal{K}_{\mathrm{Ai},\tau}(x,y)\end{pmatrix}. \tag{5.5.15}$$

The latter limit extends to resolvents as well.

The $\mathscr{g}$ and $\tilde{\hbar}$ functions have the following limits:

$$\lim_{u\to\infty}e^{\frac{2}{3}u^3+uX}\mathscr{g}_1^{\delta,u}(X)=0,\qquad \lim_{u\to\infty}e^{-\frac{2}{3}u^3-uX}\tilde{\mathscr{g}}_2^{\delta,u}(X)=\Psi_{-\tau}(x) \tag{5.5.16}$$

and

$$\begin{aligned} &\lim_{u\to\infty}e^{-\frac{2}{3}u^3-uY}\tilde{\hbar}_1^{\delta,u}(Y)=-\Psi_{-\tau}(y),\\ &\lim_{u\to\infty}e^{\frac{2}{3}u^3+uY}\tilde{\hbar}_2^{\delta,u}(Y)=e^{\frac{2}{3}\tau^3+s\tau}\int_{\langle}\frac{dz}{2\pi\mathrm{i}}\int_{\rangle z,-\tau}\frac{dw}{2\pi\mathrm{i}}\frac{e^{\frac{z^3}{3}-z(y+\tau^2)}}{e^{\frac{w^3}{3}-w(s+\tau^2)}}\frac{1}{(z-w)(w+\tau)}\\ &\qquad+\int_{-\tau\langle}\frac{dz}{2\pi\mathrm{i}}e^{\frac{z^3}{3}-z(y+\tau^2)}\frac{1}{z+\tau}=\Phi_{-\tau}(y) \end{aligned} \tag{5.5.17}$$

where the last equality is obtained by computing the residue at $w = -\tau$.

To conclude, the inner product on the right of (5.2.33) collapses, in the limit $u \to \infty$, to $\langle \Psi_{-\tau} | (\mathbb{1} - \mathcal{K}_{\mathrm{Ai},\tau})^{-1} \Phi_{-\tau} \rangle$: as $\mathscr{g}_1^{\delta,u} \to 0$ with $\tilde{\mathscr{h}}_1^{\delta,u}$ staying finite, the respective inner product is zero in the limit; moreover $\tilde{\mathscr{g}}_2^{\delta,u}, \tilde{\mathscr{h}}_2^{\delta,u}$ and $J^{-1}\overline{\mathcal{A}}$ converge to the desired quantities. Combining with $\mathscr{e}^{\delta,u} \to \mathcal{R}_{-\tau}$, we conclude that

$$\lim_{u\to\infty} F_{0,\,\mathrm{half}}^{(\delta,u)}(S) = F_{\mathrm{BR},-\tau}(s) = F_{\mathrm{BR},\tau}(s) \tag{5.5.18}$$

with the last equality coming from the fact that the Baik–Rains distribution is symmetric under $\tau \to -\tau$. □

5.6 Some determinantal computations

In this section we prove (5.3.31). For this we need to show that

$$\langle Y_1 | Z_2 \rangle = \langle Y_2 | Z_1 \rangle = 0 \text{ and } \langle Y_1 | Z_1 \rangle = \langle Y_2 | Z_2 \rangle \tag{5.6.1}$$

where $Z_i, Y_i, i = 1, 2$ are defined in (5.3.26). For brevity and consistency, let us rename

$$A = (\mathbb{1} - \overline{G})^{-1}, \qquad h_1 = g_1, \qquad h_2 = \tilde{g}_2, \qquad f_1 = f_+^\beta. \tag{5.6.2}$$

We have

$$\langle Y_1 | Z_2 \rangle = \langle f_1 | A_{12} f_1 \rangle \tag{5.6.3}$$

and

$$\langle Y_2 | Z_1 \rangle = \left\langle -h_1 \quad h_2 \,\middle|\, \begin{matrix} A_{11}h_2 + A_{12}h_1 \\ A_{21}h_2 + A_{22}h_1 \end{matrix} \right\rangle. \tag{5.6.4}$$

To show that both are zero, we need the following result.

Proposition 5.6.1. *We have:*

$$A_{12}(x,y) = -A_{12}(y,x), \qquad A_{21}(x,y) = -A_{21}(x,y), \qquad A_{11}(x,y) = A_{22}(y,x). \tag{5.6.5}$$

Proof. Assume for now that we know the norm of $\overline{G}$ is less than 1. Then we can expand A as a Neumann series $(\mathbb{1} - \overline{G})^{-1} = \mathbb{1} + \overline{G} + \overline{G}^2 + \overline{G}^3 + \dots$. We will prove by induction that, for any $n \geq 1$

$$\overline{G}_{12}^n(x,y) = -\overline{G}_{12}^n(y,x), \qquad \overline{G}_{21}^n(x,y) = -\overline{G}_{21}^n(x,y), \qquad \overline{G}_{11}^n(x,y) = \overline{G}_{22}^n(x,y). \tag{5.6.6}$$

The base case is true by the definition of $\overline{G} = J^{-1}\overline{K}$ and by the fact that

$$\overline{K}_{12}(x,y) = -\overline{K}_{21}(y,x), \qquad \overline{K}_{11}(x,y) = -\overline{K}_{11}(y,x), \qquad \overline{K}_{22}(x,y) = -\overline{K}_{22}(y,x). \tag{5.6.7}$$

Now we proceed with the inductive step: assuming that (5.6.6) holds for $B = \overline{G}^n$, for $C = \overline{G}^{n+1}$ we have

$$\begin{aligned} C_{11}(x,y) &= \left(B_{11}\overline{G}_{11} + B_{12}\overline{G}_{21}\right)(x,y) \\ &= \int_s^\infty dz B_{11}(x,z)\overline{G}_{11}(z,y) + \int_s^\infty dz B_{12}(x,z)\overline{G}_{21}(z,y) \\ &= \int_s^\infty dz B_{22}(z,x)\overline{G}_{22}(y,z) + \int_s^\infty dz B_{12}(z,x)\overline{G}_{21}(y,z) \\ &= C_{22}(y,x). \end{aligned} \tag{5.6.8}$$

Moreover,

$$
\begin{aligned}
C_{12}(x,y) &= \left(B_{11}\overline{G}_{12} + B_{12}\overline{G}_{22}\right)(x,y) \\
&= \int_s^\infty dz B_{11}(x,z)\overline{G}_{12}(z,y) + \int_s^\infty dz B_{12}(x,z)\overline{G}_{22}(z,y) \\
&= -\int_s^\infty dz B_{22}(z,x)\overline{G}_{12}(y,z) - \int_s^\infty dz B_{12}(z,x)\overline{G}_{11}(y,z) \\
&= -C_{12}(y,x)
\end{aligned}
\tag{5.6.9}
$$

where we have used that $C = B\overline{G} = \overline{G}B$. An analogous computation holds for C_{21}.

Now it may happen that the norm of $\overline{G}$, of which we remain ignorant, is ≥ 1. We then replace $\overline{G}$ by $\omega\overline{G}$ for $\omega > 0$ small enough to make $\omega\overline{G}$ of subunit norm. The argument above applies to the corresponding $A(\omega) = (\mathbb{1} - \omega\overline{G})^{-1}$. We then analytically continue in ω, as the spectrum of any trace class operator (and in particular of $\overline{G}$) is discrete without non-zero accumulation points. □

Using Proposition 5.6.1 in equation (5.6.3), we have

$$
\begin{aligned}
\langle Y_1 | Z_2 \rangle &= \int_s^\infty \int_s^\infty dx dy f_1(x) A_{12}(x,y) f_1(y) \\
&= -\int_s^\infty \int_s^\infty dx dy f_1(x) A_{12}(y,x) f_1(y) \\
&= 0.
\end{aligned}
\tag{5.6.10}
$$

For the same reason, (5.6.4) becomes:

$$
\langle Y_1 | Z_2 \rangle = -\langle h_1 | A_{11} h_2 \rangle - \langle h_1 | A_{12} h_1 \rangle + \langle h_2 | A_{22} h_1 \rangle + \langle h_2 | A_{21} h_2 \rangle = 0 \tag{5.6.11}
$$

since the last two terms are zero, and the first two are equal by Proposition 5.6.1:

$$
\begin{aligned}
\langle h_2 | A_{22} h_1 \rangle &= \int_s^\infty \int_s^\infty dx dy h_2(x) A_{22}(x,y) h_1(y) \\
&= \int_s^\infty \int_s^\infty dx dy h_2(x) A_{11}(y,x) h_1(y) \\
&= \langle h_1 | A_{11} h_2 \rangle .
\end{aligned}
\tag{5.6.12}
$$

Now, we show that $\langle Y_1 | Z_1 \rangle = \langle Y_2 | Z_2 \rangle$. The two are explicitly given by

$$
\langle Y_1 | Z_1 \rangle = \langle f_1 | A_{11} h_2 \rangle + \langle f_1 | A_{12} h_1 \rangle , \qquad \langle Y_2 | Z_2 \rangle = -\langle h_1 | A_{12} f_1 \rangle + \langle h_2 | A_{22} f_1 \rangle . \tag{5.6.13}
$$

By Proposition 5.6.1, as $A_{11}(x,y) = A_{22}(y,x)$ and $A_{12}(x,y) = -A_{12}(y,x)$, it follows that $\langle f_1 | A_{11} h_2 \rangle = \langle h_2 | A_{22} f_1 \rangle$ and that $\langle f_1 | A_{12} h_1 \rangle = -\langle h_1 | A_{12} f_1 \rangle$ proving the desired equality.

Appendix

A.1 Bounds on LPP distribution

In the proofs we use known results for the point-to-point and point-to-line LPP with exponential random variables, which we recall here.

Bounds on point-to-point LPP

Proposition A.1.1. *For $\eta \in (0,\infty)$ define $\mu = (\sqrt{\eta \ell} + \sqrt{\ell})^2$, $\sigma = \eta^{-1/6}(1+\sqrt{\eta})^{4/3}$, and the rescaled random variable*

$$L_\ell^{\mathrm{resc}} := \frac{L_{(0,0)\to(\eta\ell,\ell)} - \mu}{\sigma \ell^{1/3}}. \tag{A.1.1}$$

(a) Limit law

$$\lim_{\ell\to\infty} \mathbb{P}(L_\ell^{\mathrm{resc}} \le s) = F_{\mathrm{GUE}}(s), \tag{A.1.2}$$

with F_{GUE} the GUE Tracy-Widom distribution function.
(b) Bound on upper tail: there exist constants s_0, ℓ_0, C, c such that

$$\mathbb{P}(L_\ell^{\mathrm{resc}} \ge s) \le Ce^{-cs} \tag{A.1.3}$$

for all $\ell \ge \ell_0$ and $s \ge s_0$.
(c) Bound on lower tail: there exist constants s_0, ℓ_0, C, c such that

$$\mathbb{P}(L_\ell^{\mathrm{resc}} \le s) \le Ce^{-c|s|^{3/2}} \tag{A.1.4}$$

for all $\ell \ge \ell_0$ and $s \le -s_0$. The constants C, c can be chosen uniformly for η in a bounded set.

(a) was proven in Theorem 1.6 of [Joh00b]. Using the relation with the Laguerre ensemble of random matrices (Proposition 6.1 of [BBP06]), or to TASEP described above, the distribution is given by a Fredholm determinant. An exponential decay of its kernel leads directly to *(b)*. See e.g. Proposition 4.2 of [FN15] or Lemma 1 of [BFP14] for an explicit statement. *(c)* was proven in [BFP14] (Proposition 3 together with (56)). In the present language it is reported in Proposition 4.3 of [FN15] as well.

Bounds for point-to-line LPP

Proposition A.1.2. *Let $\mathcal{L} = \{(k,-k), k \in \mathbb{Z}\}$. Consider the rescaled LPP from $\mathcal{L}$ to (ℓ,ℓ) given by*

$$L_\ell^{\mathcal{L},\mathrm{resc}} = \frac{L_{\mathcal{L}\to(\ell,\ell)} - 4\ell}{2^{4/3}\ell^{1/3}}. \tag{A.1.5}$$

(a) Limit law
$$\lim_{\ell\to\infty} \mathbb{P}(L_\ell^{\mathcal{L},\mathrm{resc}} \leq s) = F_{\mathrm{GOE}}(2^{2/3}s). \tag{A.1.6}$$
(b) Bound on upper tail: there exists constants s_0, ℓ_0, C, c such that
$$\mathbb{P}(L_\ell^{\mathcal{L},\mathrm{resc}} \geq s) \leq Ce^{-cs} \tag{A.1.7}$$
for all $\ell \geq \ell_0$ and $s \geq s_0$.
(c) Bound on lower tail: there exists constants s_0, ℓ_0, C, c such that
$$\mathbb{P}(L_\ell^{\mathcal{L},\mathrm{resc}} \leq s) \leq Ce^{-c|s|^{3/2}} \tag{A.1.8}$$
for all $\ell \geq \ell_0$ and $s \leq -s_0$.

(a) was obtained in [Sas05, BFPS07] in terms of TASEP, which can be directly rewritten in term of LPP (the complete proof is present in [BF08]). For general slopes of $\mathcal{L}$ it was shown in [FO18]. (b) this tails follows from the asymptotic analysis on the correlation kernel made in [BF08]. (c) It follows from (A.1.4) since $\mathbb{P}(L_{\mathcal{L}\to(\ell,\ell)} \leq x) \leq \mathbb{P}(L_{(0,0)\to(\ell,\ell)} \leq x)$.

Here we state and give short proofs for the bounds on the tails of the one-point distribution of the point-to-random-line and stationary LPP with $\rho = 1/2$, used in Theorem 4.2.5.

Bounds on LPP with random initial condition

Proposition A.1.3. *Define $L^\sigma_{\mathcal{L}\to(\ell,\ell)} = \max_k\{L_{(k,-k)\to(\ell,\ell)} + h^0(k,-k)\}$ with h^0 as in (4.2.6), and consider the rescaled LPP time*
$$L_\ell^{\sigma,\mathrm{resc}} = \frac{L^\sigma_{\mathcal{L}\to(\ell,\ell)} - 4\ell}{2^{4/3}\ell^{1/3}}. \tag{A.1.9}$$
Then, there exists constants s_0, ℓ_0, C, c such that:
(a) Bound on upper tail:
$$\mathbb{P}(L_\ell^{\sigma,\mathrm{resc}} \geq s) \leq Ce^{-cs} \tag{A.1.10}$$
for all $\ell \geq \ell_0$ and $s \geq s_0$.
(b) Tail on lower tail:
$$\mathbb{P}(L_\ell^{\sigma,\mathrm{resc}} \leq s) \leq Ce^{-c|s|^{3/2}} \tag{A.1.11}$$
for all $\ell \geq \ell_0$ and $s \leq -s_0$.

Proof. (a) Define $J(u) = u(2\ell)^{2/3}(1,-1)$ and $W_\ell(u) = h^0(J(u))/(2^{4/3}\ell^{1/3})$. By Donsker's theorem, $u \mapsto B_\ell(u)$ converges weakly to a two-sided Brownian motion with diffusion coefficient $2\sigma^2$. Further, define
$$L_\ell^{\mathrm{pp}}(u) := \frac{L_{J(u)\to(\ell,\ell)} - 4\ell}{2^{4/3}\ell^{1/3}}. \tag{A.1.12}$$
Then, we can write
$$L_\ell^{\sigma,\mathrm{resc}} = \max_u\{L_\ell^{\mathrm{pp}}(u) + W_\ell(u)\} \leq \max_u\{L_\ell^{\mathrm{pp}}(u) + u^2/2\} + \max_u\{B_\ell(u) - u^2/2\}. \tag{A.1.13}$$
Thus,
$$\mathbb{P}(L_\ell^{\sigma,\mathrm{resc}} \geq s) \leq \mathbb{P}(\max_u\{L_\ell^{\mathrm{pp}}(u) + u^2/2\} \geq s/2) + \mathbb{P}(\max_u\{B_\ell(u) - u^2/2\} \geq s/2). \tag{A.1.14}$$

By computations based ob Doob maximal inequality (used for instance in (4.4.36)), one obtains $\mathbb{P}(\max_u\{B_\ell(u)-u^2/2\}\geq s/2)\leq Ce^{-cs^2}$ for some constants $C,c>0$. To bound the first term without new estimates, remark that for any M we can bound

$$\begin{aligned}\mathbb{P}(\max_u\{L_\ell^{\rm pp}(u)+u^2/2\}\geq s/2)\leq \mathbb{P}(\max_u L_\ell^{\rm pp}(u)\geq s/4-M^2/2)\\ +\mathbb{P}(\max_{|u|>M}\{L_\ell^{\rm pp}(u)+u^2/2\}\geq s/4)\end{aligned}\tag{A.1.15}$$

The exponential decay in s for the second term is just a special case of (4.4.19) (set $\tau=0$) and it holds for all $M\geq M_0$, for some finite M_0. We fix $M=M_0$ and then, using the fact that $\max_u L_\ell^{\rm pp}(u)=L_\ell^{\mathcal{L},{\rm resc}}$, by (A.1.7) we have exponential decay in s for the first term as well.

(b) It follows from (A.1.4) since $\mathbb{P}(L^\sigma_{\mathcal{L}\to(\ell,\ell)}\leq x)\leq \mathbb{P}(L_{(0,0)\to(\ell,\ell)}\leq x)$. □

Bounds on stationary LPP

Proposition A.1.4. *Let $\rho=1/2$. Then there exists constants s_0,ℓ_0,C,c such that:*
(a) Bound on upper tail:

$$\mathbb{P}(L^{\mathcal{B}}_{(0,0)\to(\ell,\ell)}\geq 4\ell+2^{4/3}s\ell^{1/3})\leq Ce^{-cs}\tag{A.1.16}$$

for all $\ell\geq\ell_0$ and $s\geq s_0$.
(b) Bound on lower tail:

$$\mathbb{P}(L^{\mathcal{B}}_{(0,0)\to(\ell,\ell)}\leq 4\ell+2^{4/3}s\ell^{1/3})\leq Ce^{-c|s|^{3/2}}\tag{A.1.17}$$

for all $\ell\geq\ell_0$ and $s\leq -s_0$.

Proof. (a) One can write $L^{\mathcal{B}}_{(0,0)\to(\ell,\ell)}=\max\{L^{|,\rho}(\ell,\ell),L^{-,\rho}(\ell,\ell)\}$, where $L^{|,\rho}(\ell,\ell)$ (resp. $L^{-,\rho}(\ell,\ell)$) are the LPP with one-sided perturbation only on $i=0$ (resp. $j=0$). Then,

$$\mathbb{P}(L^{\mathcal{B}}_{(0,0)\to(\ell,\ell)}\geq x)\leq \mathbb{P}(L^{|,\rho}(\ell,\ell)\geq x)+\mathbb{P}(L^{-,\rho}(\ell,\ell)\geq x).\tag{A.1.18}$$

By choosing $x=4\ell+s2^{4/3}\ell^{1/3}$, Lemma 3.3 of [FO18] (based on the estimates on the correlation kernel in [BBP06]) gives exponential decay in s for all $s\geq s_0$.

(b) It follows from (A.1.4), since $\mathbb{P}(L^{\mathcal{B}}_{(0,0)\to(\ell,\ell)}\leq x)\leq \mathbb{P}(L_{(0,0)\to(\ell,\ell)}\leq x)$. □

Lemma A.1.5. *Let $\rho=1/2$ and define $I(u)=(\ell-2u\ell^{2/3},\ell+2u\ell^{2/3})$. Then, for any $\alpha>0$, we have*

$$\mathbb{P}(|L^{\mathcal{B}}_{(0,0)\to I(K)}-L^{\mathcal{B}}_{(0,0)\to I(0)}|\geq\alpha\ell^{1/3})\leq 4e^{-\alpha^2/(16K)}\tag{A.1.19}$$

for all ℓ large enough. Furthermore,

$$\mathbb{P}(\max_{u\geq K}L^{\mathcal{B}}_{(0,0)\to I(u)}-L^{\mathcal{B}}_{(0,0)\to I(K)}-\beta u^2\ell^{1/3}\geq\alpha\ell^{1/3})\leq Ce^{-\frac{(\alpha+\beta K^2)^2}{16K}},\tag{A.1.20}$$

for a constant C and for all $\beta>0$ and $\alpha>-\beta K^2$ and ℓ large enough.

Proof. The process $K \mapsto Y_K := L^{\mathcal{B}}_{(0,0)\to I(K)} - L^{\mathcal{B}}_{(0,0)\to I(0)}$ is a martingale [BCS06] given by a sum of i.i.d. zero mean random variables $Z_j - 2$, with $Z_j \sim \mathrm{Exp}(1/2)$. By the exponential Chebyshev inequality,

$$\begin{aligned}\mathbb{P}(|Y_K| \geq \alpha\ell^{1/3}) &\leq \mathbb{P}(Y_K \geq \alpha\ell^{1/3}) + \mathbb{P}(-Y_K \geq \alpha\ell^{1/3})\\ &\leq \inf_{t\geq 0} e^{-t\alpha\ell^{1/3}}\mathbb{E}(e^{t(Z_1-2)})^{2K\ell^{2/3}} + \inf_{t'\geq 0} e^{-t'\alpha\ell^{1/3}}\mathbb{E}(e^{-t'(Z_1-2)})^{2K\ell^{2/3}}.\end{aligned} \tag{A.1.21}$$

Using $\mathbb{E}(e^{t(Z_1-2)}) = \frac{e^{-2t}}{1-2t}$ for $t \in (0, 1/2)$ and $\mathbb{E}(e^{-t'(Z_1-2)}) = \frac{e^{2t'}}{1+2t'}$ for all $t' \geq 0$, after the minimization we obtain

$$\mathbb{P}(|Y_K| \geq \alpha\ell^{1/3}) \leq 2e^{-\alpha^2/(16K)(1+\mathcal{O}(\alpha K^{-1}\ell^{-1/3})} \leq 4e^{-\alpha^2/(16K)} \tag{A.1.22}$$

for all ℓ large enough.

For the second estimate, from the inequality

$$\begin{aligned}&\mathbb{P}(\max_{u\geq K} L^{\mathcal{B}}_{(0,0)\to I(u)} - L^{\mathcal{B}}_{(0,0)\to I(K)} - \beta u^2\ell^{1/3} \geq \alpha\ell^{1/3})\\ &\leq \sum_{m\geq 1} \mathbb{P}(\max_{u\in[Km,K(m+1)]} L^{\mathcal{B}}_{(0,0)\to I(u)} - L^{\mathcal{B}}_{(0,0)\to I(K)} \geq (\alpha + \beta K^2 m^2)\ell^{1/3})\\ &\leq \sum_{m\geq 1} \inf_{t>0} e^{-t(\alpha+\beta K^2m^2)\ell^{1/3}}\mathbb{E}(e^{t(Z_1-2)})^{2Km\ell^{2/3}}.\end{aligned} \tag{A.1.23}$$

Maximising over t and taking the sum we finally get[8]

$$\mathbb{P}\Big(\max_{u\geq K} L^{\mathcal{B}}_{(0,0)\to I(u)} - L^{\mathcal{B}}_{(0,0)\to I(-K)} - \beta u^2\ell^{1/3} \geq \alpha\ell^{1/3}\Big) \leq Ce^{-\frac{(\alpha+\beta K^2)^2}{16K}} \tag{A.1.24}$$

for a constant C and for all $\beta > 0$ and $\alpha > -\beta K^2$ and ℓ large enough. ☐

Bounds for point-to-half line LPP

Proposition A.1.6. *Let* $I(u) = (\tau N, \tau N) + u(2N)^{2/3}(1, -1)$. *Then,*

$$\mathbb{P}\Big(\max_{|u|>M} L_{I(u)\to(N,N)} > 4(1-\tau)N + 2^{4/3}(s - \gamma M^2)N^{1/3}\Big) \leq Ce^{-cM^2(1-\tau)^{-4/3}}e^{-\tilde{c}s(1-\tau)^{-1/3}} \tag{A.1.25}$$

for some constants $C, c, \tilde{c} > 0$, *which can be taken uniform in* N *and uniform for* γ *in a compact subset of* $(0, 1/(1-\tau))$.

Proof. By symmetry, it is enough to get the bound on the distribution of $\max_{u<-M} L_{I(u)\to(N,N)}$. By first shifting $I(-M)$ to the origin, and then using the mapping between LPP and TASEP, the distribution function is the same as the distribution of TASEP particle number $n = t/4 + \tilde{\tau}(t/2)^{2/3}$ at time $t = 4(1-\tau)N + 2^{4/3}N^{1/3}(s - \gamma M^2)$, starting at $x_k(0) = -2k$, $k \geq 0$.

From Proposition 3 of [BFS08] we have an explicit expression in terms of Fredholm determinant. The upper tail estimate is standard. Using Hadamard's bound it is enough to have a bound on the correlation kernel. In Section 4 of [BFS08] exponential decay of the rescaled correlation kernel has been proven. Then, simple algebraic computations give the claimed result. ☐

[8] To be precise, for $\varepsilon > 0$ small, one can bound $\mathbb{P}(L^{\mathcal{B}}_{(0,0)\to I(u)} - L^{\mathcal{B}}_{(0,0)\to I(K)} \geq (\alpha + \beta K^2u^2)\ell^{1/3})$ for all $u \geq \varepsilon K\ell^{1/3}$ using (A.1.23) and for $m \in \{1, \ldots, \varepsilon\ell^{1/3}\}$ we can minimize over t and compute the series expansion in the exponent for large ℓ.

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