Wave dynamics in random, absorptive or laseractive media

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

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Kuala Lumpur, Malaysia

Bonn, 20.06.2016

Dieser Forschungsbericht wurde als Dissertation von der Mathematisch-Naturwissenschaftlichen Fakultät der Universität Bonn angenommen und ist auf dem Hochschulschriftenserver der ULB Bonn http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

1. Gutachter:	Prof. Dr. Johann Kroha
2. Gutachter:	Prof. Dr. Hartmut Monien
Tag der Promotion:	20.07.2016
Erscheinungsjahr:	2017

Acknowledgements

First of all, I would like to thank Prof. Dr. Johann Kroha for giving me the opportunity to work on this topic at all. I have immensely profited from his intuitive approach to problems and his patience in explaining the simplest concepts to me. His absolute willingness to help out however he can in other situations which inevitably arose during the course of performing this work was also indispensable as well.

I would also like to thank Prof. Dr. Hartmut Monien for his willingness to be co-examiner of my thesis. Apart from that, his willingness to dispense hard-hitting advice and solid scientific instruction is also very much appreciated, and he has dispensed a lot of these during my long stay at the institute.

For collaboration on the work on self-consistent theory of Anderson localization of vector waves, I would like to thank Michael Golor. I am also very grateful to him for allowing me to use some plots from his code for my thesis.

I would like to also acknowledge my current colleagues at the institute, especially Ammar Nejati and Ramses Sanchez. I've had countless hours of scientific discussions with both of them from which I have benefitted immensely. I look forward to collaborating with both of you in the future.

My past colleagues, although no more in the institute now, are also missed. Roman Katzer, Mauricio Trujillo Martinez, Katinka Ballman. They all made my life in the institute so much more interesting, and they help me out when I was sometimes too stupid to realize I needed help.

Finally, I would like to thank my family, without which I would not be in Germany at all. Their unwavering support cannot be described in words.

Contents

CHAPTER 1

Introduction

1.1 General Overview

The main topic of study in this work is the properties of light propagation and transport in random and possibly nonconservative media. Thematically, the work is to be divided into three main subtopics: first, we develop a formalism to describe the diffusive behaviour of light in a dielectrically random, conservative medium. Within this scope we show that our formalism is able to calculate physically relevant quantities like the *full counting statistics* for such a medium.

Secondly, we then extend our description to the case of nonconservative media, where we show how to calculate *photonic distribution functions* from our formalism. Speaking more descriptively, we wish to consider the distribution statistics of a *random laser*, which has been studied since its postulation by Letokhov in 1969. All these quantities have been obtained under a unified framework which is known as the *nonlinear sigma model*, which is a powerful field theoretical method for the description of random systems, and in addition, in the particular case of the *Keldysh* sigma model, is equipped to deal with situations in which the system is driven out of equilibrium, such as in the case of lasing. Hence, our formalism is able to deal with the interplay of disorder and nonconservation on an equal footing.

On the other hand, the formalism described above deals only with the case of scalar wave propagation, i.e., we neglect effects from the polarization degree of freedom of light from our considerations. For most quantities under consideration in these chapters, this is fully appropriate. However, it turns out that the polarization does affect light transport in quite nontrivial ways in that it causes the effect which is better known as *antilocalization*, which acts to reverse the localizing tendencies of the random potential. The fact that a random potential tends to localize extended waves in real space is well-known under the term *Anderson localization*. This is of course a nontrivial result as it could explain the difficulty in experimental realizations of full Anderson localization in optical systems. In this work we are able to consider such systems via a different method than that of the Keldysh sigma model; we have used a combined diagrammatic-numerical method which is known as the *self-consistent theory of Anderson localization* pioneered by Vollhardt-Woelfle in the eighties. The polarization degree of freedom was mapped onto a "pseudospin" structure and we were able to simplify the resulting tensorial structure of various

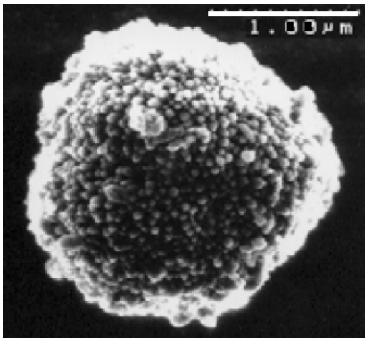
vertices that enter into the theory.

In the following sections we divide the introduction into several parts, each of which deals with one particular aspect of the work. We first describe our idea of random photonic media, in particular in the context of random lasing media. We then introduce in simple words the formalism which we use to describe such random systems, after which we mention the idea of full counting statistics of photons in passing since we calculate this quantity using our formalism in Chapter 3. We then mention the role of the polarization degree of freedom in the transport behaviour of photons, and we finally describe how we treat nonconservative systems in our formalism in the context of lasing.

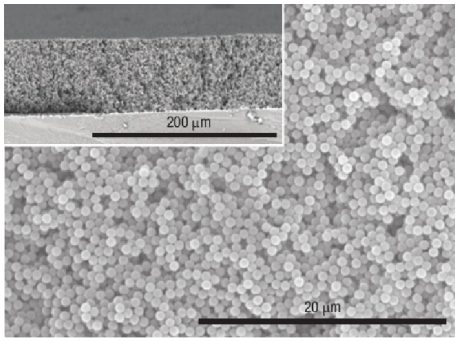
1.1.1 Random Photonic Medium

The main system of study in this work is that of a *random* photonic medium, which we imagine as a random collection of dielectric scatterers described by a space dependent disorder potential $V(\mathbf{r})$. $V(\mathbf{r})$ is in turn distributed according the a prescribed functional distribution $P[V(\mathbf{r})]$. Given that $V(\mathbf{r})$ is a spatially dependent random variable, it is always possible to define a distribution $P[V(\mathbf{r})]$; this can be seen from the fact that the random function $V(\mathbf{r})$ dictates that we need to contend with a ensemble of different realizations of $V(\mathbf{r})$, and these different realizations should then obey a certain distribution function. The main difficulty lies in dealing with the complexity of $P[V(\mathbf{r})]$.

Each configuration of V will be microscopically distinct and thus be characterized by different sets of physical properties. In order to obtain meaningful physical quantities from this ensemble of configurations one needs to *average* measured or calculated physical quantities over $P[V(\mathbf{r})]$. In this work we employ two different types of probability distributions, which have been chosen for ease of application in the respective formalisms: a *Gaussian* distributed disorder potential for calculations involving the sigma model, and a *binary* distributed disorder for calculations involving the effects of polarization on light propagation.



ZnO clusters from [1]



Photonic glass from [2]

Due to the interest in physical properties of random photonic media, a multitude of random photonic medium has been studied experimentally. It is particularly easy to find materials that are photonically random enough such that light waves are strongly scattered; air is our most common example. However, it is much more difficult to obtain materials with strong *enough* scattering such that the physically interesting regime of small mean free paths can be reached. However, these materials can be synthesized in the laboratory, and these have ranged from powders of titanium oxide [3] to alumina spheres [4]. More examples of such random photonic materials can be found in the review articles [5] and [6].

In most of these materials, the medium is conservative and does not change the intensity of light propagating in the medium. However, it has been long known that randomness in *nonconservative* medium leads to interesting effects [7], and the interest in such random photonic systems was rekindled when lasing in a zinc oxide powder was discovered [1]. This led to the discovery of many more random lasing materials, from metallic powder to paint. Two examples of such seemingly innocous random lasing materials are shown in 1.1.1.

In addition, there have also been proposals to construct disordered photonic crystals [8], where the scatterers are still arranged such as to retain spatial uniformity, but the values of the dielectric constant (which acts as the scattering potential for photons) on each scattering site should vary randomly. We use this model of disorder in our chapter on the effects of light polarization on Anderson localization of light.

1.1.2 Disorder formalism

In order to describe both disorder and nonequilibrium (and also nonlinearity), we use the Keldysh nonlinear sigma model ([9], [10], [11], [12]). In general, the nonlinear sigma model comes in three flavors, depending on the formalism used to perform the disorder averaging: the replica [13], supersymmetric [14], [15] and the Keldysh sigma models.

In simple terms, the nonlinear sigma model [16], [17] is a field-theoretical formalism which defines an *action* describing the dynamics of the so-called $\hat{Q}$-matrices, which are the remaining degrees of freedom left after one performs averaging with respect to a particular distribution of disorder potential $P[V(\mathbf{r})]$. These $\hat{Q}$ matrices can then be suitably parameterized in order to obtain physical quantities of interest. In this work we have consistently used a particular parameterization that enables us to describe the diffusive modes which are the main propagating modes in a disordered medium; however, a different parameterization could also be used to obtain other quantities, for example, the *Boltzmann-Langevin equation*.

In addition to performing the disorder averaging, the Keldysh formalism allows us to treat systems which are driven out of equilibrium, which is the case in random lasing systems. In fact, the particular characteristic of the Keldysh formalism which enable it to treat nonequilibrium systems is the same property which is exploited in order to enable the calculation of disorder averages in the functional formalism. It is clear to see that in the Keldysh nonlinear sigma model we have a versatile tool which enables us to treat disorder and nonequilibrium, in addition to nonlinearity, on the same footing.

1.1.3 Full Counting Statistics

The method developed in order to perform the disorder averaging in the functional formalism can be adapted to calculate transport statistics of a disordered optical waveguide. This is a concept borrowed from the study of similar quantities in electronic transport through a disordered wire. Counting statistics of photons in a disordered environment have been studied previously, but not in the context of the nonlinear sigma model, and not including the effects of gain. Our formalism enables us to take this into account, and we show that it does not affect the transmission statistics as compared to a disordered conservative waveguide.

Experimental measurement of transport statistics for photons is difficult, but conceptually simple to understand. Such a possible setup is shown in Fig. 1.1:

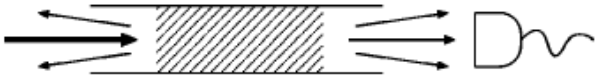


Figure 1.1: Experimental setup for photon counting experiment. From [18]

Here photons are made to propagate through a waveguide with disordered nonconservative dielectric medium, and photon counts can be registered by placing a photodetector at the end. In this manner it is possible to determine the statistics governing, for example, the arrival time of photons at the counter.

In this work we have calculated, in the field theoretical formalism, the cumulants of arbitrary order of light waves propagating through a disordered photonic waveguide. We could include the effects of static absorption / gain in our theory, and thus we could obtain corrections to the values of the bare cumulants due to these effects. Hence, we believe that our calculations will be able to shed light on the effect of energy nonconservation on transport properties in general. In addition, the study of nonlinear effects on these properties could also be straightforwardly performed, although we have left these studies for future work.

1.1.4 Role of Polarization Degree of Freedom

It has been long known that the spin degree of freedom of electrons serves as an extra channel through which coherence can be destroyed, and such decoherence effects play an important role in the studies of *Anderson localization*, where waves propagating in random media have been shown to display a change of behaviour from extended plane wave functions to spatially localized, peaked functions with increasing disorder strength. This change in behaviour is driven by wave interference effects, where diagrams favouring a return to the origin of the propagating wave dominates the behaviour of the overall system. Hence it is intuitive to see that effects which destroy this coherence will also be detrimental to the localizing contributions to the transport properties.

We show in this work that the polarization degree of freedom, analogously to that of the spin of electrons, provides such an extra channel through which decoherence occurs. Specifically, taking into account the polarization causes the probability of return back to the origin to be reduced by a factor of 2 compared to the case where only scalar waves were considered. This reduction in the return probability is commonly known as *antilocalization*. This phenomena is illustrated in Fig. 1.2 below.

1.1.5 Random Lasing

We also consider in this work the case of a system which is externally pumped such as to produce gain, i.e., a laser. Due to the external pumping a laser is an intrinsically nonequilibrium system, and the Keldysh formalism is the ideal computational tool for such systems which are driven out of equilibrium. In our case, we have an added complication in that the gain medium in the laser is random, which implies that wave transport through the system takes place via diffusion due to

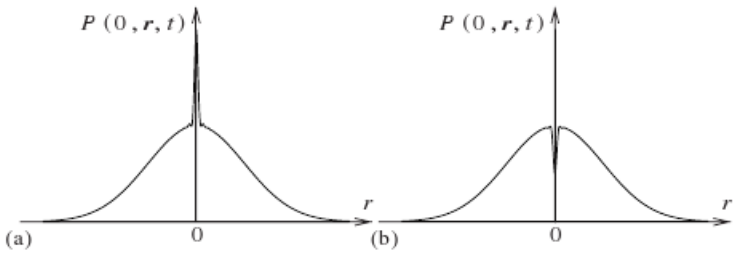


Figure 1.2: Visualization of antilocalization. On the left-hand side the probability of return to origin $p(0, \mathbf{r}, t)$ is enhanced in the absence of polarization; on the right-hand side it is decreased due to taking polarization into account. From [19]

multiple scattering of photons. This is an important point which forms the basis for the study of random lasers since it is the diffusive motion which enables photons to stay in the laser cavity long enough to produce gain.

Using the formalism to be developed here, we could derive a set of equations which describe the distribution function of photons in a random laser. This is analogous to Fokker-Planck equations which describe similar distribution functions in a regular laser, where the difference lie in the source of the randomness.

1.2 Outline of the Thesis

This thesis is divided into 6 chapters. The first chapter gives an overview of the work and introduces the main subthemes individually. Chapter 2 is a technical introduction to the mathematical formalism used in this work. Chapter 3 then starts with the first application of the formalism to a conservative disordered photonic system. We show how the nonlinear sigma model including dissipation can be derived and how to calculate correlation functions from the sigma model. In Chapter 4 we apply the previously derived sigma model to the calculation of transmission statistics of a disordered nonconservative photonic waveguide. In Chapter 5 we extend the model to include nonlinearity, which corresponds physically to an external source of energy input. This is of course crucial for the description of a random laser. In Chapter 6 we study the effects of light polarization on possible localization of light waves, using a different formalism compared to the rest of the thesis. Hence this chapter represents a significant departure compared to the rest of the thesis. The final section contains some Appendices which outline in detail certain omitted calculations in the text.

CHAPTER 2

Mathematical Formalism

The description of an intrinsically nonequilibrium system such as a laser, whose characteristics have been described in detail in Chapter 1, requires the use of different tools than those applied for equilibrium systems. These tools usually come under the rubric "Keldysh techniques", which actually describes a large set of field theoretical ideas custom-made to deal with the fact that a physical system evolves into states which might not be the same as those it initially started from. In this chapter we will introduce these ideas and explain how, in the context of our work, they are also simultaneously appropriate for the description of disordered systems.

In what follows, our system of interest denotes one which initially finds itself in a particular initial state. In principle, the specification of this state is unimportant, although it is usually assumed to be one which is trivially describable e.g., noninteracting and in a state of equilibrium. We also assume that the point of time in which this system finds itself in this state lies far back in the past (or at $t = -\infty$). This is to ensure that whatever structure or complexities of the system can be assumed to be "smoothed out" and we can assume an equilibrium initial state. We then evolve our system "slowly" enough such that during the evolution the system is only brought out of its state of equilibrium incrementally, i.e., the system evolves infinitesimally slowly. The purpose of this evolution is to bring the system to another state which is of interest, e.g., in which the constituents of the systems are interacting with one another. This new state is necessarily different from the one the system started with.
We then assume that at a specific time point (e.g., at $t = t_0$) we perform our observation (measurement) of the properties of our system. At this specific time the system is then assumed to possess the full complexity which is of interest. After this time point the system is then evolved *backwards* towards its starting time, at $t = -\infty$. This backward evolution is again assumed to proceed infinitesimally slowly, in the exact same manner as the forward evolution. Hence we end up in the same state as we started from, which mathematically enables the calculation of quantities of interest, since the initial and final states are assumed known.

The implication of this "adiabatic" evolution is that when the system is evolved from an initial to a final state, and then back to the initial state, the only net change can be the acquiring of a constant phase. This is a nontrivial statement and will be explored more carefully below.

2.0.1 Mathematical Justification

We quantify the hand-waving argumentation above in the following [12]. We assume that in the initial state, at $t = -\infty$, our system is specified by the density matrix $\hat{\rho}(-\infty)$. The evolution of $\hat{\rho}(t)$ is governed by the (possibly time-dependent) Hamiltonian $\hat{H}(t)$. Hence we can characterize the initial state of our system with the Hamiltonian $\hat{H}(-\infty)$, which we assume to be noninteracting.

The evolution of the system can be described by the usual *von Neumann* equation

$$\frac{\partial \hat{\rho}(t)}{\partial t} = -i\left[\hat{H}(t), \hat{\rho}(t)\right] \tag{2.1}$$

where $\hbar = 1$. (2.1) can be formally solved by use of the unitary evolution operator $\hat{\mathcal{U}}_{t_1,t_2}$ which works to propagate the system from time point t_1 to time point t_2. The formal solution has the form

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\left[\hat{\mathcal{U}}_{t,-\infty}\right]^{\dagger} = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}_{-\infty,t} \tag{2.2}$$

where † denotes Hermitian conjugation. $\hat{\mathcal{U}}_{t_1,t_2}$ obeys the evolution equations

$$\partial_t \hat{\mathcal{U}}_{t,t'} = -i\hat{H}(t)\hat{\mathcal{U}}_{t,t'}; \qquad \partial_{t'}\hat{\mathcal{U}}_{t,t'} = i\hat{\mathcal{U}}_{t,t'}\hat{H}(t') \tag{2.3}$$

which can be solved formally by iterative evolution by infinitesimal time steps $\delta_t = \frac{t-t'}{N}$

$$\begin{aligned}\hat{\mathcal{U}}_{t,t'} &= \lim_{N\to\infty} e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} \dots e^{-i\hat{H}(t-N\delta_t)\delta_t} e^{-i\hat{H}(t')\delta_t} \\ &= \mathbb{T}\exp\left(-i\int_t^{t'} \hat{H}(t)\,\mathrm{d}t\right)\end{aligned}, \tag{2.4}$$

where $\mathbb{T}$ denotes the time-ordering operator which is needed in this case because in general the Hamiltonian at different times do not commute with one another.
In order to obtain physical quantities in general one needs to perform averages of the form

$$\langle\hat{O}(t)\rangle \equiv \frac{\mathrm{Tr}\left\{\hat{O}\hat{\rho}(t)\right\}}{\mathrm{Tr}\left\{\hat{\rho}(t)\right\}} = \frac{1}{\mathrm{Tr}\left\{\hat{\rho}(t)\right\}}\mathrm{Tr}\left\{\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\right\} \tag{2.5}$$

and in (2.5) one can clearly see where the various quantities defined above come into play. In words, this expression describes evolution of the density matrix $\hat{\rho}(-\infty)$ at $t = -\infty$ to a state at time t via the operator $\hat{\mathcal{U}}_{t,-\infty}$, at which point the observable is measured, and subsequent evolution from time t to time $t = -\infty$. In other words, this description of time evolution is equivalent to that of time evolution in the Heisenberg picture.

In the case of equilibrium, this *forward-backward* evolution of the system is avoided by a simple assumption: that the infinitesimal (or "adiabatic") evolution which was described above has only the trivial effect of leaving a *phase* difference [20] when the system is evolved from the initial to the final and back again to the intial state. More concretely, for $T = 0$ we denote the ground state of the *full interacting* system as $|\mathrm{GS}\rangle$, which for the purpose of this explanation is

assumed to be the end state of the adiabatic evolution described above, we then have

$$|\mathrm{GS}\rangle = \hat{\mathcal{U}}_{t,-\infty}|0\rangle \tag{2.6}$$

where $|0\rangle$ is the *noninteracting* ground state, or in other words, the initial state from which we started. Hence in this notation, the type of expectation values which need to be computed in order to evaluate expressions like (2.5) have the common form

$$\langle \mathrm{GS}|\hat{O}|\mathrm{GS}\rangle = \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \tag{2.7}$$

For adiabatic time evolution we have in addition the relation

$$\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle = e^{iL}|0\rangle \tag{2.8}$$

As already mentioned above, physical idea behind this expression is that, although evolving, the system is assumed to be in its *evolving ground state* at all times, and given the normalization $\langle 0|0\rangle$ the only effect of this evolution is an additional phase factor $e^{i\Phi}$. The Hermitian conjugated (2.6) holds in addition: $\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty} = \langle 0|e^{i\Phi}$, and one can proceed to deduce that

$$\begin{aligned}
\langle GS|\hat{O}|GS\rangle &= \langle 0|\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \\
&= e^{-i\Phi}\langle 0|e^{iL}\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \\
&= e^{-i\Phi}\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}\hat{\mathcal{U}}_{-\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle \\
&= \frac{\langle 0|\hat{\mathcal{U}}_{+\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}|0\rangle}{\langle 0|\hat{\mathcal{U}}_{+\infty,-\infty}|0\rangle}
\end{aligned} \tag{2.9}$$

(2.9) describes the evolution of a system in an *adiabatic* manner from $t = -\infty$ up to a certain point at time t, at which point the system is measured and quantities calculated. Evolution of the system then proceeds further towards $t = +\infty$. In this manner we need only to consider the evolution of the system in one time direction. However, the elimination of the *backward* part of the time evolution implies two significant limitations of the procedure:

1. The denominator in (2.9) presents a difficulty when applying the procedure to disordered systems. Physical quantities in a disordered system are *disorder realization* dependent, i.e., they depend sensitively on the particular realization, out of an infinite number of them, of the disorder potential. Hence meaningful values are only obtained when *ensemble averaging* is performed on the relevant quantities, e.g., the matrix element in (2.9). However, the presence of the denominator in the expression complicates the procedure of disorder averaging in a considerable manner, and hence for a disordered system this way of obtaining physical quantities is not the best one.

2. Also more importantly, this procedure is not extendible to systems which are driven out of equilibrium. This can be trivially seen by the fact that a Hamiltonian with non-adiabatic time dependence $\hat{H}(t)$ (from external fields, for example) or arbitrary boundary conditions will in general be driven away from equilibrium, and hence the assumption of adiabatic

time-evolution does not hold. In addition, even if we assume that the fields are switched off at some point in the future and we can wait an arbitrarily long time, there is no guarantee that the final state reached by the system would be the same as that from which it started from. Hence a description in the way of (2.9) would not be valid.

From the above it can be seen that leaving out the backward evolution might not be the best idea in dealing with the important cases of disorder and nonequilibrium. The obvious solution to this is of course to explicitly take the backward evolution into account, but this requires a systematic reconsideration of the formalism shown above.
We start by reinserting the backward evolution into the expression (2.9). This can be done by using the trivial identities $\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}$ and $\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{U}}_{t,+\infty} = \hat{\mathcal{U}}_{-\infty,+\infty}$. Inserting the first into (2.9) and using the second yields

$$\langle\hat{O}\rangle(t) = \frac{1}{\mathrm{Tr}\,\hat{\rho}(-\infty)}\mathrm{Tr}\left\{\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{O}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\right\} \tag{2.10}$$

(2.10) exactly corresponds to a time evolution of the form shown in Fig. 2.1 The central quantity

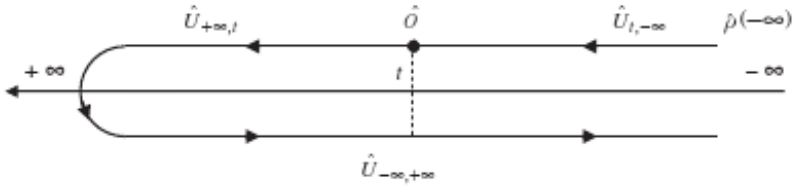


Figure 2.1: Keldysh time evolution. From [12]

in (2.10) is the evolution operator along the *closed loop* $\hat{\mathcal{U}}_C = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,-\infty}$. If we assume that the Hamiltonian is the same on both the forward and backward contours then a backward-forward evolution brings the system back to the exact same initial state, since intuitively any changes that are accunulated on the forward evolution (a phase, for example) is then directly unwound on the backward evolution. Mathematically, this means that the partition function Z has the property that

$$Z \equiv \frac{\mathrm{Tr}\,\{\hat{\mathcal{U}}_C\hat{\rho}(-\infty)\}}{\mathrm{Tr}\,\{\hat{\rho}(-\infty)\}} \tag{2.11}$$

is identically equal to 1. Of course, we are interested not in the bare partition function, but in calculating physical quantities. In terms of the partition function, this can be achieved by breaking the similarity between the forward and backward evolution paths by inserting a *source* term to the Hamiltonian $\hat{H}_V^{\pm}(t) \equiv \hat{H}(t) \pm \hat{O}V(t)$, where the plus (minus) signs refer to the forward (backward) contour directions. Since the partition function is now different between the forward and backward propagating contours, $Z \neq 1$ and we obtain from the definition (2.11) the *generating*

function

$$Z[V] \equiv \frac{\mathrm{Tr}\,\{\hat{\mathcal{U}}_C[V]\hat{\rho}(-\infty)\}}{\mathrm{Tr}\,\{\hat{\rho}(-\infty)\}} \tag{2.12}$$

from which physical quantities can be obtained via the differentiation

$$\langle\hat{O}\rangle = \frac{i}{2}\frac{\delta Z[V]}{\delta V(t)}\bigg|_{V=0} \tag{2.13}$$

2.1 Functional Formalism

In this section we will outline the construction of a functional integral formalism in the Keldysh formalism. This is particularly necessary if we want to average over realizations of disorder in some manner, as is done in the nonlinear sigma model. Starting from bosonic coherent states, we construct the partition function defined on the Keldysh contour and show how to derive the noninteracting Green's function. Due to the particular form of the evolution contour we see that the Green's function also possess special structures such that additional information relating to nonequilibrium distribution functions can be obtained. At the end of the chapter we will then show how the Keldysh action can be derived from the Green's functions and how external source terms can be incorporated into the action.

2.1.1 Partition function

As a simple example we consider the case of bosonic particles occupying a single quantum state ω_0 in this section. The Hamiltonian reads

$$\hat{H}(\hat{b}^\dagger, \hat{b}) = \omega_0 \hat{b}^\dagger \hat{b}$$

where the operators $\hat{b}^\dagger$ and $\hat{b}$ are the usual bosonic creation and annihilation operators. The central quantity in the functional formalism is the partition function. Its usual definition is simple and was given in (2.11); We note that the initial density matrix $\hat{\rho}$ can be chosen to be the equilibrium one, having the form $\hat{\rho}_0 = \exp\{-\beta(\hat{H} - \mu\hat{N})\} = \exp\{\beta(\omega_0 - \mu)\hat{b}^\dagger\hat{b}\}$ which then yields the simple form for the denominator

$$\mathrm{Tr}\,\hat{\rho}_0 = \sum_{n=0}^{\infty} e^{-\beta(\omega_0-\mu)n} = [1 - \rho(\omega_0)]^{-1} \tag{2.14}$$

which is a simple arithmetic sum. Here the factor $\rho(\omega_0) = e^{-\beta(\omega_0-\mu)}$. (2.14) is a time-independent constant and normalizes the partition function in the case of no interactions and / or disorder, since these are assumed to be turned on (off) adiabatically as the system is evolved forward (backward) via the time evolution operator. As mentioned in (2.12) and (2.13), in order to obtain observables we need to include a source term in the Hamiltonian, and computation of observables amounts, in the functional language, to the differentiation of the partition function with respect to the source term, and finally setting the source term to zero. We imagine this in more colorful

language to the attaching of a "handle" to the black box of the partition function, with which such required observables can then cranked out.

It is simple to see why such a procedure would give us the required physical quantities. If we assume the partition function with the form of the Hamiltonian including source term written in the form $\hat{O}\hat{V}(t)$ as stated above (2.13) then differentiating with respect to $\hat{V}(t)$ "brings down" the observable operator $\hat{O}$ inside the trace of the partition function due to the presence of the exponential; subsequently setting the source term to zero then gives an expression corresponding to the thermal average of the physical quantity with respect to the sourceless Hamiltonian. However, in the Keldysh formalism, we need to be aware of certain subtleties in the structure of the source term.

2.1.2 Partition function on the Keldysh contour

Having defined our contour and the peculiarities associated with it, we will now see how to perform time evolution upon the contour and how to relate this to the mechanics of performing functional integration. For this purpose an important concept from the functional formalism, that of bosonic *coherent states* will be employed and we will see how to apply it to the problem of calculating the partition function in the Keldysh framework.

Bosonic coherent states

A set of *bosonic* coherent states are defined in the usual manner as a right eigenstate of the (bosonic) annihilation operator $\hat{b}$ [12]

$$\hat{b}\,|\phi\rangle = \phi\,|\phi\rangle\,; \quad \langle\phi|\,\hat{b}^\dagger = \bar{\phi}\,\langle\phi| \tag{2.15}$$

where the complex number ϕ parameterizes the state and also denotes the complex eigenvalue of the $\hat{b}$ operator, and the overbar denotes complex conjugation. Consequently, the matrix element of a normal-ordered sequence of operators $\hat{H}(\hat{b}^\dagger, \hat{b})$ is then given by

$$\langle\phi|\,\hat{H}(\hat{b}^\dagger, \hat{b})\,|\phi\rangle = H(\bar{\phi}, \phi')\langle\phi|\phi\rangle \tag{2.16}$$

It can additionally be shown that the coherent states can be defined in the following way [21]

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{\phi^n}{n!}(\hat{b}^\dagger)^n|0\rangle = e^{\phi\hat{b}^\dagger}|0\rangle \tag{2.17}$$

where $|0\rangle$ is the vacuum state, in which $\hat{b}|0\rangle = 0$. Another striking feature of the coherent states are that they are not mutually orthogonal and form an overcomplete basis [21] which can be seen from the overlap

$$\langle\phi|\phi'\rangle = \sum_{n,n'=0}^{\infty} \frac{\bar{\phi}^n \phi'^{n'}}{\sqrt{n!n'!}}\langle n|n'\rangle = \sum_{n=0}^{\infty} \frac{(\bar{\phi}\phi')^n}{n!} = e^{\bar{\phi}\phi'} \tag{2.18}$$

which follows from the orthonormality of the pure number states. A final important characteristic of the boson coherent states is that the resolution of unity can be written in terms of them in the form

$$\hat{\mathbb{1}} = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} |\phi\rangle\langle\phi| \tag{2.19}$$

Finally, we mention that the trace of an arbitrary operator $\hat{O}$ can be simply expressed in terms of the coherent states

$$\begin{aligned} \mathrm{Tr}\,\{\hat{O}\} &\equiv \sum_{n=0}^{\infty} \langle n|\hat{O}|n\rangle = \sum_{n=0}^{\infty} \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle n|\hat{O}|\phi\rangle\langle\phi|n\rangle \\ &= \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \sum_{n=0}^{\infty} \langle\phi|n\rangle\langle n|\hat{O}|\phi\rangle = \int d[\bar{\phi}, \phi] e^{-|\phi|^2} \langle\phi|\hat{O}|\phi\rangle \end{aligned} \tag{2.20}$$

Partition function in the Keldysh formalism

In this section we will give an overview of how to include the Keldysh time contour into the usual formalism of functional integration. In principle, this task is quite trivial since the main ideas involved in the original formulation of the functional integral can be adapted to the case in which the Keldysh time contour needs to be taken into account. To start, we state an important point in the construction of the Keldysh functional integral: that the interactions and disorder are switched on (and off) on the forward (and backward) part of the contour sometime after (before) $t = -\infty$. This is of course the same assumption implicit in the usual Murray-Gell-Mann construction of field theory, with the difference that no assumption needs to be made on the switching off the interactions and disorder part of the time evolution, since, by construction, the system is forced to return to its initial state. We recall the diagram of the Keldysh contour C as presented in the previous section (Fig. 2.1). We divide C into $(2N-2)$ time intervals of length δ_t such that $t_1 = t_{2N} = -\infty$ and $t_N = t_{N+1} = +\infty$ (Fig. 2.1). Using the resolution of unity, one can proceed, as is done in the usual formulation of the functional integral, by first splitting up the contour into a finite number of discrete time slices and then inserting the resolution of unity (2.19) at each point of separation between one time slice with another; these points are then labeled with the integers $j = 1, 2, \ldots, 2N$ along the contour. Here we will illustrate this point with a particular term of order $N = 3$ originating the partition function, $Z^{\mathrm{denom}} = \mathrm{Tr}\left\{\hat{\mathcal{U}}_C \hat{\rho}\right\}$:

$$\langle\phi_6|\hat{\mathcal{U}}_{-\delta t}|\phi_5\rangle\langle\phi_5|\hat{\mathcal{U}}_{-\delta t}|\phi_4\rangle\langle\phi_4|\hat{\mathbb{1}}|\phi_3\rangle\langle\phi_3|\hat{\mathcal{U}}_{+\delta t}|\phi_2\rangle\langle\phi_2|\hat{\mathcal{U}}_{+\delta t}|\phi_1\rangle\langle\phi_1|\hat{\rho}_0|\phi_6\rangle \tag{2.21}$$

where $\hat{\mathcal{U}}_{\pm\delta_t}$ is the time evolution operator as defined in (2.4) for the time interval δ_t in the positive (negative) time direction and $|\phi_i\rangle$ are the bosonic coherent states defined in the previous secion. For infinitesimal time interval δ_t we perform an expansion of the exponentials of the partition

function in powers of δ_t; concretely, for any arbitrary factor in (2.21) we can write

$$\begin{aligned}\langle\phi_j|\hat{\mathcal{U}}_{\pm\delta_t}|\phi_{j-1}\rangle &\equiv \langle\phi_j|e^{i\hat{H}(b^\dagger,b)\delta_t}|\phi_{j-1}\rangle \approx \langle\phi_j|\left(1\mp i\hat{H}(b^\dagger,b\ )\delta_t\right)|\phi_{j-1}\rangle \\ &= \langle\phi_j|\phi_{j-1}\rangle\left(1\mp iH(\bar{\phi}_j,\phi_{j-1})\delta_t\right) \\ &\approx e^{\bar{\phi}_j\phi_{j-1}}e^{\mp iH(\bar{\phi}_j,\phi_{j-1})\delta_t}\end{aligned} \tag{2.22}$$

which holds up to linear order in δ_t and for any normal-ordered Hamiltonian. For simplicity we can again specialize to a simple Hamiltonian describing a single bosonic level

$$\hat{H}(\hat{b}^\dagger,\hat{b}) = \omega_0\hat{b}^\dagger\hat{b}\ . \tag{2.23}$$

Employing the particular identity of the coherent states [12] we obtain then

$$\langle\phi_1|e^{-\beta(\omega_0-\mu)\hat{b}^\dagger\hat{b}}|\phi_{2N}\rangle = \exp\left\{\bar{\phi}_1\phi_{2N}\rho(\omega_0)\right\}, \tag{2.24}$$

and collecting all the exponential factors along the contour one finds then for the partition function

$$Z = \frac{1}{\mathrm{Tr}\ \{\hat{\rho}_0\}}\int\prod_{j=1}^{2N}d[\bar{\phi}_j,\phi_j]\exp\left(\mathrm{i}\sum_{j,j'=1}^{2N}\bar{\phi}_jG^{-1}_{jj'}\phi_{j'}\right) \tag{2.25}$$

where for the case of $N=3$ one obtains the $2N\times 2N$ matrix $\mathrm{i}G^{-1}_{jj'}$

$$\mathrm{i}G^{-1}_{jj'} \equiv \left[\begin{array}{ccc|ccc} -1 & & & & & \rho(\omega_0) \\ h_- & -1 & & & & \\ & h_- & -1 & & & \\ \hline & & 1 & -1 & & \\ & & & h_+ & -1 & \\ & & & & h_+ & -1 \end{array}\right] \tag{2.26}$$

and $h_\mp \equiv 1\mp\mathrm{i}\omega_0\delta_t$. The discrete matrix form of the propagator in the exponent of the partition function is required in order to employ the Gaussian integration identity

$$Z[\bar{J},J] = \int\prod_{j=1}^{N}d[\bar{z}_j,z_j]e^{-\sum_{ij}^N\bar{z}_i\hat{A}_{ij}z_j+\sum_j^N[\bar{z}_jJ_j+\bar{J}_jz_j]} = \frac{e^{\sum_{ij}^N\bar{J}_i(\hat{A^{-1}})_{ij}J_j}}{\det\hat{A}}. \tag{2.27}$$

We see that if we associate $\hat{A} = -\mathrm{i}\hat{G}^{-1}$, then the determinant in the denominator of (2.27) of the matrix (2.26) has the form

$$\det\left[-\mathrm{i}\hat{G}^{-1}\right] = 1-\rho(\omega_0)(h_-h_+)^{N-1} = 1-\rho(\omega_0)(1+\omega_0^2\delta_t^2)^{N-1} \approx 1-\rho(\omega_0)e^{\omega_0^2\delta_t^2(N-1)} \xrightarrow{N\to\infty} 1-\rho(\omega_0) \tag{2.28}$$

which then leads to the normalization

$$Z = \frac{1}{\mathrm{Tr}\,\hat{\rho}_0} \frac{1}{\det\left[-\mathrm{i}\hat{G}^{-1}\right]} = 1 \tag{2.29}$$

The formal expression for the partition function, obtained from taking the limit $N \to \infty$ of (2.25) can be written analogously by first writing down the explicit discrete form of the action

$$S[\bar{\phi}, \phi] = \sum_{j=2}^{2N} \delta t_j \Big[\mathrm{i}\bar{\phi}_j \frac{\phi_j - \phi_{j-1}}{\delta t_j} - \omega_0 \bar{\phi}\phi_{j-1}\Big] + \mathrm{i}\bar{\phi}_1\Big[\phi_1 - \mathrm{i}\rho(\omega_0)\phi_{2N}\Big] \tag{2.30}$$

where $\delta t_j \equiv t_j - t_{j-1} = \pm\delta_t$ corresponding to time differences on the forward and backward branches. Then the continuum corresponds to writing $\phi_j \to \phi(t)$ and the action acquires the form

$$S[\bar{\phi}, \phi] = \int_C dt\, \bar{\phi}(t)\hat{G}^{-1}\phi(t) \tag{2.31}$$

Note that the above representation is valid for any type of continuous fields, in particular for electromagnetic fields, which are the quantities of interest in this work. In addition, although the Maxwell equations are usually expressed in the original electromagnetic fields, we will express our equations frequently in terms of the vector potential $\mathbf{A}(\mathbf{r}, \omega)$. Their relationship, in the Coulomb gauge and in the frequency representation, takes the simple form

$$\mathbf{E}(\mathbf{r}, \omega) = i\omega\mathbf{A}(\mathbf{r}, \omega) \tag{2.32}$$

We will also ignore the vector nature of the fields in this work, except for Chapter 6. Specifically, in general the polarization of electromagnetic waves represents an additional degree of freedom which for certain cases should be taken into account. However, for our work, which mainly deals with the diffusive nature of wave propagation in the absence of any time-reversal symmetry breaking scattering potentials, we shall ignore this additional degree of freedom and represent all our fields as *scalar* quantities, hence $\mathbf{E}(\mathbf{r}, \omega) \Rightarrow E(\mathbf{r}, \omega)$ and $\mathbf{A}(\mathbf{r}, \omega) \Rightarrow A(\mathbf{r}, \omega)$. However, Chapter 4 contains material which is thematically-wise orthogonal to the rest of the thesis, since in that chapter we deal specifically with the effect of the *polarization* degree of freedom on transport properties, namely the diffusion coefficient.

CHAPTER 3

The Photonic Dissipative Nonlinear Sigma Model

3.1 Introduction

The transport of waves through disordered matter has been a topic of recurring interest ever since the discovery of the Anderson localization in electronic systems [22]. Analogous phenomena have been subsequently studied for the transport of classical [23–26], matter [27, 28], and even seismic waves [29].

The research on classical-wave propagation in disordered media has been motivated by the conjectured possibility of the localization of light. The results, such as the enhancement of dwell times due to resonant scatterers and, hence, lower energy-transport velocities [30] and the correction term in the Ward identity due to frequency-dependent scattering potentials [31], have shown that, while retaining many similarities, the behavior of light in disordered media differs from that of electrons in several important aspects. One of these aspects concerns the propagation of light in nonconservative disordered media. Such systems can be physically realized, for example, as random lasers [5, 32, 33], which have received much attention recently. A promising research direction in this context are theories that combine description of wave propagation through disordered medium with the nonlinear laser equations [34–36].

The properties of light diffusion in absorbing media was studied using the photon transport equation [37–39]. In particular, it was argued that, in the parameter range of validity of the diffusion equation, the diffusion coefficient is close to its value in the conservative medium. The treatment of light propagation starting from the wave equation has been mainly conducted via the self-consistent diagrammatic theory [40, 41]. Interesting results, such as corrections to the bare diffusion coefficient due to the additional terms in the Ward identity [42, 43] and dynamics of Anderson localization in quasi-one-dimensional geometry [44] and open three-dimensional media [45] have been obtained by these methods. An alternative description of classical wave propagation is provided by the so called effective models of disordered systems, commonly known as the nonlinear sigma model [16, 17] (NLSM). Being originally developed for electronic systems, the (supersymmetric) NLSM describing light propagation in a *conservative* disordered

medium was derived in Refs. [23, 46]. Later, the effects of an open boundary on the diffusion coefficient were studied [47] by using a similar model. Unlike the self-consistent theory of transport, effective models have not yet been applied to optical systems with absorption or gain. The effective models can be useful, e.g., in describing special properties of light localization in such systems [48–50].

In this chapter we formulate the Keldysh nonlinear sigma model [11] for the propagation of electromagnetic waves in *nonconservative* disordered media in the diffusive regime. Systems with absorption or gain are relatively simple to treat in the Keldysh formalism, which makes it possible to define an action needed for the field-theoretical description. By following the general scheme as outlined in Ref. [11] for electronic systems, we derive an effective NLSM action where we obtain a term due to nonconservativeness of the medium. A similar contribution was found in Ref. [47]; in that case the term originated from the openness of the system.

Furthermore, by using the standard methods [11], we show that the light propagation can be described by a diffusion equation for a nonconservative medium. The conditions under which the NLSM yields the diffusion equation are found to be equivalent to the restrictions imposed in the theory of transport equation [37]. Similarly, the diffusion coefficient that we derive is almost independent of the absorption or gain under these conditions. For the amplifying medium, we discuss the applicability of the linear-gain approximation and determine the threshold of random lasing.

3.2 Equations of light propagation in disordered media

We start this chapter with an elementary derivation of the equation of motion for classical electromagnetic waves in a statically nonconservative and disordered medium. In such media the dielectric constant has the following form:

$$\begin{aligned} \epsilon(\mathbf{r},\omega) &= \bar{\epsilon}(\omega) + \delta\epsilon(\mathbf{r}) \\ &= \bar{\epsilon}'(\omega) + \mathrm{i}\bar{\epsilon}''(\omega) + \delta\epsilon(\mathbf{r}) \end{aligned} \tag{3.1}$$

where we write our complex dielectric constant in two parts: a complex but homogenous part, $\bar{\epsilon}$, where the overhead bar denotes a background value of the dielectric constant which is independent of the spatial variable $\mathbf{r}$. This part of ϵ is divided into real $\bar{\epsilon}'$ and imaginary parts $\bar{\epsilon}''$, respectively. The spatially varying, random part of the dielectric constant is denoted by $\delta\epsilon(\mathbf{r})$. For each position $\mathbf{r}$ it is drawn from a known probability distribution $P(\delta\epsilon(\mathbf{r}))$.

3.2.1 Helmholtz equation

In order to describe the dynamics of wave propagation, we will write down the Helmholtz equation for the medium in question. The material in this section can be found from several sources; we use the presentation in [19].

We first assume a source-free situation, $\vec{\nabla}\cdot\mathbf{E} = 0$ and $\vec{\nabla}\cdot\mathbf{B} = 0$ in which Maxwell's equations

for fields oscillating at frequency ω then read

$$\nabla \times \mathbf{E}_\omega(\mathbf{r}) = \mathrm{i}\omega \mathbf{B}_\omega(\mathbf{r}) \tag{3.2}$$
$$\nabla \times \mathbf{H}_\omega(\mathbf{r}) = -\mathrm{i}\omega \mathbf{D}_\omega(\mathbf{r}) \tag{3.3}$$

where the fields obey the usual relations

$$\mathbf{D}_\omega(\mathbf{r}) = \epsilon(\mathbf{r}, \omega)\mathbf{E}_\omega(\mathbf{r}) = [\bar{\epsilon}(\omega) + \delta\epsilon(\mathbf{r})]\mathbf{E}_\omega(\mathbf{r}) \tag{3.4}$$
$$\mathbf{B}_\omega(\mathbf{r}) = \mu_0 \mathbf{H}_\omega(\mathbf{r}) \tag{3.5}$$

Substitution of (3.4) into (3.2) yields the usual form of the Helmholtz equation

$$-\nabla^2 \mathbf{E}_\omega(\mathbf{r}) - \frac{\omega^2}{c^2}\frac{\delta\epsilon(\mathbf{r})}{\epsilon_0}\mathbf{E}_\omega(\mathbf{r}) = \frac{\bar{\epsilon}}{\epsilon_0}\frac{\omega^2}{c^2}\mathbf{E}_\omega(\mathbf{r}) \tag{3.6}$$

Hence in what follows we write the electric field as a scalar quantity $E_\omega(\mathbf{r})$. The term proportional to $\delta\epsilon(\mathbf{r})$ gives rise to multiple, random scattering. Rewriting (3.6) slightly we can write it as a wave equation for $E_\omega(\mathbf{r})$:

$$\Delta E_\omega(\mathbf{r}) + k_0^2(1 + \mu(\mathbf{r}))E_\omega(\mathbf{r}) = 0 \tag{3.7}$$

where $\mu(\mathbf{r}) = \delta\epsilon(\mathbf{r})/\bar{\epsilon}$ is the normalized fluctuation of the dielectric constant, and $k_0 = \bar{n}\omega/c$, $\bar{n} = \sqrt{\bar{\epsilon}/\epsilon_0}$.

A very important characteristic of (3.7), common to all second order wave equations, is the energy dependence of the scattering potential $V(\mathbf{r}) = -k_0^2\mu(\mathbf{r})$ on the square of the frequency. In electronic systems, when the electron energy is decreased localization is enhanced; however, from the frequency dependence of the scattering potential for scalar waves we see that at low frequencies the scattering strength is decreased, while at high frequencies, we are in the regime of geometric optics, in which interference effects are less relevant (Rayleigh regime). Hence the frequency dependence results in a decrease of the disorder scattering strength with decreasing frequency, which translates to weaker disorder and hence reduction of localization.

3.3 Keldysh approach to light propagation

Our emphasis in this chapter will be to understand how to deal with the inhomogeneity represented by the random scattering potential in (3.7). As explained in the previous chapter, an important characteristic of the Keldysh formalism enables us to write the partition function starting from the classical wave equation in a disordered medium in a simple manner. We will exploit this fact to derive the disorder-averaged partition function, and then demonstrate its use to derive various physical quantities of interest. We also show how *linear* loss can be taken into account in a natural way.

3.3.1 Partition function for non-conservative medium

We consider the two-dimensional wave equation

$$[\nabla^2 + \epsilon(\mathbf{r},\omega)\,\omega^2]A_\omega(\mathbf{r}) = 0, \tag{3.8}$$

where $\epsilon(\mathbf{r},\omega)$ is the complex, random dielectric function For real $\epsilon(\mathbf{r},\omega)$, this equation, and its complex conjugate, can be obtained by setting to zero the functional derivatives

$$\frac{\delta S}{\delta A} = 0, \quad \frac{\delta S}{\delta A^*} = 0 \tag{3.9}$$

of the action

$$\begin{aligned} S[A,A^*] &= \frac{1}{16\pi}\int d\mathbf{r}\,\frac{d\omega}{2\pi}\left[\epsilon(\mathbf{r},\omega)\,\omega^2|A_\omega(\mathbf{r})|^2 - |\nabla A_\omega(\mathbf{r})|^2\right] \\ &= \frac{1}{16\pi}\int d\mathbf{r}\,\frac{d\omega}{2\pi}A^*_\omega(\mathbf{r})\left[\epsilon(\mathbf{r},\omega)\,\omega^2 + \nabla^2\right]A_\omega(\mathbf{r}) \end{aligned},$$

treating A and A^* as independent functions. The action can be rewritten in the representation-free operator notation as

$$S[A,A^\dagger] = \frac{1}{16\pi}A^\dagger G^{-1}A, \tag{3.10}$$

where the inverse Green function operator $G^{-1} = \epsilon(\mathbf{r},\omega)\,\omega^2 + \nabla^2$ in the $(\mathbf{r},\omega)$ representation and A ($A^\dagger$) is the Hilbert-space vector $A_\omega(\mathbf{r})$ [$A^*_\omega(\mathbf{r})$].

In order to construct the quantum Hamiltonian, one expresses the energy of the system in terms of the vector potential. $A_\omega(\mathbf{r})$ and $A^*_\omega(\mathbf{r})$ are then expanded in the normal modes of the system, the expansion coefficients become the photon annihilation and creation operators.

The partition function can be written in the form of a functional integral over the fields (i.e., the classical functions) A and $A^\dagger$. To this end, we represent the classical action along the Keldysh contour as

$$\begin{aligned} S_C &= \frac{1}{16\pi}[A^\dagger_+ G^{-1}A_+ - A^\dagger_- G^{-1}A_-] \\ &= \frac{1}{16\pi}[(A^{\mathrm{cl}})^\dagger G^{-1}A^{\mathrm{q}} + (A^{\mathrm{q}})^\dagger G^{-1}A^{\mathrm{cl}}] \end{aligned}, \tag{3.11}$$

where the subscripts "±" denote the fields on the forward and backward branches of the contour and the so called classical and quantum fields are defined by

$$A^{\mathrm{cl}} = \frac{1}{\sqrt{2}}(A_+ + A_-), \quad A^{\mathrm{q}} = \frac{1}{\sqrt{2}}(A_+ - A_-). \tag{3.12}$$

The minus sign in front of the $A^\dagger_- G^{-1}A_-$ term in Eq. (3.11) takes care of the time reversal on the backward branch, whereas A_- is the representation-free (vector) notation for the function $A_-(\mathbf{r},t)$ with the forward time ordering. It is convenient to consider A^{cl} and A^{q} as components of a single

field

$$\hat{A} = \begin{pmatrix} A^{\mathrm{cl}} \\ A^{\mathrm{q}} \end{pmatrix} \tag{3.13}$$

in the Keldysh space, which is twice the size of the original Hilbert space. (We will furnish the vectors and operators in this space with a hat.) Then the contour action can be written in the form (dropping the subscript "C")

$$S[\hat{A}, \hat{A}^\dagger] = \frac{1}{16\pi} \hat{A}^\dagger \hat{G}^{-1} \hat{A}, \tag{3.14}$$

where $\hat{G}^{-1}$ has a 2×2 matrix structure with zeros on the diagonal and equal off-diagonal elements.

In order to use $S[\hat{A}, \hat{A}^\dagger]$ in the functional integral for Z, the operator $\hat{G}^{-1}$ has to be regularized [11] by imposing the causality structure on its matrix:

$$\hat{G}^{-1} = \begin{pmatrix} 0 & (G^{-1})^{\mathrm{A}} \\ (G^{-1})^{\mathrm{R}} & (G^{-1})^{\mathrm{K}} \end{pmatrix}, \tag{3.15}$$

$$(G^{-1})^{\mathrm{R,A}}_{\omega}(\mathbf{r}) = \epsilon(\mathbf{r}, \omega)\, \omega^2 + \nabla^2 \pm i0^+, \tag{3.16}$$

$$(G^{-1})^{\mathrm{K}} = (G^{-1})^{\mathrm{R}} F - F (G^{-1})^{\mathrm{A}}. \tag{3.17}$$

Here, $(G^{-1})^{\mathrm{R,A,K}}$ are the retarded, advanced, and Keldysh components of the inverse Green function operator. The operator F that parameterizes $(G^{-1})^{\mathrm{K}}$ depends on the thermal distribution. Eq. (3.16) is written under assumption of real $\epsilon(\mathbf{r}, \omega)$. In the medium with absorption, it is generalized to

$$(G^{-1})^{\mathrm{R,A}}_{\omega}(\mathbf{r}) = \epsilon'(\mathbf{r}, \omega)\, \omega^2 + \nabla^2 \pm i\epsilon''(\mathbf{r}, \omega)\, \omega^2, \tag{3.18}$$

$$\epsilon'(\mathbf{r}, \omega) = \mathrm{Re}\,[\epsilon(\mathbf{r}, \omega)], \quad \epsilon''(\mathbf{r}, \omega) = \mathrm{Im}\,[\epsilon(\mathbf{r}, \omega)], \tag{3.19}$$

where $\epsilon'' > 0$. In the case of the gain medium, $\epsilon'' < 0$, the time-dependent Green function is exponentially diverging, and its Fourier transform to the frequency domain does not exist. The frequency representation can be defined with the help of the Laplace transform, which is equivalent to introducing a fictitious absorption to the system. When the results of a calculation are transformed back to the time representation, they should not depend on the fictitious absorption. This means that one can perform the calculations in the frequency domain assuming $\epsilon'' > 0$, and obtain the final results by analytic continuation to $\epsilon'' < 0$.

The functional-integral representation of the partition function becomes

$$Z = \mathcal{N} \int D[\hat{A}, \hat{A}^\dagger]\, e^{iS[\hat{A}, \hat{A}^\dagger]}, \tag{3.20}$$

where $\mathcal{N}$ is a nonessential normalization constant that ensures $Z = 1$ and the measure is defined by

$$D[\hat{A}, \hat{A}^\dagger] = \prod_{\substack{\mathbf{r},\omega, \\ j=\mathrm{cl,q}}} \frac{d(\mathrm{Re}\, A^j_\omega(\mathbf{r}))\, d(\mathrm{Im}\, A^j_\omega(\mathbf{r}))}{\pi}. \tag{3.21}$$

Here and below we set to unity the step size for the grid used to discretize the integral.

We note that the Keldysh formalism is especially appropriate for the description of systems with absorption or gain as it naturally takes into account the finite $\epsilon''(\mathbf{r},\omega)$ in the causality structure of the inverse Green function, Eqs. (3.15), (3.16), and (3.17).

3.3.2 Disorder average

We shall study the effect of disorder in the refractive index and assume the absorption or gain in the system to be spatially uniform. Hence, we represent the dielectric constant in the form

$$\epsilon(\mathbf{r},\omega) = \epsilon'(\omega) + \Delta\epsilon'(\mathbf{r},\omega) + i\epsilon''(\omega) \tag{3.22}$$

with the averages over disorder realizations $\langle\Delta\epsilon'(\mathbf{r},\omega)\rangle = 0$ and $\langle\Delta\epsilon'(\mathbf{r},\omega)\,\Delta\epsilon'(\mathbf{r}',\omega)\rangle \propto \delta(\mathbf{r}-\mathbf{r}')$.

In order to define the scattering time τ, let us, for a moment, neglect $\epsilon''(\omega)$. The wave equation (3.8) can be interpreted as a time-independent Schrödinger equation with the energy $\mathcal{E}(\omega) = \epsilon'(\omega)\,\omega^2$ and the potential energy $V(\mathbf{r},\omega) = -\Delta\epsilon'(\mathbf{r},\omega)\,\omega^2$. The scattering time τ and other characteristic time scales of the system are assumed to be much larger than ω_0^{-1}, where ω_0 is the typical optical frequency. The slowly varying amplitude $\widetilde{A}(\mathbf{r},t) = A(\mathbf{r},t)\exp(i\omega_0 t)$ satisfy the approximate equation

$$i\widetilde{\hbar}\frac{\partial\widetilde{A}}{\partial t} = [-\nabla^2 + V(\mathbf{r},\omega_0) - \mathcal{E}(\omega_0)]\widetilde{A}, \tag{3.23}$$

which is the time-dependent Schrödinger equation with the "optical Planck constant"

$$\widetilde{\hbar} = \frac{d\mathcal{E}(\omega_0)}{d\omega_0}. \tag{3.24}$$

The scattering time can now be defined by analogy with the quantum scattering time via the correlation function [19]

$$\langle V(\mathbf{r},\omega_0)\,V(\mathbf{r}',\omega_0)\rangle = \frac{\widetilde{\hbar}}{2\pi\nu\tau}\,\delta(\mathbf{r}-\mathbf{r}'), \tag{3.25}$$

where $\nu = dn/d\mathcal{E}$ is the quantum density of states (per unit volume). Note that $\nu = \nu_0/\widetilde{\hbar}$, where $\nu_0 = dn/d\omega$ is the standard optical density of modes.

A disorder average of the partition function can be obtained by evaluating the functional integral

$$\langle Z\rangle = \int D[V]\,Z\,\exp\left[-\frac{\pi\nu\tau}{\widetilde{\hbar}}\int d\mathbf{r}\,V^2(\mathbf{r},\omega_0)\right], \tag{3.26}$$

$$D[V] = \prod_{\mathbf{r}}\sqrt{\frac{\nu\tau}{\widetilde{\hbar}}}\,dV(\mathbf{r},\omega_0). \tag{3.27}$$

The disorder-dependent part of the action is

$$\Delta S[\hat{A}, \hat{A}^\dagger, V] = -\frac{1}{16\pi}\hat{A}^\dagger V \hat{\gamma}\hat{A}, \quad \hat{\gamma} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.28}$$

where $V(\mathbf{r}, \omega_0)$ appears as an operator V diagonal in $\mathbf{r}$; F is assumed to be diagonal in $\mathbf{r}$, as well. By completing the square, we obtain the disorder contribution to the partition function

$$\langle e^{i\Delta S} \rangle = \exp\left[-\frac{\widetilde{\hbar}}{4\pi\nu\tau}\int d\mathbf{r}\left(\frac{1}{16\pi}\hat{A}^\dagger(\mathbf{r})\hat{\gamma}\hat{A}(\mathbf{r})\right)^2\right]. \tag{3.29}$$

The short-hand notation $\hat{A}(\mathbf{r})$ is used for the Keldysh-space vector $\hat{A}$ with the fixed index $\mathbf{r}$, i.e., it is a vector in the space with the reduced dimensionality; in this notation, $\hat{A}^\dagger(\mathbf{r})\hat{A}(\mathbf{r})$ involves a summation over the remaining indices, e.g., ω and the Keldysh index.

The negative sign in the exponent (3.29) is essential for the properties of nonlinear sigma model in the optical medium. In contrast to a fermionic system, the sign cannot be changed by commuting the fields.

3.4 Nonlinear sigma model

3.4.1 Hubbard-Stratonovich transformation. Saddle point

The term of the fourth-order in the fields in Eq. (3.29) can be converted to a second-order term with the help of the Hubbard-Stratonovich transformation yielding

$$\exp\left[-\frac{\widetilde{\hbar}}{4\pi\nu\tau}\int d\mathbf{r}\left(\frac{1}{16\pi}\hat{A}^\dagger(\mathbf{r})\hat{\gamma}\hat{A}(\mathbf{r})\right)^2\right] = \mathcal{N}_Q\int D[\hat{Q}]\exp\left[-\frac{\pi\nu\widetilde{\hbar}}{4\tau}\mathrm{Tr}\,\hat{Q}^2 + i\frac{\widetilde{\hbar}}{32\pi\tau}\hat{A}^\dagger\hat{\gamma}\hat{Q}\hat{A}\right],$$
$$\mathrm{Tr}\,\hat{f} \equiv \sum_{j=\mathrm{cl,q}}\int d\mathbf{r}\,\frac{d\omega}{2\pi}\,f^{jj}_{\omega\omega}(\mathbf{r}) \tag{3.30}$$

The auxiliary field $\hat{Q}$ is the Hermitian operator diagonal in $\mathbf{r}$. The measure $D[\hat{Q}]$ is defined over the independent matrix elements by analogy to Eq. (3.21). The normalization constant $\mathcal{N}_Q$ is determined by setting $\hat{A} = 0$ and $\hat{A}^\dagger = 0$. The negative coefficient in front of $\mathrm{Tr}\,\hat{Q}^2$ determines the scale of $\hat{Q}$ and can be chosen freely. The present choice leads to the simple form of matrix $\hat{\Lambda}$ introduced in (3.37). In order to prove Eq. (D.50) more easily, one can define the matrix $\hat{\mathcal{A}}(\mathbf{r}) = \hat{A}(\mathbf{r}) \otimes \hat{A}^\dagger(\mathbf{r})$ where the tensor product applies to the Keldysh and ω subspaces. Then one represents

$$\hat{A}^\dagger\hat{\gamma}\hat{Q}\hat{A} = \mathrm{Tr}\,(Q\,\hat{\mathcal{A}}\,\hat{\gamma}). \tag{3.31}$$

Now the field $\hat{Q}$ can be integrated out after completing the square.

Using Eqs. (3.29) and (D.50) in Eq. (3.20) and integrating out the fields $\hat{A}$ and $\hat{A}^\dagger$, we obtain

the disorder-averaged partition function

$$\langle Z\rangle = \widetilde{\mathcal{N}}_Q \int D[\hat{Q}]\, e^{iS[\hat{Q}]}, \tag{3.32}$$

$$iS[\hat{Q}] \equiv -\mathrm{Tr}\left[\frac{\pi\nu\widetilde{\hbar}}{4\tau}\hat{Q}^2 + \ln\left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)\right], \tag{3.33}$$

where $\hat{G}_0^{-1}$ is the inverse Green function operator that does not include the disordered part of the dielectric constant and all $\hat{Q}$-independent factors are included in the normalization constant $\widetilde{\mathcal{N}}_Q$.

In the limit of large scattering time, the main contribution to $\langle Z\rangle$ comes from the neighborhood of a saddle point. The saddle-point equation

$$\hat{Q}(\mathbf{r})\hat{\gamma} = -\frac{1}{\pi\nu}\left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)^{-1}_{\mathbf{rr}} \tag{3.34}$$

follows from the condition of stationary variation of $S[\hat{Q}]$ with respect to $\hat{\gamma}\hat{Q}$. In the $(\mathbf{k},\omega)$ representation,

$$(G_0^{-1})^{\mathrm{R,A}}_{\omega}(\mathbf{k}) = \mathcal{E}(\omega) - k^2 \pm i\epsilon''(\omega)\,\omega^2 \tag{3.35}$$

is diagonal. We will look for the solutions $Q^{\mathrm{R,A}}_{\omega}$ in the $(\mathrm{cl},\mathrm{cl})$ and (q,q) blocks of $\hat{Q}$, respectively, which are uniform in $\mathbf{r}$ and diagonal in ω. Equation (3.34) yields for these blocks:

$$Q^{\mathrm{R,A}}_{\omega} = -\frac{1}{\pi\nu}\sum_{\mathbf{k}} \frac{1}{\mathcal{E}(\omega) - k^2 \pm i\epsilon''(\omega)\,\omega^2 + \frac{\widetilde{\hbar}}{2\tau}Q^{\mathrm{R,A}}_{\omega}}. \tag{3.36}$$

The sum over the modes can be converted into an integral over $\nu\, d\mathcal{E}$, where $\mathcal{E} = k^2$. In the limit $\omega\tau \gg 1$ and $\epsilon'' \ll \epsilon'$, the lower integration limit can be extended to $-\infty$. Then $(Q_0^{\mathrm{R,A}})_{\omega} = \pm i$ is the solution. The full matrix can be written in the form

$$\hat{Q}_0 = i\hat{\Lambda}, \quad \hat{\Lambda} = \begin{pmatrix} 1^{\mathrm{R}} & 2F \\ 0 & -1^{\mathrm{A}} \end{pmatrix}, \tag{3.37}$$

which includes the regularization in $1^{\mathrm{R,A}}_{\omega} = e^{\pm i\omega 0^+}$ and the Keldysh block. The regularization leads to an important property $\mathrm{Tr}\,\hat{Q}_0^2 = 0$.

We note that the saddle point $\hat{Q}_0$ lies outside of the manifold of Hermitian matrices $\hat{Q}$. The diagonal part of $\hat{Q}_0$ is anti-Hermitian; this property can be traced back to the negative sign in the exponent in Eq. (3.29). The $\hat{Q}$ manifold can be continuously deformed to pass through the saddle point by making the transformation $\hat{Q} \mapsto e^{i\phi}\hat{Q}$ in the neighborhood of $\hat{Q} = \hat{\Lambda}$. As ϕ changes from 0 to $\pi/2$, the logarithm argument in Eq. (3.33) has no zero eigenvalues if $\epsilon'' > 0$. Thus, no singularities are crossed by $\exp(iS[\hat{Q}])$ during the deformation.

3.4.2 Effective action

The main contribution to $\langle Z \rangle$ arises from the fluctuations about the saddle point that satisfy the condition $\mathrm{Tr}\,\hat{Q}^2 = 0$. Such fluctuations produce weak variations of the action $S[\hat{Q}]$ (3.33). The matrices $\hat{Q}$ having the above property can be represented in the general form

$$\hat{Q}(\mathbf{r}) = i\hat{R}^{-1}(\mathbf{r})\,\hat{\Lambda}\hat{R}(\mathbf{r}), \tag{3.38}$$

where $\hat{R}$ is diagonal in the $\mathbf{r}$ representation. In what follows we present the results of the calculation and refer the reader to the Appendix for details. After substituting the parameterization (3.38) in Eq. (3.33) and omitting the $\hat{Q}$-independent contribution, we arrive at

$$\begin{aligned} iS[\hat{Q}] &= -\mathrm{Tr}\,\ln\left(\hat{1} + \hat{\mathcal{G}}\hat{\gamma}\hat{R}[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}]\right) \\ &\approx -\mathrm{Tr}\left(\hat{\mathcal{G}}\hat{\gamma}\hat{R}[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}]\right) + \frac{1}{2}\mathrm{Tr}\left(\hat{\mathcal{G}}\hat{\gamma}\hat{R}[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}]\right)^2 \end{aligned} \tag{3.39}$$

where

$$\hat{\mathcal{G}} = \left(\hat{G}_0^{-1} + i\frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{\Lambda}\right)^{-1} \tag{3.40}$$

is the disorder-dependent Green function operator [see Sec. 3.5.2]. The action is expanded in the fluctuations about the saddle point, which are described by the commutator $[\hat{\gamma}\hat{G}_0^{-1}, \hat{R}^{-1}]$; at the saddle point $\hat{R} = \hat{1}$ the commutator vanishes. The disorder-free inverse Green function consists of the conservative and nonconservative parts:

$$\hat{\gamma}\,(\hat{G}_0^{-1})_\omega(\mathbf{k}) = [\mathcal{E}(\omega) - k^2]\hat{1} + i\epsilon''(\omega)\,\omega^2\hat{\Lambda}. \tag{3.41}$$

The latter results in a nontrivial contribution to the commutator due to the Keldysh structure of $\hat{\Lambda}$. There are three leading-order contributions to $S[\hat{Q}]$. Using the $\mathcal{E}(\omega)$ part of $\hat{\gamma}\hat{G}_0^{-1}$ in the linear term in Eq. (3.39) we arrive at

$$iS_1[\hat{Q}] \simeq i\pi\nu\widetilde{\hbar}\,\mathrm{Tr}\,(\partial_t\hat{Q}), \quad (\partial_t\hat{Q})_{tt} \equiv (\partial_t\hat{Q}_{tt'})_{t'=t}. \tag{3.42}$$

The contribution of the k^2 part of $\hat{\gamma}\hat{G}_0^{-1}$ to the linear term of Eq. (3.39) is neglected compared to its contribution to the second-order term, which gives

$$iS_2[\hat{Q}] \simeq -\frac{\pi}{4}\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\,(\partial_{\mathbf{r}}\hat{Q})^2. \tag{3.43}$$

To derive this result, we used the property [19]

$$\frac{\widetilde{\hbar}}{2\pi\nu}\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R}}_{\omega_0}(\mathbf{k})\,\mathcal{G}^{\mathrm{A}}_{\omega_0}(\mathbf{k}) \simeq \left(\frac{1}{\tau} + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}\right)^{-1} \equiv \bar{\tau} \tag{3.44}$$

yielding the effective scattering time $\bar{\tau}$ and defined the effective diffusion coefficient in two dimensions,

$$\bar{D} = \frac{1}{2} v^2 \bar{\tau} = \frac{2\epsilon' \omega_0^2}{\widetilde{\hbar}^2} \bar{\tau}, \tag{3.45}$$

where v is the group velocity of light in the medium. In Sec. 3.5 we show that $\bar{\tau}$ and $\bar{D}$ are the relevant parameters to describe the diffusive light propagation [see Eq. (3.77)]. Finally, the nonconservative part of $\hat{\gamma}\hat{G}_0^{-1}$ yields, in the linear order in the commutator,

$$iS_3[\hat{Q}] = \pi\nu\epsilon''\omega_0^2 \operatorname{Tr}(i\hat{\Lambda}\hat{Q} + \hat{\Lambda}^2). \tag{3.46}$$

The contributions $S_{1,2,3}[\hat{Q}]$ sum up to yield the NLSM effective action

$$\boxed{iS[\hat{Q}] = -\pi\nu_0 \operatorname{Tr}\left[-i\partial_t \hat{Q} + \frac{\bar{D}}{4}(\partial_{\mathbf{r}}\hat{Q})^2 - \frac{\epsilon''\omega_0^2}{\widetilde{\hbar}}(i\hat{\Lambda}\hat{Q} + \hat{\Lambda}^2)\right]}. \tag{3.47}$$

The action vanishes at the saddle point, $S[i\hat{\Lambda}] = 0$. The key assumption behind the NLSM is the smallness of the action for fluctuations of $\hat{Q}$ restricted to the manifold $\operatorname{Tr}\hat{Q}^2 = 0$, compared the action for arbitrary fluctuations about the saddle point. The terms $S_{1,2}[\hat{Q}]$, which also appear in the NLSM for disordered fermionic systems [11], depend only on the derivatives of $\hat{Q}$. Therefore, the dominant contribution to the partition function comes from the fluctuations $\hat{Q}_{tt'}(\mathbf{r})$ [or $\hat{R}_{tt'}(\mathbf{r})$] that are slowly varying functions of $\mathbf{r}$ and $(t + t')/2$. These "massless modes" are associated with the diffusive light propagation. The assumption of slow variation justifies the neglection of higher-order terms in the expansion (3.39). The contribution $S_3[\hat{Q}]$ results from the nonconservative nature of the medium. It is, in general, comparable to the to the "massive" $\operatorname{Tr}\hat{Q}^2$ term, unless the rate of absorption or gain is smaller than the scattering rate:

$$\frac{|\epsilon''|\omega_0^2}{\widetilde{\hbar}} \ll \frac{1}{\tau}. \tag{3.48}$$

This condition specifies the regime when the light propagation is diffusive. If this requirement is not fulfilled, the massive fluctuations beyond the NLSM have to be taken into account.

3.5 Light diffusion

In this section we calculate the disorder-averaged Green-function correlator. In particular, we consider the contribution that arises from the fluctuations of the field $\hat{Q}$ in the neighborhood of the saddle point. The correlator possesses a diffusion-pole structure modified by finite correlation length due to absorption or gain.

3.5.1 Fluctuations about the saddle point

We consider the parameterization

$$\hat{Q} = i\hat{U}e^{-\hat{W}/2}\hat{\sigma}_z e^{\hat{W}/2}\hat{U}^{-1}, \quad \hat{U} = \hat{U}^{-1} = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}, \tag{3.49}$$

where $\hat{\sigma}_z$ is the Pauli matrix. Because $\hat{\Lambda} = \hat{U}\hat{\sigma}_z\hat{U}^{-1}$ (if the regularization of unit operators is neglected), this parameterization is equivalent to Eq. (3.38) with $\hat{R} = \hat{U}\exp(\hat{W}/2)\,\hat{U}^{-1}$. It can be verified by explicit calculation that the diagonal blocks of $\hat{W}$ do not contribute to $S[\hat{Q}]$ and the Green-function correlator, at least, up to the second order in $\hat{W}$. We, therefore, represent this field in the form

$$\hat{W} = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}. \tag{3.50}$$

The specific choice of $\hat{W}$ as a Hermitian matrix is justified by the requirement of convergence of the functional integral for the partition function (see below). The operator w is diagonal in the $\mathbf{r}$ representation. By expanding the parameterization (3.49) in the powers of $\hat{W}$ we find the first- and second-order deviations from the saddle point,

$$\delta\hat{Q}^{(1)} = i\begin{pmatrix} -Fw^\dagger & -w - Fw^\dagger F \\ w^\dagger & w^\dagger F \end{pmatrix}, \tag{3.51}$$

$$\delta\hat{Q}^{(2)} = \frac{i}{2}\begin{pmatrix} ww^\dagger & ww^\dagger F + Fww^\dagger \\ 0 & -ww^\dagger \end{pmatrix}. \tag{3.52}$$

We note that only the latter matrix has the causality structure; however, the fluctuations of $\hat{Q}$ are not required to obey causality. By using $\delta\hat{Q}^{(1,2)}$ in Eq. (3.47) we can calculate fluctuations of the effective action. The first-order variation of $S[\hat{Q}]$ depends on the derivatives [1] of the distribution function F generated by the first two terms in Eq. (3.47); the third term yields an identically vanishing first-order contribution. The saddle-point equation (3.36) determines the retarded and advanced sectors of $\hat{Q}$. By setting to zero the variation of the effective action near the saddle point, we obtain the Usadel equation

$$(-\partial_{\bar{t}} + \bar{D}\partial_{\mathbf{r}}^2)\,F_{\omega_0}(\mathbf{r}, \bar{t}) = 0 \tag{3.53}$$

for $F_{tt'}(\mathbf{r})$ in the mixed representation of the slow time variable $\bar{t} = (t + t')/2$ and the large frequency ω, conjugate to $t - t'$. The second-order variation is

$$\begin{aligned} i\delta S^{(2)}[w, w^\dagger] &= -\pi\nu_0 \mathrm{Tr}\left[-i\partial_t\,\delta\hat{Q}^{(2)} + \frac{\bar{D}}{4}(\partial_{\mathbf{r}}\,\delta\hat{Q}^{(1)})^2 + i\frac{\bar{D}}{2}(\partial_{\mathbf{r}}\hat{\Lambda})(\partial_{\mathbf{r}}\,\delta\hat{Q}^{(2)}) - i\frac{\epsilon''\omega_0^2}{\widetilde{\hbar}}\hat{\Lambda}\delta\hat{Q}^{(2)}\right] \\ &= -\frac{\pi\nu_0}{2}\sum_{\omega\omega'\mathbf{k}} |w_{\omega\omega'}(\mathbf{k})|^2\left[-i(\omega - \omega') + \bar{D}k^2 + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}\right], \end{aligned} \tag{3.54}$$

[1] To transfer the differentiation from w and $w^\dagger$ to F, integration by parts can be used.

where $w(\mathbf{k})$ is the Fourier transform of $w(\mathbf{r})$. Of the two terms with spatial gradients, the second term has a zero trace. The first term yields the $\bar{D}k^2$ contribution to $\delta S^{(2)}$, as well as the additional correction

$$i\delta S_F^{(2)}[w^\dagger] = -\frac{\pi\nu_0}{2}\bar{D}\,\mathrm{tr}[w^\dagger(\partial_{\mathbf{r}}F)]^2, \tag{3.55}$$

where "tr" denotes the trace of operators that do not have the Keldysh matrix structure. This correction vanishes when $F(\mathbf{r})$ is uniform, which we will assume. With the help of Eq. (3.54), the disorder-averaged partition function can be approximated by the functional integral

$$\langle Z\rangle \approx \mathcal{N}_w \int D[w,w^\dagger]\, e^{i\delta S^{(2)}[w,w^\dagger]}, \tag{3.56}$$

where $\mathcal{N}_w$ is a normalization constant. For a medium with gain, the divergence of the integral for the modes with

$$k < k_{\min} \equiv \sqrt{-\frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}\bar{D}}} \tag{3.57}$$

indicates that the long-scale fluctuations become unstable due to onset of lasing (see Sec. 3.5.3). Thus, in the long-wavelength limit the linear-gain theory breaks down and the nonlinear effects have to be taken into account [35, 36].

3.5.2 Disorder-averaged correlator

Green functions and their combinations can be expressed in terms of derivatives of the partition function with respect to the source fields:

$$G^{jk}(1,2) = -\frac{i}{16\pi}\,\frac{\delta^2 Z[\hat{J},\hat{J}^\dagger]}{\delta[J^j(1)]^*\,\delta J^k(2)}\bigg|_{\hat{J}=\hat{J}^\dagger=0}, \tag{3.58}$$

$$G^{jk}(1,2)\,G^{lm}(3,4) + G^{jm}(1,4)\,G^{lk}(3,2) = -\frac{1}{(16\pi)^2}\,\frac{\delta^4 Z[\hat{J},\hat{J}^\dagger]}{\delta[J^j(1)]^*\,\delta J^k(2)\,\delta[J^l(3)]^*\,\delta J^m(4)}\bigg|_{\hat{J}=\hat{J}^\dagger=0},$$

$$Z[\hat{J},\hat{J}^\dagger] = \mathcal{N}\int D[\hat{A},\hat{A}^\dagger]\, e^{iS[\hat{A},\hat{A}^\dagger]+\hat{J}^\dagger\hat{A}+\hat{A}^\dagger\hat{J}}, \tag{3.59}$$

where $j,k,l,m = \mathrm{cl},\mathrm{q}$ and $1,2,\ldots$ are full sets of coordinates in some representation, e.g., $1 = (\mathbf{k}_1,\omega_1)$, etc. By inverting the matrix (4.11), we identify the sectors of the Green function as $G^{\mathrm{cl,q}} = G^{\mathrm{R}}$, $G^{\mathrm{q,cl}} = G^{\mathrm{A}}$, $G^{\mathrm{cl,cl}} = G^{\mathrm{K}} \neq [(G^{-1})^{\mathrm{K}}]^{-1}$, and $G^{\mathrm{q,q}} = 0$.

The disorder-averaged Green functions and correlators are obtained by using the above expres-

sions with the disorder-averaged partition function [2]

$$\langle Z[\hat{J},\hat{J}^\dagger]\rangle = \widetilde{\mathcal{N}}_Q \int D[\hat{Q}]\, \exp\Big(iS\,[\hat{Q}] + 16\pi i \hat{J}^\dagger \hat{\mathcal{G}}_{\hat{Q}} \hat{J}\Big), \tag{3.60}$$

$$\hat{\mathcal{G}}_{\hat{Q}} \equiv \left(\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q}\right)^{-1}. \tag{3.61}$$

We find, in particular,

$$\langle G^{\mathrm{R,A,K}}(1,2)\rangle = \langle \mathcal{G}^{\mathrm{R,A,K}}_{\hat{Q}}(1,2)\rangle_{\hat{Q}}, \tag{3.62}$$

$$\langle G^{\mathrm{R}}(1,2)\, G^{\mathrm{A}}(3,4)\rangle = \langle \mathcal{G}^{\mathrm{R}}_{\hat{Q}}(1,2)\, \mathcal{G}^{\mathrm{A}}_{\hat{Q}}(3,4) + \mathcal{G}^{\mathrm{K}}_{\hat{Q}}(1,4)\, \mathcal{G}^{\mathrm{Q}}_{\hat{Q}}(3,2)\rangle_{\hat{Q}}, \tag{3.63}$$

where the average $\langle\cdots\rangle_{\hat{Q}}$ over $\hat{Q}$ is performed with the exponential weight $\exp(iS\,[\hat{Q}])$. Equation (3.62) shows that $\hat{\mathcal{G}} = \hat{\mathcal{G}}_{i\hat{\Lambda}}$ [Eq. (3.40)] is the disorder-averaged Green function in the lowest-order saddle-point approximation. The component $\mathcal{G}^{\mathrm{Q}}_{\hat{Q}} \equiv \mathcal{G}^{\mathrm{q,q}}_{\hat{Q}}$ in Eq. (3.63) is, in general, non-zero when $\hat{Q}$ does not have the causality structure. This observation is essential for the following calculation.

We calculate the Green-function correlator

$$\mathcal{R}(1,2,3,4) \equiv \langle G^{\mathrm{R}}(1,2)\, G^{\mathrm{A}}(3,4)\rangle - \langle G^{\mathrm{R}}(1,2)\rangle\langle G^{\mathrm{A}}(3,4)\rangle \tag{3.64}$$

by expansion about the saddle point. The lowest-order correction to the Green function (3.61) is

$$\hat{\mathcal{G}}_{\hat{Q}} - \hat{\mathcal{G}} \simeq -\frac{\widetilde{\hbar}}{2\tau}\hat{\mathcal{G}}\hat{\gamma}\delta\hat{Q}^{(1)}\hat{\mathcal{G}} = i\frac{\widetilde{\hbar}}{2\tau}\begin{pmatrix} \mathcal{G}^{\mathrm{R}} w \mathcal{G}^{\mathrm{A}} + F\mathcal{G}^{\mathrm{A}} w^\dagger \mathcal{G}^{\mathrm{R}} F & F\mathcal{G}^{\mathrm{A}} w^\dagger \mathcal{G}^{\mathrm{R}} \\ -\mathcal{G}^{\mathrm{A}} w^\dagger \mathcal{G}^{\mathrm{R}} F & -\mathcal{G}^{\mathrm{A}} w^\dagger \mathcal{G}^{\mathrm{R}} \end{pmatrix}. \tag{3.65}$$

The Gaussian averages with the action (3.54) are as follows:

$$\langle w\rangle_w = \langle w^\dagger\rangle_w = 0, \quad \langle w^\dagger(1,2)\, w^\dagger(3,4)\rangle_w = 0 \tag{3.66}$$

$$\langle w(1,2)\, w^\dagger(3,4)\rangle_w = \frac{2}{\pi\nu_0}\frac{\delta_{\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_4-\mathbf{k}_3}\delta_{\omega_1,\omega_4}\delta_{\omega_2,\omega_3}}{\bar{D}(\mathbf{k}_1-\mathbf{k}_2)^2 - i(\omega_1-\omega_2) + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}}. \tag{3.67}$$

Therefore, the leading contribution to the correlator comes from the K-Q term in Eq. (3.63), which is given by the product of diagonal blocks in Eq. (3.65). We find

$$\mathcal{R}(1,2,3,4) = \frac{\widetilde{\hbar}}{2\pi\nu\tau^2}\mathcal{G}^{\mathrm{R}}(1)\,\mathcal{G}^{\mathrm{R}}(2)\,\mathcal{G}^{\mathrm{A}}(3)\,\mathcal{G}^{\mathrm{A}}(4)\frac{\delta_{\mathbf{k}_1-\mathbf{k}_4,\mathbf{k}_2-\mathbf{k}_3}\delta_{\omega_1,\omega_2}\delta_{\omega_4,\omega_3}}{\bar{D}(\mathbf{k}_1-\mathbf{k}_4)^2 - i(\omega_1-\omega_4) + \frac{2\epsilon''\omega_0^2}{\widetilde{\hbar}}}. \tag{3.68}$$

The correlator has a diffusion pole with the diffusion coefficient $\bar{D}$. The pole is modified by the ϵ'' term that arises from the corresponding contribution in the effective action (4.25). This term

[2] Equation (3.60) is derived analogously to Eq. (3.32) by adding the source terms, as in Eq. (3.59), before integrating out the fields $\hat{A}$ and $\hat{A}^\dagger$.

defines the absorption rate

$$\frac{1}{\tau_a} \equiv \frac{2\epsilon'' \omega_0^2}{\widetilde{\hbar}}, \tag{3.69}$$

negative for gain.

3.5.3 Discussion

The pole structure of the correlator implies that the light intensity I in the medium satisfies the diffusion equation with the nonconservative term:

$$\left(\partial_t - \bar{D}\nabla^2 + \tau_a^{-1}\right) I = 0, \tag{3.70}$$

$$\bar{D} = \frac{1}{2} v^2 \bar{\tau} = \frac{1}{2} v^2 \left(\frac{1}{\tau} + \frac{1}{\tau_a}\right)^{-1}. \tag{3.71}$$

We compare this equation with

$$\left[\frac{\tau}{1 + 2\tau/\tau_a}\partial_t^2 + \partial_t - D'\nabla^2 + \tau_a^{-1}\frac{1 + \tau/\tau_a}{1 + 2\tau/\tau_a}\right] I = 0, \tag{3.72}$$

$$D' \equiv \frac{1}{2} v^2 \left(\frac{1}{\tau} + \frac{2}{\tau_a}\right)^{-1}, \tag{3.73}$$

that follows from the photon transport equation (see Eq. (15) of Ref. [37]). According to Ref. [37], the light propagation is diffusive if the second derivative with respect to time in Eq. (3.72) can be neglected. This is the case when

$$\tau \ll \Delta t, \tag{3.74}$$

where Δt is the characteristic time scale of intensity variation. The reaction of the medium on a fluctuation of intensity will be determined by the shortest time scale, so that $\Delta t \lesssim |\tau_a|$ can be assumed. Therefore, when neglecting the corrections of the order of $\tau/\Delta t$ in Eq. (3.72), we also have to neglect the contributions of the order of τ/τ_a. In particular, it is consistent with the diffusion approximation to set

$$D' \simeq D \equiv \frac{1}{2} v^2 \tau. \tag{3.75}$$

The independence of absorption for the diffusion coefficient was also supported by the numerical evidence in Ref. [37]. It is worth commenting on the claim [38, 39] that the diffusion coefficient in the medium with absorption must be equal to D even for $\tau/\tau_a \sim 1$. A closer look at the derivation of the diffusion coefficient from the transport equation in Ref. [38] reveals that the time-derivative terms neglected in Eqs. (A9) and (A11) of that article would yield the diffusion coefficient

$$D'' = \frac{1}{2} v^2 \tau \left(1 - 2\frac{\tau}{\tau_a}\right) \simeq D' \tag{3.76}$$

were they taken into account. Thus, the (approximate) independence of the diffusion coefficient of absorption is a consequence of the self-consistent application of the diffusion-approximation

conditions (3.48) and (3.74).

The NLSM effective action (3.47) is derived under the condition (3.74) as well. This condition guarantees the slow variation of $\hat{Q}$, and makes it possible to neglect the contribution of $\mathcal{E}(\omega)$ part of $\hat{\gamma}\hat{G}_0^{-1}$ [Eq. (3.41)] to the second-order term in Eq. (3.39). This contribution would result in a second-time-derivative term in the effective action. Again, the diffusion approximation requires that we set

$$\bar{D} \simeq D, \quad \bar{\tau} \simeq \tau \tag{3.77}$$

in the NLSM expressions. Thus, the NLSM and the theory of transport equation agree in the diffusive regime.

In the medium with gain, the the diffusive relaxation competes with the amplification. Because the long-scale intensity fluctuations disperse slower, they become unstable, and the random lasing sets in. The cutoff wavenumber $k_{\min}$ (3.57) determines the critical sample size

$$l = \sqrt{D\,|\tau_{\mathrm{a}}|} \tag{3.78}$$

above which the system is lasing and the linear-gain theory does not apply. Alternatively, the above expression yields the lasing-threshold value of $|\tau_{\mathrm{a}}|$ if l is given.

3.6 Conclusions

We obtained the functional-integral form of the partition function for an optical medium with linear absorption or gain. Keldysh technique is particularly suitable for description of nonconservative systems because it provides a natural representation for the action. The *disorder-averaged* partition function is expressed as a functional integral over the auxiliary matrix field $\hat{Q}$. Within the framework of nonlinear sigma model, we considered the fluctuations about the saddle point that fulfill the condition $\operatorname{Tr}\hat{Q}^2 = 0$. We found that the effective action $S[\hat{Q}]$ for these fluctuations contains an extra term due to absorption or gain.

With the help of the nonlinear-sigma-model partition function, we computed the disorder-averaged Green-function correlator. The leading contribution from the vicinity of the saddle point has the diffusion-pole structure modified by a finite absorption/gain rate. The diffusion coefficient is found to be approximately independent of the absorption or gain in agreement with the theory of photon transport equation. In the medium with gain, the linear theory is not applicable in the long-wavelength limit. If the sample size exceeds a certain critical length, the random lasing sets in.

CHAPTER 4

Application of the Nonlinear σ-Model: Full Counting Statistics

4.1 Introduction

Physics of disordered mesoscopic conductors [51, 52] made a strong impact on the research on light propagation in disordered media because of the well-known analogy between the scalar wave equation and the Schrödinger equation. Important information about the system can be gained by studying fluctuations of charge current (or energy current in optical medium), which are usually described via the *current cumulants* [53]. The calculation of these quantities normally require the knowledge of the full statistical distribution of transmission eigenvalues [54, 55].

The theory for electronic charge transmission has a long history and is well developed. Various methods were employed to compute the full counting statistics of electronic conduction through disordered wires, among them the semiclassical formalism [56], random matrix theory [57], and the Green's function formalism [58]. In particular, it was found in all these works that the electronic shot noise in disordered conductors is reduced below the Poissonian (uncorrelated) value by a universal factor of $\frac{1}{3}$. This reduction is a result of the existence in a disordered conductor of a fraction of non-weakly transmitting ("open") channels with transmission coefficients close to 1, with the remainder being weakly conducting ("closed") channels with small transmission coefficients. This property is reflected in the distribution of transmission coefficients originally derived by Dorokhov [59]. Although shot noise is not present in the transmission of classical electromagnetic waves, the transmission statistics is still governed by the Dorokhov's distribution, as we show below.

In the field of optics, transmission statistics in disordered channels was extensively studied experimentally [60–64] and slightly less theoretically [65, 66]. We are not aware of similar research done on dissipative or amplifying systems, which can be relevant for practical applications, such as random lasers [32].

In the present work we rederive the Keldysh nonlinear σ model for nonconservative medium [67] in the form that allows one to calculate the energy current fluctuations. Specifically, we include the source term in the action and follow the steps outlined in Ref. [12]. In the

stationary-phase approximation one obtains the Usadel equation which admits an exact solution for a one-dimensional conservative system [12, 58]. We solve the Usadel equation in the weakly nonconservative system by a perturbation expansion and calculate the action at the stationary point. The action generates the cumulants of the energy transmitted during a fixed time interval. We apply the general theory in the special cases of thermal fluctuations in equilibrium and fluctuations over disorder realizations in the pumped system at zero temperature. The latter are related to the fluctuations of the transmission coefficients described by the Dorokhov's distribution. We also find quantitative agreement with previous diagrammatic calculations [65] of the second- and third-order cumulants in the conservative medium.

4.2 Full Counting Statistics

4.2.1 General theory of counting statistics

In this section we give an elementary introduction to the theory of counting statistics. We note that the formalism to be presented below was originally written down to describe transport of electrons. What we have done is to attempt to apply the same formalism to the transport of photons, i.e., elementary excitations of the electromagnetic field. Hence, in this section we will give a simple overview of the theory of counting statistics which is independent of the physical quantities it applies to, i.e., electrons or photons. We imagine a conductor of length L. This could be a common copper wire, in the case of electrons, or a length of fibre optic cable with dielectric inhomogeneities for photons. For purpose of this section we will collectively call the excitations which propagate through these objects, "particles". We are interested in obtaining a quantity like the number of particles N passing through a certain point on the conductor in a time interval Δt. This number N_r is, of course, a random quantity, since the particle transfer is a stochastic process, potentially influenced by imperfections of the measuring device, irregularities in the conductor, or even intrinsically, due to the quantum mechanical nature of the process. For the moment we assume that the conductor is still essentially a "clean" one, i.e., its physical characteristics is not dominated by the amount of "irregularities" present in the system beyond what is practically achievable.

For an experimentalist, there are hence two ways to proceed: on the one hand, he can take many measurements, and calculate the average number $\langle N_r \rangle$, where $\langle \dots \rangle$ denotes averaging over the various values obtained from the different experimental realizations. Although also an interesting quantity, we will not be concerned with the computation of this quantity in this chapter.

On the other hand, he can try to characterize this stochasticity via the calculation of its *statistics*, i.e., by trying to determine the probability of measuring a certain value of N_r. This quantity is concerned with the *distribution* of measured values P_{N_r}, which can be obtained, in principle, by making a number of identical measurements (say, M_{tot} times) and counting the number of times the measurement yields the value N_r, say M_N times. In the limit $M_N \to \infty$ one obtains P_N from which important quantities can be calculated, for example the usual *average*

$$\langle N \rangle = \sum_N N P_N \tag{4.1}$$

but also some information about the statistics of the the process, for example the *variance*, or in more descriptive language, the *second cumulant* of N_r, which measures the degree of deviation from the average

$$\langle\langle N^2\rangle\rangle = \left\langle (N - \langle N\rangle)^2 \right\rangle = \sum_N N^2 P_N - \left(\sum_N N P_N\right)^2 \tag{4.2}$$

However, the quantities defined in (4.1) and (4.2) are still only a partial picture of the information which can be derived from a study of the statistics. Let us define the *characteristic function*

$$\Lambda(\chi) = \left\langle e^{\mathrm{i}\chi N} \right\rangle \tag{4.3}$$

(4.3) is especially suited for characterizing statistics of the distribution p_n. Specifically, we can express the *irreducible correlators* (or *cumulants*) in terms of (4.3) by simply writing

$$\ln \chi(\lambda) = \sum_{k=1}^{\infty} m_k \frac{(i\lambda)^k}{k!}, \qquad m_k \equiv \langle\langle N - \overline{N}\rangle\rangle \tag{4.4}$$

and noticing that the action of successively differentiating $\ln \Lambda(\chi)$ with respect to $\mathrm{i}\chi$ (the "counting field") will generate expressions (4.1) and (4.2) in succession. Hence we see that (4.3) is a useful quantity for the characterization of the statistics of particle transfer. In the following section (and indeed, in the rest of the chapter) we will be concerned with the calculation of various versions of (4.3).

In the following, we will specialize to the case of electronic transport, by way of illustration. We further motivate the presentation of the characteristic function via two well-understood examples. We consider first a Poisson process, which describes, among other stochastic processes, charge transport at very low transmission with uncorrelated transmission events [68]. This process is described by the Poisson distribution

$$p_k = \begin{cases} e^{-\bar{n}} \bar{n}^k / k!, & k \leq 0, \\ 0, & k < 0. \end{cases} \tag{4.5}$$

and we obtain for the characteristic function

$$\chi(\lambda) = \exp\left((e^{i\lambda} - 1)\bar{n}\right), \tag{4.6}$$

where $\bar{n} = It/e$ is the average number of particles transmitted during time t, with I the time-averaged current and e the elementary charge. We see also that all cumulants of the Poisson distribution are identical: $m_k = \bar{n}$. Physically, the Poisson distribution describes the transfer of particles across a channel in one direction from one reservoir to another in a fully *uncorrelated* manner. Transmission is assumed to be very low, such that one can restrict to single particle transfers for any time interval δt under consideration.

Another example is the well known *binomial statistics*. A binomial distribution represents a process in which a fixed number N attempts are made of a particular trial, with the probabilities p of a "successful" and $q = 1 - p$ of an "unsuccessful" trial. It is known for the binomial distribution

that the probability of obtaining k successes out of N attempts is given by the combinatorial number $C_N^k = N!/(N-k)!k!$. Hence similarly to (4.5) we can write the distribution and hence the characteristic function for the binomial process as

$$p_k = C_N^k p^k q^{N-k} \qquad \text{and} \qquad \chi(\lambda) = (pe^{i\lambda} + q)^N \tag{4.7}$$

The binomial distribution describes an ideally transmitting channel at zero temperature, which means that the electrons are in ideal wave states and hence the momentum in the transport direction is a well-defined quantum number, which is constant. This picture then describes particle transfer in a *fully correlated* state.

This concludes the introductory section in which the general concept of FCS and its role in characterizing transport properties was explained. In the following sections we will concentrate on the application of the photonic sigma model in the calculation of the FCS and how this in turn is able to yield information on the energy transport through a disordered, one-dimensional waveguide.

4.3 Nonlinear sigma model with source

4.3.1 Keldysh field theory

In this subsection we will recall some notation first presented in the previous chapter in order to show how to correctly include a source term in the original action. We consider a classical electromagnetic wave with transverse magnetic (TM) polarization in a two-dimensional medium. Following the previous chapter, we introduce two complex fields, $A_\omega(\mathbf{r})$ and $A^*_\omega(\mathbf{r})$, whose real parts correspond to the normal component of vector potential at the position $\mathbf{r}$ and frequency ω. The physically relevant frequencies for the fields are restricted to the neighborhood of the typical optical frequency $\omega = \omega_0 > 0$, which is assumed to be the largest frequency scale in the system. Within the Keldysh formalism [9, 10], each field acquires two components as $A_\pm$ and $A^\dagger_\pm$ for the positive (+) and negative (−) time directions.

To access the full counting statistics we will again derive the functional-integral form of the partition function,

$$Z = \int D[\hat{A}, \hat{A}^\dagger]\, e^{iS[\hat{A},\hat{A}^\dagger]}, \quad \hbar = 1, \tag{4.8}$$

with the measure

$$D[\hat{A}, \hat{A}^\dagger] = \mathcal{N} \prod_{\substack{\mathbf{r},\omega,\\ j=\mathrm{cl,q}}} \frac{d(\mathrm{Re}\, A^j_\omega(\mathbf{r}))\, d(\mathrm{Im}\, A^j_\omega(\mathbf{r}))}{\pi}. \tag{4.9}$$

All information about the system is contained in the action

$$S[\hat{A}, \hat{A}^\dagger] = \frac{1}{16\pi} \hat{A}^\dagger \hat{G}^{-1} \hat{A}, \tag{4.10}$$

where the inverse Green's function operator has the form

$$\hat{G}^{-1} = \begin{pmatrix} 0 & (G^{-1})^{\mathrm{A}} \\ (G^{-1})^{\mathrm{R}} & (G^{-1})^{\mathrm{K}} \end{pmatrix} \tag{4.11}$$

(we remind that matrix products involve integration over continuous variables). The retarded, advanced, and Keldysh components of $\hat{G}^{-1}$ are given by (we set the velocity of light $c = 1$)

$$(G^{-1})^{\mathrm{R,A}}_{\omega}(\mathbf{r}) = \epsilon'(\mathbf{r},\omega)\,\omega^2 + \partial^2_{\mathbf{r}} \pm i\epsilon''(\mathbf{r},\omega)\,\omega^2, \tag{4.12}$$

$$(G^{-1})^{\mathrm{K}} = (G^{-1})^{\mathrm{R}} F_0 - F_0\,(G^{-1})^{\mathrm{A}}, \tag{4.13}$$

$$\epsilon'(\mathbf{r},\omega) \equiv \mathrm{Re}\,[\epsilon(\mathbf{r},\omega)], \quad \epsilon''(\mathbf{r},\omega) \equiv \mathrm{Im}\,[\epsilon(\mathbf{r},\omega)], \tag{4.14}$$

where $\epsilon(\mathbf{r},\omega)$ is the dielectric constant of the medium and

$$F_0 = 2n_0 + 1 = \coth\frac{\omega}{2T} \quad (k_{\mathrm{B}} = 1) \tag{4.15}$$

is related to the photon occupation number n_0, which is given here for the thermal equilibrium at temperature T. The normalization constant $\mathcal{N}$ in Eq. (4.9) depends on the discretization of continuous variables and ensures the property $Z = 1$, which reflects the fact that the system arrives to its initial state as a result of the forward-backward evolution.

Till now we have only put previously defined quantities in context of the further addition of a source term, which is necessary ion order to compute averages of physical observables. The source term for the energy current density is (cf. Ref. [12])

$$\Delta S = \mathrm{Tr}\left(\boldsymbol{\lambda}\cdot\mathbf{j}^{\mathrm{cl}}\right), \tag{4.16}$$

where $\mathbf{j}^{\mathrm{cl}}$ is the classical field component of the current

$$\mathbf{j}(\mathbf{r},t) = -\frac{1}{16\pi}\left[(\partial_{\mathbf{r}}A^*)\,(\partial_t A) + (\partial_t A^*)\,(\partial_{\mathbf{r}}A)\right] \tag{4.17}$$

and the source field $\boldsymbol{\lambda}(\mathbf{r},t)$ is the quantum component of the field $(0,\boldsymbol{\lambda})^{\mathrm{T}}$ with the zero classical component in the Keldysh space. Here and below the trace includes integration over continuous variables. The average current is then given by the functional derivative of the (logarithm of) partition function with respect to the source,

$$\langle\mathbf{j}(\mathbf{r},t)\rangle = -i\frac{\delta\ln Z[\boldsymbol{\lambda}]}{\delta\boldsymbol{\lambda}(\mathbf{r},t)}\bigg|_{\boldsymbol{\lambda}\equiv 0} = -i\frac{\delta Z[\boldsymbol{\lambda}]}{\delta\boldsymbol{\lambda}(\mathbf{r},t)}\bigg|_{\boldsymbol{\lambda}\equiv 0}. \tag{4.18}$$

The fact that $\boldsymbol{\lambda}$ is a quantum field is responsible for $Z[\boldsymbol{\lambda}] \neq 1$, in general, whereas $Z[\boldsymbol{\lambda} = 0] = 1$ by construction.

Performing the Fourier transform we approximate $\partial_t A \mapsto -i\omega A \approx -i\omega_0 A$ and $\partial_{\mathbf{r}} A \mapsto i\mathbf{k}A \approx i\omega_0\sqrt{\epsilon'}\boldsymbol{\kappa}(\mathbf{k})A$, where $\boldsymbol{\kappa}(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$. Then the sum of contributions (4.10) and (4.16) yields the

total action

$$S[\hat{A}, \hat{A}^\dagger, \lambda] = \frac{1}{16\pi} \hat{A}^\dagger \left(\hat{G}^{-1} + \sqrt{2\epsilon'} \omega_0^2 \boldsymbol{\kappa} \cdot \boldsymbol{\lambda} \hat{1} \right) \hat{A}, \tag{4.19}$$

where $\hat{1}$ is the unit operator in the Keldysh space. (4.19) is derived in detail in Appendix A.1.1.

4.3.2 Nonlinear sigma model

We neglect the dispersion of the dielectric constant

$$\epsilon(\mathbf{r}, \omega) \simeq \epsilon' + \Delta\epsilon'(\mathbf{r}) + i\epsilon'' \tag{4.20}$$

and assume that its real part has a random component with $\langle \Delta\epsilon'(\mathbf{r}) \rangle = 0$ and

$$\langle \Delta\epsilon'(\mathbf{r}) \, \Delta\epsilon'(\mathbf{r}') \rangle = \frac{2\epsilon'^2}{\pi \nu_0 \omega_0^2 \tau} \delta(\mathbf{r} - \mathbf{r}'), \tag{4.21}$$

where ν_0 is the optical density of modes at the frequency ω_0, τ is the scattering time, and the averages are taken over disorder realizations. The disorder-averaged partition function including the source term has now the form

$$\boxed{\begin{aligned} \langle Z[\lambda] \rangle &= \int D[\hat{Q}] \, e^{iS[\hat{Q}, \lambda]}, \\ iS[\hat{Q}, \lambda] &= -\mathrm{Tr} \left[\frac{\pi \nu_0}{4\tau} \hat{Q}^2 + \ln \left(\hat{G}_0^{-1} + \frac{\epsilon' \omega_0}{\tau} \hat{\gamma} \hat{Q} + \sqrt{2\epsilon'} \omega_0^2 \boldsymbol{\kappa} \cdot \boldsymbol{\lambda} \hat{1} \right) \right], \quad \hat{\gamma} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}} \tag{4.22}$$

We see from (4.22) that the partition function is again expressed via the functional integral over the field $\hat{Q}$ which has a 2×2 matrix structure in the Keldysh space; here $\hat{G}_0^{-1}$ is the inverse Green's function operator in a medium with dielectric constant ϵ' without disorder. The random $\Delta\epsilon'(\mathbf{r})$ was neglected in the source term of Eq. (4.19), because its contribution to $S[\hat{Q}, \lambda]$ would be small compared to the disorder-free source term in the limit $\omega_0 \tau \gg 1$.

We find that the stationary point of the action (4.22) has the form

$$\underline{\hat{Q}} = i\hat{\Lambda}, \quad \hat{\Lambda} = \begin{pmatrix} 1^{\mathrm{R}} & 1^{\mathrm{R}} F + F \, 1^{\mathrm{A}} \\ 0 & -1^{\mathrm{A}} \end{pmatrix}. \tag{4.23}$$

As in the previous chapter, the operator F is the so called distribution function and is to be determined later. In thermal equilibrium, $F = F_0$ [Eq. (4.15)] is uniform in space. The retarded and advanced unit operators, $1_\omega^{\mathrm{R,A}} = e^{\pm i\omega\varepsilon}$, where $\varepsilon \ll \omega_0^{-1}$ is an arbitrary constant, oscillate at high frequencies $\omega \gtrsim \varepsilon^{-1}$, which leads to the property $\mathrm{Tr} \, \underline{\hat{Q}}^2 = 0$. Note that $\underline{\hat{Q}}^2 = -\hat{1}$.

The dominant contribution to $\langle Z[\lambda] \rangle$ arises from the trace-preserving fluctuations of $\hat{Q}$ about the stationary point that satisfy the conditions

$$\mathrm{Tr} \, \hat{Q}^2 = 0, \quad \hat{Q}^2 = -\hat{1}. \tag{4.24}$$

Fluctuations of this type are the "massless" modes which do not affect the $\hat{Q}^2$ term in $S[\hat{Q}, \lambda]$, and result in a weaker variation of the action compared to arbitrary, "massive" fluctuations. The massless modes describe the diffusive light propagation. Following the general prescription in Ref. [12], we generalize the expression for the effective action of massless modes [67] in the presence of source:

$$\boxed{iS[\hat{Q}, \lambda] \simeq -\pi\nu_0 \mathrm{Tr}\left[-i\partial_t\hat{Q} + \frac{D}{4}(\hat{\partial}_{\mathbf{r}}\hat{Q})^2 - \frac{i}{2\tau_a}\hat{Q}\hat{\Lambda}_0\right]}, \tag{4.25}$$

where $(\partial_t\hat{Q})_{tt} \equiv (\partial_t\hat{Q}_{tt'})_{t'=t}$,

$$D = \frac{\tau}{2\epsilon'} \tag{4.26}$$

is the diffusion coefficient, the covariant derivative is defined as

$$\hat{\partial}_{\mathbf{r}}\hat{Q} = \partial_{\mathbf{r}}\hat{Q} - i\frac{\omega_0}{\sqrt{2}}[\lambda\hat{\gamma}, \hat{Q}]_- \tag{4.27}$$

($[\,,\,]_-$ being the commutator), the absorption rate (negative for amplification) is

$$\frac{1}{\tau_a} = \frac{\epsilon''\omega_0}{\epsilon'}, \tag{4.28}$$

and $\hat{\Lambda}_0$ is given by $\hat{\Lambda}$ [Eq. (4.23)] with $F = F_0$. Explicit derivations of the covariant derivative occuring in (4.25) is given in Appendix B.3.

We note that the disorder-free inverse Green function can be written as

$$\hat{\gamma}\hat{G}_0^{-1} = (\epsilon'\omega^2 + \partial_{\mathbf{r}}^2)\,\hat{1} + i\epsilon''\omega_0^2\,\hat{\Lambda}_0, \tag{4.29}$$

i.e., $\hat{\Lambda}_0$ describes the thermal bath responsible for dissipation (if $\epsilon'' > 0$). This expression is slightly different from the one derived in the previous chapter [1]. The $\hat{Q}$-independent contributions to the action are omitted. (4.25), which is the *nonlinear σ model* in the presence of sources, is derived under the assumptions of weak variation of $\hat{Q}$ in time and space and weak absorption,

$$\frac{\tau}{|\tau_a|} \ll 1. \tag{4.30}$$

Corrections to the diffusion coefficient D due to absorption are neglected within this order of approximation (see the discussion in Ref. [67]).

[1] In Ref. [67] we did not discriminate between $\hat{\Lambda}$ and $\hat{\Lambda}_0$, hence, the results of this reference are valid only in the case of thermal equilibrium.

4.4 Transmission statistics

4.4.1 Stationary-phase approximation

We apply the general method to compute fluctuatons of energy transmitted through a quasi-one-dimensional disordered system. Let the light be continuously pumped into the channel at one end, $x = 0$, and the outgoing energy $E = t_0\, j(L)$ accumulated during a time interval t_0 be measured at the other end, $x = L$. [In the stationary regime the one-dimensional current $j(x)$ has no time dependence.] The source field

$$\lambda(x,t) = \begin{cases} \frac{\eta}{\sqrt{2}l}, & L-l \le x \le L \text{ and } 0 \le t \le t_0 \\ 0, & \text{otherwise} \end{cases}, \quad l \to 0, \tag{4.31}$$

is constructed in such a way that it couples to the current averaged over a narrow interval of length l near the channel's end during the time t_0 [2]. This form of the source field generalizes the uniform field of Ref. [12], which was sufficient for a lossless medium where $j(x) = \text{const}$. The disorder-averaged partition function $\langle Z(\eta)\rangle$ defines the cumulants C_m of energy E via the expansion of its logarithm in the counting variable η:

$$\ln\langle Z(\eta)\rangle = \sum_{m=1}^{\infty} \frac{(i\eta)^m}{m!}\, C_m. \tag{4.32}$$

In particular, $C_1 = \langle E\rangle$ and for $m = 2, 3$ the cumulants are equal to the central moments, $C_m = \langle (E - \langle E\rangle)^m\rangle$. For $m \ge 4$ the mth central moment can be expressed in terms of $C_{m'}$'s with $m' \le m$.

With the help of the gauge transformation

$$\hat{Q}_\eta(x) = \begin{cases} \hat{Q}(x), & 0 \le x \le L-l \\ e^{-i\alpha(x)\hat{\gamma}}\, \hat{Q}(x)\, e^{i\alpha(x)\hat{\gamma}}, & L-l \le x \le L \end{cases}, \tag{4.33}$$

$$\alpha(x) = \frac{x - L + l}{2l}\,\omega_0\eta, \quad \alpha \equiv \alpha(L) = \frac{\omega_0}{2}\eta, \tag{4.34}$$

having the property $\hat{\partial}_x\hat{Q} = \partial_x\hat{Q}_\eta$, the explicit source contribution is eliminated from the action (4.25), which now reads

$$iS[\hat{Q}_\eta] = -\pi\nu_0 \mathrm{Tr}\left[-i\partial_t\hat{Q}_\eta + \frac{D}{4}(\partial_{\mathbf{r}}\hat{Q}_\eta)^2 - \frac{i}{2\tau_a}\hat{Q}_\eta(\hat{\Lambda}_0)_\eta\right]. \tag{4.35}$$

By allowing $\hat{Q}_\eta$ to fluctuate under the constraints (4.24) and setting the linear variation of $S[\hat{Q}_\eta]$ to zero we obtain the Usadel equation

$$\boxed{-i(\partial_t + \partial_{t'})(\underline{\hat{Q}}_\eta)_{tt'} + D\,\partial_x(\underline{\hat{Q}}_\eta\partial_x\underline{\hat{Q}}_\eta) - \frac{i}{2\tau_a}[(\hat{\Lambda}_0)_\eta,\, \underline{\hat{Q}}_\eta]_- = 0} \tag{4.36}$$

[2] Note that $j^{\mathrm{cl}}/\sqrt{2} = (j_+ + j_-)/2$

for the stationary-point configuration $\hat{\underline{Q}}_\eta$. Without the source ($\eta = 0$), the Usadel equation reduces to the equation

$$(-\partial_t + D\partial_x^2)F - \tau_a^{-1}(F - F_0) = 0 \tag{4.37}$$

for the distribution function $F_\omega(x,t)$ obtained from the matrix $F_{t't''}(x)$ by making the Fourier transform in the fast variable $t' - t''$ and keeping the slow variable $t = (t' + t'')/2$.

In general, $\hat{\underline{Q}}_\eta$ is not of the form (4.23). The source enters the Usadel equation via the boundary conditions. We will model the pump at $x = 0$ by the time-independent non-equilibrium distribution function within the frequency band of width $\Delta\omega \ll \omega_0$:

$$F_\omega(0) = \begin{cases} F_*, & |\omega - \omega_0| < \Delta\omega/2 \\ F_0, & |\omega - \omega_0| > \Delta\omega/2 \end{cases}, \tag{4.38}$$

where $F_* \geq F_0$ and F_0 is the equilibrium distribution (4.15) at $\omega = \omega_0$. At the other end, $x = L$, the channel is assumed to be in contact with the thermal bath. Thus, the boundary conditions are

$$\begin{aligned} \hat{\underline{Q}}_\eta(0) &= \hat{\underline{Q}}(0) = i\Lambda_*, \\ \hat{\underline{Q}}_\eta(L) &= i(\Lambda_0)_\eta(L) = ie^{-i\alpha\hat{\gamma}}\Lambda_0\, e^{i\alpha\hat{\gamma}}, \end{aligned} \tag{4.39}$$

where Λ_* depends on $F_\omega(0)$.

We evaluate the functional integral for $\langle Z(\eta)\rangle$ by the stationary-phase approximation:

$$\langle Z(\eta)\rangle \simeq a(\eta)\, e^{iS(\eta)} \simeq e^{iS(\eta)}, \quad S(\eta) \equiv S[\hat{\underline{Q}}_\eta], \tag{4.40}$$

where we neglect the η dependence of the prefactor and the normalization $\langle Z(0)\rangle = 1$ is guaranteed by the form of $\hat{\underline{Q}}_{\eta=0}$ (4.23). Thus, to obtain the cumulants we need to find the time-independent solution of the Usadel equation with boundary conditions (4.39), and calculate the action $S(\eta)$.

4.4.2 Weakly nonconservative medium

We solve the Usadel equation in the limit of small absorption/amplification retaining only the leading order in the small parameter L^2/l_a^2, where the absorption length squared (negative for amplification) is defined as

$$l_a^2 = D\tau_a. \tag{4.41}$$

The second-order Usadel equation is equivalent to the system of two first-order equations:

$$\begin{aligned} \partial_x \hat{J} &= \frac{i}{2\tau_a}[(\hat{\Lambda}_0)_\eta, \hat{\underline{Q}}_\eta]_-, \\ \hat{J} &= \hat{\underline{Q}}_\eta \partial_x \hat{\underline{Q}}_\eta \quad \left[= -(\partial_x \hat{\underline{Q}}_\eta)\, \hat{\underline{Q}}_\eta\right], \end{aligned} \tag{4.42}$$

where the last equality follows from the property $\partial_x(\underline{\hat{Q}}_\eta^2) = 0$. Equation (4.42) can be formally integrated to yield

$$\underline{\hat{Q}}_\eta(x) = \underline{\hat{Q}}_\eta(0)\left[e^{\int_0^x dx'\,\hat{J}(x')}\right], \tag{4.43}$$

where the exponential is ordered in x' increasingly from left to right. In the zeroth order in absorption one finds $\hat{J}(x) = \text{const}$, which leads to [12]

$$\underline{\hat{Q}}_\eta^{(0)}(x) = \underline{\hat{Q}}_\eta(0)\, e^{\hat{J}_0 x}, \tag{4.44}$$

$$\hat{J}_0 \equiv L^{-1}\ln[-\underline{\hat{Q}}_\eta(0)\,\underline{\hat{Q}}_\eta(L)]. \tag{4.45}$$

In the next order we write $\hat{J}(x) = \hat{J}_0 + \Delta\hat{J}(x)$ and $\underline{\hat{Q}}_\eta(x) = \underline{\hat{Q}}_\eta^{(0)}(x) + \underline{\hat{Q}}_\eta^{(1)}(x)$. By expanding the ordered exponential in Eq. (4.43) in $\Delta\hat{J}(x)$ we obtain

$$\underline{\hat{Q}}_\eta^{(1)}(x) \simeq \underline{\hat{Q}}_\eta(0)\int_0^x dx'\, e^{\hat{J}_0 x'}\,\Delta\hat{J}(x')\, e^{\hat{J}_0(x-x')}. \tag{4.46}$$

The requirement $\underline{\hat{Q}}_\eta^{(1)}(L) = 0$, that follows from the boundary conditions, yields the property

$$\int_0^L dx\, e^{\hat{J}_0 x}\,\Delta\hat{J}(x)\, e^{-\hat{J}_0 x} = 0. \tag{4.47}$$

The gradient contribution to the action contains

$$\begin{aligned}\mathrm{Tr}\,(\partial_x\underline{\hat{Q}}_\eta)^2 &= -\mathrm{Tr}\left[(\partial_x\underline{\hat{Q}}_\eta)\,\underline{\hat{Q}}_\eta^2\,\partial_x\underline{\hat{Q}}_\eta\right] = \mathrm{Tr}\,\hat{J}^2\\ &\simeq \mathrm{Tr}\,\hat{J}_0^2 + 2\mathrm{Tr}\,(\hat{J}_0\Delta\hat{J}) = \mathrm{Tr}\,\hat{J}_0^2,\end{aligned} \tag{4.48}$$

according to the property (4.47). Thus, $\Delta\hat{J}(x)$ does not contribute to the action, which now takes the form

$$iS(\eta) \simeq -\pi\nu_0\mathrm{Tr}\left[\frac{D}{4}\hat{J}_0^2 - \frac{i}{2\tau_a}(\hat{\Lambda}_0)_\eta\,\underline{\hat{Q}}_\eta^{(0)}\right]. \tag{4.49}$$

The nonconcervative part of the action can be simplified in the limit $l \to 0$ by tracing over x explicitly and disregarding the interval $L - l \le x \le L$; the remaining integral can be evaluated with $l = 0$ (note that $\hat{J}_0$ does not depend on l) yielding

$$\boxed{iS_{\mathrm{nc}}(\eta) = \frac{i\pi\nu_0}{2\tau_a}\,\mathrm{Tr}_{\omega,\mathrm{K}}\left[\hat{\Lambda}_0\,\underline{\hat{Q}}(0)\,\hat{J}_0^{-1}\left(e^{\hat{J}_0 L} - \hat{1}\right)\right]}, \tag{4.50}$$

where the trace is performed in the ω and Keldysh subspaces. Below we apply the general expressions in the cases of thermal equilibrium and transport at zero temperature.

4.4.3 Special cases

Thermal fluctuations

We consider the fluctuations of transmitted energy in the absence of pumping, $F_* = F_0$, at finite temperature T. The leading contribution to the action (4.49) is

$$iS_0(\eta) = -\frac{\pi\nu_0 D t_0}{4L}\int_0^\infty \frac{d\omega}{2\pi}\, 2(\ln\lambda)^2, \tag{4.51}$$

where λ is one of the two eigenvalues

$$\lambda_{1,2} = 1 + X \pm \sqrt{X(X+2)}, \quad \lambda_1\lambda_2 = 1, \tag{4.52}$$

$$X \equiv 2\,(F_0^2 - 1)\,\sin^2\alpha, \tag{4.53}$$

of the matrix $\hat\Lambda_0\, e^{-i\alpha\hat\gamma}\hat\Lambda_0\, e^{i\alpha\hat\gamma}$ appearing in the logarithm in Eq. (4.45). According to Eq. (4.15) only frequencies $\omega \lesssim T$ contribute to the integral (4.51) [3]. The nonconcervative contribution reads

$$iS_{\rm nc}(\eta) = -\frac{\pi\nu_0}{2\tau_{\rm a}} t_0 L\int_0^\infty \frac{d\omega}{2\pi}\, 2\left[\frac{\sqrt{X(X+2)}}{\ln\lambda_1} - 1\right], \tag{4.54}$$

where the unity is subtracted in order to account for the high-frequency regularization due to $1^{\rm R,A}$. To justify this result, let us, first, set $T = 0$ ($F_0 = 1$). In this case the matrix $\hat J_0$ cannot be diagonalized, but can be brought to the upper triangular form with $1^{\rm R,A}$ on the diagonal by the rotation $\hat{\mathcal{P}}^{-1}\hat J_0\hat{\mathcal{P}}$, where

$$\hat{\mathcal{P}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}^{\rm T}. \tag{4.55}$$

The regularization makes the trace in Eq. (4.50) vanish, which results in $S_{\rm nc}(\eta; T = 0) = 0$. For finite temperature we can write

$$\begin{aligned} S_{\rm nc}(\eta; T) &= S_{\rm nc}(\eta; T) - S_{\rm nc}(\eta; T = 0) \\ &= \tilde S_{\rm nc}(\eta; T) - \tilde S_{\rm nc}(\eta; T = 0), \end{aligned} \tag{4.56}$$

where the tilde indicates that the regularization is ignored when the action is calculated. The second equality is based on the fact that the regularization is only important at frequencies $\omega \gg T$, but the high-frequency contribution to the action does not depend on T, because $F_0 \to 1$ for $\omega \to \infty$. Equation (4.54) then follows from Eq. (4.56)

It can be easily seen that the expansion of $\ln\langle Z(\eta)\rangle \simeq iS(\eta)$ contains only even powers of η. Hence, the odd-order cumulants vanish in thermal equilibrium, which is a consequence of the equivalence of positive and negative direction of the current. The odd-order cumulants are useful for studying the nonequilibrium properties at finite temperatures, because they are not obscured by thermal fluctuations [69].

[3] To regularize the integral at low frequencies we should avoid fixing ω in $\alpha = \omega\eta/2$.

Transport at zero temperature

For the medium at zero temperature in the presence of pumping ($F_* > F_0 = 1$) the leading contribution to the action becomes

$$iS_0(\eta) = -\frac{\nu_0 D t_0 \Delta\omega}{4L}(\ln\lambda)^2. \tag{4.57}$$

Here λ is one of the eigenvalues

$$\begin{aligned} \lambda_{1,2} &= 1 - Y \pm \sqrt{Y(Y-2)}, \\ Y &\equiv (F_* - 1)(e^{2i\alpha} - 1) \end{aligned}, \tag{4.58}$$

of the matrix

$$\hat{B} \equiv \hat{\Lambda}_* \, e^{-i\alpha\hat{\gamma}} \hat{\Lambda}_0 \, e^{i\alpha\hat{\gamma}}. \tag{4.59}$$

It is convenient to calculate the nonconservative part (4.50) by diagonalizing the matrix $\hat{J}_0$. This is achieved with the transformation $\hat{\mathcal{R}}^{-1}\hat{J}_0\hat{\mathcal{R}}$, where the rotation matrices

$$\hat{\mathcal{R}} = (\hat{R}^{(1)}, \hat{R}^{(2)}), \quad \hat{\mathcal{R}}^{-1} = \begin{pmatrix} (\hat{L}^{(1)})^\dagger \\ (\hat{L}^{(2)})^\dagger \end{pmatrix} \tag{4.60}$$

are defined by the biorthogonal right and left eigenvectors of matrix $\hat{B}$, $\hat{R}^{(j)}$ and $\hat{L}^{(j)}$, corresponding to the eigenvalue λ_j. After some algebra we arrive at

$$iS_{\rm nc}(\eta) = -\frac{\nu_0 t_0 \Delta\omega L}{4\tau_{\rm a}} \sum_{j=1}^{2} \left\{ \frac{\lambda_j - 1}{\ln\lambda_j} \left[1 + 2(F_* - 1)\left(L_1^{(j)}\right)^* R_2^{(j)} \right] - 1 \right\}, \tag{4.61}$$

where, again, the regularization makes it necessary to subtract unity.

The linear contribution in the $S(\eta)$ expansion yields the first-order cumulant, the average transmitted energy, which is proportional to the average current

$$\langle j \rangle \simeq \langle j_0 \rangle \left(1 - \frac{L^2}{6 l_{\rm a}^2} \right). \tag{4.62}$$

The average current in conservative medium,

$$\langle j_0 \rangle = \nu_0 \, \Delta\omega \, \omega_0 D n_* L^{-1}, \tag{4.63}$$

is linear with the photon occupation number (pump intensity) $n_* = (F_* - 1)/2$ and inversely proportional to the length of the channel. To clarify the meaning of these expressions, we derive them, alternatively, from the nonconservative diffusion equation (4.37), which, in the stationary regime, takes the form

$$\partial_x^2 n - l_{\rm a}^{-2} n = 0. \tag{4.64}$$

The solution for the occupation number satisfying the boundary conditions $n(0) = n_*$ and

$n(L) = 0$ is

$$n(x) = n_* \frac{\sinh\left[(L-x)/l_\mathrm{a}\right]}{\sinh\left(L/l_\mathrm{a}\right)}. \tag{4.65}$$

Expanding the energy current $j(L) = -\omega_0 D n'(L)$ in L/l_a we obtain Eqs. (4.62) and (4.63) (up to the number of modes $\nu_0\,\Delta\omega$).

Higher-order cumulants can be calculated by expanding Eqs. (4.57) and (4.61) in $i\eta$, possibly, with the help of a symbolic manipulation software. Specifically, we find

$$\frac{C_2/\omega_0^2}{C_1^{(0)}/\omega_0} \simeq \frac{2}{3} n_* \left(1 - \frac{11}{10}\frac{L^2}{l_\mathrm{a}^2}\right), \tag{4.66}$$

$$\frac{C_3/\omega_0^3}{C_1^{(0)}/\omega_0} \simeq \frac{16}{15} n_*^2 \left(1 - \frac{191}{336}\frac{L^2}{l_\mathrm{a}^2}\right), \tag{4.67}$$

where $C_1^{(0)} = \langle j_0 \rangle\, t_0$ and we take into account that $n_* \gg 1$ for a classical electromagnetic wave. The ratios C_m/ω_0^m describe fluctuations of the number of transmitted photons E/ω_0. The principal terms in Eqs. (4.66) and (4.67) agree with the results of diagrammatic calculation for the square pump profile incident on a disordered slab [65]. The absorption (amplification) leads to decrease (increase) of fluctuations.

The results for the conservative medium ($l_\mathrm{a}^{-2} = 0$) are in agreement with the Dorokhov's distribution [59] of the transmission coefficients,

$$P(\mathcal{T}) \simeq \frac{P_0}{\mathcal{T}\sqrt{1-\mathcal{T}}}, \tag{4.68}$$

for sufficiently narrow pumping bandwidth $\Delta\omega$ and short measuring time t_0. The distribution was used to derive the average shot noise for electron transport [57]. $P(\mathcal{T})$ needs to be regularized at $\mathcal{T} \to 0$ to be normalizable, but can be used directly to calculate the moments of $\mathcal{T}$ and, hence, of the transmitted energy $E = n_*\,\omega_0 \mathcal{T}$. The constant

$$P_0 = \frac{\nu_0 \Delta\omega D t_0}{2L} \tag{4.69}$$

can be determined by comparing the average $\langle E \rangle = n_*\,\omega_0 \langle \mathcal{T} \rangle$ with $C_1^{(0)}$. Then, for $P_0 \ll 1$, averaging with $P(\mathcal{T})$ reproduces the leading-order terms in Eqs. (4.66) and (4.67). After estimating the one-dimensional density of modes as $\nu_0 \sim 1$, we can recast the condition of small P_0 in the form

$$\Delta k l_0 \frac{l_0}{L} \sim \Delta k L \frac{t_0}{t_\mathrm{Th}} \ll 1, \tag{4.70}$$

where $\Delta k = \Delta\omega$ is the wave number pumping bandwidth, $l_0 = \sqrt{D t_0}$ is the distance that the energy diffuses in time t_0, and the Thouless time $t_\mathrm{Th} = L^2/D$ is the time that it takes to diffuse the distance L.

Despite the similarities, the fluctuations of transmitted energy of the classical wave and the fluctuations of transmitted electron charge (shot noise) are of different origin. In the latter case,

the number of transmitted electrons fluctuates for a given realization of $\mathcal{T}$, and the average noise is obtained after integrating with $P(\mathcal{T})$. In the classical case, the energy fluctuations result from the fluctuations of $\mathcal{T}$, whereas no fluctuations occur for a given disorder realization.

4.5 Conclusions

In the framework of Keldysh nonlinear σ model for classical electromagnetic waves in nonconservative disordered medium we derived an action that includes the source term for the energy current. Within the stationary-phase approximation we obtained a generating function for the cumulants of the energy transmitted through the weakly nonconservative one-dimensional disordered system. The odd-order cumulants for thermal fluctuations vanish in the absence of pumping, which is a consequence of the symmetry of the system. In the pumped system the fluctuations over the ensemble of disorder realizations can be related to the Dorokhov's distribution of transmition coefficients. Our results for a conservative medium quantitatively agree with previous diagrammatic calculations of the low-order cumulants. The absorption (amplification) causes reduction (increase) of fluctuations. The photon concentration, or energy density, in the nonconservative medium is shown to obey the diffusion equation with relaxation term; the gradient of the concentration determines the average current.

CHAPTER 5

Application of the Nonlinear σ-Model: Random Lasing

5.1 Description of laser systems

In this appendix we will present detailed derivations of equations describing the dynamics of laser systems. In general, a "laser system" consists of a multitude of "subsystems" which interact with one another, which interplay then conspires to produce highly-coherent electromagnetic radiation. These subsystems can be easily distinguished: an external pump, a "gain" medium, a confined volume from which photons might escape (the "resonator"). We will not give a full detailed description of all these components and how they work together (for good references we refer to standard references like [70, 71]), in the experimental sense; instead, we will simply summarize, in a concise manner, the corresponding facts about each subsystem which is relevant to our presentation of the theory.

5.1.1 Gain and amplification

The gain material is the most important part of a lasing system as the output from a laser depends crucially on the spectral properties of the lasing medium in the resonator. Gain media of typical laser systems are made up of atoms with very particular configurations of energy levels; this is required in order to support a state of *population inversion* in the different atomic energy levels, the maintenance of which constitutes the most important requirement of continuous laser output. Population inversion is the situation in which, effectively, more atoms have been excited to some higher energy state than which remained in a lower energy one. In principle, for this purpose we might imagine a simple two-level scheme, and then assume a large enough pumping strength such that more atoms are excited to the second (higher) energy level than which remains in the ground level.

This operating scheme is however not feasible because of *stimulated emission*. Specifically, as the higher energy level is populated more and more, it is to be expected that the intensity of radiation from atoms de-excited again into the lower energy level will increase. However, this

picture only takes into account *spontaneous emission*; it is known that via stimulated emission, there exists another channel for radiation which is actually instigated by stimulation from radiation which actively interacts with the atom; one can get a good physical picture by imagining the atom as a miniature passive resonant antenna that is set oscillating by the applied signal itself [70]. This channel of radiation will of course deplete the occupation of higher energy levels in direct proportionality to the radiation intensity, which means that one would expect that at some point the radiated intensity and the input energy would saturate and no further increase in relative difference of occupation numbers between the higher and lower energy levels is possible.

This difference in occupation between higher and lower energy states can be obtained by carefully constructing atomic energy level schemes in which lifetimes of atoms in a higher-energy state is much longer compared to the rate at which the population is depleted; this effect can be achieved in many different ways, but two of the most common ones for achieving lasing in solid-state lasers is the scheme with *three-* or *four*-level atoms. A scheme involving three atomic energy levels is schematically shown in Fig. 5.1 while a four-level system in Fig. 5.2 [71] In the

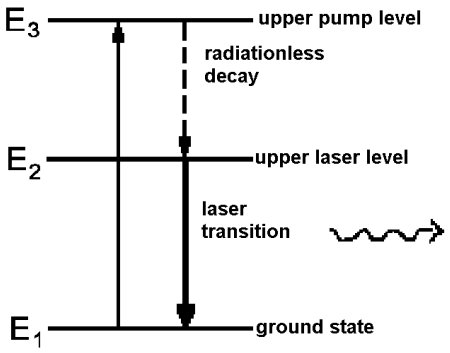


Figure 5.1: Energy level scheme for a 3-level laser. From [71].

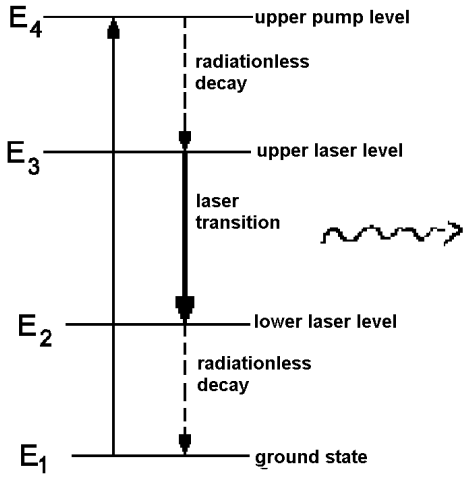


Figure 5.2: Energy level scheme for a 4-level laser. From [71]

three-level system [71], some pumping process acts between the lowest (first) and the third level; this fills up the third level. However, as mentioned before the same pumping process that fills up the third level can also depopulate it directly back to the first level via stimulated emission. However, atoms excited to the third level are also able to depopulate via other mechanisms, such as spontaneous emission or collisions with the lattice (which usually results in the generation of a phonon [70]; this second type of population decay of the higher energy level does not radiate in the electromagnetic spectrum and is termed *nonradiative relaxation*). These alternative channels for relaxation then tend to start filling up the *second* atomic level, which has the characteristic that it has a particularly long lifetime as compared to atomic lifetime in the third level. Hence an effective population inversion is achieved between the second and first levels. For this system it can be shown that in order to maintain a steady-state population inversion the pumping rate into

level three must simply be larger than the decay rate between the second and first level:

$$P > \Gamma_{12} \tag{5.1}$$

The four-level laser system works in a similar manner, but in this case the pumping induces population of the third atomic level, from which the atoms then undergo fast decay to the second level. The second level has a relatively long lifetime and hence provides the population difference between the first and second levels for a steady-state population inversion. The fact that the decay from the first to ground (0) level is also fast contributes to the magnitude of the inversion.

5.1.2 Pumping of medium atoms

The operation of a laser requires first and foremost an external energy source, or the external "pump". This represents the energy input to the system which is required to energetically excite atoms of the gain material; these excited atoms are then responsible for the light output from the cavity when they are eventually de-excited leading to radiation in the form of both *spontaneous* and *stimulated* emission. A term corresponding to the pumping appears in the laser equations, which strength is a function of various physical parameters particular to the system under consideration.

Another notable characteristic which is important and is currently a main topic of research into random lasing is possible spatial inhomoegeneity of the pumping which is represented, in our formalism, by a strong spatial dependence of the population inversion parameter $\Delta n_\omega(\mathbf{r})$. The question of spatially dependent pumping [72] has arisen as an active field of research in recent years (for a comprehensive review see [73]), and is also experimentally important due to the impossibility of achieving full homogeneity in the pump profile in experimental situations.

It has been shown in numerical simulations [72] that relatively localized *pumping spots* yields an output intensity profile that is also sharply peaked. We show in this chapter that we are able to capture the physics of spatially dependent pumping via a position dependent population inversion parameter. In addition, by exploiting known solutions of the equation that we obtain from our theory, we expect that if we allow for such a possibility we will be able to mimic the experimental observations. This is still a work in progress.

5.1.3 Randomness of medium

The main feature of note, from which arises most of the interesting properties of random lasers, is the medium. In a random laser the medium is spatially disordered, i.e., light waves propagate through a volume which contains a randomly-distributed ensemble of scattering centers (for light waves these correspond to a spatially random distributed of dielectric coefficients). Hence as light propagates through this volume its direction of propagation is multiply changed in a random manner, and hence we say that light waves in a random medium is *multiply scattered*.

A direct result of multiple scattering is that is translates to motion which is *diffusive* in nature. It is exactly this diffusion of light in random lasers, coupled with external amplification, which provides the mechanism of lasing. Letokhov [7] showed already in the sixties that in a diffusive

medium with amplification the total gain is proportional to the volume (since it is of order $O(L^3)$) whereas the loss is proportional to the area ($O(L^2)$). Hence large order increases in the light intensity characteristic of laser gain can be achieved in such media.

Despite of the obvious apparent differences between the gain mechanisms governing a regular and a random laser it has been found that generally there exists a significant number of similarities between regular and random lasers, especially with respect to properties like photon statistics and noise emission. In the latter, it is known that regular lasers exhibit excess noise originating from interference between spontaneous and stimulated emission ("Petermann factor")][70]; it was shown by Beenakker and coworker [18] via a random matrix calculation that the same noise can also be expected for a random laser. Another interesting calculation involved the second-order coherence in [74] ; there it was shown that, as compared to other chaotic sources of light, which commonly display Bose-Einstein statistics (strictly speaking, they display "bunching" behavior), the output from a random laser display Poissonnian statstics, just like in the case of a regular laser, despite a random laser also being a "chaotic" light source.

Of course, due to the random nature of the medium in which the generated light intensity propagates, there are also some differences in other aspects of the laser emission as compared to those from a regular laser. The main neglected aspect in all the calculations mentioned above is that of *coherence effects*, i.e., coherent interactions between modes of the random laser. In a regular laser, the mode structure consists of standing-wave patterns consistent with a regular cavity; in a random laser, the spatial profile of the modes is dominated by a *speckle pattern*, which manifests as a granular distribution of intensity. A related phenomena is of course the possibility of *Anderson localization*, in which coherent interference of light results in the formation of randomly shaped but closed modes with exponentially decaying amplitudes. However, localization is only expected to occur in strongly scattering medium, and no conclusive evidence for localization of light has been found thus far.

5.1.4 Semiclassical Theory of Random Lasing

In this section, for completeness, we will derive in a concise manner the form of the wave equation which we need for a description of the laser. We start from the set of Maxwell's equations

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{5.2}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \dot{\mathbf{D}} \tag{5.3}$$

where the usual symbols have been applied: $\mathbf{E}$: electric field strength, $\mathbf{B}$: magnetic induction, $\mathbf{H}$ magnetic field strength, $\mathbf{j}$: external current and finally $\mathbf{D}$: displacement vector. the derivation of the lasing action starting from the Maxwell's equations. The expressions displayed in this section are given in [33], section 3.1. We will (mostly) work in the ω representation. We start from the equation of motion of the electric field

$$-\epsilon(\mathbf{r},\omega)\omega^2 E_\omega(\mathbf{r}) - \nabla^2 E_\omega(\mathbf{r}) = 4\pi\omega^2 P_\omega, \tag{5.4}$$

where $E_\omega(\mathbf{r})$ is the electric field in medium, $P_\omega(\mathbf{r})$ the polarization function which acts as a source which generates the field. (5.4) is in turn coupled to the matter fields via further expressions governing the polarization $P_\omega(\mathbf{r})$ and population inversion of the laser system (here it is assumed that we have the simplest model of a 2-level atomic system):

$$P_\omega(\mathbf{r}) = i\frac{d^2}{\hbar\gamma_\perp}D(\omega)\frac{1}{2\pi}\int_0^\infty d\omega'\ E_\omega(\mathbf{r})\Delta n_{\omega-\omega'}, \tag{5.5}$$

$$\Delta n_\omega(\mathbf{r}) = 2\pi\Delta n_0(\mathbf{r})\delta(\omega) - i\frac{D_\parallel(\omega)}{\hbar\gamma_\parallel}\int_0^\infty d\omega'\left(E^*_{\omega'-\omega}P_{\omega'} - E_{\omega'+\omega}P^*_{\omega'}\right), \tag{5.6}$$

In the matter equations (5.5) and (5.6) one finds the parameters that define the (effective) lasing system. Here $\Delta n_\omega(\mathbf{r})$ is the population inversion, $\Delta n_0(\mathbf{r})$ the unsaturated population inversion, which is directly proportional to the pump strength, $\epsilon(\mathbf{r},\omega)$ is the dielectric constant (includes disorder and absorption), and d is the magnitude of the atomic dipole matrix element. $D(\omega)$ and $D_\parallel(\omega)$ are combinations of the following matter parameters:

$$D(\omega) \equiv \frac{1}{1 - i\frac{\omega-\nu}{\gamma_\perp}}, \tag{5.7}$$

$$D_\parallel(\omega) \equiv \frac{1}{1 - i\frac{\omega}{\gamma_\parallel}}, \tag{5.8}$$

We give a short description of the different matter parameters which appear in (5.7) and (5.8).

- ν is the atomic transition frequency; this is the resonant frequency of the 2-level atomic system (here homogeneous broadening is assumed; "homogeneous broadening" is where the level broadening due to atomic lifetime of the different levels behave the same way for all atoms in the system [75]. Hence the atomic transition frequency defines a lifetime of the higher energy level of the form $\tau = 1/\nu$). In the following it is assumed that the time dependence of the field and polarization is determined by fast oscillations which are close to the atomic frequency ν and residual slow time dependence; for practical purposes this means that only the Fourier components $E_\omega(\mathbf{r})$ and $P_\omega(\mathbf{r})$ in the close vicinity of ν contribute significantly to the dynamics [33].

- $\gamma_\perp$ and $\gamma_\parallel$ are the polarization and population inversion relaxation rates, respectively. Specifically, the population decay rate $\gamma_\parallel$ is the rate of relaxation of the population inversion (see eq. (34) of [33] and the text below it), or in other words the relaxation of the quantity $\rho_{bb} - \rho_{aa}$, where ρ_{ab} are the density matrix elements corresponding to the two-level atom with levels a and b, a being the lower and b the upper levels in our case ; in this picture it is easy to see that ρ_{aa} (ρ_{bb}) and ρ_{ab} are the probability of the atom being in the level $a(b)$ and the complex dipole moment, respectively. On this note we say that the polarization relaxation rate $\gamma_\perp$ is defined as the decay rate of the quantity ρ_{ab}.

Given the form of the wave and matter equations (5.4), (5.5) and (5.6) we can see that a cubic-in-$E_\omega(\mathbf{r})$ term can be generated by suitably cutting off the self-consistency in (5.5) and (5.6) and

substituting the resulting expresion into (5.6). This procedure is outlined in the following. We first take only the first term on the r.h.s of (5.6) as the simplest approximation to $\Delta n_\omega(\mathbf{r})$; this corresponds to an initial, constant population inversion as caused by the pump. This is substituted into (5.5) to give the 1st order (in $E_\omega(\mathbf{r})$) approximation to $P_\omega(\mathbf{r})$:

$$P_\omega^{(1)}(\mathbf{r}) = -i\frac{d^2}{\hbar\gamma_\perp}D(\omega)\Delta n_0 E_\omega(\mathbf{r}). \tag{5.9}$$

Substitution of (5.9) into the 2nd term on the r.h.s of (5.5) yields the 2nd order in $E_\omega(\mathbf{r})$ term for the population inversion:

$$\Delta n_\omega^{(2)}(\mathbf{r}) = -\frac{d^2\Delta n_0}{\hbar^2\gamma_\perp\gamma_\parallel}D_\parallel(\omega)\frac{1}{\pi}\int_0^\infty d\omega' \ \left[E^*_{\omega'-\omega}E_{\omega'}D(\omega') + E_{\omega'+\omega}E^*_{\omega'}D^*(\omega')\right]. \tag{5.10}$$

Finally, substitution of (5.10) back into r.h.s of (5.6) yields the required cubic order in $E_\omega(\mathbf{r})$ term for the polarization

$$P_\omega^{(3)}(\mathbf{r}) = i\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\frac{\Delta n_0}{\hbar\gamma_\parallel}D(\omega)\frac{1}{2\pi^2}\int_0^\infty d\omega'\int_0^\infty d\omega'' \ E_{\omega'}D_\parallel(\omega-\omega') \\ \times\left[D(\omega'')E^*_{\omega''+\omega'-\omega}E_{\omega''} + D^*(\omega'')E_{\omega''-\omega'+\omega}E^*_{\omega''}\right] \tag{5.11}$$

and the substitution of (5.11) into the r.h.s. of (5.4) gives finally the desired equation for the dynamics to 3rd order in $E_\omega(\mathbf{r})$

$$-\epsilon(\mathbf{r})\omega^2E_\omega(\mathbf{r}) - \nabla^2E_\omega(\mathbf{r}) = -i4\pi\frac{d^2}{\hbar\gamma_\perp}\omega^2D(\omega)\frac{1}{2\pi^2}\int_0^\infty d\omega'\int_0^\infty d\omega'' \ E_{\omega'}D_\parallel(\omega-\omega') \\ \times\left[D(\omega'')E^*_{\omega''+\omega'-\omega}E_{\omega''} + D^*(\omega'')E_{\omega''-\omega'+\omega}E^*_{\omega''}\right], \tag{5.12}$$

In order to obtain a clean expression for the nonlinearity, an additional approximation has been made. This is the *constant inversion approximation* ([33], sec. VI.C). This involves the assumption of small population pulsations, or $|D_\parallel(\omega-\omega')| << 1$. Since the difference $\omega-\omega'$ is typically of order $\gamma_\perp$, the constant inversion approximation implies that $\gamma_\perp >> \gamma_\parallel$. In practical terms this means for the function $D_\parallel(\omega)$ the following simplification:

$$D_\parallel(\omega) \approx \gamma_\parallel\left[\pi\delta(\omega) + i\mathcal{P}\frac{1}{\omega}\right], \tag{5.13}$$

can be made, and we neglect the principal value part of the expression (5.13). This simplifies (5.12) such that we can write it in the following manner:

$$-\epsilon(\mathbf{r})\omega^2 E_\omega(\mathbf{r}) - \nabla^2 E_\omega(\mathbf{r}) = -i4\pi\frac{d^2}{\hbar\gamma_\perp}\Delta n_0(\mathbf{r})\omega^2\mathcal{D}(\omega)E_\omega(\mathbf{r}) + \tag{5.14}$$

$$+ i4\pi\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\frac{\Delta n_0(\mathbf{r})}{\hbar}\omega^2\mathcal{D}(\omega)E_\omega(\mathbf{r})\frac{1}{\pi}\int_0^\infty d\omega'\ \mathcal{L}(\omega')|E_{\omega'}|^2, \tag{5.15}$$

where

$$\mathcal{L}(\omega) \equiv \mathrm{Re}(\mathcal{D}(\omega)) = \frac{1}{1+\frac{(\omega-\nu)^2}{\gamma_\perp^2}} \tag{5.16}$$

In order to simplify further calculations we define the quantities

$$\widetilde{E}_\omega(\mathbf{r}) \equiv \mathcal{D}(\omega)E_\omega(\mathbf{r}) \tag{5.17}$$

$$\widetilde{E}^*_\omega(\mathbf{r}) \equiv \mathcal{D}^*(\omega)E^*_\omega(\mathbf{r}) \tag{5.18}$$

which enables us to write (5.15) in the slightly different form

$$-\epsilon(\mathbf{r})\omega^2 E_\omega(\mathbf{r}) - \nabla^2 E_\omega(\mathbf{r}) = -i4\pi\frac{d^2}{\hbar\gamma_\perp}\Delta n_0(\mathbf{r})\omega^2\widetilde{E}_\omega(\mathbf{r}) + i4\pi\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\frac{\Delta n_0(\mathbf{r})}{\hbar}\omega^2\widetilde{E}_\omega(\mathbf{r})\frac{1}{\pi}\int_0^\infty d\omega'\ |\widetilde{E}_{\omega'}|^2, \tag{5.19}$$

Further defining the "generalized" dielectric constant:

$$\widetilde{\epsilon}(\mathbf{r},\omega) \equiv \epsilon(\mathbf{r}) - i4\pi\frac{d^2}{\hbar\gamma_\perp}\Delta n_0(\mathbf{r})\mathcal{D}(\omega), \tag{5.20}$$

with which we can rewrite (5.14)

$$-\widetilde{\epsilon}(\mathbf{r},\omega)\omega^2 E_\omega(\mathbf{r}) - \nabla^2 E_\omega(\mathbf{r}) = i\frac{4}{\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\Delta n_0(\mathbf{r})\widetilde{E}_\omega(\mathbf{r})\int_0^\infty d\omega'|\widetilde{E}'_\omega(\mathbf{r})|^2 \tag{5.21}$$

We note that in the Coulomb gauge the electric field is related to the vector potential as $E_\omega(\mathbf{r}) = i\omega A_\omega(\mathbf{r})$, which translates similarly to the tilded quantities: $\widetilde{E}_\omega(\mathbf{r}) = i\omega\widetilde{A}_\omega(\mathbf{r})$, where similarly

$$\widetilde{A}_\omega(\mathbf{r}) \equiv \mathcal{D}(\omega)A_\omega(\mathbf{r}) \tag{5.22}$$

$$\widetilde{A}^*_\omega(\mathbf{r}) \equiv \mathcal{D}^*(\omega)A_\omega(\mathbf{r}). \tag{5.23}$$

From the considerations above we can then write down the action of the fields $\widetilde{E}_\omega(\mathbf{r})$

$$S = \frac{1}{8\pi}\int d\mathbf{r}\int_0^\infty \frac{d\omega}{\pi}\Big\{\widetilde{E}^*_\omega(\mathbf{r})\Big[\widetilde{\epsilon}(\mathbf{r},\omega)\omega^2+\nabla^2\Big]E_\omega(\mathbf{r})\Big\}+ \\ + i\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\int d\mathbf{r}\Delta n_0(\mathbf{r})\int_0^\infty\frac{d\omega}{\pi}\int_0^\infty\frac{d\omega'}{\pi}|\widetilde{E}_\omega(\mathbf{r})|^2|\widetilde{E}_{\omega'}(\mathbf{r})|^2 \quad (5.24)$$

or in terms of the "modified" vector potential $\widetilde{A}_\omega(\mathbf{r})$

$$S = \frac{1}{8\pi}\int d\mathbf{r}\int_0^\infty \frac{d\omega}{\pi}\Big\{\widetilde{A}^*_\omega(\mathbf{r})\Big[\widetilde{\epsilon}(\mathbf{r},\omega)\omega^2+\nabla^2\Big]A_\omega(\mathbf{r})\Big\}+ \\ + i\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\int d\mathbf{r}\Delta n_0(\mathbf{r})\int_0^\infty\frac{d\omega}{\pi}\int_0^\infty\frac{d\omega'}{\pi}\,\omega^2|\widetilde{A}_\omega(\mathbf{r})|^2\omega'^2|\widetilde{A}_{\omega'}(\mathbf{r})|^2 \quad (5.25)$$

The action (5.25) will be the starting point for the derivation of the nonlinear sigma model. We recall that the disorder is contained in the random dielectric constant. Physically, (5.25) describes the behavior of a laser in a disordered medium, operating in the multimode regime. Pumping of the system is given by the population inversion $\Delta n_0(\mathbf{r})$, and multimodality is seen from the integral over ω on the right hand side of (5.25). There the quartic term gives the saturation which ensures that energy in the system saturates at some point and the laser operates in a steady-state regime.

5.2 Hubbard Stratonovich transformation

From the previous section we have seen that it is possible to obtain a quartic term in the equation of motion describing the electric field $\mathbf{E}(\mathbf{r}, t)$. We first note that we will again work in the Coulomb gauge, i.e., we will write the electric fields in terms of the vector potential fields $A_\omega(\mathbf{r})$, where $E_\omega(\mathbf{r}) = i\omega A_\omega(\mathbf{r})$. Writing the quartic action in terms of $A_\omega(\mathbf{r})$ then yields

$$S = \frac{1}{8\pi}\int d\mathbf{r}\int_0^\infty \frac{d\omega}{\pi}\Big\{\widetilde{A}^*_\omega(\mathbf{r})\Big[\widetilde{\epsilon}(\mathbf{r},\omega)\omega^2 + \nabla^2\Big]A_\omega(\mathbf{r})\Big\} + \\ + i\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\int d\mathbf{r}\,\Delta n_0(\mathbf{r})\int_0^\infty\frac{d\omega}{\pi}\int_0^\infty\frac{d\omega'}{\pi}\;\omega^2\,\omega'^2\,|\widetilde{A}_\omega(\mathbf{r})|^2|\widetilde{A}_{\omega'}(\mathbf{r})|^2. \quad (5.26)$$

On the right-hand side, there appears the product $\mathcal{L}(\omega)\mathcal{L}(\omega)$, where $\mathcal{L}(\omega)$ is the real part of the atomic propagator.

$$\mathcal{L}(\omega) \equiv \mathrm{Re}(\mathcal{D}(\omega)) = \frac{1}{1+\frac{(\omega-\nu)^2}{\gamma_\perp^2}} \quad (5.27)$$

We see that (5.26) contains a term which is of quartic order in the modified vector potential $\widetilde{A}_\omega(\mathbf{r})$, in the form

$$S_{int} = i\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2\int d\mathbf{r}\,\Delta n_0(\mathbf{r})\int_0^\infty\frac{d\omega}{\pi}\int_0^\infty\frac{d\omega'}{\pi}\;\omega^2\,\omega'^2\,|\widetilde{A}_\omega(\mathbf{r})|^2|\widetilde{A}_{\omega'}(\mathbf{r})|^2 \quad (5.28)$$

We can deal with this term by performing a Hubbard-Stratonovich (HS) transformation; the details are given in Appendix D. As stated there, we assume homogeneous external pumping in this work. This procedure then yields the quadratic action (we set $\hbar = 1$)

$$S_{int} = -\frac{d^2\sqrt{\Delta n_0}}{2\gamma_\perp}\int d\mathbf{r}\int\frac{d\omega d\omega'}{\pi^2}\widetilde{A}_\omega(\mathbf{r})\omega\Phi_{\omega\omega'}(\mathbf{r})\omega'\widetilde{A}^*_{\omega'}(\mathbf{r}) \quad (5.29)$$

$$\equiv -\widetilde{\alpha}\int d\mathbf{r}\int\frac{d\omega d\omega'}{\pi^2}\mathcal{D}(\omega)A_\omega(\mathbf{r})\,\omega\Phi_{\omega\omega'}(\mathbf{r})\omega'\widetilde{A}^*_{\omega'}(\mathbf{r}) \quad (5.30)$$

The Hubbard-Stratonovich transformed action will have the form

$$S = \frac{1}{8\pi}\int d\mathbf{r}\int\frac{d\omega}{\pi}\frac{d\omega}{\pi}\,\widetilde{A}^*_\omega(\mathbf{r})\Big(g_{\omega\omega'}(\mathbf{r}) - \alpha(\omega)\Phi_{\omega\omega'}(\mathbf{r})\Big)A_{\omega'}(\mathbf{r}) \quad (5.31)$$

where we have defined

$$g_{\omega\omega'}(\mathbf{r}) \equiv \delta(\omega-\omega')\Big[\widetilde{\epsilon}(\mathbf{r},\omega)\,\omega^2 + \nabla^2\Big] \quad (5.32)$$

and

$$\alpha(\omega) \equiv \mathcal{D}(\omega)\widetilde{\alpha} \quad (5.33)$$

$$= \frac{d^2\nu^2}{\gamma_\perp}\sqrt{\Delta n_0}\,\mathcal{D}(\omega) \quad (5.34)$$

The definition of (5.32) follows that as given in the second chapter. We have also approximated ω, ω' by the value of the atomic transition frequency ν. In the case of inhomegeneous population inversion, we can show (in Appendix D) that in this case the quadratic action is given by

$$S_{int} = -\frac{d^2}{2\sqrt{\mathcal{P}}\gamma_\perp}\int d\mathbf{r}\int\frac{d\omega d\omega'}{\pi^2}\,\Delta n_0(\mathbf{r})\widetilde{A}_\omega(\mathbf{r})\,\omega\Phi_{\omega\omega'}(\mathbf{r})\omega'\widetilde{A}^*_{\omega'}(\mathbf{r}) \tag{5.35}$$

$$\equiv -\int d\mathbf{r}\int\frac{d\omega d\omega'}{\pi^2}A_\omega(\mathbf{r})\,\alpha_{\rm ih}(\mathbf{r},\omega)\widetilde{A}^*_{\omega'}(\mathbf{r}) \tag{5.36}$$

where $\mathcal{P}$ is a constant normalization factor proportional to the integral of $\Delta n_0(\mathbf{r})$ over all space, and

$$\alpha_{\rm ih}(\mathbf{r},\omega) \equiv \frac{d^2\nu^2}{2\sqrt{\mathcal{P}}\gamma_\perp}\mathcal{D}(\omega)\Delta n_0(\mathbf{r}) \tag{5.37}$$

and we can write our action in the form

$$S = \frac{1}{8\pi}\int d\mathbf{r}\int\frac{d\omega}{\pi}\int\frac{d\omega'}{\pi}\widetilde{A}^*_\omega(\mathbf{r})\Big(g_{\omega\omega'}(\mathbf{r}) - \alpha(\mathbf{r},\omega)\Phi_{\omega\omega'}(\mathbf{r})\Big)A_{\omega'}(\mathbf{r}) \tag{5.38}$$

5.3 Disordered action

In the previous section we have derived, via the use of a Hubbard-Stratonovich transformation, the disordered action including nonlinearity. For simplicity we will consider the case of homogeneous pumping (population inversion), which is then given by the action

$$iS[\hat{Q}] \equiv -\frac{\pi\nu\widetilde{\hbar}}{4\tau}\mathrm{Tr}\,[\hat{Q}^2] - \mathrm{Tr}\,[\hat{\Phi}^\dagger\hat{\gamma}\hat{\Phi}] + \mathrm{Tr}\,\ln\Big[\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q} - \alpha\hat{\Phi}\Big], \tag{5.39}$$

where

$$\hat{\Phi}_{\omega\omega'}(\mathbf{r}) = \begin{pmatrix} \phi^q_{\omega\omega'}(\mathbf{r}) & \phi^{cl}_{\omega\omega'}(\mathbf{r}) \\ \phi^{cl}_{\omega\omega'}(\mathbf{r}) & \phi^q_{\omega\omega'}(\mathbf{r}) \end{pmatrix} \tag{5.40}$$

is the explict form of the Hubbard-Stratonovich field used in the decoupling of the interaction term. The derivation of (5.39) follows closely to that of the linear action derived in Chapter 2; the additional term $\alpha\hat{\Phi}$ is derived in detail in Appendix D. Using (5.39) we can derive the diffusive form of the sigma model. The derivation of this action is very similar to that of the linear case, which has been treated in Chapter 2. The derivation of the diffusive σ-model including the HS field is done in Appendix D and we obtain, finally

$$iS[\hat{Q},\hat{\Phi}] = \frac{i\nu}{2}\mathrm{Tr}\,[\hat{\Phi}\hat{\gamma}\hat{\Phi}] - \frac{\pi\nu}{4}\mathrm{Tr}\,\Big\{D(\nabla_{\mathbf{r}}\hat{Q})^2 - 4\partial_t\hat{Q} + \beta\hat{Q}\hat{\Lambda}_0 - 4\alpha\hat{\Phi}\hat{Q}\Big\} \tag{5.41}$$

where

$$\hat{\Phi}_{\omega\omega'}(\mathbf{r}) = \begin{pmatrix} \phi^{cl}_{\omega\omega'}(\mathbf{r}) & \phi^q_{\omega\omega'}(\mathbf{r}) \\ \phi^q_{\omega\omega'}(\mathbf{r}) & \phi^{cl}_{\omega\omega'}(\mathbf{r}) \end{pmatrix} \tag{5.42}$$

where we note the "correct" structure of $\hat{\Phi}$. β is the strength of the pumping, given by the combination of physical coefficients defined in (5.20). We note that for the derivation of (5.41) we have discarded its dependence on both ω and $\mathbf{r}$; hereafter the pump strength will be a constant parameter describing the strength of the external driving of the laser. Similar simplifications have been performed on the saturation parameter α.

(5.41) is the diffusive nonlinear sigma model of a random laser. This expression is now the starting point of a multitude of possible computations; in this work we will only touch upon one (calculation of the nonequlibrium distribution function). The task which then follows is then to find the effects of fluctuations away from the saddlepoint (4.23). This requires the choosing of a suitable parameterization of the $\hat{Q}$ matrix according to the physics of the system. In general, since we have multiple elastic scattering in our disordered medium, we mainly want to look at the diffusive modes of the system. These modes, called *diffusons*, can be studied by the following parameterization of the $\hat{Q}$ matrix

$$\hat{Q} = \hat{\mathcal{U}} \circ e^{\hat{\mathcal{W}}/2} \circ \hat{\sigma}_3 \circ e^{-\hat{\mathcal{W}}/2} \circ \hat{\mathcal{U}}^{-1} \tag{5.43}$$

where

$$\hat{\mathcal{W}} = \begin{pmatrix} 0 & w \\ w^\dagger & 0 \end{pmatrix}, \qquad \hat{\mathcal{U}} = \hat{\mathcal{U}}^{-1} = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}, \tag{5.44}$$

and w, $w^\dagger$ represent diffusion modes of the system.

5.4 Stationary saddlepoint in presence of nonlinearity

Before we consider the diffusive fluctuations in presence of nonlinearity we need to determine the how the stationary saddlepoint (4.23) is modified by nonlinearity. In this section we will perform sketches of computations to illustrate the procedure; the explict calculations are to be found in Appendix D. We start with the form of the action including the Hubbard-Stratonovich field (5.41). Varying with respect to the classical and quantum components of the HS field $\phi^c l$ and ϕ^q we obtain the following set of equations[1]

$$i\frac{\delta S[\hat{Q}, \phi^{cl}, \phi^q]}{\delta \phi^{cl}} \stackrel{!}{=} 0 \Rightarrow -2\underline{\phi}^q - \alpha\left(\mathbf{1} - \alpha\hat{G}_{cl}\hat{\gamma}\underline{\phi}^q\right)^{-1} \circ \left(\frac{\delta}{\delta\phi^{cl}}\hat{G}_{cl}\hat{\gamma}\right) \circ \underline{\phi}^q = 0 \tag{5.45}$$

$$i\frac{\delta S[\hat{Q}, \phi^{cl}, \phi^q]}{\delta \phi^{q}} \stackrel{!}{=} 0 \Rightarrow -2\underline{\phi}^{cl} - \alpha\left(\mathbf{1} - \alpha\hat{G}_{cl}\hat{\gamma}\underline{\phi}^q\right)^{-1} \circ \hat{G}_{cl}\hat{\gamma} = 0 \tag{5.46}$$

where the notation $\hat{G}_{cl}\hat{\gamma} = \begin{pmatrix} \left\{[G_0^{-1}]^R + \frac{\tilde{\hbar}}{2\tau}\underline{\underline{Q}}^R - \phi^{cl}\right\}^{-1} & G_0^K \\ 0 & \left\{[G_0^{-1}]^A + \frac{\tilde{\hbar}}{2\tau}\underline{\underline{Q}}^A - \phi^{cl}\right\}^{-1} \end{pmatrix} \equiv \begin{pmatrix} \underline{\underline{G}}^R_{cl} & \underline{\underline{G}}^K_{cl} \\ 0 & \underline{\underline{G}}^A_{cl} \end{pmatrix}$

and we have used that the partition function in the absence of the quantum component of the HS field is unity $Z[\phi^{cl}, \phi^q = 0] = 1$. We note that the Keldysh component of the Green's function

[1] The notation $A \circ B$ denotes convolution over time and space coordinates of fields A and B.

$\underline{\underline{G}}^K_{cl}$ can be parameterized in the usual way as $\underline{\underline{G}}^K_{cl} = \underline{\underline{G}}^R_{cl} \circ F - F \circ \underline{\underline{G}}^A_{cl}$, and in the case with nonlinearity the Green's functions $\underline{\underline{G}}^{(R,A)}_{cl}$ are functionals of the classical component of the HS field, $\underline{\underline{G}}^{(R,A)}_{cl} \equiv \underline{\underline{G}}^{(R,A)}_{cl}[\underline{\phi}^{cl}]$

Evaluating the second term of (5.45) explicitly we have

$$\begin{aligned} -2\underline{\phi}^q - \left[\frac{1}{\det(\dots)} \circ \begin{pmatrix} 1 & \underline{\phi}^q \underline{\underline{G}}^R_{cl} \\ \underline{\phi}^q \underline{\underline{G}}^A_{cl} & 1 - \underline{\phi}^q \underline{\underline{G}}^K_{cl} \end{pmatrix}\right] \circ \begin{pmatrix} 0 & \underline{\phi}^q \\ \underline{\phi}^q & 0 \end{pmatrix} &\stackrel{!}{=} 0 \\ -\left[\frac{1}{\det(\dots)} \circ \begin{pmatrix} \underline{\phi}^q \underline{\underline{G}}^R_{cl} & \odot \\ \otimes & \underline{\phi}^q \underline{\underline{G}}^A_{cl} \end{pmatrix}\right] \circ \underline{\phi}^q &\stackrel{!}{=} 2\underline{\phi}^q \\ -\left[\frac{1}{\det(\dots)} \circ \underline{\phi}^q \circ \left(\underline{\underline{G}}^R_{cl} + \underline{\underline{G}}^A_{cl}\right)\right] \circ \underline{\phi}^q &\stackrel{!}{=} 2\underline{\phi}^q \end{aligned} \tag{5.47}$$

The structure of (5.47) is important since, written out explicitly, the l.h.s. has the form

$$\begin{aligned} &\left[\frac{1}{\det(\dots)} \circ \underline{\phi}^q \circ \left(\underline{\underline{G}}^R_{cl} + \underline{\underline{G}}^A_{cl}\right)\right] \circ \underline{\phi}^q = \\ &\quad = \int d\mathbf{r} \int dt \left[\frac{1}{[\det(\dots)]_{tt}} \underline{\phi}^q(\mathbf{r},t) \left(\underline{\underline{G}}^R_{cl}(\mathbf{r},\mathbf{r};t,t) + \underline{\underline{G}}_{cl}(\mathbf{r},\mathbf{r};t,t)\right) \underline{\phi}^q(\mathbf{r},t)\right] \end{aligned} \tag{5.48}$$

where the equality in the time dependencies are due to the requirements of the trace (integration over temporal and spatial variables). Specifically, the determinant, being a operation in the space of the 2×2 Keldysh space will retain the same time dependence as the original matrix, while all the other quantities in the square brackets will have their time dependencies imposed by the trace. Since it is a known fact that due to the time causal characteristics of the retarded and advanced Green's functions, we have the relation $G^R(t,t) + G^A(t,t) = 0$, the $\phi^q(\mathbf{r},t)$ at saddlepoint vanishes.

The classical component of the HS field ϕ^{cl} can be determined similarly. We look at (5.46) and writing out everything explicitly we obtain

$$\begin{aligned} 2\underline{\phi}^{cl} &= -\alpha \left[\frac{1}{\det(\dots)} \circ \begin{pmatrix} 1 & \underline{\phi}^q \underline{\underline{G}}^R_{cl} \\ \underline{\phi}^q \underline{\underline{G}}^A_{cl} & 1 - \underline{\phi}^q \underline{\underline{G}}^K_{cl} \underline{\phi}^q \end{pmatrix}\right] \circ \begin{pmatrix} \underline{\underline{G}}^R_{cl} & \underline{\underline{G}}^K_{cl} \\ 0 & \underline{\underline{G}}^A_{cl} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\alpha \left[\frac{1}{\det(\dots)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \circ \begin{pmatrix} \underline{\underline{G}}^K_{cl} & \underline{\underline{G}}^R_{cl} \\ \underline{\underline{G}}^A_{cl} & 0 \end{pmatrix} \\ &= -\alpha\, \underline{\underline{G}}^K_{cl}(\mathbf{r},\mathbf{r};t,t) \Rightarrow \underline{\phi}^{cl}(\mathbf{r};t) = -\frac{\alpha}{2} \underline{\underline{G}}^K_{cl}(\mathbf{r},\mathbf{r};t,t) \end{aligned} \quad , \tag{5.49}$$

Hence we obtain the value of the classical component of the HS field. We note that due to the fact that the Green's functions $\underline{\underline{G}}^R_{cl}[\phi^q]$ and $\underline{\underline{G}}^A_{cl}[\phi^q]$ are functionals of the saddlepoint value of the HS field, (5.49) is actually a self-consistent expression for $\underline{\phi}^{cl}$. Written out in full, (5.49) is given by

$$\boxed{\underline{\phi}^{cl}(\mathbf{r},t) = -\frac{\alpha}{2} \int dt' \left[G^R_0(t,t';\mathbf{r},\mathbf{r};[\underline{\phi}^{cl}]) F(t',t;\mathbf{r},\mathbf{r}) - F(t,t';\mathbf{r},\mathbf{r}) G^A_0(t',t;\mathbf{r},\mathbf{r};[\underline{\phi}^{cl}])\right]} \tag{5.50}$$

(5.50) can then be Wigner transformed[2] into the usual form

$$\underline{\phi}^{cl}(\mathbf{R},T) = -\frac{\alpha}{2}\sum_{\mathbf{k}}\int d\epsilon\left[G_e^R(\mathbf{R},T;\mathbf{k},\epsilon) - G_e^A(\mathbf{R},T;\mathbf{k},\epsilon)\right]F(\mathbf{R},T;\epsilon) \tag{5.52}$$

where the "effective" Green's functions are defined as

$$G_e^{R,A}(\mathbf{R},T;\mathbf{k},\epsilon) \equiv \frac{1}{\epsilon - \varepsilon(\mathbf{k}) \pm \frac{i}{2\tau} - \underline{\phi}^{cl}(\mathbf{R},T)}. \tag{5.53}$$

To be able to determine $\underline{\phi}^{cl}$, we need a second equation that relates $F(\mathbf{r},\mathbf{r}';t,t')$ and $\underline{\phi}^{cl}$; this can be obtained by choosing a suitable parameterization of the diffusive fluctuations around saddlepoint and varying; this expression is called the *Usadel* equation.

5.5 Diffusive motion

In order to obtain the diffusive motion of light in disordered media we expand the action (5.41) up to 2nd order in the diffusive modes. To facilitate this expansion we proceed as follows: the matrix $\hat{Q}$, in the parameterization (5.43), can be expanded in powers of the modes d, $\bar{d}$ up to quadratic order in these modes to obtain:

$$\hat{Q} \approx i\hat{\Lambda} + \delta\hat{Q}^{(1)} + \delta\hat{Q}^{(2)} \tag{5.54}$$

where the superscripts denote the powers of the d, $\bar{d}$ modes in each term. This expansion in the diffusive modes has already been discussed in an earlier chapter. The expressions $\delta\hat{Q}^{(1)}$ and $\delta\hat{Q}^{(2)}$ are as follows:

$$\delta\hat{Q}^{(1)} = i\hat{\mathcal{U}}\hat{\sigma}_z\hat{\mathcal{W}}\hat{\mathcal{U}}^{-1} = i\begin{pmatrix} -F\circ w^\dagger & -w - F\circ w^\dagger\circ F \\ \bar{d} & w^\dagger\circ F \end{pmatrix} \tag{5.55}$$

and

$$\delta\hat{Q}^{(2)} = \frac{i}{2}\hat{\mathcal{U}}\hat{\mathcal{W}}^2\hat{\sigma}_z\hat{\mathcal{U}}^{-1} = \frac{i}{2}\begin{pmatrix} w\circ w & w\circ w^\dagger\circ F + F\circ w^\dagger\circ w \\ 0 & -w^\dagger\circ w \end{pmatrix} \tag{5.56}$$

Keeping both orders we perform a systematic expansion. The explicit calculations are shown in Appendix D and we will only show the final result in this section. The total action to 2nd order in

[2] A Wigner transform is defined as the following transformation of a field depending on one spatial variable $\mathbf{r}$ and two time variables t, t':

$$F_{tt'}(\mathbf{r}) = \int \frac{d\epsilon}{2\pi} F_\varepsilon(\mathbf{r},T)e^{-i\epsilon(t-t')} \tag{5.51}$$

where T (and also $\mathbf{R}$, in our case) are "slowly-varying" quantities and hence represent our actual physical variables.

the diffusons is given by

$$iS[w,w^\dagger,\hat{\Phi}] = \frac{\pi\nu}{4}\mathrm{tr}\Big\{D\left[2w^\dagger\nabla_{\mathbf{r}}^2 w + w^\dagger(\nabla_{\mathbf{r}}F)w^\dagger(\nabla_{\mathbf{r}}F)\right] + 8\left[(\partial_t w)w^\dagger\right] - i\left[ww^\dagger F + Fww^\dagger\right]\underline{\phi}^q + \\ + \left[ww^\dagger - w^\dagger w\right]\underline{\phi}^{cl} - -2D(\nabla_{\mathbf{r}}F)(\nabla_{\mathbf{r}}w^\dagger) + 2[w^\dagger(\partial_t F)] + 4i[\underline{\phi}^q\left[(w^\dagger - w) + 2F - Fw^\dagger F\right] + \\ + \underline{\phi}^{cl}(w^\dagger F - Fw^\dagger)]\Big\} \quad (5.57)$$

To obtain the usual form of the diffusion pole we neglect the term $w^\dagger(\nabla_{\mathbf{r}}F)w^\dagger(\nabla_{\mathbf{r}}F)$. In this form, we note that the expansion in powers of w, $w^\dagger$ has been done around the saddlepoint value $i\hat{\Lambda}$, i.e., the stationary saddlepoint, at which the saddlepoint values of ϕ^{cl} and ϕ^q have been determined. In other words, we should regard (5.57) as a formal expression. In order to obtain the complete correct expression of (5.57), we need to find equations which determine the saddlepoint values $\underline{\phi}^{cl}$ and $\underline{\phi}^q$. We note that this has been partially done above in a previous section; there we see that by varying the original (non-diffusive) action we obtain the following values

$$\underline{\phi}^{cl}(\mathbf{r},t) = -\frac{\alpha}{2}\underline{\underline{G}}^K(\mathbf{r},\mathbf{r};t,t) \quad (5.58)$$

$$\underline{\phi}^q(\mathbf{r},t) = 0 \quad (5.59)$$

We will show below that the values of $\underline{\phi}^{cl}$ and $\underline{\phi}^q$ do not change in the presence of fluctuations; in other words, at the *common* saddlpoint manifold of ϕ^{cl} and ϕ^q we find that the saddlepoint values of the ϕ fields are unaltered (however, if we allow for *different* saddlepoint manifolds, i.e., if we allow, for example, that the diffusive fluctuations are at saddlepoint while those of the HS are not, then this might change the fact that the saddlepoint values of ϕ are unaltered, but we did not examine this possibility).

5.5.1 Action for the lifetime term

The Usadel equation is derived by taking into account the action expanded to first order in the diffusive modes which we are interested in and varying with respect to them in order to find the most probable fluctuations around the stationary saddlepoint; these fluctuations are now (weakly) spatial and temporal dependent. We proceed by substituting (5.55) into the action (5.41) and keeping only terms which are of first order in the diffusive modes. This expression is then varied w.r.t. d, $\bar{d}$, ϕ^q and ϕ^{cl}. This procedure yields 4 equations which determine these quantities in the most probable configuration. The action expanded up to the correct order is as follows:

$$iS^{(1)}[w,w^\dagger,\hat{\Phi}] = -\frac{\pi\nu}{4}\mathrm{tr}\Big\{2D(\nabla_{\mathbf{r}}F)(\nabla_{\mathbf{r}}w^\dagger) - 2[w^\dagger(\partial_t F)] + \tau_p^{-1}(w^\dagger - w)F - \\ - 4i[\underline{\phi}^q\left[(w^\dagger - w) + 2F - Fw^\dagger F\right] - \underline{\phi}^{cl}(w^\dagger F - Fw^\dagger)]\Big\} \quad (5.60)$$

The 4 determining equations are then obtained as follows:

$$\begin{aligned} i\frac{\delta S^{(1)}[\hat{Q},d,\bar{d}]}{\delta w^{\dagger}_{\tau\tau'}} &= -2D\left(\nabla^2_{\mathbf{r}}F_{\tau\tau'}(\mathbf{r},\mathbf{r})\right) + 2\partial_\tau F_{\tau\tau'}(\mathbf{r},\mathbf{r}) - \tau_p^{-1}F_{\tau\tau'}(\mathbf{r},\mathbf{r}) - \\ &\quad - 4i\left[\left(\underline{\phi}^{cl}(\tau)-\underline{\phi}^{cl}(\tau')\right)F_{\tau\tau'}(\mathbf{r},\mathbf{r}) - \underline{\phi}^{q}(\tau)\delta_{\tau\tau'} - F\circ\underline{\phi}^{q}\circ F\right] \stackrel{!}{=} 0 \\ &\quad \Rightarrow D(\nabla^2_{\mathbf{r}}F_{\tau\tau'}(\mathbf{r})) - \partial_\tau F_{\tau\tau'}(\mathbf{r}) + \tau_p^{-1}F = 4i\left(\underline{\phi}^{cl}(\tau)-\underline{\phi}^{cl}(\tau')\right)F_{\tau\tau'}(\mathbf{r}) \end{aligned} \tag{5.61}$$

$$\begin{aligned} i\frac{\delta S^{(1)}[d,\bar{d},\hat{\Phi}]}{\delta w_{\tau\tau'}} &= 4i\underline{\phi}^{q}(\tau)\delta_{\tau\tau'} \stackrel{!}{=} 0 \\ &\Rightarrow \underline{\phi}^{q}(\tau) = 0 \end{aligned} \tag{5.62}$$

For the saddlepoint equations determining the HS fields $\phi^{cl}(\mathbf{r},t)$ and $\phi^{q}(\mathbf{r},t)$ we need to take the term in the action which is quadratic in the ϕs. Doing this, we obtain the last 2 equations

$$\begin{aligned} &i\frac{\delta S^{(1)}[w,w^{\dagger},\hat{\Phi}]}{\delta\phi^{cl}} \stackrel{!}{=} 0 \\ &\Rightarrow \frac{2}{\alpha}\underline{\phi}^{q}(\mathbf{r},\tau) = -4i\int dt'\,(\underset{\sim}{w}^{\dagger}_{\tau t'}(\mathbf{r})F_{t'\tau}(\mathbf{r}) - F_{\tau t'}(\mathbf{r})\underset{\sim}{w}^{\dagger}_{t'\tau}(\mathbf{r})) = 0 \end{aligned} \tag{5.63}$$

$$\begin{aligned} &i\frac{\delta S^{(1)}[w,w^{\dagger},\hat{\Phi}]}{\delta\phi^{q}} \stackrel{!}{=} 0 \\ &\Rightarrow \frac{2}{\alpha}\underline{\phi}^{cl}(\mathbf{r},\tau) = -4i(\underset{\sim}{w}^{\dagger}_{\tau\tau}(\mathbf{r}) - \underset{\sim}{w}_{\tau\tau}(\mathbf{r}) + 2F_{\tau\tau}(\mathbf{r})) - \int dt'\,dt''\,F_{\tau t'}(\mathbf{r})\underset{\sim}{w}^{\dagger}_{t't''}(\mathbf{r})F_{t''\tau}(\mathbf{r}) = 2F_{\tau\tau}(\mathbf{r}) \end{aligned} \tag{5.64}$$

In (5.63) we see that the the quantum component of the HS field $\underline{\phi}^{q}$ is indeed zero, since we have expanded around the saddle point value at $\underset{\sim}{w}^{\dagger},\underset{\sim}{w}=0$. From the (5.63) and (5.64) we see that the saddlepoint values for ϕ^{cl} and ϕ^{q} even in the presence of diffusive fluctuations retains their same value as obtained in the absence of fluctuations (the saddlepoint value $\underline{\phi}^{cl}$ has the form indicated because we have expanded the diffusive fluctuations around the stationary saddlepoint, at which the parameterization for the Keldysh component of the $\hat{\Lambda}$ matrix is given by $\Lambda^{K}=\Lambda^{R}\circ F - F\circ\Lambda^{A}$, and where $\Lambda^{R}=-\Lambda^{A}=1$). Using the result from (5.61) we see that the interaction will affect the form of the Usadel equation only via the classical component of the HS field $\hat{\underline{\Phi}}$; hence, we obtain an equation describing the dynamics of the field $F(\mathbf{r},\mathbf{r};t,t')$

$$\boxed{\begin{aligned} D\nabla^2_{\mathbf{r}}F(\mathbf{r},\mathbf{r};t,t') - \frac{1}{2}\Big(\partial_t F(\mathbf{r},\mathbf{r};t,t') + \partial_{t'}F(\mathbf{r},\mathbf{r};t,t')\Big) + \tau_p^{-1}F(\mathbf{r},\mathbf{r};t,t') = \\ = \left(\underline{\phi}^{cl}(\mathbf{r},t)+\underline{\phi}^{cl}(\mathbf{r},t')\right)F(\mathbf{r},\mathbf{r};t,t') \end{aligned}} \tag{5.65}$$

(5.65) is the kinetic equation determining the dynamics of the function $F(\mathbf{r},\mathbf{r};t,t')$ and the classical component of the HS field $\underline{\phi}^{cl}(\mathbf{r},t)$. In order for F to yield a physically meaningful quantity we need to transform it in the Wigner representation, but since (5.65) is coupled to the HS field $\phi(\mathbf{r},t)$ we will proceed by first writing down the integro-differential equation.

5.6 Nonlinear integro-differential equation for the laser distribution function

5.6.1 Introduction

In this section we outline our attempt at a solution of the set of coupled nonlinear equations (5.50) and (5.65). However, before that we need to introduce some simplifications and additional details which are crucial for their solution.

First, we write (5.50) and (5.65) for the physical quantity $F(z, z; t, t')$ as a single nonlinear integro-differential equation and transform this in the Wigner representation, which has then the functional dependence $\widetilde{F}(\mathbf{R}, T; \epsilon)$. In this representation we are able to deduce the behavior of the photonic occupation as a function of the energy ϵ, which is important for the determination of linewidth. In addition, the dependence of F on the spatial variable also contains information on the size of the lasing spot, which we show how to obtain analytically in the next section.

Secondly, we note that we will restrict our considerations to one dimension, i.e., our equations will describe only dynamics in the z-direction. Space in the $x - y$ plane will be assumed to be translationally invariant, i.e., we can perform a Fourier transform in those directions. In doing this we introduce an additional variable $\mathbf{q}_\parallel$ into the problem , which has the meaning of the parallel momentum in the $x - y$ plane. Hence physically, our system of consideration consists of a thin slab which stretches indefinitely in the $x - y$ directions, and has a fixed thickness in the z-direction.

Finally, we need to impose boundary conditions in order to obtain solutions to our equations. In our problem we impose boundary conditions similar to conventional Neumann conditions, where instead of the value of the solution ($u(x = 0) = a, u(x = L) = b$) is provided at the boundaries (as in Dirichlet boundary conditions), the values of the derivative ($u'(x = 0) = a, u'(x = L) = b$) at the boundaries are provided instead. In our case, we impose a slightly different version of the Neumann conditions, where instead of specifying the *values* of the derivatives at the boundaries, we require that the derivatives at the boundaries are proportional to the *solution* itself at the boundaries.

5.6.2 Wigner transform and first approximation of coupled set of integro-differential equations

In this section we will perform further analytical simplification on the coupled set of equations (5.50) and (5.65). Specfically, we will write these as a single integro-differential equation which we then transform in the Wigner representation.

We first simplify the problem to 1D, because later we adopt a particular spatial form for our system which assumes homogeneity in both x, y-directions, and diffusive motion only in the z-direction. We will write this 1D version of the distribution function as $\widetilde{F}(z, z; t, t')$, or the Wigner-transformed version $\widetilde{f}(z, q; T, \epsilon)$, where $q \equiv |\mathbf{q}_\parallel|$, and $\mathbf{q}_\parallel$ is the wave vector in the $x - y$ plane.

In $1D$ (5.65) and (5.50) take the forms

$$\boxed{\begin{aligned} D\partial_z^2 \widetilde{F}(z,z;t,t') - \frac{1}{2}\Big(\partial_t \widetilde{F}(z,z;t,t') + \partial_{t'}\widetilde{F}(z,z;t,t')\Big) + \tau_p^{-1} F(\mathbf{r},\mathbf{r};t,t') = \\ = \Big(\underline{\phi}^{cl}(z,t) + \underline{\phi}^{cl}(z,t')\Big)\widetilde{F}(z,z;t,t') \end{aligned}} \tag{5.66}$$

and

$$\boxed{\underline{\phi}^{cl}(z,t) = -\frac{\alpha}{2}\int dt' \left[G^R(z,z;t,t')\widetilde{F}(z,z;t',t) - \widetilde{F}(z,z;t,t')G^A(z,z;t',t)\right]} \tag{5.67}$$

where $\underline{\phi}^{cl}(z,t)$ is the saddlepoint value of the classical component of the Hubbard-Stratonovich matrix field $\mathbf{\Phi}(z,t)$. We have shown that the quantum component of $\mathbf{\Phi}(z,t)$ is identically zero, at least within the saddlepoint manifold. In order to separate out the physically relevant time and length scales, which are typically much longer and slowly varying as compared to "microscopic" scales of the system (atomic transition frequency between energy levels, frequency of light oscillations, etc), we turn to the Wigner representation of our quantities.

The Wigner transformation involves the rewriting the time and length scales present in our system in terms of *sums* and *differences*:

$$Z \equiv \frac{z+z'}{2}, \qquad \Delta z \equiv z' - z \tag{5.68}$$

$$T \equiv \frac{t+t'}{2}, \qquad \Delta t \equiv t' - t \tag{5.69}$$

We can then perform Fourier transform with respect to the "fast" variables, where the transform has the form

$$\widetilde{F}(z,z;t,t') = \widetilde{F}_{T-\frac{\Delta t}{2},T+\frac{\Delta t}{2}}(z,z) \tag{5.70}$$

$$= \int \frac{d\epsilon}{2\pi} f(Z;T,\epsilon)\, e^{-i\epsilon t} \tag{5.71}$$

where we have suppressed the dependence of $\widetilde{f}$ on the difference of z. We note that since $f(Z;T,\epsilon)$ is a spatially local quantity, we have, with respect to the spatial variable Z the simple relation $Z = z$ (however, in the following we will keep the capitalized notation in order to emphasize what we are working with Wigner variables). We obtain for (5.66)

$$D\partial_Z^2 f(Z;T,\epsilon) - \partial_T f(Z;T,\epsilon) + \tau_p^{-1} f(Z;T,\epsilon) = \int d\varepsilon' \left[f(Z;T,\varepsilon + \frac{1}{2}\varepsilon') - f(Z;T,\varepsilon - \frac{1}{2}\varepsilon')\right]\underline{\phi}^{cl}(Z,\varepsilon')e^{i\varepsilon' T} \tag{5.72}$$

and for the 2nd equation

$$\underline{\phi}^{cl}(Z,\varepsilon) = -\frac{\alpha}{2}\int dT \int d\varepsilon' \left[G^R(Z;T,\varepsilon' - \frac{1}{2}\varepsilon) - G^A(Z;T,\varepsilon' + \frac{1}{2}\varepsilon)\right] f(Z;T,\varepsilon') \tag{5.73}$$

where the retarded / advanced Green's functions have the form(s)

$$G^{R(A)}(Z;T,\varepsilon) \equiv \frac{1}{\varepsilon - \xi_{\mathbf{k}} \pm \frac{i}{2\tau} - \underline{\phi}^{cl}(Z,T)} \tag{5.74}$$

Substituting (5.73) into (5.72) yields the nonlinear integro-differential equation of our problem, expressed in terms of the Wigner variables

$$\begin{aligned} D\partial_Z^2 f(Z;T,\Omega) - \partial_T f(Z;T,\Omega) + \tau_p^{-1} f(Z;T,\Omega) = \\ = \int \frac{d\omega'}{2\pi} \int dT \int \frac{d\Omega}{2\pi} e^{i\varepsilon T} \left[G^R(z,z;T+T',\Omega_+) - G^A(z,z;T+T',-\Omega_+)\right] \times \\ \times f(Z;T,\Omega)\Big(f(Z;T,\omega-\omega') - f(Z;T,\omega'-\omega)\Big) \end{aligned} \tag{5.75}$$

As have been mentioned before, the point of performing a Wigner transformation is to separate the "fast" and "slow" variables of the problem such that we can study the typically macroscopic quantities while getting rid of small rapid fluctuations. In terms of the equations under study this translates to expansions of our functions $G^{R(A)}(z,z;T,\Omega \pm \frac{1}{2}\omega)$ and $f(z,z;T,\Omega \pm \frac{1}{2}\omega)$ in powers of ω and retaining terms up to the appropriate order. The details of this calculation can be found in Appendix D. Here we will simply give the final expression below

$$\begin{aligned} D\frac{\partial^2}{\partial Z^2} f(Z;T,\omega) - \frac{\partial}{\partial T} f(Z;T,\omega) + \tau_p^{-1} f(Z;T,\omega) = \\ = \frac{\alpha}{2} \int \frac{d\Omega}{2\pi} \Bigg\{ \Big[f(Z;T,\Omega)\,\mathrm{Im}G^R(Z;T,\Omega)\Big]\Big[f(Z;T,\omega) + f(Z;T,-\omega)\Big] + \\ + \left[\left(\frac{\partial}{\partial T} f(Z;T,\Omega)\right)\Big(\mathrm{Im}G^R(Z;T,\Omega)\Big)\right] \frac{\partial}{\partial \omega}\left[f(Z;T,\omega) + f(Z;T,-\omega)\right] \Bigg\} \end{aligned} \tag{5.76}$$

We note that (5.76) can be written in the usual finite differential form as described above, but in this work we will only take into account the first term on the right-hand side (term with no differential operators) in order to make the connection with the nonlocal Fisher equation stemming from the set of equations expressed in the original variables.

Alternative approach to approximation of integro-differential equation

In this section we mention an additional simplification of our problem. We again look at (5.76). If we now proceed to discard terms which are of second order in the derivatives with respect to the slowly varying variables, we obtained the simple expression

$$\begin{aligned} D\frac{\partial^2}{\partial Z^2} f(Z;T,\omega) - \frac{\partial}{\partial T} f(Z;T,\omega) + \tau_p^{-1} f(Z;T,\omega) = \\ = \frac{\alpha}{2} \int \frac{d\Omega}{2\pi} \Big[f(Z;T,\Omega)\,\mathrm{Im}G^R(Z;T,\Omega)\Big]\Big[f(Z;T,\omega) + f(Z;T,-\omega)\Big] \end{aligned} \tag{5.77}$$

Keeping in mind that our final expressions must be functions of the slow, or macroscopic variables, we can now assume that our distribution functions are independent of the fast, or microscopic variables, i.e.,

$$f(Z;T,\Omega) \approx f(Z;T). \tag{5.78}$$

In this approximation we can then write (5.77) in the form of a *local* Fisher equation, i.e.,

$$\boxed{D\frac{\partial^2}{\partial Z^2}f(Z;T) - \frac{\partial}{\partial T}f(Z;T) + \tau_p^{-1}f(Z;T) = \alpha\left(\int \frac{d\Omega}{2\pi}\,\mathrm{Im}G^R(Z;T,\Omega)\right)f(Z;T)^2}. \tag{5.79}$$

which now presents the form of a simple local Fisher equation with a (Z,T) dependent prefactor which can be interpreted as the spectral function.

5.6.3 Diffusive motion in a thin slab

Fourier transform in the parallel direction

The description of transport as presented in previous sections applies to the diffusive behavior of the distribution function $f(Z,T;\epsilon)$, which applies only in the single dimension z. In a realistic situation we will also need to know how to deal with the other two dimensions and to correctly include the dynamics in the $x-y$ plane in our expressions; this issue will be dealt with in this section. The simplest possible configuration is the thin slab, which is a particularly simple high-dimensional physical configuration. Our spatial dimension of interest is aligned along the z-axis, leaving the space translational invariance unbroken in the $x-y$ direction. This implies that we can perform a straightforward Fourier transform along both the x- and y-axes but leave the z-axis in its original form, where we expect our solution for $f(z,\mathbf{q}_{\|},T;\epsilon)$ will show the relevant dynamics, where here $\mathbf{q}_{\|}$ is the momentum vector in the $x-y$ plane.

We can see how to include the effect of the spatial homogeneity in the $x-y$ direction by going back to the diffusion equation in three-dimensional space and including the "collision integral" from the right-hand side of (5.86)

$$D\nabla_{\mathbf{R}}\widetilde{f}(\mathbf{R},T;\epsilon) - \partial_T\widetilde{f}(\mathbf{R},T;\epsilon) + \beta\widetilde{f}(\mathbf{R},T;\epsilon) = I[\widetilde{f}(\mathbf{R},T;\epsilon)^2] \tag{5.80}$$

where here we temporarily denote the full 3-dimensional spatial vector as $\mathbf{R}\equiv(X,Y,Z)$; we will then move to more appropriate notation as necessary. The $\nabla_{\mathbf{R}}$ operator in (5.80) can be trivially rewritten as

$$\nabla_{\mathbf{R}}\widetilde{f}(\mathbf{R},T;\epsilon) \equiv \left(\nabla_{\|}^2 + \frac{\partial^2}{\partial Z^2}\right)\widetilde{f}(Z,\vec{X},T;\epsilon) \tag{5.81}$$

where on the right-hand side of (5.81) we have rewritten the function $\widetilde{f}(\mathbf{R},T;\mathbf{k}.\epsilon)$ as depending on the Z-coordinate and a $\vec{X}$ vector which we assume to point in the $x-y$ plane only. The nabla-operator in the parallel direction is space translational invariant, i.e., it admits of a Fourier transform of the form

$$\nabla_{\|}^2 \xrightarrow{\text{F.T.}} -|\vec{q}_{\|}|^2 \tag{5.82}$$

Both the neglect of the time dependence and the taking of the Fourier transform in the $x-y$ axes results in the following modification to the diffusion equation, (5.80)

$$-Dq_{\parallel}^2 f(Z,q_{\parallel},T;\epsilon)+D\partial_Z^2 f(Z,q_{\parallel},T;\epsilon)=I[f(Z,q_{\parallel},T;\epsilon)^2] \tag{5.83}$$

which yields for (5.65) and (**??**):

$$\begin{aligned}-Dq_{\parallel}^2 f(Z,q_{\parallel},T;\epsilon)+D\partial_Z^2 f(q_{\parallel},Z,T;\epsilon)-\partial_T f(Z,q_{\parallel},T;\epsilon)+\beta f(Z,q_{\parallel},T;\epsilon)=\\ =\left(\underline{\phi}^{cl}(Z,T)-\underline{\phi}^{cl}(Z,T')\right)f(Z,q_{\parallel},T;\epsilon)\end{aligned} \tag{5.84}$$

and

$$\underline{\phi}^{cl}(Z,T)=-\frac{\alpha}{2}\int d\epsilon\sum_{\mathbf{k}}\left[G_e^R(Z,T;\mathbf{k},\epsilon)-G_e^A(Z,T;\mathbf{k},\epsilon)\right]f(Z,q_{\parallel},T;\epsilon). \tag{5.85}$$

here written in the original space and time variables. Substituting (5.85) into (5.84) gives the single equation

$$\begin{aligned}&-Dq_{\parallel}^2 f(Z,q_{\parallel},T;\epsilon)+D\partial_Z^2 f(q_{\parallel},Z,T;\epsilon)-\partial_T f(Z,q_{\parallel},T;\epsilon)+\beta f(Z,q_{\parallel},T;\epsilon)=\\ &=-\alpha\int d\omega\sum_{\mathbf{k}}\left(\mathrm{Im}\,G_e^R(\mathbf{R},T;\mathbf{k},\omega)f(q_{\parallel},Z,T;\omega)\right)\left(f(q_{\parallel},Z,T;\epsilon)+f(q_{\parallel},Z,T;-\epsilon)\right).\end{aligned} \tag{5.86}$$

where $\mathrm{Im}\,G_e^R(Z,T;\mathbf{k},\epsilon)\equiv G_e^R(\mathbf{R},T;\mathbf{k},\epsilon)-G_e^A(\mathbf{R},T;\mathbf{k},\epsilon)$. We can perform the additional simplification mentioned in the last subsection by discarding the frequency dependence of the f functions (Eq. (5.78)) to obtain the final expression

$$\boxed{\begin{aligned}-Dq_{\parallel}^2 f(Z,q_{\parallel},T)+D\partial_Z^2 f(q_{\parallel},Z,T)-\partial_T f(Z,q_{\parallel},T)+\beta f(Z,q_{\parallel},T)=\\ =-\alpha\int d\omega\left(\sum_{\mathbf{k}}\mathrm{Im}\,G_e^R(\mathbf{R},T;\mathbf{k},\omega)\right)f(q_{\parallel},Z,T)^2.\end{aligned}} \tag{5.87}$$

(5.87) is a general form of an equation usually called the *nonlocal Fisher-Kolmogorov-Petrovskii-Piskunov* (Fisher-KPP) equation ([76], [77]). This is a well-studied equation which has been applied to the theory of population growth in ecology, econophysics and pattern formation in bacterial cultures [78]. In general, it is suited to the description of effects of competition between enhancement and depletion of resources which are required for sustained growth of a population (of individual animals, value of stocks, or modes in a multimode random laser).

The first term in (5.87) has the form of a constant prefactor multiplied by the solution $f(Z,q_{\parallel},T)$, which is now a function of three variables. The second and third terms, along with the right-hand side, corresponds to the original three-dimensional diffusion equation (5.80), but now with dynamics in only the dimension z dimension. In the next section we will describe how we combine the solution of the $1D$ version of (5.80) (which of course gives a solution with the functional dependence $f(\mathbf{R},T)$) with the first term in (5.83) to obtain the overall solution $f(Z,q_{\parallel},T)$.

Boundary conditions

As for the solution of any second-order differential equation, boundary conditions are required. In our problem we impose a particular form of the usual Neumann boundary conditions: whereas in the usual case the Neumann conditions states that, at the boundaries $x = 0$ and $x = L$ (assuming a one-dimensional domain $x \in [0, L]$)

$$\frac{d}{dx}u(x,t)\bigg|_{x=0} = 0, \qquad \frac{d}{dx}u(x,t)\bigg|_{x=L} = 0, \tag{5.88}$$

in our problem the coupling of the system at the boundaries should be proportional to the solution for the system at the boundaries, i.e.,

$$\frac{d}{dx}u(x,t)\bigg|_{x=0} = \alpha u(x=0,t), \qquad \frac{d}{dx}u(x,t)\bigg|_{x=L} = \beta u(x=L,t), \tag{5.89}$$

where α and β are physical parameters describing the strength of coupling of the system to the environment. In physical terms, our boundary conditions enforces that the loss of energy from our system must always be proportional to the actual energy intensity in the system. This is then important to ensure the existence of stable solutions to our integro-differential equation. Numerically, this set of boundary conditions are also simple to implement, and we will do so in the next section.

5.6.4 Numerical approach to solve the nonlinear integro-differential equation

Crank-Nicolson scheme

We employ a usual Crank-Nicolson (CN) scheme for the solution of (5.87). This method is of second order in the spatial and first order in the time coordinates, respectively. In this section we will see how to write down a CN method involving a integral term on the right hand side. In finite differentiation, the 2nd order spatial differential operator (in 1-dimension) can be approximated as

$$\frac{\partial^2}{\partial Z^2} f(q_\parallel, Z, T) \approx \frac{w_{i+1}^{j,k} - 2w_i^{j,k} + w_{i-1}^{j,j}}{h^2} \tag{5.90}$$

and the first order in the time derivative is then approximated by

$$\frac{\partial}{\partial T} f(q_\parallel, Z, T) \approx \frac{w_i^{j+1,k} - w_i^{j,k}}{k} \tag{5.91}$$

where $w_i^{j,k}$ denotes the value of the required solution at time, space and energy discretization step (x_i, t_j, ϵ_k). Note that the k-index is not affected in the dicretization of the spatial and time derivative operators; in (5.90) and (5.91) it is a simple index. However, the k-index comes into play in the right-hand side of (5.86) due to the presence of the convolution term.

In the Crank-Nicolson procedure one combines the time stepping in the form of (5.90) (usually

called the *forward differencing* scheme) with the alternative (*backward differencing*) form

$$\frac{\partial^2}{\partial Z^2} f(q_\parallel, Z, T) \approx \frac{w_{i+1}^{j+1,k} - 2w_i^{j+1,k} + w_{i-1}^{j+1,k}}{h^2} \tag{5.92}$$

to yield the expression

$$w_i^{j+1,k} - w_i^{j,k} = \lambda\Big[w_{i+1}^{j,k} - 2w_i^{j,k} + w_{i-1}^{j,k} + w_{i+1}^{j+1,k} - 2w_i^{j+1,k} + w_{i-1}^{j+1,k}\Big] \tag{5.93}$$

where $\lambda \equiv \frac{Dk}{2h^2}$. Up till now we have not taken the nonlinear term into account. We implemented a simple trapezoidal qudrature rule, which we can write as

$$\int_a^b dx\, f(x) \approx \frac{1}{2} h_I \sum_{I=0}^{N-1} f(x_I) \tag{5.94}$$

where $h_I \equiv \frac{b-a}{N_x - 1}$, where N is the number of discretization points. Hence the integral part of the equation can be discretized as

$$\int_{-\infty}^{+\infty} d\omega\, \mathrm{Im}\, G_e^R(\mathbf{R}, T; \mathbf{k}, \omega) \approx \frac{1}{2} m_I \sum_{k=0}^{N_\omega - 1} \mathrm{Im}\, G_e^R(\mathbf{R}_i, T_j; \mathbf{k}, \omega_k) \tag{5.95}$$

where $m_I = \frac{b-a}{N_t - 1}$. Putting together everything we see that for values of $i = 1, 2, \ldots, N_x - 2$, $j = 1, 2, \ldots, N_t - 2$ and $k = 0, 1, \ldots, N_\omega - 1$ we obtain the discretized form of (5.86) including nonlinearity and also pumping in the CN scheme:

$$\begin{aligned}\Bigg[1 + \lambda + \frac{1}{2} k m_I \beta \Bigg(\sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} w_i^{j,K}\Bigg) - \frac{1}{2}\gamma k\Bigg] w_i^{j+1,k} - \frac{1}{2}\lambda\left(w_{i+1}^{j+1,k} + w_{i-1}^{j+1,k}\right) - \\ - \Bigg\{\Bigg[1 - \lambda - \frac{1}{2} k m_I \beta \sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} + \frac{1}{2}\gamma k\Bigg] w_i^{j,k} + \frac{1}{2}\lambda\left(w_{i+1}^{j,k} + w_{i-1}^{j,k}\right)\Bigg\} = 0\end{aligned} \tag{5.96}$$

In order to solve for $w_i^{j,k}$ at arbitrary space, time and energy discretization points (x_i, t_j), we need to provide initial and boundary conditions. We will explain the finite discretization of the ("Neumann") boundary conditions in the next subsection, but the necessity of the initial conditions can be seen as follows. We start from an *Ansatz* to the solution at initial time point $t = t_0$

$$\mathbf{w}^0 = \begin{pmatrix} w_0^{0,k} & w_1^{0,k} & w_2^{0,k} & \ldots w_{N_x-1}^{0,k} \end{pmatrix} \tag{5.97}$$

and substituting this into the term in the second curly brackets in (5.96), on obtains a coupled set of nonlinear equations which can be solved using a root-finding algorithm. Doing this yields a solution at time point $t = t_1$, which we again substitute in the part in the curly brackets in (5.96). Performing this operation for successive time steps yields the solution vector $\mathbf{w}^i$ at an arbitrary time step i.

Boundary conditions

In order to be able to solve the set of coupled equations in (5.66) and (5.67) correctly, we need to impose initial and boundary conditions. We have already mentioned how the *Ansatz* to the solution at time point $t = t_0$ is essential to obtaining a solution w_i^j in the CN scheme. On the other hand, in order to obtain physically meaningful solutions, we also need to impose boundary conditions. In a previous section we have explained and physically motivated our particular choice of boundary conditions; these were defined in (5.88). In this section we will explain how to include (5.88) into our finite differencing scheme.

In order to incorporate (5.88) into our discretized equations, we first note that our equations (5.96) at space coodinates $x = x_0$ and $x = x_{N_x-1}$ involves the quantities $w = w_{-1}^{j,k}$ and $w = w_{N_x}^{j,k}$, respectively, which lie outside of our vector of solutions (which carry space indices in the range $i = 0, 1, \ldots, N_x - 1$). The values w_{-1} and w_{N_x} are hence to be determined additionally from the boundary conditions. To do this we first discretize (5.88). This yields

$$\left.\frac{\partial}{\partial x} w\right|_{i=0} \approx \frac{w_1^{j,k} - w_{-1}^{j,k}}{2h} = \alpha w_0^{j,k} \tag{5.98}$$

$$\left.\frac{\partial}{\partial x} w\right|_{i=N_x-1} \approx \frac{w_{N_x}^{j,k} - w_{N_x-2}^{j,k}}{2h} = \beta w_{N_x-1}^{j,k} \tag{5.99}$$

We can then solve for the unknown values $w_{-1}^{j,k}$ and $w_{N_x}^{j,k}$ in terms of known quantities, i.e.,

$$w_{-1}^{j,k} = w_1^{j,k} - 2h\alpha w_0^{j,k} \tag{5.100}$$

$$w_{N_x}^{j,k} = w_{N_x-2}^{j,k} + 2h\beta w_{N_x-1}^{j,k} \tag{5.101}$$

The finite difference equations at the points $i = 0$ and $i = N_x - 1$ are, after taking (5.100) and (5.101) into account are, respectively:

$$\left[1 + \lambda(1 + h\alpha) + \frac{1}{2} k m_I \beta \sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} - \frac{1}{2}\gamma k + \frac{1}{2} D q_\parallel^2 k\right] w_0^{j+1} - \lambda w_0^{j+1} - \\ - \left\{\left[1 - \lambda(1 + h\alpha) - \frac{1}{2} k m_I \beta \sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} + \frac{1}{2}\gamma k - \frac{1}{2} D q_\parallel^2 k\right] w_0^j + \lambda w_1^j\right\} = 0 \tag{5.102}$$

and

$$\left[1 + \lambda(1 - h\beta) + \frac{1}{2} k m_I \beta \sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} - \frac{1}{2}\gamma k + \frac{1}{2} D q_\parallel^2 k\right] w_{N_x-1}^{j+1} - \lambda w_{N_x-2}^{j+1} - \\ - \left\{\left[1 - \lambda(1 - h\beta) - \frac{1}{2} k m_I \beta \sum_{K=0}^{N_\omega - 1} (\mathrm{Im}G)_i^{j,K} + \frac{1}{2}\gamma k - \frac{1}{2} D q_\parallel^2 k\right] w_{N_x-1}^j + \lambda w_{N_x-2}^j\right\} = 0 \tag{5.103}$$

where we have also inserted the dependence on the momentum in the $x-y$ plane ($q_\parallel$) in the discretized equations. The set of equations (5.102), (5.96) and (5.103) form a set of $N_x - 1$ coupled nonlinear equations which then needs to be solved numerically.

For our problem, we want to study the behavior of the solutions $w_i^{j,k}$ in the steady-state regime, which translates to the solution at $t \to +\infty$. In the CN scheme, this can be implemented rather trivially by performing the time-stepping up to a point where the solution does not change.

5.7 Comments and Future Work

In this section we would like to comment on the nonlocal Fisher equation which we have derived in the previous sections. We note that this equation yields interesting solutions when different assumptions are made on the gain (pump) and the nonlocal terms. All solutions in this section have been taken from [79].

First, we see that if we allow for a nonlocal pumping coefficient (which, as is described in Appendix D, can be included in our derivation of the Fisher equation) and impose a specific pump profile, we can expect to obtain solutions which are of the form described in [72], hence in agreement with numerical simulations in $1D$. This can be seen in Fig. 5.3, where the x-dependent pumping coefficient is given by the bimodal form

$$b(x) = \frac{1}{(1+50(x-15)^2)^{10}} + \frac{0.9}{(1+50(x-85)^2)^5} \tag{5.104}$$

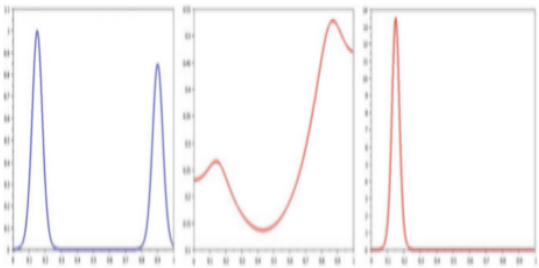


Figure 5.3: Solution of the Fisher equation (right-most figure) for a particular form of pump function $a(x)$ (left-most figure). This result is to be compared with numerical simulations from [73]. Figure from [79].

Although (5.104) looks like a very specific pump profile which is not reproducible, it has been proven in [79] that the Fisher equation with a *local* pump profile is able to yield "spike" solutions, which would explain the observation of spatially narrow output spectrum in numerical simulations.

In addition, treating the nonlocal term carefully, we can expect a solution of the form shown in Fig. 5.4, where we see a striking visualization of lasing spots, which are localized regions of large output intensity. We note that this solution of the Fisher equation only depends on the nonlocality of the saturation term, and hence does not require the presence of a spatially dependent pumping.

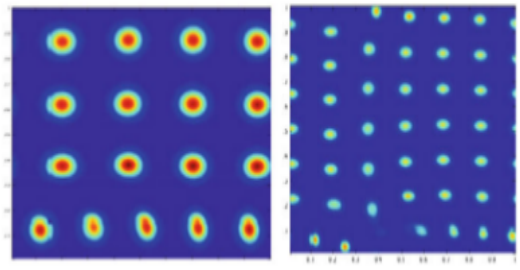


Figure 5.4: Spatial pattern formation due to nonlocal Fisher equation. Expected lasing spots can be clearly seen. Figure on the right shows spot pattern for very small diffusion constant D. Figure from [79].

This of course points at a universal feature of random lasers, which we could then reveal as being the solution of a particular form of Fisher's equation.

5.8 Conclusion

In this chapter we have described in detail the derivation of the photonic distribution function $f(q_\parallel, Z, T)$ for random lasers via the nonlinear sigma model. This function is characterized by the wavevector $q_\parallel$ originating from the Fourier transform of in the $x - y$ plane, where translational invariance is assumed. In addition, it is also assumed to vary slowly in space and time, which behavior is described by its dependence on the slow variables (Z, T). The calculation of this quantity is a nice demonstration of the kind of nonequilibrium quantities which can be obtained from the Keldysh nonlinear sigma model. $f(q_\parallel, Z, T)$ results from the solution of a set of coupled equations relating $f(q_\parallel, Z, T)$ and the Hubbard Stratonovich field $\underline{\phi}^{cl}(Z, \epsilon)$, (5.72) and (5.73). These equations are derived from the saddlepoint equations for the diffusive modes $w^\dagger$, w and the HS field $\phi^{cl}(Z, \epsilon)$, which are in turn arrived at from the diffusive nonlinear sigma model including nonlinearity, (5.41). The solution of (5.72) and (5.73) has to be done numerically, and we outlined our computational strategy. Unfortunately, due to complexity of the occuring equations, no numerical results have yet been obtained. The equation to be evaluated, well-known in other contexts as the *Fisher* equation, shows a rich variety of solutions [79] and work is continuing to extract numerically the solutions corresponding to our regime of interest.

CHAPTER 6

Effects of Light Polarization on Localization of Light

6.1 Introduction

Propagation of classical waves in a disordered medium has been the focus of intensive theoretical and experimental investigations since the publication of the paper [80] by John in which the existence of photon mobility edges in systems with disorder was postulated. This was followed by the equally seminal 1987 paper [81] (similar work was done independently by Yablonovitch [82]) where the possibility of localization of electromagnetic waves in systems with interplay of "order and disorder" ("disordered superlattice microstructures") was first discussed. The most important property of such structures is the possibility to restrict phase space for photon propagation near critical frequencies demarcating the the photonic "pseudogap" region [81]. In other words, these could be seen as the electromagnetic analogue of electronic bandgaps in anomalous semiconductors [83]. Experimentally, such systems have also been comprehensively studied in recent years [8, 84] and the important physical quantity, the diffusion coefficient $D(\omega)$ of propagating light waves, measured and shown to exhibit localization characteristics [3, 4, 85–89]. Theoretical studies of this problem have mainly consisted of diagrammatic self-consistent [40, 41] calculations of $D(\omega)$ [26, 35, 36, 42, 43] and field theoretical descriptions of light transport which in addition takes into account the effects of absorption [67] and openness of media [47]; in most of these works the the additional degree of freedom represented by light polarization was not considered, usually due to reasons of simplicity. Hence, the effect of the vector nature of light on its transport properties has not been fully understood.

We show in this chapter that polarization actually plays an important role in the transport of light through random dielectric medium, i.e., it accounts for the difficulty to conclusively account for strong localization of light in experiments. Although other dephasing mechanisms like absorption play important roles, we find that, due to it being an intrinsic property of light waves, polarization represents an intrinsic channel for dephasing which is not easily subtracted from in experiments. We show that this can be clearly seen by drawing a formal analogy with the electronic phenomena of antilocalization [19, 90–92]. In systems with spin, application of

an external magnetic field will cause coupling of the spin degrees of freedom to the applied field, which yields an additional term in the Hamiltonian of the system [19] This additional term incorporates the spin structure of the problem. Properly diagonalized in the singlet-triplet eigenvector space, it has been shown [90, 91] to induce attenuation in the diffusion probability by a factor of 2 at the origin. This phenomena has been confirmed in various experiments [93–95], perturbative [90, 96, 97] and renormalization group [98–100] calculations.

We propose a suitable model of a disordered photonic crystal which is tractable numerically via a diagrammatic self-consistent calculation [40, 41] with additional suitable formalism which to take into account the vectorial degrees of freedom for electromagnetic waves. Setting up the self-consistent equations in this formalism yields a set of complicated matrix equations which needs to be solved. Fortunately the matrix structure can be simplified considerably using a averaging procedure, and as a result we obtain scalar equations for all quantities except for the important two-particle interaction vertices. These could however be diagonalised in their respective eigenbases, namely the singlet and triplet bases. We have then essentially three coupled equations which we then solve numerically for the diffusion coefficient. We show some numerical results at the end and our conclusions.

6.2 Formalism

6.2.1 Helmholtz equation

We again start from the Helmholtz equation in frequency and real space ((3.7))

$$\nabla \times \nabla \times \boldsymbol{E}_{n\boldsymbol{k}}(\boldsymbol{r}) = \frac{\omega_{n\boldsymbol{k}}^2}{c^2} \varepsilon(\boldsymbol{r}) \boldsymbol{E}_{n\boldsymbol{k}}(\boldsymbol{r}), \tag{6.1}$$

where we have retained the full vectorial structure of the equation, unlike in the previous chapters where we have only considered the scalar version of (6.1) $E_{n\boldsymbol{k}\sigma}(\boldsymbol{r})$ is the electric field in the medium, and the additional subscripts n and $\boldsymbol{k}$ denotes the band index and wavevector, respectively. Here we restrict our model to take the form of a translational invariant crystal lattice, where randomness will be introduced via the values of the dielectric constant on each lattice site.

We again write the random, real-valued dielectric function $\varepsilon(\boldsymbol{r})$ as the sum of a *periodic* $\varepsilon_p(\boldsymbol{r})$ and a *random* (disordered) $\Delta\varepsilon(\boldsymbol{r})$ part. $\varepsilon_p(\boldsymbol{r})$ shares the periodicity of the crystal lattice and only contributes to the background dielectric constant of the crystal. The wavefunction of electromagnetic waves in such a system is described by the usual Bloch wavefunctions [83] $E_{n\boldsymbol{k}}(\boldsymbol{r})$ which obeys (6.1) with $\varepsilon(\boldsymbol{r})$ replaced by $\varepsilon_p(\boldsymbol{r})$.

To take into account the additional degree of freedom due to the different polarizations, we exploit the transverse nature of light polarization, in which

$$\boldsymbol{k} \cdot \boldsymbol{D}_{n\boldsymbol{k}} = 0 \tag{6.2}$$

where $\boldsymbol{D}_{n\boldsymbol{k}}$ is the electric displacement field which is related to the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ in a linear,

conservative system in the following manner

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0 \varepsilon(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r},t) \tag{6.3}$$

(6.2) and (6.3) effectively means that only 2 components of the 3-dimensional polarization vector is linearly independent in a perfect crystal. This allows us to establish an analogy between the polarization and electronic spin degrees of freedom, or in other words, to impose a "pseudospin" structure to single-particle quantities, which includes, for example, the Green's function [42, 101]. In what follows we will denote this additional degree of freedom via an additional index σ.

Randomness is introduced as a perturbation to the periodicity of the Helmholtz equation, or in other words, the disordered Helmholtz equation is given by:

$$\left(\frac{\omega^2}{c^2}\varepsilon_p(\boldsymbol{r}) - \nabla \times \nabla \times\right) \boldsymbol{E}_\omega(\boldsymbol{r}) + \frac{\omega^2}{c^2}\Delta\varepsilon(\boldsymbol{r})\boldsymbol{E}_\omega(\boldsymbol{r}) = 0. \tag{6.4}$$

Hence disorder is introduced *substitutionally*, i.e., the crystal lattice keeps its structural order, but the basis changes from site to site in a random fashion. We note that the squared frequency dependence of the disorder term is probably the most important characteristic of light propagation in disordered systems and also the source of new effects which are unknown to electronic systems. In particular, it leads to vanishing disorder effects for frequencies $\omega \to 0$, whereas in contrast for electronic systems they play a major role in the low energy region.

6.2.2 Photonic Wannier basis

The set of electric Bloch functions which takes into account the additional mode/pseudospin index, fulfills the following orthogonality relation,

$$\int d\boldsymbol{r}\, \boldsymbol{E}^*_{n\boldsymbol{k}\sigma}(\boldsymbol{r})\, \varepsilon_p(\boldsymbol{r})\, \boldsymbol{E}_{n'\boldsymbol{k}'\sigma'}(\boldsymbol{r}) = \frac{(2\pi)^3}{u}\delta_{nn'}\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\sigma\sigma'}, \tag{6.5}$$

where u is the volume of one unit cell in the crystal. In the case of a disordered crystal, one would expect that the electromagnetic wave functions would be primarily localized and in such a case the extended nature of the Bloch wavefunctions are then not suitable for their description. We use instead the Wannier wavefunctions, obtained from the Bloch wavefunctions via a simple Fourier transform with respect to $\boldsymbol{k}$ [102],

$$\boldsymbol{W}_{i\sigma}(\boldsymbol{r}) = \frac{u}{(2\pi)^3}\int_{\mathrm{BZ}} d\boldsymbol{k}\, e^{-i\boldsymbol{k}\boldsymbol{R}_i}\boldsymbol{E}_{\boldsymbol{k}\sigma}(\boldsymbol{r}). \tag{6.6}$$

Owing to the unitarity of the Fourier transform, the Wannier functions will obey the same orthogonality relation as the Bloch functions,

$$\int d\boldsymbol{r}\, \boldsymbol{W}^*_{i\sigma}(\boldsymbol{r})\varepsilon_p(\boldsymbol{r})\boldsymbol{W}_{j\sigma'}(\boldsymbol{r}) = \delta_{ij}\,\delta^{\sigma\sigma'}. \tag{6.7}$$

We remark that due to the local nature of the disorder the Wannier functions forms a natural set of basis functions upon which we can act with our disorder Hamiltonian. We also note that the use of the Wannier basis also simplifies matters as we will then assume nearest neighbour hopping of the polarization modes, which is in itself a small restriction since the dispersion relation of a disordered crystal in three dimensions is very close to that of the ordered one, and moreover one expects that the main contribution to transport would come from the $|\vec{k} + \vec{k}'|$ contribution of the Cooperon.

In contrast to the case of the perfectly periodic crystal, we note that the addition of disorder implies one important fact: the Wannier functions now couple different pseudospin indices on different sites, which implies that the hopping process would be able to change the polarization. We can see this if we look at the form of the Hamiltonian of our system. We first note that the form of (6.4) implies that we can write this in the form

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{V} \equiv \nabla \times \nabla \times -\frac{\omega^2}{c^2}\Delta\varepsilon \tag{6.8}$$

where $\boldsymbol{H}_0$ denotes the part of the Hamiltonian containing only the periodic potential ε_p, and $\boldsymbol{V}$ denotes then part which contains the disorder. The crux of the description above is that we will then write the operator equation (6.8) in the Wannier basis which then has the form

$$\mathbf{H} = -t \sum_{\substack{\langle i,j\rangle \\ \sigma}} |i,\sigma\rangle\langle j,\sigma| + \sum_{\substack{i \\ \sigma\sigma'}} V_{ii}^{\sigma\sigma'} |i,\sigma\rangle\langle i,\sigma'|, \tag{6.9}$$

where $|i,\sigma\rangle$ is the Wannier basis vectors in the bracket formalism, with the appropriate indices i,σ; t is the usual hopping parameter, $\langle i, j\rangle$ denotes nearest neighbour hopping, and $V_{ii}^{\sigma\sigma'}$ are matrix elements of the following scattering matrix

$$\mathbf{V}_i = -\frac{\omega^2}{c^2}\Delta\varepsilon_i \begin{pmatrix} \mathcal{V}_1 & \mathcal{T} \\ \mathcal{T}^* & \mathcal{V}_2 \end{pmatrix}, \tag{6.10}$$

with the scattering parameters $\mathcal{V}_i$ and $\mathcal{T}$ are matrix elements of of the Hamiltonian in the Wannier basis corresponding to the spin conserving and nonconserving channels, respectively.

6.3 Analytical Approaches

The particular nature of our model and the accompanying formalism developed to describe it will be detailed in the following sections. These approaches deals with several different aspects of the problem:

1. Taking into account the effects of disorder via the *coherent potential approximation* in which the *self-energy* resulting from multiple scattering of propagating waves off our disorder potential, is seen to modify the wave behavior via the Green's function. The self-energy can then be obtained via CPA.

2. Deriving a transport theory which describes these propagating waves in a random medium,

while taking into account the vector structure of the waves. This step is performed by writing down the quantity which is important for our case, namely the *intensity propagator* $\mathbf{I}(\mathbf{r}, \mathbf{r}'; t, t')$ and from there, obtaining the well-known *Boltzmann equation*.

3. As our waves propagate in a random medium, we expect diffusive behavior to arise from our transport theory calculations. We keep this physical *Ansatz* in mind to derive, applying various standard approximations, a *self-consistent* expression for the diffusion coefficient, more commonly known under the label *Vollhardt-Woelfle theory of localization*.

4. Taking into account the vector nature of propagating waves means that for all our previous mentioned quantities there will be an associated matrix structure in the vectorial (polarization) space. In order to derive physical quantities we need to deal with this matrix structure. For this purpose we apply a mode averaging procedure to our equations, which subsquently simplifies such that they can be diagonalized and physical quantities extracted from them.

6.4 Coherent Potential Approximation

6.4.1 Green's Function Formalism

In order to analyze the transport properties of a disordered crystal, we first define our vectorial Green's functions. Starting from the periodic operator $\mathbf{H}_0$ of Eq. (6.8) (where only the periodic part of the dielectric constant $\varepsilon_p(\mathbf{r})$ is present), the defining equation is given by

$$\left(\frac{\omega^2}{c^2}\bar{\varepsilon} - \nabla \times \nabla \times\right) G_0(\omega, \boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{6.11}$$

where we defined $\bar{\varepsilon} \equiv \frac{1}{V}\int d\mathbf{r}\, \varepsilon_p(\mathbf{r})$ as the the average dielectric constant of the ordered medium. The imaginary part of $G_0(\omega, \mathbf{r}, \mathbf{r}')$ is directly related to the system's density of states [103].

By switching to momentum (or Bloch) basis $|\boldsymbol{k}, \sigma\rangle$, one is able to find an expression for the periodic Green's function

$$\begin{aligned} \mathbf{G}_0(\omega) &= \left(\frac{\omega^2}{c^2}\bar{\varepsilon} - \mathbf{H}_0\right)^{-1} \mathbf{1} = \left(\frac{\omega^2}{c^2}\bar{\varepsilon} - \mathbf{H}_0\right)^{-1} \sum_{\boldsymbol{k},\sigma} \boldsymbol{\varepsilon}_p |\boldsymbol{k}, \sigma\rangle\langle \boldsymbol{k}, \sigma| \boldsymbol{\varepsilon}_p \\ &\approx \sum_{\boldsymbol{k},\sigma} \frac{\boldsymbol{\varepsilon}_p |\boldsymbol{k}, \sigma\rangle\langle \boldsymbol{k}, \sigma| \boldsymbol{\varepsilon}_p}{\omega^2 \bar{\varepsilon}/c^2 - \omega^2_{\boldsymbol{k}\sigma} \bar{\varepsilon}/c^2}, \end{aligned} \tag{6.12}$$

where we used the completeness of the electric Bloch states and their eigenfunction character with respect to $\mathbf{H}_0$. In the last step, we made the approximation $\mathbf{H}_0|\boldsymbol{k}, \sigma\rangle \approx \frac{\omega^2}{c^2}\bar{\varepsilon}$ to get rid of the position dependence of the dielectric function inside the denominator.

From eq. (6.12), the matrix elements of $\mathbf{G}_0$ can be calculated in a straightforward way to be

$$\begin{aligned}(G_0)^{\sigma\sigma'}_{\boldsymbol{k}\boldsymbol{k}'}(\omega) \equiv \langle \boldsymbol{k}, \sigma | \mathbf{G}_0 | \boldsymbol{k}', \sigma' \rangle &= \frac{1}{\omega^2 \bar{\varepsilon}/c^2 - \omega^2_{\boldsymbol{k}\sigma}\bar{\varepsilon}/c^2} \frac{(2\pi)^3}{u}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta^{\sigma\sigma'} \\ &\equiv (G_0)^{\sigma}_{\boldsymbol{k}} \frac{(2\pi)^3}{u}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta^{\sigma\sigma'}.\end{aligned} \tag{6.13}$$

One can see from eqs. (6.12) and (6.13) that the Green's function exhibits poles at the positions of the eigenvalues. This behavior can be used to extract the density of states from the Green's function. The density of states $N(\omega)$ for the photonic crystal with a continuous eigenvalue spectrum $\omega_{\boldsymbol{k}\sigma}$ is defined as

$$N_\sigma(\omega) = \frac{u}{(2\pi)^3} \int d\boldsymbol{k}\, \delta(\omega - \omega_{\boldsymbol{k}\sigma}), \tag{6.14}$$

so that the number of states in a frequency interval $[\omega, \omega + d\omega]$ is given by $N(\omega)\, d\omega$.

Consider now the $\boldsymbol{k}$-summation over $(G_0)_{\boldsymbol{k}}$ from eq. (6.12), and neglect the pseudospin σ without loss of generality. One can transform the integral over the pole by introducing a small imaginary part $i\eta$ in the denominator and then using the identity

$$\lim_{\eta\to 0} \frac{1}{\omega^2 - \omega^2_{\boldsymbol{k}\sigma} + i\eta} = \mathcal{P}\frac{1}{\omega^2 - \omega^2_{\boldsymbol{k}\sigma}} - i\pi\, \delta(\omega^2 - \omega^2_{\boldsymbol{k}\sigma}), \tag{6.15}$$

where $\mathcal{P}$ denotes Cauchy's principal value. One obtains

$$\begin{aligned}\int d\boldsymbol{k}\, (G_0)^{\sigma}_{\boldsymbol{k}} &= \lim_{\eta\to 0} \int d\boldsymbol{k}\, \frac{c^2/\bar{\varepsilon}}{\omega^2 - \omega^2_{\boldsymbol{k}\sigma} + i\eta} \\ &= \mathcal{P}\int d\boldsymbol{k}\, \frac{c^2/\bar{\varepsilon}}{\omega^2 - \omega^2_{\boldsymbol{k}\sigma}} - i\pi \frac{c^2}{\bar{\varepsilon}} \int d\boldsymbol{k}\, \delta(\omega^2 - \omega^2_{\boldsymbol{k}\sigma}).\end{aligned} \tag{6.16}$$

If one now transforms the delta function as $\delta(\omega^2 - \omega^2_{\boldsymbol{k}\sigma}) = \delta(\omega - \omega_{\boldsymbol{k}\sigma})/2\omega$ (excluding negative frequencies), one can relate the density of states from eq. (6.14) to the imaginary part of the Green's function by

$$N^\sigma(\omega) = -\frac{2\omega\bar{\varepsilon}}{\pi c^2} \frac{u}{(2\pi)^3} \int d\boldsymbol{k}\, \mathrm{Im}\, G^{\sigma\sigma}_{\boldsymbol{k}}(\omega). \tag{6.17}$$

The additional ω-factor is a consequence of the ω^2-dependence of our equations.

6.4.2 Configurational Averaging

It was shown in sec. 6.2 that for the disordered crystal it is possible to separate the Hamiltonian in the form $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$. When the eigenvalues for the periodic problem $\mathbf{H}_0$ are known, Green's functions provide a method to calculate the disordered system properties from the ordered ones. Using the Green's function $\mathbf{G}_0$ of the periodic problem $\mathbf{H}_0$, i.e. $(z - \mathbf{H}_0)\mathbf{G}_0 = \mathbf{1}$, then using

eq. (6.1) we can easily solve for **G**,

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\mathbf{T}\mathbf{G}_0, \tag{6.18}$$

where we introduced the scattering matrix **T** as

$$\begin{aligned}\mathbf{T} &= \mathbf{V} + \mathbf{V}\mathbf{G}_0\mathbf{V} + \mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V} + \dots \\ &= \mathbf{V}[\mathbf{1} - \mathbf{G}_0\mathbf{V}]^{-1}.\end{aligned} \tag{6.19}$$

From the above equation it can be seen that the scattering matrix describes all possible sequences of wave scatterings and propagations inside the crystal. Being functions of the disorder potential **V**, the scattering matrix **T** and the Green's functions **G** are hence random quantities; as has been done in the previous chapters, the usual procedure in this case is to perform a *configurational* average over **V**, or concretely over possible realizations of the disorder potential, which are however identical from a macroscopic point of view (see Fig. 6.1). Taking the Green's function for the disordered system **G**, which will originally depend on the whole set of $\{\varepsilon_i\}$, its configurationally averaged counterpart is given by

$$\langle \mathbf{G} \rangle_c \equiv \int d\varepsilon_1 \int d\varepsilon_2 \cdots \int d\varepsilon_N \, P(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N)\, \mathbf{G}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N), \tag{6.20}$$

where the probability function $P(\varepsilon_1, \dots, \varepsilon_N)$ describes the distribution of dielectric values on every site. We will be interested in the special case of independently and identically distributed ε-values, i.e. we neglect any correlation between the sites. The probability then factorizes into the product of single site probabilities,

$$P(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N) = \prod_i^N P(\varepsilon_i). \tag{6.21}$$

If one applies configurational averaging to eq. (6.18), one obtains the **Dyson equation**

$$\langle \mathbf{G} \rangle = \mathbf{G}_0 + \mathbf{G}_0 \langle \mathbf{V}\mathbf{G} \rangle \equiv \mathbf{G}_0 + \mathbf{G}_0 \mathbf{\Sigma} \langle \mathbf{G}. \tag{6.22}$$

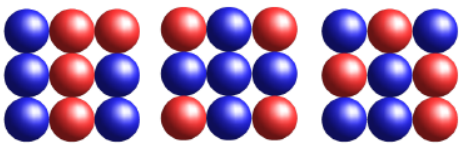


Figure 6.1: Three different possible disorder configurations with identical probability distributions of scatterer types.

With the introduction of the self-energy $\mathbf{\Sigma}$, there exists a useful relation between the periodic and disordered Green's functions and the self-energy which can be derived from eq. (6.22):

$$\langle \mathbf{G} \rangle^{-1} = \mathbf{G}_0^{-1} - \mathbf{\Sigma}. \tag{6.23}$$

6.4.3 The Effective Potential

From (6.23) we see that to obtain the configurationally averaged Green's function $\langle \mathbf{G} \rangle^{-1}$ one simply needs to compute the self energy $\mathbf{\Sigma}$. A method to calculate the self-energy $\mathbf{\Sigma}$ in disordered systems is given by the *coherent potential approximation* (CPA), which is based on the concepts of an effective medium and the *single site approximation*. The effective medium approach assumes that wave propagation in an disordered medium can be locally (i.e., within the coherence length, fig. 6.2) described, as if it propagates in an *ordered*, so-called *effective* medium. This effective medium is derived from the periodic medium by a shift by a *coherent potential* $\Sigma_i(\omega)$. We refer to [101], [104] for more in-depth expositions of these concepts. Here we will only state the expressions appropriate for our problem.

The Hamiltonian of the effective medium is then $\mathbf{H}_e = \mathbf{H}_0 + \mathbf{\Sigma}$ and its Green's function can be determined from the ordered one by

$$\mathbf{G}_\mathrm{e} = \mathbf{G}_0 + \mathbf{G}_0 \mathbf{\Sigma}(\omega) \mathbf{G}_\mathrm{e} = \left[\mathbf{G}_0^{-1} - \mathbf{\Sigma}(\omega) \right]^{-1}. \tag{6.24}$$

The disorder potential $\mathbf{V}$ of the disordered crystal is now expressed relative to this effective medium, $\bar{\mathbf{V}} \equiv \mathbf{V} - \mathbf{\Sigma}$. The right choice of $\mathbf{\Sigma}$ is determined such that the configurationally averaged scattering matrix relative to the effective medium $\bar{\mathbf{T}}$ vanishes (CPA condition):

$$\langle \bar{\mathbf{T}}(\mathbf{\Sigma}) \rangle \overset{!}{=} 0. \tag{6.25}$$

If this is achieved, performing the average on eq. (6.18) yields

$$\begin{aligned} \langle \mathbf{G} \rangle &= \langle \mathbf{G}_\mathrm{e} \rangle + \langle \mathbf{G}_\mathrm{e} \bar{\mathbf{T}} \mathbf{G}_\mathrm{e} \rangle \\ &= \mathbf{G}_\mathrm{e} + \mathbf{G}_\mathrm{e} \langle \bar{\mathbf{T}} \rangle \mathbf{G}_\mathrm{e} \\ &= \mathbf{G}_\mathrm{e}. \end{aligned} \tag{6.26}$$

Here, we exploited that the effective Green's function $\mathbf{G}_\mathrm{e}$ itself describes an periodic medium and therefore is not affected by configuration averaging. One can see that with condition (6.25) the

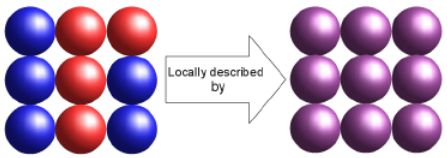


Figure 6.2: Effective medium: Within the wave's coherence length, the disorded photonic crystal can be described by a periodic crystal.

configurationally averaged full Green's function is equivalent to the effective Green's function and the coherent potential is identical to the self-energy. Eventually, this procedure placed the focus on finding an expression for the total scattering matrix $\bar{\mathbf{T}}$, so that from eq. (6.25) a value for the self-energy $\boldsymbol{\Sigma}$ can be obtained. Expanding the total scattering matrix in terms of the single site scattering $\mathbf{t}_i$, which comprise all successive scatterings at a single site i, and performing the configurational averaging, one obtains the CPA equation for the self-energy $\boldsymbol{\Sigma}$,

$$\langle \bar{\mathbf{t}} \rangle_\mathrm{c} = \int d\varepsilon_i \, P(\varepsilon_i) \, (\mathbf{V}_i(\varepsilon_i) - \boldsymbol{\Sigma}) \Big[\mathbf{1} - \mathbf{G}_\mathrm{e}^\mathrm{D}(\boldsymbol{\Sigma}) \, (\mathbf{V}_i(\varepsilon_i) - \boldsymbol{\Sigma}) \Big]^{-1} \stackrel{!}{=} 0. \tag{6.27a}$$

or, more conviniently for numerical work,

$$\boldsymbol{\Sigma} \stackrel{!}{=} \int d\varepsilon_i \, P(\varepsilon_i) \, \mathbf{V}_i(\varepsilon_i) \Big[\mathbf{1} - \mathbf{G}_\mathrm{e}^\mathrm{D}(\boldsymbol{\Sigma}) \, (\mathbf{V}_i(\varepsilon_i) - \boldsymbol{\Sigma}) \Big]^{-1}. \tag{6.27b}$$

If one includes (pseudo)spin into the CPA, the disorder potential $\mathbf{V}_i$ has a 2×2 matrix structure, which will be adopted by the self-energy and by the effective Green's function eventually. To be more specific, using the disorder potential from eq. (6.10) the expression $\mathbf{V}_i - \boldsymbol{\Sigma}$ becomes

$$\mathbf{V}_i - \boldsymbol{\Sigma} = -\frac{\omega^2}{c^2} \Delta\varepsilon_i \begin{pmatrix} \mathcal{V}_1 & \mathcal{T} \\ \mathcal{T}^* & \mathcal{V}_2 \end{pmatrix} - \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix}. \tag{6.28}$$

The diagonal part of the effective Green's function has to be rewritten as

$$\begin{aligned} \mathbf{G}_\mathrm{e}^\mathrm{D}(\omega, \boldsymbol{\Sigma}) &\equiv \frac{u^2}{(2\pi)^6} \int d\boldsymbol{k} \int d\boldsymbol{k}' e^{i\boldsymbol{k}\boldsymbol{R}_i} \, (\mathbf{G}_\mathrm{e})_{\boldsymbol{k}\boldsymbol{k}'}(\omega, \boldsymbol{\Sigma}) \, e^{-i\boldsymbol{k}'\boldsymbol{R}_i} \\ &= \frac{u}{(2\pi)^3} \int d\boldsymbol{k} \int d\boldsymbol{k}' e^{i\boldsymbol{k}\boldsymbol{R}_i} \, (\mathbf{G}_\mathrm{e})_{\boldsymbol{k}}(\omega, \boldsymbol{\Sigma}) \, \delta(\boldsymbol{k} - \boldsymbol{k}') \, e^{-i\boldsymbol{k}'\boldsymbol{R}_i} \\ &= \frac{u}{(2\pi)^3} \int d\boldsymbol{k} \, \Big[(\mathbf{G}_0)_{\boldsymbol{k}}^{-1} - \boldsymbol{\Sigma}(\omega) \Big]^{-1} \\ &= \frac{u}{(2\pi)^3} \int d\boldsymbol{k} \left[\frac{\bar{\varepsilon}}{c^2} \begin{pmatrix} \omega^2 - \omega_{\boldsymbol{k}1}^2 & 0 \\ 0 & \omega^2 - \omega_{\boldsymbol{k}2}^2 \end{pmatrix} - \begin{pmatrix} \Sigma^{11}(\omega) & \Sigma^{12}(\omega) \\ \Sigma^{21}(\omega) & \Sigma^{22}(\omega) \end{pmatrix} \right]^{-1}, \end{aligned} \tag{6.29}$$

where eqs. (6.13) and (6.24) have been used. From the local effective Green's function one can calculate the density of states via

$$\mathbf{N}(\omega) = -\frac{2\omega\bar{\varepsilon}}{\pi c^2} \operatorname{Im} \mathbf{G}_\mathrm{e}^\mathrm{D}(\omega), \tag{6.30}$$

also showing a 2×2-matrix structure.

6.5 Transport Theory

We are concerned with intensity transport of light in disordered photonic crystals. Such a quantity can be obtained via the two-point correlation function, which in our system is a 4th-rank tensor

given by

$$\mathbf{I}(\boldsymbol{r},\boldsymbol{r}';t,t') \equiv \langle \mathbf{G}(\boldsymbol{r},\boldsymbol{r}',t)\otimes\mathbf{G}^{*}(\boldsymbol{r},\boldsymbol{r}',t)\rangle_{\mathrm{c}}, \tag{6.31}$$

where $\mathbf{G}(\boldsymbol{r},\boldsymbol{r}',t)$ is a 2×2 matrix propagator and $\langle\dots\rangle_{\mathrm{c}}$ indicates mode averaging (see next section) in addition to the usual configurational average. We show in the Appendix C.1 that the quantity (6.31) can be rewritten to reflect, in a diagrammatical manner, the infinite sequence of wave propagation and scattering describing the dynamics of wave motion in a random medium, and this dynamics can be correctly extracted in a diagrammatic fashion from the Bethe-Salpeter (BS) equation in $\boldsymbol{K}$ and frequency Ω space

$$\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K}) = \Big((\mathbf{G}_{\mathrm{e}})^{+}_{\boldsymbol{k}}\otimes(\mathbf{G}_{\mathrm{e}})^{-}_{\boldsymbol{k}}\Big)\left[\mathbf{1}\otimes\mathbf{1} + \frac{u}{(2\pi)^3}\int d\boldsymbol{k}''\boldsymbol{\gamma}_{\boldsymbol{k}\boldsymbol{k}''}\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}''}(\Omega,\boldsymbol{K})\right]. \tag{6.32}$$

where $\boldsymbol{\Phi}_{\boldsymbol{k},\boldsymbol{k}'} \equiv \langle \mathbf{G}^{+}_{\boldsymbol{k}}\otimes\mathbf{G}^{-}_{\boldsymbol{k}'}\rangle_{\mathrm{c}}$ and $\boldsymbol{\Phi}_{\boldsymbol{k}} \equiv \frac{u}{(2\pi)^3}\int d\boldsymbol{k}'\,\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}\boldsymbol{k}'}$. However, in the form (6.32) the BS equation is physically not very relevant, since it is difficult to interpret what physical quantities are described by $\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$. It is however, possible to derive from (6.32) a much more well-known quantity which directly provides us with information about the nature of wave transport in our medium, namely the *diffusion coefficient* $D(\omega)$. The framework which allows the computation of $D(\omega)$ from $\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$ is commonly known as the *self-consistent theory*, since the required quantity $D(\omega)$ can be obtained from solving a self-consistent integral equation. We sketch below the steps involved in the derivation of $D(\omega)$ from $\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$. An illustrative rewriting of the BS equation can be performed by use of the following matrix identity (only valid for commuting matrices) to rewrite the tensor product of the effective Green's functions in terms of single particle quantities:

$$\mathbf{G}^{+}\otimes\mathbf{G}^{-} = \underbrace{\Big[\mathbf{1}\otimes(\mathbf{G}^{-})^{-1} - (\mathbf{G}^{+})^{-1}\otimes\mathbf{1}\Big]^{-1}}_{\equiv\left[-\boldsymbol{\Lambda}(\mathbf{G}^{-1})\right]^{-1}}\underbrace{\big[(\mathbf{G}^{+})\otimes\mathbf{1} - \mathbf{1}\otimes(\mathbf{G}^{-})\big]}_{\equiv\boldsymbol{\Delta}\mathbf{G}}. \tag{6.33}$$

Using this identity and bringing its first factor to the left hand side, the BS equation can be rewritten as

$$-\boldsymbol{\Delta}\Big(\mathbf{G}^{-1}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})\Big)\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K}) = \boldsymbol{\Delta}\mathbf{G}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})\left[\mathbf{1}\otimes\mathbf{1} + \frac{u}{(2\pi)^3}\int d\boldsymbol{k}''\boldsymbol{\gamma}^{\omega}_{\boldsymbol{k}\boldsymbol{k}''}(\Omega,\boldsymbol{K})\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}''}(\Omega,\boldsymbol{K})\right]. \tag{6.34}$$

Due to the similar structure with the Boltzmann equation known from kinetic theory, the above equation is often called a *generalized Boltzmann equation*. It governs the transport of the intensity tensor component $\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$. The BS equation is solved in the usual way of expanding the intensity tensor component $\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$ up to first order in the current vertex (otherwise known as the moment expansion) [41, 105], from which we obtain the important transport equations: the continuity equation,

$$-i\Omega\mathbf{S}^{\omega}(\Omega,\boldsymbol{K}) - i|\boldsymbol{K}|\mathbf{J}^{\omega}(\Omega,\boldsymbol{K}) = \mathbf{j}(\Omega,\boldsymbol{K}), \tag{6.35}$$

governing the relationship between the energy density $\mathbf{S}^{\omega}(\Omega,\boldsymbol{K}) = \frac{u}{(2\pi)^3}\int d\boldsymbol{k}\,\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$ and the energy density flux $\mathbf{J}^{\omega}(\Omega,\boldsymbol{K}) \equiv \frac{u}{(2\pi)^3}\int d\boldsymbol{k}\,(\hat{\boldsymbol{K}}\cdot\boldsymbol{v}_{\boldsymbol{k}})\boldsymbol{\Phi}^{\omega}_{\boldsymbol{k}}(\Omega,\boldsymbol{K})$, where $\mathbf{j}(\Omega,\boldsymbol{K})$ denotes a possible source term. The derivation of (6.35) required the use of a Ward identity for classical waves [42],

which has a nontrivial right-hand side due to the fact that the conserved quantity for classical waves is energy instead of particle number. The form of this equation is detailed in the subsection below.

The other required equation is the diffusion equation, or Fick's law, commonly written in the form

$$-i|\boldsymbol{K}|\mathbf{D}(\omega)\mathbf{S}^{\omega}(\Omega,\boldsymbol{K})+\mathbf{J}^{\omega}(\Omega,\boldsymbol{K})=0. \tag{6.36}$$

Here, $\mathbf{D}(\omega)$ denotes a quantity which can be interpreted as the diffusion constant. Taking the hydrodynamic limit ($\Omega, \boldsymbol{K} \to 0$), Eq. (6.35) and (6.36) can be solved for the quantity $\mathbf{S}^{\omega}(\Omega,\boldsymbol{K})$, which in turn yields $\mathbf{D}(\omega)$ in the form

$$\mathbf{D}(\omega)=[\mathbf{1}\otimes\mathbf{1}+\mathbf{M}(\omega)]^{-1}\mathbf{D}_{\mathrm{L}}\equiv\frac{\mathbf{D}_{\mathrm{L}}}{\mathbf{1}\otimes\mathbf{1}+\mathbf{M}(\omega)}, \tag{6.37}$$

where $\mathbf{D}_{\mathrm{L}}$ is the so-called *ladder diffusion constant* originating from incoherent transport

$$\begin{aligned}\mathbf{D}_{\mathrm{L}}&=\frac{i\tilde{D}}{2\omega}\left[\mathbf{1}\otimes\mathbf{1}+\boldsymbol{\delta}\right]^{-1}\\&=\frac{i}{4\omega}\frac{\int_{\boldsymbol{k}}(\hat{\boldsymbol{K}}\cdot\boldsymbol{v}_{k})^{2}\Delta G_{\boldsymbol{k}}^{2}(0,0)}{\int_{\boldsymbol{k}''}\Delta G_{\boldsymbol{k}''}(0,0)}\left[\mathbf{1}\otimes\mathbf{1}+\boldsymbol{\delta}\right]^{-1}.\end{aligned} \tag{6.38}$$

and

$$\boldsymbol{\delta}\equiv-\frac{1}{2\omega}\frac{\partial}{\partial\Omega}\mathbf{f}(\Omega)\Big|_{\Omega=0}\left[\Pi\Sigma(0)\,\mathbf{1}\otimes\mathbf{1}+\Pi G_{0}\boldsymbol{\gamma}_{\mathrm{L}}(0)\right]. \tag{6.39}$$

is a renormalization factor stemming from the nonzero right-hand side of the Ward identity.

$\mathbf{M}(\omega)$ is called the *relaxation kernel* and has the form

$$\mathbf{M}(\omega)\propto\int d\boldsymbol{k}\int d\boldsymbol{k}'\,(\hat{\boldsymbol{K}}\cdot\boldsymbol{v}_{\boldsymbol{k}})\,\mathrm{Im}\,G_{\boldsymbol{k}}\,\boldsymbol{\gamma}^{\omega}_{\boldsymbol{k}\boldsymbol{k}'}(\mathrm{Im}\,G_{\boldsymbol{k}'})^{2}\,(\hat{\boldsymbol{K}}\cdot\boldsymbol{v}_{\boldsymbol{k}'}). \tag{6.40}$$

From the form of (6.40) we can see that the presence of the relaxation kernel $M(\omega)$ renormalizes the quantity D_L. Physically, D_L should be interpreted as the "bare" diffusion constant, i.e., the quantity which can be calculated by using classical expressions, e.g., the Einstein relation. In other words, D_L does not take into account coherent effects which are important in the studies of wave localization.

The relaxation kernel $M(\omega)$ represents the correction to D_L. As can be seen from the explicit expression (6.40) $M(\omega)$ is strongly influenced by the *irreducible vertex* $\boldsymbol{\gamma}^{\omega}_{\mathbf{k}\mathbf{k}'}$. In priciple, $\boldsymbol{\gamma}^{\omega}_{\mathbf{k}\mathbf{k}'}$ incorporates *all possible* wave scattering events since it is diagrammatically an infinite sum of terms, each corresponding to a particular configuration of such events. Hence it is impossible to obtain in its entirety. However, by chooses particular terms which are physically relevant, the integral in (6.40) can be solved (at least numerically) and we can then obtain the values for (6.37) corresponding to these terms. In the studies of wave localization account is taken usually of two particular choices for $\gamma^{\omega}_{\mathbf{k}\mathbf{k}'}$: the so-called *ladder* $\boldsymbol{\gamma}^{L}$ and *crossed* $\boldsymbol{\gamma}^{C}_{\mathbf{k}\mathbf{k}'}$ diagrams. We refer to Appendix C.3 for further information on these quantities, but we point out that the ladder vertex $\boldsymbol{\gamma}_L$ is independent of momenta $\mathbf{k}$, $\mathbf{k}'$; i.e., it is homogeneous in momentum space. In addition,

we also refer to Appendix C.5 for the computational details regarding the derivation of (6.38) and (6.40).

From presentation of the formalism till now we see that we work with quantities that possess degrees of freedom in our "pseudospin" space which manifests via the 2×2 matrix structure of the Green's function and self-energy, for example. However, as seen in the next section, $\boldsymbol{\gamma}^{\omega}_{\mathbf{k}\mathbf{k}'}$ has the form of a tensor product of two 2×2 quantities, thus making it a 4×4 matrix, thus making the solution of (6.37), even in the usual approximations a particularly difficult task. In the next section we will describe our method for reducing the dimensionality of these quantities.

6.5.1 Self-Consistent Theory of Localization

It is well known from standard sources [41, 105] that even taking into account both the ladder $\boldsymbol{\gamma}^{L}$ and crossed vertices $\boldsymbol{\gamma}^{\mathrm{C}}_{\mathbf{k}\mathbf{k}'}$ is only an adequate choice if we work in the weak scattering limit, i.e., the limit when the mean free path l between two scatterings is much larger than the wavelength λ. Only then configurational averaging will cancel all interference contributions except those coming from coherent backscattering.

However, for the opposite case, the strong scattering limit ($l \ll \lambda$), many other interference contribution are possible, since two waves will not accumulate phase differences that quickly and hence are more resistant to averaging. Analytical approaches are unable to access this regime of scattering strength and numerical methods are needed. The most well-known of these is the so-called *self-consistent theory* of Vollhardt and Woelfle [40]. A short description will be given below.

The authors of [40] extended the crossed vertex by all diagrams which are double crossed with maximally crossed diagrams (figure 6.3). It can be shown [41] that this new vertex $\boldsymbol{\gamma}^{\mathrm{SC}}$ is given by a similar expression as the crossed vertex, only now the Diffusion constant D_{L} has to be replaced with the full diffusion constant D,

$$\boldsymbol{\gamma}^{\mathrm{SC}}_{kk'} = \boldsymbol{\gamma}_{\mathrm{L}}\, \mathbf{S}(\Omega, \boldsymbol{k} + \boldsymbol{k}')\, \boldsymbol{\gamma}_{\mathrm{L}} \propto \frac{1}{-i\Omega + D|\boldsymbol{k} + \boldsymbol{k}'|^2}. \tag{6.41}$$

If this is inserted into the relaxation kernel $\mathbf{M}$, one obtains a self-consistent equation for the Diffusion constant D via eq. (6.37), hence the name *self-consistent theory of localization*.

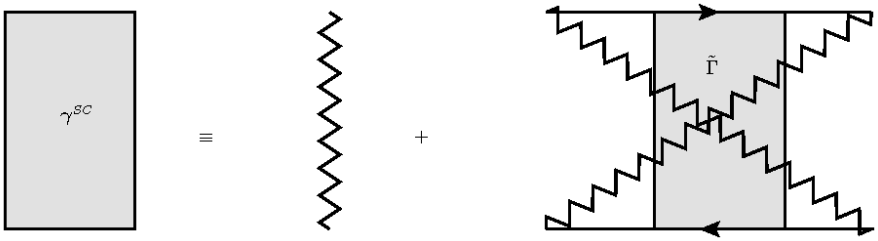


Figure 6.3: Constituent parts of the self-consistent irreducible vertex. $\tilde{\mathbf{\Gamma}}$ contains *all* diagrams, which are not entirely crossed by an interaction line.

6.6 Disorder and Mode Averaging Procedure

In our model, we will treat all polarization modes equally with respect to their scattering strength and their dispersion inside the crystal. That is why it is reasonable to expect that we can average over the modes without losing important information. In order to define *mode averaging* $\langle \dots \rangle_\sigma$, we use the fact that we can expand any hermitian 2×2 matrix in a basis made of the unit matrix $\mathbf{1}$ and the Pauli matrices $\boldsymbol{\sigma}_i$, which are defined by

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.42}$$

With the help of the Pauli matrices, we can expand a 2×2 matrix $\mathbf{A}$, which describes single particle effects, as follows,

$$\mathbf{A} = A_0\,\mathbf{1} + A_1\boldsymbol{\sigma}_1 + A_2\boldsymbol{\sigma}_2 + A_3\boldsymbol{\sigma}_3 \equiv A_0\mathbf{1} + \boldsymbol{A}\cdot\vec{\boldsymbol{\sigma}}. \tag{6.43}$$

Here, $\vec{\boldsymbol{\sigma}}$ denotes a three-dimensional vector containing the Pauli matrices as its entries and $\boldsymbol{A} = (A_1, A_2, A_3)^{\mathrm{T}}$. The term $\boldsymbol{A}\cdot\vec{\boldsymbol{\sigma}}$ contains the information about the anisotropy in mode space and will be dropped after mode averaging:

$$\langle \mathbf{A} \rangle_\sigma = \langle A_0\,\mathbf{1} + \boldsymbol{A}\cdot\vec{\boldsymbol{\sigma}} \rangle_\sigma \equiv A_0\,\mathbf{1} = \frac{1}{2}\mathrm{tr}(\mathbf{A})\,\mathbf{1} \tag{6.44}$$

As one can see, the obtained result is just the averge of the matrix' diagonal elements multiplied by the unit matrix. This implies that, in principle, it can be treated as a scalar quantity. We apply the same reasoning to all our equations, with the result that we obtain scalar equations for most of the quantities, with the exception of the irreducible scattering vertex tensors, which we need to treat separately. These are not reducible to scalar equations as they couple different pseudospin modes. However, due to the fact that they are concrete representations of the $SU(2)$ algebra which govern the symmetries of electronic spins, they can be diagonalized in the same eigenbases which are used in the study of spin-orbit scattering in disordered semiconductor systems [96, 97], namely the singlet and triplet bases.

6.7 Singlet/Triplet Structure of the Interaction Vertex

We recap the central logic of our ideas up to now: that the polarization of light, which we have mapped onto a pseudospin model resulting in a 2×2-matrix structure of the effective Green's function $\mathbf{G}_\mathrm{e}$ and the self-energy $\mathbf{\Sigma}$ in polarization mode space, can be shown to affect the transport properties of light in random media. Quantitatively, this effect manifests itself in the different values of the diffusion coefficient in different "pseudospin" channels. In this section we show how to decouple the previously obtained matrix quantities such that the correct matrix form of the diffusion coefficient in the different channels can be obtained. We recall that the equations describing intensity transport were then derived by considering the tensor product of two Green's functions, $\mathbf{G} \otimes \mathbf{G}^*$. Thus, the resulting equations and quantities will have a 4×4 matrix structure in polarization mode space. Performing the tensor product gives us matrices in the product space $|\sigma\tau\rangle$ of two modes, with a standard choice of basis vectors $|11\rangle, |12\rangle, |21\rangle, |22\rangle$. We will then diagonalize these 4×4 matrix quantities by employing the appropriate choice of eigenbases.

6.7.1 Vertex Diagonalization

The diagonalization of interaction vertices or other two-particle quantities with a 4×4 matrix structure has to be done in consistency with the mode averaging. For that, we consider an interaction vertex $\boldsymbol{\gamma}$ which is built from two scattering potential matrices $\boldsymbol{v}$, i.e., $\boldsymbol{\gamma} = \boldsymbol{v} \otimes \boldsymbol{v}^*$. The mode averaged vertex $\langle \boldsymbol{\gamma} \rangle_\sigma$ is then given by

$$\begin{aligned} \langle \boldsymbol{\gamma} \rangle_\sigma &= \langle \mathbf{v} \otimes \mathbf{v} \rangle_\sigma = \langle (v_0 \, \mathbf{1} + \boldsymbol{v} \cdot \vec{\boldsymbol{\sigma}}) \otimes (v_0^* \, \mathbf{1} + \boldsymbol{v}^* \cdot \vec{\boldsymbol{\sigma}}^*) \rangle_\sigma \\ &= |v_0|^2 \mathbf{1} \otimes \mathbf{1} + |\boldsymbol{v}|^2 \vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}}^* \end{aligned} \tag{6.45}$$

Obviously, elements proportional to $\boldsymbol{\sigma}_i \otimes \boldsymbol{\sigma}_j^*$, $i \neq j$, which represent some type of mode anisotropy, have been dropped by this procedure. This is not the case for terms proportional to $\boldsymbol{\sigma}_i \otimes \boldsymbol{\sigma}_i^*$, which are mode-isotropic, as it is indicated by the Pauli matrix identity $(\boldsymbol{\sigma}_i)^2 = \mathbf{1}$. As it turns out, the pseudospin structure in eq. (6.45) is identical to the spin structure in the problem of spin-orbit coupling in disordered electronic systems [19]. We can take advantage of this correspondence and switch to the more suitable *singlet/triplet basis*, in which the mode averaged quantites become diagonal. This will now be done for the ladder vertex $\boldsymbol{\gamma}^\mathrm{L}$ and the crossed vertex $\boldsymbol{\gamma}^\mathrm{C}_{\mathbf{k}\mathbf{k}'}$.

Ladder Vertex

We start with the mode-averaged ladder vertex γ^{L} in the product basis and then change to its eigenbasis:

$$\begin{aligned}\langle \boldsymbol{\gamma}^{\mathrm{L}} \rangle_\sigma &= \gamma_{\mathrm{L}}^0 \, \mathbf{1} \otimes \mathbf{1} + \gamma_{\mathrm{L}}^m \, \vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}}^* \\ &= \gamma_{\mathrm{L}}^0 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \gamma_{\mathrm{L}}^m \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_{\mathrm{L}}^S & 0 & 0 & 0 \\ 0 & \gamma_{\mathrm{L}}^T & 0 & 0 \\ 0 & 0 & \gamma_{\mathrm{L}}^T & 0 \\ 0 & 0 & 0 & \gamma_{\mathrm{L}}^T \end{pmatrix}_{\mathrm{L}},\end{aligned} \tag{6.46}$$

where the subscript "L" on the matrix denotes the eigenbasis of the ladder vertex given by

$$\text{singlet:} \qquad |S_{\mathrm{L}}\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle), \qquad \text{with} \quad \gamma_{\mathrm{L}}^S = \gamma_{\mathrm{L}}^0 + 3\gamma_{\mathrm{L}}^m \tag{6.47}$$

$$\text{triplet:} \qquad |T_{\mathrm{L}}\rangle = \begin{cases} |12\rangle \\ |21\rangle \\ \frac{1}{\sqrt{2}}(|22\rangle - |11\rangle) \end{cases}, \qquad \text{with} \quad \gamma_{\mathrm{L}}^T = \gamma_{\mathrm{L}}^0 - \gamma_{\mathrm{L}}^m. \tag{6.48}$$

The singlet eigenvalue $\boldsymbol{\gamma}_{\mathrm{L}}^S$ always describes the so-called *Goldstone mode* of the system and accounts for energy conservation. The effects of polarization mode flipping will be contained in the triplet eigenvalue γ_{L}^T instead, whose triple degeneracy is a consequence of the isotropy of the configurationally averaged crystal.

Cooperon Vertex

In sec. 6.5.1, it has been shown that the cooperon vertex $\boldsymbol{\gamma}_{\mathrm{C}}$ can be expressed in terms of ladder vertices after performing time-reversal. During this transformation, however, the mode indices of the lower channel line have also been flipped. As a result, the eigenbasis of this vertex is a different one as in the ladder case. More precisely, exchanging the mode indices $\tau \leftrightarrow \tau'$ in figure C.4 affects the Pauli vector product as follows:

$$(\vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}}^*)^{\sigma\sigma'\tau\tau'} \xrightarrow{\text{twist}} (\vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}}^*)^{\sigma\sigma'\tau'\tau} = (\vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}})^{\sigma\sigma'\tau\tau'}.$$

With this slight modification, the mode structure of the cooperon vertex reads

$$\langle \boldsymbol{\gamma}^{\mathrm{C}} \rangle_{\sigma} = \gamma_{\mathrm{C}}^{0}\, \mathbf{1} \otimes \mathbf{1} + \gamma_{\mathrm{C}}^{m}\, \vec{\boldsymbol{\sigma}} \otimes \vec{\boldsymbol{\sigma}} \tag{6.49}$$

$$= \gamma_{\mathrm{C}}^{0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \gamma_{\mathrm{C}}^{m} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.50}$$

$$= \begin{pmatrix} \gamma_{\mathrm{C}}^{S} & 0 & 0 & 0 \\ 0 & \gamma_{\mathrm{C}}^{T} & 0 & 0 \\ 0 & 0 & \gamma_{\mathrm{C}}^{T} & 0 \\ 0 & 0 & 0 & \gamma_{\mathrm{C}}^{T} \end{pmatrix}_{\mathrm{C}}. \tag{6.51}$$

Here, the subscript "C" denotes the eigenbasis of the cooperon vertex given by

$$\text{singlet:} \qquad |S_{\mathrm{C}}\rangle = \frac{1}{\sqrt{2}}(|12\rangle - |21\rangle), \qquad \text{with} \quad \gamma_{\mathrm{C}}^{S} = \gamma_{\mathrm{C}}^{0} - 3\gamma_{\mathrm{C}}^{m} \tag{6.52}$$

$$\text{triplet:} \qquad |T_{\mathrm{C}}\rangle = \begin{cases} |11\rangle \\ |22\rangle \\ \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) \end{cases}, \qquad \text{with} \quad \gamma_{\mathrm{C}}^{T} = \gamma_{\mathrm{C}}^{0} + \gamma_{\mathrm{C}}^{m}. \tag{6.53}$$

As we will see in the next section, the possibility to diagonalize our two-particle quantities will considerably simplify the solution of the Bethe-Salpeter equation.

6.8 Numerical Results

In this section we shall present various numerical results of expressions previously described. In section 6.8.1 we present waterfall plots of the density of modes as calculated using the coherent potential approximation described in Sec. 6.4 via the expression

$$N(\omega) = -\frac{2\omega}{\pi}\mathrm{Im}G \tag{6.54}$$

where the Green's function refers to the expression given in (6.29). In Sec. 6.8.2 we then show the results for the central quantity in our work: the diffusion constant D. Results are given for D in three different channels: onefrom the singlet, and two corresponding to different eigenvalues from the triplet channel. These are denoted as D^S, D^{T0} and D^{T1}, respectively.

6.8.1 Density of Modes

$\mathcal{T} = 0$

The density of modes, obtained from (6.30) via the CPA, is shown in Fig. 6.4 in the form of a waterfall plot. The DOMs at different values of the disorder strength, here represented by the concentration of one of the sphere types, say p_A, as a fraction of the total scatterer concentration at fixed values of the dielectric constants of the A and B scatterers (ε_A and ε_B, respectively), are plotted against the light frequency ω. Hence in this plot a fraction of $p_A = 0.5$ represents the one with the largest disorder strength. Most importantly, in Fig. 6.4 we set the pseudospin flipping parameter $\mathcal{T} = 0$. This corresponds in our model to purely "potential scattering" of the light fields, able to alter the intensity but not polarization degrees of freedom. We first note that for the ordered case in which $p_A = 1.0$ or $p_A = 0.0$ the DOM shows clearly visible van Hove singularities which are to be expected for ordered systems. On the other hand, introduction of disorder leads to broadening of the DOM until a dip appears at a certain value of ω. This dip eventually develops into a full gap, in the range of which light fields do not propagate.

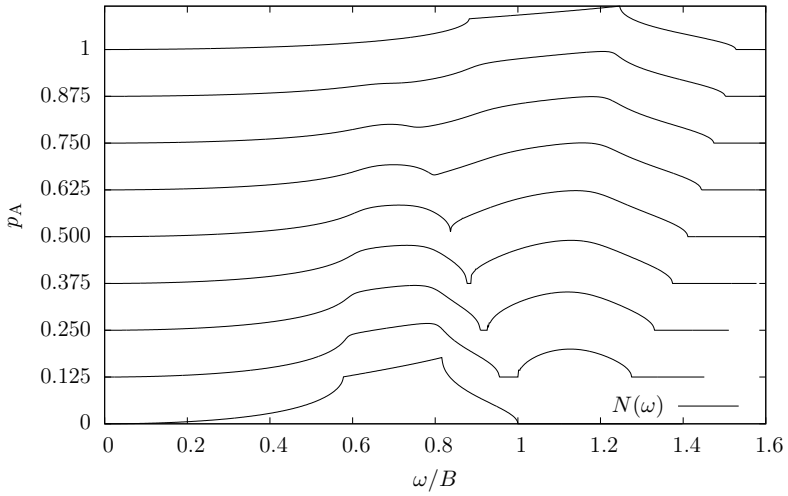


Figure 6.4: Waterfall plot of the density of modes.

$\mathcal{T} \neq 0$

Now we can study the effect of a finite pseudospin flipping parameter, $\mathcal{T} \neq 0$, on the density of modes. The behavior of the DOM with varying values of $\mathcal{T}$ is shown in 6.5. The behavior of the DOM can be summarized in the following manner: the increase in magnitude of $\mathcal{T}$ implies a corresponding increase in the disorder strength. This increase brings about a general broadening of the DOM. In addition, we see that a finite $\mathcal{T}$ gives a upper band edge value which is generally smaller than that at zero $\mathcal{T}$.

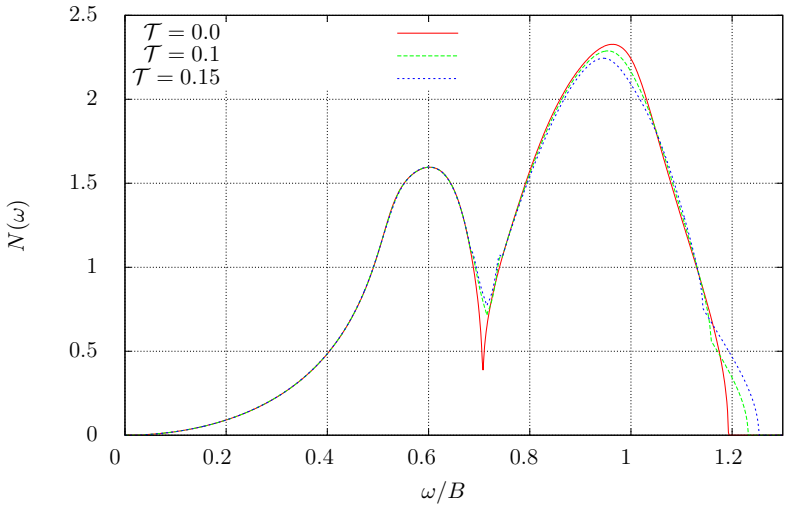


Figure 6.5: Density of modes in presence of finite $\mathcal{T}$.

6.8.2 Diffusion constant

We now move onto the results for the diffusion constant, which is the central physical quantity of interest. For the diffusion constant we differentiate between contributions from "classical" diffusion, i.e., that comes from the ladder (Diffuson) diagrams, and those coming from coherence effects, which are due to contributions from the so-called maximally crossed or Cooperon diagrams. It is precisely in this second set of contributions that the most significant effects of light polarization appears, as we see below. Hence in this section we first look at results of the classical diffusion constant, and then move onto those for the crossed diagrams.

Classical diffusion constant

As follows from our analytical analysis above, once we enable pseudospin flipping, we need to consider the diffusion constant in both the singlet, here denoted D^S and triplet D_i^T, $i = 0, 1$ channels, where i differentiates the two values of the energy eigenvalues in the triplet channel. We start from the ladder diffusion constant, D^L. This is shown in Fig. 6.6 We discern several expected

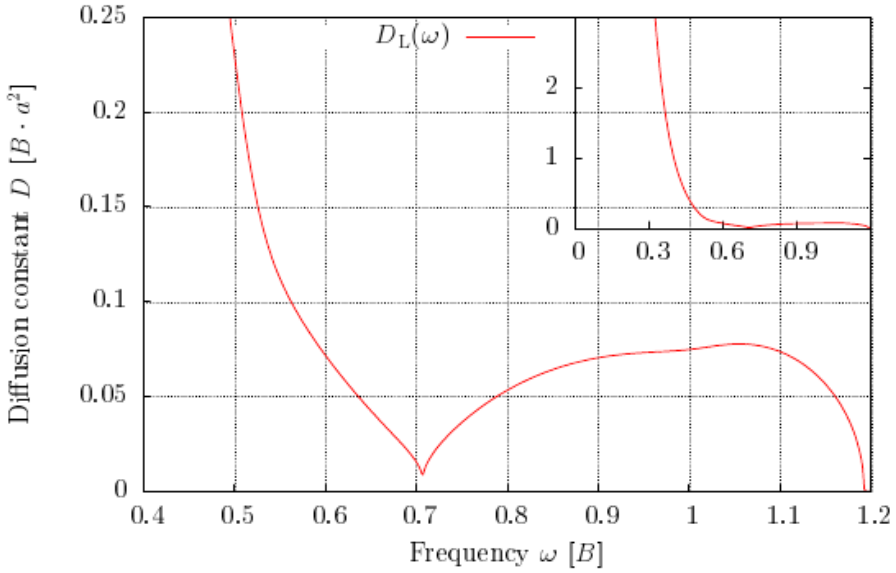


Figure 6.6: Classical diffusion constant.

features in the plot above: at low frequencies the curve follows an expected ω^{-4} behavior, which is in the regime of Rayleigh scattering. Since the disorder potential in the wave equation for classical waves at smaller frequencies will be scattered less and hence obeys nearly free propagation, which yields the Rayleigh regime. At higher frequencies we encounter a pseudogap region at frequency $\omega/B \approx 0.7$ where a minimum of the diffusion constant appears, where diffusion drops to a very low but nonzero value. The diffusion constant then stays a low values until vanishing at around $\omega/B \approx 1.2$, which is the value of the band edge in our model.

Full diffusion constant

We first look at results showing the behavior of the diffusion constant in the singlet channel, D^S, as a function of the flipping parameter $\mathcal{T}$. This is shown in the figures below. Starting with the

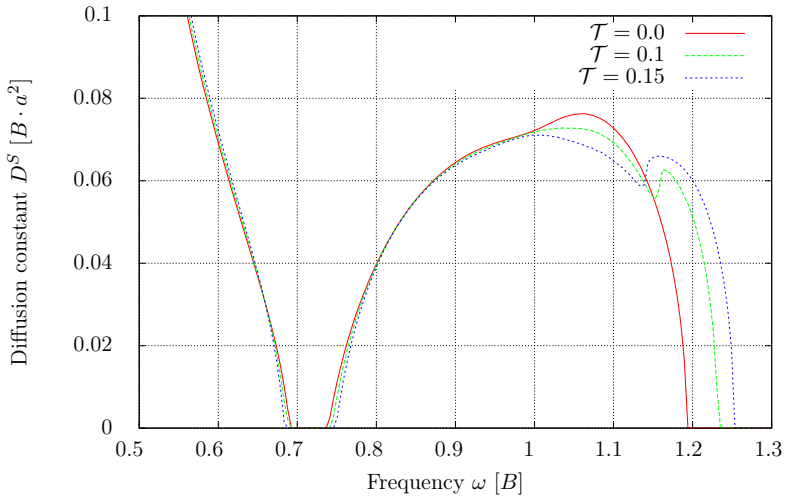


Figure 6.7: Diffusion constant in the singlet channel.

diffusion constant in the singlet channel, we can compare figure 6.7 with 6.6 and conclude that the inclusion of coherent effects correctly leads to an overall reduction of the diffusion constant in comparison with the ladder diffusion D_L. A striking observation here is the appearance of vanishing values of D^S in the frequency range $0.7 < \omega/B < 0.75$, which now implies localized states in this region. We note however that the singlet mode represents actually the Goldstone mode of the system, and hence represents conservation of intensity (particle number). Given that $\mathcal{T}$ is certainly particle-number conserving, we do not actually expect large deviations between the behavior of D^L and D^S, and this is confirmed from 6.7 for the range of frequencies $\omega/B < 1.0$. Some deviations are seen for frequencies outside of this range, which still needs to be explained in full.

We now move onto the diffusion constant in the triplet channel. As already noted in previous sections, in the triplet channel the diffusion constant takes on different values corresponding to the two different eigenvalues resulting from the diagonalization of the scattering vertex in the triplet channel, here denoted D^{T0} and D^{T1}. These are shown in the plots below:

We see that in the absence of mode flipping, both values of D^T are identical to that of D^S, which is to be expected since in this case there is also conservation of polarization (pseudospin) in addition to that of intensity. However, in the presence of mode flipping, it can be seen that the diffusion constant *increases* as compared to that of D^S which is a signature of *anti-localization*, in which the localizing effects of the disorder potential is decrease overall. For the case of vector waves (as we have tried to model in this work) the observed anti-localization effect can be thought to result from the randomizing of polarizations due to mode flipping. As (weak) localization is due to coherent interference between electromagnetic waves, we expect that by introducing

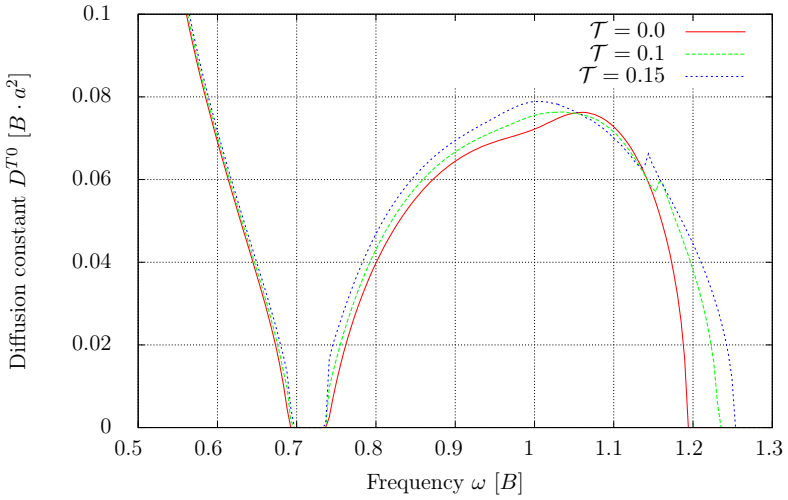


Figure 6.8: D^{T0} as a function of $\mathcal{T}$.

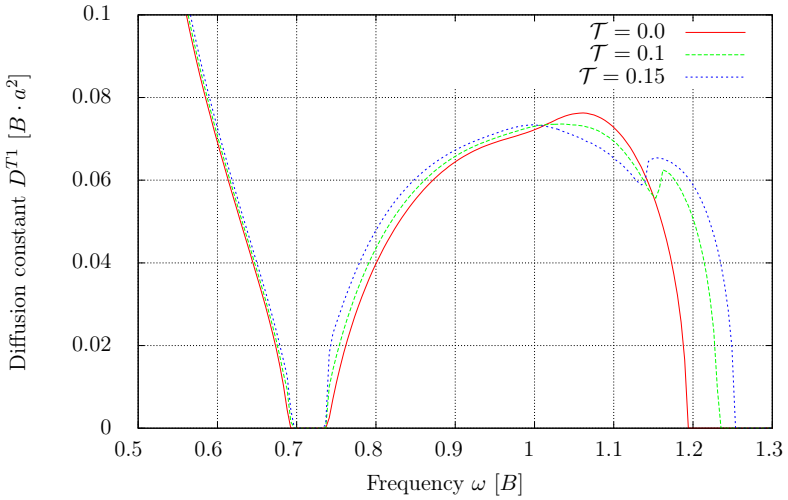


Figure 6.9: D^{T1} as a function of $\mathcal{T}$.

an external "handle" to increase decoherence effects in the system we will decrease effects of localization. For vector waves, mode flipping is exactly such a handle, which can be introduced experimentally via materials with polarization-dependent optical scattering probabilities. A summary of the results plotted above at a fixed value of $\delta\epsilon$ and p_A is shown in 6.10. This is the main result of this work.

6.8.3 Phase diagram

Taking into account all the results above we can plot a phase diagram representing the transport behavior of the system for various values of the parameter $\mathcal{T}$ across a range of frequencies

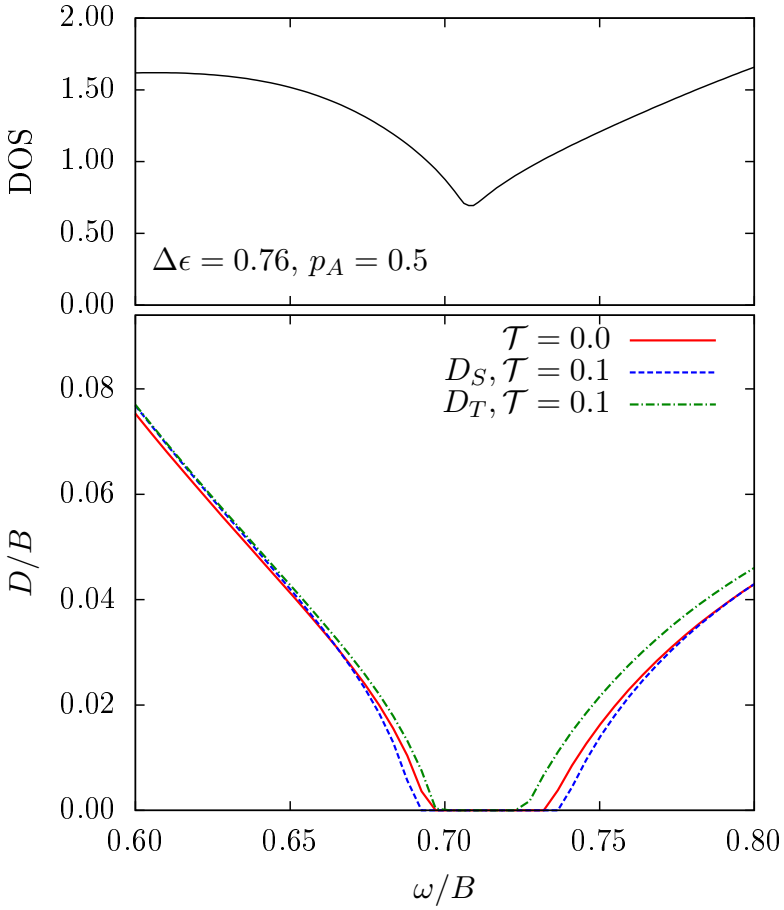


Figure 6.10: Behavior of diffusion constants in different channels.

ω/B. This is shown in 6.11 below: In order to have an experimentally realizable system where transport of light waves can be controlled, to some extent, via the effect of disorder, the phase diagram in Fig. 6.11 is very useful. It shows that a gap appears around the range of frequencies $0.70 < \omega/B < 0.75$, which is confirmed by the lower plot in Fig. 6.10. In experimental studies of light propagation in disordered media, one would hope to be able to accurately locate the frequencies of interest in the near region of the gap, in order that localization properties could be enhanced. On the other hand, due to the frequency-dependence of the disorder potential, one is restricted to a range of frequencies not too near to $\omega = 0$. In our model we see that we are able to control various regimes of transport properties via changing the value of the mode flipping parameter $\mathcal{T}$ around the frequency range of interest (red and green curves in Fig. 6.11). It is also clearly visible from the phase diagram that the switching on of the mode flipping parameter decreases the range of frequencies in which modes of the light field are localized.

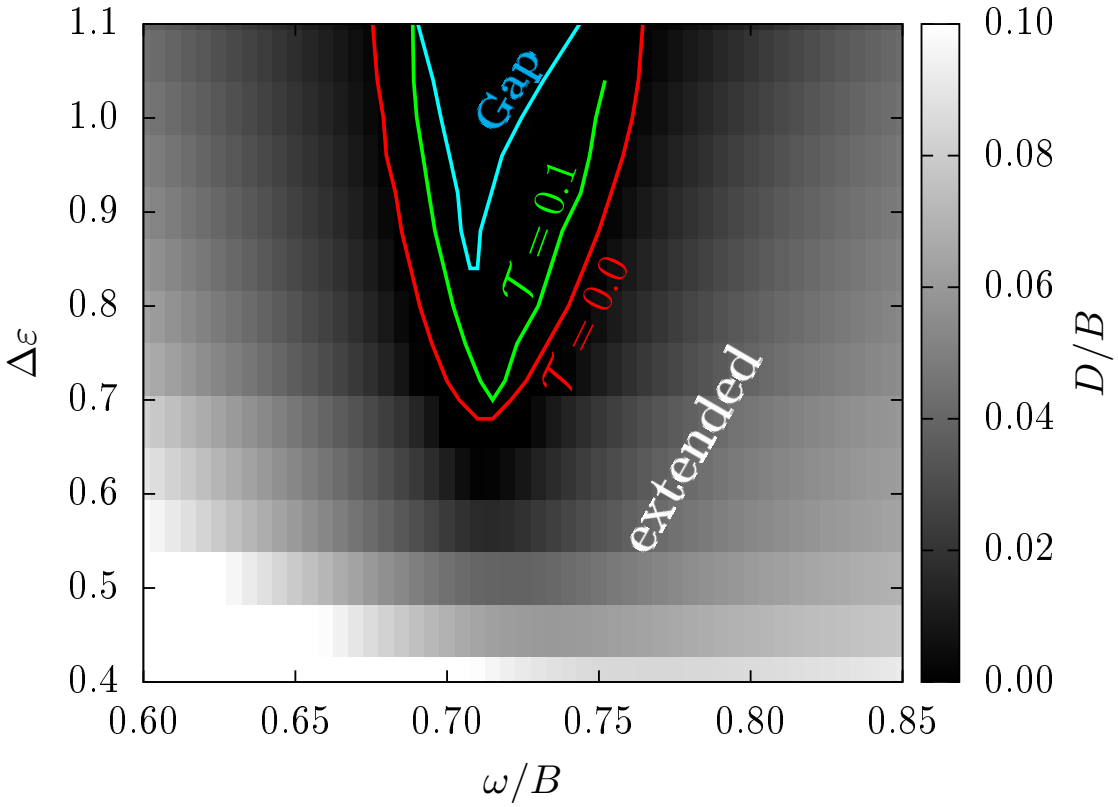


Figure 6.11: Phase Diagram

6.9 Connection to Experiments: Classical Intensity

6.9.1 Introduction

Conclusions derived from the analytical and numerical results shown above can be applied as a useful tool in the experimental search for strong Anderson localization of electromagnetic waves in random dielectric media. Specifically, they can play a role as a definitive signature for differentiating between absorption and localization of waves in such media. This is an important point because current experimental studies often face difficulties to conclusively establish the existence (or not) of Anderson localization in obtained data. In this section we will sketch how our results can help to overcome this difficulty.

6.9.2 Classical Intensity

In this section we introduce the basic quantity measured in experiments to determine transport properties of a particular medium: the classical wave intensity. Traditionally, for classical waves including polarization we associate this quantity with the sum of diagrams of the form shown in the Appendix (C.2) where $\alpha,\beta,\gamma,\delta$ represent the pseudospin directions carried by

each propagating line. As was discussed in section E these are ladder (Diffuson) diagrams, which we can represent in the singlet-triplet eigenbasis. The formalism in Sec. (6.5) carries through completely, now with the requirement that we need to form the irreducible vertex of the Bethe-Salpeter equation ((6.32)) with a particular set of polarization indices, corresponding to the diagram C.2. (6.32) thus has the tensorial component structure of the form

$$\mathbf{\Phi}^{L}_{\alpha\alpha,\beta\beta}(\Omega,\boldsymbol{K}) = \left((\mathbf{G}_{\mathrm{e}})^{+}_{\boldsymbol{k}} \otimes (\mathbf{G}_{\mathrm{e}})^{-}_{\boldsymbol{k}}\right)\left[\mathbf{1}\otimes\mathbf{1} + \frac{u}{(2\pi)^3}\sum_{\mu}\int d\boldsymbol{k}''\boldsymbol{\gamma}^{L}_{\alpha\alpha,\mu\mu}\mathbf{\Phi}^{L}_{\mu\mu,\beta\beta}(\Omega,\boldsymbol{K})\right]. \tag{6.55}$$

for the Diffuson, and for the Cooperon has a similar equation with the corresponding index structure $\mathbf{\Phi}^{C}_{\alpha\beta,\beta\alpha}(\mathbf{q})$. We note that in "matching" the pairings of *identical* pseudospin indices between scattering events to calculate the probability of classical diffusion, the dimensions of our vertices are reduced from 4 to 2; for example, the matched vertices of the Diffuson have the form

$$\langle\boldsymbol{\gamma}^{\mathrm{L}}\rangle_{(\sigma)} = \gamma^{0}_{L}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \gamma^{m}_{L}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} \tag{6.56}$$

in the original product basis. One can apply a similar reasoning as detailed above and diagonalize (6.56) in the singlet-triplet basis. Doing this, it can be shown [19] that by taking into proper account of sums over pseudospin components in the Diffuson and Cooperon BS equations, one can conclude that the diffusion probability away from the origin is reduced by a factor of 2 compared to the situation in which the pseudospin structure is not taken into account (scalar diffusion). Absorption, on the other hand, affects both coherent and incoherent parts of the diffusion probability equally [19]. Hence it is tempting, especially in experiments, to set up measurements such that the polarization is taken into account correctly, in the sense that the diffusion probability in the singlet-triplet channels be measured and compared with the diffusion probability of incoherent scalar waves.

6.10 Conclusion

In this chapter we examined propagation of light in random dielectric material of binary type. We take into account the polarization degree of freedom of light by a mapping to a fictitious "pseudospin" space, which is allowed due to the transverse nature of vectorial light propagation. Our ultimate aim is a calculation of the *diffusion coefficient* $\boldsymbol{D}(\omega)$ using the self-consistent formalism of Vollhardt-Woelfle. For this purpose we need to evaluate several important quantities: the self-energy $\mathbf{\Sigma}(\omega)$, and the ladder and crossed irreducible vertices, $\boldsymbol{\gamma}^{L}_{\boldsymbol{k}\boldsymbol{k}'}$ and $\boldsymbol{\gamma}^{C}_{\boldsymbol{k}\boldsymbol{k}'}$)(corresponding to diagrams in Appendix C.3), respectively. We simplify the complexity resulting from the polarization structure by means of mode averaging, which renders unimportant quantities scalar while preserving the important tensor structure of the vertices. These are then diagonalized in the appropriate subspaces and the eigenvalues then used in the calculation of $\boldsymbol{D}(\omega)$. We discover that, as expected, the inclusion of coherent effects (crossed vertices) radically modifies the transport behaviour as compared to the pure diffusive (ladder vertices) case. In addition, when we take into account the effect of different polarization channels on the diffusion coefficient (we call these

different channels the *singlet* and *triplet* channels) we see that in the singlet channel the diffusive behavior is unaffected since there is conservation of polarization in this channel [106], while in the triplet channel we can show a marked increase in the diffusion coefficient as compared to the singlet case, which is a signature of *antilocalization*. Hence we show that by properly addressing the polarization degree of freedom, we are able to explain the difficulty of experimental realization of full localization of light in random media. We discuss some experimental implications of this at the end.

CHAPTER 7

Conclusion

In this thesis we considered the behavior of light propagating in dielectrically disordered and energetically nonconservative material. The two main physical attributes of interest in this problem, namely disorder and energy nonconservation, can be dealt with in one stroke via the use of the mathematical formalism commonly known as the Keldysh technique. We have approached the work described in this thesis in a systematic, stepwise fashion, and this is reflected in the ordering of chapters.

First, we derived in the Keldysh formalism a field theory of light propagation in disordered, nonconservative media. In this early part of the work the nonconservation is provided by simple static absorption. This field theoretical formulation is commonly known as the nonlinear sigma model. We also show how to calculate physical quantities like correlation functions from the sigma model, and how a source term can be included in the action of the field theory. This represents the contents of Chapter 3.

In the next part, represented in the thesis by Chapter 4 we applied the derived field theory to a nontrivial application: the calculation of full counting statistics. We derived within the framework of the nonlinear sigma model a generating functional for the cumulants of energy transmitted through a weakly nonconservative one-dimensional disordered system. We find fluctuations of transmittance which is in accordance to Dorokhov's distribution of transmission coefficients. Our numerical results also agree quantitatively with previous diagrammatic results of low order cumulants.

In Chapter 5 we come to the main part of the work, namely the application of the field theoretical formalism to random lasing. Here the emphasis is on description of a pumped photonic system which undergoes gain via the mechanism of spontaneous and stimulated emission of photons. We are able to calculate, again in the context of the field theory, the photonic distribution function $f(z, \mathbf{q}, T)$ as a function of spatial coordinate z, wavevector $\mathbf{q}$ and Wigner time coordinate T. $f(z, \mathbf{q}, T)$ also depends crucially on the pumping strength α and the imposed boundary conditions. We find that the resulting equation governing $f(z, \mathbf{q}, T)$, at the saddlepoint of our nonlinear sigma model, takes the form of a *nonlocal Fisher* equation, which is a nonlinear reaction-diffusion equation describing the interplay of birth / pumping and competition / saturation. The Fisher equation allows for a variety of solutions, depending on the specifics of the problem. We are

presently evaluating this equation numerically.

In the final Chapter 6 we depart from the methodology of the previous chapters, which were concerned with the consideration of scalar waves in which the vector nature of light waves do not play a role. In this chapter we specifically consider the effect of the vector nature of light on wave properties, specifically whether polarization increases or decreases the propensity of light waves in disordered dielectric media to become localized (Anderson localization). In this study we map the light polarization to a "pseudospin" degree of freedom which we then treat with techniques adapted from classical studies of electronic spin. We find that the polarization of light waves does in fact contribution to a diminished probability of return to the origin, the value of which determines of course the ease for the occurrence of Anderson localization.

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Appendix

APPENDIX A

Photonic Dissipative Nonlinear σ-Model

A.1 Derivation of the effective action

A.1.1 Derivation of Eq. (3.39)

After substitution of the parameterization (3.38) in Eq. (3.33) we obtain

$$\begin{aligned} iS[\hat{Q}] &= -\mathrm{Tr}\,\ln\left[\hat{R}\hat{\gamma}\hat{R}^{-1}\left(\hat{R}\hat{\gamma}\hat{G}_0^{-1}\hat{R}^{-1}+i\frac{\widetilde{\hbar}}{2\tau}\hat{\Lambda}\right)\right] \\ &= -\mathrm{Tr}\,\ln\hat{\gamma}-\mathrm{Tr}\,\ln\left(\hat{\gamma}\hat{\mathcal{G}}^{-1}+\hat{R}[\hat{\gamma}\hat{G}_0^{-1},\hat{R}^{-1}]\right) \end{aligned} .$$

By separating $\mathrm{Tr}\,\ln(\hat{\gamma}\hat{\mathcal{G}}^{-1})$ and dropping the $\hat{Q}$-independent terms we arrive at Eq. (3.39).

A.1.2 Derivation of Eq. (3.42)

The conservative part of $\hat{\gamma}\hat{G}_0^{-1}$, when substituted in the first trace in Eq. (3.39), yields

$$\begin{aligned} iS_1[\hat{Q}] &= -\mathrm{Tr}\sum_{\substack{\omega,\omega'\\ \mathbf{k},\mathbf{k}'}}\hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma}\hat{R}_{\omega\omega'}(\mathbf{k}-\mathbf{k}')\,\hat{R}^{-1}_{\omega'\omega}(\mathbf{k}'-\mathbf{k})\Big[\mathcal{E}(\omega')-\mathcal{E}(\omega)-k'^2+k^2\Big] \\ &\simeq -\mathrm{Tr}\sum_{\substack{\omega,\Delta\omega\\ \mathbf{k},\Delta\mathbf{k}}}\hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma}\hat{R}_{\omega,\omega+\Delta\omega}(-\Delta\mathbf{k})\,\hat{R}^{-1}_{\omega+\Delta\omega,\omega}(\Delta\mathbf{k})\Big[\widetilde{\hbar}\Delta\omega-(2\mathbf{k}+\Delta\mathbf{k})\cdot\Delta\mathbf{k}\Big]' \end{aligned}$$

where $\Delta\omega=\omega'-\omega$ and $\Delta\mathbf{k}=\mathbf{k}'-\mathbf{k}$. We note that $\hat{R}$ is peaked at small wave vectors in the k representation. The sum

$$\sum_{\mathbf{k}}\hat{\mathcal{G}}_\omega(\mathbf{k})\hat{\gamma}=-i\pi\nu\hat{\Lambda} \tag{A.1}$$

follows from the saddle-point condition; furthermore, $\sum_{\mathbf{k}} \hat{\mathcal{G}}_\omega(\mathbf{k})\,\mathbf{k} = 0$ due to the symmetry. After calculating the $\Delta\mathbf{k}$ sum we arrive at

$$\begin{aligned} iS_1[\hat{Q}] &= i\pi\nu\,\mathrm{Tr}\sum_{\omega,\Delta\omega}\int d\mathbf{r}\,\hat{\Lambda}\hat{R}_{\omega,\omega+\Delta\omega}(\mathbf{r})\Big[\widetilde{\hbar}\Delta\omega + \partial_{\mathbf{r}}^2\Big]\hat{R}^{-1}_{\omega+\Delta\omega,\omega}(\mathbf{r}) \\ &= \pi\nu\,\mathrm{Tr}\int d\mathbf{r}\,dt\,dt'\,i\hat{\Lambda}\hat{R}_{t't}(\mathbf{r})\Big[i\widetilde{\hbar}\partial_t + \partial_{\mathbf{r}}^2\Big]\hat{R}^{-1}_{tt'}(\mathbf{r}) \end{aligned} .$$

Applying the representation (3.38) we obtain Eq. (3.42) from the ∂_t part. The $\partial_{\mathbf{r}}^2$ part is neglected compared to $iS_2[\hat{Q}]$; the latter contribution is multiplied by $\widetilde{\hbar}\bar{D} \sim \omega_0\bar{\tau} \gg 1$.

A.1.3 Derivation of Eq. (3.43)

We substitute the k^2 part of $\hat{\gamma}\hat{G}_0^{-1}$ in the second trace in Eq. (3.39) to get

$$\begin{aligned} iS_2[\hat{Q}] &= \frac{1}{2}\mathrm{Tr}\sum_{\mathbf{k}_1\ldots\mathbf{k}_4}\hat{\mathcal{G}}(\mathbf{k}_1)\hat{\gamma}\,\hat{R}(\mathbf{k}_1-\mathbf{k}_2)\,\hat{R}^{-1}(\mathbf{k}_2-\mathbf{k}_3)\hat{\mathcal{G}}(\mathbf{k}_3)\hat{\gamma}\,\hat{R}(\mathbf{k}_3-\mathbf{k}_4)\,\hat{R}^{-1}(\mathbf{k}_4-\mathbf{k}_1)(k_2^2-k_3^2)(k_4^2-k_1^2) \\ &\simeq 2\mathrm{Tr}\sum_{\substack{\bar{\mathbf{k}}\,\Delta\bar{\mathbf{k}}\\ \Delta\mathbf{k}\,\Delta\mathbf{k}'}}\hat{\mathcal{G}}(\bar{\mathbf{k}})\hat{\gamma}\,\hat{R}\left(\Delta\bar{\mathbf{k}} - \frac{\Delta\mathbf{k}+\Delta\mathbf{k}'}{2}\right)\hat{R}^{-1}(\Delta\mathbf{k}')\hat{\mathcal{G}}(\bar{\mathbf{k}})\hat{\gamma}\,\hat{R}\left(-\Delta\bar{\mathbf{k}} - \frac{\Delta\mathbf{k}+\Delta\mathbf{k}'}{2}\right)\times \\ &\quad\times\hat{R}^{-1}(\Delta\mathbf{k})(\bar{\mathbf{k}}\cdot\Delta\mathbf{k})(\bar{\mathbf{k}}\cdot\Delta\mathbf{k}') \end{aligned} ,$$

where $\bar{\mathbf{k}} = \sum_{i=1}^4 \mathbf{k}_i/4$ and we neglected the contributions of higher order in $\Delta\bar{\mathbf{k}} = (\mathbf{k}_1+\mathbf{k}_4-\mathbf{k}_2-\mathbf{k}_3)/2$, $\Delta\mathbf{k} = \mathbf{k}_4-\mathbf{k}_1$, and $\Delta\mathbf{k}' = \mathbf{k}_2-\mathbf{k}_3$. We use the representation

$$\hat{\mathcal{G}}\hat{\gamma} = \frac{1}{2}\mathcal{G}^{\mathrm{R}}(\hat{1}+\hat{\Lambda}) + \frac{1}{2}(\hat{1}-\hat{\Lambda})\mathcal{G}^{\mathrm{A}} \tag{A.2}$$

and the well-known relations [see Eqs. (3.45) and (3.44)]

$$\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R}}_{\omega}(\mathbf{k})\,\mathcal{G}^{\mathrm{A}}_{\omega'}(\mathbf{k})\,k_\alpha k_\beta \simeq \frac{1}{2}\pi\nu\widetilde{\hbar}\bar{D}\delta_{\alpha\beta}, \tag{A.3}$$

$$\sum_{\mathbf{k}}\mathcal{G}^{\mathrm{R(A)}}_{\omega}(\mathbf{k})\,\mathcal{G}^{\mathrm{R(A)}}_{\omega'}(\mathbf{k})\,k_\alpha k_\beta \simeq 0, \tag{A.4}$$

to find

$$\begin{aligned} iS_2[\hat{Q}] &= -\frac{1}{2}\pi\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\left[(\hat{1}+\hat{\Lambda})\hat{R}\,(\partial_{\mathbf{r}}\hat{R}^{-1})\cdot(\hat{1}-\hat{\Lambda})\hat{R}\,(\partial_{\mathbf{r}}\hat{R}^{-1})\right] \\ &= \frac{1}{4}\pi\nu\widetilde{\hbar}\bar{D}\,\mathrm{Tr}\left[\partial_{\mathbf{r}}(\hat{R}^{-1}\hat{\Lambda}\hat{R})\right]^2 \end{aligned} , \tag{A.5}$$

from which Eq. (3.43) follows.

A.1.4 Derivation of Eq. (3.46)

The nonconservative part of $\hat{\gamma}\hat{G}_0^{-1}$, being substituted in the first trace in Eq. (3.39) yields

$$iS_3[\hat{Q}] \simeq -i\epsilon''\omega_0^2 \,\mathrm{Tr} \sum_{\mathbf{k}\mathbf{k}'} \hat{\mathcal{G}}(\mathbf{k})\hat{\gamma}\hat{R}(\mathbf{k}-\mathbf{k}')\Big[\hat{\Lambda}\hat{R}^{-1}(\mathbf{k}'-\mathbf{k}) - \hat{R}^{-1}(\mathbf{k}'-\mathbf{k})\,\hat{\Lambda}\Big]. \tag{A.6}$$

We change the variable $\mathbf{k}' = \mathbf{k} + \Delta\mathbf{k}$ and apply Eq. (A.1). After cyclically moving the operators under the trace we obtain Eq. (3.46).

APPENDIX B

Appplication of the Nonlinear σ-Model: Full Counting Statistics

B.1 Introduction

In this appendix we give the explicit calculations for the full counting statistics paper. We will fill in some derivation details not given in the main text, and in particular, the full derivation of the nonlinear σ-model action including a current source term will be given. The inclusion of the source term is necessary as a "handle" with which the moments of the current can be generated from the action.

B.2 Disordered action with current

The current can be written classically as follows (we set $c = 1$)

$$\mathbf{j} = \frac{1}{4\pi}\mathbf{E} \times \mathbf{H} \tag{B.1}$$

For simplicity, we align our electric field in the z-direction such that

$$E_z = -\dot{A} \tag{B.2}$$

which then restricts the magnetic field such that

$$\mathbf{H} = \nabla \times \mathbf{A} \Rightarrow H_x = \partial_y A, \;\; H_y = -\partial_x A \tag{B.3}$$

In terms of the above-defined quantities we can write the current:

$$\mathbf{j} = -\frac{1}{4\pi}\dot{\mathbf{A}}\partial_\perp \mathbf{A} : \;\; j_x = -\frac{1}{4\pi}\dot{A}\partial_x A, \;\; j_y = -\frac{1}{4\pi}\dot{A}\partial_y A \tag{B.4}$$

Using the definition of the Fourier transform

$$\mathbf{A}_R(t) = \mathrm{Re}\underbrace{\left\{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\mathbf{A}_\omega\, e^{-i\omega t}\right\}}_{\mathbf{A}(t)} \tag{B.5}$$

and from here on we use complex $\mathbf{A}(t)$. We also assume that relevant frequencies are in a narrow interval around the atomic transition frequency $\nu = \omega_0$. We can thus define the current using the complex $\mathbf{A}(t)$:

$$\mathbf{j} = -\frac{1}{16\pi}\left(\dot{\mathbf{A}}^*(t)\nabla_\perp\mathbf{A}(t) + (\nabla_\perp\mathbf{A}^*)\,\dot{\mathbf{A}}\right) \tag{B.6}$$

and the addtional term to the action coupling the source term has the form

$$S_j = \int_C dt \int d\mathbf{r}\,\boldsymbol{\lambda}(\mathbf{r},t)\cdot\mathbf{j}(\mathbf{r},t) \tag{B.7}$$

where here $\boldsymbol{\lambda}(\mathbf{r},t)$ is the source term, here representing an analog to the vector potential in the case of electrons. The $\int_C$ integration can be rewritten in a familiar way,

$$S_j = -\frac{1}{16\pi}\int_{-\infty}^{\infty} dt \int d\mathbf{r}\left\{\boldsymbol{\lambda}_+\cdot\left[\dot{\mathbf{A}}_+^*\,(\partial_\perp\mathbf{A}_+) + (\partial_\perp\mathbf{A}_+^*)\,\mathbf{A}_+\right] - \boldsymbol{\lambda}_-\cdot\left[\dot{\mathbf{A}}_-^*\partial_\perp\mathbf{A}_- + (\partial_\perp\mathbf{A}_-^*)\,\mathbf{A}_-\right]\right\} \tag{B.8}$$

Performing the Keldysh change of variables

$$\lambda_\pm = \frac{\lambda^{cl}\pm\lambda^q}{\sqrt{2}}, \quad \mathbf{A}_\pm = \frac{\mathbf{A}^{cl}\pm\mathbf{A}^q}{\sqrt{2}} \tag{B.9}$$

gives the integrand in (B.8) in the Keldysh (cl, q) space as

$$I = \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}\mathbf{A}^{\mathrm{cl}} & \mathbf{A}^{\mathrm{q}}\end{pmatrix}}_{\widehat{\mathbf{A}}^\dagger}\underbrace{\begin{pmatrix}\vec{\lambda}^q & \vec{\lambda}^{cl}\\ \vec{\lambda}^{cl} & \vec{\lambda}^q\end{pmatrix}}_{\widehat{\vec{\lambda}}}\underbrace{\begin{pmatrix}\mathbf{A}^{\mathrm{cl}}\\ \mathbf{A}^{\mathrm{q}}\end{pmatrix}}_{\widehat{\mathbf{A}}} + \frac{1}{\sqrt{2}}(\nabla_\perp\widehat{\mathbf{A}})^\dagger\cdot\widehat{\vec{\lambda}}\cdot\widehat{\mathbf{A}} \tag{B.10}$$

$$\approx -\sqrt{2}\omega_0^2\sqrt{\epsilon'}\hat{\mathbf{A}}^\dagger(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{\mathbf{A}} \tag{B.11}$$

where we made the following approximations

$$\partial_t \approx \mp i\omega_0 \tag{B.12}$$

$$\nabla_\perp \approx \pm i\omega_0\sqrt{\epsilon'}\vec{\kappa}, \quad |\vec{\kappa}| = 1 \tag{B.13}$$

We can represent $\hat{\vec{\lambda}}$ as

$$\hat{\vec{\lambda}} = \vec{\lambda}^q\hat{\mathbb{1}} + \vec{\lambda}^{cl}\widehat{\gamma}, \quad \widehat{\gamma} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \tag{B.14}$$

and we have finally, analogously to the simple free action described in the first chapter, the following action including external sources

$$S = S_0 + S_j = \frac{1}{16\pi}\widehat{\mathbf{A}}^{\dagger}\Big(\hat{G}_0^{-1} + \sqrt{2}\omega_0^2\sqrt{\epsilon'}(\vec{\kappa}\cdot\widehat{\vec{\lambda}})\Big)\widehat{\mathbf{A}} \tag{B.15}$$

B.3 Keldysh sigma model with current source term

In this section we will derive in detail the Keldysh sigma model action including source term as was shown in Chapter 4. We start from the original Tr ln action as was derived in Chapter 3, Eq. (3.33). Including the source term, the term has the form

$$\mathrm{Tr}\,\ln\Big(\widehat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau} + \sqrt{2}\omega_0^2\sqrt{\epsilon'}(\vec{\kappa}\cdot\widehat{\vec{\lambda}})\Big) \tag{B.16}$$

and using the usual parameterization of the $\hat{Q}$ matrix

$$\hat{Q} = i\hat{R}^{-1}\hat{\Lambda}\hat{R} \tag{B.17}$$

we can rewrite (B.18) such to obtain

$$\mathrm{Tr}\,\ln\Big(\widehat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau} + \sqrt{2}\omega_0^2\sqrt{\epsilon'}(\vec{\kappa}\cdot\widehat{\vec{\lambda}})\Big) = \mathrm{Tr}\,\ln\Big(\hat{R}[\hat{\gamma}\hat{G}_0^{-1},\hat{R}^{-1}] + \widehat{\gamma\mathcal{G}}^{-1} + \sqrt{2}\omega_0^2\sqrt{\epsilon'}\hat{R}\widehat{\gamma}(\vec{\kappa}\cdot\widehat{\vec{\lambda}})\hat{R}^{-1}\Big) \tag{B.18}$$

where $\hat{\gamma}\hat{\mathcal{G}}^{-1} = \hat{\gamma}\hat{G}_0^{-1} + i\frac{\tilde{\hbar}}{2\tau}\hat{\Lambda}$. Now if we pull out the $\hat{Q}$-independent $\hat{\gamma}\hat{\mathcal{G}}^{-1}$ and expand the ln we obtain the following sequence of expressions

$$\begin{aligned}\mathrm{Tr}\,\ln\Big(\mathbb{1} + \underbrace{\hat{\mathcal{G}}\hat{\gamma}\hat{R}\big[\hat{\gamma}\hat{G}_0^{-1},\hat{R}^{-1}\big]}_{(\dots)_0} + \sqrt{2}\omega_0^2\sqrt{\epsilon'}\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\Big) \approx \quad (\dots)_0 \quad + \\ + \underbrace{\sqrt{2}\omega_0^2\sqrt{\epsilon'}\mathrm{Tr}\Big(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\Big)}_{\mathbf{1}} - \underbrace{\omega_0^4\epsilon'\mathrm{Tr}\Big(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\hat{\vec{\kappa}}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\Big)^2}_{\mathbf{2}} - \underbrace{\sqrt{2}\omega_0^2\sqrt{\epsilon'}\mathrm{Tr}\Big(\hat{\mathcal{G}}\hat{\gamma}\hat{R}[\hat{\gamma}\hat{G}_0^{-1},\hat{R}^{-1}]\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\Big)}_{\mathbf{3}}\end{aligned} \tag{B.19}$$

We will look at the individual terms in (B.19).

B.3.1 Term 1

We first write out the expression in term **1** in Fourier transformed form

$$\begin{aligned}\sqrt{2}\omega_0^2\sqrt{\epsilon'}\mathrm{Tr}\Big(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\Big) &= \mathrm{Tr}\sum_{\mathbf{k}_1,\mathbf{k}_2}\hat{\mathcal{G}}(\mathbf{k}_1)\hat{\gamma}\hat{R}(\mathbf{k}_1-\mathbf{k}_2)\hat{\gamma}\vec{\kappa}(\mathbf{k}_2)\cdot(\hat{\vec{\lambda}}\hat{R}^{-1})(\mathbf{k}_1-\mathbf{k}_2) \\ &\approx \mathrm{Tr}\sum_{\mathbf{k}}\hat{\mathcal{G}}(\mathbf{k})\vec{\kappa}(\mathbf{k})\cdot\sum_{\Delta\mathbf{k}}\hat{\gamma}\hat{R}(\Delta\mathbf{k})\hat{\gamma}(\hat{\vec{\lambda}}\hat{R}^{-1})(-\Delta\mathbf{k}) = 0\end{aligned}, \tag{B.20}$$

since the sum over $\vec{\kappa}(\mathbf{k})$ gives zero. We have used the notation $\mathbf{k} = \frac{\mathbf{k}_1+\mathbf{k}_2}{2}$, $\Delta\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$, and neglected $\Delta\mathbf{k}$ in $\hat{\mathcal{G}}$ and $\vec{\kappa}$.

B.3.2 Term 2

For term **2** we again make use of the decomposition first introduced in Chapter 3

$$\hat{\mathcal{G}}\hat{\gamma} = \frac{1}{2}\mathcal{G}^R(\mathbb{1} + \hat{\Lambda}) + \frac{1}{2}(\mathbb{1} - \hat{\Lambda})\hat{\mathcal{G}}^A \tag{B.21}$$

In which case the terms which contain the products $\mathcal{G}^R\mathcal{G}^A$ and $\mathcal{G}^A\mathcal{G}^R$[1] will yield the expression

$$\begin{aligned}
&\mathrm{Tr}\left(\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\lambda}})\hat{R}^{-1}\right)^2 = \\
&= \frac{1}{2}\mathrm{Tr}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4}\mathcal{G}^R(\mathbf{k}_1)(\mathbb{1}+\hat{\Lambda})\hat{R}(\mathbf{k}_1-\mathbf{k}_2)\hat{\gamma}\vec{\kappa}(\vec{\mathbf{k}}_2)\cdot(\hat{\vec{\lambda}}\hat{R}^{-1})(\mathbf{k}_2-\mathbf{k}_3)\mathcal{G}^A(\mathbf{k}_3)(\mathbb{1}-\hat{\lambda})\hat{R}(\mathbf{k}_3-\mathbf{k}_4)\hat{\gamma}\vec{\kappa}(\mathbf{k}_4)\cdot \\
&\quad\cdot(\hat{\vec{\lambda}}\hat{R}^{-1})(\mathbf{k}_4-\mathbf{k}_1) \\
&\approx \frac{1}{2}\sum_{\mathbf{k}}\mathcal{G}^R(\mathbf{k})\kappa_\alpha(\mathbf{k})\mathcal{G}^A(\mathbf{k})\kappa_\beta(\mathbf{k})\mathrm{Tr}\left[\hat{R}\hat{\gamma}\hat{\lambda}_\alpha\hat{R}^{-1}\hat{R}\hat{\gamma}\hat{\lambda}_\beta\hat{R}^{-1} - \hat{\Lambda}\hat{R}\hat{\gamma}\hat{\lambda}_\alpha\hat{R}^{-1}\hat{\Lambda}\hat{R}\hat{\gamma}\hat{\lambda}_\beta\hat{R}^{-1}\right]
\end{aligned} \tag{B.22}$$

Using the fact that[2]

$$\sum_{\mathbf{k}}\mathcal{G}^R\kappa_\alpha\mathcal{G}^A\kappa_\beta = \frac{2\pi\nu\bar{\tau}}{\tilde{\hbar}}\frac{\delta_{\alpha\beta}}{d} = \frac{\pi\nu\tilde{\hbar}}{2\epsilon'\omega_0^2}\bar{D}\delta_{\alpha\beta} \tag{B.23}$$

where $\bar{D} = \frac{v^2\bar{\tau}}{d}$ and the *group velocity* is given by $v = \frac{d\omega}{dk} = \frac{d\omega}{d\sqrt{\epsilon'\omega^2}} = \frac{2\sqrt{\epsilon'\omega^2}}{\tilde{\hbar}}$, we then arrive at the final form for Term **2**

$$\frac{\pi\nu\tilde{\hbar}}{4\epsilon'\omega_0^2}\bar{D}\mathrm{Tr}\left[(\hat{\gamma}\hat{\vec{\lambda}})\cdot(\hat{\gamma}\hat{\vec{\lambda}}) + (\hat{\gamma}\hat{\vec{\lambda}}\hat{Q})\cdot(\hat{\gamma}\hat{\vec{\lambda}}\hat{Q})\right] \tag{B.24}$$

B.3.3 Term 3

For term **3** we use the fact that

$$\hat{\gamma}(\hat{G}_0)_\omega(\mathbf{k}) = \left[\varepsilon(\omega) - k^2\right]\mathbb{1} + i\epsilon''(\omega)\omega^2\hat{\Lambda} \tag{B.25}$$

[1] Terms containing products of the form $\mathcal{G}^R\mathcal{G}^R$ and $\mathcal{G}^A\mathcal{G}^A$ will integrate out to zero due to placement of poles on the same sides of the complex plane

[2] These expressions have been proven and used in Chapter 3

where $\varepsilon(\omega) = \epsilon'\omega^2$. We look at (B.25) excluding the $\varepsilon(\omega)$ term, which contributes to the time derivative. Hence

$$\begin{aligned}
&\mathrm{Tr}\,\hat{\mathcal{G}}\hat{\gamma}\hat{R}\left[\hat{\gamma}\hat{G}_0^{-1},\hat{R}^{-1}\right]\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}(\vec{\kappa}\cdot\hat{\vec{\gamma}})\hat{R}^{-1}\rightarrow\\
\rightarrow&\mathrm{Tr}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_4}\hat{\mathcal{G}}(\mathbf{k}_1)\hat{\gamma}\hat{R}(\mathbf{k}_1-\mathbf{k}_2)\Big[-k_2^2\hat{R}^{-1}(\mathbf{k}_2-\mathbf{k}_3)+\hat{R}^{-1}(\mathbf{k}_2-\mathbf{k}_3)k_3^2+\\
&+i\epsilon''\omega_0^2\Big(\hat{\Lambda}\hat{R}(\mathbf{k}_2-\mathbf{k}_3)-\hat{R}^{-1}(\mathbf{k}_2-\mathbf{k}_3)\hat{\Lambda}\Big)\Big]\hat{\mathcal{G}}(\mathbf{k}_3)\hat{\gamma}\hat{R}(\mathbf{k}_3-\mathbf{k}_4)\hat{\gamma}\vec{\kappa}(\mathbf{k}_4)\cdot(\hat{\vec{\lambda}}\hat{R}^{-1}(\mathbf{k}_4-\mathbf{k}_1))
\end{aligned}\quad, \tag{B.26}$$

and using the same decomposition as above,

$$\hat{\mathcal{G}}(\mathbf{k}_i)=\frac{1}{2}\mathcal{G}^R(\mathbf{k}_i)(\mathbb{1}+\hat{\Lambda})+\frac{1}{2}(\mathbb{1}-\hat{\Lambda})\hat{\mathcal{G}}^A(\mathbf{k}_i) \tag{B.27}$$

while discarding terms that integrate to zero gives the approximate expression

$$\begin{aligned}
&\approx\frac{1}{2}\sum_{\mathbf{k}}\mathcal{G}^R(\mathbf{k})k_\alpha\mathcal{G}^A(\mathbf{k})\kappa_\beta(\mathbf{k})\sum_{\pm}\mathrm{Tr}\left(\mathbb{1}\pm\hat{\Lambda}\right)\hat{R}(i\partial_{x_\alpha}\hat{R}^{-1})(\mathbb{1}\mp\hat{\Lambda})\hat{R}\hat{\gamma}\lambda_\beta\hat{R}^{-1}=\\
&=i\frac{\pi\nu\tilde{\hbar}\bar{D}}{2\sqrt{\epsilon'\omega_0^2}}\mathrm{Tr}\left[\hat{\gamma}\hat{\vec{\lambda}}\cdot\left((\partial_{\mathbf{r}}\hat{R}^{-1})\hat{R}+\hat{Q}\partial_{\mathbf{r}}\hat{Q}+\hat{R}^{-1}\hat{\Lambda}^2(\partial_{\mathbf{r}}\hat{R})\right)\right]\\
&=i\frac{\pi\nu\tilde{\hbar}\bar{D}}{2\sqrt{\epsilon'\omega_0^2}}\mathrm{Tr}\left[\hat{\gamma}\hat{\vec{\lambda}}\cdot\hat{Q}\partial_{\mathbf{r}}\hat{Q}\right]
\end{aligned}\quad, \tag{B.28}$$

where we have used that $\sum_{\mathbf{k}}\hat{\mathcal{G}}^R(\mathbf{k})\hat{\mathcal{G}}^A(\mathbf{k})\kappa_\beta(\mathbf{k})=0$ and $\mathrm{Tr}\,(\vec{\lambda}\cdot\hat{Q}\hat{R}^{-1}\partial_{\mathbf{r}}\hat{\Lambda}\hat{R})=0$ to obtain the second line.

B.3.4 Final action with current source

We can now combine our previously obtained free action (in absence of current source term) with the terms **1**, **2** and **3** to obtain a *covariant derivative* in the disordered action of the form

$$\begin{aligned}
iS[\hat{Q}]&=i\tilde{S}[\hat{Q}]-\pi\nu_0\frac{\bar{D}}{4}\mathrm{Tr}\left[(\partial_{\mathbf{r}}\hat{Q})^2-\omega_0^2\left(\hat{\gamma}\hat{\vec{\lambda}}\right)^2-\omega_0^2\left(\hat{\gamma}\hat{\vec{\lambda}}\hat{Q}\right)^2-2i\sqrt{2}\omega_0\hat{\gamma}\hat{\vec{\lambda}}\cdot\hat{Q}\partial_{\mathbf{r}}\hat{Q}\right]\\
&=\left(\partial_{\mathbf{r}}\hat{Q}-i\frac{\omega_0}{\sqrt{2}}[\hat{\gamma}\hat{\vec{\lambda}},\hat{Q}]\right)\left(\partial_{\mathbf{r}}\hat{Q}-i\frac{\omega_0}{\sqrt{2}}[\hat{\gamma}\hat{\vec{\lambda}},\hat{Q}]\right)\\
&\equiv(\hat{\partial}_{\mathbf{r}}\hat{Q})^2
\end{aligned}\quad. \tag{B.29}$$

APPENDIX C

Effect of Light Polarization on Light Localization

C.1 Mathematical Description of Intensity Transport

In this section we give a short description of how (pulse) light intensity can be described in a mathematically correct manner with the help of diagrammatics. Due to the electric field's vector nature, the Green's function will have a 2nd-rank tensor structure. The corresponding intensity tensor $\mathbf{I}(\boldsymbol{r}, \boldsymbol{r}', t)$ of the field is a measure of its energy density and given by the tensor product of two Green's functions, resulting in a 4th-rank tensor

$$\mathbf{I}(\boldsymbol{r}, \boldsymbol{r}', t) \equiv \langle \mathbf{G}(\boldsymbol{r}, \boldsymbol{r}', t) \otimes \mathbf{G}^*(\boldsymbol{r}, \boldsymbol{r}', t) \rangle_{\mathrm{c}}, \tag{C.1}$$

where $\langle \dots \rangle_{\mathrm{c}}$ indicates the configurational average. For further analysis, the intensity tensor can be transformed into frequency domain and expressed in terms of spectral Green's functions (position dependence is suppressed for simplicity during argument)

$$\begin{aligned}
\mathbf{I}(\omega') &\equiv \int dt\, e^{i\omega' t}\, \mathbf{I}(t) = \int dt\, e^{i\omega' t} \langle \boldsymbol{G}(t) \otimes \boldsymbol{G}^*(t) \rangle_{\mathrm{c}} \\
&= \int dt \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi}\, e^{i\omega' t} e^{-i\omega_1 t} e^{i\omega_2 t} \langle \mathbf{G}(\omega_1) \otimes \mathbf{G}^*(\omega_2) \rangle_{\mathrm{c}} \\
&= \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi}\, 2\pi\delta(\omega' - \Omega) \langle \mathbf{G}(\omega_1) \otimes \mathbf{G}^*(\omega_2) \rangle_{\mathrm{c}} \\
&= \int \frac{d\omega}{2\pi} \left\langle \mathbf{G}\left(\omega + \tfrac{\Omega}{2}\right) \otimes \mathbf{G}^*\left(\omega - \tfrac{\Omega}{2}\right) \right\rangle_{\mathrm{c}} \equiv \int \frac{d\omega}{2\pi}\, \mathbf{S}^{\omega}(\Omega, \boldsymbol{r}, \boldsymbol{r}').
\end{aligned} \tag{C.2}$$

After writing out the Fourier transforms of the Green's functions, we introduced the center-of-mass frequency $\omega \equiv (\omega_1 + \omega_2)/2$ and relative frequency $\Omega \equiv \omega_1 - \omega_2$, and eventually performed the time integration. The intensity tensor $\mathbf{S}^{\omega}$ is often called *density-density correlation function*,

and will be transformed to momentum space in order to allow a diagrammatic analysis [101]:

$$\begin{aligned}\mathbf{S}^{\omega}(\Omega,\boldsymbol{r},\boldsymbol{r}') &= \left\langle \mathbf{G}\left(\omega+\tfrac{\Omega}{2},\boldsymbol{r},\boldsymbol{r}'\right)\otimes\mathbf{G}^{*}\left(\omega-\tfrac{\Omega}{2},\boldsymbol{r},\boldsymbol{r}'\right)\right\rangle_{\mathrm{c}} \\ &= \frac{u^3}{(2\pi)^9}\int d\boldsymbol{K}\, e^{i\boldsymbol{K}(\boldsymbol{r}-\boldsymbol{r}')}\int d\boldsymbol{k}\int d\boldsymbol{k}'\left\langle \mathbf{G}^{+}_{\boldsymbol{k}\boldsymbol{k}'}(\Omega,\boldsymbol{K})\otimes\mathbf{G}^{-}_{\boldsymbol{k}'\boldsymbol{k}}(\Omega,\boldsymbol{K})\right\rangle_{\mathrm{c}} \\ &\equiv \frac{u}{(2\pi)^3}\int d\boldsymbol{K}\, e^{i\boldsymbol{K}(\boldsymbol{r}-\boldsymbol{r}')}\,\mathbf{S}^{\omega}(\Omega,\boldsymbol{K})\end{aligned} \tag{C.3}$$

where u denotes the crystal's unit cell volume. Here we introduced the *retarded* Green's function $\mathbf{G}^{+}$ and the *advanced* Green's function $\mathbf{G}^{-}$ as

$$\mathbf{G}^{+}_{\boldsymbol{k}\boldsymbol{k}'}(\Omega,\boldsymbol{K}) \equiv \mathbf{G}(\underbrace{\omega+\tfrac{\Omega}{2}}_{\omega_+},\underbrace{\boldsymbol{k}+\tfrac{\boldsymbol{K}}{2}}_{\boldsymbol{k}_+},\underbrace{\boldsymbol{k}'+\tfrac{\boldsymbol{K}}{2}}_{\boldsymbol{k}'_+}) \tag{C.4}$$

$$\mathbf{G}^{-}_{\boldsymbol{k}'\boldsymbol{k}}(\Omega,\boldsymbol{K}) \equiv \mathbf{G}^{*}(\underbrace{\omega-\tfrac{\Omega}{2}}_{\omega_-},\underbrace{\boldsymbol{k}-\tfrac{\boldsymbol{K}}{2}}_{\boldsymbol{k}_-},\underbrace{\boldsymbol{k}'-\tfrac{\boldsymbol{K}}{2}}_{\boldsymbol{k}'_-}) \tag{C.5}$$

The density-density correlation function $\mathbf{S}^{\omega}(\Omega,\boldsymbol{K})$ describes the intensity component which is modulated with frequency Ω and wave vector $\boldsymbol{K}$. As it can be seen from the corresponding Fourier transforms, these are conjugate variables to the propagation time t and the distance $|\boldsymbol{r}-\boldsymbol{r}'|$, respectively. Since diffusive transport is expected on large time and length scales, we will be interested in the case $\Omega,|\boldsymbol{K}|\to 0$, the so-called *hydrodynamic limit*. Using eq. (6.18), one can express the averaged tensor product of two Green's functions product in terms of the effective Green's functions

$$\begin{aligned}\langle\mathbf{G}^{+}\otimes\mathbf{G}^{-}\rangle_{\mathrm{c}} &= \langle(\mathbf{G}^{+}_{\mathrm{e}}+\mathbf{G}^{+}_{\mathrm{e}}\bar{\mathbf{T}}^{+}\mathbf{G}^{+}_{\mathrm{e}})\otimes(\mathbf{G}^{-}_{\mathrm{e}}+\mathbf{G}^{-}_{\mathrm{e}}\bar{\mathbf{T}}^{-}\mathbf{G}^{-}_{\mathrm{e}})\rangle_{\mathrm{c}} \\ &= \mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}+\mathbf{G}^{+}_{\mathrm{e}}\langle\bar{\mathbf{T}}^{+}\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}\bar{\mathbf{T}}^{-}\rangle_{\mathrm{c}}\mathbf{G}^{-}_{\mathrm{e}} \\ &= \mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}+(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\langle\bar{\mathbf{T}}^{+}\otimes\bar{\mathbf{T}}^{-}\rangle_{\mathrm{c}}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}) \\ &\equiv \mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}+(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\boldsymbol{\Gamma}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}}).\end{aligned} \tag{C.6}$$

Here, $\bar{\mathbf{T}}$ denotes again the total scattering matrix relative to the effective medium and the superscripts label the considered frequency, i.e., $\bar{\mathbf{T}}^{+}\equiv\bar{\mathbf{T}}(\omega+\Omega/2)$ and $\bar{\mathbf{T}}^{-}\equiv\bar{\mathbf{T}}^{*}(\omega-\Omega/2)$. $\mathbf{G}_{\mathrm{e}}$ describes propagation in the effective medium, which is periodic and, hence, not affected by disorder averaging. Also the identity $(\mathbf{AB})\otimes(\mathbf{CD})=(\mathbf{A}\otimes\mathbf{C})(\mathbf{B}\otimes\mathbf{D})$ of the tensor product has been exploited. The new quantity $\boldsymbol{\Gamma}$ is called *total interaction vertex* and includes all possible interferences between two waves. It can always be written as a sum of sequences of a *irreducible scattering vertex* $\boldsymbol{\gamma}$. These contain all interaction diagrams which cannot be turned into separate diagrams by cutting two propagator lines vertically. Then, the total vertex $\boldsymbol{\Gamma}$ can be decomposed as follows,

$$\begin{aligned}\boldsymbol{\Gamma} &= \boldsymbol{\gamma}+\boldsymbol{\gamma}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\boldsymbol{\gamma}+\boldsymbol{\gamma}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\boldsymbol{\gamma}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\boldsymbol{\gamma}+\ldots \\ &= \boldsymbol{\gamma}+\boldsymbol{\gamma}(\mathbf{G}^{+}_{\mathrm{e}}\otimes\mathbf{G}^{-}_{\mathrm{e}})\boldsymbol{\Gamma}.\end{aligned} \tag{C.7}$$

Plugging this into eq. (C.6) yields

$$\begin{aligned}\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle_c &= \mathbf{G}_e^+ \otimes \mathbf{G}_e^- + (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\gamma}(\mathbf{G}_e^+ \otimes \mathbf{G}_e^-) + (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\gamma}(\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\Gamma}(\mathbf{G}_e^+ \otimes \mathbf{G}_e^-) \\ &= \mathbf{G}_e^+ \otimes \mathbf{G}_e^- + (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\gamma}\left[\mathbf{G}_e^+ \otimes \mathbf{G}_e^- + (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\Gamma}(\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\right] \\ &= \mathbf{G}_e^+ \otimes \mathbf{G}_e^- + (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\boldsymbol{\gamma}\langle \mathbf{G}^+ \otimes \mathbf{G}^- \rangle_c. \end{aligned} \tag{C.8}$$

Introducing $\boldsymbol{\Phi}_{k,k'} \equiv \langle \mathbf{G}^+_{kk'} \otimes \mathbf{G}^-_{k'k} \rangle_c$ and using the fact that the effective Green's function $\mathbf{G}_e$ is diagonal in momentum space (eq. (6.29)), the above equation can be written as

$$\boldsymbol{\Phi}^{\omega}_{kk'}(\Omega, \boldsymbol{K}) = \left((\mathbf{G}_e)^+_k \otimes (\mathbf{G}_e)^-_k\right)\left[\frac{(2\pi)^3}{u}\,\delta(\boldsymbol{k}-\boldsymbol{k}') + \frac{u}{(2\pi)^3}\int d\boldsymbol{k}''\,\boldsymbol{\gamma}^{\omega}_{kk''}(\Omega, \boldsymbol{K})\boldsymbol{\Phi}^{\omega}_{k''k'}(\Omega, \boldsymbol{K})\right]. \tag{C.9}$$

This equation is known as the *Bethe-Salpeter equation* (BS) and can be understood as the two-particle equivalent of the Dyson equation (6.22). It is an integral equation for the intensity tensor's spectral components $\boldsymbol{\Phi}^{\omega}_{kk'}$, taking only the effective Green's function $\mathbf{G}_e$ and the irreducible vertex $\boldsymbol{\gamma}$ as inputs. The Bethe-Salpeter equation can be depicted diagrammatically as seen in figure C.1. For our purposes it will be useful to transform the BS equation by integrating over $\boldsymbol{k}'$ first. With the notation $\boldsymbol{\Phi}_k \equiv \frac{u}{(2\pi)^3}\int d\boldsymbol{k}'\,\boldsymbol{\Phi}^{\omega}_{kk'}$ the Bethe-Salpeter equation then reads

$$\boldsymbol{\Phi}^{\omega}_{k}(\Omega, \boldsymbol{K}) = \left((\mathbf{G}_e)^+_k \otimes (\mathbf{G}_e)^-_k\right)\left[\mathbf{1}\otimes\mathbf{1} + \frac{u}{(2\pi)^3}\int d\boldsymbol{k}''\,\boldsymbol{\gamma}_{kk''}\boldsymbol{\Phi}^{\omega}_{k''}(\Omega, \boldsymbol{K})\right]. \tag{C.10}$$

C.2 Ward Identity

In the evaluation of the Boltzmann equation, we made use of the Ward Identity. The Ward identity is a consequence of energy conservation and establishes a useful connection between single particle quantities like Green's functions and self-energies on one hand, the two-particle interaction vertex $\boldsymbol{\gamma}$ on the other hand. The expression for the Ward identity in nonconservative media was first derived some time ago [42]. For our special case of the disordered photonic medium with a pseudospin structure in the disorder potential, the Ward Identity was derived in a later work [107] and reads:

$$\boldsymbol{\Delta\Sigma}(\Omega) - \int_{k'} \boldsymbol{\Delta}\mathbf{G}_{k'}(\Omega, \boldsymbol{K})\boldsymbol{\gamma}_{k'k}(\Omega, \boldsymbol{K}) = \mathbf{f}(\Omega)\left[\boldsymbol{\Pi\Sigma}(\Omega) - \int_{k'} \boldsymbol{\Pi}\mathbf{G}_{k'}(\Omega, \boldsymbol{K})\boldsymbol{\gamma}_{k'k}(\Omega, \boldsymbol{K})\right], \tag{C.11}$$

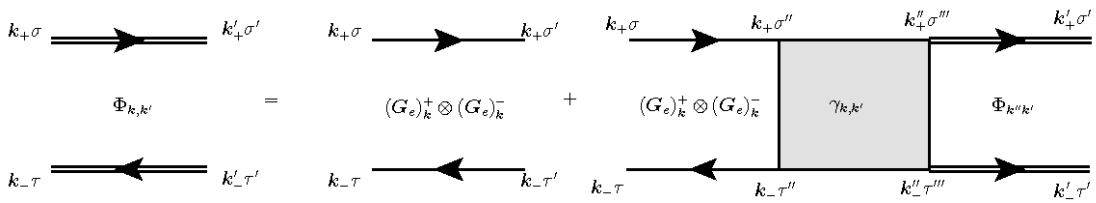

Figure C.1: Bethe-Salpeter equation in diagrammatic form.

where the appearing quantities are defined as follows:

$$\mathbf{\Delta G}_{\boldsymbol{k}}(\Omega, \boldsymbol{K}) \equiv (\mathbf{G}_{\mathrm{e}}^{+})_{\boldsymbol{k}} \otimes \mathbf{1} - \mathbf{1} \otimes (\mathbf{G}_{\mathrm{e}}^{-})_{\boldsymbol{k}} \tag{C.12}$$

$$\mathbf{\Delta\Sigma}(\Omega) \equiv \mathbf{\Sigma}(\omega_{+}) \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{\Sigma}^{*}(\omega_{-}) \tag{C.13}$$

$$\mathbf{\Pi G}_{\boldsymbol{k}}(\Omega, \boldsymbol{K}) \equiv (\mathbf{G}_{\mathrm{e}}^{+})_{\boldsymbol{k}} \otimes \mathbf{1} + \mathbf{1} \otimes (\mathbf{G}_{\mathrm{e}}^{-})_{\boldsymbol{k}} \tag{C.14}$$

$$\mathbf{\Pi\Sigma}(\Omega) \equiv \mathbf{\Sigma}(\omega_{+}) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{\Sigma}^{*}(\omega_{-}) \tag{C.15}$$

$$\mathbf{f}(\Omega) \equiv [\mathbf{V}(\omega_{+}) \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{V}^{*}(\omega_{-})]^{-1}[\mathbf{V}(\omega_{+}) \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{V}^{*}(\omega_{-})]. \tag{C.16}$$

Here$(\mathbf{G}_{\mathrm{e}}^{\pm})_{\mathbf{k}}$ denotes the effective Green's function, with the subscript and $\pm$ in the superscript denoting $\mathbf{k} \pm \frac{\mathbf{K}}{2}$ and $\omega_{\pm} \equiv \omega \pm \frac{\Omega}{2}$, respectively. This has been calculated in Sec. 6.4.3. The right hand side of the Ward Identity does not appear when considering electronic systems and is an outcome of the squared frequency dependence of our disorder potential. In the ladder approximation $\boldsymbol{\gamma}_{\boldsymbol{k}\boldsymbol{k}'} \approx \boldsymbol{\gamma}_{\mathrm{L}}$ and the limit $\Omega, |\boldsymbol{K}| \to 0$, the Ward Identity reads

$$\mathbf{\Delta\Sigma} - \mathbf{\Delta G}_0 \boldsymbol{\gamma}_{\mathrm{L}}(0) = \mathbf{f}(\Omega)[\mathbf{\Pi\Sigma} + \mathbf{\Pi G}_0 \boldsymbol{\gamma}_{\mathrm{L}}(0)], \tag{C.17}$$

with the integrated Green's functions $\mathbf{\Delta G}_0 \equiv \int_{\boldsymbol{k}} \mathbf{\Delta G}_{\boldsymbol{k}}$ and $\mathbf{\Pi G}_0 \equiv \int_{\boldsymbol{k}} \mathbf{\Pi G}_{\boldsymbol{k}}$. From eq. (C.17), one can easily derive an expression for the interaction vertex in ladder approximation $\boldsymbol{\gamma}_{\mathrm{L}}$ in terms of known single particle quantities:

$$\boldsymbol{\gamma}_{\mathrm{L}}(\Omega) = [\mathbf{\Delta G}_0 + \mathbf{f}(\Omega)\mathbf{\Pi G}_0]^{-1}[\mathbf{\Delta\Sigma} - \mathbf{f}(\Omega)\mathbf{\Pi\Sigma}] \tag{C.18}$$

The prefactor $\mathbf{f}(\Omega)$ can be expanded for frequencies $\Omega \ll \omega$ and diagonalized according to the procedure explained in Sec. 6.6. We obtain for the singlet/triplet eigenvalues $f^{S/T}$

$$f^{S/T}(\Omega) \approx \frac{\Omega}{\omega}\left(1 \pm \frac{\mathcal{T}^2}{\mathcal{V}^2}\right) + O\left(\Omega^3\right). \tag{C.19}$$

C.3 Evaluation of the Irreducible Vertex

C.3.1 Ladder Approximation

In order to estimate an expression for the irreducible vertex $\boldsymbol{\gamma}$ which was introduced in the previous chapter, we start at the definition of the total vertex $\boldsymbol{\Gamma}$ from eq. (C.6) and expanding in terms of single-site scattering events, we have

$$\begin{aligned}
\boldsymbol{\Gamma} &= \langle \bar{\mathbf{T}}^{+} \otimes \bar{\mathbf{T}}^{-} \rangle_{\mathrm{c}} \\
&= \left\langle \Big[\sum_i \bar{\mathbf{t}}_i^{+} + \sum_i \sum_{j \neq i} \bar{\mathbf{t}}_i^{+} \mathbf{G}_{\mathrm{e}}^{+} \bar{\mathbf{t}}_j^{+} + \dots \Big] \otimes \Big[\sum_k \bar{\mathbf{t}}_k^{-} + \sum_k \sum_{l \neq k} \bar{\mathbf{t}}_k^{-} \mathbf{G}_{\mathrm{e}}^{-} \bar{\mathbf{t}}_l^{-} + \dots \Big] \right\rangle_{\mathrm{c}} \\
&= \left\langle \Big[\sum_i \bar{\mathbf{t}}_i^{+} \Big(\mathbf{1} + \sum_{j \neq i} \mathbf{G}_{\mathrm{e}}^{+} \bar{\mathbf{t}}_j^{+} + \dots \Big) \Big] \otimes \Big[\sum_k \bar{\mathbf{t}}_k^{-} \Big(\mathbf{1} + \sum_{l \neq k} \bar{\mathbf{t}}_l^{-} \mathbf{G}_{\mathrm{e}}^{-} + \dots \Big) \Big] \right\rangle_{\mathrm{c}} \\
&\equiv \sum_i \sum_k \boldsymbol{\Gamma}_{ik}.
\end{aligned} \tag{C.20}$$

In this form the total vertex $\boldsymbol{\Gamma}$ is written as the sum of all possible two-wave interaction events. Since we are working within the CPA, we know that the single site scattering matrices vanish, $\langle \bar{\mathbf{t}}_i^{\pm} \rangle_{\mathrm{c}} = 0$ (CPA condition). However, this will in general not be true for higher moments like $\langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_i^{-} \rangle_{\mathrm{c}}$ or $\langle (\bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-})(\mathbf{G}_{\mathrm{e}}^{+} \otimes \mathbf{G}_{\mathrm{e}}^{-})(\bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_j^{-}) \rangle_{\mathrm{c}}$. If one again neglects correlations between different scatterings at different sites, one can express the configurationally averaged products as the product of averaged scattering tensors,

$$\langle (\bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-})(\mathbf{G}_{\mathrm{e}}^{+} \otimes \mathbf{G}_{\mathrm{e}}^{-})(\bar{\mathbf{t}}_j^{+} \otimes \bar{\mathbf{t}}_l^{-}) \rangle_{\mathrm{c}} \approx \langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-} \rangle_{\mathrm{c}} (\mathbf{G}_{\mathrm{e}}^{+} \otimes \mathbf{G}_{\mathrm{e}}^{-}) \langle \bar{\mathbf{t}}_j^{+} \otimes \bar{\mathbf{t}}_l^{-} \rangle_{\mathrm{c}}. \tag{C.21}$$

This is of course identical with the single-site approximation used in the derivation of the CPA in sec. 6.4.3. Furthermore, one can drop all terms with an odd number of scattering matrices $\bar{\mathbf{t}}$, since in the averaging process there will always be an isolated matrix which averages to zero. The total vertex components $\boldsymbol{\Gamma}_{ik}$, which describe all scattering sequences starting at site i and k for the retarded wave and the advanced wave, respectively, can then be written as

$$\begin{aligned}
\boldsymbol{\Gamma}_{ik} &= \left\langle \Big[\bar{\mathbf{t}}_i^{+} + \bar{\mathbf{t}}_i^{+} \sum_{j \neq i} \mathbf{G}_{\mathrm{e}}^{+} \bar{\mathbf{t}}_j^{+} + \dots \Big] \otimes \Big[\bar{\mathbf{t}}_k^{-} + \bar{\mathbf{t}}_k^{-} \sum_{l \neq k} \mathbf{G}_{\mathrm{e}}^{-} \bar{\mathbf{t}}_l^{-} + \dots \Big] \right\rangle_{\mathrm{c}} \\
&\overset{\mathrm{SSA}}{\approx} \langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-} \rangle_{\mathrm{c}} + \langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-} \rangle_{\mathrm{c}} \sum_{j \neq i} \sum_{l \neq k} (\mathbf{G}_{\mathrm{e}}^{+} \otimes \mathbf{G}_{\mathrm{e}}^{-}) \langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_k^{-} \rangle_{\mathrm{c}} + \dots \\
&= \langle \bar{\mathbf{t}}_i^{+} \otimes \bar{\mathbf{t}}_i^{-} \rangle_{\mathrm{c}} \, \delta_{ik} \Big[\mathbf{1} \otimes \mathbf{1} + \sum_{j \neq i} \sum_{l \neq k} (\mathbf{G}_{\mathrm{e}}^{+} \otimes \mathbf{G}_{\mathrm{e}}^{-}) \boldsymbol{\Gamma}_{jl} \Big].
\end{aligned} \tag{C.22}$$

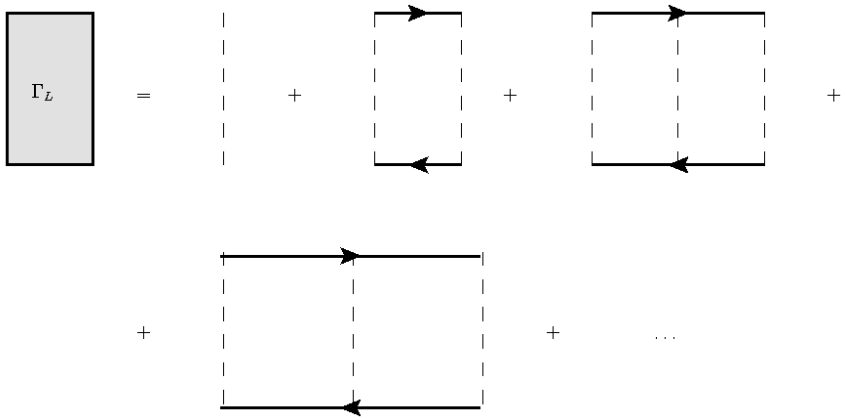

Figure C.2: The total interaction vertex Γ in the ladder approximation.

To get rid of the summation restrictions in the above expression, we can add the missing term on both sides of the equation and obtain

$$\Big[\mathbf{1}\otimes\mathbf{1} + \langle \bar{\mathbf{t}}_i^+ \otimes \bar{\mathbf{t}}_i^- \rangle_c (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\Big](\mathbf{\Gamma}_L)_{ik} = \langle \bar{\mathbf{t}}_i^+ \otimes \bar{\mathbf{t}}_i^- \rangle_c \, \delta_{ik}\Big[\mathbf{1}\otimes\mathbf{1} + \sum_j \sum_l (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)(\mathbf{\Gamma}_L)_{jl}\Big]$$

$$\Leftrightarrow \qquad (\mathbf{\Gamma}_L)_{ik} = (\boldsymbol{\gamma}^L)_i \delta_{ik}\Big[\mathbf{1}\otimes\mathbf{1} + \sum_j \sum_l (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)(\mathbf{\Gamma}_L)_{jl}\Big],$$

$$\text{with} \quad (\gamma^L)_i \equiv \Big[\mathbf{1}\otimes\mathbf{1} + \langle \bar{\mathbf{t}}_i^+ \otimes \bar{\mathbf{t}}_i^- \rangle_c (\mathbf{G}_e^+ \otimes \mathbf{G}_e^-)\Big]^{-1} \langle \bar{\mathbf{t}}_i^+ \otimes \bar{\mathbf{t}}_i^- \rangle_c .$$

Comparing above result with the decomposition of the total vertex $\mathbf{\Gamma}$ in eq. (C.7), it is possible to identify γ^L with the irreducible scattering vertex within the single-site approximation. In this regime all scattering elements $\langle \bar{\mathbf{t}}_i^+ \otimes \bar{\mathbf{t}}_i^- \rangle_c$ have the same strength, resulting in a completely isotropic irreducible vertex $(\boldsymbol{\gamma}^L)_{kk'} = \boldsymbol{\gamma}^L$. This isotropy leads to a vanishing relaxation kernel $M(\omega)$, thus γ^L describes classical diffusive (i.e. incoherent) transport. For this reason it is called *diffuson*.

The total scattering vertex Γ_L can then easily be written in terms of diffusons, as seen in fig. C.2. Due to their similarity with ladders lying on the side, this group of diagrams is called *ladder diagrams*, and the corresponding approximation which takes only these diagrams into account is called *ladder approximation*.

C.3.2 Coherent Backscattering and the Cooperon Vertex

Staying within the ladder approximation is not that interesting, since the characteristic property of waves – their ability to interfere with each other – is completely neglected in that case. As it turns out, interference effects play a role, especially the phenomenon called *coherent backscattering* has a considerable localizing influence on wave transport.

Coherent backscattering is caused by the fact that all scattering paths which scatter waves

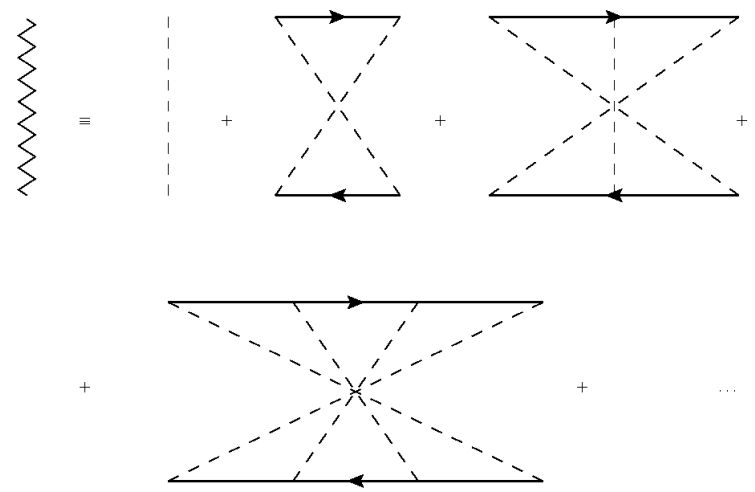

Figure C.3: The cooperon vertex γ_{C} comprises all maximally crossed diagrams.

opposite to their incident direction can have the same phase relation, resulting in constructive interference in the backwards direction. This effected is expected to slow down the diffusion process and thus is also known as *weak localization*. The corresponding diagrams contain only interaction lines which are maximally crossed (see figure **??**).

To include coherent backscattering into our theory, we have to add all of these *maximally crossed diagrams* (fig. C.3) to our irreducible vertex [108]. The modified vertex will be denoted by $\boldsymbol{\gamma}^{\mathrm{C}}$ (and a zigzag line in diagrams) and is often called *cooperon*, in reference to similar diagrams which describe cooper pairs in superconductivity [109]. If the considered system is invariant under a time-reversal operation, one can express the cooperon vertex completely in terms of ladder diagrams. As a result, one can use quantities calculated within the incoherent ladder approximation to include coherent backscattering. The individual steps needed for this transformation are depicted in figure C.4, If one performs the transformation on every constituent of the cooperon vertex, one will obtain the total vertex for the ladder approximation $\boldsymbol{\Gamma}_{\mathrm{L}}$ with $\boldsymbol{K}$ replaced by $\boldsymbol{k} + \boldsymbol{k}'$

$$\gamma^{\mathrm{C}}_{\boldsymbol{k}\boldsymbol{k}'} = \boldsymbol{\Gamma}_{\mathrm{L}}(\Omega, \boldsymbol{k} + \boldsymbol{k}') \propto \frac{1}{-i\Omega + D_{\mathrm{L}}|\boldsymbol{k} + \boldsymbol{k}'|^2} \tag{C.23}$$

Apparently, the contribution from the cooperon vertex diverges in the case $\boldsymbol{k} = -\boldsymbol{k}'$, showing the effect of coherent backscattering. The cooperon vertex can now be inserted into equation (6.40) to calculate its effects on the diffusion process.

C.4 Mode Structure of the Diffusion Constant

As we will show in this section, it is possible to overcome the incovenient matrix structure of eq. (6.40) in polarization mode space by consequent diagonalization of the tensor quantities. For this we will make use of the methods developed in section 6.7.1, i.e., we will switch from the

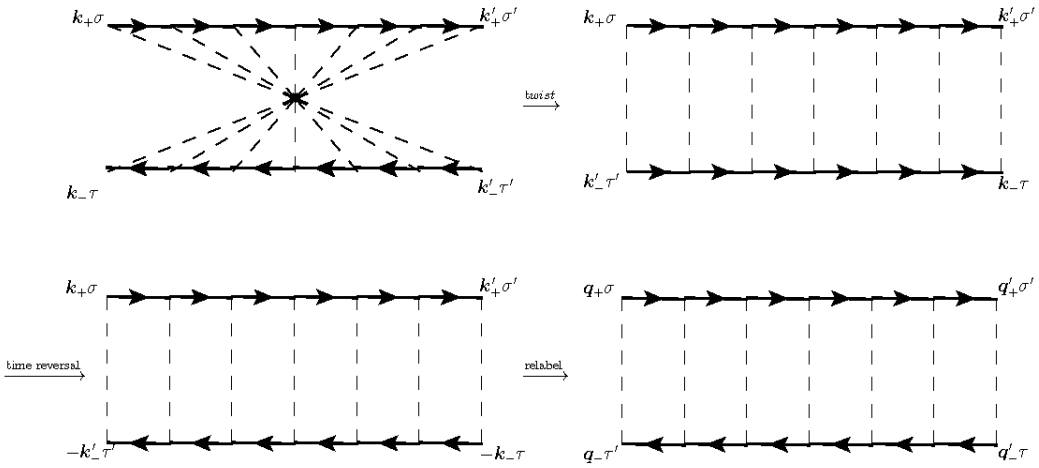

Figure C.4: Transformation of a maximally crossed diagram into a ladder diagram.

original product basis to a more suitable singlet/triplet basis. Most of these calculations have been first performed in [106].

In our case, the mode structure of the ladder diffusion constant $\mathbf{D}_\mathrm{L}$ is adopted from the renormalization factor $\boldsymbol{\delta}$, which in turn contains the ladder vertex $\boldsymbol{\gamma}_\mathrm{L}$ and a factor $\mathbf{f}(\Omega)$, see eq. (6.38) and (6.39). These can be diagonalized in the same basis with eigenvalues $\gamma_\mathrm{L}^{S/T}$ (eq. (6.47) and (6.48)) and $f^{S/T}$ (eq. (C.19)). This structure is carried over to $\mathbf{D}_\mathrm{L}$, so that we have a diagonal singlet/triplet structure for the ladder diffusion constant in the ladder eigenbasis (6.47) and (6.48):

$$\mathbf{D}_\mathrm{L} = \begin{pmatrix} D_\mathrm{L}^S & 0 & 0 & 0 \\ 0 & D_\mathrm{L}^T & 0 & 0 \\ 0 & 0 & D_\mathrm{L}^T & 0 \\ 0 & 0 & 0 & D_\mathrm{L}^T \end{pmatrix}_\mathrm{L}, \qquad \text{with} \quad D_\mathrm{L}^{S/T} = \frac{i}{4} \frac{\int_{\boldsymbol{k}} (\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k)^2 \Delta G_{\boldsymbol{k}}^2(0,0)}{\omega\,(1+\delta^{S/T})\Delta G_0}. \tag{C.24}$$

The mode structure of the renormalizing factor $\mathbf{R} \equiv \mathbf{1} \otimes \mathbf{1} + \mathbf{M}$ is determined by the irreducible vertex $\boldsymbol{\gamma}_{\boldsymbol{k}\boldsymbol{k}'}$ inside the relaxation kernel $\mathbf{M}$. In the case of the cooperon vertex $\boldsymbol{\gamma}^\mathrm{C}_{\boldsymbol{k}\boldsymbol{k}'}$, it has been shown in sec. 6.7.1 that the vertex can be diagonalized with singlet and triplet eigenvalues, however only in a different eigenbasis than the ladder vertex.

For further considerations, it is necessary to have both constituents of eq. (6.40) available in the same basis. If we choose to work in the ladder eigenbasis (6.47) and (6.48), we will have to transform the cooperon-diagonal quantity $\mathbf{R}$ into the ladder eigenbasis. To do this, consider the change of basis matrices $\mathbf{U}_\mathrm{L}$ and $\mathbf{U}_\mathrm{C}$, which perform the change from the pseudospin product

basis to the ladder eigenbasis or cooperon eigenbasis, respectively,

$$\mathbf{U}_{\mathrm{L}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{U}_{\mathrm{C}} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{C.25}$$

The change of basis matrix $\mathbf{U}_{\mathrm{C2L}}$ which transforms from the cooperon eigenbasis to the ladder eigenbasis is hence given by

$$\mathbf{U}_{\mathrm{C2L}} \equiv \mathbf{U}_{\mathrm{L}}\mathbf{U}_{\mathrm{C}}^{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}. \tag{C.26}$$

With the help of $\mathbf{U}_{\mathrm{C2L}}$ and some basic linear algebra we can find out how the quantity $\mathbf{R}$, which is diagonal in the cooperon eigenbasis with a singlet eigenvalue R^S and a triplet eigenvalue R^T, will appear in the ladder eigenbasis:

$$\mathbf{U}_{\mathrm{C2L}}\mathbf{R}\mathbf{U}_{\mathrm{C2L}}^{-1} = \mathbf{U}_{\mathrm{C2L}} \begin{pmatrix} R^S & 0 & 0 & 0 \\ 0 & R^T & 0 & 0 \\ 0 & 0 & R^T & 0 \\ 0 & 0 & 0 & R^T \end{pmatrix}_{\mathrm{C}} \mathbf{U}_{\mathrm{C2L}}^{-1} = \begin{pmatrix} R^T & 0 & 0 & 0 \\ 0 & \frac{R^T+R^S}{2} & \frac{R^T-R^S}{2} & 0 \\ 0 & \frac{R^T-R^S}{2} & \frac{R^T+R^S}{2} & 0 \\ 0 & 0 & 0 & R^T \end{pmatrix}_{\mathrm{L}}. \tag{C.27}$$

As we can see, the singlet eigenvalue R^S is now mixed with the triplet eigenvalue R^T inside the space which actually belongs to the ladder triplet eigenstates. A nice feature of this kind of structure is the fact that it commutes with ladder diagonal quantities, i.e, $\mathbf{R}\mathbf{D}_{\mathrm{L}} = \mathbf{D}_{\mathrm{L}}\mathbf{R}$. We can therefore understand eq. (**??**) as the product of a singlet/triplet diagonal quantity $\mathbf{D}_{\mathrm{L}}$ and another quantity $\mathbf{R}$ with a structure as in eq. (C.27). However, the product of both quantities can also be diagonalized, resulting in three different values for the full diffusion constant D:

$$\mathbf{D} = \begin{pmatrix} D^S & 0 & 0 & 0 \\ 0 & D^{T0} & 0 & 0 \\ 0 & 0 & D^{T1} & 0 \\ 0 & 0 & 0 & D^{T1} \end{pmatrix}_{\mathrm{LC}} = \begin{pmatrix} D_{\mathrm{L}}^T R^S & 0 & 0 & 0 \\ 0 & D_{\mathrm{L}}^S R^T & 0 & 0 \\ 0 & 0 & D_{\mathrm{L}}^T R^T & 0 \\ 0 & 0 & 0 & D_{\mathrm{L}}^T R^T \end{pmatrix}_{\mathrm{LC}}. \tag{C.28}$$

Here, the subscript "LC" denotes the new eigenbasis with the corresponding eigenstates

$$|S_{\mathrm{D}}\rangle = \frac{1}{\sqrt{2}}(|21\rangle - |12\rangle) \qquad \text{with} \quad D^S = D_{\mathrm{L}}^T R^S, \tag{C.29}$$

$$|T_{\mathrm{D}}^0\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \qquad \text{with} \quad D^{T0} = D_{\mathrm{L}}^S R^T, \tag{C.30}$$

$$|T_{\mathrm{D}}^{\pm 1}\rangle = \begin{cases} \frac{1}{\sqrt{2}}(|22\rangle - |11\rangle) \\ \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle) \end{cases} \qquad \text{with} \quad D^{T1} = D_{\mathrm{L}}^T R^T. \tag{C.31}$$

Equation (C.28) displays three scalar equations which can be solved iteratively to obtain three diffusion constants D^S, D^{T0} and D^{T1}.

The appearance of a third eigenvalue for the diffusion constant can be understood using symmetry arguments. As was mentioned in section 6.7, the threefold degeneracy of the triplet eigenvalue can be seen as a consequence of the ladder approximation's isotropic character. Including coherent backscattering into our theory, however, allows constructive interference only to happen in a certain direction and thus breaks the isotropic symmetry of our system. This symmetry breaking then reduces the threefold degeneracy to a twofold one. Since coherent backscattering does not break the inversion symmetry of the system, some degeneracy is still conserved. Our choice of superscripts for the diffusion constants is influenced by the analogy to the quantum mechanical spin coupling, where the triplet eigenstates can be further characterized by the magnetic quantum number m which can take on values $m \in \{-1, 0, +1\}$. Since states with $m = \pm 1$ are degenerate in systems with inversion symmetry, we also denote the degenerate diffusion constant with D^{T1}.

C.5 Diffusion Constant Calculation

Once we obtained the self-energies from CPA, we can move on to calculate the transport quantities derived in chapter 6. In particular, we are interested in the set of full diffusion constants D^S, D^{T0} and D^{T1}. Most of these calculations have been performed in [106].

C.5.1 Implementation

Irreducible Ladder Vertex & Green's Functions

For their calculation, we need an expression for the irreducible ladder vertex $\boldsymbol{\gamma}_{\mathrm{L}}$ which appears inside the renormalization factor $\boldsymbol{\delta}$ (eq. (6.39)) and the relaxation kernel $\mathbf{M}$ (eq. (6.40)). In the case $\Omega \to 0$, it can be obtained via the Ward Identity (eq. (C.17)) from the CPA self-energy $\boldsymbol{\Sigma}$ and Green's function $\mathbf{G}_{\mathrm{e}}^{\mathrm{D}}$ by

$$\boldsymbol{\gamma}_{\mathrm{L}}(0) = \left\langle \left[\mathbf{G}_{\mathrm{e}}^{\mathrm{D}} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{G}_{\mathrm{e}}^{\mathrm{D}*}\right]^{-1} \left[\boldsymbol{\Sigma} \otimes \mathbf{1} - \mathbf{1} \otimes \boldsymbol{\Sigma}^*\right] \right\rangle_{\sigma}. \tag{C.32}$$

If we diagonalize $\boldsymbol{\gamma}_{\mathrm{L}}(0)$ as described in section 6.7.1, we obtain (using MATHEMATICA) for its singlet and triplet eigenvalues $\gamma_{\mathrm{L}}^{S/T}$ the following expressions:

$$\begin{aligned}\gamma_{\mathrm{L}}^{S} &= \frac{(\mathrm{Im}\,G)^3\mathrm{Im}\,\Sigma - (\mathrm{Im}\,G)^2(\mathrm{Re}\,G'\,\mathrm{Re}\,\Sigma' + 2\mathrm{Im}\,G'\,\mathrm{Im}\,\Sigma') + \mathrm{Im}\,G\,\mathrm{Im}\,\Sigma(2(\mathrm{Re}\,G')^2 + (\mathrm{Im}\,G')^2)}{((\mathrm{Im}\,G)^2 + (\mathrm{Re}\,G')^2)((\mathrm{Im}\,G)^2 - (\mathrm{Im}\,G')^2)} \\ \gamma_{\mathrm{L}}^{T} &= \frac{\mathrm{Im}\,G\,\mathrm{Im}\,\Sigma + \mathrm{Re}\,G'\,\mathrm{Re}\,\Sigma'}{(\mathrm{Im}\,G)^2 + (\mathrm{Re}\,G')^2},\end{aligned} \tag{C.33}$$

where G denotes the diagonal component of $\mathbf{G}_{\mathrm{e}}^{\mathrm{D}}$ and G' the off-diagonal one. One can easily check that both eigenvalues become identical to $\gamma_{\mathrm{L}} = \frac{\mathrm{Im}\,\Sigma}{\mathrm{Im}\,G}$ for disabled mode flipping ($G', \Sigma' = 0$), in consistency with the scalar theory [].

Besides the irreducible vertex $\boldsymbol{\gamma}_{\mathrm{L}}$, the Green's functions $\Delta G_{\boldsymbol{k}}(0,0)$ and $\Delta G_{\boldsymbol{k}}^2(0,0)$ also appear several times, their exact structure shall be discussed now.

From its definition in eq. (C.12), we can see that $\Delta G_{\boldsymbol{k}}(0,0)$ has the following form

$$\begin{aligned}\Delta G_{\boldsymbol{k}}(0,0) &= \frac{1}{2}\left(\frac{1}{\omega^2 - e(\boldsymbol{k}) - \Sigma_1} - \frac{1}{\omega^2 - e(\boldsymbol{k}) - \Sigma_1^*}\right) + \frac{1}{2}\left(\frac{1}{\omega^2 - e(\boldsymbol{k}) - \Sigma_2} - \frac{1}{\omega^2 - e(\boldsymbol{k}) - \Sigma_2^*}\right) \\ &= \frac{1}{2}\frac{2i\,\mathrm{Im}\,\Sigma_1}{(\omega^2 - e(\boldsymbol{k}) - \mathrm{Re}\,\Sigma_1)^2 + (\mathrm{Im}\,\Sigma_1)^2} + \frac{1}{2}\frac{2i\,\mathrm{Im}\,\Sigma_2}{(\omega^2 - e(\boldsymbol{k}) - \mathrm{Re}\,\Sigma_2)^2 + (\mathrm{Im}\,\Sigma_2)^2},\end{aligned}$$

where $\Sigma_{1/2} \equiv \Sigma \pm \Sigma'$ have been defined. Apparently, $\Delta G_{\boldsymbol{k}}(0,0)$ describes the sum of two Lorentzian functions centered at $e(\boldsymbol{k}) = \omega^2 - \mathrm{Re}\,\Sigma_{1/2}$ and with a peak width given by $\mathrm{Im}\,\Sigma_{1/2}$. Also, its integrated quantity ΔG_0 appears in the calculation, it can be evaluated as

$$\begin{aligned}\Delta G_0 &\equiv \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\,\Delta G_{\boldsymbol{k}}(0,0) \\ &= i\int_0^1 de\,\tilde{N}(e)\left(\frac{\mathrm{Im}\,\Sigma_1}{(\omega^2 - e - \mathrm{Re}\,\Sigma_1)^2 + (\mathrm{Im}\,\Sigma_1)^2} + \frac{\mathrm{Im}\,\Sigma_2}{(\omega^2 - e - \mathrm{Re}\,\Sigma_2)^2 + (\mathrm{Im}\,\Sigma_2)^2}\right).\end{aligned} \tag{C.34}$$

The three-dimensional integration can be reduced to a one-dimensional one with the help of $\tilde{N}(e)$, since $\Delta G_{\boldsymbol{k}}(0,0)$ only depends on the momentum $\boldsymbol{k}$ via the squared dispersion $e(\boldsymbol{k})$.

The exact form of the squared Green's function $\Delta G_{\boldsymbol{k}}^2(0,0)$ can be derived from eq. (**??**), and is given by

$$\Delta G_{\boldsymbol{k}}^2(0,0) = \frac{1}{2}\left(\frac{2i\,\mathrm{Im}\,\Sigma_1}{(\omega^2 - e(\boldsymbol{k}) - \mathrm{Re}\,\Sigma_1)^2 + (\mathrm{Im}\,\Sigma_1)^2}\right)^2 + \frac{1}{2}\left(\frac{2i\,\mathrm{Im}\,\Sigma_2}{(\omega^2 - e(\boldsymbol{k}) - \mathrm{Re}\,\Sigma_2)^2 + (\mathrm{Im}\,\Sigma_2)^2}\right)^2. \tag{C.35}$$

This expression obviously contains the sum of two *squared* Lorentzians, with the same peak center and width as in the case of $\Delta G_{\boldsymbol{k}}(0,0)$.

The singlet and triplet values for the ladder diffusion constant are then calculated from eq. (C.24). The expressions again comprise three-dimensional momentum integrals, but this time

over the squared Green's function $\Delta G_{\boldsymbol{k}}^2(0,0)$ with a squared current term $(\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k)^2$:

$$\begin{aligned} D_{\mathrm{L}}^{S/T} &= \frac{i}{4\omega\,(1+\delta^{S/T})\Delta G_0} \frac{1}{(2\pi)^3} \int d\boldsymbol{k}\,(\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k)^2 \Delta G_{\boldsymbol{k}}^2(0,0) \\ &= \frac{i}{4\omega\,(1+\delta^{S/T})\Delta G_0} \int_0^1 de\,\Delta G_{e(\boldsymbol{k})}^2(0,0) F(e). \end{aligned} \tag{C.36}$$

Once again, the problem can be reduced to a one-dimensional integration, if we have access to the special function $F(e)$, defined via

$$F(e) \equiv \frac{1}{(2\pi)^3} \int d\boldsymbol{k} (\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k)^2 \delta(e - e(\boldsymbol{k})). \tag{C.37}$$

More information about the appearance and evaluation of the auxiliary function $F(e)$ can be found in appendix B.

Relaxation Kernel

In order to calculate the full diffusion constant $\mathbf{D}$ from eq. (6.37), it is crucial to evaluate the integral M_{int} inside the relaxation kernel $\mathbf{M}$ in an efficient way. In our case the relaxation kernel has the structure

$$M_{\mathrm{int}}(\Omega) \equiv \int d\boldsymbol{k} \int d\boldsymbol{k}'\,(\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k) \frac{\Delta G_{\boldsymbol{k}}(0,0)\Delta G_{\boldsymbol{k}'}^2(0,0)}{-i\Omega/D + |\boldsymbol{k}+\boldsymbol{k}'|^2} (\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_{k'}), \tag{C.38}$$

where D is one of the interesting diffusion constants. This presents a six-dimensional momentum integration of a quite complicated function, which can exhibit singular behavior either caused by the Lorentz-shaped Green's functions $\Delta G_{\boldsymbol{k}}, \Delta G_{\boldsymbol{k}}^2$ or, more importantly, by values of momenta $\boldsymbol{k} \approx -\boldsymbol{k}'$. In the above form, an accurate numerical calculation is not possible in reasonable time. Fortunately, with some approximations it can be reduced to a two-dimensional integration as we will show now (idea taken from [101]).

To begin with, one can always execute the full $\boldsymbol{k}$-integral in two steps. First, we perform an integral over the surface $S(e)$ in $\boldsymbol{k}$-space, which is given by all $\boldsymbol{k}$-points that yield the same value for $e(\boldsymbol{k})$. The result can then be integrated over all possible e-values. Thus, we replace

$$\int d\boldsymbol{k} \to \int_0^1 de \int_e \frac{dS}{|\boldsymbol{v}_k|}. \tag{C.39}$$

The constant frequency surface area $S_0(e)$ for a given value of e would then be calculated from

$$S_0(e) \equiv \int_e dS = \int d\boldsymbol{k}\,|\boldsymbol{v}_{\boldsymbol{k}}|\,\delta(e - e(\boldsymbol{k})). \tag{C.40}$$

In order to benefit from this procedure, one needs an inverse relation of the squared dispersion $e(\boldsymbol{k})$. Since the dispersion is not an injective function, an exact inversion is generally not possible. However, one can give an approximation by defining the average momentum length $\bar{k}(e)$ for a

given value e by

$$\bar{k}(e) \equiv \frac{\int_e dS\ |\boldsymbol{k}|}{S_0(e)}. \tag{C.41}$$

Again, both functions $\bar{k}(e)$ and $S_0(e)$ have to be calculated and stored only once, so they can be accessed when required. Furthermore we use a linear approximation for the gradient of the squared dispersion,

$$\boldsymbol{v_k} \equiv \nabla_{\boldsymbol{k}} e(\boldsymbol{k}) = \frac{1}{6}\begin{pmatrix} \sin k_x \\ \sin k_y \\ \sin k_y \end{pmatrix} \approx \frac{1}{6}\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \frac{1}{6}\boldsymbol{k}. \tag{C.42}$$

Within this approximation, the current terms in the relaxation kernel yield $(\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_k) = |\boldsymbol{v_k}| \cos\theta_{Kk}$ and $(\hat{\boldsymbol{K}} \cdot \boldsymbol{v}_{k'}) = |\boldsymbol{v_{k'}}| \cos\theta_{Kk'}$, since $\hat{\boldsymbol{K}}$ denotes a unit vector. Here, θ_{Kk} and $\theta_{Kk'}$ denote the angles between $\boldsymbol{K}$ and $\boldsymbol{k}$ or $\boldsymbol{K}$ and $\boldsymbol{k'}$, respectively.

Putting everything together, we end up with the following integral

$$\int_0^1 de \int_0^1 de' \int_e dS \int_{e'} dS' \frac{\Delta G_{e(\boldsymbol{k})}(0,0)\, \Delta G^2_{e'(\boldsymbol{k})}(0,0) \cos\theta_{Kk} \cos\theta_{Kk'}}{-i\Omega/D + (\bar{k}^2(e) + \bar{k}^2(e') + 2\bar{k}(e)\bar{k}(e')\cos\theta_{kk'})}, \tag{C.43}$$

where we introduced another angle $\theta_{kk'}$ as the angle between the in- and outgoing momenta $\boldsymbol{k}$ and $\boldsymbol{k'}$. In eq. (C.43), we approximate the constant-e surface integrations by integrations over spherical surfaces:

$$\int_e dS \to \frac{S_0(e)}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi. \tag{C.44}$$

With that, the angular part of eq. (C.43) for a fixed relative momentum direction $\boldsymbol{K}$ reads

$$\frac{S_0(e)S_0(e')}{(4\pi)^2} \int_0^\pi d\theta_{Kk} \int_0^{2\pi} d\phi_{Kk} \int_0^\pi d\theta_{Kk'} \int_0^{2\pi} d\phi_{Kk'} \frac{\cos\theta_{Kk} \cos\theta_{Kk'}}{-i\Omega/D + \bar{k}^2(e) + \bar{k}^2(e') + 2\bar{k}(e)\bar{k}(e')\cos\theta_{kk'}}.$$

The result of above integral does not depend on which one of the three momenta $\boldsymbol{k}, \boldsymbol{k'}$ or $\boldsymbol{K}$ is fixed. That is why we can choose to fix the direction of $\boldsymbol{k}$ instead of $\boldsymbol{K}$ and then integrate over the angles $\theta_{kk'}, \phi_{kk'}$ instead of $\theta_{Kk'}, \phi_{Kk'}$. Additionally, we can express the $\cos\theta_{Kk'}$ in the numerator in terms of the new integration variables with $\cos\theta_{Kk'} = \cos\theta_{Kk}\cos\theta_{kk'} + \sin\theta_{Kk}\sin\theta_{kk'}\cos\phi_{kk'}$,

and arrive at

$$\begin{aligned}
&\frac{S_0(e)S_0(e')}{(4\pi)^2}\int_0^{\pi}d\theta_{Kk}\int_0^{2\pi}d\phi_{Kk}\int_0^{\pi}d\theta_{kk'}\int_0^{2\pi}d\phi_{kk'}\frac{\cos\theta_{Kk}(\cos\theta_{Kk}\cos\theta_{kk'}+\sin\theta_{Kk}\sin\theta_{kk'}\cos\phi_{kk'})}{-i\Omega/D+\bar{k}^2(e)+\bar{k}^2(e')+2\bar{k}(e)\bar{k}(e')\cos\theta_{kk'}}\\
&=\frac{S_0(e)S_0(e')}{4}\int_0^{\pi}d\theta_{Kk}\int_0^{\pi}d\theta_{kk'}\frac{\cos^2\theta_{Kk}\cos\theta_{kk'}}{-i\Omega/D+\bar{k}^2(e)+\bar{k}^2(e')+2\bar{k}(e)\bar{k}(e')\cos\theta_{kk'}}\\
&=\frac{S_0(e)S_0(e')}{6}\int_0^{\pi}d\theta_{kk'}\frac{\cos\theta_{kk'}}{-i\Omega/D+\bar{k}^2(e)+\bar{k}^2(e')+2\bar{k}(e)\bar{k}(e')\cos\theta_{kk'}}\\
&=\frac{S_0(e)S_0(e')}{6\bar{k}(e)\bar{k}(e')}\left[1-\frac{\bar{k}^2(e)+\bar{k}^2(e')-i\Omega/D}{4\bar{k}(e)\bar{k}(e')}\ln\left(\frac{(\bar{k}(e)+\bar{k}(e'))^2-i\Omega/D}{(\bar{k}(e)-\bar{k}(e'))^2-i\Omega/D}\right)\right].
\end{aligned}\tag{C.45}$$

Here, we first performed both ϕ-integrations, followed by the integration over θ_{Kk}. The remaining integral then was evaluated by substituting $x=\cos\theta_{kk'}$ and using the formula

$$\int_{-1}^{1}dx\,\frac{x}{a+bx}=\frac{2}{b}-\frac{a}{b^2}\ln\left(\frac{a+b}{a-b}\right),\tag{C.46}$$

where we identified $a=\bar{k}^2(e)+\bar{k}^2(e')-i\Omega/D$ and $b=2\bar{k}(e)\bar{k}(e')$.

After all these considerations we are finally left with a two-dimensional integral M_{int} inside the relaxation kernel $\mathbf{M}$,

$$\begin{aligned}
M_{\text{int}}(\Omega)=\int_0^1 de\int_0^1 de'\,\Delta G_{e(\boldsymbol{k})}(0,0)\,\Delta G^2_{e'(\boldsymbol{k})}(0,0)\frac{S_0(e)S_0(e')}{6\bar{k}(e)\bar{k}(e')}\\
\times\left[1-\frac{\bar{k}^2(e)+\bar{k}^2(e')-i\Omega/D}{4\bar{k}(e)\bar{k}(e')}\ln\left(\frac{(\bar{k}(e)+\bar{k}(e'))^2-i\Omega/D}{(\bar{k}(e)-\bar{k}(e'))^2-i\Omega/D}\right)\right].
\end{aligned}\tag{C.47}$$

Even now, the evaluation of M_{int} is anything but an easy task. In our calculations, we created for every parameter set $\{\omega,\Sigma,\Sigma'\}$ different two-dimensional integration grids and weights, which were then customized for the occuring Lorentzian and logarithmic peaks.

Diffusion Constant

Now we want to find an expression for the diffusion constant in the special case of $\Omega=0$. Using matrix multiplications, we first bring eq. (6.37) to the following form:

$$\begin{aligned}
&\mathbf{D}(\Omega)=\mathbf{D}_{\text{L}}[\mathbf{1}\otimes\mathbf{1}+\mathbf{M}(\Omega)]^{-1}\\
\Leftrightarrow\ &\mathbf{D}(\Omega)[\mathbf{1}\otimes\mathbf{1}+\mathbf{D}\mathbf{D}^{-1}\mathbf{M}(\Omega)]=\mathbf{D}(\Omega)_{\text{L}}\\
\Leftrightarrow\ &\mathbf{D}(\Omega)=\mathbf{D}_{\text{L}}+\mathbf{D}(\Omega)\mathbf{M}(\Omega)
\end{aligned}\tag{C.48}$$

In the eigenbasis of the full diffusion constant and for $\Omega=0$, we can factor the inverse full diffusion constant $\mathbf{D}^{-1}(0)$ out of the relaxation kernel $\mathbf{M}(0)$, which then cancels with the diffusion constant $\mathbf{D}$ in the above equation. One can then derive the following equations for the three

different eigenvalues of the full diffusion constant $\mathbf{D}(0)$:

$$
\begin{aligned}
D^S &= D_{\mathrm{L}}^T - \frac{i\Delta G_0}{\Delta\Sigma(0)} \frac{\gamma_{\mathrm{L}}^S}{2\omega(1+\delta^S)} \frac{M_{\mathrm{int}}(0)}{\int_0^1 de\, \Delta G_e^2 F(e)} \\
D^{T0} &= D_{\mathrm{L}}^S - \frac{i\Delta G_0}{\Delta\Sigma(0)} \frac{\gamma_{\mathrm{L}}^T}{2\omega(1+\delta^T)} \frac{M_{\mathrm{int}}(0)}{\int_0^1 de\, \Delta G_e^2 F(e)} \\
D^{T1} &= D_{\mathrm{L}}^T - \frac{i\Delta G_0}{\Delta\Sigma(0)} \frac{\gamma_{\mathrm{L}}^T}{2\omega(1+\delta^T)} \frac{M_{\mathrm{int}}(0)}{\int_0^1 de\, \Delta G_e^2 F(e)}.
\end{aligned}
\tag{C.49}
$$

APPENDIX D

Application of the Nonlinear σ-Model: Random Lasing

D.1 Quantum theory of the laser

D.1.1 Introduction

The semiclassical theory of lasing, as illustrated in Chapter 5, describes major features of lasing very well, while at the same time maintaining the relative simplicity of the classical description. However, a major shortcoming of the semiclassical theory is that it can best describe lasing strictly above the lasing threshold, or strictly below it. The transition from a nonlasing situation to a lasing one cannot be fit into the semiclassical framework since all expressions derived from this theory are deterministic in nature, and the one important ingredient missing from the deterministic description is *noise*, an intrinsically stochastic quantity. Laser noise is responsible for *spontaneous* emission, which ensures that even in the absence of pumping (and hence at a point far below the lasing threshold), there will still be spontaneous excitation and decay of atoms to and from different energy levels. Radiation originating from these transitions are responsible for light emitted from normal lamps, for example. In other words, without the inclusion of spontaneous emission, a nonlasing system will remaing nonlasing for all times, and to study dynamics of lasing we would have to assume that we are already in the lasing regime.

Hence, it is then important that one sees how the full quantum mechanical derivation of the lasing equation can be done. In our work we restricted ourselves to the semiclassical case, where there are no noise terms, but for sake of completeness we will detail in the following the complete derivation of the lasing equations and thereafter restrict ourselves to the relevant regime. The entirety of this section has been adapted from the relevant sections in [75].

D.1.2 The laser Hamiltonian

In order to have a full quantum mechanical description we need to write down the Hamiltonian for our system. For that it is important to understand how the laser system can be physically modeled.

In general the laser system can be seen as being made up of an enclosed volume (the "resonator"), containing the gain material, and an outside source of energy (the "pump") to excite the gain electrons. The resonator is assumed to be leaky, i.e., energy can be dissipated out into an infinite continuum via the coupling of the resonator to heat baths. Radiation inside the resonator couples to the atoms of the gain medium, in the sense that each different mode (assuming multimode operation of the laser) couples to a different bath. The atoms are also assumed to be coupled to their own baths. In addition, we also need to deal with the randomness in the gain medium.

There are some subtleties associated with the description above: assuming coupling of each radiating mode in the resonator to its *own* bath implies absence of bath-bath coupling, which simplifies the analysis although it is known that this effect is known to incorrect for resonators with arbitrary values of the Q-factor [18]. In addition, it is also of course important to ensure that the losses from the cavity are not so large as to destroy the discrete mode structure of the fields inside the resonator [18].

The Hamiltonian which we have described above will have the following form:

$$H = H_f + H_A + H_{B_1} + H_{B_2} + H_{A-f} + H_{B_1-f} + H_{B_2-A} \tag{D.1}$$

A derivation of this Hamiltonian (quantization procedure) is given below. Here we first give and explain the individual terms in detail:

1. The first two terms comprises the "free" part of the Hamiltonian.

   a) H_f is the part of the Hamiltonian describing the propagation of free field modes. For simplicity here we can assume a regularly shaped cavity; the specific shape does not matter for our purposes as long as it supports a mode structure of the waves inside the cavity. This term also contains the coupling of the field to the disorder potential $\mathcal{V}$, which characteristics will be clarified in a later chapter. H_f hence takes the simple form
   $$H_f = \hbar \sum_\nu \omega_\nu b_\nu^\dagger b_\nu + \hbar \sum_\nu \mathcal{V}_\nu b_\nu^\dagger b_\nu \tag{D.2}$$
   where $b_\nu^\dagger$, b_ν are field creation and annihilation operators, and we have dropped the zero point energy.

   b) H_A describes the atomic Hamiltonian. We remind that lasing action requires the presence of a gain medium in which propagating light can be amplified. The behavior of this medium is described by this term H_A. The physical picture of such a gain medium has been elucidated in Sec. 5.1.4, in particular the required configurations of atomic energy levels. From there we summarize that the simplest possible picture of a gain medium is the 2-level atom; in such a picture population inversion is assumed to occur only between these two levels and no other transitions are possible. In general lasing action requires more complicated atomic structures, but for the purpose of a "toy" model it is sufficient. The 2-level atom located at $\mathbf{r}_0$ is described by the

Hamiltonian

$$H_A = \frac{1}{2}\hbar\overline{\omega}\hat{\sigma}_z(\mathbf{r}_0) \tag{D.3}$$

where $\overline{\omega}$ is the frequency difference between the 1st and 2nd atomic levels, measured with respect to the 1st level, assuming that the energy of the 1st level can be set to 0. This is the energy of the atomic transition, and is the largest energy scale in the system under consideration. $\hat{\sigma}_z(\mathbf{r}_0)$ is the z Pauli matrix describing the occupation of the atom at $\mathbf{r}_0$. Of course, this term can be trivially generalized to the situation of a collection of atoms (which is usually the case) by summing (D.3) over an additional index

$$H_A = \frac{1}{2}\hbar\overline{\omega}\sum_{\mu}\hat{\sigma}_z(\mathbf{r}_\mu) \tag{D.4}$$

or, in second-quantized form,

$$H_A = \hbar\overline{\omega}\sum_{\mu} a^{\dagger}_{2,\mu}a_{2,\mu}, \tag{D.5}$$

where the operators $a^{\dagger}_{2,\mu}, a_{2,\mu}$ create (annihilate) an atom at the second (higher) energy level.

c) The first bath term H_{B_1} denotes the bath to which the atomic levels are coupled. As this bath will have only the role of a dissipative bath we will assume that it takes the simple form of a bath of harmonic oscillators, which thus have the usual form

$$H_{B_1} = \sum_{\omega}\hbar\omega\,\mathcal{B}^{\dagger}_{\omega}\,\mathcal{B}_{\omega} \tag{D.6}$$

The bath modes have a continuum of frequencies.

d) The second bath couples to the field modes of the resonator; we can also assume the dynamics of a harmonic oscillator Hamiltonian for this bath,

$$H_{B_2} = \sum_{m}\int d\omega\,\hbar\omega\,B^{\dagger}_{m}(\omega)B_{m}(\omega) \tag{D.7}$$

The bath modes are characterized by a discrete index m as they are assumed to couple to discrete modes of the laser resonator.

2. The term H_{A-f} is the coupling between the field modes and the atomic levels. It has the form

$$H_{A-f} = \sum_{\mu}\sum_{\lambda}(A^{*}_{\mu\lambda}\hat{\sigma}^{+}_{\mu}b_{\lambda} + A_{\mu\lambda}b^{\dagger}_{\lambda}\hat{\sigma}^{-}_{\mu}) \tag{D.8}$$

where the atom-field coupling strength is given by $A_{\mu\lambda}$ in the form

$$A_{\mu\lambda} = -i\sqrt{\frac{\overline{\omega}}{2\hbar\epsilon_0}}\int d^3\mathbf{x}\,\mathbf{u}_{\lambda}(\mathbf{x}_\mu)\,\varphi^{*}_1(\mathbf{x}_\mu)\,e\mathbf{x}\,\varphi_2(\mathbf{x}_\mu)$$

where $\varphi_i(\mathbf{x})$ is the eigenfunction of the atomic level i, while $\mathbf{u}_\lambda(\mathbf{x}_\mu)$ denotes the field eigenmode λ at position $\mathbf{x}_\mu$ of the atom. For the general case of a multi-atomic medium laser operating in the multimode regime, the indices μ and λ are then summed over.

3. The terms $H_{B_2} + H_{B_2-f}$ provides for the bath-field mode coupling. This is the term which allows for coupling of resonator modes with the external continuum. There is an inherent subtlety in quantizing such "leaky" modes since it is not *a priori* clear what are the suitable eigenmodes to use in the expansion of the field operators during the quantization process. It has been shown in [110] that the bath B_1 and indeed its coupling to the resonator modes can be derived for a quantum description of the system in an exact manner, and it can be shown that the field Hamiltonian then reduces to a usual "system-bath" coupling.

$$H_{B_2} + H_{B_2-f} = \sum_m \int d\omega\, \hbar\omega\, B_m^\dagger(\omega) B_m(\omega) + \\ + \hbar \sum_\lambda \sum_m \int d\omega \left[\mathcal{W}_{\lambda,m}(\omega) b_\lambda^\dagger B_m(\omega) + \mathcal{W}_{\lambda,m}^* B_m^\dagger(\omega) b_\lambda \right] \quad \text{(D.9)}$$

The first term describes the dynamics of the bath modes, which corresponds to set of independent harmonic oscillators. The second term very closely resemble a typical "system-bath" coupling term, where the $a_\lambda^{(\dagger)}$ operators correspond "resonator" creation / annihilation operators, while $B_m^{(\dagger)}(\omega)$ are the bath operators, now frequency dependent.

4. The final terms $H_{B_1} + H_{B_1-A}$ are the terms corresponding to the 2nd (distinct) bath and the coupling of this bath to the atomic levels. This coupling induces an additional decay of the population inversion in the system. Since this bath only plays the role of a dissipative bath we will assume that it takes the simplest configuration of a harmonic oscillator bath. This also means that the atom-bath coupling term will take the usual form, as shown below

$$H_{B_1} + H_{B_1-A} = \hbar \sum_{\omega'} \omega' \mathcal{B}_{\omega'}^\dagger \mathcal{B}_{\omega'} + \hbar \sum_{k'} \left(g_{k'}^{AB}(\mathbf{r}_0) \hat{\sigma}_+ \mathcal{B}_{k'} + H.c. \right) \quad \text{(D.10)}$$

The coupling constant $g_k^{AB}(\mathbf{r}_0)$ is local at the location of our single two-level atom; in the case where we assume a collection of gain atoms there will be an additional sum over atomic positions in the second term in (D.10). At the level of this work the effect of H_{B_2-f} would be simply to contribute to the total noise content of the lasing system.

D.1.3 Equations of motion of quantities

Having defined the Hamiltonian of our system of interest, the next task is to find an expression describing the field modes of the resonator. This will involve the elimination of the various atomic and bath modes, and in the approach of Haken this is done via the equations of motion for the field modes and atomic modes. Specifically, we will derive equations of motion for the following quantities:

1. The field modes $b_\nu^\dagger$, b_ν;
2. The atomic dipole moments $\hat{\sigma}_+$ and $\hat{\sigma}_-$;
3. The inversion $\hat{\sigma}_z$.

Equation of motion for the field modes in the presence of field-bath and field-atom coupling

Since we have the explicit Hamiltonian terms for the various parts required the equation of motion for $b_\nu^\dagger$, b_ν can be derived simply. We consider the equation

$$\dot{b}_\nu = \frac{i}{\hbar}\Big[(H_f + H_{A-f} + H_{B_2-f}), b_\nu\Big] \tag{D.11}$$

$$= -i\omega_\nu b_\nu - i\mathcal{V}_\nu b_\nu + i\sum_\mu A_{\mu\nu}\hat{\sigma}_\mu^- - i\sum_m \int d\omega\, \mathcal{W}_{\nu,m}(\omega)B_m(\omega) \tag{D.12}$$

In order to write (D.12) in terms of field modes b_ν, $b_\nu^\dagger$ only we can correspondingly derive an equation of motion for the bath modes of B_2, which is

$$\dot{B}_m^\dagger(\omega) = i\omega B_m^\dagger(\omega) + i\sum_\lambda \mathcal{W}_{\lambda,m}(\omega)b_\lambda^\dagger \tag{D.13}$$

(D.13) allows immediately solution in the form

$$B_m^\dagger(\omega,t) = i\sum_\nu \int_0^t d\tau\ \mathcal{W}_{\nu,m}(\omega)b_\nu^\dagger e^{i\omega(t-\tau)} + B_m^\dagger(\omega,t=0)e^{i\omega t} \tag{D.14}$$

and substitution of (D.14) into (D.12) gives the solution

$$\dot{b}_\nu = -i\omega_\nu b_\nu - i\mathcal{V}_\nu b_\nu + i\sum_\mu A_{\mu\nu}\hat{\sigma}_\mu^- - i\sum_m \int d\omega\, \mathcal{W}_{\nu,m}(\omega)B_m(\omega,t=0)e^{-i\omega t}$$
$$- \sum_\nu \sum_m \int d\omega \int_{t_0}^t d\tau \left\{\mathcal{W}_{\nu,m}(\omega)\mathcal{W}_{\nu,m}^*(\omega)\, b_\nu^\dagger(\tau)e^{-i\omega(t-\tau)}\right\} \tag{D.15}$$

We can simplify (D.15) via several simplifications:

1. We separate the time dependence of the field modes $b_\nu^\dagger(t)$ as follows:

$$b_\nu^\dagger(t) = \mathrm{e}^{i\omega_\nu t}\widetilde{b}_\nu^\dagger(t) \tag{D.16}$$

   where the time dependence of the factor $b_\nu(t)$ is much slower than that of the exponential factor.

2. In addition, we replace the summation over m in the 2nd term on the r.h.s of (D.15) by an

integral over ω_m such that

$$\sum_m |\mathcal{W}_{\nu,m}(\omega)|^2 \Rightarrow \int_0^\infty d\omega_m \, |\widetilde{\mathcal{W}}_{\nu,\omega_m}|^2 \exp\left[i\omega_m(t-\tau)\right] \exp\left[i\omega_\nu(t-\tau)\right] \tag{D.17}$$

where the $\widetilde{\mathcal{W}}_{\nu,\omega_m}$ differ from $\mathcal{W}_{\nu\omega_m}$ by a factor stemming from the numeration of the ω_ms Given that the coupling factors $\widetilde{\mathcal{W}}_{\nu,\omega_m}$ are associated with a continuum of bath modes, we make the simplification that they are independent of ω_m and ν. We can also shift the integration variable such that $\omega \to \omega'$ such that

$$\omega_m - \omega_\nu = \omega' \tag{D.18}$$

which renders the lower limit of integration to be at $-\omega_\nu$, which we replace by $-\infty$. Hence the integral can be carried out such that

$$\int_{-\infty}^{+\infty} d\omega \, |\widetilde{\mathcal{W}}|^2 \exp\left[i\omega(t-\tau)\right] = 2\kappa\delta(t-t') \tag{D.19}$$

where $\kappa = \pi|\widetilde{\mathcal{W}}|^2$. Putting all parts together we see that the equation of motion for the field modes can now be written in the form

$$\dot{b}_\nu = -i\Big[\big(\omega_\nu + \mathcal{V}_\nu\big) + i\kappa\Big]b_\nu + i\sum_\mu A_{\mu,\nu}\hat{\sigma}_\mu^- \underbrace{- i\sum_m \int d\omega \, \mathcal{W}_{\nu,m}(\omega) B_m(\omega, t=0) e^{-i\omega t}}_{F_\nu(t)} \tag{D.20}$$

$F(t)$, which involves the bath modes B_ω at $t = 0$, is a fluctuating force. The correlation functions of this term can be determined. I will just list these here and do not derive them (they are similarly derived in the book of Haken):

$$\langle F_\lambda(t)\rangle = \langle F_\lambda^\dagger(t)\rangle = 0 \tag{D.21}$$

$$\langle F_\lambda(t) F_{\lambda'}(t')\rangle = \langle F_\lambda^\dagger(t) F_{\lambda'}^\dagger(t')\rangle = 0 \tag{D.22}$$

$$\langle F_\lambda^\dagger(t) F_{\lambda'}(t')\rangle = \overline{n}_\lambda(T) 2\kappa_\lambda \delta(t-t')\delta_{\lambda\lambda'} \tag{D.23}$$

$$\langle F_\lambda(t) F_{\lambda'}^\dagger(t')\rangle = (\overline{n}_\lambda(T)+1) 2\kappa_\lambda \delta(t-t')\delta_{\lambda\lambda'} \tag{D.24}$$

where $\overline{n}_\lambda(T)$ is the mean number of modes with frequency λ at temperature T.

Equation of motion for the atomic modes in the presence of atom-bath

EOM for the inversion $\hat{\sigma}_z$

In this section we derive the equation of motion for the atomic inversion $\hat{\sigma}_z$. This involves the calculation of the expression

$$\dot{\hat{\sigma}}_\mu^z = \frac{i}{\hbar}\Big[H_A + H_{A-f} + H_{B_1-A}, \hat{\sigma}_\mu^z\Big] \tag{D.25}$$

Using the commutation relations

$$[\hat{\sigma}_\mu^-, \hat{\sigma}_{\mu'}^+] = -\hat{\sigma}_\mu^z \delta_{\mu,\mu'} \tag{D.26}$$

$$[\hat{\sigma}_\mu^-, \hat{\sigma}_{\mu'}^z] = 2\hat{\sigma}_\mu^- \delta_{\mu,\mu'} \tag{D.27}$$

we can calculate (D.25) easily.

$$\dot{\hat{\sigma}}_\mu^z = -2i \sum_\lambda \left(A_{\mu,\lambda}^* \hat{\sigma}_\mu^+ b_\lambda - A_{\mu,\lambda} b_\lambda^\dagger \hat{\sigma}_\mu^- \right) - 2i \sum_{k'} \left(g_{k'}^{AB}(\mathbf{r}_0) B_{k'} \hat{\sigma}_\mu^+ - \left(g_{k'}^{AB}(\mathbf{r}_0) \right)^* B_{k'}^\dagger \hat{\sigma}_\mu^- \right) \tag{D.28}$$

We can also derive the equation of motion for the bath operators, which takes the familiar form

$$B_k(\omega, t) = e^{-i\omega(t-t_0)} B_k(\omega, t_0) - i \int_{t_0}^t dt' \left(g_k^{AB}(\mathbf{r}_0) \right)^* \exp(-i\omega(t' - t_0)) \hat{\sigma}_\mu^- \tag{D.29}$$

which we substitute back into the noise term to obtain, for the noise (third) term in (D.30) the expression

$$\begin{aligned} \dot{\hat{\sigma}}_\mu^z = -2i \sum_\lambda \left(A_{\mu\lambda}^* \hat{\sigma}_\mu^+ b_\lambda - A_{\mu\lambda} b_\lambda^\dagger \hat{\sigma}_\mu^- \right) - \\ \underbrace{- 2i \sum_{k'} \left[g_{k'}^{AB}(\mathbf{r}_0) e^{-i\omega'(t-t_0)} B_{k'}(\omega, t_0) \hat{\sigma}_\mu^+ - \left(g_{k'}^{AB}(\mathbf{r}_0) \right)^* e^{i\omega'(t-t_0)} B_{k'}^\dagger(\omega, t_0) \hat{\sigma}_\mu^- \right]}_{\Gamma_\mu^d} + \\ \underbrace{+ 4i \sum_{k'} \int_{t_0}^t \left[|g_{k'}^{AB}(\mathbf{r}_0)|^2 (e^{-i\omega'(t-t_0)} - e^{i\omega'(t-t_0)}) \right] \hat{\sigma}_\mu^z}_{-4i\gamma_\parallel \left(\hat{\sigma}_\mu^z(t=t_0) - \hat{\sigma}_\mu^z(t) \right)} \end{aligned} \tag{D.30}$$

EOM for the atomic dipole moments $\hat{\sigma}_\pm$

The coupling of the atomic variables to their respective baths and to the field modes will also affect the variables $\hat{\sigma}_\pm$, which in turn influences the transition between atomic levels. The equation of motion for these variables has the form

$$\dot{\hat{\sigma}}_\mu^+ = \frac{i}{\hbar} \left[\mathcal{H}_A + \mathcal{H}_{A-B_1} + \mathcal{H}_{A-f}, \hat{\sigma}_+ \right] \tag{D.31}$$

The various commutators in the above expression is quite similar to various quantities already calculated above, and hence we will use them in and show directly the final expression,

$$\dot{\hat{\sigma}}_\mu^+ = i\overline{\omega} \sigma_\mu^+ \underbrace{- i \sum_k \left(g_k^{AB}(\mathbf{r}_0) \right)^* B_k^\dagger(\omega, t_0) \sigma_\mu^z e^{i\omega(t-t_0)}}_{\Gamma^\mu} + \underbrace{\sum_k \int_{t_0}^t d\tau \, |g_k^{AB}(\mathbf{r}_0)|^2 e^{i\omega(t-t_0)} \hat{\sigma}_\mu^+}_{\gamma} - i \sum_\lambda A_{\mu,\lambda} b_\lambda^\dagger \hat{\sigma}_\mu^z \tag{D.32}$$

D.1.4 Nonlinearity and multimode operation

Having derived the equations determining the operation of the laser in (D.20), (D.30) and (D.32) we can now eliminate the atomic variables $\hat{\sigma}^{\pm}_{\mu}$, $\hat{\sigma}^{z}_{\mu}$ in order to obtain an equation for the field modes alone. We will do this again as described in Haken, for the case of multimode operation, up to cubic order in the mode operators $b^{\dagger}_{\nu}$, b_{ν}. To do this we first assume that the field modes can be written in the form $b_{\nu}(t) = B_{\nu}(\tau)\exp(-i\Omega_{\nu}t)$, $b^{\dagger}_{\nu}(t) = B^{*}_{\nu}(\tau)\exp(i\Omega_{\nu}t)$ where the time dependence of the envelope $B_{\nu}(\tau)$, $B^{*}_{\nu}(\tau)$ is slow compared to the atomic relaxation time scales γ and $\gamma_{\parallel}$. Technically, we assume $B_{\nu}(\tau)$ to be time-independent within the individual steps of this derivation. We recap here the form of the expressions we will use for the derivation below

$$\dot{b}_{\nu} = -i\Big[\big(\omega_{\nu} + \mathcal{V}_{\nu}\big) + i\kappa_{\nu}\Big]b_{\nu} + i\sum_{\mu} A_{\mu,\nu}\hat{\sigma}^{-}_{\mu} + F_{\nu}(t) \tag{D.33}$$

$$\dot{\hat{\sigma}}^{z}_{\mu} = -2i\sum_{\lambda}\left(A^{*}_{\mu\lambda}\hat{\sigma}^{+}_{\mu}b_{\lambda} - A_{\mu\lambda}b^{\dagger}_{\lambda}\hat{\sigma}^{-}_{\mu}\right) - 4\gamma_{\parallel}(\hat{\sigma}^{z}_{\mu}(t_0) - \hat{\sigma}^{z}_{\mu}(t)) + \Gamma^{d}_{\mu} \tag{D.34}$$

$$\dot{\hat{\sigma}}^{+}_{\mu} = i(\bar{\omega}_{\mu} - i\gamma)\hat{\sigma}^{+}_{\mu} - i\sum_{\lambda} A_{\lambda,\mu}b^{\dagger}_{\lambda}\hat{\sigma}^{z}_{\mu} + \Gamma^{\mu} \tag{D.35}$$

In the first step we first assume that due to pump and relaxation processes the inversion $\hat{\sigma}^{z}_{\mu}$ has reached an initial value $\sigma^{z}_{(0),\mu}$. With the values of $b_{\nu}(t)$ and $\hat{\sigma}^{z}_{(0),\mu}$ we insert these values in (D.32). Since we see that on the r.h.s. of this expression there is a sum of exponentials due to the presence of $\exp(-i\Omega_{\nu}t)$ and $\exp(i\Omega_{\nu}t)$, we assume a similar ansazt for the values of the dipole moment at the first step, which we denote $\hat{\sigma}^{+}_{(1),\mu}(t)$: $\hat{\sigma}^{+}_{(1),\mu}(t) = \sum_{\nu}\hat{B}_{\nu,\mu}e^{i\Omega_{\nu}t}$. This yields the form of (D.32) as

$$\sum_{\nu}(i\Omega_{\nu})\,\hat{B}_{\nu,\mu}e^{i\Omega_{\nu}t} = i(\overline{\omega}_{\mu} - i\gamma)\sum_{\nu}\hat{B}_{\nu,\mu}e^{i\Omega_{\nu}t} - i\sum_{\lambda}\left(A_{\lambda,\mu}\hat{B}^{*}_{\lambda}(\tau)e^{i\Omega_{\lambda}t}\sigma^{z}_{(0),\mu}\right) \tag{D.36}$$

Comparing the coefficients of the exponentials on both sides we obtain the following expression for $\hat{\sigma}^{+}_{(1),\mu}(t)$:

$$\hat{\sigma}^{+}_{(1),\mu}(t) = \sum_{\nu}\frac{\sigma^{z}_{(0),\mu}}{\overline{\omega}_{\mu} - \Omega_{\nu} - i\gamma}A_{\nu,\mu}b^{\dagger}_{\nu} \tag{D.37}$$

Having the expression for $\hat{\sigma}^{+}_{(1),\mu}(t)$ we can now compute an "improved" version of the inversion, $\hat{\sigma}^{z}_{(1),\mu}$. We can do this by assuming the a form of the inversion of the form

$$\hat{\sigma}^{z}_{(1),\mu} = \sum_{\nu,\nu'}\left[C_{\nu,\nu'}e^{i(\Omega_{\nu'}-\Omega_{\nu})t} + D_{\nu,\nu'}e^{-i(\Omega'_{\nu}-\Omega_{\nu})t}\right] \tag{D.38}$$

We however first look at the substitution of $\hat{\sigma}^{\pm}_{(1),\mu}$ on the r.h.s. of (D.30) by (D.37). This yields the expression

$$\dot{\hat{\sigma}}^{z}_{(1),\mu}(t) = 4\gamma_{\parallel}\hat{\sigma}^{z}_{(1),\mu} - 2i\left\{\sum_{\lambda,\nu}\left[\frac{A^{*}_{\mu,\lambda}A_{\nu,\mu}}{\overline{\omega}_{\mu} - \Omega_{\nu} - i\gamma}b^{\dagger}_{\nu}b_{\lambda} - \frac{A_{\mu,\lambda}A^{*}_{\nu,\mu}}{\overline{\omega}_{\mu} - \Omega_{\nu} + i\gamma}b^{\dagger}_{\lambda}b_{\nu}\right]\right\} \tag{D.39}$$

Performing the time derivative on the l.h.s on (D.38) and comparing coefficients on both sides yields the coefficients $C_{\nu,\nu'}$ and $D_{\nu,\nu'}$

$$C_{\nu,\nu'} = \frac{1}{i(\Omega_{\nu'} - \Omega_\nu) - 4\gamma_\parallel} \frac{A^*_{\mu,\lambda} A_{\nu,\mu}}{\overline{\omega}_\mu - \Omega_\nu - i\gamma} B^*_\nu(\tau) B_\lambda(\tau) \tag{D.40}$$

Similarly, we can also obtain the coefficients $D_{k''K}$

$$D_{\nu,\nu'} = \frac{1}{i(\Omega_{\nu'} - \Omega_\nu) + 4\gamma_\parallel} \frac{A^*_{\nu,\mu} A_{\lambda,\mu}}{\overline{\omega}_\mu - \Omega_\nu + i\gamma} B^*_\lambda(\tau) B_\nu(\tau) \tag{D.41}$$

Using the explicit solution for $\hat{\sigma}^z_{(1),\mu}(t)$ we will obtain an "improved" version of the dipole moment, hence denoted $\hat{\sigma}^+_{(2),\mu}(t)$. This can be obtained from the general expression

$$\dot{\hat{\sigma}}^+_{(2),\mu}(t) = -i\left(\overline{\omega} - i\gamma\right)\hat{\sigma}^+_{(2),\mu} - \sum_\lambda A_{\lambda,\mu} b^\dagger_\lambda \hat{\sigma}^z_{(1),\mu} + \Gamma^\mu \tag{D.42}$$

$$\begin{aligned} = -i\left(\overline{\omega} - i\gamma\right)\hat{\sigma}^+_{(2),\mu} - i\sum_\lambda \sum_{\nu,\nu'} A_{\lambda,\mu} b^\dagger_\lambda \Bigg\{ & \frac{A^*_{\mu,\lambda} A_{\nu,\mu}}{i(\Omega_{\nu'} - \Omega_\nu) - 4\gamma_\parallel} \frac{B^*_\nu(\tau) B_\lambda(\tau)}{\overline{\omega}_\mu - \Omega_\nu - i\gamma} e^{i(\Omega_{\nu'} - \Omega_\nu)t} + \\ & + \frac{A^*_{\nu,\mu} A_{\lambda,\mu}}{i(\Omega_{\nu'} - \Omega_\nu) + 4\gamma_\parallel} \frac{B^*_\lambda(\tau) B_\nu(\tau)}{\overline{\omega}_\mu - \Omega_\nu + i\gamma} e^{i(\Omega_{\nu'} - \Omega_\nu)t} \Bigg\} + \Gamma^\mu \end{aligned} \tag{D.43}$$

and using the following Ansatz for $\hat{\sigma}^+_{(2)}$

$$\hat{\sigma}^+_{(2),\mu}(t) = \sum_{i,j,k} \left(\mathcal{X}_{ijk} e^{i(\Omega_i + \Omega_j - \Omega_k)t} + \mathcal{Y}_{ijk} e^{i(\Omega_i - \Omega_j + \Omega_k)t} \right) \tag{D.44}$$

where the coefficients $\mathcal{X}_{ijk}$ and $\mathcal{Y}_{ijk}$ are to be determined. Differentiating the left-hand side of (D.44) and comparing with the right-hand side of (D.43) we obtain for these coefficients

$$\mathcal{X}_{ijk} = \frac{-i}{i\left(\Omega_i + \Omega_j - \Omega_k - \overline{\omega}_\mu\right) - \gamma} \frac{A_{i,\mu} A^*_{j,\mu} A_{k,\mu}}{i(\Omega_j - \Omega_k) - 4\gamma_\parallel} \frac{B^*_i(\tau) B_j(\tau) B^*_k(\tau)}{\overline{\omega}_\mu - \Omega_k - i\gamma} \tag{D.45}$$

$$\mathcal{Y}_{ijk} = \frac{-i}{i\left(\Omega_i - \Omega_j + \Omega_k - \overline{\omega}_\mu\right) - \gamma} \frac{A_{i,\mu} A_{j,\mu} A^*_{k,\mu}}{i(\Omega_i - \Omega_k) + 4\gamma_\parallel} \frac{B^*_i(\tau) B^*_j(\tau) B_k(\tau)}{\overline{\omega}_\mu - \Omega_k + i\gamma} \tag{D.46}$$

The expansion of $\hat{\sigma}_\mu(t)$ up to second order in the field operators $\hat{b}$, $\hat{b}^\dagger$ is given in total then as

$$\hat{\sigma}^+_\mu(t) = \sum_\nu \frac{\sigma^z_{(0),\mu}}{\overline{\omega}_\mu - \Omega_\nu - i\gamma} A_{\nu,\mu} b^\dagger_\nu - 2\sum_{i,j,k} \frac{\sigma^z_{(0),\mu}}{i\left(\Omega_i + \Omega_j - \Omega_k - \overline{\omega}_\mu\right) - \gamma} \frac{A_{i,\mu} A^*_{j,\mu} A_{k,\mu}}{i(\Omega_j - \Omega_k) - 4\gamma_\parallel} \frac{\hat{b}^\dagger_i \hat{b}^\dagger_j \hat{b}_k}{\overline{\omega}_\mu - \Omega_k + i\gamma} + c.c. \tag{D.47}$$

We can now substitute (D.47) into (D.15) to yield the equation of motion for the field amplitudes up to 3rd order

$$\begin{aligned}
\dot{b}_\nu &= -i\Big[\big(\omega_\nu + \mathcal{V}_\nu\big) + i\kappa_\nu\Big]b_\nu + i\sum_\mu A_{\mu,\nu}\hat{\sigma}^-_\mu + F_\nu(t) \\
&= -i\Big[\big(\omega_\nu + \mathcal{V}_\nu\big) + i\kappa_\nu\Big]b_\nu + i\sum_\mu A_{\mu,\nu}\Big(\sum_\nu \frac{\sigma^z_{(0),\mu}}{\overline{\omega}_\mu - \Omega_\nu - i\gamma}A_{\nu,\mu}b^\dagger_\nu - \\
&\quad - 2\sum_{i,j,k}\frac{\sigma^z_{(0),\mu}}{i\big(\Omega_i + \Omega_j - \Omega_k - \overline{\omega}_\mu\big) - \gamma}\frac{A_{i,\mu}A^*_{j,\mu}A_{k,\mu}}{i(\Omega_j - \Omega_k) - 4\gamma_\parallel}\frac{\hat{b}^\dagger_i\hat{b}^\dagger_j\hat{b}_k}{\overline{\omega}_\mu - \Omega_k + i\gamma} + c.c.\Big)^\dagger + F_\nu(t) \\
&= -i\Big[\big(\omega_\nu + \mathcal{V}_\nu\big) + i\kappa_\nu\Big]b_\nu + i\sum_\mu A_{\mu,\nu}\Big(\sum_\nu \frac{\sigma^z_{(0),\mu}}{\overline{\omega}_\mu - \Omega_\nu + i\gamma}A^*_{\nu,\mu}b_\nu \\
&\quad - 2\sum_{i,j,k}\frac{\sigma^z_{(0),\mu}}{i\big(\Omega_i + \Omega_j - \Omega_k - \overline{\omega}_\mu\big) - \gamma}\frac{A^*_{i,\mu}A_{j,\mu}A^*_{k,\mu}}{i(\Omega_j - \Omega_k) - 4\gamma_\parallel}\frac{\hat{b}_i\hat{b}_j\hat{b}^\dagger_k}{\overline{\omega}_\mu - \Omega_k - i\gamma} + c.c.\Big) + F_\nu(t)
\end{aligned} \tag{D.48}$$

From (D.48) we see that we obtain again a cubic nonlinearity similar to the one derived in the semiclassical formalism in (5.21). We also see that the correct form of the generalized dielectric constant (5.20) can also be obtained by including the second term in (D.48) into the first term. The cubic nonlinearity is proportional to the initial population inversion, denoted $\Delta n_0(\mathbf{r})$ in the semiclassical theory and $\hat{\sigma}^z_{(0),\mu}$ in the quantum one. One minor difference consists of the multimodal form of (D.48), which is to be contrasted with the semiclassical theory; however this difference can be negated by setting the mode indices $i = j = k$ in (D.48).

The one major difference between (5.21) and (D.48) is actually present in the form of the *noise* term $F_\nu(t)$, which is completely absent from (5.21). As was clear from the derivation, these terms arise from the coupling of our system components (field modes, medium atoms) to external baths, and serve to drive *spontaneous* emission and in addition provides a lifetime atoms in the higher energy states. Hence we see that in the semiclassical theory we are unable to account for the lifetime of atoms due to spontaneous emission; as is well known [111] the reduction to the semiclassical picture eliminates terms that lead to a nonzero laser linewidth; however, in the our study of *random* lasers, we are interested in the finite linewidth generated by *multiple scattering*, which is of course still finite in the semiclassical picture. Hence in derive the formalism which follows we will concentrate on the semiclassical formulation as presented in Sec. 5.1.4 and leave the quantum equation of motion for the field modes (D.48) for future work.

D.2 Hubbard-Stratonovich Transformation

In this section of the Appendix we will give a more detailed derivation of the Hubbard-Stratonovich (HS) transformation which was used to decouple the interaction term of the action. Specifically, here we derive the parameters α_x and α_s. We start from the interacting term as shown in (D.49) and reproduced below

$$S_{int} = i\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2 \int d\mathbf{r}\,\Delta n_0(\mathbf{r}) \int_0^\infty \frac{d\omega}{\pi} \int_0^\infty \frac{d\omega'}{\pi}\; \omega^2\,\omega'^2\,|\widetilde{A}_\omega(\mathbf{r})|^2|\widetilde{A}_{\omega'}(\mathbf{r})|^2 \tag{D.49}$$

and with the associated HS identity

$$e^{iS_{int}} = \exp\Big\{-\frac{1}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2 \int d\mathbf{r}\,\Delta n_0(\mathbf{r}) \int_0^\infty \frac{d\omega}{\pi} \int_0^\infty \frac{d\omega'}{\pi}\; \omega^2\,\omega'^2\,|\widetilde{A}_\omega(\mathbf{r})|^2|\widetilde{A}_{\omega'}(\mathbf{r})|^2\Big\} \tag{D.50}$$

$$\begin{aligned} &= \mathcal{N}_{\bar\phi,\phi} \int \mathcal{D}[\bar\phi,\phi] \exp\Big\{-\alpha_s^2 \int d\mathbf{r}\,\Delta n_0(\mathbf{r}) \int \frac{d\omega d\omega'}{\pi^2}\Big[\bar\phi_{\omega\omega'}(\mathbf{r})\phi_{\omega'\omega}(\mathbf{r}) + \\ &\qquad + \alpha_x\omega\omega'\widetilde{A}^*_\omega(\mathbf{r})\phi_{\omega\omega'}(\mathbf{r})\widetilde{A}_{\omega'}(\mathbf{r})\Big]\Big\} \end{aligned} \tag{D.51}$$

We note that the form of the HS identity is slightly different depending on whether the population inversion (external pumping) $\Delta n_0(\mathbf{r})$ is considered homogeneous ($\Delta n_0(\mathbf{r}) = \Delta n_0$) or retains its full spatial dependence. In both cases, however, the most important factor is to find fields $\bar\phi_{\omega\omega'}(\mathbf{r}), \phi_{\omega\omega'}(\mathbf{r})$ that satisfies the condition

$$\mathcal{N}_{\bar\phi,\phi} \int \mathcal{D}[\bar\phi,\phi] \exp\Big\{-\alpha_s^2 \int d\mathbf{r}\,\Delta n_0(\mathbf{r}) \int \frac{d\omega d\omega'}{\pi^2}\bar\phi_{\omega\omega'}(\mathbf{r})\phi_{\omega'\omega}(\mathbf{r})\Big\} \stackrel{!}{=} 1 \tag{D.52}$$

where $\bar\phi, \phi$ are generally *independent* fields. We would like to cancel the quartic term (D.49). For the time being let us assume a spatially-independent population inversion. Then it is easy to see via simple substitution that this can be achieved if we shift the fields $\bar\phi_{\omega',\omega}(\mathbf{r}), \phi_{\omega',\omega}(\mathbf{r})$ in the following manner

$$\bar\phi_{\omega\omega'}(\mathbf{r}) \mapsto \bar\phi_{\omega\omega'}(\mathbf{r}) + i\alpha_x\widetilde{A}_\omega(\mathbf{r})\omega\omega'\widetilde{A}^*_{\omega'}(\mathbf{r}) \tag{D.53}$$

$$\phi_{\omega'\omega}(\mathbf{r}) \mapsto \phi_{\omega'\omega}(\mathbf{r}) + i\alpha_x\widetilde{A}_{\omega'}(\mathbf{r})\omega'\omega\widetilde{A}^*_{\omega'}(\mathbf{r}) \tag{D.54}$$

with the condition

$$\alpha_x^2\alpha_s^2 = \frac{\Delta n_0}{4\hbar}\left(\frac{d^2}{\hbar\gamma_\perp}\right)^2 \tag{D.55}$$

We see in this case, provided that suitable values of α_x and α_s could be found, we only need to introduce *one* HS field. We first determine the coefficient α_s, as this is associated with the

normalization condition (D.52). As is usually done, we can discretize (D.52):

$$\begin{aligned}
&\mathcal{N}_{\bar{\phi},\phi} \int \mathcal{D}[\bar{\phi},\phi] \exp\Big\{ -\alpha_s^2 \int d\mathbf{r} \int \frac{d\omega d\omega'}{\pi^2} \bar{\phi}_{\omega\omega'}(\mathbf{r}) \phi_{\omega'\omega}(\mathbf{r}) \Big\} \\
&= \mathcal{N}_{\bar{\phi},\phi} \int \mathcal{D}[\bar{\phi}^i_{\omega,\omega'}, \phi^i_{\omega,\omega'}] \exp\Big\{ -\alpha_s^2 \sum_{\omega,\omega'} \sum_i \bar{\phi}^i_{\omega\omega'} \phi^i_{\omega'\omega} \Big\} \\
&= \mathcal{N}_{\bar{\phi},\phi} \prod_{\omega,\omega'} \prod_i \int \mathcal{D}[\bar{\phi}^i_{\omega,\omega'}, \phi^i_{\omega,\omega'}] \exp\Big\{ -\alpha_s^2 \bar{\phi}^i_{\omega\omega'} \phi^i_{\omega'\omega} \Big\} \\
&\propto \mathcal{N}_{\bar{\phi},\phi} \det(2\alpha_s^2)^{-1/2} \stackrel{!}{=} 1
\end{aligned} \tag{D.56}$$

where the normalization constant $\mathcal{N}_{\bar{\phi},\phi}$ can be freely chosen to satisfy (D.56). Hence we choose it such that $\alpha_s = 1$, and in this manner we can determine this parameter. Once α_s is fixed, we can then find α_x from (D.55), which can then be trivially determined to be

$$\alpha_x = \frac{d^2 \sqrt{\Delta n_0}}{2\hbar\gamma_\perp} \tag{D.57}$$

In the case where the spatial dependence of the pumping is not constant, we can use considerations similar to the above to derive the coefficients α_x and α_s. In this case, we obtain

$$\alpha_x = \frac{1}{2\sqrt{\mathcal{P}}} \left(\frac{d^2}{\gamma_\perp} \right) \tag{D.58}$$

$$\alpha_s = \pm \sqrt{\mathcal{P}} \tag{D.59}$$

However these values are only approximate and more study is needed to convince of the validity of this approach. For the work in this thesis a homogeneous pumping is thus assumed.

D.3 Derivation of term containing the Hubbard-Stratonovich field

In this section we will give a short derivation of the term containing the Hubbard-Stratonovich field in the diffusive action, starting from (D.60). The first two terms on the right-hand side of (D.60) correspond to the free Green's function and Q-field contributions resulting from disorder averaging, respectively. These terms have been described in Chapter 2 and we have seen that we can derive a diffusive form of the action. Including the contribution from the HS field is relatively straightforward, and we see that from the last term in (D.60) we obtain

$$\begin{aligned}
-\alpha \mathrm{Tr}\ \ln[\hat{\Phi}] &= -\alpha \mathrm{Tr}\ \ln\left\{\hat{\gamma}\hat{R}^{-1}\left[\hat{R}\hat{\gamma}\hat{\Phi}\hat{R}^{-1}\right]\hat{R}\right\} \\
&\Rightarrow -\alpha \mathrm{Tr}\ \ln\left[\hat{\mathcal{G}}\hat{\gamma}\hat{R}\hat{\gamma}\hat{\Phi}\hat{R}^{-1}\right] \\
&\Rightarrow -i\pi\nu\alpha \mathrm{tr}\ \ln\left[\hat{\Lambda}\hat{R}\hat{\gamma}\hat{\Phi}\hat{R}^{-1}\right] \\
&= -i\pi\nu\alpha \mathrm{tr}\ \ln\left[\hat{\gamma}\hat{\Phi}\hat{R}^{-1}\hat{\Lambda}\hat{R}\right] \\
&= -i\pi\nu\alpha \mathrm{tr}\ \ln\left[\hat{\gamma}\hat{\Phi}\hat{Q}\right] \equiv -i\pi\nu\alpha \mathrm{tr}\ \ln\left[\hat{\Phi}\hat{Q}\right]
\end{aligned}$$

D.4 Calculation of the saddlepoint equations in presence of nonlinearity

In this section we will perform a detailed calculation of the saddle point equations including the nonlinearity. We start from the full action containing the Hubbard-Stratonovich field $\hat{\Phi}_{\omega\omega'}(\mathbf{r})$, which has the form

$$iS[\hat{Q}] \equiv -\frac{\pi\nu\widetilde{\hbar}}{4\tau}\mathrm{Tr}\,[\hat{Q}^2] - \mathrm{Tr}\,[\hat{\Phi}^\dagger\hat{\gamma}\hat{\Phi}] + \mathrm{Tr}\ \ln\left[\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q} - \alpha\,\hat{\Phi}\right], \tag{D.60}$$

We concentrate on the third term on the right-hand side. This term can be written explicitly as

$$\begin{aligned}
&\mathrm{Tr}\ \ln\left[\hat{G}_0^{-1} + \frac{\widetilde{\hbar}}{2\tau}\hat{\gamma}\hat{Q} - \alpha\,\hat{\Phi}\right] \\
&= \mathrm{Tr}\ \ln\left[\begin{pmatrix} 0 & \left[G_0^A\right]^{-1} \\ \left[G_0^R\right]^{-1} & \left[G_0^K\right]^{-1} \end{pmatrix} + \frac{\widetilde{\hbar}}{2\tau}\begin{pmatrix} 0 & Q^A \\ Q^R & Q^K \end{pmatrix} - \alpha\begin{pmatrix} \phi^q & \phi^{cl} \\ \phi^{cl} & \phi^q \end{pmatrix}\right] \\
&= \mathrm{Tr}\ \ln\Bigg[\underbrace{\begin{pmatrix} 0 & \left[G_0^A\right]^{-1} + \frac{\widetilde{\hbar}}{2\tau}Q^A - \alpha\phi^{cl} \\ \left[G_0^R\right]^{-1} + \frac{\widetilde{\hbar}}{2\tau}Q^R - \alpha\phi^{cl} & \frac{\widetilde{\hbar}}{2\tau}Q^K \end{pmatrix}}_{\hat{\gamma}\hat{G}_{cl}^{-1}} - \alpha\underbrace{\begin{pmatrix} \phi^q & 0 \\ 0 & \phi^q \end{pmatrix}}_{\hat{\Phi}^q}\Bigg] \\
&= \mathrm{Tr}\ \ln\left[\hat{\gamma}\hat{G}_{cl}^{-1}\left(1 - \alpha\hat{G}_{cl}\hat{\gamma}\hat{\Phi}^q\right)\right] \\
&= \mathrm{Tr}\ \ln\hat{\gamma}\hat{G}_{cl}^{-1} \mp \mathrm{Tr}\ \ln\left(1 - \alpha\hat{G}_{cl}\hat{\gamma}\hat{\Phi}^q\right)
\end{aligned} \tag{D.61}$$

We look at the second term in the expression above (the first term is proportional to the determinant of the inverse Green's function containing only the *classical* components of the Hubbard-Stratonovich field, and is hence unity. Taking the log gives zero). Written out explicitly this term is

$$\begin{aligned}
\mathcal{S}' &= \mathrm{Tr}\,\ln\left(\mathbb{1} - \alpha\hat{G}_{cl}\hat{\Phi}^q\right) \\
&\equiv \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\,d\mathbf{r}'\int dt\,dt'\;\ln\left(\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') - \alpha\hat{G}_{cl}(\mathbf{r},\mathbf{r}';t,t')\hat{\gamma}\hat{\Phi}^q(\mathbf{r}',\mathbf{r};t',t)\right)\right\} \\
&= \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\,d\mathbf{r}'\int dt\,dt'\;\ln\left(\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') - \alpha\hat{G}_{cl}(\mathbf{r},\mathbf{r}';t,t')\hat{\gamma}\hat{\Phi}^q(\mathbf{r}',t')\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t')\right)\right\} \\
&= \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\int dt\;\ln\left(\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t') - \alpha\hat{G}_{cl}(\mathbf{r},\mathbf{r};t,t)\hat{\gamma}\hat{\Phi}^q(\mathbf{r},t)\right)\right\}
\end{aligned}$$

where we imposed that $\hat{\Phi}^q(\mathbf{r},\mathbf{r}';t,t') = \hat{\Phi}^q(\mathbf{r},t)\delta_{\mathbf{r},\mathbf{r}'}\delta(t-t')$. Taking the functional derivative of (D.62) with respect to $\phi^{cl}(\mathbf{x},\tau)$ gives

$$\begin{aligned}
&\frac{\delta\mathcal{S}'}{\delta\phi^{cl}(\mathbf{x},\tau)} \\
&= \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\int dt\left[(\pm i\alpha)\underbrace{\left(\mathbb{1} - \alpha\hat{G}_{cl}(\mathbf{r},\mathbf{r};t,t)\hat{\Phi}^q(\mathbf{r},t)\right)^{-1}}_{\hat{f}^{-1}(\mathbf{r},t;\mathbf{r},t)}\begin{pmatrix}\left[G^R_{cl}\right]' & \left[G^K_{cl}\right]' \\ 0 & \left[G^A_{cl}\right]'\end{pmatrix}\begin{pmatrix}0 & \phi^q(\mathbf{r},t) \\ \phi^q(\mathbf{r},t) & 0\end{pmatrix}\right]\right\} \\
&= \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\int dt\left[\frac{(\pm i\widetilde{\alpha})}{\det\hat{f}(\mathbf{r},t;\mathbf{r},t)}\begin{pmatrix}1 & \mp\alpha G^R_{cl}\circ\phi^q \\ \mp i\alpha G^A_{cl}\circ\phi^q & 1\pm i\alpha G^K_{cl}\circ\phi^q\end{pmatrix}\begin{pmatrix}\left[G^R_{cl}\right]' & \left[G^K_{cl}\right]' \\ 0 & \left[G^A_{cl}\right]'\end{pmatrix}\times\right.\right. \\
&\qquad\left.\left.\times\begin{pmatrix}0 & \phi^q(\mathbf{r},t) \\ \phi^q(\mathbf{r},t) & 0\end{pmatrix}\right]\right\} \\
&= \mp\mathrm{tr}_K\left\{\int d\mathbf{r}\int dt\left[\frac{(\pm i\alpha)}{\det\hat{f}(\mathbf{r},t;\mathbf{r},t)}\begin{pmatrix}\left[G^K_{cl}\right]'\circ\phi^q \mp i\alpha G^R_{cl}\circ\phi^q\circ\left[G^R_{cl}\right]'\circ\phi^q & \oplus \\ \otimes & i\alpha G^R_{cl}\circ\phi^q\circ\left[G^R_{cl}\right]'\circ\phi^q\end{pmatrix}\right]\right\} \\
&= \mp\int d\mathbf{r}\int dt\left[\frac{(\pm i\alpha)}{\det\hat{f}(\mathbf{r},t;\mathbf{r},t)}\Big(\underbrace{\left[G^K_{cl}\right]'\circ\phi^q}_{\mathbf{A}} \mp \underbrace{i\alpha G^R_{cl}\circ\phi^q\circ\left[G^A_{cl}\right]'\circ\phi^q + i\alpha G^A_{cl}\circ\phi^q\circ\left[G^R_{cl}\right]'\circ\phi^q}_{\mathbf{B}}\Big)\right]
\end{aligned} \tag{D.62}$$

We first look at the term **A**. Let us parameterize the Keldysh component of the Green's function matrix by the distribution function F in the usual manner: $G^K_{cl} = G^R_{cl}\circ F - F\circ G^A_{cl}$. Hence term **A** can be interpreted in the following manner (the prime should be intepreted as the operation of

$\frac{\delta}{\delta\phi(\mathbf{x},\tau)}$):

$$
\begin{aligned}
&\int d\mathbf{r} \int dt\, \frac{(\pm i\alpha)}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)} \left[G^K_{cl}\right]' \circ \phi^q \\
&\equiv \int d\mathbf{r} \int dt\, dt'\, \frac{(\pm i\alpha)}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)} \left[G^R_{cl}(\mathbf{r},t;\mathbf{r}',t')F(\mathbf{r}',t';\mathbf{r},t) - F(\mathbf{r},t;\mathbf{r}',t')G^A_{cl}(\mathbf{r}',t';\mathbf{r},t)\right]' \phi^q(\mathbf{r},t) \\
&= \int d\mathbf{r} \int dt\, \frac{(\pm i\alpha)}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)} \left[-\Big(G^R_{cl}(\mathbf{r},t;\mathbf{r},t)\Big)^2 F(\mathbf{r},t;\mathbf{r},t) + F(\mathbf{r},t;\mathbf{r}',t')\Big(G^A_{cl}(\mathbf{r},t;\mathbf{r},t)\Big)^2\right] \phi^q(\mathbf{r},t)
\end{aligned}
$$

But since in the development of the formalism we have demanded that $G^R_{cl}(t,t) + G^A_{cl}(t,t) = 0$ (at equal times!), it is easy to see that the last expression in (D.63) vanishes. Term **B** can be treated in a similar manner. We write this down explicitly

$$
\begin{aligned}
&\int d\mathbf{r} \int dt \left[\frac{(\pm i\alpha)^2}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)}\Big(G^R_{cl} \circ \widetilde{\phi}^q \circ \left[G^A_{cl}\right]' \circ \widetilde{\phi}^q + G^A_{cl} \circ \phi^q \circ \left[G^R_{cl}\right]' \circ \phi^q\Big)\right] \\
&= \int d\mathbf{r} \int dt \int dt' \int dt'' \Big[- \frac{(\pm i\alpha)^2}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)}\Big(G^R_{cl} \circ \phi^q \circ \Big(G^A_{cl}\Big)^2 \circ \phi^q + G^A_{cl} \circ \phi^q \circ \Big(G^R_{cl}\Big)^2 \circ \phi^q\Big)\Big] \\
&= \int d\mathbf{r} \int dt \Big[- \frac{(\pm i\alpha)^2}{\det \hat{f}(\mathbf{r},t;\mathbf{r},t)} G^R_{cl} \circ \phi^q \circ G^A_{cl} \circ \Big(G^A_{cl} + G^R_{cl}\Big) \circ \phi^q\Big] = 0
\end{aligned}
\tag{D.63}
$$

where in the last step we have used the fact that for equal times $G^R_{cl}(t,t) + G^A_{cl}(t,t) = 0$ to exchange $G^R_{cl} \to -G^A_{cl}$ and $G^A_{cl} \to -G^R_{cl}$. In the last step we have simply extracted the common factors. The resulting sum in the brackets is then zero. From the above calculations we see that at the saddlepoint the quantum component of the Hubbard-Stratonovich field vanishes $\widetilde{\phi}^q_{cl}(\mathbf{r},t) = 0$

D.5 Diffusive motion

In this section we will derive explicitly the action up to the 2nd power of the diffusive modes $w^\dagger$ and w . We remind that these quantities have the functional dependencies

$$w^\dagger = w^\dagger_{tt'}(\mathbf{r}) \tag{D.64}$$

$$w = w_{tt'}(\mathbf{r}) \tag{D.65}$$

The action including nonlinearity has the form

$$iS[\hat{Q},\hat{\Phi}] = \frac{i\nu}{2}\mathrm{Tr}\left[\hat{\Phi}\hat{\sigma}_x\hat{\Phi}\right] - \frac{\pi\nu}{4}\mathrm{Tr}\left\{D(\nabla_{\mathbf{r}}\hat{Q})^2 - 4i\partial_t\hat{Q} - 4i\widetilde{\alpha}\hat{\widetilde{\Phi}}\hat{Q}\right\} \tag{D.66}$$

Since we need to express the action in both first and second order in w , $w^\dagger$ we first write down the expressions that contribute respectively at those orders. The expansion of the $\hat{Q}$ matrix up to second order in the diffusive modes can be written symbolically as

$$\hat{Q} \approx i\hat{\Lambda} + \delta\hat{Q}^{(1)} + \delta\hat{Q}^{(2)} \tag{D.67}$$

We first look at contributions coming from the terms $\mathbb{1}_A$ and $\mathbb{2}_A$ to the respective actions $iS^{(1)}[\hat{Q},\hat{\Phi}]$ and $iS^{(2)}[\hat{Q},\hat{\Phi}]$. Substituting (D.67) into the spatial derivative term of (D.66) yields

$$\begin{aligned}
D\,\mathrm{Tr}\,(\nabla_{\mathbf{r}}\hat{Q})^2 &= D\,\mathrm{Tr}\left\{\nabla_{\mathbf{r}}\left[\hat{i}\Lambda + \delta\hat{Q}^{(1)} + \delta\hat{Q}^{(2)}\right]\right\}^2 \\
&= D\,\mathrm{Tr}\left\{\left[i\nabla_{\mathbf{r}}\hat{\Lambda} + \nabla_{\mathbf{r}}\delta\hat{Q}^{(1)} + \nabla_{\mathbf{r}}\delta\hat{Q}^{(2)}\right]^2\right\} \\
&= D\,\mathrm{Tr}\left\{-\left(\nabla_{\mathbf{r}}\hat{\Lambda}\right)^2 + 2i\left(\nabla_{\mathbf{r}}\hat{\Lambda}\right)\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right) + \right. \\
&\quad \left. + 2i\left(\nabla_{\mathbf{r}}\hat{\Lambda}\right)\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(2)}\right) + \left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right)^2 + 2\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right)\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(2)}\right) + \left(\delta\hat{Q}^{(2)}\right)^2\right\}
\end{aligned} \tag{D.68}$$

To first order in diffusive modes the action has the form

$$iS^{(1)}[\hat{Q},\hat{\Phi}] = \frac{i\nu}{2}\mathrm{Tr}\left[\hat{\Phi}\hat{\sigma}_x\hat{\Phi}\right] - \frac{\pi\nu}{4}\mathrm{Tr}\left\{\underbrace{2iD(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)})(\nabla_{\mathbf{r}}\hat{\Lambda})}_{\mathbb{1}_A} - \underbrace{4i\partial_t\delta\hat{Q}^{(1)}}_{\mathbb{1}_B} - \underbrace{4i\widetilde{\alpha}\hat{\widetilde{\Phi}}\delta\hat{Q}^{(1)}}_{\mathbb{1}_C}\right\} \tag{D.69}$$

while the same quantity to second order is

$$\begin{aligned}
iS^{(2)}[\hat{Q},\hat{\Phi}] = \frac{i\nu}{2}\mathrm{Tr}\left[\hat{\Phi}\hat{\sigma}_x\hat{\Phi}\right] - \frac{\pi\nu}{4}\mathrm{Tr}\Big\{&\underbrace{D(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)})^2 + 2iD(\nabla_{\mathbf{r}}\delta\hat{Q}^{(2)})(\nabla_{\mathbf{r}}\hat{\Lambda})}_{\mathbb{2}_A} - \underbrace{4i\partial_t\delta\hat{Q}^{(2)}}_{\mathbb{2}_B} - \\
&- \underbrace{4i\widetilde{\alpha}\hat{\widetilde{\Phi}}\delta\hat{Q}^{(2)}}_{\mathbb{2}_C}\Big\}
\end{aligned} \tag{D.70}$$

In Keldysh space, each term in (D.68) has the following forms

$$\mathrm{Tr}\left(i\nabla_{\mathbf{r}}\hat{\Lambda}\right)^2 = \mathrm{Tr}\left[\begin{pmatrix} 0 & -4\,(\nabla_{\mathbf{r}}F)^2 \\ 0 & 0 \end{pmatrix}\right] = 0 \tag{D.71}$$

$$\begin{aligned} 2i\mathrm{Tr}\left[\left(\nabla_{\mathbf{r}}\hat{\Lambda}\right)\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right)\right] &= -2\mathrm{Tr}\left[\begin{pmatrix} 2\,(\nabla_{\mathbf{r}}F)\left(\nabla_{\mathbf{r}}w^{\dagger}\right) & 2F\left((\nabla_{\mathbf{r}}w^{\dagger})F + w^{\dagger}(\nabla_{\mathbf{r}}F)\right) \\ 0 & 0 \end{pmatrix}\right] \\ &= -4\,\mathrm{tr}\left[\,(\nabla_{\mathbf{r}}F)\left(\nabla_{\mathbf{r}}w^{\dagger}\right)\right] = -4\,\mathrm{tr}\left[\left(\nabla_{\mathbf{r}}^2F\right)w^{\dagger}\right] \end{aligned} \tag{D.72}$$

$$2i\,\mathrm{Tr}\left[\left(\nabla_{\mathbf{r}}\hat{\Lambda}\right)\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(2)}\right)\right] = -2\,\mathrm{Tr}\left[\begin{pmatrix} 0 & -2(\nabla_{\mathbf{r}}F)\left((\nabla_{\mathbf{r}}w^{\dagger})w + w^{\dagger}(\nabla_{\mathbf{r}}w)\right) \\ 0 & 0 \end{pmatrix}\right] = 0 \tag{D.73}$$

The first and third terms in the sequence of equations above do not contribute to the trace, while the second term proportional to $\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}$ is of 1st order in the diffusive modes. This term then contributes to $\mathbb{1}_A$ of $iS^{(1)}[\hat{Q},\hat{\Phi}]$. The final term in the series which contribute to (D.70), proportional to $(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)})^2$ is calculated below

$$\begin{aligned} &\mathrm{Tr}\left[\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right)^2\right] = \\ &= -\mathrm{Tr}\left[\begin{pmatrix} \left[\nabla_{\mathbf{r}}(Fw^{\dagger})\right]^2 - \left(\nabla_{\mathbf{r}}w + \nabla_{\mathbf{r}}(Fw^{\dagger}F)\right)(\nabla_{\mathbf{r}}w^{\dagger}) & \oplus \\ \odot & -(\nabla_{\mathbf{r}}w^{\dagger})(\nabla_{\mathbf{r}}w + \nabla_{\mathbf{r}}(Fw^{\dagger}F)) + [\nabla_{\mathbf{r}}(w^{\dagger}F)]^2 \end{pmatrix}\right] \\ &= -\mathrm{tr}\Big[2(\nabla_{\mathbf{r}}F)(\nabla_{\mathbf{r}}w^{\dagger}) + [\nabla_{\mathbf{r}}(Fw^{\dagger})]^2 + [\nabla_{\mathbf{r}}(w^{\dagger}F)]^2 - \\ &\quad - 2(\nabla_{\mathbf{r}}w^{\dagger})\left[\nabla_{\mathbf{r}}w + ((\nabla_{\mathbf{r}}F)w^{\dagger}F + F(\nabla_{\mathbf{r}}w^{\dagger})F + Fw^{\dagger}(\nabla_{\mathbf{r}}F))\right]\Big] \end{aligned}$$

Performing partial integrations we are then left with the following expression (reinstating the diffusion constant D)

$$\begin{aligned} -2\,D\,\mathrm{tr}\left[\left(\nabla_{\mathbf{r}}\delta\hat{Q}^{(1)}\right)^2\right] &= -2\,D\,\mathrm{tr}\left[(\nabla_{\mathbf{r}}F)(\nabla_{\mathbf{r}}w^{\dagger}) - (\nabla_{\mathbf{r}}w^{\dagger})(\nabla_{\mathbf{r}}w) + (\nabla_{\mathbf{r}}w^{\dagger})F(\nabla_{\mathbf{r}}w^{\dagger})F\right] \\ &\Longrightarrow -2\,D\,\mathrm{tr}\left[w^{\dagger}\nabla_{\mathbf{r}}^2w\right] \end{aligned} \tag{D.74}$$

This term[1] contributes to 2_A of $iS^{(2)}[\hat{Q},\hat{\Phi}]$. We now look at the time derivative terms $\mathbb{1}_B$ and 2_B. The former results in

$$\begin{aligned} -4i\,\mathrm{Tr}\left[\partial_t\delta\hat{Q}^{(1)}\right] &= 4\,\mathrm{tr}\left[-\partial_t F_{tt'}w^{\dagger}_{t't} + (\partial_t w^{\dagger}_{tt'})F_{t't}\right] \\ &= -4\,\mathrm{tr}\left[w^{\dagger}_{t't}\left(\partial_t + \partial'_t\right)F_{tt'}\right] \end{aligned} \tag{D.75}$$

[1] In calculating this term we have discarded a term which is of the form $\mathrm{tr}\left[w^{\dagger}(\partial_{\mathbf{r}}F)w^{\dagger}(\partial_{\mathbf{r}}F)\right]$.

while the term in $\mathbb{2}_B$ has the form

$$\begin{aligned} -4i\,\mathrm{Tr}\left[\partial_t\left(\delta\hat{Q}^{(2)}\right)\right] &= 2\,\mathrm{tr}\left[(\partial_t w_{tt'})w^\dagger_{t't} - (\partial_t w^\dagger_{tt'})w_{t't}\right] \\ &= 2\,\mathrm{tr}\left[w^\dagger_{t't}(\partial_t + \partial'_t)w_{tt'}\right] \end{aligned} \tag{D.76}$$

The final term in the expansion in powers of the diffusive modes come from the last term in (D.66). The complete contribution has the form

$$\begin{aligned} i\pi\nu\widetilde{\alpha}\,\mathrm{Tr}\left[\hat{Q}\hat{\tilde{\Phi}}\right] &= i\pi\nu\widetilde{\alpha}\,\mathrm{Tr}\left[\left(i\hat{\Lambda} + \delta\hat{Q}^{(1)} + \delta\hat{Q}^{(2)}\right)\hat{\tilde{\Phi}}\right] \\ &= i\pi\nu\widetilde{\alpha}\,\mathrm{Tr}\Big[i\begin{pmatrix} \mathbb{1}^R & 2F \\ 0 & -\mathbb{1}^A \end{pmatrix}\circ\begin{pmatrix} \widetilde{\phi}^{cl} & \widetilde{\phi}^{q} \\ \widetilde{\phi}^{q} & \widetilde{\phi}^{cl} \end{pmatrix} + i\begin{pmatrix} \mathbb{1}^R & 2F \\ 0 & -\mathbb{1}^A \end{pmatrix}\circ\begin{pmatrix} -Fw^\dagger & -w - Fw^\dagger F \\ w^\dagger & w^\dagger F \end{pmatrix} + \\ &\qquad + \frac{i}{2}\begin{pmatrix} w & w^\dagger & ww^\dagger F + Fw^\dagger w \\ 0 & & -w^\dagger w \end{pmatrix}\circ\begin{pmatrix} \widetilde{\phi}^{cl} & \widetilde{\phi}^{q} \\ \widetilde{\phi}^{q} & \widetilde{\phi}^{cl} \end{pmatrix}\Big] \\ &= -\pi\nu\widetilde{\alpha}\,\mathrm{tr}\Big[\Big(2F - (d + Fw^\dagger F) + w^\dagger + \frac{1}{2}\left(ww^\dagger F + Fww^\dagger\right)\Big)\phi^q + \\ &\qquad + \Big(-Fw^\dagger + w^\dagger F + \frac{1}{2}\left(ww^\dagger - w^\dagger w\right)\Big)\phi^{cl}\Big] \end{aligned}$$

Hence the total action in 1st order in $w, w^\dagger$ is given by

$$\begin{aligned} iS^{(1)}[w, w^\dagger, \widetilde{\phi}^{cl}, \widetilde{\phi}^{q}] = \frac{i\nu_0}{2}\mathrm{Tr}\left[\hat{\Phi}\hat{\sigma}_x\hat{\Phi}\right] + \pi\nu_0\mathrm{tr}\Big\{D\left(\nabla^2_{\mathbf{r}}F\right)w^\dagger + w^\dagger\left(\partial_t + \partial_{t'}\right)F - \\ - \widetilde{\alpha}\Big[\left(2F - \left(w - w^\dagger\right) - Fw^\dagger F\right)\widetilde{\phi}^{q} + \left(-Fw^\dagger + w^\dagger F\right)\widetilde{\phi}^{cl}\Big]\Big\} \end{aligned} \tag{D.77}$$

and in 2nd order

$$\begin{aligned} iS^{(2)}[w, w^\dagger, \widetilde{\phi}^{cl}, \widetilde{\phi}^{q}] = \frac{i\nu_0}{2}\mathrm{Tr}\left[\hat{\Phi}\hat{\sigma}_x\hat{\Phi}\right] + \pi\nu_0\mathrm{tr}\Big\{Dw^\dagger\nabla^2_{\mathbf{r}}w - w^\dagger\left(\partial_t + \partial_{t'}\right)w - \\ - \widetilde{\alpha}\Big[\left(ww^\dagger F + Fww^\dagger\right)\widetilde{\phi}^{q} + \left(ww^\dagger + w^\dagger w\right)\widetilde{\phi}^{cl}\Big]\Big\} \end{aligned} \tag{D.78}$$

D.6 Saddlepoint action

We look at the prefactor of the $\widetilde{\phi}^q(\mathbf{r},t)$ term in the previous expression. The lower case trace is given explicitly by

$$\begin{aligned}
&-\pi\nu\widetilde{\alpha}\ \mathrm{tr}\Big[\Big(2F-(d+Fw^\dagger F)+w^\dagger+\frac{1}{2}\Big(ww^\dagger F+Fww^\dagger\Big)\Big)\phi^q\Big]\\
&=-\pi\nu\widetilde{\alpha}\Big\{\int d\mathbf{r}\int dt\,dt'\Big[2F_{tt'}(\mathbf{r})\phi^q_t(\mathbf{r})-w_{tt'}(\mathbf{r})\phi^q(\mathbf{r})-\\
&\qquad-\int dt''F_{tt'}(\mathbf{r})w^\dagger_{t't''}(\mathbf{r})F_{t''t}(\mathbf{r})\phi^q_t(\mathbf{r})+w^\dagger_{tt}(\mathbf{r})\phi^q_t(\mathbf{r})+\\
&\quad+\frac{1}{2}\int dt''\Big(w_{tt'}(\mathbf{r})w^\dagger_{t't''}(\mathbf{r})F_{t''t}(\mathbf{r})\phi^q_t(\mathbf{r})+F_{tt'}(\mathbf{r})w_{t't''}(\mathbf{r})w^\dagger_{t''t}(\mathbf{r})\phi^q_t\Big)\Big]\Big\}\\
&\equiv\mathcal{X}
\end{aligned}\tag{D.79}$$

The lower case trace for the prefactor of the $\widetilde{\phi}^{cl}(\mathbf{r},t)$ can be similarly written

$$\begin{aligned}
&-\pi\nu\widetilde{\alpha}\ \mathrm{tr}\Big[\Big(-Fw^\dagger+w^\dagger F+\frac{1}{2}\Big(ww^\dagger-w^\dagger w\Big)\Big)\phi^{cl}\Big]\\
&=-\pi\nu\widetilde{\alpha}\Big\{\int d\mathbf{r}\int dt\,dt'\Big[-F_{tt'}w^\dagger_{t't}(\mathbf{r})\phi^{cl}_t(\mathbf{r})+w^\dagger_{tt'}(\mathbf{r})F_{t't}(\mathbf{r})\phi^{cl}_t(\mathbf{r})+\\
&\quad+w_{tt'}(\mathbf{r})w^\dagger_{t't}(\mathbf{r})\phi^{cl}_t(\mathbf{r})-w^\dagger_{tt'}(\mathbf{r})w_{t't}(\mathbf{r})\phi^{cl}_t(\mathbf{r})\Big]\Big\}\\
&\equiv\mathcal{Y}
\end{aligned}\tag{D.80}$$

Performing the functional derivative of (D.79) with respect to $w^\dagger_{\tau\tau'}(\mathbf{r})$ gives the expression

$$\begin{aligned}
\frac{\delta}{\delta w^\dagger_{\tau\tau'}(\mathbf{x})}\mathcal{X}=-\pi\nu\widetilde{\alpha}\Big[&-\int dtF_{\tau't}(\mathbf{x})\phi^q_t(\mathbf{x})F_{t\tau}(\mathbf{x})-\phi^q_\tau(\mathbf{x})\delta_{\tau\tau'}+\\
&+\frac{1}{2}\int dt\Big(F_{\tau't}(\mathbf{x})\phi^q_t(\mathbf{x})w_{t\tau}(\mathbf{x})+\phi^q_{\tau'}(\mathbf{x})F_{\tau't}(\mathbf{x})d_{t\tau}(\mathbf{x})\Big)\Big]
\end{aligned}\tag{D.81}$$

and of (D.80) yields

$$\frac{\delta}{\delta w^\dagger_{\tau\tau'}(\mathbf{x})}\mathcal{Y}=-\pi\nu\widetilde{\alpha}\Big[F_{\tau'\tau}(\mathbf{x})\Big(\phi^{cl}_\tau(\mathbf{x})-\phi^{cl}_\tau(\mathbf{x})\Big)+w_{\tau'\tau}(\mathbf{x})\Big(\phi^{cl}_{\tau'}(\mathbf{x})-\phi^{cl}_\tau(\mathbf{x})\Big)\Big]\tag{D.82}$$

We note that due to the vanishing of $\phi^q(\mathbf{r},t)$ at the saddlepoint, the right-hand side of (D.81) vanishes as well. As for (D.82), the vanishing of the fluctuations at saddlepoint renders the second term on the right-hand side to be zero, and hence we are left with the first term, which is the source term.

Finally, we also need to differentiate with respect to $w_{\tau\tau'}(\mathbf{x})$. For both $\mathbb{X}$ and $\mathbb{Y}$ terms, we obtain

the combined expression

$$\frac{\delta}{\delta w_{\tau\tau'}(\mathbf{x})}(\mathcal{X}+\mathcal{Y}) = -\pi\nu\widetilde{\alpha}\Big[-\phi^q_{\tau'\tau}(\mathbf{x})+ \\ + w^\dagger_{\tau'\tau}(\mathbf{x})(\phi^{cl}_\tau(\mathbf{x})-\phi^{cl}_{\tau'}(\mathbf{x})) + \int dt w^\dagger_{\tau' t}(\mathbf{x})\left(\phi^q_t(\mathbf{x})F_{t\tau}(\mathbf{x}) + F_{t\tau}(\mathbf{x})\phi^q_\tau(\mathbf{x})\right)\Big] \quad \text{(D.83)}$$

It is then easy to see from the vanishing of $\phi^q(\mathbf{r},t)$ and of the fluctuations at saddlepoint that the right-hand side of (D.83) vanishes and hence does not contribute to the evolution equation for $F_{\tau\tau'}(\mathbf{x},\mathbf{x})$.

D.7 Derivation of the Wigner-transformed nonlinear integro-differential equation

In this appendix we record the full explicit calculation of the Wigner transform for the two main expressions, to be given below.

D.7.1 General expressions for the Wigner transformation

In this section we will fix notation and display some elementary steps in the calculation of the Wigner transformation. In the following sections we will apply the relations derived in this section to our own expressions. We consider a function depending on 2 sets of spatial-time variables $f(x_1, t_1; x_2, t_2)$. The *Wigner* or mixed variables are the sums and differences of the respective spatial and time variables

$$R = \frac{x_1 + x_2}{2} \qquad r = x_1 - x_2 \tag{D.84}$$

$$T = \frac{t_1 + t_2}{2} \qquad t = t_1 - t_2. \tag{D.85}$$

and hence we can rewrite our function in the equivalent forms

$$\begin{aligned} f(x_1, t_1; x_2, t_2) &= f\left(R + \frac{1}{2}r, T + \frac{1}{2}t; R - \frac{1}{2}r, T - \frac{1}{2}t\right) \\ &= \widetilde{f}(R, T; r, t) \end{aligned} \tag{D.86}$$

We can now perform a Fourier transform with respect to the "fast" variables r, t. To do this it is often convenient to introduce the following abbreviation

$$X \equiv (T, R), \quad x \equiv (t, r) \tag{D.87}$$

such that we can write the transform in the following way

$$F(X, p) = \int e^{-ipx} f\left(X + \frac{1}{2}x, X - \frac{1}{2}x\right) \tag{D.88}$$

For the case of a convolution calculation, the following steps will then be performed

$$G(x_1, x_1') \equiv \int dx_2\, A(x_1, x_2)\, B(x_2, x_1') \tag{D.89}$$

which can be written then in Wigner coordinates as

$$\begin{aligned}\widetilde{G}(X,x) &= \int dx_2\, A\left(X+\frac{1}{2}x, x_2\right) B\left(x_2, X-\frac{1}{2}x\right) \\ &= \int dx_2\, \widetilde{A}\left(\frac{1}{2}\left(X+\frac{1}{2}x+x_2\right), X+\frac{1}{2}x-x_2\right) \widetilde{B}\left(\frac{1}{2}\left(x_2+X-\frac{1}{2}x\right), x_2-\left(X-\frac{1}{2}x\right)\right)\end{aligned} \tag{D.90}$$

Performing a shift $x_2 \Rightarrow x_2 - \left(X - \frac{1}{2}x\right)$ we obtain the expression

$$\widetilde{G}(X,x) = \int dx_2 \widetilde{A}\left(X+\frac{1}{2}x_2, x-x_2\right) \widetilde{B}\left(X-\frac{1}{2}x+\frac{1}{2}x_2, x_2\right) \tag{D.91}$$

and the Fourier transform with respect to (x,t) can be performed, resulting in the expression

$$\begin{aligned}G(X,p) &= \int dx\, e^{-ixp} \int dx_2\, \widetilde{A}\left(X+\frac{1}{2}x_2, x-x_2\right) \widetilde{B}\left(X-\frac{1}{2}x+\frac{1}{2}x_2, x_2\right) \\ &= \int dx\, e^{-ixp} \int dx_2 \int \frac{dp'}{(2\pi)^4}\, e^{-ip'(x-x_2)} A\left(X+\frac{1}{2}x_2, p'\right) \\ &\qquad \times \int \frac{dp''}{(2\pi)^4}\, e^{-ip''x_2} B\left(X-\frac{1}{2}x+\frac{1}{2}x_2, p''\right)\end{aligned} \tag{D.92}$$

D.7.2 WT for 1st Equation

We begin with the differential equation with source term, reproduced below:

$$D\partial_Z^2 f_{tt'}(Z) - \partial_T f_{tt'}(Z) = \left(\underset{\sim}{\phi}_t^{cl}(z) + \underset{\sim}{\phi}_{t'}^{cl}(z)\right) f_{tt'}(Z). \tag{D.93}$$

To simplify the calculation of the Wigner transform we write the sum $\left(\phi_t^{cl}(z) + \phi_{t'}^{cl}(z)\right) \equiv \Psi(z,t;z,t')$, which then yields for (D.93) the total expression

$$D\partial_Z^2 f_{tt'}(Z) - \partial_T f_{tt'}(Z) = \Psi(z,t;z,t') f_{tt'}(Z). \tag{D.94}$$

The Wigner transformation involves the rewriting the time and length scales present in our system in terms of *sums* and *differences*:

$$Z \equiv \frac{z+z'}{2}, \qquad \Delta z \equiv z - z' \tag{D.95}$$

$$T \equiv \frac{t+t'}{2}, \qquad \Delta t \equiv t - t' \tag{D.96}$$

We can then perform Fourier transform with respect to the "fast" variables, where the transform has the form

$$\begin{aligned} f_{tt'}(Z) &= f_{T+\frac{\Delta t}{2}, T-\frac{\Delta t}{2}}(Z) \\ &= \int \frac{d\epsilon}{2\pi} \widetilde{f}_\epsilon(Z,T)\, e^{-i\epsilon\Delta t} \end{aligned} \tag{D.97}$$

where we have written $\widetilde{f}_\varepsilon(Z,T) \equiv \widetilde{f}(Z,\varepsilon,T)$. We note that since $f_{tt'}(z)$ is a spatially local quantity, we have, with respect to the spatial variable Z the simple relation $Z = z$. We first look at the right-hand side of (D.94). We perform the Wigner transform by first multiplying (D.94) by $e^{i\Omega\Delta t}$ and integrating over ε. Then

$$
\begin{aligned}
& i \int d\Delta t \; e^{i\omega\Delta t} \Psi(z,t;z,t') f_{tt'}(Z) \\
&= i \int d\Delta t \; e^{i\omega\Delta t} \widetilde{\Psi}(z,T;z,\Delta t) \widetilde{f}(z,T;z,\Delta t) \\
&= i \int \frac{d\omega' d\omega''}{(2\pi)^2} \int d\Delta t \, e^{i\omega\Delta t} \Big\{ \Big[e^{-i\omega'\Delta t} \widetilde{\Psi}(z,z;T,\omega') \Big] \times \\
&\qquad \times \Big[e^{-i\omega''\Delta t} \widetilde{f}(z,z;T,\omega'') + e^{-i\omega''(-\Delta t)} \widetilde{f}(z,z;T,\omega'') \Big] \Big\} \\
&= i \int \frac{d\omega' d\omega''}{2\pi} \int d\Delta t \; e^{-i\omega'\Delta t} e^{i\omega\Delta t} \widetilde{\Psi}(z,z;T,\omega') \times \\
&\qquad \times \left\{ e^{-i(-\omega+\omega'+\omega'')\Delta t} \widetilde{f}(z,z;T,\omega'') + e^{-i(-\omega+\omega'-\omega'')\Delta t} \widetilde{f}(z,z;T,\omega'') \right\} \\
&= i \int \frac{d\omega' d\omega''}{2\pi} \; \widetilde{\Psi}(z,z;T,\omega') \times \\
&\qquad \times \left\{ \delta(-\omega+\omega'+\omega'') \widetilde{f}(z,z;T,\omega'') + \delta(-\omega+\omega'-\omega'') \widetilde{f}(z,z;T,\omega'') \right\} \\
&= i \int \frac{d\omega'}{2\pi} \; \widetilde{\Psi}(z,z;T,\omega') \left\{ \widetilde{f}(z,z;T,\omega-\omega') + \widetilde{f}(z,z;T,\omega'-\omega) \right\}
\end{aligned}
\tag{D.98}
$$

On the left-hand side, one can proceed analogously. We write

$$
\begin{aligned}
\int d\Delta t \; e^{i\omega\Delta t} \left(D\partial_Z^2 f_{tt'}(Z) - \partial_T f_{tt'}(Z) \right) &= \int d\Delta t \; e^{i\omega\Delta t} \left(D\partial_Z^2 \widetilde{f}_{T,\Delta t}(Z) - \partial_T \widetilde{f}_{T,\Delta t}(Z) \right) \\
&= D\partial_Z^2 \widetilde{f}_{T,\Omega}(Z) - \partial_T \widetilde{f}_{T,\Omega}(Z)
\end{aligned}
\tag{D.99}
$$

Hence one obtains for the complete expression

$$
D\partial_X^2 f_{T,\Omega}(Z) - \partial_T f_{T,\Omega}(Z) = i \int \frac{d\omega'}{2\pi} \; \widetilde{\Psi}(z,z;T,\omega') \left\{ \widetilde{f}(z,z;T,\omega-\omega') + \widetilde{f}(z,z;T,\omega'-\omega) \right\}
$$

The function $\Psi(z,z;T,\omega)$ in ω-space is given generically as

$$\begin{aligned}
\widetilde{\Psi}(z,z;T,\omega) &= \int d\Delta t e^{i\omega'\Delta t}\widetilde{\Psi}(z,z;T,\Delta t) \\
&= \int d\Delta t e^{i\omega'\Delta t}\left(\widetilde{\phi}^{cl}(z,t)+\widetilde{\phi}^{cl}(z,t')\right) \\
&= \int d\Delta t e^{i\omega'\Delta t}\left(\widetilde{\phi}^{cl}(z,T+\frac{1}{2}\Delta t)+\widetilde{\phi}^{cl}(z,T-\frac{1}{2}\Delta t)\right) \\
&= \int d\Delta t e^{i\omega'\Delta t}\left\{\left[\int \frac{d\Omega}{2\pi}e^{-i\Omega(T+\frac{1}{2}\Delta t)}\widetilde{\phi}(z,\Omega)\right]+\left[\int \frac{d\Omega}{2\pi}e^{-i\Omega(T-\frac{1}{2}\Delta t)}\widetilde{\phi}(z,\Omega)\right]\right\} \\
&= \int \frac{d\Omega}{2\pi}e^{-i\Omega T}\int d\Delta t\left[e^{-i(\Omega-\omega')\Delta t}\widetilde{\phi}(z,\Omega)+e^{-i(-\Omega-\omega')\Delta t}\widetilde{\phi}(z,\Omega)\right] \\
&= \int \frac{d\Omega}{2\pi}e^{-i\Omega T}\left(\widetilde{\phi}(z,\Omega)\delta(\Omega-\omega')(2\pi)+\widetilde{\phi}(z,\Omega)\delta(-\Omega-\omega')(2\pi)\right) \\
&= e^{-i\omega' T}\widetilde{\phi}(z,\omega')+e^{i\omega' T}\widetilde{\phi}(z,-\omega') \equiv \widetilde{\Psi}(z,z;T,\omega')
\end{aligned} \tag{D.100}$$

and hence to continue we have to find $\widetilde{\phi}(z,z;T,\omega)$.

D.7.3 WT for 2nd Equation

The second equation in our set of equations to be solved is the integral equation

$$\underline{\phi}_t^{cl}(z) = -\frac{\alpha}{2}\int dt' \left[G^R(z,z;t,t')f(z;t',t) - f(z;t,t')G^A(z,z;t',t)\right], \tag{D.101}$$

which can be transformed in a similar manner: on the left hand side we multiply by $e^{i\varepsilon t}$ and integrate over t

$$\int \frac{d\varepsilon}{2\pi} e^{i\varepsilon t}\phi_t(Z) = \phi_\varepsilon(Z)$$

which gives for the right-hand side

$$\begin{aligned}
&\int dt \int dt'\, e^{i\omega t}\Big\{G^R(z,z;t,t')f(z;t',t) - f(z;t,t')G^A(z,z;t',t)\Big\}\\
&= \int dT \int d\Delta t\, e^{i\omega t}\Big\{G^R(z,z;T,\Delta t)f(z;T,\Delta t) - f(z;T,\Delta t)G^A(z,z;T,\Delta t)\Big\}\\
&= \int dT\, d\Delta t\, e^{i\omega t}\Big\{\left[\int \frac{d\Omega}{2\pi} e^{-i\Delta t\Omega} G^R(z,z;T,\Omega)\right]\left[\int \frac{d\Omega'}{2\pi} e^{i\Delta t\Omega'} f(z;T,\Omega')\right] -\\
&\qquad\qquad -\left[\int \frac{d\Omega}{2\pi}\, e^{-i\Delta t\Omega} f(Z;T,\Omega)\right]\left[\int \frac{d\Omega'}{2\pi}\, e^{i\Delta t\Omega'} G^A(z,z;T,\Omega')\right]\Big\}\\
&= \int dT\, d\Delta t \int \frac{d\Omega}{2\pi}\frac{d\Omega'}{2\pi} e^{i\omega(T+\frac{1}{2})\Delta t}\Big\{e^{i\left(\Omega'-\Omega\right)\Delta t} G^R(z,z;T,\Omega)f(Z;T,\Omega') -\\
&\qquad\qquad - e^{-i\Delta t\left(\Omega'+\Omega\right)} f(Z;T,\Omega)G^A(z,z;T,\Omega')\Big\}\\
&= \int dT\, d\Delta t \int \frac{d\Omega}{2\pi}\frac{d\Omega'}{2\pi}\, e^{i\omega T}\Big\{e^{i\left(\Omega'-\Omega+\frac{1}{2}\omega\right)\Delta t} G^R(z,z;T,\Omega)f(Z;T,\Omega') -\\
&\qquad\qquad - e^{-i\left(\Omega'+\Omega-\frac{1}{2}\omega\right)\Delta t} f(Z;T,\Omega)G^A(z,z;T,\Omega')\Big\}\\
&= \int dT \int \frac{d\Omega}{2\pi}\, e^{i\omega T}\Big\{G^R(z,z;T,\Omega)f(Z;T,\Omega-\frac{1}{2}\omega) - f(Z;T,\Omega)G^A(z,z;T,-\Omega+\frac{1}{2}\omega)\Big\}\\
&= \int dt \int \frac{d\Omega}{2\pi}\, e^{i\omega T}\Big(G^R(z,z;T,\Omega+\frac{1}{2}\varepsilon)f(Z;T,\Omega) - f(Z;T,\Omega)G^A(z,z;T,\Omega+\frac{1}{2}\varepsilon)\Big)\\
&= \int dt \int \frac{d\Omega}{2\pi}\, e^{i\omega T}\Big(G^R(z,z;T,\Omega+\frac{1}{2}\omega) - G^A(z,z;T,-\Omega+\frac{1}{2}\omega)\Big)f(Z;T,\Omega)
\end{aligned}$$

D.7.4 WT for full expression

We will now combine the WT results from the previous 2 sections to obtain the full Wigner transformed expression. From the previous sections we have

$$D\partial_X^2 f_{T,\Omega}(Z) - \partial_T f_{T,\Omega}(Z) = \int \frac{d\varepsilon}{2\pi}\, \phi_\varepsilon^{cl}(Z)\left(f_{T,\Omega-\frac{1}{2}\varepsilon}(Z) - f_{T,\Omega+\frac{1}{2}\varepsilon}(Z)\right)e^{-i\varepsilon T} \tag{D.102}$$

and

$$\phi_\varepsilon^{cl}(Z) = \int dT \int \frac{d\Omega}{2\pi}\, e^{i\varepsilon T}\left[G^R(z,z;T,\Omega-\frac{1}{2}\varepsilon) - G^A(z,z;T,\Omega+\frac{1}{2}\varepsilon)\right] f(Z;T,\Omega) \tag{D.103}$$

Substituting (D.103) into (D.102) we obtain the full expression

$$\begin{aligned} &D\partial_X^2 f_{T,\Omega}(Z) - \partial_T f_{T,\Omega}(Z) = \\ &\quad = \int \frac{d\omega'}{2\pi} \int dT \int \frac{d\Omega}{2\pi}\, e^{i\varepsilon T}\left[G^R(z,z;T+T',\Omega_+) - G^A(z,z;T+T',-\Omega_+)\right]\times \\ &\qquad\qquad \times f_{T,\Omega}(Z)\left(f_{T,\omega-\omega'}(Z) - f_{T,\omega'-\omega}(Z)\right) \end{aligned} \tag{D.104}$$

where we have defined $\Omega_\pm \equiv \Omega \pm \frac{1}{2}\omega$ To proceed further we can expand the Green's functions in powers of the "small" frequency ω, i.e. we have the following relations:

$$G^R(z,z;T'+T,\Omega+\frac{1}{2}\omega) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\omega\right)^n \frac{\partial^n}{\partial\Omega^n} G^R(z,z;T'+T,\Omega) \tag{D.105}$$

$$G^A(z,z;T'+T,-\Omega+\frac{1}{2}\omega) = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\left(\frac{1}{2}\omega\right)^n \frac{\partial^n}{\partial\Omega^n} G^A(z,z;T'+T,-\Omega) \tag{D.106}$$

$$f(z,z;T,\omega-\omega') = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\omega'^n \frac{\partial^n}{\partial\omega^n} f(z,z;T,\omega) \tag{D.107}$$

$$f(z,z;T,-\omega+\omega') = \sum_{n=0}^{\infty}\frac{1}{n!}(-1)^n\omega'^n \frac{\partial^n}{\partial\omega^n} f(z,z;T,-\omega) \tag{D.108}$$

Putting everything together one we obtain the following expression on the right hand side:

$$\begin{aligned}
&\frac{\alpha}{2}\int\frac{d\omega'}{2\pi}\int dT\int\frac{d\Omega}{2\pi}e^{i\omega' T}f(z,z;T'+T,\omega)\Big\{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\omega'\right)^n\times\\
&\times\frac{\partial^n}{\partial\Omega^n}\Big[G^R(z,z;T'+T,\Omega)-(-1)^nG^A(z,z;T'+T,-\Omega)\Big]\times\\
&\times\sum_{m=0}^{\infty}\frac{1}{m!}(-1)^m(\omega')^m\frac{\partial^m}{\partial\omega^m}\Big[f(z,z;T,\omega)+f(z,z;T,\omega)\Big]\Big\}\delta(T)
\end{aligned}\tag{D.109}$$

$$\begin{aligned}
=&\frac{\alpha}{2}\int\frac{d\omega'}{2\pi}\int dT\int\frac{d\Omega}{2\pi}e^{i\omega' T}\frac{\partial^m}{\partial T^m}f(z,z;T'+T,\omega)\times\\
&\times\Big\{\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\left(-\frac{1}{2}\right)^n(-i)^{n+m}\frac{\partial^m}{\partial\omega^m}\Big[f(z,z;T,\omega)+f(z,z;T,\omega)\Big]\times\\
&\times\frac{\partial^n}{\partial T^n}\frac{\partial^n}{\partial\Omega^n}\Big[G^R(z,z;T'+T,\Omega)-(-1)^nG^A(z,z;T'+T,-\Omega)\Big]\Big\}\delta(T)
\end{aligned}\tag{D.110}$$

$$\begin{aligned}
=&\frac{\alpha}{2}\int\frac{d\omega'}{2\pi}\int dT\int\frac{d\Omega}{2\pi}e^{i\omega' T}\Big\{\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\left(-\frac{1}{2}\right)^n(-i)^{n+m}\frac{\partial^m}{\partial\omega^m}\Big[f(z,z;T,\omega)+f(z,z;T,\omega)\Big]\times\\
&\times\frac{\partial^n}{\partial T^n}\frac{\partial^n}{\partial\Omega^n}\Big\{\sum_{j=0}^{\infty}\frac{1}{j!}T^j\frac{\partial^j}{\partial T'^j}\Big[G^R(z,z;T',\Omega)-(-1)^nG^A(z,z;T',-\Omega)\Big]\Big\}\Big\}\delta(T)\times\\
&\times\frac{\partial^m}{\partial T^m}\Big[\sum_{l=0}^{\infty}\frac{1}{l!}T^l\frac{\partial^l}{\partial T'^l}f(z,z;T',\Omega)\Big]
\end{aligned}\tag{D.111}$$

From the final expression above it is easy to see that when the following is fulfilled:

$$m>l\quad\text{or}\quad n>j\tag{D.112}$$

$$m<l\quad\text{or}\quad n<j\tag{D.113}$$

then all terms except of the one corresponding to $m=l$ and $n=j$ vanishes. If we now keep only up to first order in the "slow" derivatives $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\Omega}$ and $\frac{\partial}{\partial\omega}$, then we obtain the final expression for the nonlinear integro-differential equation

$$\begin{aligned}
&D\frac{\partial^2}{\partial z^2}f(z,z;T,\omega)-\frac{\partial}{\partial T}f(z,z;T,\omega)=\\
&\qquad=\frac{\alpha}{2}\int\frac{d\Omega}{2\pi}\Big\{\Big[f(z,z;T,\Omega)\mathrm{Im}G^R(z,z;T,\Omega)\Big]\Big[f(z,z;T,\omega)+f(z,z;T,-\omega)\Big]+\\
&\qquad+\left[\left(\frac{\partial}{\partial T}f(z,z;T,\Omega)\right)\Big(\mathrm{Im}G^R(z,z;T,\Omega)\Big)\right]\frac{\partial}{\partial\omega}\left[f(z,z;T,\omega)+f(z,z;T,-\omega)\right]\Big\}
\end{aligned}\tag{D.114}$$

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