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## On a Hadamard-type fractional turbulent flow model with deviating arguments in a porous medium\*

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**Abstract.** In this paper, we concern a Hadamard-type fractional-order turbulent flow model with deviating arguments. By using some standard fixed point theorems, the uniqueness, existence and nonexistence of solutions of the fractional turbulent flow model are investigated. Our results are new and are well illustrated with the aid of three examples.

**Keywords:** fractional turbulent flow model, Hadamard fractional derivation, deviating arguments.

### 1 Introduction

Fractional differential equation models are important and useful in a number of fields such as blood flow phenomena, earthquake, rheology, electrodynamics of a complex medium, aerodynamics, viscoelasticity, and optics and signal processing, and so on [10, 21, 23, 28, 38, 47]. With the development of fractional differential equation models in theory and practice, the subject has acquired great achievements. For some recent works on the subject, readers can see [4–9, 12, 13, 15, 19, 25, 27, 29, 31, 33, 36, 37, 40–42, 44–46] and the references therein.

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Differential equations with deviating arguments are often applied to investigations connected with economic, mechanics, mathematical physics, etc. [1, 16, 32]. In recent years, many researchers are taking notice of Hadamard-type fractional differential equation, not only Riemann–Liouville or Caputo-type fractional differential equation. It is another kind of fractional derivatives and be introduced by Hadamard in 1892 [18]. Hadamard derivative has predominant properties, and its definition includes logarithmic function of arbitrary. For the development of Hamamard fractional calculus, we refer to [2, 3, 11, 17, 20–22, 26, 34, 35].

As we all know, turbulent flow in a porous medium is a basic mechanics problem. In [24], in order to study this type of problem, Leibenson put forward the  $p$ -Laplacian equation as follows:

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)),$$

where  $\phi_p(s) = |s|^{p-2}s$ , which is defined as  $p$ -Laplacian operator, and  $\phi_p^{-1}(s) = \phi_q(s)$ ,  $1/p + 1/q = 1$ .

In [14], Chen, Liu and Hu studied the existence of solutions of the following fractional differential equation with  $p$ -Laplacian operator at resonance:

$$\begin{aligned} D_{0+}^{\beta} \phi_p(D_{0+}^{\alpha} x(t)) &= f(t, x(t), D_{0+}^{\alpha} x(t)), \quad t \in [0, 1], \\ D_{0+}^{\alpha} x(0) &= D_{0+}^{\alpha} x(1) = 0, \end{aligned}$$

where  $0 < \alpha, \beta \leq 1$ ,  $1 < \alpha + \beta \leq 2$ ,  $D_{0+}^{\alpha}$  is a Caputo fractional derivative. By using the coincidence degree theory, a new existence result for the above problem is given.

In [39], Zhang studied the existence of solutions of the following two point boundary value problems with the fractional  $p$ -Laplacian operator:

$$\begin{aligned} D_{0+}^{\beta} \phi_p(D_{0+}^{\alpha} x(t)) &= f(t, x(t), D_{0+}^{\alpha} x(t)), \quad t \in [0, 1], \\ D_{0+}^{\alpha} x(0) &= x(1) = 0, \end{aligned}$$

where  $0 < \alpha, \beta \leq 1$ ,  $D_{0+}^{\alpha}$  is a Caputo fractional derivative, and  $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous. With the help of Schauder fixed point theorem, author establishes two theorems on the existence of solutions for two point boundary value problems with the fractional  $p$ -Laplacian operator.

In [43], Zhang, Liu and Wu investigated the fractional-order model of turbulent flow in a porous medium

$$\begin{aligned} -D_t^{\beta} \phi_p(-D_t^{\alpha} x)(t) &= f(x(t), D_t^{\gamma} x(t)), \quad t \in (0, 1), \\ D_t^{\alpha} x(0) &= D_t^{\alpha+1} x(1) = D_t^{\alpha} x(1) = 0, \\ D_t^{\gamma} x(0) &= 0, \quad D_t^{\gamma} x(1) = \int_0^1 D_t^{\gamma} x(s) dA(s), \end{aligned}$$

where  $D_t^{\alpha}, D_t^{\beta}, D_t^{\gamma}$  are the standard Riemann–Liouville derivatives,  $\int_0^1 x(s) dA(s)$  is denoted by a Riemann–Stieltjes integral, and  $0 < \gamma \leq 1 < \alpha \leq 2 < \beta < 3$ ,  $\alpha - \gamma > 1$ .

$A$  is a function of bounded variation, and  $dA$  can be a signed measure. By employing the fixed point theorem of the mixed monotone operator, authors obtain the uniqueness of positive solution of the above fractional-order turbulent flow model.

Inspired by the above mentioned work, in this paper, we study the uniqueness, the existence and nonexistence of solutions for a fractional-order turbulent flow model with deviating arguments

$$\begin{aligned} -{}^H D^\beta \phi_p(-{}^H D^\alpha u(t)) &= f(t, u(t), u(\theta(t))), \\ {}^H D^\alpha u(1) = {}^H D^\alpha u(e) &= 0, \quad u(1) = 0, \\ {}^H D^{\alpha-1} u(1) &= \eta {}^H D^{\alpha-1} u(e), \end{aligned} \tag{1}$$

where  $1 < \alpha, \beta \leq 2$ ,  $\eta \in \mathbb{R}$ ,  $J = [1, e]$ ,  $\theta \in C[J, J]$ ,  $t \in J$ ,  $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ ,  ${}^H D^\alpha$ ,  ${}^H D^\beta$  denote Hadamard fractional derivative of order  $\alpha$  and  $\beta$ , respectively.

Since the fractional-order turbulent flow model involves Hadamard fractional derivative and deviating arguments has been seldom studied, in this paper, we study the uniqueness, the existence and nonexistence of solutions of the Hadamard-type fractional-order turbulent flow model with deviating arguments, i.e. problem (1). By using the Schauder's fixed point theorem and the contraction mapping principle, three new results on the Hadamard-type fractional-order turbulent flow model (1) are obtained.

We also use the following properties of the  $p$ -Laplacian operator:

(L1) If  $1 < q < 2$ ,  $uv > 0$ ,  $|u|, |v| \geq m > 0$ , then

$$|\phi_q(u) - \phi_q(v)| \leq (q - 1)m^{q-2}|u - v|.$$

(L2) If  $q \geq 2$ ,  $|u|, |v| \leq M$ , then

$$|\phi_q(u) - \phi_q(v)| \leq (q - 1)M^{q-2}|u - v|.$$

## 2 Preliminaries

For the convenience of the reader, we recall some basic definitions about Hadamard fractional calculus and important lemmas.

**Definition 1.** (See [21].) The Hadamard fractional integral of order  $q$  for a function  $g$  is defined as

$${}^H I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

**Definition 2.** (See [21].) The Hadamard derivation of fractional order  $q$  for a function  $g : [1, \infty) \rightarrow \mathbb{R}$  is defined as

$${}^H D^q g(t) = \frac{1}{\Gamma(n - q)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds,$$

$n - 1 < q < n$ ,  $n = [q] + 1$ , where  $[q]$  denotes the integer part of the real number  $q$ , and  $\log(\cdot) = \log_e(\cdot)$ .

**Lemma 1.** (See [21].) *Let  $q > 0$  and  $x \in C[1, \infty) \cap L^1[1, \infty)$ . Then the Hadamard fractional differential equation  ${}^H D^q x(t) = 0$  has the solutions*

$$x(t) = \sum_{i=1}^n c_i (\log t)^{q-i},$$

and the following formula holds:

$${}^H I^q {}^H D^q x(t) = x(t) + \sum_{i=1}^n c_i (\log t)^{q-i},$$

where  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ , and  $n - 1 < q < n$ .

**Lemma 2.** *For any  $\sigma \in C[1, e]$ , the unique solution of the linear Hadamard fractional-order turbulent flow model*

$$\begin{aligned} -{}^H D^\beta \phi_p(-{}^H D^\alpha u(t)) &= \sigma(t), \quad t \in J, \\ {}^H D^\alpha u(1) &= {}^H D^\alpha u(e) = 0, \quad u(1) = 0, \\ {}^H D^{\alpha-1} u(1) &= \eta {}^H D^{\alpha-1} u(e) \end{aligned} \quad (2)$$

is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ &\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s}. \end{aligned} \quad (3)$$

*Proof.* By Lemma 1, we get

$$\phi_p(-{}^H D^\alpha u(t)) = -\frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \sigma(s) \frac{ds}{s} + c_0 (\log t)^{\beta-1} + c_1 (\log t)^{\beta-2},$$

where constants  $c_0, c_1 \in \mathbb{R}$ .

Thus,

$${}^H D^\alpha u(t) = -\phi_q \left[ -\frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \sigma(s) \frac{ds}{s} + c_0(\log t)^{\beta-1} + c_1(\log t)^{\beta-2} \right],$$

where constants  $c_0, c_1 \in \mathbb{R}$ .

Employing the boundary value condition  ${}^H D^\alpha u(1) = {}^H D^\alpha u(e) = 0$ , we have

$$c_1 = 0, \quad c_0 = \frac{1}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-1} \sigma(s) \frac{ds}{s}.$$

Thus, one has

$${}^H D^\alpha u(t) = \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \sigma(s) \frac{ds}{s} - \frac{(\log t)^{\beta-1}}{\Gamma(\beta)} \int_1^e (1 - \log s)^{\beta-1} \sigma(s) \frac{ds}{s} \right].$$

Applying Lemma 1 again, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s} + c_2(\log t)^{\alpha-1} \\ &\quad + c_3(\log t)^{\alpha-2}, \end{aligned} \tag{4}$$

where constants  $c_2, c_3 \in \mathbb{R}$ .

By condition  $u(1) = 0$ , we get  $c_3 = 0$ . This, together with (4), gives

$$\begin{aligned} {}^H D^{\alpha-1} u(t) &= \int_1^t \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s} + c_2 \Gamma(\alpha). \end{aligned}$$

The boundary value condition  ${}^H D^{\alpha-1} u(1) = \eta {}^H D^{\alpha-1} u(e)$  implies

$$\begin{aligned} c_2 &= \frac{1}{(1 - \eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \sigma(\tau) \frac{d\tau}{\tau} \right] \frac{ds}{s}. \end{aligned}$$

Then, substituting  $c_2$  and  $c_3$  into (4), we get equation (3).

The proof is complete. □

### 3 Uniqueness results

To transform problem (1) into a fixed point problem, we define the operator  $T : C[1, e] \rightarrow C[1, e]$  as

$$\begin{aligned} (Tu)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ &\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s}. \end{aligned}$$

To state and prove the uniqueness results, we pose the following conditions:

(H1) The function  $f : [1, e] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is jointly continuous.

(H2) There exists a function  $k_f \in L([1, e], \mathbb{R})$  such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq k_f(t)(|u_1 - v_1| + |u_2 - v_2|)$$

for all  $(t, u_1, u_2), (t, v_1, v_2) \in [1, e] \times \mathbb{R} \times \mathbb{R}$ .

For convenience, we define an operator  $T_0 : C[1, e] \rightarrow C[1, e]$  as follows:

$$\begin{aligned} T_0u(s) &= \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right]. \end{aligned}$$

**Lemma 3.** Assume conditions (H1), (H2) hold and  $q > 2$ . Then there exists a constant  $k > 0$  such that

$$|T_0u(s) - T_0v(s)| \leq k\|u - v\|,$$

where  $k = (q-1)2^q \|k_f\|^{q-1} (2C + M)^{q-2} / (\Gamma(\beta+1))^{q-1}$ .

*Proof.* Define a ball  $B_C = \{u \in C[1, e]: \|u\| \leq C\}$ , and suppose  $q > 2$

$$\begin{aligned} & \left| \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ & \quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right| \\ & \leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \\ & \quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \\ & \leq \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} k_f(\tau) (|u(\tau)| + |u(\theta(\tau))| + |f(\tau, 0, 0)|) \frac{d\tau}{\tau} \\ & \quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} k_f(\tau) (|u(\tau)| + |u(\theta(\tau))| + |f(\tau, 0, 0)|) \frac{d\tau}{\tau} \\ & \leq \frac{(\log s)^\beta}{\Gamma(\beta + 1)} \|k_f\| (2\|u\| + M) + \frac{(\log s)^{\beta-1}}{\Gamma(\beta + 1)} \|k_f\| (2\|u\| + M) \\ & \leq \frac{2\|k_f\|(2C + M)}{\Gamma(\beta + 1)}, \end{aligned}$$

where  $M = \max_{\tau \in [1, e]} |f(\tau, 0, 0)|$ .

Now using the property (L2), we get

$$\begin{aligned} & |(T_0 u)(s) - (T_0 v)(s)| \\ & = \left| \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \right. \\ & \quad \left. \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \right. \\ & \quad \left. - \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f(\tau, v(\tau), v(\theta(\tau))) \frac{d\tau}{\tau} \right. \right. \\ & \quad \left. \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} f(\tau, v(\tau), v(\theta(\tau))) \frac{d\tau}{\tau} \right] \right| \end{aligned}$$

$$\begin{aligned}
&\leq (q-1) \left( \frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)} \right)^{q-2} \\
&\quad \times \left| \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} [f(\tau, u(\tau), u(\theta(\tau))) - f(\tau, v(\tau), v(\theta(\tau)))] \frac{d\tau}{\tau} \right. \\
&\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} [f(\tau, u(\tau), u(\theta(\tau))) - f(\tau, v(\tau), v(\theta(\tau)))] \frac{d\tau}{\tau} \right| \\
&\leq (q-1) \left( \frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)} \right)^{q-2} \\
&\quad \times \left| \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} [k_f(\tau)(|u(\tau) - v(\tau)| + |u(\theta(\tau)) - v(\theta(\tau))|)] \frac{d\tau}{\tau} \right. \\
&\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} [k_f(\tau)(|u(\tau) - v(\tau)| + |u(\theta(\tau)) - v(\theta(\tau))|)] \frac{d\tau}{\tau} \right| \\
&\leq 2(q-1)\|k_f\| \left( \frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)} \right)^{q-2} \\
&\quad \times \left| \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} \|u - v\| \frac{d\tau}{\tau} - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \|u - v\| \frac{d\tau}{\tau} \right| \\
&\leq \frac{(q-1)2^{q-1}\|k_f\|^{q-1}(2C+M)^{q-2}}{(\Gamma(\beta+1))^{q-2}} \left[ \frac{(\log s)^\beta}{\Gamma(\beta+1)} \|u - v\| + \frac{(\log s)^{\beta-1}}{\Gamma(\beta+1)} \|u - v\| \right] \\
&\leq \frac{(q-1)2^q\|k_f\|^{q-1}(2C+M)^{q-2}}{(\Gamma(\beta+1))^{q-1}} \|u - v\| = k\|u - v\|. \quad \square
\end{aligned}$$

**Theorem 1.** Assume (H1), (H2) hold and  $q > 2$ . If

$$k \left[ \frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\eta)\Gamma(\alpha)} \right] < 1, \quad (5)$$

then problem (1) has a unique solution.

*Proof.* For any  $u, v \in B_C$ , in view of (H1) and (H2) combined with Lemma 3, we have

$$\begin{aligned}
& |(Tu)(t) - (Tv)(t)| \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} T_0 u(s) \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e T_0 u(s) \frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} T_0 v(s) \frac{ds}{s} - \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e T_0 v(s) \frac{ds}{s}
\end{aligned}$$



$$\begin{aligned} &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} k \|u - v\| \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e k \|u - v\| \frac{ds}{s} \\ &\leq k \left[ \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \right] \|u - v\| \\ &\leq k \left[ \frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\eta)\Gamma(\alpha)} \right] \|u - v\|. \end{aligned}$$

In view of (5),  $T$  is a contraction mapping. By the Banach contraction mapping principle, the operator  $T$  has a fixed point, which implies that Hadamard-type fractional turbulent flow model (1) has a unique solution.  $\square$

### 4 Existence results

Now, we are in the position to present a existence result, which is based on the following Schaefer fixed point theorem.

**Theorem 2.** (See [30].) *Let  $X$  be a Banach space. Assume that  $\Omega$  is an open bounded subset of  $X$  with  $\theta \in \Omega$ , and let  $T : \bar{\Omega} \rightarrow X$  be a completely continuous operator such that  $\|Tu\| \leq \|u\|$  for all  $u \in \partial\Omega$ . Then  $T$  has a fixed point in  $\bar{\Omega}$ .*

**Theorem 3.** *Assume that:*

(H3) *There exist non-decreasing functions  $\varphi, \psi: [0, \infty) \rightarrow [0, \infty)$  and the functions  $a(t), b(t) \in C[1, e]$  such that*

$$|f(t, u, v)| \leq a(t)\varphi(|u|) + b(t)\psi(|v|), \quad t \in [1, e], \quad u, v \in \mathbb{R}.$$

(H4) *There exists a constant  $\mathcal{M} > 0$  such that*

$$\begin{aligned} &\frac{\mathcal{M}}{2^{q-1} [\|a\|\varphi(\|u\|) + \|b\|\psi(\|u\|)]^{q-1}} \\ &\times \frac{1}{\left[ \frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right]} > 1. \end{aligned}$$

*Then problem (1) has at least one solution.*

*Proof.* Firstly, we prove that the operator  $T$  is completely continuous. Clearly, continuity of the operator  $T$  follows from the continuity of  $f$ . Let  $\Omega \subset C[1, e]$  be bounded. For any  $u, v \in \Omega$ , there exist positive constant  $L$  such that  $|f(t, u, v)| \leq L$ . We can get

$$\begin{aligned} |(Tu)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\ &\quad \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\
& + \left. \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \phi_q \left( \frac{2L}{\Gamma(\beta+1)} \right) \frac{ds}{s} \\
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left( \frac{2L}{\Gamma(\beta+1)} \right) \frac{ds}{s} \\
& = \frac{2^{q-1} L^{q-1} (\log t)^\alpha}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{2^{q-1} L^{q-1} (\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
& \leq \frac{2^{q-1} L^{q-1}}{\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \left( \frac{1}{\alpha} + \frac{1}{1-\eta} \right) \\
& = \mathcal{N} \quad (\text{constant}).
\end{aligned}$$

Therefore,  $\|Tu\| \leq \mathcal{N}$ . For all  $t_1, t_2 \in [1, e]$  and  $1 \leq t_1 \leq t_2 \leq e$ , we have

$$\begin{aligned}
& |(Tu)(t_2) - (Tu)(t_1)| \\
& = \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[ \left( \log \frac{t_2}{s} \right)^{\alpha-1} - \left( \log \frac{t_1}{s} \right)^{\alpha-1} \right] \right. \\
& \quad \times \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\
& \quad \left. \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right. \\
& \quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( \log \frac{t_2}{s} \right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\
& \quad \left. \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \right. \\
& \quad \left. + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \right.
\end{aligned}$$

$$\begin{aligned}
 & \left| -\frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \Big| \\
 \leq & \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[ \left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \phi_q \left[ \frac{2L}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
 & + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \phi_q \left[ \frac{2L}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
 & + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{2L}{\Gamma(\beta+1)} \right] \frac{ds}{s} \Big| \\
 = & \left[ \frac{(\log t_2)^\alpha - (\log t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \right] \phi_q \left[ \frac{2L}{\Gamma(\beta+1)} \right].
 \end{aligned}$$

Thus, by the Arzela–Ascoli theorem, the operator  $T$  is completely continuous.

Next, we consider the set  $V = \{u \in C[1, e]: u = \mu Tu, \mu \in (0, 1)\}$  and show that the set  $V$  is bounded.

$$\begin{aligned}
 |u| &= |\mu(Tu)(t)| \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\
 &\quad \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 &\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\
 &\quad \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} |f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)] \frac{d\tau}{\tau} \right. \\
 &\quad \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)] \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
 &\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)] \frac{d\tau}{\tau} \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)] \frac{d\tau}{\tau} \frac{ds}{s} \\
= & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{(\log s)^\beta [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \right. \\
& + \left. \frac{(\log s)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{(\log s)^\beta [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \right. \\
& + \left. \frac{(\log s)^{\beta-1} [a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
\leq & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{2^{q-1} [\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(\Gamma(\beta+1))^{q-1}} \frac{ds}{s} \\
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \frac{2^{q-1} [\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(\Gamma(\beta+1))^{q-1}} \frac{ds}{s} \\
= & \frac{(\log t)^\alpha 2^{q-1} [\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \\
& + \frac{(\log t)^{\alpha-1} 2^{q-1} [\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
\leq & 2^{q-1} [\|a\|\varphi(\|u\|) + \|b\|\psi(\|u\|)]^{q-1} \\
& \times \left[ \frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right].
\end{aligned}$$

Thus,

$$\frac{\|u\|}{2^{q-1} [\|a\|\varphi(\|u\|) + \|b\|\psi(\|u\|)]^{q-1} \left[ \frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right]} \leq 1.$$

This, together with condition (H4), gives  $\|u\| < \mathcal{M}$ . So, the set  $V$  is bounded. Thus, by the Schaefer's fixed point theorem, the operator  $T$  has at least one fixed point, which implies that Hadamard-type fractional turbulent flow model has at least one solution.  $\square$

## 5 Nonexistence results

In this section, we introduce a parameter in problem (1), then we give some sufficient conditions of nonexistence of solution to a class of Hadamard-type fractional turbulent

flow model with a parameter. Precisely, we study the following problem:

$$\begin{aligned} -{}^H D^\beta \phi_p(-{}^H D^\alpha u(t)) &= \lambda f(t, u(t), u(\theta(t))), \\ {}^H D^\alpha u(1) &= {}^H D^\alpha u(e) = 0, \quad u(1) = 0, \\ {}^H D^{\alpha-1} u(1) &= \eta {}^H D^{\alpha-1} u(e), \end{aligned} \tag{6}$$

where  $\lambda$  is a positive parameter.

**Theorem 4.** *If  $\lim_{|u|+|v|\rightarrow 0^+} f(t, u, v)/(|u| + |v|)^{p-1}$  and  $\lim_{|u|+|v|\rightarrow \infty} f(t, u, v)/(|u| + |v|)^{p-1}$  exist, then there exists  $\lambda_0 > 0$  such that the Hadamard fractional turbulent flow model (6) has no solution for  $0 < \lambda < \lambda_0$ .*

*Proof.* Since  $\lim_{|u|+|v|\rightarrow 0^+} f(t, u, v)/(|u| + |v|)^{p-1}$  and  $\lim_{|u|+|v|\rightarrow \infty} f(t, u, v)/(|u| + |v|)^{p-1}$  exist, then there exist positive numbers  $\varepsilon_1, \varepsilon_2, r, N$  such that

- (i) for  $0 < \|u\| < r$ ,  $|f(t, u(t), u(\theta(t)))| < 2^{p-1}\varepsilon_1\|u\|^{p-1}$ .
- (ii) for  $\|u\| > N$ ,  $|f(t, u(t), u(\theta(t)))| < 2^{p-1}\varepsilon_2\|u\|^{p-1}$ .

Without loss of generality, let  $r < N$  and

$$\varepsilon_3 = \max \left\{ \varepsilon_1, \varepsilon_2, \max \left\{ \frac{f(t, u(t), u(\theta(t)))}{[|u(t)| + |u(\theta(t))|]^{p-1}} : r < \|u\| < N \right\} \right\}.$$

Thus, for all  $t \in (1, e)$  and  $u \in \mathbb{R}$ , we have  $|f(t, u(t), u(\theta(t)))| < 2^{p-1}\varepsilon_3\|u\|^{p-1}$ .

Let us consider

$$\begin{aligned} (T_\lambda u)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \lambda f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \lambda f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s} \\ &\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \lambda f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right. \\ &\quad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} \lambda f(\tau, u(\tau), u(\theta(\tau))) \frac{d\tau}{\tau} \right] \frac{ds}{s}. \end{aligned}$$

If  $u$  is a solution of the fractional turbulent flow model (6), then  $u = T_\lambda u$ . In fact,

$$\begin{aligned} |(T_\lambda u)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} |\lambda f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\ &\quad \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1 - \log s)^{\beta-1} |\lambda f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} |\lambda f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right. \\
& + \left. \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} |\lambda f(\tau, u(\tau), u(\theta(\tau)))| \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{d\tau}{\tau} \right. \\
& + \left. \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{1}{\Gamma(\beta)} \int_1^s \left( \log \frac{s}{\tau} \right)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{d\tau}{\tau} \right. \\
& + \left. \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{d\tau}{\tau} \right] \frac{ds}{s} \\
& = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \phi_q \left[ \frac{(\log s)^\beta 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right. \\
& + \left. \frac{(\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
& + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \phi_q \left[ \frac{(\log s)^\beta 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right. \\
& + \left. \frac{(\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right] \frac{ds}{s} \\
& \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(\Gamma(\beta+1))^{q-1}} \frac{ds}{s} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^e \frac{2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(\Gamma(\beta+1))^{q-1}} \frac{ds}{s} \\
& = \frac{(\log t)^\alpha 2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{(\log t)^{\alpha-1} 2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
& \leq \left[ \frac{2^q (\lambda \varepsilon_3)^{q-1}}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{2^q (\lambda \varepsilon_3)^{q-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right] \|u\| \\
& = M (\lambda \varepsilon_3)^{q-1} \|u\|,
\end{aligned}$$

where  $M = 2^q(2-\eta)/((1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1})$ .

Let  $\lambda_0 = (\varepsilon_3 M^{1/(q-1)})^{-1}$ . For  $0 < \lambda < \lambda_0$ , we have

$$\|T_\lambda u\| \leq M(\lambda \varepsilon_3)^{q-1} \|u\| < M(\lambda_0 \varepsilon_3)^{q-1} \|u\| = \|u\|,$$

which is a contradiction. Therefore,  $u$  is not a solution of problem (6). □

### 6 Examples

*Example 1.* Consider the following Hadamard fractional boundary value problem:

$$\begin{aligned} -{}^H D^{3/2} [\phi_{3/2}(-{}^H D^{3/2} u(t))] &= \frac{t^2 + 1}{20} \frac{|u(t)|}{|u(t)| + 1} + \frac{t^2}{25} e^{-u(\theta(t))}, \quad t \in [1, e], \\ {}^H D^{3/2} u(1) = {}^H D^{3/2} u(0) &= 0, \quad u(1) = 0, \quad {}^H D^{1/2} u(1) = \frac{1}{3} {}^H D^{1/2} u(e), \end{aligned} \tag{7}$$

where  $p = 3/2, q = 3, \alpha = 3/2, \beta = 3/2, \eta = 1/3, \theta \in C([1, e], [1, e])$ .

Note

$$\begin{aligned} &|f(t, u_1, u_2) - f(t, v_1, v_2)| \\ &= \left| \frac{t^2 + 1}{20} \left( \frac{|u_1|}{|u_1| + 1} - \frac{|v_1|}{|v_1| + 1} \right) + \frac{t^2}{25} (e^{-u_2} - e^{-v_2}) \right| \\ &\leq \frac{t^2 + 1}{20} (|u_1 - v_1| + |u_2 - v_2|), \end{aligned}$$

we find that  $k_f(t) = (t^2 + 1)/20$ , then choose  $0 < k < 0.409$ , we have

$$k \left[ \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{(1 - \eta)\Gamma(\alpha)} \right] < 2.4449k \leq 1,$$

we can choose  $k < 0.409$ .

Thus, the hypotheses of Theorem 1 hold. Therefore, the conclusion of Theorem 1 applies to Hadamard fractional boundary value problem (7).

*Example 2.* We consider the following Hadamard fractional boundary value problem:

$$\begin{aligned} -{}^H D^{3/2} [\phi_4(-{}^H D^{7/4} u(t))] &= \frac{|u(t)|^5}{t^2(|u(t)|^5 + 5)} + \frac{e^2}{5} e^{u^2(\sqrt{t})}, \quad t \in [1, e], \\ {}^H D^{7/4} u(1) = {}^H D^{7/4} u(0) &= 0, \quad u(1) = 0, \quad {}^H D^{3/4} u(1) = \frac{1}{4} {}^H D^{3/4} u(e), \end{aligned} \tag{8}$$

where  $p = 4, q = 4/3, \alpha = 7/4, \beta = 3/2, \eta = 1/4, \theta(t) = \sqrt{t}$ , and  $f(t, u, v) = |u|^5 / (t^2(|u|^5 + 5)) + (e^2/5)e^{v^2}$ . We can see

$$|f(t, u, v)| \leq \frac{|u|^5}{t^2(|u|^5 + 5)} + \frac{e^2}{5} e^{|u|^2},$$

then  $a(t) = 1/t^2, b(t) = e^2/5$ , and condition (H3) holds.

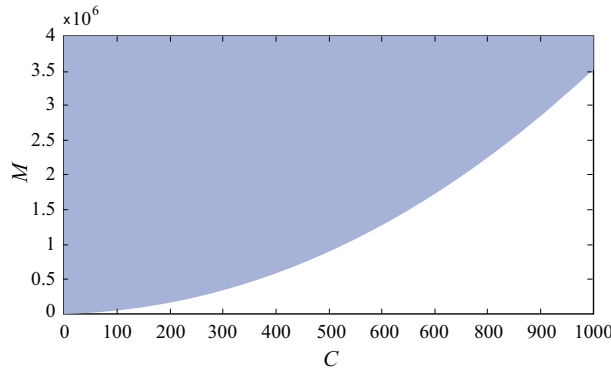


Figure 1

By a simple computation,

$$\frac{\mathcal{M}}{2^{q-1} \left[ \frac{|u|^5}{t^2(|u|^5+5)} + \frac{e^2}{5} e^{|u|^2} \right]^{q-1} \left[ \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)\Gamma(\beta+1)^{q-1}} \right]} > 1,$$

i.e.

$$\frac{\mathcal{M}}{2.3748 \left[ \frac{\|C\|^5}{\|C\|^5+5} + \frac{e^2}{5} e^{\|C\|^2} \right]} > 1,$$

see Fig. 1.

If we choose appropriate  $\mathcal{M}$  and  $C$ , condition (H4) is holds. Then all the assumptions of Theorem 3 are hold. Therefore, problem (8) has at least one solution.

*Example 3.* We consider the following Hadamard fractional turbulent flow model:

$$\begin{aligned} -{}^H D^{5/4} [\phi_3(-{}^H D^{3/2} u(t))] &= \frac{\lambda t(|u(t)| + |u(t^2)|)^{5/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]}, \quad t \in [1, e], \\ {}^H D^{3/2} u(1) = {}^H D^{3/2} u(0) &= 0, \quad u(1) = 0, \quad {}^H D^{1/2} u(1) = \frac{1}{2} {}^H D^{1/2} u(e), \end{aligned} \tag{9}$$

where  $p = 3, q = 3/2, \alpha = 3/2, \beta = 5/4, \eta = 1/2, \theta(t) = t^2, \lambda$  is a positive parameter, and

$$f(t, u, v) = \frac{t(|u(t)| + |u(t^2)|)^{5/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]}.$$

We have

$$\begin{aligned} \lim_{|u|+|v| \rightarrow 0^+} \frac{f(t, u, v)}{(|u| + |v|)^{p-1}} &= \lim_{|u(t)|+|u(t^2)| \rightarrow 0^+} \frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]} = 0, \\ \lim_{|u|+|v| \rightarrow \infty} \frac{f(t, u, v)}{(|u| + |v|)^{p-1}} &= \lim_{|u(t)|+|u(t^2)| \rightarrow \infty} \frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]} = 0, \end{aligned}$$



i.e.

- (i) for all  $\varepsilon_1 > 0$  (choose  $\varepsilon_1 = 1/500$ ), there is  $r = (500\varepsilon_1)^2/2 = 0.5$  such that if  $0 < \|u\| < r$ ,

$$\left| \frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]} - 0 \right| < \varepsilon_1;$$

- (ii) for all  $\varepsilon_2 > 0$  (choose  $\varepsilon_2 = 1/10^8$ ), there is  $N = (e/(500\varepsilon_2))^{2/3}/2 \approx 1939$  such that if  $\|u\| > N$ ,

$$\left| \frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[ (|u(t)| + |u(t^2)|)^2 + 1]} - 0 \right| < \varepsilon_2,$$

and if  $\|u\| \in (r, 1)$ ,

$$\frac{f(t, u, v)}{(|u| + |v|)^{p-1}} < \frac{\sqrt{2}t}{500},$$

if  $\|u\| \in (1, N)$ ,

$$\frac{f(t, u, v)}{(|u| + |v|)^{p-1}} < \frac{t}{1000\sqrt{2}}.$$

Therefore,  $\varepsilon_3 = \max\{1/500, 1/10^8, 0.0076\} = 0.0076$ . Since

$$M = \frac{2^q(2 - \eta)}{(1 - \eta)\Gamma(\alpha)(\Gamma(\beta + 1))^{q-1}} \approx 8.99,$$

then  $\lambda_0 = (\varepsilon_3 M^{1/(q-1)})^{-1} = 1.6280$ .

Thus, all the assumptions of Theorem 4 are satisfied. Consequently, the conclusion of Theorem 4 implies that problem (9) has no solution for  $0 < \lambda < 1.6280$ .

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