The Klein–Gordon Equation with Modified Coulomb Potential Plus Inverse-Square–Root Potential in Three-Dimensional Noncommutative Space

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Abstract

In present work, the three-dimensional modified Klein-Gordon equation (MKGE) is analytically solved under modified Coulomb potential plus inverse-square–root potential, in the symmetries of noncommutative quantum mechanics (NCQM), using the generalized Bopp's shift method. The new energy shift (ground state, first excited state and n^{th} excited state) is obtained via first-order perturbation theory in the 3-dimensional noncommutative real space (NC: 3D-RS) symmetries instead of solving MKGE with the Weyl Moyal star product. It is found that the perturbative solutions of discrete spectrum for studied potential depended on the parabolic cylinder functions, the Gamma function, the discreet atomic quantum numbers (j,l,s,m) and the potential parameters (a and b), in addition to noncommutativity parameters (θ and σ).

Keywords: Klein-Gordon equation, Coulomb potential plus inverse-square–root potential, noncommutative space phase, and Bopp's shift method.

Introduction

It is well known that the search for solutions of the three fundamental dynamical (Schrödinger, Klein-Gordon, and Dirac) equations plays important role in various fields such as nuclear, molecular, heavy-quarkonium mesons and so on. This is achieved, using various potentials schemes such as the Coulombic potential, the pseudoharmonic potential, the Kratzer potential, and the inverse-square-root potential, which has both scattering states and bound states [1-5]. Furthermore, when a particle is in a strong potential field, the relativistic effects are needed to generate corrections on the non-relativistic bound state in relativistic quantum mechanics. Thus, the Klein-Gordon and Dirac equations become a reality. The Shi-Hai Dong [6] in their work solved the relativistic Schrödinger equation (the Klein-Gordon equation) with Coulomb potential plus inverse-square–root potential $V(r) = -\dfrac{a}{r} + \dfrac{b}{\sqrt{r}}$, using the Ansatz method and obtained energy eigenvalues and the wave functions, he has obtained the relativistic closed-form solutions both in 3-dimensions and in two dimensions. Recently, there has been an increased interest in finding exact solutions of Schrödinger, Klein-Gordon and Dirac equations for various potential schemes in a large symmetry known by noncommutative quantum mechanics NCQM for the purpose of obtaining more explanations that are accurate and finding other new applications at non-sciences scales. The main objective is to develop the research article [6] and expanding it in the new space phase NCQM in order to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. On the other hand, to explore the possibility of creating new applications and more profound interpretations in the sub-atomics and nano scales using new version the modified Coulomb potential plus inverse-square–root potential, which has the following form:

$$V_{eff-cs}(r) = 2(E+M)\left(-\frac{a}{r} + \frac{b}{\sqrt{r}}\right) + \frac{l(l+1)}{r^2} \rightarrow V_{eff-cs}(\hat{r}) = V_{eff-cs}(r) + \left\{\frac{l(l+1)}{r^4} - (E+M)\left(\frac{a}{r^3} - \frac{b}{2}\frac{1}{r^{5/2}}\right)\right\}\vec{L}\vec{\theta}$$

(1)

It should be noted that, the noncommutativity was introduced firstly by Heisenberg W. in 1930 [7] and then formulated by Syndre H. in 1947 [8]. The algebraic structure of new space-phase (NCQM) is based on new canonical commutations relations, in both Schrödinger picture and Heisenberg picture, respectively, as follows (the natural units $c = \hbar = 1$ are employed throughout this paper) [9-18]:

$$[x_\mu, p_\nu] = i\delta_{\mu\nu}\hbar \rightarrow \left[\hat{x}_\mu \overset{*}{,} \hat{p}_\nu\right] = \left[\hat{x}_\mu(t), \hat{p}_\nu(t)\right] = i\delta_{\mu\nu}\hbar \Rightarrow \Delta\hat{x}_\mu \Delta\hat{p}_\nu \geq \frac{\delta_{\mu\nu}}{2}$$

$$[x_\mu, x_\nu] = 0 \rightarrow \left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right] = \left[\hat{x}_\mu(t), \hat{x}_\nu(t)\right] = i\theta_{\mu\nu} \Rightarrow \Delta\hat{x}_\mu \Delta\hat{x}_\nu \geq \frac{\theta_{\mu\nu}}{2}$$

(2)

The indices $\mu, \nu \equiv \overline{1,3}$. This means that the principle of uncertainty for Heisenberg generalized to include another new uncertainty related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ in addition to the ordinary uncertainty of $(\hat{x}_\mu, \hat{p}_\nu)$. The very small parameter $\theta^{\mu\nu}$ (compared to the energy) has elements of the antisymmetric real matrix and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f, g)(x) \rightarrow (\hat{f}, \hat{g})(\hat{x})$ to the new form $\hat{f}(\hat{x})\hat{g}(\hat{x}) \equiv (f * g)(x)$ in (NC: 3D-RS) symmetries [17-25]:

$$(fg)(x) \rightarrow (f * g)(x) = \exp\left(i\theta_{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}\right)f(x_\mu)g(x_\nu) \approx fg(x) - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f \, \partial_\nu^x g \bigg|_{x_\mu = x_\nu} + O(\theta^2)$$

(3)

In which $O(\theta^2)$ stands for the second and higher-order term of θ. The second and the third terms in the above equation represent the effects of (space-space) noncommutativity properties. However, the new operators $\hat{\xi}(t) = (\hat{x}_\mu \vee \hat{p}_\mu)(t)$ in the Heisenberg picture depend on the corresponding new operators $\hat{\xi}_\mu = \hat{x}_\mu \vee \hat{p}_\mu$ in the Schrödinger picture from the following projections relations:

$$\xi_\mu(t) = \exp(i\hat{H}_{cs}(t - t_0))\xi_\mu \exp(-i\hat{H}_{cs}(t - t_0)) \Rightarrow \hat{\xi}_\mu(t) = \exp(i\hat{H}_{nc-cs}(t - t_0)) * \hat{\xi}_\mu * \exp(-i\hat{H}_{nc-cs}(t - t_0))$$

(4)

Here $\xi_\mu = x_\mu \vee p_\mu$ and $\xi_\mu(t) = (x_\mu \vee p_\mu)(t)$, while the dynamics of new systems $\dfrac{d\xi_\mu(t)}{dt}$ are described from the following motion equations in NCQM:

$$\frac{d\xi_\mu(t)}{dt} = \left[\xi_\mu(t), \hat{H}_{cs}\right] \Rightarrow \frac{d\hat{\xi}_\mu(t)}{dt} = \left[\hat{\xi}_\mu(t) \overset{*}{,} \hat{H}_{nc-cs}\right]$$

(5)

Here $\hat{H}_{cs}$ is the quantum Hamiltonian operator for Coulomb potential plus inverse-square–root potential in the relativistic quantum mechanics while $\hat{H}_{nc-cs}$ is the new Hamiltonian operator in NRCQM for modified Coulomb potential plus inverse-square–root potential. This paper consists of four sections and the organization scheme is given as follows: In next section, the theory part, we briefly review the Klein-Gordon equation with modified Coulomb potential plus inverse-square–root potential on based to ref. [6]. The Section 3 is devoted to studying the MKGE by applying the Bopp's shift method and obtained the effective potential. Then, we apply the standard perturbation theory to find the energy shift of ground state, first excited state and the nth excited state which produced by the effects of modified spin-orbital and modified Zeeman interactions and we discuss some particulars cases. Finally, in the last section, summary and conclusions are presented.

Materials and Methods

Overview of the eigenfunctions and the energy eigenvalues for the Coulomb potential plus inverse-square–root potential in RQM

As already mentioned, our aim is to obtain the spectrum of the Coulomb potential plus inverse-square–root potential $V(r) = -\dfrac{a}{r} + \dfrac{b}{\sqrt{r}}$ in three-dimensional relativistic noncommutative quantum mechanics 3DRQM. To achieve the goal, it is useful to summarize, the Klein-Gordon (KG) equation with equal scalar and vector potentials $V(r) = S(r)$ for a particle of rest mass M in three-dimensional relativistic quantum mechanics 3DRQM [6]:

$$\left\{ \nabla^2 + \left(E^2 - M^2\right) - 2(E+M)\left(-\frac{a}{r} + \frac{b}{\sqrt{r}}\right) \right\}\Psi(r,\theta,\varphi) = 0 \Rightarrow \left\{ \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \left(E^2 - M^2\right) - 2(E+M)\left(-\frac{a}{r} + \frac{b}{\sqrt{r}}\right) - \frac{l(l+1)}{r^2} \right\}R(r) = 0$$

(6)

Where $\Psi(r,\theta,\varphi) = R_l(r)Y_l^m(\theta,\varphi)$ denote to the complete wave function. For removing the derivation of the first order, we introduce $R_l(r) = \dfrac{U_l(r)}{r}$, thus Eq. (6) become:

$$\left\{ \frac{d^2}{dr^2} + \left(E^2 - M^2\right) - 2(E+M)\left(-\frac{a}{r} + \frac{b}{\sqrt{r}}\right) - \frac{l(l+1)}{r^2} \right\}U(r) = 0 \qquad (7)$$

Based in ref. [6], the complete wave function is given by:

$$\Psi^{(p)}(r,\theta,\varphi) = \begin{cases} a_0 r^\nu \exp\left(-\gamma_0 r - \eta_0 r^{1/2}\right)Y(\theta,\varphi) & \text{for} \quad p = 0 \\ \left(a_0 + a_1 r^{1/2}\right)r^\nu \exp\left(-\gamma_1 r - \eta_1 r^{1/2}\right)Y(\theta,\varphi) & \text{for} \quad p = 1 \\ \left(a_0 + a_1 r^{1/2} + \ldots + a_p r^{p/2}\right)r^\nu \exp\left(-\gamma_p r - \eta_p r^{1/2}\right)Y(\theta,\varphi) & \text{for} \quad p \end{cases} \qquad (8)$$

Where a_i can be determined by the normalization condition, $\gamma_p = \sqrt{M^2 - E_p^{\,2}}$, $\eta_p = \dfrac{bE_p}{\sqrt{M^2 - E_p^{\,2}}}$ and $\nu = -1/2 + \sqrt{(l+1/2)^2 - a^2}$. In addition, the energy E_p of the relativistic Coulomb potential plus inverse-square–root potential is obtained from the following relation [6]:

$$b^2 M^2 + 2aE_p\left(M^2 - E_p^{\,2}\right) = \left(p + 2 + \nu\right)\left(M^2 - E_p^{\,2}\right)\sqrt{M^2 - E_p^{\,2}} \qquad (9)$$

Solution of MKG for modified Coulomb potential plus inverse-square–root potential

In this section, we shall give an overview or a brief preliminary for modified Coulomb potential plus inverse-square–root potential in (NC: 3D-RS) symmetries. To perform this task the physical form of modified Klein-Gordon equation (MKGE), it is necessary to apply the notion of the Weyl Moyal star product on the differential equation satisfied by the radial wave function $U_l(r)$ in Eq. (7), thus, the radial wave function $U_l(r)$ in (NC: 3D-RS) symmetries becomes, see refs. [10-11]:

$$\left\{\frac{d^2}{dr^2}+\left(E^2-M^2\right)-2(E+M)V(r)-\frac{l(l+1)}{r^2}\right\}U_l(r)=0\Rightarrow\left\{\frac{d^2}{dr^2}+\left(E^2-M^2\right)-2(E+M)V(r)-\frac{l(l+1)}{r^2}\right\}*U_l(r)=0$$

(10)

The Bopp's shift method has been successfully applied to RNCQM and NRNCQM problems using modified Dirac equation MDE, MKGE, and modified Schrödinger equation MSE. This method has produced very promising results for a number of situations having physical, chemical interest. The method reduces three modified fundamental equations (MDE, MKGE, and MSE) to the (DE, KGE, and SE), respectively, under the similtaniously translation in space. It based on the following new commutator [24-29]:

$$\left[\hat{x}_\mu,\hat{x}_\nu\right]=\left[\hat{x}_\mu(t),\hat{x}_\nu(t)\right]=i\theta_{\mu\nu}$$

(11)

The new generalized positions and momentum coordinates $\left(\hat{x}_\mu,\hat{p}_\nu\right)$ in (NC: 3D-RS) are defined in terms of the commutative counterparts $\left(x_\mu,p_\nu\right)$ in QM via, respectively [26-33]:

$$\left(x_\mu,p_\nu\right)\Rightarrow\left(\hat{x}_\mu,\hat{p}_\nu\right)=\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu,p_\mu\right)$$

(12)

The above equation allows us to obtain the operator $r^2\Rightarrow\hat{r}^2=r^2-\vec{\mathbf{L}}\vec{\Theta}$ in (NC-3D: RS) symmetries. The two couplings $\vec{\mathbf{L}}\vec{\Theta}$ equal $\left(L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}\right)$ and (L_x, L_y and L_z) are the three components of angular momentum operator $\vec{L}$ while $\Theta_{\mu\nu}=\theta_{\mu\nu}/2$. Thus, the reduced like Schrödinger equation (without star product) can be written as:

$$\left\{\frac{d^2}{dr^2}+\left(E^2-M^2\right)-2(E+M)V(r)-\frac{l(l+1)}{r^2}\right\}*U_l(r)=0\Rightarrow\left\{\frac{d^2}{dr^2}+\left(E^2-M^2\right)-2(E+M)V(\hat{r})-\frac{l(l+1)}{\hat{r}^2}\right\}U_l(r)=0$$

(13)

The new operator of $V_{cs}(\hat{r})$ can be expressed as [10-11]:

$$V_{cs}(\hat{r})\equiv V_{cs}\left(\sqrt{\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu-\frac{\theta_{\mu\nu}}{2}p_\nu\right)}\right)=V_{cs}(r)-\frac{\vec{\mathbf{L}}\vec{\theta}}{2r}\frac{\partial V_{cs}(r)}{\partial r}+O\left(\theta^2\right)$$

(14)

We have $\dfrac{\partial V_{cs}(r)}{\partial r}=\dfrac{a}{r^2}-\dfrac{b}{2}\dfrac{1}{r^{3/2}}$ and $\dfrac{1}{\hat{r}^2}\approx\dfrac{1}{\hat{r}^2}+\dfrac{\vec{\mathbf{L}}\vec{\theta}}{r^4}+O\left(\theta^2\right)$, this allow us to write the modified radial part of KGE in (NC: 3D-RS) symmetries:

$$\left\{\frac{d^2}{dr^2}+\left(E^2-M^2\right)-2(E+M)V(r)-\frac{l(l+1)}{r^2}-\vec{\mathbf{L}}\vec{\theta}\left[\frac{l(l+1)}{r^4}-(E+M)\left(\frac{a}{r^3}-\frac{b}{2}\frac{1}{r^{5/2}}\right)\right]\right\}U_l(r)=0\text{ (15)}$$

Moreover, to illustrate this equation in a simple mathematical way, it is useful to enter the following symbols $V_{eff-cs}(r)=2(E+M)\left(-\dfrac{a}{r}+\dfrac{b}{\sqrt{r}}\right)+\dfrac{l(l+1)}{r^2}$ and $E_{eff-cs}=M^2-E^2$, thus, the radial equation (14) becomes:

$$\left\{ \frac{d^2}{dr^2} - \left[E_{eff-cs} + V_{eff-cs}(r) + V_{pert-sc}(r) \right] \right\} U_l(r) = 0 \tag{16}$$

With:

$$V_{pert-sc}(r) = \left[\frac{l(l+1)}{r^4} - (E+M)\left(\frac{a}{r^3} - \frac{b}{2} \frac{1}{r^{5/2}} \right) \right] \vec{\mathbf{L}}\vec{\theta} \tag{17}$$

The additive part of effective potential is proportional to the infinitesimal vector $\vec{\theta} = \theta_{11} e_x + \theta_{12} e_y + \theta_3 e_z$. Thus, we can consider $V_{pert-sc}(r)$ as a perturbation terms compared with the parent potential (effective potential operator) $V_{eff-cs}(r)$ in (NC: 3D-RS) symmetries. The purpose here is to give a complete prescription to determine the energy level of ground state, first excited state, and n^{th} excited state, by applying the perturbative theory, in the case of NRRQM. In the first-order perturbation theory the expectation value of r^{-3}, $r^{-5/2}$ and r^{-4} with respect to the exact solution of Eq. (15), are given by:

$$\langle 0|r^{-3}|0 \rangle = a_0^2 \int_0^{+\infty} r^{2\nu-1} \exp(-2\gamma_0 r - 2\eta_0 r^{1/2}) dr$$

$$\langle 0|r^{-5/2}|0 \rangle = a_0^2 \int_0^{+\infty} r^{2\nu-1/2} \exp(-2\gamma_0 r - 2\eta_0 r^{1/2}) dr \tag{18}$$

$$\langle 0|r^{-4}|0 \rangle = a_0^2 \int_0^{+\infty} r^{2\nu-2} \exp(-2\gamma_0 r - 2\eta_0 r^{1/2}) dr$$

$$\langle 1|r^{-3}|1 \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2})^2 r^{2\nu+2-3} \exp(-2\gamma_1 r - 2\eta_1 r^{1/2}) dr$$

$$\langle 1|r^{-5/2}|1 \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2})^2 r^{2\nu+2-5/2} \exp(-2\gamma_1 r - 2\eta_1 r^{1/2}) dr \tag{19}$$

$$\langle 1|r^{-4}|1 \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2})^2 r^{2\nu+2-4} \exp(-2\gamma_1 r - 2\eta_1 r^{1/2}) dr$$

and

$$\langle p|r^{-3}|p \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2} + ... + a_p r^{p/2})^2 r^{2\nu+2-3} \exp(-2\gamma_p r - 2\eta_p r^{1/2}) dr$$

$$\langle p|r^{-5/2}|p \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2} + ... + a_p r^{p/2})^2 r^{2\nu+2-5/2} \exp(-2\gamma_p r - 2\eta_p r^{1/2}) dr \tag{20}$$

$$\langle p|r^{-4}|p \rangle = \int_0^{+\infty} (a_0 + a_1 r^{1/2} + ... + a_p r^{p/2})^2 r^{2\nu+2-4} \exp(-2\gamma_p r - 2\eta_p r^{1/2}) dr$$

Now, we introduce a new variable $r = \zeta^2$. This, allow us to simplify Eqs. (18) and (19) to the form:

$$\langle 0|r^{-3}|0\rangle = 2a_0{}^2 \int_0^{+\infty} \zeta^{4v-1} \exp\!\left(-2\gamma_0\zeta^2 - 2\eta_0\zeta\right) d\zeta$$

$$\langle 0|r^{-5/2}|0\rangle = 2a_0{}^2 \int_0^{+\infty} \zeta^{4v+1-1}\,\zeta^{v+3-3/2} \exp\!\left(-2\gamma_0\zeta^2 - 2\eta_0\zeta\right) d\zeta \qquad (21)$$

$$\langle 0|r^{-4}|0\rangle = 2a_0{}^2 \int_0^{+\infty} \zeta^{4v-2-1} \exp\!\left(-2\gamma_0\zeta^2 - 2\eta_0\zeta\right) d\zeta$$

$$\langle 1|r^{-3}|1\rangle = 2\int_0^{+\infty} \left(a_0{}^2 \zeta^{4v-1} + a_1{}^2 \zeta^{4v+2-1} + 2a_0a_1\zeta^{4v+1-1}\right)\exp\!\left(-2\gamma_1 r - 2\eta_1 r^{1/2}\right) dr$$

$$\langle 1|r^{-5/2}|1\rangle = 2\int_0^{+\infty} \left(a_0{}^2 \zeta^{4v+1-1} + a_1{}^2 \zeta^{4v+3-1} + 2a_0a_1\zeta^{4v+2-1}\right)\exp\!\left(-2\gamma_1 r - 2\eta_1 r^{1/2}\right) dr \qquad (22)$$

$$\langle 1|r^{-4}|1\rangle = 2\int_0^{+\infty} \left(a_0{}^2 \zeta^{4v-4-1} + a_1{}^2 \zeta^{4v-1} + 2a_0a_1\zeta^{4v-1-1}\right)\exp\!\left(-2\gamma_1 r - 2\eta_1 r^{1/2}\right) dr$$

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi = \delta_{ll'}\delta_{mm'}$. It is convenient to apply the following special integral [34]:

$$\int_0^{+\infty} x^{v-1\cdot}\exp\!\left(-\lambda x^2 - \gamma x\right)dx = (2\lambda)^{-\frac{v}{2}}\Gamma(v)\exp\!\left(\frac{\gamma^2}{8\lambda}\right)D_{-v}\!\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \qquad (23)$$

Where $D_{-v}\!\left(\dfrac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(v)$ denote to the parabolic cylinder functions and Gamma function. After straightforward calculations we can obtain the explicitly results:

$$\langle 0|r^{-3}|0\rangle = 2a_0{}^2 (4\gamma_0)^{-\frac{4v}{2}}\Gamma(4v)\exp\!\left(\frac{\eta_0{}^2}{4\gamma_0}\right)D_{-4v}\!\left(\frac{\eta_0}{\sqrt{\gamma_0}}\right)$$

$$\langle 0|r^{-5/2}|0\rangle = 2a_0{}^2 (4\gamma_0)^{-\frac{4v+1}{2}}\Gamma(4v+1)\exp\!\left(\frac{\eta_0{}^2}{4\gamma_0}\right)D_{-(4v+1)}\!\left(\frac{\eta_0}{\sqrt{\gamma_0}}\right) \qquad (24)$$

$$\langle 0|r^{-4}|0\rangle = 2a_0{}^2 (4\gamma_0)^{-\frac{4v-2}{2}}\Gamma(4v-2)\exp\!\left(\frac{\eta_0{}^2}{4\gamma_0}\right)D_{-(4v-2)}\!\left(\frac{\eta_0}{\sqrt{\gamma_0}}\right)$$

And

$$\langle 1|r^{-3}|1\rangle = \exp\!\left(\frac{\eta_1{}^2}{4\gamma_1}\right)\left\{\begin{array}{l} 2a_0{}^2 (4\gamma_1)^{-\frac{4v}{2}}\Gamma(4v)D_{-(4v)}\!\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + 2a_1{}^2 (4\gamma_1)^{-\frac{4v+2}{2}}\Gamma(4v+2)D_{-(4v+2)}\!\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + \\[2mm] + 4a_0a_1(4\gamma_1)^{-\frac{4v+1}{2}}\Gamma(4v+1)D_{-(4v+1)}\!\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) \end{array}\right\} \qquad (25.1)$$

$$\left\langle 1\left|r^{-5/2}\right|1\right\rangle = \exp\left(\frac{\eta_1^2}{4\gamma_1}\right)\left\{\begin{array}{l} 2a_0^2(4\gamma_1)^{-\frac{4\nu+1}{2}}\Gamma(4\nu+1)D_{-(4\nu+1)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + 2a_1^2(4\gamma_1)^{-\frac{4\nu+3}{2}}\Gamma(4\nu+3)D_{-(4\nu+3)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + \\[3mm] + 4a_0 a_1(4\gamma_1)^{-\frac{4\nu+2}{2}}\Gamma(4\nu+2)D_{-(4\nu+2)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) \end{array}\right\} \quad (25.2)$$

$$\left\langle 1\left|r^{-4}\right|1\right\rangle = \exp\left(\frac{\eta_1^2}{4\gamma_1}\right)\left\{\begin{array}{l} 2a_0^2(4\gamma_0)^{-\frac{4\nu-4}{2}}\Gamma(4\nu-4)D_{-(4\nu-4)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + 2a_1^2(4\gamma_1)^{-\frac{4\nu}{2}}\Gamma(4\nu)D_{-(4\nu)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) + \\[3mm] + 4a_0 a_1(4\gamma_1)^{-\frac{4\nu-1}{2}}\Gamma(4\nu-1/4)D_{-(4\nu-1)}\left(\frac{\eta_1}{\sqrt{\gamma_1}}\right) \end{array}\right\} \quad (25.3)$$

We have two principals cases, the first one, correspond to replace $\vec{L}\vec{\theta}$ by $\theta\vec{L}\vec{S}$ with ($\theta = \sqrt{\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2}$),

we have chosen the vector $\vec{\theta}$ parallel to the spin $\vec{S}$, then, we replace $\theta\vec{L}\vec{S}$ by $\frac{\theta}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$. The set

($H_{so-cs}(r,\theta)$, J^2, L^2, S^2 and J_z) forms a complete set of conserved quantities, the eigenvalues of the spin orbital coupling operator are $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - s(s+1)\}$, with $|l - s| \leq j \leq |l + s|$. This, allow us to obtain the energy shift $\Delta E(n = 0, j, l, s)$, $\Delta E(n = 1, j, l, s)$ and $\Delta E(n, j, l, s)$ due to the spin-orbital complying induced by $V_{pert-sc}(r)$ in (NC: 3D-RS) symmetries as follows:

$$\Delta E(n = 0, j, l, s) = k(l)\theta\left\{\frac{l(l+1)}{\langle 0|r^4|0\rangle} - (E + M)\left(\frac{a}{\langle 0|r^{-3}|0\rangle} - \frac{b}{2}\frac{1}{\langle 0|r^{5/2}|0\rangle}\right)\right\}$$

$$\Delta E(n = 1, j, l, s) = k(l)\theta\left\{\frac{l(l+1)}{\langle 1|r^4|1\rangle} - (E + M)\left(\frac{a}{\langle 1|r^{-3}|1\rangle} - \frac{b}{2}\frac{1}{\langle 1|r^{5/2}|1\rangle}\right)\right\} \quad (26)$$

$$\Delta E(n, j, l, s) = k(l)\theta\left\{\frac{l(l+1)}{\langle n|r^4|n\rangle} - (E + M)\left(\frac{a}{\langle n|r^{-3}|n\rangle} - \frac{b}{2}\frac{1}{\langle n|r^{5/2}|n\rangle}\right)\right\}$$

The second case corresponds to replacing both ($\vec{L}\vec{\theta}$ and θ_{12}) by ($\sigma_{12}BL_z$ and $\sigma_{12}B$) in addition to use $\langle n|L_z|m\rangle = m\delta_{nm}$ (with $-l \leq m \leq +l$). This, allow us to obtain the energy shift $\Delta E(n = 0, m)$, $\Delta E(n = 1, m)$ and $\Delta E(n, m)$ due to the modified Zeeman effect induced by $V_{pert-sc}(r)$ in (NC: 3D-RS) symmetries as follows:

$$\Delta E(n=0,m) = B\left\{\frac{l(l+1)}{\langle 0|r^4|0\rangle} - (E+M)\left(\frac{a}{\langle 0|r^{-3}|0\rangle} - \frac{b}{2}\frac{1}{\langle 0|r^{5/2}|0\rangle}\right)\right\}\sigma m$$

$$\Delta E(n=1,m) = B\left\{\frac{l(l+1)}{\langle 1|r^4|1\rangle} - (E+M)\left(\frac{a}{\langle 1|r^{-3}|1\rangle} - \frac{b}{2}\frac{1}{\langle 1|r^{5/2}|1\rangle}\right)\right\}\sigma m \qquad (27)$$

$$\Delta E(n,m) = B\left\{\frac{l(l+1)}{\langle n|r^4|n\rangle} - (E+M)\left(\frac{a}{\langle n|r^{-3}|n\rangle} - \frac{b}{2}\frac{1}{\langle n|r^{5/2}|n\rangle}\right)\right\}\sigma m$$

Results and Discussion

The superposition principal permitted to deduce the additive energy shift $\Delta E(n=0,j,l,s,m)$, $\Delta E(n=1,j,l,s,m)$ and $\Delta E(n,j,l,s,m)$ of ground state, first excited state and the nth excited state due to the spin-orbital complying and modified Zeeman effect which induced by $V_{pert-sc}(r)$ in (NC: 3D-RS) symmetries as follows:

$$\Delta E(n=0,j,l,s,m) = (k(l)\theta + B\sigma m)\left\{\frac{l(l+1)}{\langle 0|r^4|0\rangle} - (E+M)\left(\frac{a}{\langle 0|r^{-3}|0\rangle} - \frac{b}{2}\frac{1}{\langle 0|r^{5/2}|0\rangle}\right)\right\}$$

$$\Delta E(n=1,j,l,s,m) = (k(l)\theta + B\sigma m)\left\{\frac{l(l+1)}{\langle 1|r^4|1\rangle} - (E+M)\left(\frac{a}{\langle 1|r^{-3}|1\rangle} - \frac{b}{2}\frac{1}{\langle 1|r^{5/2}|1\rangle}\right)\right\} \qquad (28)$$

$$\Delta E(n,j,l,s,m) = (k(l)\theta + B\sigma m)\left\{\frac{l(l+1)}{\langle n|r^4|n\rangle} - (E+M)\left(\frac{a}{\langle n|r^{-3}|n\rangle} - \frac{b}{2}\frac{1}{\langle n|r^{5/2}|n\rangle}\right)\right\}$$

On the other hand, it is evident to consider the quantum number m takes $(2l+1)$ values and we have also two values for a fermionic particle with $s=1/2$ ($j=l\pm1/2$). Thus every state in usually three-dimensional space of energy for fermionic particle under modified Coulomb potential plus inverse-square–root potential will be $(2(2l+1))$ sub-states. To obtain the total complete degeneracy of energy level of the modified internal energy potential in (NC-3D: RS) symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$\sum_{i=0}^{n-1}(2l+1) = n^2 \rightarrow 2\left(\sum_{i=0}^{n-1}(2l+1)\right) \equiv 2n^2 \qquad (29)$$

Let us now look at two special cases, the first one correspond $a = Ze^2 \neq 0$ and $b=0$, which give the effective Colombian potential in (NC-3D: RS) symmetries $V_{pert-sc}(r,a,b=0)$ and the corresponding like radial Schrödinger equation which is exactly compatible with the results of [10]:

$$V_{pert-sc}(r,a,b=0) = \left[\frac{l(l+1)}{r^4} - (E+M)\frac{a}{r^3}\right]\vec{\mathbf{L}}\vec{\theta}$$

$$\left\{\frac{d^2}{dr^2} + (E^2 - M^2) - 2(E+M)V(r) - \frac{l(l+1)}{r^2} - \vec{\mathbf{L}}\vec{\theta}\left[\frac{l(l+1)}{r^4} - (E+M)\frac{a}{r^3}\right]\right\}R(r) = 0$$

(30)

While the second special case $a = 0$ and $b \neq 0$, which correspond the effective inverse-square–root potential in (NC-3D: RS) symmetries $V_{pert-sc}(r, a=0, b)$ and the like radial Schrödinger equation:

$$V_{pert-sc}(r,a,b=0) = \left[\frac{l(l+1)}{r^4} + \frac{b(E+M)}{2}\frac{1}{r^{5/2}}\right]\vec{\mathbf{L}}\vec{\theta}$$

$$\left\{\frac{d^2}{dr^2} + (E^2 - M^2) - 2(E+M)V(r) - \frac{l(l+1)}{r^2} - \vec{\mathbf{L}}\vec{\theta}\left[\frac{l(l+1)}{r^4} - \frac{b(E+M)}{2}\frac{1}{r^{5/2}}\right]\right\}R(r) = 0$$

(31)

If we consider $(\theta, \sigma) \rightarrow (0,0)$, we recover the results of relativistic commutative quantum mechanics obtained in ref. [6] for the Coulomb potential plus inverse-square–root potential, which means that our calculations are correct.

Conclusions

In this paper, we have investigated the MKGE for the modified Coulomb potential plus inverse-square–root potential in the (NC-3D: RS) symmetries. The energy shift for ground state $\Delta E(n=0, j, l, s, m)$, first excited state $\Delta E(n=1, j, l, s, m)$ and n^{th} excited $\Delta E(n, j, l, s, m)$ due to the noncommutativity property is obtained via first-order perturbation theory and expressed by parabolic cylinder functions, Gamma function, the discreet atomic quantum numbers (j, l, s, m) and the potential parameters (a and b), in addition to noncommutativity parameters (Θ and σ). This behavior is similar to the Zeeman effect and spin-orbital coupling in which a magnetic field is applied locally to the system and the spin-orbital couplings which are automatically induced with the perturbed potential shift in (NC: 3D-RS) symmetries. Therefore, we can conclude that the MKGE, in the symmetries of RNCQM, can be extended to describe a dynamic state of a particle with spin non-null.

Acknowledgements

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