Structure of New Solitary Solutions for The Schwarzian Korteweg De Vries Equation And (2+1)-Ablowitz-Kaup-Newell-Segur Equation

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Abstract

In this research, we introduce and represent the modified Khater method on two basic models in the optical fiber. These two models describe the dynamics of the wave movement in the optical fiber. It is a new modification of new recent method which developed by Mostafa M. A. Khater in 2017. We implement this new modified technique on Schwarzian Korteweg de Vries equation and (2+1)-Ablowitz-Kaup-Newell-Segur equation. This modification of Khater method produces more closed solutions than many other methods. Schwarzian Korteweg de Vries (SKdV) equation has a closed relationship with (2+1)-Ablowitz- Kaup-Newell-Segur equation. Schwarzian Korteweg de Vries equation prescribes the location in a micro-segment of space and motion of the isolated waves in varied fields which localized in a tiny portion of space. It is a great and basic system in fluid mechanics, nonlinear optics, plasma physics, and quantum field theory.

Keywords: Schwarzian Korteweg de Vries equation; (2+1)-Ablowitz-Kaup-Newell-Segur equation; Modified Khater method; Optical traveling wave solutions; Exact, solitary and approximate solutions.

1. Introduction

Partial differential equations (PDEs) that's the important part of the math that praises his names to many strands of science and that's because of his potential and abilities to characterize many cosmic and natural phenomena like physics and chemistry and biology, fluid mechanics, hydrodynamics, optics, plasma physics and other strands of science and knowledge. Especially, when Zabusky & Kruskal (1965) introduced the mean of the soliton. This is because of its analytical and descriptive capabilities for these different models where many recent techniques have emerged and scientists have developed ways to access the exact and individual solutions to these models. For example of these methods:

The $\left(\frac{G'}{G}\right)$ –expansion method, the $e^{-\phi(\xi)}$–expansion method, modified Kudryashov methods, modified $\left(\frac{G'}{G}\right)$ –expansion method, the $\left(\frac{G'}{G},\frac{1}{G}\right)$ –expansion method, extended $e^{-\phi(\xi)}$–expansion method, the extended tanh-function method, the Kudryashov, Novel $\left(\frac{G'}{G}\right)$ –expansion method, the improved $\tan\left(\frac{\phi}{2}\right)$-expansion method, modified simple equation method, Khater method , Adomian decomposing method and so [1]- [24].

Khater method is looked like one of the latest methods in this zone as it just detected from just one year and also it features the results of some techniques so that, these techniques can be examined as a particular condition of Khater technique.

Schwarzian Korteweg de Vries equation was displayed by Krichever and Novikov in the following form: see [25]

$$\frac{\Psi_t}{\Psi_x}+\left(\frac{\Psi_{xx}}{\Psi_x}\right)_x-\frac{1}{2}\left(\frac{\Psi_{xx}}{\Psi_x}\right)^2=0. \tag{1.1}$$

where Ψ satisfies Newton's equation of motion in a cubic potential. This equation has also another form as below:

$$S_t + \frac{1}{4}\, S_{x\,x\,z} - \frac{S_x\, S_{x\,z}}{2\,S} - \frac{S_{x\,x}\, S_z}{4\,S} + \frac{S_x^2\, S_z}{2\,S^2} - \frac{S_x}{8}\int \left(\frac{S_x^2}{S^2}\right)_z dx = 0. \tag{1.2}$$

This equation plays an important and vital role in a nonlocal form and a right-moving soliton. Schwarzian Korteweg de Vries is so closed to (2+1) Ablowitz–Kaup–Newell–Segur (AKNS) equation.

In this research, we implement a modified Khater method on these two modules. We demonstrate the basic steps of this new method. Through this study, the reader observes the extent of rapprochement between both methods but the only difference between the two methods is how much convergence solutions are given using modified Khater method for approximate solutions which speeds up modified Khater method is the successful extension of Khater method.

The vestiges of this paper are regulated as below: In section **2**, we give the structure of modified Khater method. In section 3, we implement the modified Khater method to gain new structure of the exact and solitary traveling wave solutions of the two mentioned models indicated above. In section **4**, conclusions are given.

2. Structure of modified Khater method:

Examination the nonlinear partial differential equation be in the below:

$$Q\big(u, u_x, u_y, u_z, u_t, u_{x\,x}, u_{x\,y}, u_{x\,z}, u_{x\,t}, \dots\big) = 0, \tag{2.1}$$

where Q is the polynomial function of u. Utilizing the wave transformation:

$$u(x, y, z, t) = u(\xi), \xi = x + y + z - ct. \tag{2.2}$$

We convert a nonlinear partial differential equation (NLPDE) into the nonlinear ordinary differential equation (NLODE) to be in the following form:

$$\Psi(u, u', u'', u''', \dots) = 0, \tag{2.3}$$

where Ψ is the function of $u(\xi)$. Balancing the highest order derivative term and nonla inear term which involved in Eq. (2.3).

Step 1. According to the modified Khater method, the general exact solution of an ordinary differential equation in the below:

$$u(\xi) = \sum_{i=0}^{N} a_i\, a^{i\,f(\xi)}. \tag{2.4}$$

Such that $f(\xi)$ depends on the following auxiliary equation:

$$f'(\xi) = \frac{1}{\ln(a)}\left[\alpha\, a^{-f(\xi)} + \beta + \sigma\, a^{f(\xi)}\right], \tag{2.5}$$

where (α, β, σ) are arbitrary constant will identify later.

Step 2. Evaluate the positive integer N in Eq. (2.4) and that by using the balance technique.

$$D\left[\frac{d^{\epsilon}u(\xi)}{d\,\xi^{\epsilon}}\right]=N+\epsilon,\ D\left[u^{\epsilon}\left(\frac{d^{\epsilon}u(\xi)}{d\,\xi^{\epsilon}}\right)^{s}\right]=\epsilon\,N+s(N+\epsilon).$$

Step 3. Replacement Eq. (2.4) into Eq. (2.4) and collecting all the terms of the same power of $\left[a^{i\,f(\xi)}, i = 0,1,2,\dots\right]$. We obtain a system of algebraic equations. Solving this system by utilization any computer program to obtain all previous parameters.

3. Application:

In this section, we implement modified Khater method for these two models (the Schwarzian Korteweg de Vries equation and (2+1)-Ablowitz-Kaup-Newell-Segur equation) and also we show the exact traveling wave solutions and solitary wave solutions of each one of these models.

3.1. The Schwarzian Korteweg de Vries equation:

Consider the Schwarzian Korteweg de Vries equations [26]- [30] in the following form:

$$\begin{cases}4\,u^2\,v_x-4\,u\,u_x v+u^2\,u_{x\,x\,z}-u\,u_{x\,x}u_z-3\,u\,u_x u_{x\,z}+3\,u_x^2 u_z-\,u^4\,u_z=0,\\ u_t-v_t=0.\end{cases}\tag{3.1.1}$$

Using the wave transformation $[u(x,z,t)=u(\xi), v(x,z,t)\xi=v(\xi), \xi=x+z-c\,t]$ and integration the second equation with zero constant of integration in the system then substituting the obtained equation into the first equation of the system, we obtain:

$$u^2\,u'''-4\,u\,u'u''+\,3\,u'^2-u^4\,u'=0.\tag{3.1.2}$$

Balancing the highest order derivative term and nonlinear term $[u^2\,u'''\&u^4\,u']\Rightarrow\ (N=1)$. According to the general solution of the suggested method (modified Khater method), we get the general exact solution of Eq. (3.1.2) in the following form:

$$u(\xi)=a_0+a_1\,a^{f(\xi)}.\tag{3.1.3}$$

Substituting Eq. (3.1.3) and its derivatives into Eq. (3.1.2). Collecting the coefficient of the same power of $\left[a^{i\,f(\xi)}, i=0,1,2,\dots\right]$ and equating the result equations with zero. We get the system of algebraic equation. Solving this system by any computer program or even manually, we obtain:

Case 1.

$$\sigma=\frac{a_0\,(\beta-a_0)}{\alpha},\ a_0=a_0\,,a_1=\frac{a_0\,(\beta-a_0)}{\alpha}.$$

Consequently, we obtain the exact traveling solution in the form:

$$u(\xi)=a_0\left[1+\frac{(\beta-a_0)}{\alpha}\ \ a^{f(\xi)}\right].\tag{3.1.4}$$

Therefore, the solitary traveling wave solutions are in the following form:

When $(\beta^2-4\alpha\sigma<0\ \&\ \sigma\neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right], \tag{3.1.5}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right]. \tag{3.1.6}$$

When $(\beta^2 - 4\alpha\sigma > 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right], \tag{3.1.7}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\, coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\,\xi\right)\right)\right]. \tag{3.1.8}$$

When $(\beta^2 + 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\, tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right)\right], \tag{3.1.9}$$

$$u(\xi) = a_0\left[1\, \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\, tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\,\xi\right)\right)\right]. \tag{3.1.10}$$

When $(\beta^2 + 4\alpha^2 < 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.11}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.12}$$

When $(\beta^2 - 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.13}$$

$$(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.14}$$

When $(\beta^2 - 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.15}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.16}$$

When ($\alpha\sigma < 0\ \&\ \sigma \neq 0\ \&\ \beta = 0$):

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right], \tag{3.1.17}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right]. \tag{3.1.18}$$

When ($\alpha\sigma > 0\ \&\ \sigma \neq 0\ \&\ \beta = 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,\frac{\sqrt{\alpha\,\sigma}}{\sigma}\, tan\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right], \tag{3.1.19}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta - a_0)}{\alpha}\,\sqrt{\frac{\alpha}{\sigma}}\, cot\left(\sqrt{\alpha\,\sigma}\,\xi\right)\right]. \tag{3.1.20}$$

When ($\beta = 0\ \&\alpha = -\sigma$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right)\right]. \tag{3.1.21}$$

When ($\beta = k\ \&\alpha = 2\,k\&\sigma = 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(e^{k\,\xi} - 2\right)\right]. \tag{3.1.22}$$

When ($\beta = 0\ \&\alpha = \sigma$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\,[tan(\alpha\,\xi + C)]\right]. \tag{3.1.23}$$

When ($\sigma = 0$):

$$u(\xi) = a_0\left[1 + \frac{(\beta - a_0)}{\alpha}\left(e^{\beta\,\xi} - \frac{\alpha}{\beta}\right)\right]. \tag{3.1.24}$$

Where ($C\ \&K$) are arbitrary constants.

Case 2.

$$\sigma = -\frac{a_0\,(\beta + a_0)}{\alpha},\ a_0 = a_0\,, a_1 = \frac{a_0\,(\beta + a_0)}{\alpha}.$$

Where $[C\ \&K]$ are arbitrary constants.

Consequently, we obtain the exact traveling solution in the form:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\ \ a^{f(\xi)}\right]. \tag{3.1.25}$$

Therefore, the solitary traveling wave solutions are in the following form:

When $(\beta^2 - 4\alpha\sigma < 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right], \tag{3.1.26}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right]. \tag{3.1.27}$$

When $(\beta^2 - 4\alpha\sigma > 0\ \&\ \sigma \neq 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right], \tag{3.1.28}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\ coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\ \xi\right)\right)\right]. \tag{3.1.29}$$

When $(\beta^2 + 4\alpha^2 > 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\ tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\ \xi\right)\right)\right], \tag{3.1.30}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\ tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\ \xi\right)\right)\right]. \tag{3.1.31}$$

When $(\beta^2 + 4\alpha^2 < 0\ \&\ \sigma \neq 0\ \&\ \sigma = -\alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.32}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + \alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.33}$$

When $(\beta^2 - 4\alpha^2 > 0 \,\&\, \sigma \neq 0 \,\&\, \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1. 34}$$

$$(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\, cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.35}$$

When $(\beta^2 - 4\alpha^2 > 0 \,\&\, \sigma \neq 0 \,\&\, \sigma = \alpha)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right], \tag{3.1.36}$$

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\left(-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\, coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right)\right]. \tag{3.1.37}$$

When $(\alpha\sigma < 0 \,\&\, \sigma \neq 0 \,\&\, \beta = 0)$:

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right], \tag{3.1.38}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{-\alpha\,\sigma}}{\sigma}\, coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right)\right]. \tag{3.1.39}$$

When $(\alpha\sigma > 0 \,\&\, \sigma \neq 0 \,\&\, \beta = 0)$:

$$u(\xi) = a_0\left[1 + \frac{(\beta + a_0)}{\alpha}\,\frac{\sqrt{\alpha\,\sigma}}{\sigma}\, tan(\sqrt{\alpha\,\sigma}\,\xi)\right], \tag{3.1.40}$$

$$u(\xi) = a_0\left[1 - \frac{(\beta + a_0)}{\alpha}\,\sqrt{\frac{\alpha}{\sigma}}\, cot(\sqrt{\alpha\,\sigma}\,\xi)\right]. \tag{3.1.41}$$

When $(\beta = 0 \,\& \alpha = -\sigma)$:

$$u(\xi) = a_0 \left[1 + \frac{(\beta + a_0)}{\alpha} \left(\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right)\right]. \tag{3.1.42}$$

When $(\beta = k \,\& \alpha = 2\,k \& \sigma = 0)$:

$$u(\xi) = a_0 \left[1 + \frac{(\beta + a_0)}{\alpha} \left(e^{k\,\xi} - 2\right)\right]. \tag{3.1.43}$$

When $(\beta = 0 \,\& \alpha = \sigma)$:

$$u(\xi) = a_0 \left[1 + \frac{(\beta + a_0)}{\alpha} \left[tan(\alpha\,\xi + C)\right]\right]. \tag{3.1.44}$$

When $(\sigma = 0)$:

$$u(\xi) = a_0 \left[1 + \frac{(\beta + a_0)}{\alpha} \left(e^{\beta\,\xi} - \frac{\alpha}{\beta}\right)\right]. \tag{3.1.45}$$

Where $(C \,\& K)$ are arbitrary constants.

3.2. The (2+1)-Ablowitz-Kaup-Newell-Segur equation:

Examine the (2+1)-Ablowitz-Kaup-Newell-Segue equation [26] and [31] - [34] in the following form:

$$4\,u_{x\,t} + u_{x\,x\,x\,z} + 8\,u_{x\,z} u_x + 4\,u_z\,u_{x\,x} = 0. \tag{3.2.1}$$

Using the wave transformation $[u(x,z,t) = u(\xi), v(x,z,t)\xi = v(\xi), \xi = x + z - c\,t]$, we get:

$$-4\,c\,u' + u''' + 6\,u'^2 = 0. \tag{3.2.2}$$

Balancing the highest order derivative term and nonlinear term $[\,u''' \& \,u'^2] \Rightarrow (N = 1)$. According to the general solution of the suggested method (modified Khater method), we get the general exact solution of Eq. (3.2.2) is the same to general exact solution of Eq. (3.1.2) . Substituting Eq. (3.1.3) and its derivatives into Eq. (3.2.2). Collecting the coefficient of the same power of $\left[a^{i\,f(\xi)}, i = 0,1,2, \dots\right]$ and equating the result equations with zero. We get the system of algebraic equation. Solving this system by any computer program or even manually, we obtain:

$$\alpha = \frac{\beta^2 - 4\,c}{4\,\sigma}, a_0 = a_0, a_1 = -\sigma.$$

Consequently, the exact traveling wave solution is in below:

$$u(\xi) = a_0 - \sigma\,a^{f(\xi)}. \tag{3.2.3}$$

Therefore, the solitary traveling wave solutions are in the following form:

When $(\beta^2 - 4\alpha\sigma < 0 \,\&\, \sigma \neq 0)$:

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right)\right], \tag{3.2.4}$$

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma}\cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right)\right]. \tag{3.2.5}$$

When $(\beta^2 - 4\alpha\sigma > 0 \ \& \ \sigma \neq 0)$:

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\xi\right)\right], \tag{3.2.6}$$

$$u(\xi) = a_0 - \sigma\left[\frac{-\beta}{2\sigma} - \frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2\sigma}\coth\left(\frac{\sqrt{(\beta^2 - 4\alpha\sigma)}}{2}\xi\right)\right]. \tag{3.2.7}$$

When $(\beta^2 + 4\alpha^2 > 0 \ \& \ \sigma \neq 0 \ \& \ \sigma = -\alpha)$:

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\tanh\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\xi\right)\right], \tag{3.2.8}$$

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 + 4\alpha^2}}{2\alpha}\coth\left(\frac{\sqrt{\beta^2 + 4\alpha^2}}{2}\xi\right)\right]. \tag{3.2.9}$$

When $(\beta^2 + 4\alpha^2 < 0 \ \& \ \sigma \neq 0 \ \& \ \sigma = -\alpha)$:

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2\alpha}\tan\left(\frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2}\xi\right)\right], \tag{3.2.10}$$

$$u(\xi) = a_0 - \sigma\left[\frac{\beta}{2\alpha} - \frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2\alpha}\cot\left(\frac{\sqrt{-(\beta^2 + 4\alpha^2)}}{2}\xi\right)\right]. \tag{3.2.11}$$

When $(\beta^2 - 4\alpha^2 < 0 \ \& \ \sigma \neq 0 \ \& \ \sigma = \alpha)$:

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\tan\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\xi\right)\right], \tag{3.2.12}$$

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} + \frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2\alpha}\cot\left(\frac{\sqrt{-(\beta^2 - 4\alpha^2)}}{2}\xi\right)\right]. \tag{3.2.13}$$

When $(\beta^2 - \alpha^2 > 0 \ \& \ \sigma \neq 0 \ \& \ \sigma = \alpha)$:

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\tanh\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right], \quad (3.2.14)$$

$$u(\xi) = a_0 - \sigma\left[-\frac{\beta}{2\alpha} - \frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2\alpha}\coth\left(\frac{\sqrt{(\beta^2 - 4\alpha^2)}}{2}\,\xi\right)\right]. \quad (3.2.15)$$

When $(\alpha\sigma < 0 \,\&\, \sigma \neq 0 \,\&\, \beta = 0)$:

$$u(\xi) = a_0 + \sqrt{-\alpha\,\sigma}\tanh\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right), \quad (3.2.16)$$

$$u(\xi) = a_0 + \sqrt{-\alpha\,\sigma}\coth\left(\frac{\sqrt{-\alpha\,\sigma}}{2}\,\xi\right), \quad (3.2.17)$$

When $(\alpha\sigma > 0 \,\&\, \sigma \neq 0 \,\&\, \beta = 0)$:

$$u(\xi) = a_0 - \sqrt{\alpha\,\sigma}\,tan\left(\sqrt{\alpha\,\sigma}\,\xi\right), \quad (3.2.18)$$

$$u(\xi) = a_0 + \sqrt{\alpha\,\sigma}\,tan\left(\sqrt{\alpha\,\sigma}\,\xi\right). \quad (3.2.19)$$

When $(\beta = 0 \,\&\alpha = -\sigma)$:

$$u(\xi) = a_0 - \sigma\left[\frac{e^{2\,\alpha\,\xi} + 1}{e^{2\,\alpha\,\xi} - 1}\right]. \quad (3.2.20)$$

When $(\alpha = \sigma = 0)$:

$$u(\xi) = a_0 - \sigma e^{\beta\,\xi}. \quad (3.2.21)$$

When $(\beta^2 = \alpha\,\sigma)$:

$$u(\xi) = a_0 + \frac{-2\alpha\sigma\,(\beta\,\xi + 2)}{\beta^2\,\xi}. \quad (3.2.22)$$

When $(\beta = k \,\&\alpha = 2\,k\&\sigma = 0)$:

$$u(\xi) = a_0 - \sigma\left[e^{k\,\xi} - 2\right]. \quad (3.2.23)$$

When $(\beta = k \,\&\sigma = 2\,k\&\alpha = 0)$:

$$u(\xi) = a_0 - \frac{\sigma e^{k\,\xi}}{1 - e^{k\,\xi}}. \quad (3.2.24)$$

When $(\alpha = 0)$:

$$u(\xi) = a_0 - \frac{\sigma\beta\,e^{\beta\,\xi}}{2 - \sigma\,e^{\beta\,\xi}}. \quad (3.2.25)$$

When $(\beta = \sigma = 0)$:

$$u(\xi) = a_0 - \sigma\,\alpha\,\xi. \qquad (3.2.26)$$

When $(\beta = \alpha = 0)$:

$$u(\xi) = a_0 + \frac{\sigma}{\sigma\,\xi}. \qquad (3.2.27)$$

When $(\beta = 0\ \&\alpha = \sigma)$:

$$u(\xi) = a_0 - \sigma tan(\alpha\,\xi + C). \qquad (3.2.28)$$

When $(\sigma = 0)$:

$$u(\xi) = a_0 - \sigma\left[e^{\beta\,\xi} - \frac{\alpha}{\beta}\right]. \qquad (3.2.29)$$

Where k, C are arbitrary constant.

4. Figures:

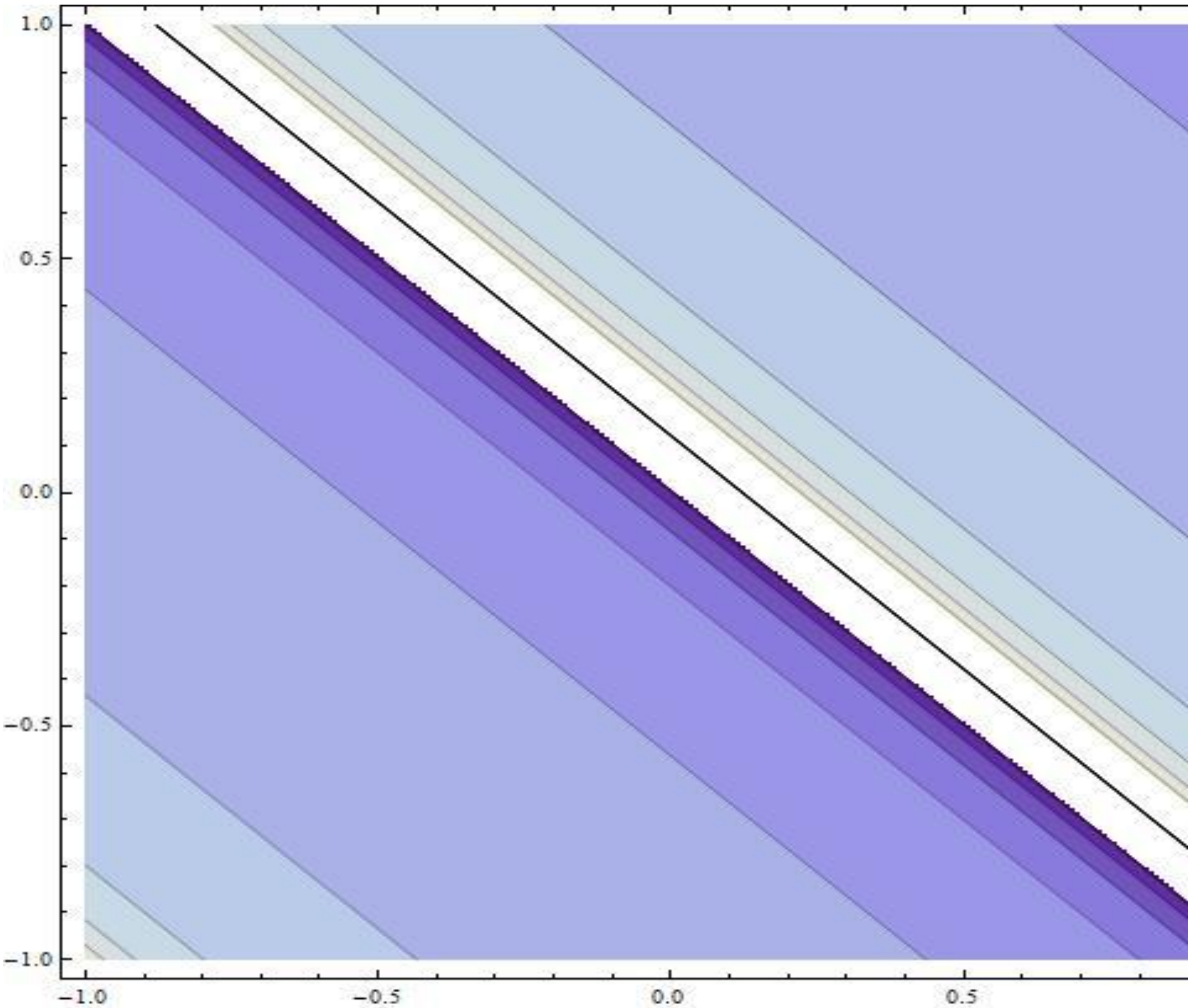

Fig.1 Contour plot of Eq. (3.1.5) when ($\alpha = 3, \beta = 2, \sigma = 4, a_0 = 5, z = 1, c = -1$).

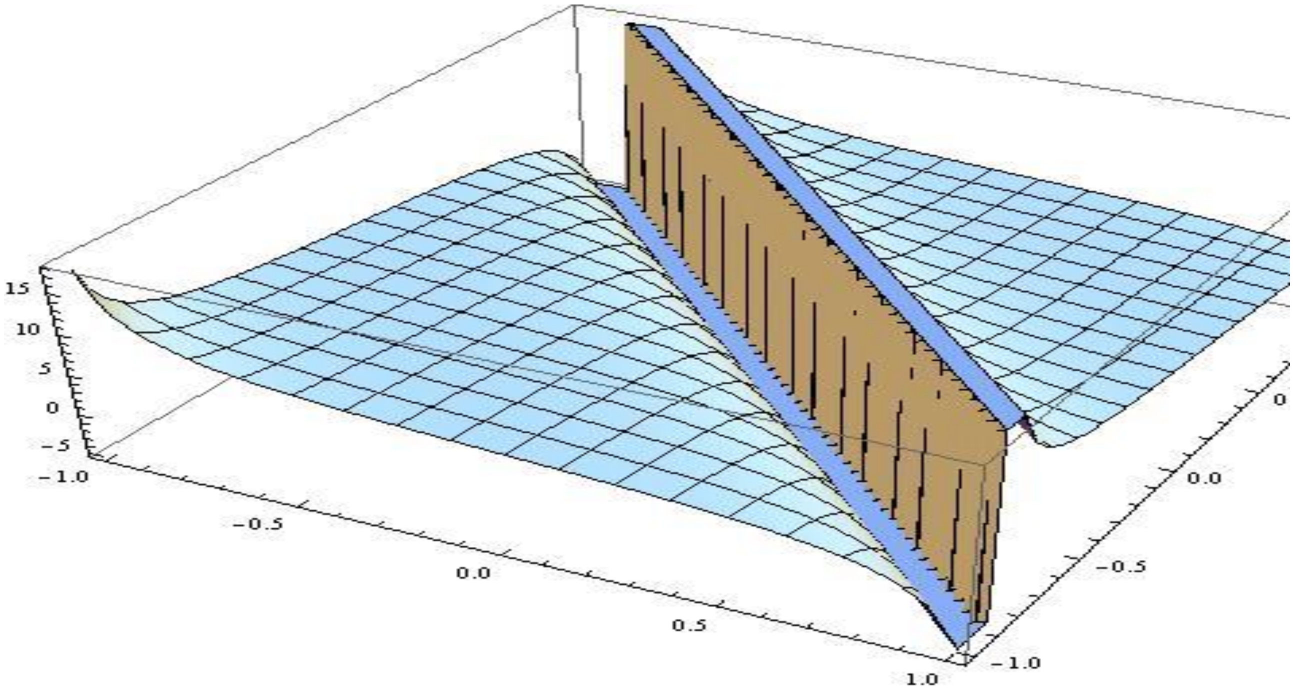

Fig.2 3D-plot of Eq. (3.1.5) when ($\alpha = 3, \beta = 2, \sigma = 4, a_0 = 5, z = 1, c = -1$).

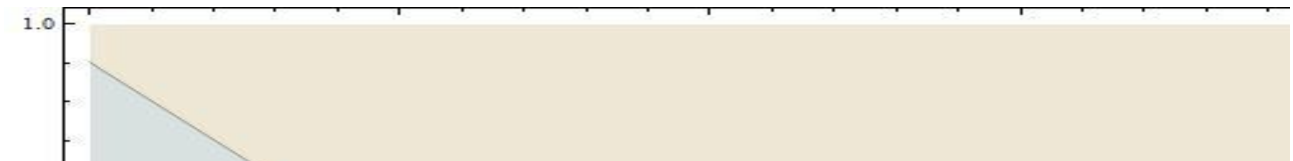

Fig.3 Contour plot of Eq. (3.1.6) when ($\alpha = 1, \beta = 3, \sigma = 2, a_0 = 5, z = 1, c = -1$).

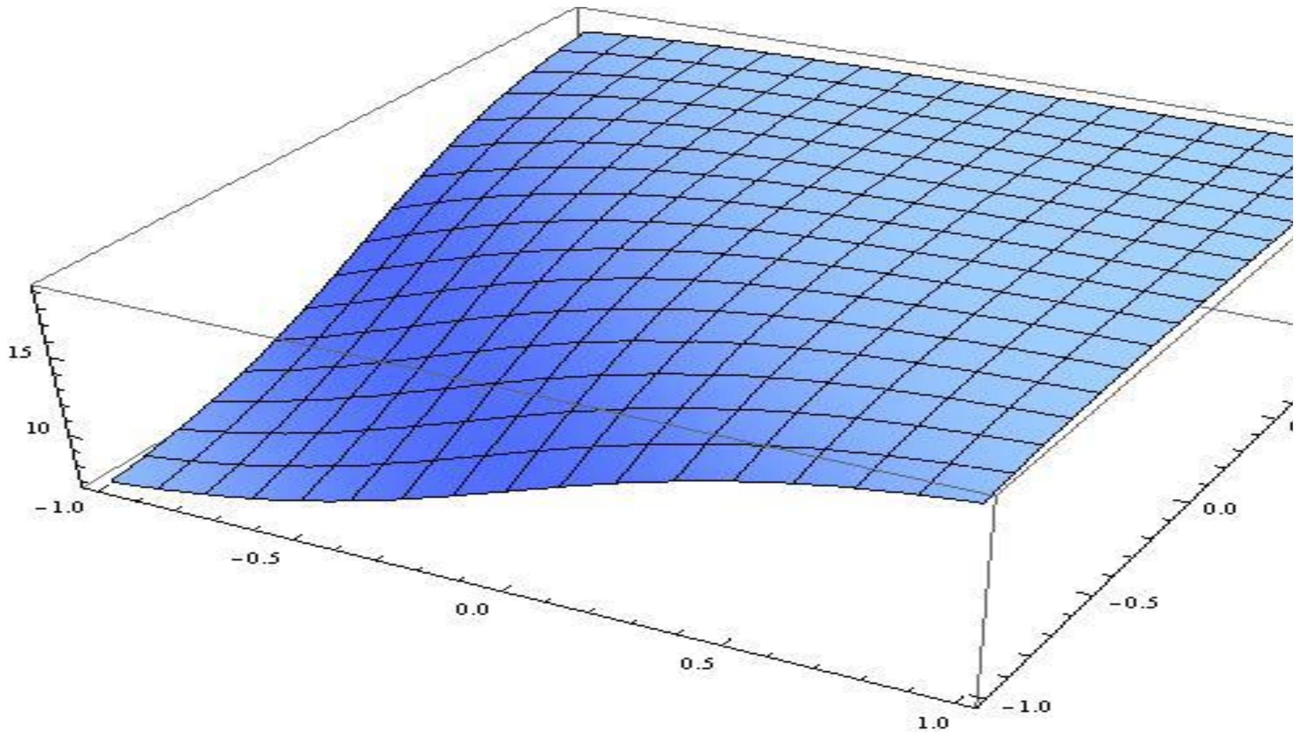

Fig.4 3D-plot of Eq. (3.1.6) when ($\alpha = 1, \beta = 3, \sigma = 2, a_0 = 5, z = 1, c = -1$).

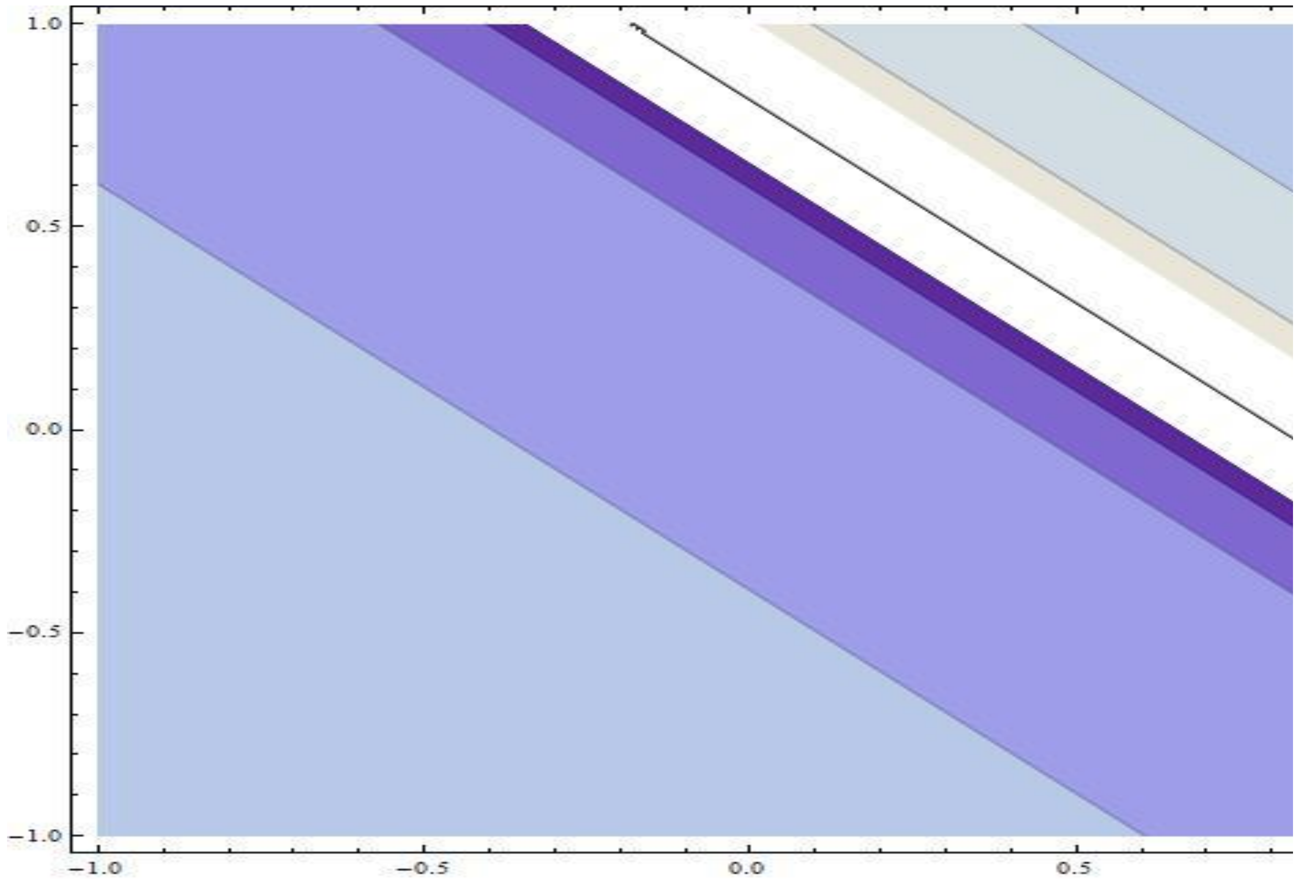

Fig.5 Contour plot of Eq. (3.1.11) when ($\alpha = 2, \beta = 1, \sigma = -2, a_0 = 5, z = 1, c = -1$).

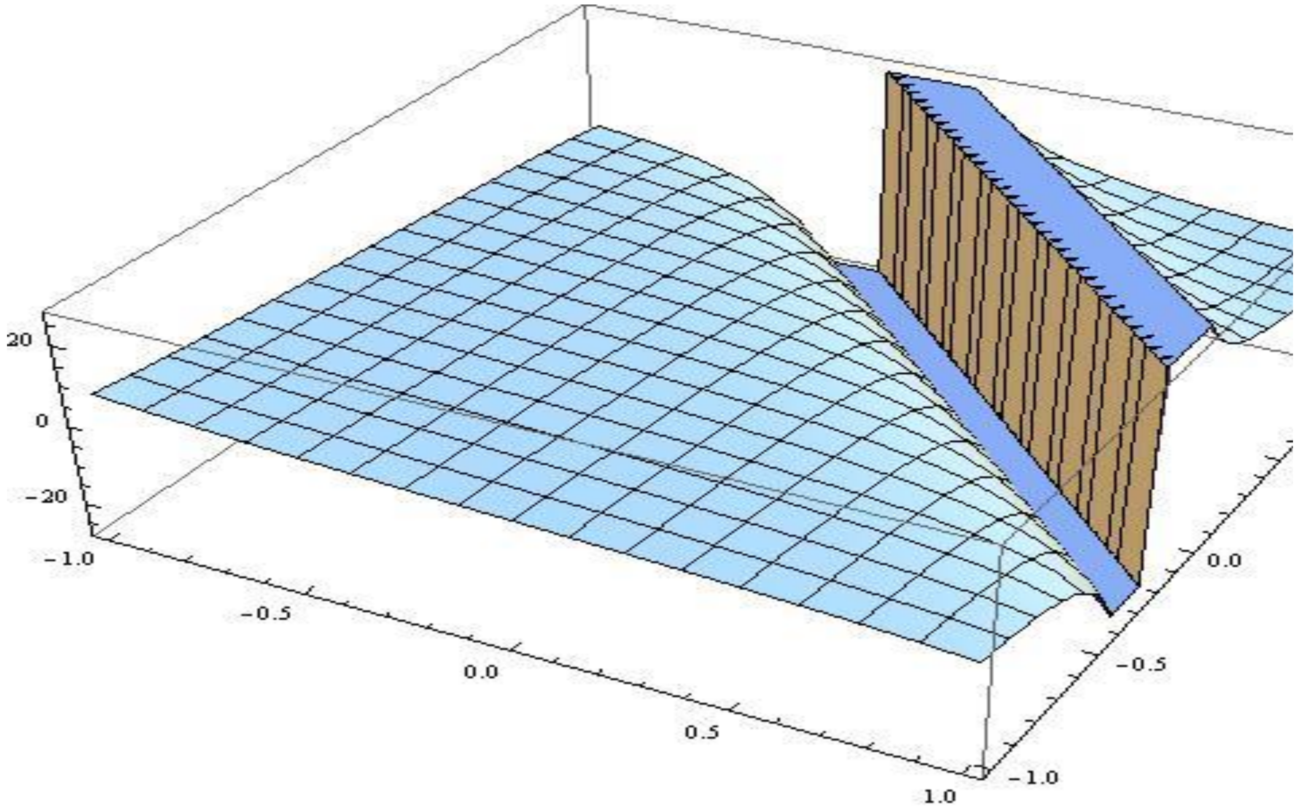

Fig.6 3D-plot of Eq. (3.1.11) when ($\alpha = 2, \beta = 1, \sigma = -2, a_0 = 5, z = 1, c = -1$).

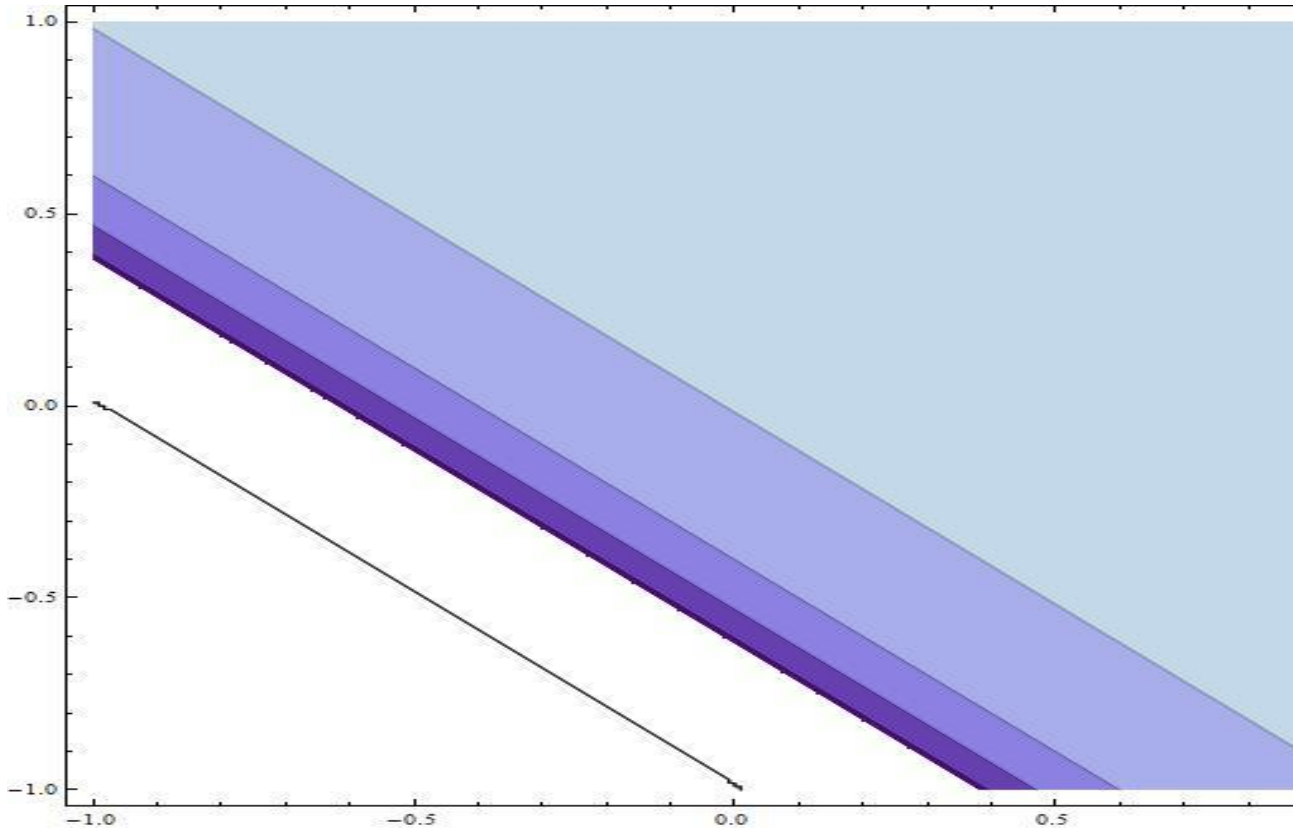

Fig.7 Contour plot of Eq. (3.1.21) when ($\alpha = -2, \beta = 0, \sigma = 2, a_0 = 5, z = 1, c = -1$).

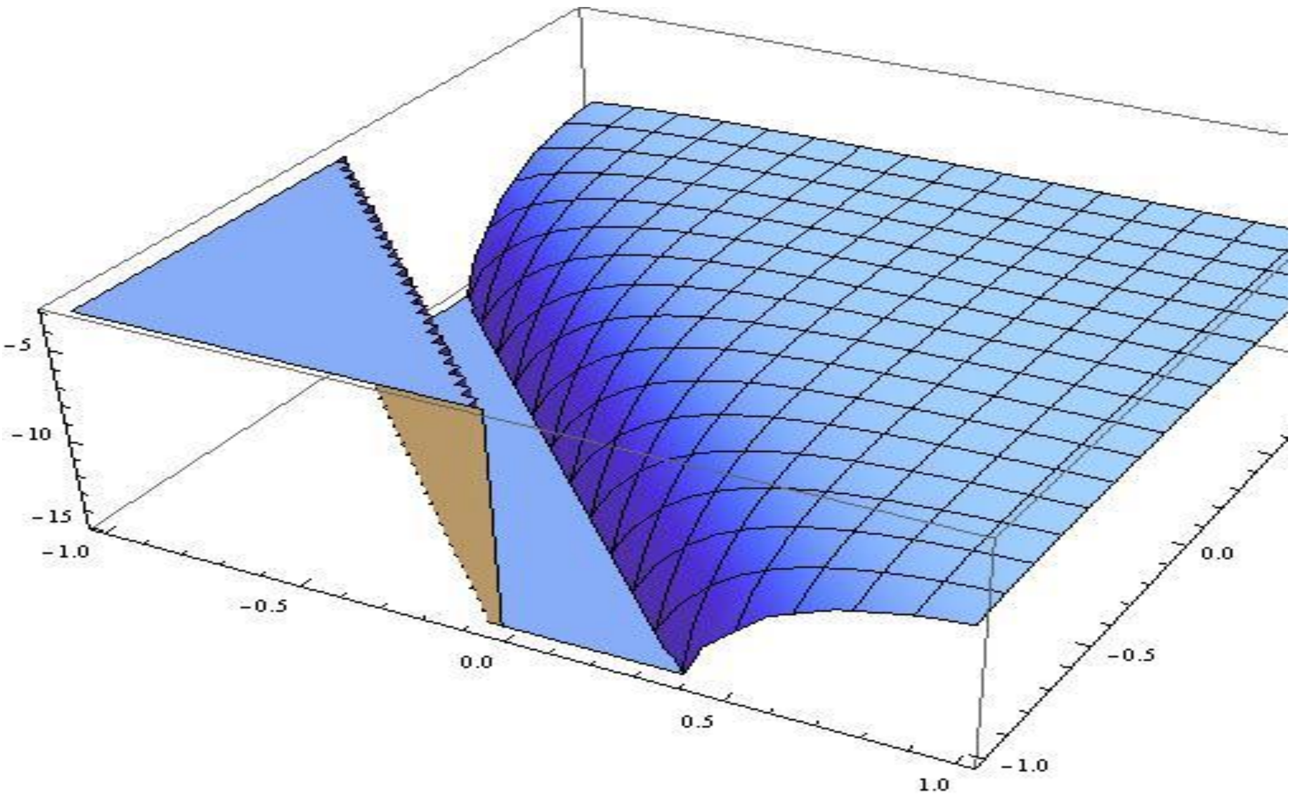

Fig.8 3D-plot of Eq. (3.1.21) when ($\alpha = -2, \beta = 0, \sigma = 2, a_0 = 5, z = 1, c = -1$).

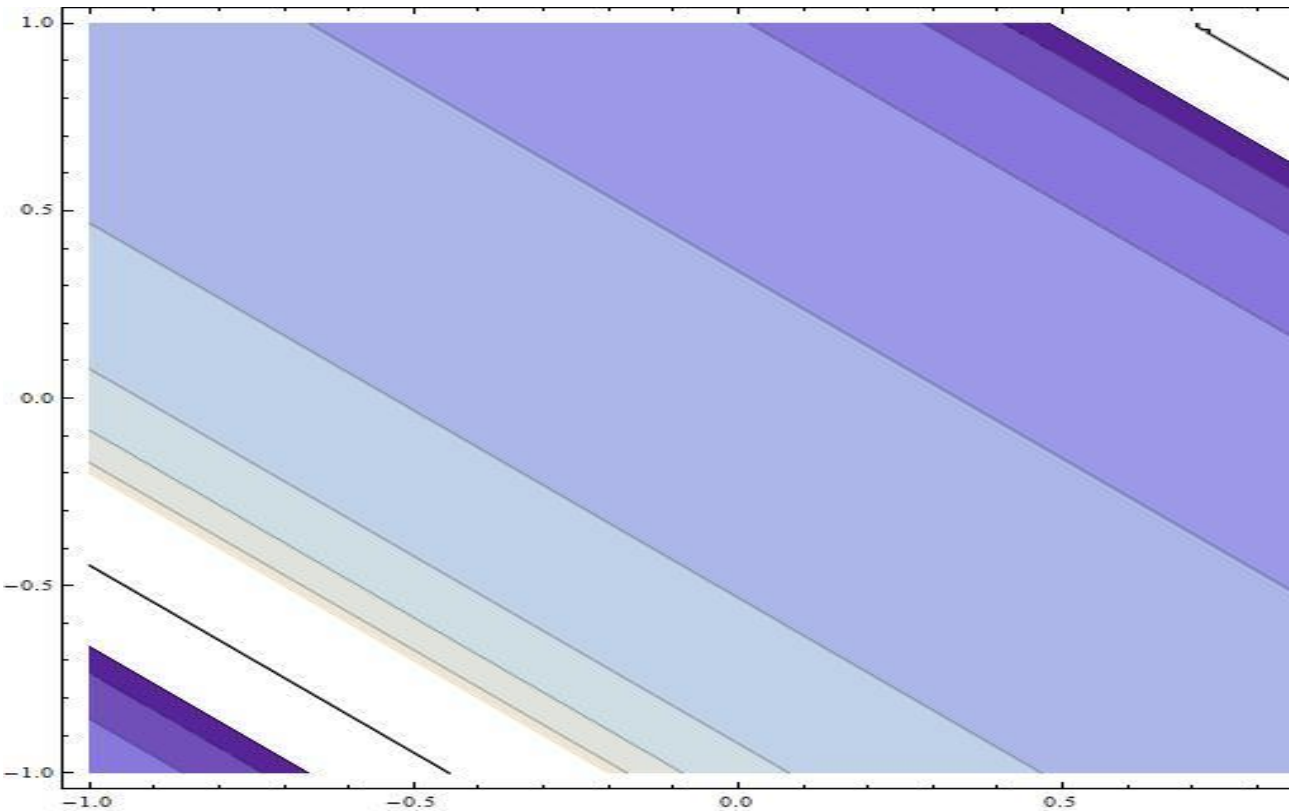

Fig.9 Contour plot of Eq. (3.1.25) when ($\alpha = 1, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

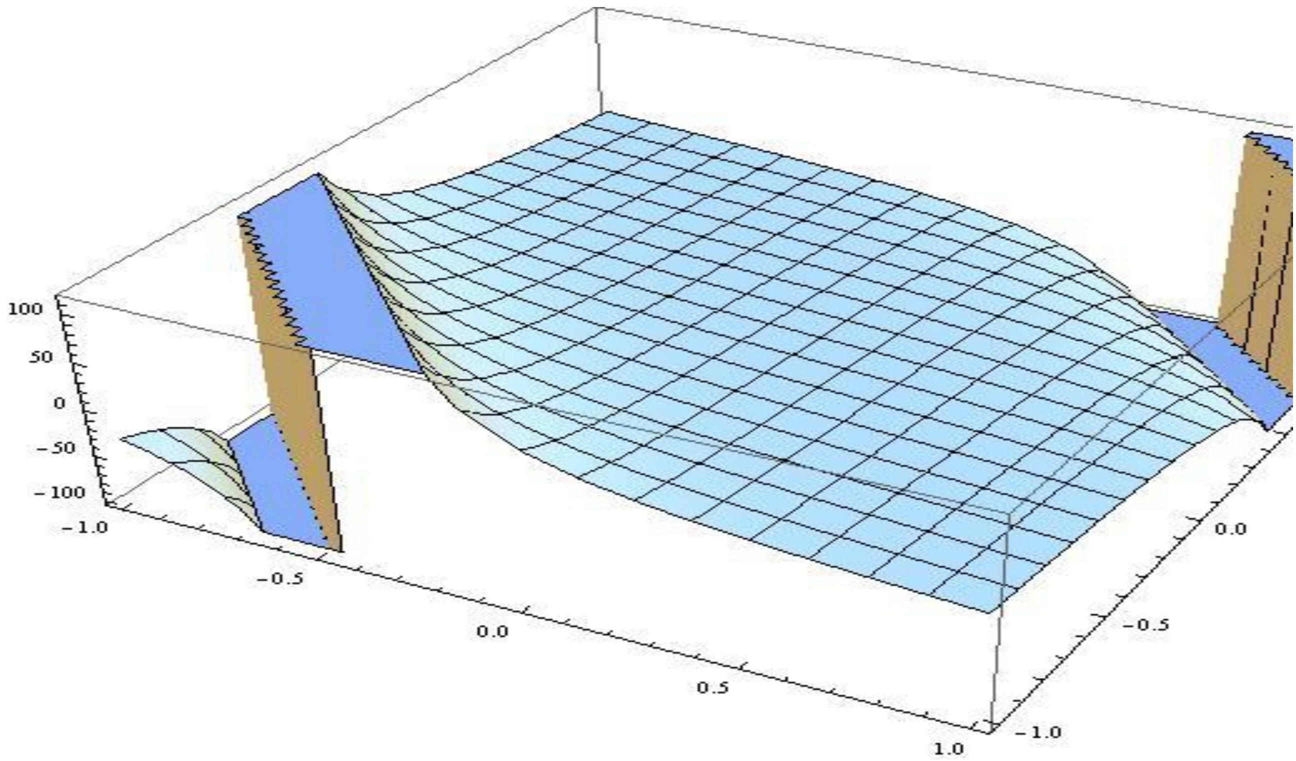

Fig.10 3D-plot of Eq. (3.1.25) when ($\alpha = 1, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

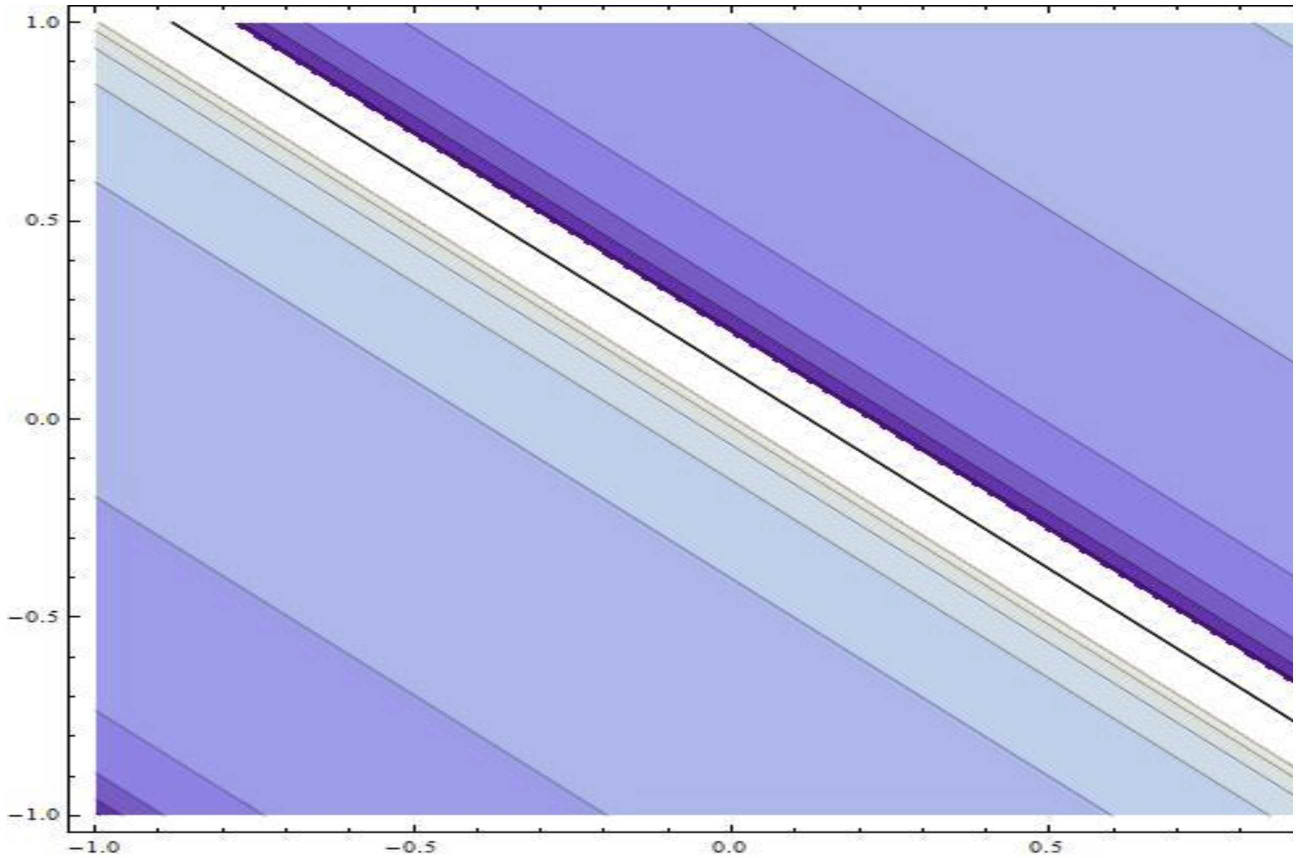

Fig.11 Contour plot of Eq. (3.1.42) when ($\alpha = 2, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

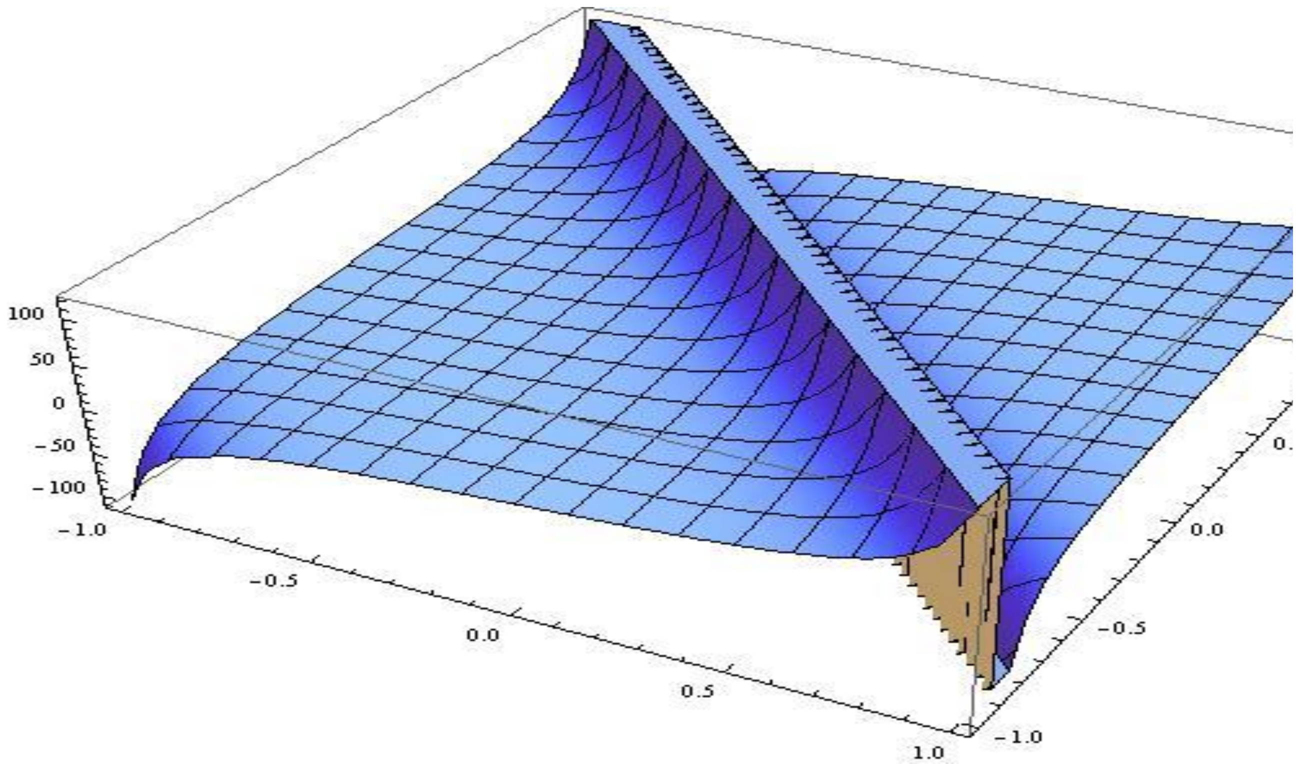

Fig.12 3D-plot of Eq. (3.1.42) when ($\alpha = 2, \beta = 0, \sigma = 1, a_0 = 5, z = 1, c = -1$).

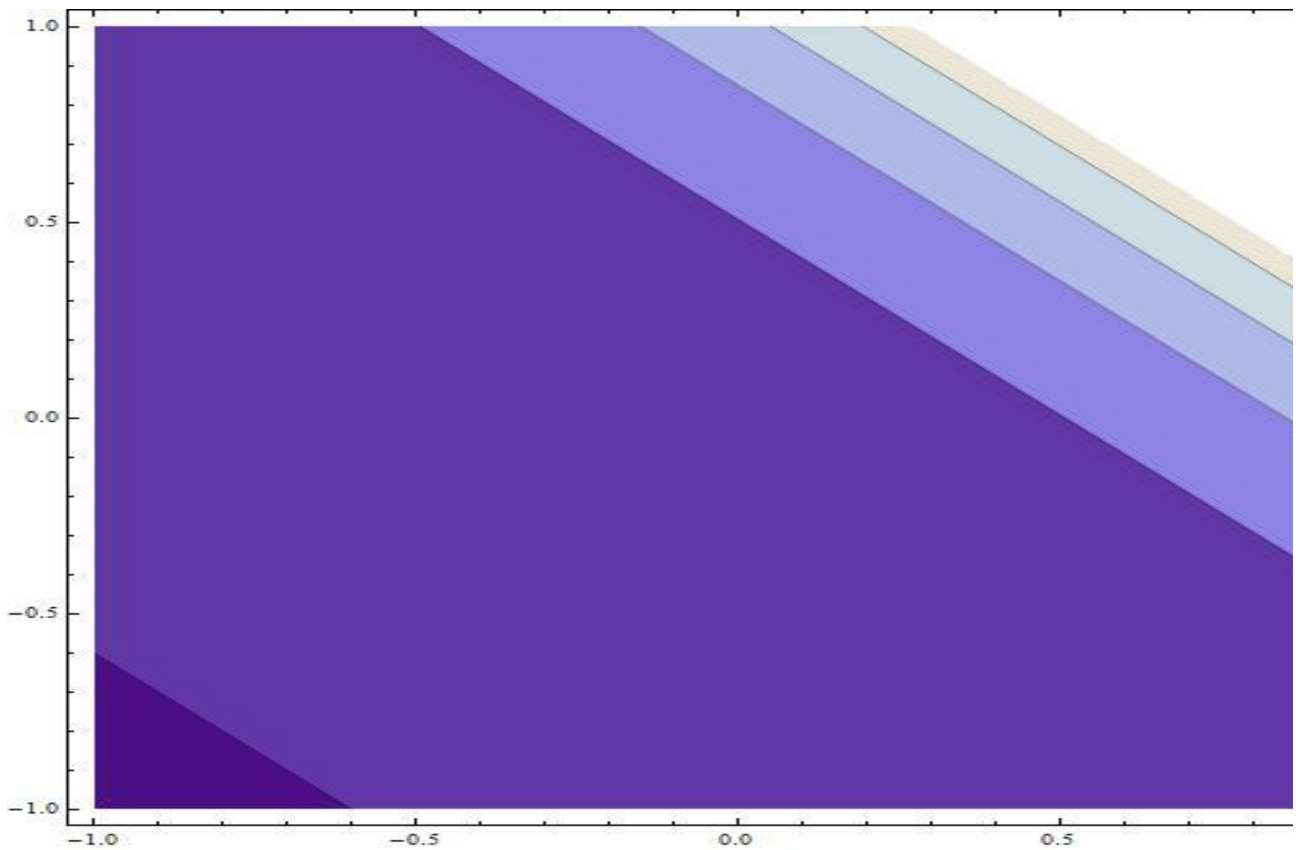

Fig.13 Contour plot of Eq. (3.1.51) when ($\alpha = 1, \beta = 2, \sigma = 0, a_0 = 5, z = 1, c = -1$).

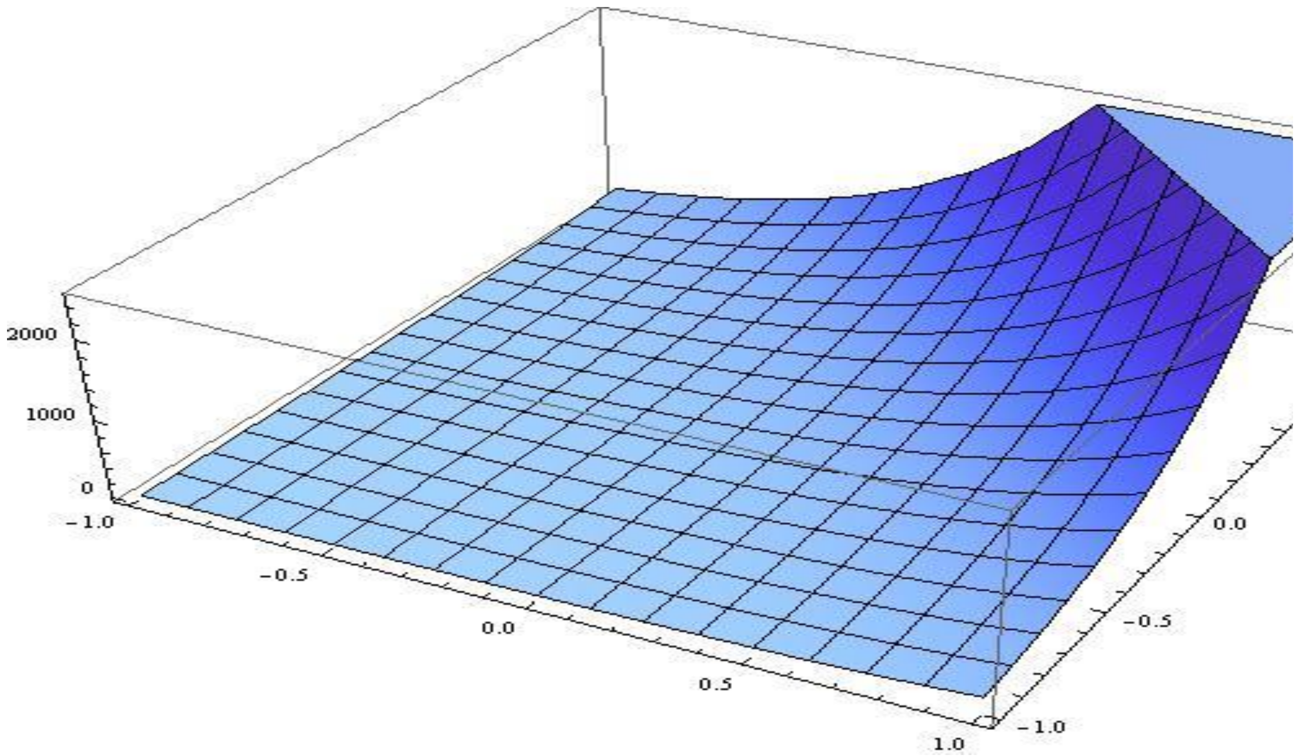

Fig.14 3D-plot of Eq. (3.1.51) when ($\alpha = 1, \beta = 2, \sigma = 0, a_0 = 5, z = 1, c = -1$).

5. Conclusion:

In this paper, we introduce a new modification of the Khater method. Khater method is considered as one of the most powerful generalized methods in nonlinear partial differential equation field. Especially, it concludes all solutions that can be obtained by using many different methods. We implement the modified Khater method on two significant modules in mathematical physics. We find a new form of solitary traveling solutions for Schwarzian Korteweg de Vries is so closed to (2+1) Ablowitz-Kaup-Newell-Segur (AKNS) equation. We plot some of our obtained solutions Fig. [1] - Fig. [14] to show the solitary and contour plot of these solutions. The earned solitary solutions show the physical features of each model. This renders examination the capabilities of these models and how they are applied in normal life. This helps in the progress and well-being of mankind.

6. Acknowledgements

(Mostafa Khater) I would like to dedicate this paper to my mother and the soul of my father; he was there for the beginning of this degree and did not make it to the end. His love, support, and constant care will never be forgotten. He is very much missed.

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