On special FC_n-cell modules

Ahmed B. Hussein

Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq

ahmedbaqer@uomustansiriyah.edu.iq

(Received: February/2020, Accepted: April/2020)

Abstract In this paper, we study a specified family of Fuss-Catalan algebra cell modules.We define the set of the basis diagrams and we give the general form for the Gram matrix related to this family. In addition, we state when these modules are irreducible by finding the determinant of the Gram matrices.Finally, we define a homomorphism between certain cell modules.

1 Introduction

Bisch and Jones [2] was first introduced the Fuss-Catalan algebras as a generalization of the Temperley-Lieb algebras. They proved that the Fuss-Catalan algebra which is denoted by $\mathrm{FC}_n(a,b)$ is generically semisimple when a and b are complex numbers [1, Corollary 2.2.5] . In addition, they computed the structure of these algebras when they are semisimple. In [7], the author prove some interesting results related to $\mathrm{FC}_n(a,b)$. Firstly, this family of algebras satisfies the definition of cellular algebras that have been given by Graham and Leher [5], and defining their cell modules. Secondly, it satisfies the axioms of tower of recollement that has been defined by Cox, Martin, Parker and Xi [3], moreover, $\mathrm{FC}_n(a,b)$ are quasi-hereditary when a and b are non-zero complex numbers.

The set of generators for $\mathrm{FC}_n(a_1,a_2,\ldots,a_k)$ and the isomorphism between the abstract definition and the diagram definition has been given by Landau [8]. Furthermore, Francesco [4], has used the Fuss-Catalan algebras to introduce new integrable lattice models and new hyperbolic solutions to the Yang-Baxter equation. The dimension of $\mathrm{FC}_n(a,b)$ is the Fuss-Catalan numbers so the name.

We give a brief summary of this paper. In section 2, we review the definition of the Fuss-Catalan algebras and some known results when it is semi-simple. In section 3, we discuss the cellularity of $\mathrm{FC}_n(a,b)$ and we define the Fuss-Catalan algebra cell modules, moreover, the dimension of the cell modules and the definition of the bilinear form corresponding to each cell module has been given. In section 4, we give the theorem that state FC_n is a tower of recollement and defining the indexing set for the simple FC_n-modules. In section 5, we introduce a collection of cell modules $\Delta_n(\mu_n)$ with certain label μ_n and their dimensions. We describe the basis diagrams and we define a bilinear form related to each cell module. Further, the values of a and b such that these modules are irreducible have been determined by constructing the Gram matrices and calculating their determinants. In section 6, using the theory of tower of recollement from section 4, we define homomorphisms between two sets of cell modules. Finally, in section 7, we define a family of cell modules, $\Delta_n(\hat{\mu}_n)$, such that their basis are basis of $\Delta_n(\mu_n)$ after rotating them about the y-axis, so, all results for $\Delta_n(\mu_n)$ can be passed to $\Delta_n(\hat{\mu}_n)$.

2 Fuss-Catalan algebras.

In this section, we shall introduce some definitions.

Definition 2.1 ([1, p.96]). An (n, n)-planar diagram D is a diagram consisting of two horizontal, parallel lines with n vertices drawn on each line. The vertices are connected by strings either to vertices on the same line or to vertices on the parallel line provided that D has no crossings and strings do not leave the strip in the plane defined by the top and the bottom lines of the diagram.

Definition 2.2 ([1, p.97]). A colouring of a (kn, kn)-planar diagram D is an identical assignment of colours a_1, a_2, …, a_k to each of the top and the bottom vertices of D such that the colouring is of the form

$$(a_1a_2\cdots a_k)(a_ka_{k-1}\cdots a_1)\cdots(a_1a_2\cdots a_k) \text{ if n is odd}$$

$$(a_1a_2\cdots a_k)(a_ka_{k-1}\cdots a_1)\cdots(a_ka_{k-1}\cdots a_1) \text{ if n is even.}$$

In addition, only vertices with the same colours have connecting strings.

Let $\mathcal{B}_{k,n}$ be the set of all coloured planar (nk, nk)-diagrams, since diagrams in $\mathcal{B}_{k,n}$ have same number of vertices and same colouring, thus multiplication on $\mathcal{B}_{k,n}$ is defined by the concatenation of diagrams. That is, for $D_1, D_2 \in \mathcal{B}_{k,n}$, the diagram $D_1D_2 = (a_1)^{r_1}(a_2)^{r_2}\ldots(a_k)^{r_k}D_3$ where a_1, a_2, …, a_k are complex numbers, D_3 is a diagram in $\mathcal{B}_{k,n}$ obtained by placing D_1 above D_2 and removing the r_i closed loops formed by strings with colour a_i from the new diagram. In general, multiplication on $\mathcal{B}_{k,n}$ is not commutative.

Definition 2.3 ([1, Definition 2.1.2]). Fix k complex numbers (the colours) $a_1, a_2, \ldots, a_k$, $k \geq 1$, and denote by $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ the complex linear span of $\mathcal{B}_{k,n}, n \geq 1$. We set $\mathrm{FC}_{k,0} = \mathbb{C}$. Clearly, $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$ is then an associative algebra over $\mathbb{C}$ with multiplication being the multiplication of diagrams as explained above, extended linearly and respecting the distributivity law to all of $\mathrm{FC}_{k,n}(a_1, a_2, \ldots, a_k)$.

There is another definition of the Fuss-Catalan algebras which is defined by the generators and relations [8, Definition 1], and an isomorphism between these two definitions is introduced in [8, Theorem 6].

The dimension of the Fuss-Catalan algebras $\mathrm{FC}_{k,n}$ is the number of elements in the set $\mathcal{B}_{k,n}$ and it is given by the Fuss-Catalan numbers.

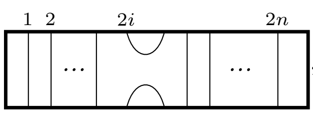
Proposition 2.4 ([1, Corollary 2.1.7]). *We have*

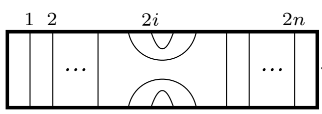
$$\dim \mathrm{FC}_{k,n} = \frac{1}{kn+1}\binom{(k+1)n}{n}$$

for $k \geq 1$ and $n \geq 1$.

In this paper, we interested in the algebras $\mathrm{FC}_{2,n}(a, b)$ such that the parameters a and b are non-zero complex numbers. In addition, we use the notations FC_n for $\mathrm{FC}_{2,n}(a, b)$ and $\mathcal{B}_n$ for $\mathcal{B}_{2,n}$.

Proposition 2.5 ([1, Proposition 4.1.3]). *The algebra FC_n is generated by the diagrams 1, ${}_1U_i$ and ${}_2U_i$, where $1 \leq i \leq n-1$, and*

${}_1U_i =$ 

, ${}_2U_i =$ 

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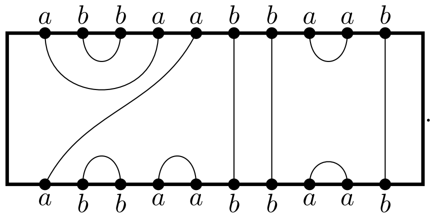
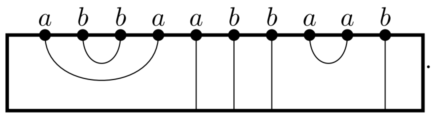
Definition 2.6 ([1, p.106])**.** The colouring of a basis diagram in FC_n is $a(yx)^{\frac{n}{2}-1}ya$ for n even, and $a(yx)^{\frac{n-1}{2}}b$ for n odd, where $x = aa$ and $y = bb$.

We introduce some notations, for a diagram $D \in \mathcal{B}_n$, a string that connects two vertices in apposite lines is called a *through string*, whereas a string that connects two vertices in the same line is called a *non-through string*. We denote by $\ell(D)$ the number of through strings in D. The *label* of D is a well defined word on a and b represents the associated colours of the through strings reading from left to right. The upper half diagram of D that represents the vertices in the top line with their strings is called the *initial part* of D, while the lower half diagram that represents the vertices in the bottom line with their strings is called the *final part* of D. The initial and the final parts of D have label similar to the label of D, and they are uniquely determined by D.

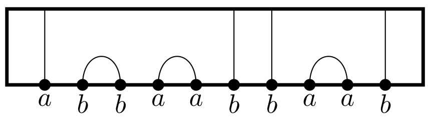
Example 2.7. The label of the diagram $D \in \mathrm{FC}_5$ is $abbb = ab^3$, and $\ell(D) = 4$.



The initial part of D is the diagram



and the final part of D is the diagram



Lemma 2.8 ([1, Lemma 3.1.4])**.** *A label of a diagram $D \in \mathcal{B}_n$ is either empty (only possible if n is even) or of the form $aw(x, y)a$ if n is even and $aw(x, y)b$ if n is odd, where w is a word on $x = aa$ and $y = bb$.*

Definition 2.9 ([1, Definition 3.1.12])**.** If λ is a word on a and b, we define $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ to be the number of distinct initial parts of the diagrams in FC_n having label λ with the convention that $\begin{bmatrix} n \\ \lambda \end{bmatrix} = 0$ if λ is not a label.

Theorem 2.10 ([1, Theorem 3.1.15])**.** *If a and b are such that $\mathrm{FC}_n(a, b)$ is semi-simple, we have*

$$\mathrm{FC}_n(a, b) \cong \bigoplus_{\lambda : \left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right] > 0} M_{\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]}(\mathbb{C}),$$

where the sum is over all labels λ of diagrams in FC_n.

Let $D, D' \in \mathrm{FC}_n$, and let M be the initial part of D. The multiplication of D' with M is defined to be the initial part of the diagram $D'D$.

Definition 2.11 ([1, Definition 3.1.16]). Let λ be a word in a and b with $\begin{bmatrix} n \\ \lambda \end{bmatrix} > 0$ and let ℓ be number of letters in λ. Let V_λ be the complex vector space of dimension $\begin{bmatrix} n \\ \lambda \end{bmatrix}$ having the distinct initial parts M of diagrams in $\mathcal{B}_n$ with label λ as a basis. Let $D \in \mathcal{B}_n$ and define a representation π_λ of $\mathrm{FC}_n(a,b)$ on V_λ by

$$\pi_\lambda(D)M = \begin{cases} DM, & \text{if } \ DM \text{ has } \ell \text{ through strings,} \\ 0, & \text{otherwise,} \end{cases}$$

for $M \in V_\lambda$ (where we replace closed loops in DM with a or b as usual).

Theorem 2.12 ([1, Theorem 3.1.17]). *If a and b are such that $\mathrm{FC}_n(a,b)$ is semi-simple, then all representations π_λ as defined above are irreducible and any irreducible representation of $\mathrm{FC}_n(a,b)$ is equivalent to a π_λ.*

3 Cellularity of $\mathrm{FC}_n(a,b)$

For a given cellular algebra, the theory of cellular algebras, that introduced by Graham and Lehrer [5], determine the full set of irreducible modules.

Definition 3.1 ([5, Definition 1.1]). Let R be a commutative ring with identity. A cellular algebra over R is an associative (unital) algebra A, together with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ where

(C1) Λ is a partially ordered set and for each $\lambda \in \Lambda$ there is a finite set $\mathcal{W}(\lambda)$ such that $\mathcal{B} = \{C^\lambda_{S,T} \mid \lambda \in \Lambda \text{ and } S,T \in \mathcal{W}(\lambda)\}$ is an R-basis of A.

(C2) If $\lambda \in \Lambda$ and $S,T \in \mathcal{W}(\lambda)$. Then $*$ is an R-linear anti-isomorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S,T \in \mathcal{W}(\lambda)$ then for any element $a \in A$ we have

$$aC^\lambda_{S,T} = \Big(\sum_{S' \in \mathcal{W}(\lambda)} r_a(S',S) C^\lambda_{S',T} \Big) + r'$$

where $r_a(S',S) \in R$ is independent of T, and r' is a linear combination of basis elements with upper label $\mu < \lambda$.

If A is a cellular algebra, then the basis $\mathcal{B}$ in Definition 3.1 is called a *cellular basis.*

Definition 3.2 ([5, Definition 2.1]). Let A be a cellular algebra with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$. For each $\lambda \in \Lambda$ define the left A-module $\Delta(\lambda)$ as a free R-module with basis $\{C_S \mid S \in \mathcal{W}(\lambda)\}$ and A-action defined by

$$aC_S = \sum_{S' \in \mathcal{W}(\lambda)} r_a(S',S) C_{S'} \qquad (a \in A, \ S \in \mathcal{W}(\lambda))$$

where $r_a(S',S) \in R$ as defined in definition 3.1. This module called the *cell module* of A labelled by λ.

Defining the action of A on $\Delta(\lambda)$ by $C_S a = \sum_{S' \in \mathcal{W}(\lambda)} r_{a^*}(S',S) C_{S'}$, then the cell module $\Delta(\lambda)$ is a right A-module.

Definition 3.3. For each $\lambda \in \Lambda$, define A^λ to be the R-submodule of A with basis

$$\{C^\mu_{S,T} \mid \mu \in \Lambda, \mu < \lambda \text{ and } S,T \in \mathcal{W}(\mu)\}.$$

Definition 3.4 ([5, Definition 2.3]). For $\lambda \in \Lambda$, define a bilinear form $\Phi_\lambda : \Delta(\lambda) \times \Delta(\lambda) \to R$ by the equation

$$C^{\lambda}_{S_1,T_1} C^{\lambda}_{S_2,T_2} \equiv \Phi_\lambda(C_{T_1}, C_{S_2}) C^{\lambda}_{S_1,T_2} \mod A^{\lambda}$$

where $S_1, S_2, T_1, T_2 \in \mathcal{W}(\lambda)$.

Proposition 3.5 ([5, Proposition 2.4]). *Let $\lambda \in \Lambda$, and $x, y \in \Delta(\lambda)$. Then*

(i) *The form Φ_λ is symmetric, i.e. we have $\Phi_\lambda(x, y) = \Phi_\lambda(y, x)$.*

(ii) *For all $a \in A$, we have $\Phi_\lambda(a^* x, y) = \Phi_\lambda(x, ay)$.*

Definition 3.6 ([5, Definition 3.1]). Let A be a cellular algebra with cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$. For $\lambda \in \Lambda$, define

$$\operatorname{rad}(\lambda) = \{x \in \Delta(\lambda) \mid \Phi_\lambda(x, y) = 0 \text{ for all } y \in \Delta(\lambda)\}.$$

Proposition 3.7 ([5, Proposition 3.2]). *Let $\lambda \in \Lambda$. Then*

(i) $\operatorname{rad}(\lambda)$ *is an A-submodule of $\Delta(\lambda)$.*

(ii) *If $\Phi_\lambda \neq 0$, the quotient $\Delta(\lambda)/\operatorname{rad}(\lambda)$ is absolutely irreducible.*

(iii) *If $\Phi_\lambda \neq 0$,* $\operatorname{rad}(\lambda)$ *is the Jacobson radical of the cell module $\Delta(\lambda)$.*

For a cellular algebra A, the following theorem, that was introduced by Graham and Lehrer [5], classify the irreducible cell modules.

Theorem 3.8 ([5, Theorem 3.4]). *Let R be a field and let $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ be a cell datum for the R-algebra A. For each $\lambda \in \Lambda$ define the cell module $\Delta(\lambda)$ and bilinear form Φ_λ on $\Delta(\lambda)$ as in definitions 3.2 and 3.4 respectively. Suppose $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. Then the set $\{L_\lambda = \Delta(\lambda)/\operatorname{rad}(\lambda) \mid \lambda \in \Lambda_0\}$ is a complete set of non-isomorphic irreducible A-modules.*

Theorem 3.9 ([5, Theorem 3.8]). *Let R be a field and let $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ be a cell datum for the R-algebra A. Then the following are equivalent:*

(i) *The algebra A is semisimple.*

(ii) *The non-zero cell modules of A are irreducible and pairwise inequivalent.*

(iii) *The form Φ_λ is non-degenerate, that is,* $\operatorname{rad}(\lambda) = 0$ *for each $\lambda \in \Lambda$.*

Now, we identify a cell datum $(\Lambda, \mathcal{W}, \mathcal{B}, *)$ such that $\mathrm{FC}_n(a, b)$ are cellular.

Definition 3.10 ([1, Definition 3.1.5]). If λ and λ' are words on an alphabet, we say $\lambda \leq \lambda'$ if λ is obtained from λ' by removing some (or no) letters of λ'.

For example, we have $abba \leq abbbba$.

Definition 3.11. A cell datum $(\Lambda, \mathcal{W}_n(\lambda), \mathcal{B}_n, *)$ is defined as following: the label set for the diagrams in $\mathcal{B}_n$ is denoted by Λ, and the set of all distinct initial parts for diagrams in $\mathcal{B}_n$ having label λ is denoted by $\mathcal{W}_n(\lambda)$. Define $\mathcal{B}_n = \{C^{\lambda}_{S,T} \mid \lambda \in \Lambda \text{ and } S, T \in \mathcal{W}_n(\lambda)\}$ where $C^{\lambda}_{S,T}$ is a diagram obtained by replacing S above T after turning T upside down. Furthermore, define $*$ on FC_n to be the function that flips a diagram upside down.

The author [7, Lemmas 4.3 and 4.4] show that the pair $(\Lambda, \leq)$ forms a partially ordered set and $*$ is an anti-isomorphism.

Theorem 3.12 ([7, Theorem 4.5])**.** *The Fuss-Catalan algebras* $\mathrm{FC}_n(a, b)$ *are cellular algebras with cell datum* $(\Lambda, \mathcal{W}_n(\lambda), \mathcal{B}_n, *)$ *that defined in Definition 3.11.*

Definition 3.13. For each $\lambda \in \Lambda$, there is a cell module $\Delta_n(\lambda)$ with basis $\mathcal{W}_n(\lambda)$ such that for all $D \in \mathrm{FC}_n$ and $M \in \mathcal{W}_n(\lambda)$, the action is defined by

$$D \cdot M = \begin{cases} DM, & \text{if } \ \ell(DM) = \ell(M), \\ 0, & \text{otherwise.} \end{cases}$$

Comparing the bases and the actions in Definitions 3.13 and 2.11, we get

Proposition 3.14 ([2, Section 2])**.** *Suppose that* $\Delta_n(\lambda)$ *be a cell module with* $\lambda = a^{l_1} b^{l_2} \cdots z^{l_p}$ *where* $z = a$ *if* n *even and* $z = b$ *if* n *odd. The dimension of* $\Delta_n(\lambda)$ *is* $\begin{bmatrix} n \\ \lambda \end{bmatrix}$, *(defined in Definition 2.9), and is equal to:*

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{cases} \frac{s}{3(m-r)+s}\binom{3(m-r)+s}{m-r} & \text{if } \ m \geq r \\ 0 & \text{if } \ m < r, \end{cases}$$

where $s = \frac{3l-2p+4}{2}$, *and* $r = \frac{l-p+1}{2}$ *if* $n = 2m$ *while* $r = \frac{l-p}{2}$ *if* $n = 2m + 1$.

4 Towers of algebras

For $n \geq 0$, let e_n be idempotents in finite dimensional algebras A_n. A labeling set for the simple A_n-modules is denoted by Λ_n, and a labeling set for the simple $A_n/A_n e_n A_n$-modules is denoted by Λ^n. In addition, we define $\Lambda_n^m = \Lambda^m \subset \Lambda_n$ if $n \geq m$, and $\Lambda_n^m = \emptyset$ If $n < m$, where $m, n \in \mathbb{N}$ and $n - m$ is even.

Proposition 4.1. *Let* $\Delta_n(\lambda)$ *be a cell module for the algebra* FC_n. *Then the following are equivalent:*

(i) $\dim \Delta_n(\lambda) = 1$.

(ii) No basis diagram for $\Delta_n(\lambda)$ *has the patterns*  *and* .

(iii) $\Delta_{n-2}(\lambda) = 0$.

See [1, Proposition 3.2.5].

Theorem 4.2 ([6, Theorem 6.2g])**.** *Let* $\{L(\lambda) \mid \lambda \in \Lambda\}$ *be a full set of simple* A*-modules, and set* $\Lambda^e = \{\lambda \in \Lambda \mid eL(\lambda) \neq 0\}$. *Then* $\{eL(\lambda) \mid \lambda \in \Lambda^e\}$ *is a full set of simple* eAe*-modules. Further, the simple modules* $L(\lambda)$ *with* $\lambda \in \Lambda \setminus \Lambda^e$ *are a full set of simple* A/AeA*-modules.*

Remark 4.3. From Theorem 4.2, we have $\Lambda^e = \Lambda_{n-2}$ and $\Lambda^n = \Lambda_n \setminus \Lambda_{n-2}$. By Proposition 4.1, if $\dim \Delta_n(\lambda) = 1$ then $\dim \Delta_{n-2}(\lambda) = 0$ and $\Delta_n(\lambda)$ is simple. Thus $\lambda \in \Lambda_n$ and $\lambda \notin \Lambda_{n-2}$, so, $\lambda \in \Lambda^n$.

Theorem 4.4 ([7, Theorem 6.16])**.** *The Fuss-Catalan algebras* FC_n *form a tower of recollement.*

A family of algebras A_n that satisfied the axioms introduced by Cox, Martin, Parker and Xi [3], to study the representation theory of towers of algebras, is called a tower of recollement. These axioms are powerful technique to determine the simple A_n-modules and when the algebra is semi-simple. In addition, finding morphisms for A_n-modules can be reduced to the case when one of them is simple as stated in this theorem.

Theorem 4.5 ([3, Theorem 1.1]). *Let A_n be a tower of recollement.*

(i) *For all pairs of weights $\lambda \in \Lambda_n^m$ and $\mu \in \Lambda_n^l$ we have*

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \cong \begin{cases} \mathrm{Hom}(\Delta_m(\lambda), \Delta_m(\mu)) & \text{if } l \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Suppose that for all $n \geq 0$ and pairs of weights $\lambda \in \Lambda_n^n$ and $\mu \in \Lambda_n^{n-2}$ we have $\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 0$ then each of the algebras A_n is semisimple.*

5 Gram determinant

For $n \geq 4$, we study a certain family of $\mathrm{FC}_n(a, b)$- cell modules, namely $\Delta_n(\mu_n)$ such that

$$\mu_n = \begin{cases} a^2(abba)^{m-2} & \text{if } n = 2m \\ a^2(abba)^{m-2}ab & \text{if } n = 2m+1. \end{cases} \tag{1}$$

For this family, we describe the bases diagrams with the dimensions. Moreover, we define a bilinear form to construct the Gram matrices and we calculate their Gram determinants to specify for which values of a and b these modules are irreducible. For the first values of n, we listed the label μ_n in Table 1 .

n	4	5	6	7	8
μ_n	a^2	$a^2(ab)$	$a^2(ab^2a)$	$a^2(ab^2a^2b)$	$a^2(ab^2a^2b^2a)$

Table 1: The label μ_n

Proposition 5.1. *Consider the label μ_n that defined in (1). Then, for $n \geq 4$, we have*

$$\dim \Delta_n(\mu_n) = 2n - 4.$$

Proof. We have two cases the even and the odd case for μ_n.

Case I: Let $n = 2m$ where $m \geq 2$. Then $\mu_n = a^2(abba)^{m-2}$, and the length of μ_n is $l(\mu_n) = 4(m-2)+2$. We can write μ_n in the form $\mu_n = awa$, where $w = (a^2b^2)^{m-2}$. Suppose that w_a is the number of a^2's in w and w_b is the number of b^2's in w. We can see that $w_a = w_b = m-2$, thus the number of parts, p, of μ_n is $p = w_a + w_b + 1 = 2m-3$. Now, we use Theorem 3.14 to calculate the dimension of $\Delta_n(\mu_n)$. We have

$$r = m - 1 \qquad \text{and} \qquad s = 4m - 4 = 2n - 4.$$

Hence,

$$\begin{bmatrix} n \\ \mu_n \end{bmatrix} = \frac{s}{3+s}\binom{3+s}{1} = 2n - 4.$$

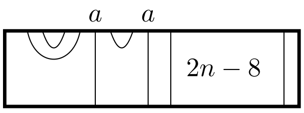
Case II: Suppose that $n = 2m+1$ where $m \geq 2$. Then $\mu_n = a^2(abba)^{m-2}ab$, and the length of μ_n is $l(\mu_n) = 4(m-2)+4$. We can write μ_n in the form $\mu_n = a^3wb$, where $w = (b^2a^2)^{m-2}$. Suppose that w_a is the number of a^2's in w and w_b is the number of b^2's in w. We can see that $w_a = w_b = m-2$, thus the number of parts of μ_n is $p = 1 + w_a + w_b + 1 = 2m-2$. By using Theorem 3.14 to calculate the dimension of $\Delta_n(\mu_n)$, we have

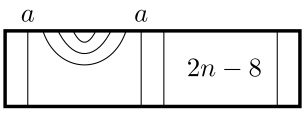
$$r = m-1 \qquad \text{and} \qquad s = 4m-2 = 2n-4.$$

Hence,

$$\begin{bmatrix} n \\ \mu_n \end{bmatrix} = \frac{s}{3+s}\binom{3+s}{1} = s = 2n-4. \quad \square$$

Definition 5.2. For $i = 0, 1, \ldots, 2n-5$, we define the half diagrams M_i as following:

$M_0 =$ 

, $M_1 =$ 

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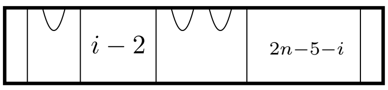
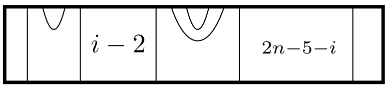
$M_i =$ 

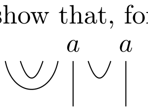
if $i = 2, 4, \ldots, 2n-6$,

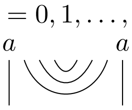
$M_i =$ 

if $i = 3, 5, \ldots, 2n-5$.

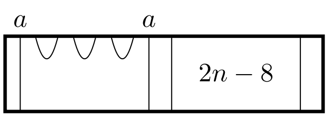
Remark 5.3. Since the identity diagram of FC_n has $2n$ through strings, thus its label is $(abba)^{\frac{n}{2}}$ if n even, while its label is $(abba)^{\frac{n-1}{2}}ab$ if n odd.

Proposition 5.4. *Let μ_n be as defined in (1), then, for $n \geq 4$, the basis diagrams for the cell module $\Delta_n(\mu_n)$ is the set $\{M_i \mid i = 0, 1, \ldots, 2n-5\}$, where the diagrams M_i are as defined in Definition 5.2.*

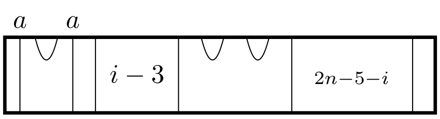
Proof. The number of diagrams in the set $\{M_i \mid i = 0, 1, \ldots, 2n-5\}$ is $2n-4$, and all diagrams are distinct. So, it sufficient to show that, for all $i = 0, 1, \ldots, 2n-5$, the diagrams M_i has label μ_n. For the diagrams M_0 and M_1, the subdiagrams 

and 

of M_0 and M_1 respectively, have label a^2. The subdiagram of M_0 and M_1 that have $2n-8$ through strings can be considered as the identity diagram of FC_{n-4}, so, by Remark 5.3, this subdiagram has label $(abba)^{\frac{n-4}{2}}$ if n even, and it has label $(abba)^{\frac{n-5}{2}}ab$ if n odd. Hence, the label of M_0 and M_1 is μ_n. For the diagrams M_i where $i = 2, 4, \ldots, 2n-6$, if $i = 2$ then

$M_2 =$ 

has label equivalent to the label of M_0 and M_1, thus the label of M_2 is μ_n. If $i > 2$ then

$M_i =$ 

Since the four vertices of the two consecutive non-through strings in M_i have coloring either $(aabb)$ or $(bbaa)$, thus if we remove these four vertices with their non-through strings then we get a subdiagram of M_i with $i-3+2n-5-i=2n-8$ through strings and its label is the label of the identity diagram for FC_{n-4}. Hence, the label of M_i is μ_n. For the diagrams M_i where $i=3,5,\ldots,2n-5$, we can arguing as for the case i is even to show that the label of M_i is μ_n. $\square$

Since FC_n are cellular algebras, thus we can define a bilinear form associated to each cell module. Furthermore, by Theorem 3.9, a cell module is irreducible if its Gram matrix is non-degenerate. That is, if the determinant of the Gram matrix is non-zero.

Definition 5.5. Let M_1 and M_2 be basis diagrams for a cell module $\Delta_n(\lambda)$. We may define a bilinear form $\langle -,-\rangle$: $\Delta_n(\lambda)\times\Delta_n(\lambda)\to\mathbb{C}$ as follows: connecting the corresponding vertices of M_1 and M_2 after flipping M_2 upside down and placing it over M_1. If number of through strings in this connecting equals $\ell(M_1)$ then $\langle M_1,M_2\rangle=a^{r_1}b^{r_2}$ where r_1 and r_2 are number of a and b-loops that be formed, otherwise, $\langle M_1,M_2\rangle=0$. We extend bilinearly to all of $\Delta_n(\lambda)$.

Notice that, for $D_1=C^{\lambda}_{S_1,T_1}, D_2=C^{\lambda}_{S_2,T_2}\in\mathrm{FC}_n$, if $D_3=D_1D_2$, then by Definition 3.4, the bilinear form $\Phi_\lambda(C_{T_1},C_{S_2})=c$ if D_3 has label λ and $\Phi_\lambda(C_{T_1},C_{S_2})=0$ if D_3 has label $\lambda'<\lambda$, where $c\in\mathbb{C}$ is product of $a's$ and $b's$ that obtained from the connecting of T_1 with S_2. Thus, our bilinear form $\langle -,-\rangle$ is equivalent to Φ_λ, and Proposition 3.5 can be stated as following

Lemma 5.6. *For all basis diagrams M_1, $M_2\in\Delta_n(\lambda)$ and $D\in\mathrm{FC}_n$, we have*

(i) $\langle M_1,M_2\rangle=\langle M_2,M_1\rangle$

(ii) $\langle M_1,DM_2\rangle=\langle D^*M_1,M_2\rangle$

where D^* is the reflection of D about a horizontal line which was defined in Definition 3.11.

Definition 5.7. The Gram matrix corresponding to the cell module $\Delta_n(\lambda)$ that has ordered basis $(M_1,M_2,\ldots,M_r)$ is defined by, for i, $j=1$, 2, …, r,

$$G_n(\lambda)_{i,j}=\langle M_i,M_j\rangle.$$

Proposition 5.8. *Let μ_n be as defined in (1). The Gram matrix $G(\mu_n)$ for the cell module $\Delta_n(\mu_n)$ subject to the ordered basis $(M_0,M_1,\ldots,M_{2n-5})$ has the following form*

(i) $G(\mu_n)=\begin{pmatrix} ab^2 & b^2 & b & 0\\ b^2 & ab^2 & ab & b^2\\ b & ab & ab^2 & b\\ 0 & b^2 & b & ab^2\end{pmatrix}$ *if $n=4$.*

(ii) $G(\mu_n)=\begin{pmatrix} G(\mu_{n-1}) & Q_s\\ Q_s^T & D_t\end{pmatrix}$ *if $n>4$,*

where $Q_s^T=\begin{pmatrix} 0 & \ldots & 0 & bs\\ 0 & \ldots & 0 & b\end{pmatrix}_{2\times(2n-6)}$, $D_t=\begin{pmatrix} ab^2 & bt\\ bt & ab^2\end{pmatrix}$, *and $(s,t)=(a,b)$ if n even while $(s,t)=(b,a)$ if n odd.*

Proof. We will construct the Gram matrix of $\Delta_n(\mu_n)$ subject to the ordered basis $(M_0,M_1,\ldots,M_{2n-5})$ that defined in Proposition 5.4.

(i) For $n = 4$, we have $G(\mu_4) = (M_{ij}) = \langle M_i, M_j \rangle$ where $i, j = 0, 1, 2, 3$. Hence, the result.

(ii) For $n > 4$, we have

$$G(\mu_n) = \begin{array}{c|cccc|cc}
\langle -,- \rangle & M_0 & M_1 & \dots & M_{2n-8} \quad M_{2n-7} & M_{2n-6} & M_{2n-5} \\ \hline
M_0 & & & & & & \\
M_1 & & & & & & \\
\vdots & & & G_{11} & & & G_{12} \\
M_{2n-8} & & & & & & \\
M_{2n-7} & & & & & & \\ \hline
M_{2n-6} & & & G_{21} & & & G_{22} \\
M_{2n-5} & & & & & &
\end{array}$$

Let H be the set of basis diagrams of $\Delta_n(\mu_n)$ that have at least two through strings at the right end, that is, $H = \{M_0, M_1, \dots, M_{2n-7}\}$. For $i = 0, 1, \dots, 2n-7$, let M'_i be the diagram that obtained by removing the last two through strings from M_i in H. So, by Proposition 5.4, $H' = \{M'_0, M'_1, \dots, M'_{2n-7}\}$ is the set of basis diagrams for $\Delta_{n-1}(\mu_{n-1})$. In addition, the last two through strings in the diagrams of H are not contribute to the inner product. Thus $\langle M_i, M_j \rangle = \langle M'_i, M'_j \rangle$ for all $0 \le i, j \le 2n-7$, and hence, $G_{11} = G(\mu_{n-1})$. By using diagram multiplication, we can deduce that

$\langle M_{2n-6}, M_i \rangle = \langle M_{2n-5}, M_i \rangle = 0$ for all $0 \le i \le 2n-8$,

$\langle M_{2n-6}, M_{2n-7} \rangle = b^2 s$, $\langle M_{2n-5}, M_{2n-7} \rangle = b$, $\langle M_{2n-5}, M_{2n-6} \rangle = b^2 t$

$\langle M_{2n-6}, M_{2n-6} \rangle = \langle M_{2n-5}, M_{2n-5} \rangle = ab^2$,

where $(s,t) = (a,b)$ if n even while $(s,t) = (b,a)$ if n odd. Finally, we get $G_{21} = G_{12}^T = Q_s^T$ and $G_{22} = D_t$. □

Proposition 5.9. *Let μ_n be as defined in (1). For $n \ge 5$, the determinant of the Gram matrix $G(\mu_n)$ for the cell module $\Delta_n(\mu_n)$ is*

$$\det G(\mu_n) = \alpha \begin{cases} b\ (a^2-1)(a^2-2)^{\frac{n-6}{2}}(b^2-2)^{\frac{n-4}{2}}, & \text{if } n \text{ even,} \\ a\ (b^2-1)\big[(a^2-2)(b^2-2)\big]^{\frac{n-5}{2}}, & \text{if } n \text{ odd,} \end{cases} \tag{2}$$

where $\alpha = a^{n-4}\, b^{3n-7}(b^2-1)(a^2+a-1)(a^2-a-1)$.

Proof. We use the induction on n. By using Proposition 5.8, for $n = 5$, we have

$$G(\mu_5) = b \cdot \begin{pmatrix} ab & b & 1 & 0 & 0 & 0 \\ b & ab & a & b & 0 & 0 \\ 1 & a & ab & 1 & 0 & 0 \\ 0 & b & 1 & ab & b & 1 \\ 0 & 0 & 0 & b & ab & a \\ 0 & 0 & 0 & 1 & a & ab \end{pmatrix},$$

and

$$\det G(\mu_5) = a^2 b^8 (b^2-1)^2 (a^2+a-1)(a^2-a-1).$$

Suppose that this claim is true for all $m < n$. By Proposition 5.8, we have

$$G(\mu_n) = \begin{pmatrix} G(\mu_{n-1}) & Q_s \\ Q_s^T & D_t \end{pmatrix} = \left(\begin{array}{c|c} \begin{matrix} G(\mu_{n-2}) & Q_t \\ Q_t^T & D_s \end{matrix} & Q_s \\ \hline Q_s^T & D_t \end{array}\right).$$

Substituting the block matrices Q_s, Q_t^T, D_s, Q_s^T and D_t, we get

$$\det G(\mu_n) = \det \left(\begin{array}{ccccc|cc|cc} \multicolumn{5}{c|}{G(\mu_{n-2})} & \multicolumn{2}{c|}{Q_t} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \cdots & 0 & tb & ab^2 & sb & 0 & 0 \\ 0 & 0 & \cdots & 0 & b & sb & ab^2 & sb & b \\ \hline \multicolumn{5}{c|}{\multirow{2}{*}{0}} & 0 & sb & ab^2 & tb \\ \multicolumn{5}{c|}{} & 0 & b & tb & ab^2 \end{array}\right) \begin{matrix} \\ R_4 \\ R_3 \\ R_2 \\ R_1 \end{matrix}$$

If we replace R_4 by $(-t)R_3 + R_4$ and R_2 by $(-s)R_1 + R_2$, we get

$$\det G(\mu_n) = \det \left(\begin{array}{ccccc|cc|cc} \multicolumn{5}{c|}{G(\mu_{n-2})} & \multicolumn{2}{c|}{Q_t} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & x & -stb & -tb \\ 0 & 0 & \cdots & 0 & b & sb & ab^2 & sb & b \\ \hline \multicolumn{5}{c|}{\multirow{2}{*}{0}} & 0 & 0 & 0 & y \\ \multicolumn{5}{c|}{} & 0 & b & tb & ab^2 \end{array}\right) \begin{matrix} \\ R_4 \\ R_3 \\ R_2 \\ R_1 \end{matrix}$$

where $x = b(s - tab)$ and $y = b(t - sab)$. Replacing (R_1) by $\frac{-b}{x}R_4 + R_1$, we get

$$\det G_(\mu_n) = \det \left(\begin{array}{ccccc|cc|cc} \multicolumn{5}{c|}{G(\mu_{n-2})} & \multicolumn{2}{c|}{Q_t} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & x & -stb & -tb \\ 0 & 0 & \cdots & 0 & b & sb & ab^2 & sb & b \\ \hline \multicolumn{5}{c|}{\multirow{2}{*}{0}} & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{D'}} \\ \multicolumn{5}{c|}{} & 0 & 0 & \multicolumn{2}{c}{} \end{array}\right) \tag{3}$$

where $D' = \begin{pmatrix} 0 & b(t - sab) \\ tb(\frac{2s-tab}{s-tab}) & \left(\frac{tb}{s-tab} + ab^2\right) \end{pmatrix}$. The matrix in (3) can be written in block form as following

$$P = \begin{pmatrix} A & B \\ C & D' \end{pmatrix}$$

Now, $\det P = \det G(\mu_n) = \det(A - B(D')^{-1}C)\det(D')$, but C is zero matrix then $\det G(\mu_n) = \det A \det D'$. Notice that $\det A = \det G(\mu_{n-1})$, so, by the induction hypothesis, $\det A$ is as in (2) with replacing n by $n-1$. Furthermore, $ab = st$ then $\det D' = t^2b^2(s^2-1)(\frac{t^2-2}{t^2-1})$. Finally, the result follows by multiplying the determinant of A by the determinant of D' in the even and the odd case. $\square$

From Theorem 3.9, we have

Proposition 5.10. *The cell module $\Delta_n(\mu_n)$ is irreducible if the values of a and b are such that equation 2 that given in Proposition 5.9 is non-zero.*

6 Homomorphism

We shall define a homomorphism between a one dimensional cell module and $\Delta_n(\mu_n)$, and then we use Theorem 4.5 to generalize our result.

Proposition 6.1. *Let ${}_1U_i$, ${}_2U_i$ be the generators of $\mathrm{FC}_n(a,b)$, and consider the cell module $\Delta_n(\mu_n)$ that is spanned by the set $\{M_j \mid j = 0, 1, \ldots, M_{2n-5}\}$, where μ_n is as defined in (1). Then, for $j = 1, 2$, the multiplication of ${}_jU_i$ with M_j when $i = 1, 2, 3$ is as following*

	M_1	M_2
${}_1U_1$	M_2	bM_2
${}_1U_2$	aM_1	aM_2
${}_1U_3$	M_2	bM_2

	M_1	M_2
${}_2U_1$	M_0	bM_0
${}_2U_2$	abM_1	aM_1
${}_2U_3$	M_3	bM_3

and when $i > 3$, we have ${}_jU_iM_j = 0$.

Proof. By using diagram multiplication, we can show the results for $i = 1, 2, 3$. For $i > 3$, the diagram ${}_jU_iM_j$, where $j = 1, 2$, contains four non-through strings while each basis diagram for $\Delta_n(\mu_n)$ has only three non-through strings. Hence, these diagrams equal to zero in $\Delta_n(\mu_n)$.

□

Theorem 6.2. *Let $\Delta_n(\lambda)$ be the one dimensional cell module that spanned by the initial part v of ${}_1U_2$ with label λ, and consider the cell module $\Delta_n(\mu_n)$ that spanned by the set $\{M_i \mid i = 0, 1, \ldots, M_{2n-5}\}$, where μ_n is as defined in (1). Then $\theta : \Delta_n(\lambda) \to \Delta_n(\mu_n)$ defined by $\theta(v) = M_2 - bM_1$, is a non-zero module homomorphism when $b^2 = 1$.*

Proof. Let $U = \{{}_1U_1, {}_1U_2, \ldots, {}_1U_{n-1}, {}_2U_1, {}_2U_2, \ldots, {}_2U_{n-1}\}$ be the set of the generators of $\mathrm{FC}_n(a,b)$. In $\Delta_n(\lambda)$, for $i = 1, 2, \ldots, n-1$, we can see that,

$${}_rU_iv = \begin{cases} av & \text{if } i = 2 \text{ and } r = 1, \\ 0 & \text{otherwise,} \end{cases}$$

To show that θ is a homomorphism, we need to prove that $\theta(uv) = u\theta(v)$ for all $u \in U$. If $0 \neq uv \in \Delta_n(\lambda)$, then $u = {}_1U_2$, and $\theta({}_1U_2v) = \theta(av) = a\theta(v)$. On the other hand, by Proposition 6.1, we get

$$\begin{aligned} {}_1U_2\theta(v) &= {}_1U_2(M_2 - bM_1) \\ &= a(M_2 - bM_1) \\ &= a\theta(v). \end{aligned}$$

Now we turn to the case when $0 = uv \in \Delta_n(\lambda)$, this happens only when $u = {}_1U_i$ with $i \neq 2$ or when $u = {}_2U_i$. Let us discuss these two cases:

(i) For $i \neq 2$, we have ${}_1U_iv = 0$. Therefore we need to show that ${}_1U_i\theta(v) = 0$ when $i \neq 2$. We have ${}_1U_i\theta(v) = {}_1U_i(M_2 - bM_1)$. By Proposition 6.1, we get ${}_1U_i\theta(v) = 0$ for all $1 \leq i \leq n-1$ and $i \neq 2$.

(ii) For all $i = 1, 2, \ldots, n-1$, we have ${}_2U_i v = 0$. Thus we need to show that ${}_2U_i\theta(v) = 0$ as well. We have ${}_2U_i\theta(v) = {}_2U_i(M_2 - bM_1)$, by Proposition 6.1, ${}_2U_i\theta(v) = 0$ for all $1 \leq i \leq n-1$. Notice that, for $i = 2$, we have

$$\begin{aligned} {}_2U_2\theta(v) =& {}_2U_2(M_2 - bM_1) \\ =& aM_1 - ab^2M_1 \\ =& a(1-b^2)M_1 \end{aligned}$$

but $b^2 = 1$ then ${}_2U_2\theta(v) = 0$. □

From Theorem 4.5, we have

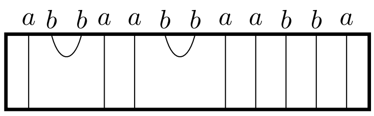
Corollary 6.3. *Let λ be the label of ${}_1U_2$ and μ_n is as defined in (1). Then for all $m \geq n$ such that $m - n$ is even we have a non-zero homomorphism $\theta : \Delta_m(\lambda) \to \Delta_m(\mu_n)$ when $b^2 = 1$.*

7 The cell module $\Delta_n(\hat{\mu}_n)$

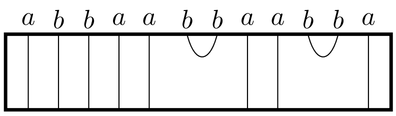
In this section, we introduce another family of cell modules that spanned by diagrams obtained by rotating the basis diagrams for $\Delta_n(\mu_n)$ about the y-axis.

Definition 7.1. For $D \in \mathrm{FC}_n$, we define $\hat{D}$ to be the diagram that represents the rotation of D about the y-axis.

For example, if

$D =$ 

then

$\hat{D} =$ 

Let λ be label of D and let $\hat{\lambda}$ be the label obtained by reading λ from right to left. Notice that, if n even then the label of $\hat{D}$ is $\hat{\lambda}$, but if n odd then the label of $\hat{D}$ is $\hat{\lambda}$ after swapping a with b and b with a. For example, if n even and $\lambda = ab^2a^4b^4a$ then $\hat{\lambda} = ab^4a^4b^2a$, and if n odd and $\lambda = ab^4a^2b^3$ then $\hat{\lambda} = a^3b^2a^4b$. Define $\Delta_n(\hat{\mu}_n)$ to be the cell module that spanned by diagrams with label $\hat{\mu}_n$. It is clear that $\Delta_n(\mu_n)$ and $\Delta_n(\hat{\mu}_n)$ have same dimension and Definition 7.1 define a FC_n- module homomorphism, thus $\Delta_n(\mu_n) \cong \Delta_n(\hat{\mu}_n)$ and all results that we have for $\Delta_n(\mu_n)$ can be applied for $\Delta_n(\hat{\mu}_n)$.

Proposition 7.2. *Consider the cell module $\Delta_n(\hat{\mu}_n)$, then we have*

(i) $\dim \Delta_n(\hat{\mu}_n) = 2n - 4$.

(ii) Let $\{M_i \mid i = 0, 1, \ldots, 2n-5\}$ be the basis of $\Delta_n(\mu_n)$ that defined in Definition 5.2, then $\Delta_n(\hat{\mu}_n)$ spanned by $\{\hat{M}_i \mid i = 0, 1, \ldots, 2n-5\}$.

(iii) For $n \geq 5$, the Gram determinants for $\Delta_n(\hat{\mu}_n)$ is

$$\det G(\mu_n) = \alpha \begin{cases} b\ (a^2-1)(a^2-2)^{\frac{n-6}{2}}(b^2-2)^{\frac{n-4}{2}}, & \text{if } n \text{ even,} \\ b\ (a^2-1)\big[(a^2-2)(b^2-2)\big]^{\frac{n-5}{2}}, & \text{if } n \text{ odd,} \end{cases} \tag{4}$$

where $\alpha = s^{n-4}\ t^{3n-7}(t^2-1)(s^2+s-1)(s^2-s-1)$ *and* $(s,t) = (a,b)$ *if n even, while* $(s,t) = (b,a)$ *if n odd.*

(iv) $\Delta_n(\hat{\mu}_n)$ is irreducible if a and b are such that equation 4 is non-zero.

(v) Let v be the initial part of the generator ${}_1U_{n-2}$ for FC_n with label λ. There is a non-zero homomorphism $\theta : \Delta_n(\lambda) \to \Delta_n(\hat{\mu}_n)$ defined by $\theta(v) = \hat{M}_2 - c\hat{M}_1$ with $c^2 = b^2 = 1$ if n even, and $c^2 = a^2 = 1$ if n odd.

(vi) Let λ be the label of ${}_1U_{n-2}$. Then there is a non-zero homomorphism $\theta : \Delta_n(\lambda) \to \Delta_n(\hat{\mu}_n)$ when $b^2 = 1$ if n even, and $a^2 = 1$ if n odd.

Proof. (i) and (ii) are clear.

(iii) If n even then every a-string (resp. b-string) in M_i it becomes a-string, (resp.b-string) in $\hat{M}_i$, so, $\langle M_i, M_j\rangle = \langle \hat{M}_i, \hat{M}_j\rangle$ for all $0 \leq i, j \leq 2n-5$, and hence, $\det G(\hat{\mu}_n) = \det G(\mu_n)$. If n odd then every a-string (resp. b-string) in M_i it becomes b-string, (resp.a-string) in $\hat{M}_i$, so, if $\langle M_i, M_j\rangle = a^{r_1}b^{r_2}$ for some integers r_1, r_2, then $\langle \hat{M}_i, \hat{M}_j\rangle = b^{r_1}a^{r_2}$ for all $0 \leq i, j \leq 2n-5$, and hence, $\det G(\hat{\mu}_n)$ is equal to $\det G(\mu_n)$ after swapping a with b and b with a.

(iv) By using Theorem 3.9.

(v) Notice that ${}_1U_{n-3} = {}_1\hat{U}_3$, ${}_1U_{n-2} = {}_1\hat{U}_2$, and ${}_1U_{n-1} = {}_1\hat{U}_1$. In addition, for $j = 1, 2$, we have ${}_jU_i\hat{M}_j \neq 0$ if $i = n-3, n-2, n-1$ and ${}_jU_i\hat{M}_j = 0$ if $1 \leq i \leq n-4$. We can argue as in the proof of Theorem 6.2 to get the result.

(vi) The result follow from Theorem 4.5. □

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