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Norms of Self-Adjoint Two-Sided Multiplication Operators in Norm-Attainable Classes

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Abstract

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. In this note, we give characterizations when the elementary operator $\mathcal{T}_{A,B} : B(H) \to B(H)$ defined by $\mathcal{T}_{A,B}(X) = AXB + BXA$, $\forall\, X \in B(H)$ and A, B fixed in $B(H)$ is self adjoint and implemented by norm-attainable operators. We extend our work by showing that the norm of the adjoint of $\mathcal{T}_{A,B}$ is equal to the norm of $\mathcal{T}_{A,B}$ when it is implemented by normal operators.

Keywords: Elementary Operators, Norm, Norm-attainability, Self-adjoint.

1 Introduction

The norm property for elementary operators has been considered in a large number of papers but it still remains interesting to many mathematicians. This is so because calculating these norms involves finding a formula that describes the norms of elementary operators in terms of their coefficients. Up-to-date, there is no known formula for calculating the norm of an arbitrary elementary operator acting on general Banach algebras. Estimating these norms from above is trivial but estimating the lower bound has proved to be difficult. For the Jordan elementary operator, Mathieu [9] proved that for prime C*-algebras, the coefficient is $\frac{2}{3}$, Cabrera and Rodriguez [4] obtained $\frac{1}{20412}$ for JB*-algebras while Stacho and Zalar [17] proved that for standard operator algebras the value is $2(\sqrt{2}-1)$. Timoney [21, 22], Blanco, Boumazgour and Ransford [3] showed that the coefficient is 1. Nyamwala [11] worked on a C*-algebra of the type $C^*(P, Q_k, 1)$ and Nyamwala and Agure [12] worked on general C*-algebras. They both obtained the maximum value of 2. Recent studies by Hong-Ke Du, Yue-Qing Wang, and Gui-Bao Gao gave a fundamental result on the elementary operator acting on B(H) while Seddik characterized normaloid operators using the injective norm. The results obtained in these studies show that the lower estimate lie between 1 and 2, that is, $\|A\|\|B\| \leq \|AXB + BXA\| \leq 2\|A\|\|B\|$. In this paper we consider elementary operators implemented by norm-attainable operators and characterize these norm-attainable operators. We also obtain the norms of the adjoint of normally represented elementary operators. This paper is arranged in the following sections : 1. Introduction; 2. Preliminaries; 3. Norm-attainability; 4. Norms of Two-Sided Multiplication Operators.

2 Preliminaries

Consider a C*-algebra $\mathcal{A}$ and let $\mathcal{T}_{A,B} : \mathcal{A} \to \mathcal{A}$. $\mathcal{T}$ is called an elementary operator if it has the following expression: $\mathcal{T}(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall\, X \in \mathcal{A}$, where A_i, B_i are fixed in $\mathcal{A}$ or $\mathcal{M}(\mathcal{A})$ where $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$.

For $A,\ B \in B(H)$ we define the particular elementary operators:

(i). the left multiplication operator $\mathcal{L}_A : B(H) \to B(H)$ by:
$\mathcal{L}_A(X) = AX,\ \forall\ X \in B(H)$.

(ii). the right multiplication operator $\mathcal{R}_B : B(H) \to B(H)$ by :
$\mathcal{R}_B(X) = XB,\ \forall\ X \in B(H)$.

(iii). the generalized derivation (implemented by $A,\ B$) by:
$\delta_{A,B} = \mathcal{L}_A - \mathcal{R}_B$.

(iv). the inner derivation(implemented by A) by:
$\delta_A(X) = AX - XA,\ \forall\ X \in B(H)$.

(v). the basic elementary operator(implemented by $A,\ B$) by:
$\mathcal{M}_{A,\ B}(X) = AXB,\ \forall\ X \in B(H)$.

(vi). the Jordan elementary operator(implemented by $A,\ B$) by:
$\mathcal{T}_{A,\ B}(X) = AXB + BXA,\ \forall\ X \in B(H)$.

The relationship between the norm of $\mathcal{T}_{A,\ B}(X) = AXB + BXA$ and the norms of A and B are known (see [17]). We shall consider the C*-algebra $B(H)$ with unit I. The algebraic numerical range of an element A in $B(H)$ is defined by $W_a(A) = \{f(A) :\ f \in \mathcal{P}(B(H))\}$ where $\mathcal{P}(B(H)) = \{f \in (B(H))^* : f(I) = 1 = \|f\|\}$

Definition 2.1 *For an operator $A \in B(H)$ we define:*

(i). Numerical range by $W(A) = \{\langle Ax, x\rangle :\ x \in H,\ \|x\| = 1\}$.

(ii). Numerical radius by $w(A) = \sup\{|z| :\ z \in W(A)\}$.

Definition 2.2 *Let $B(H)$ an algebra with involution. Then the linear functional f is called a state on $B(H)$ if f is positive and $\|f\| = f(e) = 1$ where e is a unit element in $B(H)$.*

Definition 2.3 *If $A \in B(H,\ K)$, where H and K are Hilbert spaces, then the linear operator $A^* \in B(K,\ H)$ satisfying $\langle Ax, y\rangle = \langle x, A^*y\rangle\ \forall\ x \in H$ and $\forall\ y \in K$ is called the adjoint of A.*

Definition 2.4 *A bounded operator $A \in B(H)$ is said to be self-adjoint if $A^* = A$. Thus, A is Hermitian and $\mathfrak{D}(A) = H$ ($\mathfrak{D}(A)$ is the domain of A) if and only if A is self-adjoint.*

Definition 2.5 *A bounded linear operator A on a Hilbert space H is said to be normal if it commutes with its adjoint i.e $AA^* = A^*A$.*

Definition 2.6 *Let W be a complex normed space and let Ω denote a subalgebra of $B(W)$. Ω is called a standard operator subalgebra of $B(H)$ if it contains all finite rank operators.*

From this stage and in the sequel, we denote a bounded linear operator and a norm-attainable operator in $B(H)$ by A and A_N respectively. We note that for $A,\ S \in B(H)$, A is said to be positive if $\langle Ax, x\rangle \geq 0,\ \forall\ x \in H$ and S an isometry(Co-isometry) if $S^*S = SS^* = I$ where I is an identity operator in $B(H)$.

Definition 2.7 *An operator $A \in B(H)$ is said to be norm-attainable if for there exists a unit vector $x_0 \in H$, $\|Ax_0\| = \|A\|$.*
A derivation, $\delta_{A,B}$, on a C^*-algebra $B(H)$ is said to be norm-attainable if there exists a functional $\varphi \in H^*$ such that $\|\delta_{A,B}\varphi\| = \|\delta_{A,B}\|$ and $\|\varphi\| = 1$.
An operator $\mathcal{U}_{\tilde{A},\tilde{B}}(X) = \sum_{i=1}^{n} A_i X B_i$, is said to be norm-attainable if there exists a contraction X in the unit ball, $B(H)_1$, such that $\|\mathcal{U}_{\tilde{A},\tilde{B}}(X)\| = \|\mathcal{U}_{\tilde{A},\tilde{B}}\|$, where $\tilde{A} = (A_1, ..., A_n)$ and $\tilde{B} = (B_1, ..., B_n)$ are n-tuples in $B(H)$. If $\tilde{B} = \tilde{A}$ then we have $\mathcal{U}_{\tilde{A},\tilde{A}}$ simply denoted by $\mathcal{U}_{\tilde{A}}$. For $i = 1$ we have an inner derivation $\mathcal{U}_A$.

3 Norm-attainability

In this section, we characterize norm-attainability. We begin with the following lemma.

Lemma 3.1 *Let $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1$, $\|S\zeta\| = \|S\|$, $\langle S\zeta, \zeta\rangle = 0$.*

Proof. For any x satisfying that $x \perp \{\zeta, S\zeta\}$, define X as follows

$$X : \zeta \to \zeta, \;\; S\zeta \to -S\zeta, \;\; x \to 0,$$

because $S\zeta \perp \zeta$. Since X is a bounded operator on H and $\|X\zeta\| = \|X\| = 1$,

$$\|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|.$$

It follows that $\|\delta_S\| = 2\|S\|$ via the result in [19, Theorem 1], because $\langle S\zeta, \zeta\rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable.

□

Theorem 3.2 *Let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$. There exists an operator $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$. Furthermore, there exists a vector $\eta \in H$, $\|\eta\| = 1$ such that $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta\rangle = \beta$.*

Proof. Without loss of generality, we may assume that $\|S\| = 1$. Let $x_n \in H$ $(n = 1, 2, ...)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and

$$\lim_{n\to\infty} \langle Sx_n, x_n\rangle = \beta.$$

Consider a partial isometry G and $L = \int_0^1 \beta dE_\beta$, the spectral decomposition of L. Let $S = GL$, the polar decomposition of S. Since

$$\lim_{n\to\infty} \|Sx_n\| = \|S\| = \|L\| = 1,$$

we have that $\|Lx_n\| \to 1$ as n tends to ∞ and

$$\lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lim_{n\to\infty} \langle GLx_n, x_n\rangle = \lim_{n\to\infty} \langle Lx_n, G^*x_n\rangle.$$

Now for $H = \overline{Ran(L)} \oplus KerL$, we can choose x_n such that $x_n \in \overline{Ran(L)}$ for large n. Indeed, let

$$x_n = x_n^{(1)} \oplus x_n^{(2)}, \; n = 1, 2, ...$$

Then we have that

$$Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$$

and that

$$\lim_{n\to\infty} \|x_n^{(1)}\| = 1, \lim_{n\to\infty} \|x_n^{(2)}\| = 0$$

since

$$\lim_{n\to\infty} \|Lx_n\| = 1.$$

Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we obtain

$$\lim_{n\to\infty} \left\| L\frac{1}{\|x_n^{(1)}\|}x_n^{(1)} \right\| = \lim_{n\to\infty} \left\| S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)} \right\| = 1,$$

$$\lim_{n\to\infty} \left(S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)} \right) = \beta.$$

Now assume that $x_n \in \overline{RanL}$. Since G is a partial isometry from $\overline{RanL}$ onto $\overline{RanS}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$. For L is a positive operator, $\|L\| = 1$ and for any $x \in H$,

$$\langle Lx, x\rangle \leq \langle x, x\rangle = \|x\|^2.$$

Replacing x with $L^{\frac{1}{2}}x$, we get that $\langle L^2x, x\rangle \leq \langle Lx, x\rangle$, where $L^{\frac{1}{2}}$ is the positive square root of L. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \leq \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty} \|Lx_n\| = 1$ and that

$$\|Lx_n\|^2 \leq \langle Lx_n, x_n\rangle \leq \|Lx_n\|^2 = 1.$$

Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \geq 0$, we have $\lim_{n\to\infty}\langle (I - L)x_n, x_n\rangle = 0$. thus $\lim_{n\to\infty} \|(I - L)^{\frac{1}{2}}x_n\| = 0$. Indeed,

$$\lim_{n\to\infty} \|(I - L)x_n\| \leq \lim_{n\to\infty} \|(I - L)^{\frac{1}{2}}\|.\|(I - L)^{\frac{1}{2}}x_n\| = 0.$$

For $\alpha > 0$, let $\gamma = [0, 1 - \frac{\alpha}{2}]$ and let $\rho = [1 - \frac{\alpha}{2}, 1]$. We have

$$\begin{aligned} L &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^1 \mu dE_\mu \\ &= LE(\gamma) \oplus LE(\rho). \end{aligned}$$

Next we show that $\lim_{n\to\infty} \|E(\gamma)x_n\| = 0$. If there exists a subsequence $x_{n_i}, (i = 1, 2, ...,)$ such that $\|E(\gamma)x_{n_i}\| \geq \epsilon > 0,\ (i = 1, 2, ...,)$, then since $\lim_{i\to\infty} \|x_{n_i} - Lx_{n_i}\| = \lim_{i\to\infty}(I - L)x_{n_i} = 0$, it follows that

$$\begin{aligned} \lim_{i\to\infty} \|x_{n_i} - Lx_{n_i}\| &= \lim_{i\to\infty}(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2) \\ &= 0. \end{aligned}$$

Hence we have that $\lim_{i\to\infty}(\|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0$. Now it is clear that

$$\begin{aligned} \|E\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| &\geq \|E\gamma)x_{n_i}\| - \|LE(\gamma)\|.\|E\gamma)x_{n_i}\| \\ &\geq (I - \|LE(\gamma)\|).\|E\gamma)x_{n_i}\| \\ &\geq \frac{\alpha}{2}\epsilon \\ &> 0. \end{aligned}$$

This is a contradiction. Therefore,

$$\lim_{n\to\infty} \|E(\gamma)x_n\| = 0.$$

Since

$$\lim_{n\to\infty} \langle Lx_n, x_n\rangle = 1,$$

we have that

$$\lim_{n\to\infty} \langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$$

and

$$\lim_{n\to\infty} \langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta.$$

It is easy to see that

$$\lim_{n\to\infty} \|E(\rho)x_n\| = 1,\ \lim_{n\to\infty} \left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|} \right) = 1$$

and

$$\lim_{n\to\infty} \left(L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|} \right) = \beta$$

Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$. Let

$$\begin{aligned} J &= \int_\gamma \mu dE_\mu + \int_\rho \mu dE_\mu \\ &= \int_0^{1-\frac{\alpha}{2}} \mu dE_\mu + \int_{1-\frac{\alpha}{2}+0}^{1} \mu dE_\mu \\ &= J_1 \oplus E(\rho). \end{aligned}$$

Then it is evident that

$$\|J\| = \|S\| = \|L\| = 1, Jx_n = x_n,$$

and $\|J - L\| < \frac{\alpha}{2}$. If we can find a contraction V such that $V - G < \frac{\alpha}{2}$ and $\|Vx_n\| = 1$ for a large n, $\langle Vx_n, x_n\rangle = \beta$, then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that

$$\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta,$$

$$\begin{aligned} \|S - Z\| &= \|GL - VJ\| \\ &\le \|GL - GJ\| + \|GJ - VJ\| \\ &\le \|G\|.\|L - J\| + \|G - V\|.\|J\| \\ &\le \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha. \end{aligned}$$

To finish the proof, we now construct the desired contraction V. Clearly, $\lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$, because $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$ and

$$\lim_{n\to\infty} \|x_n - Lx_n\| = 0.$$

Let $Gx_n = \phi_n x_n + \varphi_n y_n,\ \ (y_n \perp x_n,\ \|y_n\| = 1)$ then $\lim_{n\to\infty} \phi_n = \beta$, because

$$\lim_{n\to\infty} \langle Gx_n, x_n\rangle = \lim_{n\to\infty} \langle x_n, G^*x_n\rangle = \beta$$

but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so we have that $\lim_{n\to\infty} |\varphi_n| = \sqrt{1-|\beta|}^2$. Now for $\alpha > 0$, there exists an integer M such that $|\phi_M - \beta| < \frac{\alpha}{8}$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1-|\beta|}^2,\ \ |\varphi_M - \varphi_M^0| < \frac{\alpha}{8}$. We have that

$$\begin{aligned} Gx_M &= \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M \\ &= (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M. \end{aligned}$$

Let $q_M = \beta x_M + \varphi_M^0 y_M$,

$$Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M.$$

Suppose that $y \perp x_M$, then

$$\begin{aligned} \langle Gx_M, Gy\rangle &= (\phi-\beta)\langle x_M, Gy\rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy\rangle + \langle q_M, Gy\rangle \\ &= 0, \end{aligned}$$

because G^*G is a projection from H to $RanL$. It follows that

$$|\langle q_M, Gy\rangle| \leq |\phi_M - \beta|.\|y\| + |\varphi_M - \varphi_M^0|.\|y\| \leq \frac{\alpha}{4}\|y\|.$$

If we suppose that $Gy = \phi q_M + y^0,\ (y^0 \perp q_M,)$ then y^0 is uniquely determined by y. Hence we can define V as follows

$$V : x_M \to q_M,\ y \to y^0,\ \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0,$$

with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now,

$$\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1,$$

$$\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2.$$

It follows that

$$\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1,$$

for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y,\ \ \|x\| = 1,\ x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we can show that

$$\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2.$$

If $y \perp x_M,\ \ \|y\| \leq 1$ then obtain

$$\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M\rangle| = |\langle q_M, Gy\rangle| < \frac{\alpha}{4}.$$

Hence for any $x \in H,\ x = \phi x_M + \varphi_M y,\ \ \|x\| = 1$,

$$\begin{aligned} \|Gx - Vx\|^2 &= \|\phi(G-V)x_M + \varphi(G-V)y\|^2 \\ &= |\phi|^2\|(G-V)x_M\|^2 + |\varphi|^2\|(G-V)y\|^2 \\ &< |\phi|^2.\frac{\alpha^2}{16} + |\varphi|^2.\frac{\alpha^2}{16} \\ &< \frac{\alpha^2}{8}, \end{aligned}$$

which implies that

$$\|(G-V)x\| < \frac{\alpha}{2},\ \ \|x\| = 1,$$

and hence $\|(G-V)\| < \frac{\alpha}{2}$. Let $S = VJ$. Then S is what we want and this completes the proof.

□

Theorem 3.3 *(from [7]) Let $\tilde{A} = A_1, ...A_n$ and $\tilde{B} = B_1, ...B_n$ be in $B(H)$ and let $\mathcal{T}_{\tilde{A},\tilde{B}} : B(H) \to B(H)$ be defined by $\mathcal{T}_{\tilde{A},\tilde{B}}(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall\, X \in B(H)$, then*

$$\|\mathcal{T}_{\tilde{A},\tilde{B}}\| = \sup_{U \in \mathcal{U}(H)} \|\mathcal{T}_{\tilde{A},\tilde{B}}(U)\|.$$

Moreover, there is a contraction $X \in B(H)_1$ such that $\|\mathcal{T}_{\tilde{A},\tilde{B}}(X)\| = \|\mathcal{T}_{\tilde{A},\tilde{B}}\|$ if and only if there is a unitary $U \in \mathcal{U}(H)$ such that $\|\mathcal{T}_{\tilde{A},\tilde{B}}(U)\| = \|\mathcal{T}_{\tilde{A},\tilde{B}}\|$.

See [7] for proof.

Corollary 3.4 *(i) $\mathcal{U}(H)$ is the algebra of all unitaries, $B(H)_1$ ia a unit ball; (ii) An operator $\mathcal{T}_{\tilde{A},\tilde{B}}$ is said to be norm-attainable if there is a contraction $X \in B(H)_1$ such that $\|\mathcal{T}_{\tilde{A},\tilde{B}}(X)\| = \|\mathcal{T}_{\tilde{A},\tilde{B}}\|$. For more details on norm attainability see [6] and [7].*

Corollary 3.5 *If $A, B \in B(H)$ are norm-attainable then $A + B$; $A - B$; λA, $\lambda \in \mathbb{C}$ and AI (I is an identity operator in B(H)) are norm-attainable.*

Proof. The proofs follow explicitly by use of limits, triangle and Cauchy-Schwarz inequalities.

□

Corollary 3.6 *For $A \in B(H)$, A is norm-attainable if and only if its adjoint is norm-attainable.*

Proof. The proof follows immediately from [7].

□

Lemma 3.7 *A positive self-adjoint operator is normal and norm-attainable.*

Proof. The proof is trivial.

□

Remark 3.8 *An operator A being normal $\Leftrightarrow$ A is norm-attainable.*

4 Norms of Two-Sided Multiplication Operators

Lemma 4.1 *Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $M_{A,B:N} : B(H) \to B(H)$ be a norm-attainable basic elementary defined by $M_{A,B:N}(X) = AXB$, $\forall\, X \in B(H)$ where A, B are fixed in $B(H)$. Then $\|M_{A,B:N}|B(H)\| = \|A\|\|B\|$.*

Proof. By definition, $\|M_{A,B:N}|B(H)\| = \sup\{\|M_{A,B:N}(X)\| : X \in B(H), \|X\| = 1\}$.
This implies that $\|M_{A,B:N}|B(H)\| \geq \|M_{A,B:N}(X)\|, \forall X \in B(H), \|X\| = 1$. So $\forall \epsilon > 0, \|M_{A,B:N}(X)|B(H)\| - \epsilon < \|M_{A,B:N}(X)\|, \forall X \in B(H), \|X\| = 1$. But, $\|M_{A,B}(X)|B(H)\| - \epsilon < \|AXB\| \leq \|A\|\|X\|\|B\| = \|A\|\|B\|$. Since ϵ is arbitrary, this implies that

$$\|M_{A,B:N}(X)|B(H)\| \leq \|A\|\|B\|. \tag{1}$$

On the other hand, let $\xi, \eta \in H, \|\xi\| = \|\eta\| = 1, \phi, \varphi \in H^*$. Now,

$$\|M_{A,B:N}(X)|B(H)\| \geq \|M_{A,B:N}(X)\| : \forall X \in B(H), \|X\| = 1.$$

But,

$$\begin{aligned} \|M_{A,B:N}(X)\| &= \sup\{\|(M_{A,B:N}(X))\eta\| : \forall \eta \in H, \|\eta\| = 1\} \\ &= \sup\{\|(AXB)\eta\| : \eta \in H, \|\eta\| = 1\}. \end{aligned}$$

Setting $A = (\phi \otimes \xi_1), \forall \xi_1 \in H, \|\xi_1\| = 1$ and $B = (\varphi \otimes \xi_2), \forall \xi_2 \in H, \|\xi_2\| = 1$, then $\forall \eta \in H$ we have,

$$\begin{aligned} \|M_{A,B:N}|B(H)\| &\geq \|M_{A,B:N}(X)\| \geq \|(M_{A,B:N}(X))\eta\| \\ &= \|(AXB)\eta\| \\ &= \|((\phi \otimes \xi_1)X(\varphi \otimes \xi_2))\eta\| \\ &= \|(\phi \otimes \xi_1)X(\varphi(\eta)\xi_2)\| \\ &= \|(\phi \otimes \xi_1)\varphi(\eta)X(\xi_2)\| \\ &= |\varphi(\eta)|\|(\phi \otimes \xi_1)X(\xi_2)\| \\ &= |\varphi(\eta)|\|\phi(X(\xi_2))\xi_1\| \\ &= |\varphi(\eta)||\phi(X(\xi_2))|\|\xi_1\| \\ &= \|A\|\|B\|. \end{aligned}$$

□

Theorem 4.2 *Let H be a complex Hilbert space, $B(H)$ the algebra of bounded linear operators on H. Let $\mathcal{T}_{A,B} : B(H) \to B(H)$ be defined by $\mathcal{T}_{A,B}(X) = AXB + BXA, \forall X \in B(H)$ where A, B are fixed in $B(H)$ and its adjoint $\mathcal{T}^*_{A^*,B^*}(X) = A^*XB^* + B^*XA^*, \forall X \in B(H)$, then $\|\mathcal{T}_{A,B}|B(H)\| = \|\mathcal{T}^*_{A^*,B^*}|B(H)\|$ if and only if $\mathcal{T}_{A,B}$ is (i) normal and (ii) norm-attainable.*

Proof. (i). ⇒ part: Assuming the normality and norm-attainability, we need to show that the equality holds. Now for any $\omega \in H$ with $\|\omega\| = 1$ and $\varrho \in H^*$, define an operator $\varrho \otimes \omega \in B(H)$ by: $(\varrho \otimes \omega)\zeta = \varrho(\zeta)\omega, \forall \omega \in H$. Also for any $\varrho, \tau \in H^*$, define $\mathcal{T}_{\varrho\otimes\omega, \tau\otimes\omega} : B(H) \to B(H)$ by

$$\mathcal{T}_{\varrho\otimes\omega, \tau\otimes\omega}(X) = [(\varrho \otimes \omega)X(\tau \otimes \omega) + (\tau \otimes \omega)X(\varrho \otimes \omega)](\zeta), \forall \zeta \in H, X \in B(H).$$

Taking suprema over $X \in B(H)$ and some $\eta \in H$ and taking into account the result of [3] then $\forall \eta \in H \, \exists X \in B(H)$ such that $X(\omega) = \eta$.

$$\|\mathcal{T}_{\varrho\otimes\omega, \tau\otimes\omega}(X)\| \leq \sup_{\substack{\eta, \zeta \in H : \\ \|\eta\| = \|\zeta\| = 1}} [(\varrho \otimes \omega)X(\tau \otimes \omega) + (\tau \otimes \omega)X(\varrho \otimes \omega)](\zeta). \tag{2}$$

Setting $\eta = X(\omega)$ and substituting it in inequality (2) reduces it to

$$\|\mathcal{T}_{A,B}\| = \sup_{\substack{\eta,\ \zeta \in H: \\ \|\eta\| = \|\zeta\| = 1}} |\varrho(\eta)\tau(\zeta) + \tau(\eta)\varrho(\zeta)|. \tag{3}$$

Let A, $B \in B(H)$ $\exists$ ϱ, $\tau \in H^*$ such that $A = (\varrho \otimes \omega)$ and $B = (\tau \otimes \omega)$. Taking the infimum on both sides of equation (3) and since $H = \mathbb{C}^2$ and $\{e_1, e_2\}$ is an orthonormal basis for H, then by Riesz representation theorem, we have $\phi(\eta) = \langle \eta, e_1 \rangle$ with $\|\phi\| = \|e_1\| = 1$; $\varphi(\eta) = \langle \eta, e_2 \rangle$ with $|\varphi\| = \|e_2\| = 1$; $\phi(\zeta) = \langle \zeta, e_1 \rangle$ with $\|\phi\| = \|e_1\| = 1$; $\varphi(\zeta) = \langle \zeta, e_2 \rangle$ with $\|\varphi\| = \|e_2\| = 1$. Invoking these representations yields the desired results. Conversely, by Remark 3.8 and Definitions 2.3, 2.4 and 2.5, consider a self-adjoint norm-attainable operator $\mathcal{T}_{A,B:N}$(proof for self-adjointedness is elementary and is left for the reader). Since $\mathcal{T}_{A,B:N}$ is self-adjoint, we only need to show that $\mathcal{T}_{A,B:N}$ is norm-attainable implies that $\mathcal{T}^*_{A^*,B^*:N} = \mathcal{T}_{A,B:N}$ is norm-attainable. If $\mathcal{T}_{A,B:N}$ is equal to zero, then there is nothing to prove so let $\mathcal{T}_{A,B:N}$ be nonzero. If $\mathcal{T}_{A,B:N}$ is norm-attainable then there is a functional $\varphi \in H^*$ such that $\|\mathcal{T}_{A,B:N}\varphi\| = \|\mathcal{T}_{A,B:N}\|$ and $\|\varphi\| = 1$ i.e. $\|\mathcal{T}^*_{A^*,B^*:N}\mathcal{T}_{A,B:N}\varphi\| = \|(\mathcal{T}_{A,B:N})^2\|\varphi$. Let $\mu_0 = \frac{\mathcal{T}_{A,B:N}\varphi}{\|\mathcal{T}_{A,B:N}\|}$. Then μ_0 is a unit functional and $\|\mathcal{T}^*_{A^*,B^*:N}\mu_0\| = \|\mathcal{T}_{A,B:N}\| = \|\mathcal{T}^*_{A^*,B^*:N}\|$.
Indeed,

$$\begin{aligned}
\|\mathcal{T}^*_{A^*,B^*:N}\mu_0\| &= \|\mathcal{T}^*_{A^*,B^*:N}\frac{\mathcal{T}_{A,B:N}\varphi}{\|\mathcal{T}_{A,B:N}\|}\| \\
&= \frac{1}{\|\mathcal{T}_{A,B:N}\|}\|\mathcal{T}^*_{A^*,B^*:N}\mathcal{T}_{A,B:N}\varphi\| \\
&= \frac{1}{\|\mathcal{T}_{A,B:N}\|}\|(\mathcal{T}_{A,B:N})^2\|\varphi \\
&= \|\mathcal{T}_{A,B:N}\|.
\end{aligned}$$

But $\|\mathcal{T}^*_{A^*,B^*:N}\| = \|\mathcal{T}_{A,B:N}\|$. This completes the proof.

□

5 Conclusions

In this paper, we have considered elementary operators implemented by norm-attainable operators and characterize these norm-attainable operators. We haver also obtained the norms of the adjoint of normally represented elementary operators.

Conflicts of Interest

Author declares no conflict of interest.

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