Global solution and asymptotic behaviour for a wave equation type p-Laplacian with p-Laplacian damping

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Abstract

In this work we study the global solution, uniqueness and asymptotic behaviour of the nonlinear equation

$$u_{tt} - \Delta_p u - \Delta_p u_t = |u|^{r-1}u$$

where $\Delta_p u$ is the nonlinear p-Laplacian operator, $2 \leq p < \infty$. The global solutions are constructed by means of the Faedo-Galerkin approximations and the asymptotic behavior is obtained by Nakao method.

Keywords: p-Laplacian, global solution, asymptotic behaviour, p-Laplacian damping.

1 Introduction

We use the Sobolev Space with its properties as in R. A. Adams [1]. Throughout this paper we omit the space variable x of $u(x,t)$ and simply denote $u(x,t)$ by $u(t)$ when no confusion arises. C denotes various positive constants depending on the known constants and may be different at each appearance. Let $\Omega \in \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$, $2 \leq p < \infty$ and q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$, $D'(0,T)$ and $D(0,T)$, $T > 0$, will be denoted using the simple notation $\langle \cdot\,, \cdot \rangle$.

According to Poincaré's inequality, the standard norm $\| \cdot \|_{W_0^{1,p}(\Omega)}$ is equivalent to the norm $\|\nabla \cdot \|_p$ on $W_0^{1,p}(\Omega)$. Henceforth, we put $\| \cdot \|_{W_0^{1,p}(\Omega)} = \|\nabla \cdot \|_p$. We denote $\| \cdot \|_{L^2(\Omega)} = | \cdot |_2$ and the usual inner product by $(\cdot\,, \cdot)$. We denote the p-Laplacian operator by $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$, which can be extended to a monotone, bounded, hemicontinuos and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$$

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \,.\nabla v \, \mathrm{d}x$$

The existence of a global solution for wave equation of p-Laplacian type

$$u'' - \Delta_p u = 0 \tag{1}$$

without any additional dissipation term is an open problem. For $n = 1$ M. Derher [13] proved the local in time existence of solution and showed by a generic counter-example that the global in time solution can not use expected. Adding a strong damping $(-\Delta u')$ in (1) the well-posedness and asymptotic behavior was studied by J. M. Greenberg [15]. In fact, the strong damping plays an important role on the existence and stability for p-Laplacian wave equation see for instance for $n \geq 2$ [4, 5, 6, 11, 14, 18, 21, 22, 25]. Nevertheless, if the strong damping is replaced by a weaker damping (u'), then global existence and uniqueness are only know for $n = 1, 2$. See [8, 27]. In this work we consider a p-Laplacian damping. We have interested in to prove existence and uniqueness of solution and energy decay to the problem

$$u'' - \Delta_p u - \Delta_p u' = |u|^{r-1}u, \quad x \in \Omega, \quad \mathbb{R}^+, \tag{2}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega, \quad t \geq 0. \tag{4}$$

When $p = 2$ we obtain the strong damping $(-\Delta u')$, that provides the very strong dissipation. However, for $p > 2$ this effect is diminished by the fact that such a damping term is quasilinear and is, in some sense, degenerate. In addition, the degenerate nature of the p-Laplacian as an elliptic operator is known to cause serious difficulties, see [12].

The outline of the paper is as follows. In the Section, 2 we introduce some notations and the stability set created from Nehari Manifold. In the Section 3 we prove the existence of solution by Faedo-Galerkin method. By result of M. Nakao [20], energy decay in a appropriate set of stability will be given in Section 4. Finally we present the final comments.

2 The Potential Well

Is well known that the energy of a PDE system is, in some sense, split into kinetic and potential energy. Following the idea of Y. Ye [25] we are able to construct a set of stability as follows. We will prove that there is a valley or a "well" of depth d created in the potential energy. If this height d is strictly positive, we find that, for solutions with initial data in the "good part" of the well, the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause the blow-up in finite time. However in the good part of the well it remains bounded. As a result, the total energy of the solution remains finite on any time interval $[0, T)$, providing the global existence of the solution. We started by introducing the functional $J : W_0^{1,p}(\Omega) \cap W^{1,2(p-1)}(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{1}{r+1}\int_\Omega |u|^{r+1} \, \mathrm{d}x. \tag{5}$$

For $u \in W_0^{1,p}(\Omega)$ we define the functional

$$J(\lambda u) = \frac{\lambda^p}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{\lambda^{r+1}}{r+1}\int_\Omega |u|^{r+1} \, \mathrm{d}x, \quad 0 < \lambda. \tag{6}$$

The total energy of the problem (2)-(4) is defined by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega |u'(t)|^2 \, \mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u|^p \, \mathrm{d}x - \frac{1}{r+1}\int_\Omega |u|^{r+1} \, \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |u'(t)|^2 \, \mathrm{d}x + J(u(t)). \end{aligned}$$

With this notation, we point out that the critical points of the functional J are the weak solutions of the elliptic problem

$$\begin{aligned} -\Delta_p u &= |u|^{r-1}u \quad in \quad \Omega, \\ u &= 0 \quad on \quad \partial\Omega. \end{aligned}$$

Associated with the J we have the well known *Nehari Manifold* given by

$$\mathcal{N} \stackrel{\text{def}}{=} \left\{ u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega)/\{0\} \ : \ \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\right]_{\lambda=1} = 0 \right\}.$$

Equivalently,

$$\mathcal{N} = \left\{ u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega)/\{0\} \ : \ \int_\Omega |\nabla u|^p \,\mathrm{d}x = \int_\Omega |u|^{r+1} \,\mathrm{d}x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [2],

$$d \stackrel{\text{def}}{=} \inf_{u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega)/\{0\}} \sup_{0 \le \lambda} J(\lambda u).$$

It is well-known that for $1 < r \le 5$ the depth of the well d is a real constant strictly positive (e.g. [[24], Theorem 4.2]) and

$$d = \inf_{u \in \mathcal{N}} J(u).$$

We now introduce the potential well

$$\mathcal{W} = \{u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega) \ : \ J(u) < d\} \cup \{0\}$$

and partition it into two sets

$$\mathcal{W}_1 = \{u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega) \ : \ \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x > \frac{1}{r+1} \int_\Omega |u|^{r+1} \,\mathrm{d}x\} \cup \{0\},$$
$$\mathcal{W}_2 = \{u \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega) \ : \ \frac{1}{p} \int_\Omega |\nabla u|^p \,\mathrm{d}x < \frac{1}{r+1} \int_\Omega |u|^{r+1} \,\mathrm{d}x\}.$$

We will refer to $\mathcal{W}_1$ as the "good" part of the well and $\mathcal{W}_2$ as the "bad" part of the well. Then we define by $\mathcal{W}_1$ the set of stability for the problem (2)-(4).

3 Global Solution

3.1 Existence

Theorem 3.1. *Let $1 < r \le 5$. Suppose the $E(0) < d$. Given $u_0 \in \mathcal{W}_1$, $u_1 \in W_0^{1,p}(\Omega)$ there exists a function $u \colon \Omega \times (0,T) \to \mathbb{R}$ such that*

$$\begin{aligned}
&u \in L^\infty(0,T; L^{r+1}(\Omega)) \cap L^\infty(0,T; W_0^{1,p}(\Omega)),\\
&u' \in L^\infty(0,T; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)),\\
&u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{a.e. in } \Omega,\\
&\frac{\mathrm{d}}{\mathrm{d}t}(u', v) + \langle -\Delta_p u, v\rangle + \langle -\Delta_p u', v\rangle - (|u|^{r-1}u, v) = 0, \ \forall \ v \in W_0^1(\Omega) \cap W^{1,2(p-1)}(\Omega) \ \text{in } D'(0,T).
\end{aligned}$$

Proof. Denote by

$$\mathcal{K}_j = \{K \subset \{u \in L^2(\Omega) \ : \ ||u||_2 = 1\} \ : \ K \text{ is compact, symmetric and } \gamma(K) \ge j\},$$

where $\gamma(G) = \inf\{m \ : \ \exists \phi \ : \ G \to \mathbb{R}/\{0\}, \phi \text{ odd continuous function}\}$ denotes the Krasnoselski genus. In [10] it is proved that

$$\lambda_j = \inf_{G \in \mathcal{K}_j} \sup_{u \in G} ||\nabla u||_2^2$$

is a sequence of eigenvalue of the p-Laplacian. $-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,q}(\Omega)$ is a monotone, coercive and hemicontinuos operator. Minty-Browder theorem, see [26], guarantees the existence of a basis $(w_j)_{j=1}^{\infty}$ given by the solution of the stationary problem

$$\begin{aligned}-\Delta_p w_j &= \lambda_j w_j,\\ w_j(0) &= w_{0j},\end{aligned}$$

that can be extended as a basis of Galerkin, orthogonal in $W_0^{1,p}(\Omega)$ and orthonormal in $L^2(\Omega)$. Now, for each $m\in\mathbb{N}$, let us put $V_m=\mathrm{Span}\{w_1,w_2,\dots,w_m\}$. We search for a function $u_m(t)=\displaystyle\sum_{j=1}^{m}k_{jm}(t)w_j$ such that for any $v\in V_m$, $u_m(t)$ satisfies the approximate equation

$$(u_m''(t),v)+\langle -\Delta_p u_m(t),v\rangle+\langle -\Delta_p u_m'(t),v\rangle-(|u_m(t)|^{r-1}u_m(t),v)=0, \tag{7}$$

with the initial conditions $u_m(0)=u_{0m}$ and $u_m'(0)=u_{1m}$, where u_{0m} e u_{1m} are choose in V_m so that

$$w_{0m}\to u_0\ \in\ W_0^1(\Omega)\cap W^{1,2(p-1)}(\Omega)\quad\text{and}\quad u_{1m}\to u_1\ in\ W_0^1(\Omega). \tag{8}$$

Putting $v=w_i$, $i=1,2,\dots,m$, and using

$$\begin{aligned}u_m''(t) &= \sum_{j=1}^{m}k_{jm}''(t)w_j(x),\\ \Delta_p u_m(t) &= \sum_{j=1}^{m}k_{jm}(t)\Delta_p w_j(x),\\ \Delta_p u_m'(t) &= \sum_{j=1}^{m}k_{jm}'(t)\Delta_p w_j(x),\end{aligned}$$

we observe that (7) is a system of ODEs in the variable t and has a local solution $u_m(t)$ in a interval $[0,t_m)$, by virtue of Carathéodory's theorem, see [9]. In the next step we obtain the a priori estimates for the solution $u_m(t)$ so that it can be extended to the whole interval $[0,T]$, $T>0$.

A Priori Estimates: We replace $v=u_m'(t)$ in the approximate equation (7) and we get

$$(u_m''(t),u_m'(t))-\langle\Delta_p u_m(t),u_m'(t)\rangle-\langle\Delta_p u_m'(t),u_m'(t)\rangle-\big(|u_m(t)|^{r-1}u_m(t),u_m'(t)\big)=0 \tag{9}$$

Let $\theta(t)\in D(0,t_m)$. So we have,

$$\langle(u_m''(t),u_m'(t)),\theta(t)\rangle = \left\langle\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega|u_m'(t)|^2\,\mathrm{d}x,\theta(t)\right\rangle \tag{10}$$

$$\langle\langle -\Delta_p u_m(t),u_m'(t)\rangle_p,\theta(t)\rangle = \left\langle\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{p}\int_\Omega|\nabla u_m(t)|^p\,\mathrm{d}x,\theta(t)\right\rangle \tag{11}$$

$$\langle(-\Delta_p u_m'(t),u_m'(t))\,,\theta(t)\rangle = \left\langle\int_\Omega|\nabla u_m'(t)|^p\,\mathrm{d}x,\theta(t)\right\rangle \tag{12}$$

$$\big\langle\langle|u_m(t)|^{r-1}u_m(t),u_m'(t)\rangle_p,\theta(t)\big\rangle = \left\langle\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r+1}\int_\Omega|u_m(t)|^{r+1}\,\mathrm{d}x,\theta(t)\right\rangle \tag{13}$$

Replacing (10), (11), (12), (13) in (9) we obtain in $D'(0,t_m)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\int_\Omega|u_m'(t)|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla u_m(t)|^p\,\mathrm{d}x-\frac{1}{r+1}\int_\Omega|u_m(t)|^{r+1}\,\mathrm{d}x\right\}=-\int_\Omega|\nabla u_m'(t)|^p\,\mathrm{d}x \tag{14}$$

The approximate energy

$$E_m(t)=\frac{1}{2}\int_\Omega|u_m'(t)|^2\,\mathrm{d}x+\frac{1}{p}\int_\Omega|\nabla u_m(t)|^p\,\mathrm{d}x-\frac{1}{r+1}\int_\Omega|u_m(t)|^{r+1}\,\mathrm{d}x$$

satisfies

$$\begin{aligned} E_m(t) &\leq E_m(0) \\ &= \frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x + J(u_m(0)). \end{aligned}$$

We have that $J(u_m(0)) < d$. By to convergence of initial data (8), there exists a constant $C > 0$ independent of t and m such that

$$\frac{1}{2}\int_\Omega |u_m'(0)|^2\,\mathrm{d}x \leq C.$$

With the estimate $E_m(t) \leq E_m(0) \leq C$, that is,

$$\frac{1}{2}\int_\Omega |u_m'(t)|^2\,\mathrm{d}x + \frac{1}{p}\int_\Omega |\nabla u_m(t)|^p\,\mathrm{d}x - \frac{1}{r+1}\int_\Omega |u_m(t)|^{r+1}\,\mathrm{d}x \leq C$$

and so we can extend the approximate solutions $u_m(t)$ to the interval $[0,T]$, $T > 0$, see [16]. Note that

$$\int_0^T\int_\Omega |\nabla u_m'(t)|^p\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_\Omega |\nabla u_m'(t)|^p\,\mathrm{d}x\,\mathrm{d}t + E_m(T) \leq E_m(0) \leq C. \tag{15}$$

and then we have

$$\begin{aligned} u_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;L^{r+1}(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega)), && (16)\\ u_m'(t) &\quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)), && (17)\\ -\Delta_p u_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;W^{-1,q}(\Omega)), && (18)\\ -\Delta_p u_m'(t) &\quad \text{is bounded in} \quad L^q(0,T;W^{-1,q}(\Omega)), && (19)\\ |u_m(t)|^{r-1}u_m(t) &\quad \text{is bounded in} \quad L^\infty(0,T;L^{\frac{r+1}{r}}(\Omega)). && (20) \end{aligned}$$

Now we are going to obtain an estimate for $u_m''(t)$. Since our Galerkin basis was taken in the Hilbert space $L^2(\Omega)$ we can use the standard projection arguments as described in Lions [17]. Then from the approximate equation and the estimates (16)-(18) we get

$$u_m''(t) \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \tag{21}$$

Passage to the Limit: From (16)-(20) going to the suitable subsequence if necessary (which we continue to denote in the same way), there exists u such that

$$\begin{aligned} u_m(t) &\rightharpoonup u(t) \;\text{ weakly star in } L^\infty(0,T;L^{r+1}(\Omega)) \cap L^\infty(0,T;W_0^{1,p}(\Omega)), && (22)\\ u_m'(t) &\rightharpoonup u'(t) \;\text{ weakly star in } L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega)), && (23)\\ -\Delta_p u_m(t) &\rightharpoonup \mathcal{X}_1(t) \;\text{ weakly star in } L^\infty(0,T;W^{-1,q}(\Omega)), && (24)\\ -\Delta_p u_m'(t) &\rightharpoonup \mathcal{X}_2(t) \;\text{ weakly star in } L^q(0,T;W^{-1,q}(\Omega)), && (25)\\ |u_m(t)|^{r-1}u_m(t) &\rightharpoonup \mathcal{X}_3(t) \;\text{ weakly star in } L^\infty(0,T;L^{\frac{r+1}{r}}(\Omega)). && (26) \end{aligned}$$

Applying the Lions-Aubin compactness lemma [17], we get respectively from (22)-(23) and (23)-(21),

$$\begin{aligned} u_m(t) &\to u(t) \;\text{ strongly in } L^2(0,T;L^2(\Omega)), && (27)\\ u_m'(t) &\to u'(t) \;\text{ strongly in } L^2(0,T;L^2(\Omega)). && (28) \end{aligned}$$

We need to prove that $\mathcal{X}_1(t) = -\Delta_p u(t)$. We denote

$$\Gamma = \left|\int_0^T \langle -\Delta_p u_m(t), v\rangle_p - \langle -\Delta_p u(t), v\rangle_p \theta(t)\,\mathrm{d}t\right|.$$

Consider $\forall\ x, y \in \mathbb{R}^n, 2 \leq p$ the elementary inequality that is a consequence of the mean value theorem

$$\left||x|^{p-2}x - |y|^{p-2}y\right| \leq c\ \left(|x|^{p-2} + |y|^{p-2}\right)|x-y|. \tag{29}$$

Applying (29) and Hölder generalized inequality with

$$\frac{p-2}{2(p-1)} + \frac{1}{2} + \frac{1}{2(p-1)} = 1$$

we deduce

$$\begin{aligned}
\Gamma &= \left|\int_0^T \langle(-\Delta_p u_m(t)) - (-\Delta_p u(t)), v\rangle_p \theta(t)\,\mathrm{d}t\right| \\
&= \left|\int_0^T \int_\Omega \left(|\nabla u_m(t)|^{p-2}\nabla u_m(t) - |\nabla u(t)|^{p-2}\nabla u(t)\right)\nabla v\,\mathrm{d}x\,\theta(t)\,\mathrm{d}t\right| \\
&\leq c||\theta||_\infty \int_0^T \int_\Omega \left(|\nabla u_m(t)|^{p-2} + |\nabla u(t)|^{p-2}\right)|\nabla u_m(t) - \nabla u(t)||\nabla v|\,\mathrm{d}x\,\mathrm{d}t \\
&\leq C\int_0^T \left(\|\nabla u_m(t)\|_{2(p-1)}^{p-2} + \|\nabla u(t)\|_{2(p-1)}^{p-2}\right)|\nabla u_m(t) - \nabla u(t)|_2 \|\nabla v\|_{2(p-1)}\,\mathrm{d}t.
\end{aligned}$$

From

$$E_m(t) + \int_0^t ||\nabla u_m'||_p^p = E_m(0),$$

we get $E_m(t) < d$ for all $t \in [0,T], T > 0$. We will to prove that $u_m(t) \in \mathcal{W}_1$ on $[0,T], T > 0$. By contradiction argument, suppose that there exists $t_0 \in (0,t)$ such that $u_m(t_0) \notin \mathcal{W}_1$. Since $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \{\ \}$, then $u(t_0) \in \mathcal{W}_2$ so $\frac{1}{p}||\nabla u_m(t_0)||_p^p < \frac{1}{r+1}||u_m(t_0)||_{r+1}^{r+1}$. The initial data $u_0 \in \mathcal{W}_1$ implies $\frac{1}{p}||\nabla u_m(0)||_p^p > \frac{1}{r+1}||u_m(0)||_{r+1}^{r+1}$ and so, there exists $s \in (0,t_0)$ such that $\frac{1}{p}||\nabla u_m(s)||_p^p = \frac{1}{r+1}||u_m(s)||_{r+1}^{r+1}$. Consider

$$\tau = \sup\left\{s \in (0,t_0)\ :\ \frac{1}{p}||\nabla u_m(s)||_p^p = \frac{1}{r+1}||u_m(s)||_{r+1}^{r+1}\right\}.$$

We have $u_m(t) \in \mathcal{W}_2$ for all $\tau < t \leq t_0$. On the one hand, if $||\nabla u_m(\tau)||_p^p \neq 0$, then $u_m(\tau) \in \mathcal{N}$ the Nehari Manifold and

$$E_m(\tau) = \frac{1}{2}|u_m'(\tau)|_2^2 + J(u_m(\tau)) \geq J(u_m(\tau)) \geq \inf_{u_m \in \mathcal{N}} J(u_m) = d,$$

that is a contradiction with $E_m(t) < d$. Applying Sobolev imbedding with $p \geq r+1$ and Poincarè inequality we get

$$\frac{1}{p}||\nabla u_m(t)||_p^p < \frac{1}{r+1}||u_m(t)||_{r+1}^{r+1} \leq C||u_m(t)||_p^{r+1} \leq C||\nabla u_m(t)||_p^{r+1},\ \text{for all } \tau < t \leq t_0.$$

On the other hand, if $||\nabla u_m(\tau)||_p^p = 0$, we have

$$0 < \lim_{t\to\tau_+} C||\nabla u_m(t)||_p^{r+1} = C||\nabla u_m(\tau)||_p^{r+1}$$

and we obtain a contradiction again. Then $u_m(t) \in \mathcal{W}_1$ and in particular $u_m(t) \in W^{1,2(p-1)}$ from where follows that

$$\left|\int_0^T \langle -\Delta_p u_m(t), v\rangle_p - \langle -\Delta_p u(t), v\rangle_p \theta(t)\,\mathrm{d}t\right| \leq C\int_0^T |\nabla u_m(t) - \nabla u(t)|_2\,\mathrm{d}t.$$

Using (27) we get that $u_m(t) \to u(t)$ almost everewhere in $\Omega \times (0,T)$ which leads to $\mathcal{X}_1(t) = -\Delta_p u(t)$.

Applying (29), Hölder generalized inequality as before and (28) we obtain

$$\left|\int_0^T \langle -\Delta_p u_m'(t), v\rangle_p - \langle -\Delta_p u'(t), v\rangle_p \theta(t)\,\mathrm{d}t\right| \leq c\int_0^T |\nabla u_m'(t) - \nabla u'(t)|_2\,\mathrm{d}t \quad\to\quad 0.$$

from where follows by the same argument before, $\mathcal{X}_2(t) = -\Delta_p u'(t)$.

To prove that $\mathcal{X}_3(t) = |u(t)|^{r-1}u(t)$ observe that,

$$\int_0^T \int_\Omega \left||u_m(t)|^{r-1}u_m(t)\right|^{\frac{r+1}{r}} \mathrm{d}x = \int_0^T \int_\Omega |u_m(t)|^{r+1} \mathrm{d}x \leq c$$

so $|u_m(t)|^{r-1}u_m(t) \to |u(t)|^{r-1}u(t)$ almost everywhere in $\Omega \times [0,t)$. Therefore from [17] Lemma 1.3, we infer that

$$|u_m(t)|^{r-1}u_m(t) \to |u(t)|^{r-1}u(t) \ \text{ weakly in in } L^{\frac{r+1}{r}}(0,T; L^{\frac{r+1}{r}}(\Omega)) \tag{30}$$

so we have from (26) and (30) that $\mathcal{X}_3 = |u(t)|^{r-1}u(t)$.

Now, with the convergences (23),(24),(25) and (26) we can pass to the limit in the approximate equation (7) and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(u'(t),v) + \langle -\Delta_p u(t), v\rangle + \langle -\Delta u'(t), v\rangle_p - (|u(t)|^{r-1}u(t), v) = 0$$

for all $v \in W_0^{1,p}(\Omega) \cap W^{1,2(p-1)}(\Omega)$ in $D'(0,T)$ at the sense of distributions. The verification of the initial data is a routine procedure. The prove of existence is complete.

□

4 Asymptotic behaviour

We use the following result due to M. Nakao, see [[20], Lemma 2].

Lemma 4.1. *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}$, satisfying*

$$\sup_{t\leq s\leq t+1} \text{ess}\ \phi(s)^{1+\alpha} \leq C_0[\phi(t) - \phi(t+1)] \quad for\ t \geq 0,$$

where C_0 and α are positive constants. Then

$$\phi(t) \leq C(1+t)^{-\frac{1}{\alpha}},\ \forall t \geq 0,$$

where C is a positive constants.

Theorem 4.1. *Under the hypothesis of Theorem 3.1, the solution $u(t)$ of problem (2)-(4) satisfies*

$$E(t) \leq C_0(1+t)^{-\frac{1}{\alpha}},\ \forall t \geq 0,$$

where C_0 and α are positive constants.

Proof. Multiplying (2) by $u'(t)$ and integrating by over Ω we have

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega |u'(t)|_2^2 + \frac{1}{p}||\nabla u(t)||_p^p - \frac{1}{r+1}|u_m(t)|_{r+1}^{r+1} + ||\nabla u'(t)||_p^p\right] = 0$$

that is

$$\frac{d}{dt}E(t) + ||\nabla u'(t)||_p^p = 0.$$

Integrating from t to $t+1,\ t \geq 0$, we obtain

$$E(t+1) + \int_t^{t+1} ||\nabla u'(s)||_p^p \,\mathrm{d}s = E(t). \tag{31}$$

So,

$$\int_t^{t+1} |u'(s)|_2^2 \,\mathrm{d}s \leq C \int_t^{t+1} ||\nabla u'(s)||_p^p \,\mathrm{d}s = C[E(t) - E(t+1)] \stackrel{\text{def}}{=} F^2(t). \tag{32}$$

Thus, there exist $t_1 \in [t, t+\frac{1}{4}], t_2 \in [\frac{3}{4}, t+1]$ such that

$$|u'(t_i)|_2 \leq 2F(t_i),\ i = 1, 2. \tag{33}$$

Multiplying (2) by $u(t)$ and integrating by over Ω we have

$$\frac{d}{dt}(u'(t), u(t)) - |u'(t)|_2^2 + ||\nabla u(t)||_p^p - |u(t)|_{r+1}^{r+1} + \int_\Omega |\nabla u'(s)|^{p-2} \nabla u'(s)\,.\nabla u(s) \,\mathrm{d}s = 0. \tag{34}$$

Using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \int_\Omega |\nabla u'(s)|^{p-2} \nabla u'(s)\,.\nabla u(s) \,\mathrm{d}s &\leq C||\nabla u'(s)||_p^{p-1} ||\nabla u(s)||_p \\ &\leq \frac{C^q}{q} ||\nabla u'(s)||_p^{(p-1)q} + \frac{1}{p} ||\nabla u(s)||_p^p \\ &= C_1 ||\nabla u'(s)||_p^p + \frac{1}{p} ||\nabla u(s)||_p^p. \end{aligned} \tag{35}$$

Integrating (34) from t_1 to t_2 and using (32),(33) and (35) we get

$$\begin{aligned} \int_{t_1}^{t^2} \left[||\nabla u(s)||_p^p - |\nabla u(s)|_{r+1}^{r+1} \right] \mathrm{d}s &\leq C|u'(t_1)|_2 ||\nabla u(t_1)||_p + C|u'(t_2)|_2 ||\nabla u(t_2)||_p \\ &\quad + F^2(t) + C_1 \int_{t_1}^{t^2} ||\nabla u'(s)||_p^p \,\mathrm{d}s + \frac{1}{p} \int_{t_1}^{t^2} ||\nabla u(s)||_p^p \,\mathrm{d}s \\ &\leq F^2(t) + 4CF(t) \sup_{t \leq s \leq t+1} \text{ess} \sqrt[p]{E(s)} + \frac{1}{p} \int_{t_1}^{t^2} ||\nabla u(s)||_p^p \,\mathrm{d}s. \end{aligned} \tag{36}$$

Therefore,

$$\int_{t_1}^{t^2} \left[(1 - \frac{1}{p}) ||\nabla u(s)||_p^p - |\nabla u(s)|_{r+1}^{(r+1} \right] \mathrm{d}s \leq C_2 F^2(t) + 4CF(t) \sup_{t \leq s \leq t+1} \text{ess} \sqrt[p]{E(s)} \stackrel{\text{def}}{=} G^2(t). \tag{37}$$

From (32) and (38) follows that

$$\int_{t_1}^{t^2} E(s) \,\mathrm{d}s \leq C_3 [F^2(t) + G^2(t)].$$

Then, there exists $t^* \in [t_1, t_2]$ such that

$$E(t^*) \leq C_4 [F^2(t) + G^2(t)].$$

From (31) we have

$$\sup_{t \leq s \leq t+1} \text{ess}\, E(s) \leq E(t^*) + \int_t^{t+1} ||\nabla u'(s)||_p^p \,\mathrm{d}s,$$

so,

$$\begin{aligned}
\sup_{t\le s\le t+1} \text{ess } E(s) &\le C_5[F^2(t)+G^2(t)] \\
&= C_6\left[F^2(t)+4F(t)\sup_{t\le s\le t+1}\text{ess } \sqrt[p]{E(s)}\right] \\
&\le C_6F^2(t)+C_7F^q(t)+\frac{1}{p}\sup_{t\le s\le t+1}\text{ess } E(s) \\
&\le C_8F^q(t)[F^{2-q}(t)+1].
\end{aligned}$$

Then, we have

$$F^2(t) \stackrel{\text{def}}{=} \int_t^{t+1} ||\nabla u'(s)||_p^p \, \mathrm{d}s \le \int_0^{t+1} ||\nabla u'(s)||_p^p \, \mathrm{d}s = E(0)-E(t+1) \le E(0),$$

whence

$$F^{2-q}(t) = \left(F^2(t)\right)^{\frac{2-q}{2}} \le (E(0))^{\frac{2-q}{2}} = C(E(0)).$$

Therefore,

$$\sup_{t\le s\le t+1}\text{ess } E(s) \le C(E(0))(F(t))^q = C(E(0))\left(F^2(t)\right)^{\frac{q}{2}}$$

$$\sup_{t\le s\le t+1}\text{ess } E^{\frac{2}{q}}(s) \le C_0[E(t)-E(t+1)].$$

Observe that $p \ge 2$ implies

$$\frac{2}{q} > 1, \text{ that is, } \frac{2}{q} = 1+\alpha, \text{ with, } \alpha > 0.$$

Finally,

$$\sup_{t\le s\le t+1}\text{ess } E^{1+\alpha}(s) \le C_0[E(t)-E(t+1)].$$

The proof is complete. □

Final comments

The force term can be more general that $|u|^{r-1}u$, type $f \in C^1(\mathbb{R})$ with $|f(u)| \le c|u|^r$ for all $|u| \ge 1$, where $1 \le r < 6$ with the suitable Sobolev imbeddings. For instance, for $n = 3$ we have $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and the Nemytskii operator $f(u)$ is locally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$ for $1 \le r \le 3$. We call the source sub-critical when $1 \le r < 3$, critical if $r = 3$ and supercritical for $3 < r \le 5$. When $5 < r < 6$ the source is super-supercritical, see [7] and is the situation where the potential energy may not be defined in the finite energy space, so the problem itself is no longer in the framework of the potential well theory.

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