Inverse Source Problem with Many Frequencies and Attenuation for One-Dimensional Domain

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Abstract

In this article, we consider the one dimensional inverse source problem for Helmholtz equation with attenuation (damping) factor in a one layer medium. We establish a stability by using multiple frequencies at the two end points of the domain which contains the compact support of the source functions. The main result is an estimate which consists of two parts: the data discrepancy and the high frequency tail. We show that increasing stability possible using multi-frequency wave at the two endpoints.

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1 Introduction and statement of problem

We consider the one dimensional attenuated Helmholtz equation (with attenuation factor b) in a one layered medium:

$$u(x,\omega)'' + (k^2 + b)u(x,\omega) = -f_1 - bf_0 + ikf_0, \quad x \in (-1,1), \tag{1}$$

where the wave field u is required to satisfy the outgoing wave conditions:

$$u'(-1,\omega) + iku(-1,\omega) = 0, \qquad u'(1,\omega) - iku(1,\omega) = 0. \tag{2}$$

Now let

$$u^*(x,\kappa) := u(x,k), \quad \kappa^2 := k^2 + bki, \tag{3}$$

then the equation (1) becomes

$$u^{*\prime\prime} + \kappa^2 u^* = -f_1 - bf_0 + ikf_0. \tag{4}$$

Using standard notation of Helmholtz equation, $\kappa = c\omega$, $\omega > 0$ is the angular frequency and c are constants. Given $f_0, f_1 \in H^1(-1,1)$, it is well-known that the problem (3) has a unique solution considering radiation conditions (2)(see [22]);

$$u(x,\omega) = \int_{-1}^{1} G(x-y)(-f_1 - bf_0 + ikf_0)(y)dy, \tag{5}$$

where $G(x)$ is the Green function given as follows

$$G(x) = \frac{ie^{ik|x|}}{2k}. \tag{6}$$

In this research, we consider the inverse source problem when the source functions f_1, f_0 are complex functions with a compact supports contained in $(-1, 1)$. Our main goal is to recover the source functions f_1, f_0 using the boundary data $u(1, \omega)$ and $u(-1, \omega)$ with $\omega \in (0, K)$ where $K > 1$ is a positive constant.

In application, inverse source problem arises in many areas of science. It has numerous applications in acoustical and biomedical/medical imaging, antenna synthesis, geophysics, and material science ([2, 3]). It has been shown that the data of the inverse source problems for Helmholtz equations with single frequency can not guarantee the uniqueness ([15], Ch.4). On the other hand, various studies, for instance in [4], showed that the uniqueness can be regained by taking multifrequency boundary measurement in a non-empty frequency interval $(0, K)$ noticing the analyticity of wave-field on the frequency [15, 19] and [14]. Because of the wide applications, these problems have being attracted considerable attention. In the papers [1, 5, 6, 8, 10, 11, 12, 16, 17, 18, 20, 23] and [22], these kind of problems have been extensively investigated recently. The applied technique is also can be used for system of equations see for example [13] where inverse sources problems were considered regarding the classical elasticity system.

The main purpose of this article is to prove the following theorem.

Theorem 1.1. *There exists a generic constant C depending on the domain Ω such that*

$$\| f_0 \|^2_{(0)} (-1,1) + \| f_1 \|^2_{(0)} (-1,1) \leq Ce^{Cb^2}\Big(\epsilon^2 + \frac{(b^2+1)M^2}{K^{\frac{2}{3}}E^{\frac{1}{4}}+1}\Big), \tag{7}$$

for all $u \in H^2(\Omega)$ solving (1) with $K > 1$. Here

$$\epsilon^2 = \int_0^K \omega^2\big(|u(1,\omega)|^2 + |u(-1,\omega)|^2\big)d\omega,$$

$E = -ln\epsilon$ and $M = \max\big\{ \| f_0 \|_{(2)} (-1,1) + \| f_1 \|_{(1)} (-1,1), 1\big\}$ where $\| \, . \, \|_{(l)} ((-1,1))$ is the standard Sobolev norm in $H^l((-1,1))$.

To proof our main theorem, let's define the following functions:

$$f_{1p} = \begin{cases} f_1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \quad f_{1n} = \begin{cases} 0 & \text{if } x > 0, \\ f_1 & \text{if } x < 0, \end{cases}$$

$$f_{0p} = \begin{cases} f_0 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \quad f_{0n} = \begin{cases} 0 & \text{if } x > 0, \\ f_0 & \text{if } x < 0. \end{cases}$$

Remark 1.1: The estimate in (7) consists of two parts: the data discrepancy and the high frequency part. The first part is of the LIpschitz type. The second part is of logarithmic type. The second part decrease as K increases which makes the problem more stable. The estimate (7) also implies the uniqueness of the inverse source problem.

2 Proof of Theorem 1.1

2.1 Increasing Stability of Continuation to higher frequencies

Let

$$I(k) = I_1(k) + I_2(k)$$

where

$$I_1(k) = \int_0^k \omega^2 |u(-1,\omega)|^2 d\omega, \qquad I_2(k) = \int_0^k \omega^2 |u(1,\omega)|^2 d\omega, \tag{8}$$

using (5) and a simple calculation shows that

$$\omega u(1,\omega) = \int_0^1 \frac{i}{2c} e^{ic\omega(1-y)}(-f_{1p} - bf_{0p} + ikf_{0p})(y)dy \tag{9}$$

$$\omega u(-1,\omega) = \int_0^1 \frac{i}{2c} e^{ic\omega(y+1)}(-f_{1n} - bf_{0n} + ikf_{0n})(y)dy \tag{10}$$

where $y \in (-1,1)$. Functions I_1 and I_2 are both analytic with respect to the wave number $k \in \mathbb{C}$ and play important roles in relating the inverse source problems of the Helmholtz equation and the Cauchy problems for the wave equations.

Lemma 2.1. *Let $supp f_0, supp f_1 \subset (-1,1)$ and $f_0 \in H^0(-1,1)$, $f_1 \in H^0(-1,1)$. Then*

$$|I_1(k)| \leq C\Big(|k| \parallel f_1 \parallel^2_{(0)} (-1,1) + (|k|b^2 + \frac{|k|^3}{3}) \parallel f_0 \parallel^2_{(0)} (-1,1)\Big)e^{4c\left(4k_1+b\right)}, \tag{11}$$

$$|I_2(k)| \leq C\Big(|k| \parallel f_1 \parallel^2_{(0)} (-1,1) + (|k|b^2 + \frac{|k|^3}{3}) \parallel f_0 \parallel^2_{(0)} (-1,1)\Big)e^{4c\left(4k_1+b\right)}, \tag{12}$$

Proof. We have $\kappa = \kappa_1 + \kappa_2 i = \sqrt{k^2+bki}$ is complex analytic on $\mathbb{C} \setminus [0,-bi]$ and in particular on the set $\mathbb{S} \setminus [0,k]$, where $\mathbb{S}$ is the sector $\{|\arg k| < \pi/4\}$ with $k = k_1 + ik_2$. It is easy to see that $|\kappa| = |k|^{\frac{1}{2}}|k+bi|^{\frac{1}{2}} \leq 2k_1^{1/2}(k_1^{1/2}+b)^{1/2}$ and $|k| \leq \sqrt{2}k_1 \leq \sqrt{2}|\kappa|$ for any k in $\mathbb{S}$. By a simple calculation and (5), we can show that

$$I_1(k) = \int_0^k |\int_0^1 \frac{1}{2c} e^{ic\omega(y+1)}(-f_{1p} - bf_{0p} + ikf_{0p})(y)dy|^2 d\omega. \tag{13}$$

Since the integrands in (13) are analytic functions of k in $\mathbb{S}$, their integrals with respect to ω can be taken over any path in $\mathbb{S}$ joining points 0 and k in the complex plane. Using the change of variable $\omega = ks$, $s \in (0,1)$ in the line integral (3) we obtain

$$|I_1(k)| \leq \int_0^1 |k| \Big| \int_0^1 \frac{1}{2c} e^{ic\omega(y+1)}(-f_{1p} - bf_{0p} + ikf_{0p})(y)dy\Big|^2 d\omega,$$

using the following inequalities for $y \in (-1,1)$

$$|e^{\pm ic\omega(y+1)}| \leq e^{2c|\kappa_2|}$$

it is easy to drive that

$$|I_1(k)| \leq \int_0^1 |k| \int_{-1}^1 \Big(|f_1(y)| + (b + s|k|)|f_o(y)|e^{2c|\kappa_2|}dy\Big)^2 ds,$$

and integrating with respect to s, using the bound for $|\kappa|$ in $\mathbb{S}$ and trivial inequality $2ab \leq a^2 + b^2$, we complete the proof of (11).
Similarly for $y \in (-1, 1)$ we have

$$|e^{\pm ic\omega(1-y)}| \leq e^{2c|\kappa_2|},$$

using the same argument for $I_2(k)$, the proof of (12) is complete. □

Obviously functions $I_1(k), I_2(k)$ are analytic functions of $k = k_1 + ik_2 \in \mathbb{S}$ and $|k_2| \leq k_1$. The following steps are essential to link the unknowns $I_1(k)$ and $I_2(k)$ for $k \in [K, \infty)$ to the known value ϵ in (1). Clearly

$$\begin{aligned} &|I_1(k)e^{-16ck}| \leq \\ &C\Big(|k| \parallel f_1 \parallel^2_{(0)} (-1, 1) + (|k|b^2 + \frac{|k|^3}{3}) \parallel f_0 \parallel^2_{(0)} (-1, 1)\Big)e^{4cb}, \\ &\leq Cb^2e^{4cb}M^2, \end{aligned} \tag{14}$$

where $M = \max\big\{ \parallel f_1 \parallel^2_{(0)} (-1, 1) + \parallel f_0 \parallel^2_{(0)} (-1, 1), 1\big\}$. With the similar argument bound (14) is true for $I_2(k)$. Observing that

$$|I_1(k)e^{-2k}| \leq \epsilon^2, \quad |I_2(k)e^{-2k}| \leq \epsilon^2 \ \ on\ [0, K].$$

Lets define $\mu(k)$ be the harmonic measure of the interval $[0, K]$ in $\mathbb{S}\backslash[0, K]$, then as known (for example see [15], p.67), from two previous inequalities and analyticity of the function $I_1(k)e^{-2k}$ and $I_2(k)e^{-2k}$, we can derive that

$$|I_1(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2, \tag{15}$$

when $K < k < +\infty$. Similarly it also yields for

$$|I_2(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2, \tag{16}$$

consequently

$$|I(k)e^{-2k}| \leq C\epsilon^{2\mu(k)}M^2. \tag{17}$$

To achieve a lower bound of the harmonic measure $\mu(k)$, we use the following lemma. The proof can be found in [8].

Lemma 2.2. *Let $\mu(k)$ be the harmonic measure of the interval $[0, K]$ in $\mathbb{S}\backslash[0, K]$, then*

$$\begin{cases} \frac{1}{2} \leq \mu(k), & if \quad 0 < k < 2^{\frac{1}{4}}K, \\ \frac{1}{\pi}\Big(\big(\frac{k}{K}\big)^4 - 1\Big)^{\frac{-1}{2}} \leq \mu(k), & if \quad 2^{\frac{1}{4}}K < k. \end{cases} \tag{18}$$

2.1.1 Exact observability bound for wave equation with damping factor

To derive the bound for higher frequency, we consider the hyperbolic initial value problem

$$\partial_t^2 U(x,t) - U''(x,t) + b\partial_t U(x,t) = 0 \ on \ (-1,1)\times(0,\infty), \tag{19}$$

$$U(x,0) = f_0(x),\ \partial_t U(x,0) = f_1(x) \ on \ (-1,1).$$

It is easy to see that if $b = 0$, the exact observability bounds for the hyperbolic equation can be found in [8, 12, 11]. The following theorem presents a generalized result which is interesting in itself.

Theorem 2.3. *Let the observation time $4(D+1) < T < 5(D+1)$. Then there exists a generic constant C depending on the domain Ω such that*

$$\| f_0 \|_{(1)}^2 ((-1,1)) + \| f_1 \|_{(0)}^2 ((-1,1)) \leq Ce^{Cb^2}\Big(\| \partial_t U \|_{(0)}^2 (\partial\Omega\times(0,T)) + \| U \|_{(0)}^2 (\partial\Omega\times(0,T))\Big), \tag{20}$$

for all $U \in H^2((-1,1)\times(0,\infty))$ solving (19), where $\partial\Omega$ represents the set of endpoints.

Proof. The proof can be found in ([17], Lemma 3.3, Theorem 3.1). □

2.2 Increasing stability for inverse source problem

To continue estimating the reminders, we consider the following hyperbolic initial value problem

$$\partial_t^2 U - U'' + b\partial_t U = 0 \ on \ \mathbb{R}\times(0,\infty),\ U(x,0) = f_0(x),\ \partial_t U(x,0) = f_1(x) \ on \ \mathbb{R} \tag{21}$$

Defining $U(x,t) = 0$ for $t < 0$. We claim that the solution of (1) coincides with the Fourier transform of U;

$$u(x,k) = \int_{-\infty}^{\infty} U(x,t)e^{ikt}dt. \tag{22}$$

Known results in [9] (see Theorem 1.1. and Theorem 1.2.), [8] and the assumption on the functions f_0, f_1 imply that

$$\| U(.,t) \|_{(0)} \leq C(f_0,f_1)(1+t)^{-\frac{1}{4}},$$

Using the same idea and same proof in [12], (22) is straightforward.
To continue the estimate for reminders of the whole integrands in (8) for (k,∞), we need the following lemma.

Lemma 2.4. *Let u be a solution to the forward problem (1) with $f_1 \in H^1((-1,1))$ and $f_0 \in H^2((-1,1)))$ with $supp f_0, supp f_1 \subset (-1,1)$, then*

$$\int_k^{\infty} \omega^2 |u(-1,\omega)|^2 d\omega + \int_k^{\infty} \omega^2 |u(1,\omega)|^2 d\omega \tag{23}$$

$$\leq Ck^{-1}\Big((1+b^2) \| f_0 \|_{(2)}^2 (-1,1) + \| f_1 \|_{(1)}^2 (-1,1)\Big),$$

Proof. To proof the lemma, let $k_1 = k_2 = k$ in [12]. □

Now, we are ready to proof Theorem 1.1.

Proof. We can assume that $\epsilon < 1$ and $3\pi E^{-\frac{1}{4}} < 1$, otherwise the Theorem 1.1 is obvious. Let

$$k = \begin{cases} K^{\frac{2}{3}}E^{\frac{1}{4}} & \text{if} \quad 2^{\frac{1}{4}}K^{\frac{1}{3}} < E^{\frac{1}{4}} \\ K & \text{if} \quad E^{\frac{1}{4}} \leq 2^{\frac{1}{4}}K^{\frac{1}{3}}, \end{cases} \tag{24}$$

if $E^{\frac{1}{4}} \leq 2^{\frac{1}{4}}K^{\frac{1}{3}}$, then $k = K$, using the (15) and (17), we can conclude

$$|I(k)| \leq 2\epsilon^2. \tag{25}$$

If $2^{\frac{1}{4}}K^{\frac{1}{3}} < E^{\frac{1}{4}}$, we can assume that $E^{-\frac{1}{4}} < \frac{1}{4\pi}$, otherwise $C < E$ and hence $K < C$ and the bound (7) is straightforward. From (24), Lemma 2.2, (15) and the equality $\epsilon = \frac{1}{e^E}$ we obtain

$$|I_1(k)| \leq CM^2b^2e^{4b}e^{4k}e^{\frac{-2E}{\pi}\left((\frac{k}{K})^4-1\right)^{\frac{-1}{2}}}$$

$$\leq CM^2e^{4b}b^2e^{-\frac{2}{\pi}K^{\frac{2}{3}}E^{\frac{1}{2}}(1-\frac{5\pi}{2}E^{\frac{-1}{4}})},$$

using the trivial inequality $e^{-t} \leq \frac{6}{t^3}$ for $t > 0$ and the assumption at the beginning of the proof, we conclude that

$$|I_1(k)| \leq CM_0^2b^2e^{2b}\frac{1}{K^2E^{\frac{3}{2}}\left(1-\frac{5\pi}{2}E^{-\frac{1}{4}}\right)^3}. \tag{26}$$

Using (13), (25), (26), and Lemma 4.1 we obtain

$$\int_0^{+\infty}\omega^2|u(-1,\omega)|^2d\omega + \int_0^{+\infty}\omega^2|u(1,\omega)|^2d\omega \tag{27}$$

$$\leq I(k) + \int_k^{\infty}\omega^2|u(-1,\omega)|^2d\omega + \int_k^{\infty}\omega^2|u(1,\omega)|^2d\omega$$

$$\leq 2\epsilon^2 + C\Big(\frac{(b^2+1)M^2e^{4b}}{K^2E^{\frac{3}{2}}} + \frac{(b^2+1)\parallel f_0\parallel_{(2)}^2 + \parallel f_1\parallel_{(1)}^2}{K^{\frac{2}{3}}E^{\frac{1}{4}}+1}\Big).$$

Application of (27) and Theorem 3.1 finally lead to

$$\parallel f_1\parallel_{(0)}^2(\Omega) + \parallel f_0\parallel_{(1)}^2(\Omega) \leq Ce^{b^2}\Big(\parallel\partial_tU\parallel_{(0)}^2(\partial\Omega\times(0,T)) + \parallel U\parallel_{(0)}^2(\partial\Omega\times(0,T))\Big)$$

$$\leq Ce^{\alpha^2}\Big(\parallel\partial_tU\parallel_{(0)}^2(\partial\Omega\times(0,\infty)) + \parallel U\parallel_{(0)}^2(\partial\Omega\times(0,\infty))\Big)$$

$$\leq Ce^{b^2}\Big(\epsilon^2 + \frac{(b^2+1)M^2e^{8b}}{K^2E^{\frac{3}{2}}} + \frac{(b^2+1)\parallel f_0\parallel_{(2)}^2 + \parallel f_1\parallel_{(1)}^2}{K^{\frac{2}{3}}E^{\frac{1}{4}}+1}\Big),$$

due to the Parseval's identity. Since $K^{\frac{2}{3}}E^{\frac{1}{4}} < K^2E^{\frac{3}{2}}$ for $1 < K, 1 < E$, the proof is complete.

□

3 Concluding Remarks

In this paper, we studied the inverse source problem with many frequencies and damping factor in a one dimensional domain. The result showed that if K grows the estimate improves. It also showed that if we have data for all wave number, that is $k \in (0, \infty)$, the estimate will be a Lipschitz estimate. The next challenge is recovering the source function for the inverse source problem for higher dimensional domains or more than two layers mediums. These kind of problem has application in material science.

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