Existence of Positive Solutions for Higher Order Boundary Value Problems on Time Scales

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Abstract

In this paper, we establish the existence of at least one positive solution for the higher order boundary value problems on time scales by using the Krasnoselskii fixed point theorem..

Key words: Positive solutions, cone theory, time scales.

AMS (MOS) Subject Classification. 34K11; 39A10; 39A99

1 Introduction

Boundary value problems (BVPs) play a major role in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications, as well as emerging technologies such as nanotechnology and biotechnology rely on the BVPs to simulate complex phenomena at different scales for design and manufactures of high-technology products. Due to their important role in both theory and applications, the BVPs have generated a great deal of interest in recent years. Recently, there is an increasing interest in the literature on multi-point BVPs for higher-order differential equations, see for example [5-7, 10, 11, 13]. In particular, we would like to mention some results of Graef and Yang [6], Guo et al. [7], and Su and Wang [13]. The calculus on time scales was introduced by Stefan Hilger in his Ph.D thesis in order to create a theory that can unify discrete and continuous analysis. A time scale is an arbitrary nonempty closed subset of real numbers. Thus the real numbers, the integers, the natural numbers, the closed intervals, the Cantor set, i.e. are examples of time scales. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the reals. By choosing the time scale to be the set of real numbers, the general result yields a result concerning an ordinary differential equation, and by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. We may summarize the above and state that "Unification and Extension" are the two main features of the time scales calculus. Hence, the study of dynamic equations on time scales is worthwhile and has theoretical and practical values. Recently, for the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results.

Now, we will give some basic definitions on time scale calculus.

Definition 0.1. *Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ by*

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\} \ for \ all \ t \in \mathbb{T}.$$

If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function, then it is often convenient to use the notation f^σ to denote the function defined by the composition $f \circ \sigma$, that is

$$f^\sigma(t) = f \circ \sigma(t) = f(\sigma(t)),$$

for $t \in \mathbb{T}$.

We also need the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 0.2. *Assume $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$)*

$$|\,[f(\sigma(t)) - f(s)] - f^\Delta(t)\,[\sigma(t) - s]\,| \leq \epsilon|\sigma(t) - s| \ for \ all \ s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative if f at t.

Also f is delta differentiable at t iff the limit

$$lim_{s\to t}\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists a finite number.

Definition 0.3. *For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ we shall talk about second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa^2} \longrightarrow \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \longrightarrow \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$ and $\sigma^n(t)$ for $n \in \mathbb{N}$ are defined accordingly. For convenience we also put*

$$\sigma^0(t) = t, \ f^{\Delta^0} = f, \ and \ \mathbb{T}^{\kappa^0} = \mathbb{T}.$$

Definition 0.4. *Let $F : \mathbb{T} \longrightarrow \mathbb{R}$ be antiderivative of f such that $F^\Delta(t) = f(t)$ for $t \in \mathbb{T}^\kappa$. We define the definite delta integral of a function $f(t)$ by*

$$\int_r^s f(t)\Delta t = F(s) - F(r) \ for \ all \ r, s \in \mathbb{T}.$$

Throughout this paper, by an interval (a, b) we mean the intersection of the real interval (a, b) with the given time scale $\mathbb{T}$.

In this work, we shall consider the existence of positive solutions for the following n-th order boundary value problem

$$u^{\Delta^n}(t) + f(t, u(t), ..., u^{\Delta^{n-2}}(t)) = 0, \qquad t \in (a, b),\ n \geq 2 \tag{0.1}$$

$$\begin{aligned} u^{\Delta^i}(a) = 0, \quad 0 \leq i \leq n-3 \\ \alpha u^{\Delta^{n-2}}(a) - \beta u^{\Delta^{n-1}}(a) = 0, \\ \gamma u^{\Delta^{n-2}}(\sigma(b)) + \delta u^{\Delta^{n-1}}(\sigma(b)) = 0, \end{aligned} \tag{0.2}$$

where $f \in C([a, b] \times [0, \infty)^{n-1}; [0, \infty))$, $a, b \in \mathbb{T}$ and $\alpha, \gamma > 0,\ \beta, \delta \geq 0$. We assume $\sigma(b)$ is right-dense so that $\sigma^j(b) = \sigma(b),\ j \geq 1$. Assume also that $[a, \sigma(b)]$ is such that $\xi := \min\left\{t \in \mathbb{T} : t \geq \frac{\sigma(b)+3a}{4}\right\}$ and $\omega := \max\left\{t \in \mathbb{T} : t \leq \frac{3\sigma(b)-a}{4}\right\}$ both exist and satisfy $\frac{\sigma(b)+3a}{4} \leq \xi \leq \omega \leq \frac{3\sigma(b)-a}{4}$ and if $\sigma(\omega) = b$, asume $\sigma(\omega) < \sigma(b)$. Next, let $t_0 \in [\xi, \omega]$ be defined by $\displaystyle\int_\xi^\omega k(t_0, s)\Delta s = \max_{t\in[\xi,\omega]} \int_\xi^\omega k(t, s)\Delta s.$

The outline of the paper is as follows. In the second section, we present some lemmas and theorems that will be used to prove our main results. In the third section, we will establish a new theorem of existence of positive solution for (0.1)-(0.2) and we will give an example to illustrate the main results in this paper.

1 The Green's function and bounds

In this section we discuss the n-th order problem (0.1)-(0.2).

Let $d = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(b) - a)$. In order to abbreviate our discussion, throughout this paper, we shall assume that the following assumptions hold:

K(t,s) is the Green's function for the dynamic equation

$$-u^{\Delta^n}(t) = 0, \qquad t \in (a, b) \tag{1.3}$$

subject to the boundary conditions (0.2);

k(t,s) is the Green's function for the dynamic equation

$$-u^{\Delta\Delta}(t) = 0, \qquad t \in (a, b) \tag{1.4}$$

subject to the boundary conditions

$$\begin{aligned} \alpha u(a) - \beta u^{\Delta}(a) = 0, \\ \gamma u(\sigma(b)) + \delta u^{\Delta}(\sigma(b)) = 0. \end{aligned} \tag{1.5}$$

In reference [3], the Green's function for the problem (2.4)-(2.5) was found such that

$$k(t,s) = \begin{cases} \frac{1}{d}\left(\alpha(t-a)+\beta\right)\left(\gamma(\sigma(b)-\sigma(s))+\delta\right), & t \le s \\ \frac{1}{d}(\alpha(\sigma(s)-a)+\beta)(\gamma(\sigma(b)-t)+\delta), & \sigma(s) \le t. \end{cases}$$

Lemma 1.1. [Lemma 7.28, 3] *If $d > 0$ and $\alpha, \beta, \gamma, \delta \ge 0$ the Green's function $k(t,s)$ for the problem (2.4)-(2.5) satisfies*

$$k(t,s) > 0, \quad (t,s) \in (a,\sigma(b)) \times (a,b);$$
$$\text{and } k(t,s) \le k(\sigma(s),s), \quad (t,s) \in [a,\sigma(b)] \times [a,b];$$
$$\text{and } k(t,s) \ge Mk(\sigma(s),s), \quad (t,s) \in \left[\tfrac{\sigma(b)+3a}{4}, \tfrac{3\sigma(b)-a}{4}\right] \times [a,b].$$

where

$$M := min\left\{\frac{\alpha(\sigma(b)-a)+4\beta}{4\alpha(\sigma(b)-a)+4\beta}, \frac{\gamma(\sigma(b)-a)+4\delta}{4\gamma(\sigma(b)-\sigma(a))+4\delta}\right\} \tag{1.6}$$

and $M \in (0,1)$.

Lemma 1.2. *Let $K(t,s)$ and $k(t,s)$ be the Green's function respectively for the problem (1.3)-(1.2) and (1.4)-(2.5), the following holds:*

$$\frac{\Delta^{n-2}}{\Delta t^{n-2}} K(t,s) = k(t,s), \quad t,s \in (a,b),$$

we means $\frac{\Delta^{n-2}}{\Delta t^{n-2}} K(t,s)$ that n-th order delta derivation of the function $K(t,s)$ for the first variable $t \in \mathbb{T}^{k^{n-2}}$.

Proof. Let $v(t)$ be the solution of the following second order boundary value problem;

$$u^{\Delta\Delta}(t) = h(t), \quad t \in (a,b) \quad ,$$
$$\alpha u(a) - \beta u^{\Delta}(a) = 0,$$
$$\gamma u(\sigma(b)) - \beta u^{\Delta}(\sigma(b)) = 0.$$

So that,

$$v(t) = \int_a^t k(t,s)h(s)\Delta s.$$

Let $u^{\Delta^{n-2}} := v(t)$, thus we get

$$u^{\Delta^{n-2}}(t) = \int_a^t k(t,s)h(s)\Delta s.$$

If we integrate from a to t, we obtain

$$\int_a^t u^{\Delta^{n-2}}(s)\Delta s = \int_a^t \left(\int_a^s k(s,\xi)h(\xi)\Delta\xi \right) \Delta s,$$

and so if we change bounds of integration, we obtain

$$\begin{aligned} u^{\Delta^{n-3}}(t) - u^{\Delta^{n-3}}(a) &= \int_a^t \left(\int_\xi^t k(s,\xi)h(\xi)\Delta s \right) \Delta\xi \\ &= \int_a^t \left(\int_\xi^t k(s,\xi)\Delta s \right) h(\xi)\Delta\xi \\ &= \int_a^t \left(\int_s^t k(\tau,s)\Delta\tau \right) h(s)\Delta s. \end{aligned}$$

Since $u^{\Delta^{n-3}}(a) = 0$, we get

$$u^{\Delta^{n-3}}(t) = \int_a^t \left(\int_s^t k(\tau,s)\Delta\tau \right) h(s)\Delta s,$$

again we integrate from a to t, we obtain

$$\begin{aligned} \int_a^t u^{\Delta^{n-3}}(s)\Delta s &= \int_a^t \left[\int_a^s \left(\int_\xi^s k(\tau,\xi)\Delta\tau \right) h(\xi)\Delta\xi \right] \Delta s \\ u^{\Delta^{n-4}}(t) - u^{\Delta^{n-4}}(a) &= \int_a^t \left[\int_\xi^t \left(\int_\xi^s k(\tau,\xi)\Delta\tau \right) h(\xi)\Delta s \right] \Delta\xi \\ &= \int_a^t \left[\int_\xi^t \left(\int_\xi^s k(\tau,\xi)\Delta\tau \right) \Delta s \right] h(\xi)\Delta\xi, \end{aligned}$$

thus,

$$u^{\Delta^{n-4}}(t) = \int_a^t \left[\int_s^t \left(\int_s^\tau k(\nu,s)\Delta\nu \right) \Delta\tau \right] h(s)\Delta s.$$

If we will continue, we obtain

$$u(t) = \int_a^t K(t,s)h(s)\Delta s,$$

where

$$K(t,s) = \underbrace{\int \cdots \int}_{n-2} k(\nu,s)\Delta\nu ... \Delta\tau, \tag{1.7}$$

and $u(t)$ is the solution of the following $n-th$ order dynamic equation

$$u^{\Delta^n}(t) = h(t), \quad t \in (a, b),$$

with the boundary conditions (1.2).

From (2.7), we can easily see that

$$\frac{\Delta^{n-2}}{\Delta t^{n-2}} K(t,s) = K^{\Delta^{n-2}}(t,s) = k(t,s).$$

As a special case, for $n = 4$, we have

$$\frac{\Delta^2}{\Delta t^2} K(t,s) = \left[K^{\Delta}(t,s)\right]^{\Delta} = \left[\int_s^t k(\nu,s)\Delta\nu\right]^{\Delta} = k(t,s).$$

□

Also, we see that $K(t,s) \geq 0$ since $k(t,s) \geq 0$ for $t \in [a, \sigma(b)],\ s \in [a,b]$, from Lemma 2.1.

2 Main Results

In this section, we consider the existence of at least one positive solution for the problem (1.1)-(1.2).

To prove the existence of at least one positive solution of the BVP (1.1) and (1.2), we will use the following theorem which can be found in Krasnoselskii's book [9] and in Deimling's book [4].

Theorem 2.1 ([4], [9]). *Let B be a Banach space, and let P be a cone in B. Assume that Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1,\ \overline{\Omega_1} \subset \Omega_2$ and let*

$$A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P,$$

be a completely continuous operator such that either

1. *$\|Au\| \leq \|u\|,\ u \in P \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|,\ u \in P \cap \partial\Omega_2$; or*
2. *$\|Au\| \geq \|u\|,\ u \in P \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|,\ u \in P \cap \partial\Omega_2$.*

Then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

For our constructions, let $B = \{u \in C^{(n-2)}[a, \sigma(b)] : u^{\Delta^i}(0) = 0,\ 0 \leq i \leq n-3\}$ with norm $\|u\| = sup\{|u^{\Delta^{n-2}}(t)| : t \in [a, \sigma(b)]\}$, and we define the set $P \subset B$ by $P = \{u \in B : u^{\Delta^{n-2}}(t) \geq 0\ on\ [a, \sigma(b)]\ and\ u^{\Delta^{n-2}}(t) \geq M\|u\|\ for\ t \in [\xi, \sigma(\omega)]\}$. Then P is cone in B.

Lemma 2.1. *Let u in P. For $0 \leq i \leq n-2$, we have*

$$u^{\Delta^i}(t) \geq 0, \quad t \in (a, \sigma(b)),$$

and

$$u^{\Delta^i}(t) \geq M\|u\|(\omega-\xi)^{(n-2-i)}, \quad t \in [\xi, \sigma(\omega)].$$

In particular,

$$u(t) \geq M\|u\|(\omega-\xi)^{(n-2)}, \quad t \in [\xi, \sigma(\omega)].$$

Proof. We know that for $0 \leq i \leq n-3$

$$u^{\Delta^i}(t) = \int_0^t u^{\Delta^{i+1}}(s)\Delta s, \quad t \in [\xi, \sigma(\omega)].$$

Thus, for $t \in [a, \sigma(b)]$, $u^{\Delta^i}(t) \geq 0 \; for \; 0 \leq i \leq n-2$.

For $t \in [\xi, \sigma(\omega)]$, we have

$$u^{\Delta^{n-3}}(t) = \int_a^t u^{\Delta^{n-2}}(s)\Delta s \geq \int_\xi^\omega u^{\Delta^{n-2}}(s)\Delta s \geq \int_\xi^\omega M\|u\|\Delta s = M\|u\|(\omega-\xi),$$

and similarly

$$u^{\Delta^{n-4}}(t) = \int_a^t u^{\Delta^{n-3}}(s)\Delta s \geq \int_\xi^\omega u^{\Delta^{n-3}}(s)\Delta s \geq \int_\xi^\omega M\|u\|(\omega-\xi)\Delta s = M\|u\|(\omega-\xi)^2.$$

Continuing this process we obtain,

$$u^{\Delta^i}(t) \geq M\|u\|(\omega-\xi)^{(n-2-i)}, \quad t \in [\xi, \sigma(\omega)].$$

And so, by taking $i = 0$, we get

$$u(t) \geq M\|u\|(\omega-\xi)^{(n-2)}, \quad t \in [\xi, \sigma(\omega)].$$

$\square$

Theorem 2.2. *Assume that there exist two distinct positive constants* λ, η *such that*

$$f(t,u_1,u_2,...,u_{n-1}) \leq \lambda \left(\int_a^{\sigma(b)} k(\sigma(s),s)\Delta s \right)^{-1}, \tag{2.8}$$

on $[a,\sigma(b)] \times [0,\lambda]^{n-1}$, *and*

$$f(t,u_1,u_2,...,u_{n-1}) \geq \eta \left(\int_\xi^{\omega} k(t_0,s)\Delta s \right)^{-1}, \tag{2.9}$$

on $[\xi,\sigma(\omega)] \times \left[M\xi(\omega-\xi)^{n-2},\xi\right]^{n-1}$.

Then BVP (1.1)-(1.2) has at least one positive solution u *such that* $\|u\|$ *lies between* λ *and* η.

Proof. Without loss of generality, we may assume that $\lambda < \eta$. It is clear that the problem (1.1)-(1.2) has a solution $u = u(t)$ if and only if u is the solution of the operator equation

$$u(t) = \int_a^{\sigma(b)} K(t,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s := Au(t), \ \ u \in B$$

or

$$u^{\Delta^{n-2}}(t) = \int_a^{\sigma(b)} k(t,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s := (Au)^{\Delta^{n-2}}(t), \ \ u \in B.$$

It follows from the definition of P and Lemma 2.1 that

$$\begin{aligned}
\min_{t\in[\xi,\sigma(\omega)]}(Au)^{\Delta^{n-2}}(t) &= \min_{t\in[\xi,\sigma(\omega)]} \int_a^{\sigma(b)} k(t,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\geq M \int_a^{\sigma(b)} k(\sigma(s),s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\geq M \int_a^{\sigma(b)} k(t,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s.
\end{aligned}$$

Hence $\min_{t\in[\xi,\sigma(\omega)]}(Au)^{\Delta^{n-2}}(t) \geq M\|Au\|$ which implies that $A(P) \subseteq P$. Furthermore, it is easy to check that $A : P \to P$ is completely continuous. In order to complete the proof, we seperate rest of the proof into the following steps:

Step I: Let $\Omega_1 := \{u \in P : \|u\| < \lambda\}$ be an open subset of B. It follows from Lemma 2.1 that for $u \in \partial\Omega_1$,

$$
\begin{aligned}
(Au)^{\Delta^{n-2}}(t) &= \int_a^{\sigma(b)} k(t,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\leq \int_a^{\sigma(b)} k(\sigma(s),s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\leq \lambda \left(\int_a^{\sigma(b)} k(\sigma(s),s)\Delta s \right)^{-1} \left(\int_a^{\sigma(b)} k(\sigma(s),s)\Delta s \right) = \lambda = \|u\|.
\end{aligned}
$$

Hence,

$$\|Au\| \leq \|u\| \text{ for } u \in \partial\Omega_1.$$

Step II: Let $\Omega_2 := \{u \in P : \|u\| < \eta\}$ be an open subset of B. It follows from the definitions of $\|.\|$, P and Lemma 2.3 that for $i = 0, 1, ..., n-2$,

$$
\begin{cases}
0 \leq u^{\Delta^i}(t) \leq \|u\| = \eta, & t \in (a, \sigma(b)), \\
u^{\Delta^i}(t) \geq M\|u\|(\omega - \xi)^{n-2-i}, & t \in [\xi, \sigma(\omega)]
\end{cases}
$$

for $u \in \partial\Omega_2$.

Thus we get

$$
\begin{aligned}
(Au)^{\Delta^{n-2}}(t_0) &= \int_a^{\sigma(b)} k(t_0,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\geq \int_\xi^{\omega} k(t_0,s) f(s,u(s),u^{\Delta}(s),...,u^{\Delta^{n-2}}(s))\Delta s \\
&\geq \eta \left(\int_\xi^{\omega} k(t_0,s)\Delta s \right)^{-1} \left(\int_\xi^{\omega} k(t_0,s)\Delta s \right) = \eta = \|u\|.
\end{aligned}
$$

Hence,

$$\|Au\| \geq \|u\| \; for \; u \in \partial\Omega_2.$$

The proof now follows from the first part of Theorem 3.1. ∎

Example 2.1. *Let* $\mathbb{T} = \{1, \frac{3}{2}, 2\} \cup [3,4]$ *and we consider the following dynamic equation:*

$$
\begin{aligned}
-u^{\Delta^7}(t) &= \frac{\sin(u^{\Delta^4}(t))}{10} + \exp(-(u^{\Delta^2}(t)) + \sqrt{t}, \qquad t \in (1,4), \\
& u^{\Delta^i}(1) = 0, \quad 0 \leq i \leq 4
\end{aligned}
$$

$$\frac{1}{2}u^{\Delta^5}(1) - u^{\Delta^6}(1) = 0,$$

$$u^{\Delta^5}(4) + 2u^{\Delta^6}(4) = 0.$$

We can easily see that $d = \frac{7}{2}$, $\int_1^4 k(\sigma(s), s)\Delta s \cong 6,54$ *and* $\max_{t\in[2,3]} \int_2^3 k(t, s)\Delta s = \int_2^3 k(2, s)\Delta s \cong 2,85.$

Thus we get

$$f(t, u_1, ..., u_6) = \frac{\sin(u_4(t))}{10} + \exp(-(u_1^2(t)) + \sqrt{t} \leq 3,1 \leq \lambda\frac{1}{6,54} \text{ for } \lambda = 22$$

and

$$f(t, u_1, ..., u_6) = \frac{\sin(u_4(t))}{10} + \exp(-(u_1^2(t)) + \sqrt{t} \geq 0,9 \geq \eta\frac{1}{2,85} \text{ for } \eta = 2.$$

Using Theorem 3.2, the boundary value problem has at least one positive solution u *such that* $2 \leq \|u\| \leq 22.$

Conclusions

In recent years lots of studies have been conducted on higher order boundary value problems on time scales. While the relationship between the Green's function for $n-th$ order problem and $(n-1)-th$ order problem has been used in the studies done so far, in this study existence of positive solutions has been proven using the relationship between the Green's function of second order problem and the $n-th$ order problem.

Acknowledgement

The authors are grateful to the editor and anonymous referees for their constructive comments and suggestions which led to improvement of the original manuscript.

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