Wilf's formula and a generalization of the Choi – Lee – Srivastava identities

Ulrich Abel

Technische Hochschule Mittelhessen, Fachbereich MND

Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany.

e-mail: *Ulrich.Abel@mnd.thm.de*

Abstract

The identities of Choi, Lee, and Srivastava imply a formula proposed by Wilf. We show that these identities are immediate consequences of the well-known product formulas for the sine function and the cosine function. Moreover, we prove a generalization.

Keywords: Euler–Mascheroni constant, infinite product formulas.

Mathematics Subject Classification (2010): 33B10, 40A20

1 Introduction

Herbert S. Wilf [1] proposed in the problem section of *The American Mathematical Monthly* to prove the identity

$$\cosh\left(\frac{\pi}{2}\right) = \frac{\pi}{2} e^{\gamma} \prod_{k=1}^{\infty} e^{-1/k}\left(1+\frac{1}{k}+\frac{1}{2k^2}\right), \tag{1}$$

where γ denotes the Euler–Mascheroni constant. In the following there appeared several proofs ([2], cf. [3, 4]). Chen and Paris [5, Theorem 1] gave explicit expressions for infinite products of the form

$$\prod_{k=1}^{\infty} e^{-p_1/k}\left(1+\frac{p_1}{k}+\frac{p_2}{k^2}+\cdots+\frac{p_m}{k^m}\right),$$

where $p_1, \cdots, p_m \in \mathbb{C}$ and m is any positive integer (see also [6]). Choi, Lee, and Srivastava [7] derived the following generalization

$$\begin{aligned}
\sinh(\pi z) &= \pi z \left(1+z^2\right) e^{2\gamma} \prod_{k=1}^{\infty}\left(1+\frac{2}{k}+\frac{1+z^2}{k^2}\right), \\
\cosh(\pi z) &= \pi \left(\frac{1}{4}+z^2\right) e^{\gamma} \prod_{k=1}^{\infty} e^{-1/k}\left(1+\frac{1}{k}+\frac{1/4+z^2}{k^2}\right).
\end{aligned} \tag{2}$$

Recently, C. Hernández-Aguilar, J. López-Bonilla, and R. López-Vázquez [8], proved the latter identities [8, Eqs. (3) and (2)] using a certain relation involving an infinite product and the gamma function [8, Eq. (4)]. In this note we show that these identities are immediate consequences of the well-known product formulas

$$\sin(\pi z) = \pi z \prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right), \qquad \cos(\pi z) = \prod_{k=1}^{\infty}\left(1-\frac{4z^2}{(2k-1)^2}\right) \tag{3}$$

for the sine function and the cosine function, respectively. Moreover, we derive the following generalization.

Theorem 1 *For $r \in \mathbb{N}$ and $z \in \mathbb{C}$, the hyperbolic functions possess the representations*

$$\begin{aligned}
\sinh(\pi z) &= \pi z \left(\prod_{j=1}^{r}\left(j^2+z^2\right)\right) e^{2r\gamma} \prod_{k=1}^{\infty} e^{-2r/k}\left(1+\frac{2r}{k}+\frac{r^2+z^2}{k^2}\right), \\
\cosh(\pi z) &= \pi \left(\prod_{j=1}^{r}\left(\left(j-\frac{1}{2}\right)^2+z^2\right)\right) e^{(2r-1)\gamma} \prod_{k=1}^{\infty} e^{-(2r-1)/k}\left(1+\frac{2r-1}{k}+\frac{\left(r-\frac{1}{2}\right)^2+z^2}{k^2}\right).
\end{aligned}$$

In the special case $r = 1$ the formulas reduce to the identities (2), which are valid also in the cases $z = \pm i$ and $z = \pm i/2$, respectively. For $z = 1/2$, we obtain

$$\cosh\left(\frac{\pi}{2}\right) = \pi\left(\prod_{j=1}^{r}\left(j^2 - j + \frac{1}{2}\right)\right) e^{(2r-1)\gamma}\prod_{k=1}^{\infty} e^{-(2r-1)/k}\left(1 + \frac{2r-1}{k} + \frac{r^2 - r + \frac{1}{2}}{k^2}\right).$$

Wilf's formula (1) is the special case $r = 1$.

2 Proof of Theorem 1

The product representation (3) of sine implies

$$\sinh(\pi z) = -i\sin(i\pi z) = \pi z\lim_{n\to\infty} f_n(z),$$

where

$$f_n(z) = \prod_{k=1}^{n}\frac{k^2 + z^2}{k^2} = \left(\prod_{j=1}^{r}\frac{j^2 + z^2}{(n+j)^2 + z^2}\right)\prod_{k=1}^{n}\frac{(k+r)^2 + z^2}{k^2}.$$

Furthermore,

$$\prod_{k=1}^{n} e^{-1/k} = e^{-(\ln n + \gamma_n)} = \frac{1}{n}e^{-\gamma_n},$$

with positive reals γ_n tending to $\lim_{n\to\infty}\gamma_n = \gamma$. Hence,

$$f_n(z) = \left(\prod_{j=1}^{r}\frac{j^2 + z^2}{(n+j)^2 + z^2}\right)\cdot n^{2r}e^{2r\gamma_n}\prod_{k=1}^{n}\left(e^{-2r/k}\frac{(k+r)^2 + z^2}{k^2}\right).$$

The limit letting $n \to \infty$ leads to the first formula of the theorem.
Analogously, the product representation (3) of cosine implies

$$\cosh(\pi z) = \cos(i\pi z) = \lim_{n\to\infty} g_n(z),$$

where

$$g_n(z) = \prod_{k=1}^{n}\frac{(2k-1)^2 + 4z^2}{(2k-1)^2} = \left(\prod_{j=1}^{r}\frac{(2j-1)^2 + 4z^2}{(2n+2j-1)^2 + 4z^2}\right)\prod_{k=1}^{n}\frac{(2k+2r-1)^2 + 4z^2}{(2k-1)^2}.$$

As above we conclude that

$$\begin{aligned} g_n(z) &= \left(\prod_{j=1}^{r}\frac{(j-1/2)^2 + z^2}{(n+j-1/2)^2 + z^2}\right) n^{2r-1}e^{(2r-1)\gamma_n}\prod_{k=1}^{n}\left(e^{-(2r-1)/k}\frac{(k+r-1/2)^2 + z^2}{k^2}\cdot\frac{k^2}{(k-1/2)^2}\right) \\ &\to \pi\left(\prod_{j=1}^{r}\left(\left(j - \frac{1}{2}\right)^2 + z^2\right)\right) e^{(2r-1)\gamma}\prod_{k=1}^{\infty}\left(e^{-(2r-1)/k}\frac{k^2 + (2r-1)k + \left(r - \frac{1}{2}\right)^2 + z^2}{k^2}\right) \end{aligned}$$

as $n \to \infty$, since it is well-known (see, e.g., [9, (6.1.46)]), that

$$\prod_{k=1}^{n}\frac{k}{k-1/2} = \Gamma(1/2)\frac{\Gamma(n+1)}{\Gamma(n+1/2)} \sim \sqrt{\pi n} \qquad (n\to\infty).$$

This completes the proof.

Conclusion

The above note presents a generalization of the identities by Choi, Lee, and Srivastava. We show that these identities are immediate consequences of the well-known product formulas for the sine function and the cosine function. They imply a formula proposed by Herbert S. Wilf.

Acknowledgment

The author is grateful to Georg Arends for a careful proofreading. Furthermore, he wants to thank the anonymous referees for their helpful comments which led to an improved version of the paper.

Conflicts of Interest

The author declares that there is no conflict of interests.

Funding Statement

The author states that funding is not applicable for this paper.

References

[1] Herbert S. Wilf, Problem 10588, Amer. Math. Monthly **104** (1997) 456.

[2] J. López-Bonilla, R. López-Vázquez, On an identity of Wilf for the Euler-Mascheroni's constant, Prespacetime Journal **9** (2018), 516–518.

[3] Junesang Choi, Tae Young Seo, Evaluation of some infinite series, Indian J. Pure Appl. Math. **28** (1997) 791–796.

[4] Chao-Ping Chen, Junesang Choi, Two infinite product formulas with two parameters, Integral Transforms and Special Functions **24** (2013), 357–363.
doi.org/10.1080/10652469.2012.693081

[5] Chao-Ping Chen and Richard B. Paris, Generalizations of two infinite product formulas, Integral Transforms Spec. Funct. **25** (2014), 547–551.
http://dx.doi.org/10.1080/10652469.2014.885965

[6] Chao-Ping Chen and Richard B. Paris, On the asymptotics of products related to generalizations of the Wilf and Mortini problems, Integral Transforms Spec. Funct. **27** (2015), 281–288.
doi.org/10.1080/10652469.2015.1118627

[7] Junesang Choi, Jungseob Lee, H. M. Srivastava, A generalization of Wilf's formula, Kodai Math. J. **26** (2003), 44–48.

[8] C. Hernández-Aguilar, J. López-Bonilla, R. López-Vázquez, Identities of Choi–Lee–Srivastava involving the Euler-Mascheroni's constant, MathLAB Journal **1** (2018) 296–298.

[9] M. Abramowitz, A. Stegun, *Handbook of Mathematical Functions*, Appl. Math. Ser. 55, National Bureau of Standards, Washington D. C., 1972.