Solutions And Formulae For Some Systems Of Difference Equations

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Abstract

This paper is written to provide some solutions to the following systems of difference equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1+x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\pm1\pm y_{n-1}x_{n-3}\right)},\quad n=0,1,...,$$

where the initial data $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are arbitrary non zero real numbers.

Keywords: system of difference equations**,** recursive equation, boundedness, periodicity**.**
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1 Introduction

Our major and essential objective in this paper is to solve and deal with the following dynamic systems of recursive equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1+x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\pm1\pm y_{n-1}x_{n-3}\right)},\quad n=0,1,...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are real numbers.

It should be known that the theory of discrete dynamic systems emerges and arises as discrete analogous and also as numerical solutions of some systems of differential and delay differential equations which describe some natural phenomena in biology, physics, economy, etc. Researchers and scholars have discovered various properties of difference equations and systems of difference equations and they publish a massive number of papers on this field. Take, for instance, the following ones. In [2] Asiri et al. explained and interpreted the periodic solutions of the following system:

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{\pm 1 \pm x_{n-2}y_{n-1}x_n}.$$

Kurbanli et al. [21] solved the following system of recursive equations:

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

El-Metwally et al. [12] highlighted the solutions and the periodic solutions of the following system:

$$x_{n+1} = \frac{y_{n-2}}{-1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{\pm 1 \pm x_{n-2}y_{n-1}x_n}.$$

In [24] Touafek et al. discussed the periodic nature and analyzed some solutions of

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}.$$

Elsayed [14] explored some specific solutions to the following systems:

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 \pm x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1}x_n}.$$

The difference equations

$$x_{n+1} = \frac{Ax_n x_{n-3}}{Bx_n \pm Cx_{n-2}},$$

were deeply studied by Elsayed and Gafel, see [16]. Moreover, Din [5] made a significant contributions to solve the following dynamic system of difference equations:

$$x_{n+1} = \frac{ay_n}{b + cy_n}, \qquad y_{n+1} = \frac{dy_n}{e + fx_n}.$$

Cinar [4] gave an explanation of the periodicity of nonnegative solutions of the following system:

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

More results on dynamic systems of difference equations can be discovered in refs [18]-[20].

2 First System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(1+y_{n-1}x_{n-3})}$

This section is considered to give some forms of solutions to the following system of rational difference equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1 + x_{n-1}y_{n-3}\right)}, \ y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(1 + y_{n-1}x_{n-3}\right)}, \ n = 0, \ 1, \ldots . \qquad (1)$$

The initial conditions of this system are arbitrary real numbers. In the next theorem, we will present some forms of solutions to system (1).

Theorem 1 *Assume that* $\{x_n, y_n\}$ *is a solution to system (1) and let* $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c,\ x_0 = d,\ y_{-3} = \alpha,\ y_{-2} = \beta,\ y_{-1} = \gamma$ *and* $y_0 = \omega$*. Then, for* $n = 0,\ 1,\ \ldots$ *we have*

$$x_{4n-3} = \frac{c^n \alpha^n \prod_{i=0}^{n-1} [(2i) a\gamma + 1]}{a^{n-1} \gamma^n \prod_{i=0}^{n-1} [(2i+1) c\alpha + 1]}, \qquad x_{4n-2} = \frac{d^n \beta^n \prod_{i=0}^{n-1} [(2i) b\omega + 1]}{b^{n-1} \omega^n \prod_{i=0}^{n-1} [(2i+1) d\beta + 1]},$$

$$x_{4n-1} = \frac{c^{n+1} \alpha^n \prod_{i=0}^{n-1} [(2i+1) a\gamma + 1]}{a^n \gamma^n \prod_{i=0}^{n-1} [(2i+2) c\alpha + 1]}, \qquad x_{4n} = \frac{d^{n+1} \beta^n \prod_{i=0}^{n-1} [(2i+1) b\omega + 1]}{b^n \omega^n \prod_{i=0}^{n-1} [(2i+2) d\beta + 1]}.$$

$$y_{4n-3} = \frac{a^n \gamma^n \prod_{i=0}^{n-1} [(2i) c\alpha + 1]}{c^n \alpha^{n-1} \prod_{i=0}^{n-1} [(2i+1) a\gamma + 1]}, \qquad y_{4n-2} = \frac{b^n \omega^n \prod_{i=0}^{n-1} [(2i) d\beta + 1]}{d^n \beta^{n-1} \prod_{i=0}^{n-1} [(2i+1) b\omega + 1]},$$

$$y_{4n-1} = \frac{a^n \gamma^{n+1} \prod_{i=0}^{n-1} [(2i+1) c\alpha + 1]}{c^n \alpha^n \prod_{i=0}^{n-1} [(2i+2) a\gamma + 1]}, \qquad y_{4n} = \frac{b^n \omega^{n+1} \prod_{i=0}^{n-1} [(2i+1) d\beta + 1]}{d^n \beta^n \prod_{i=0}^{n-1} [(2i+2) b\omega + 1]}.$$

Proof. The results hold for $n = 0$. Now, we suppose that $n > 1$ and assume that the results hold for $n - 1$. That is

$$x_{4n-7} = \frac{c^{n-1} \alpha^{n-1} \prod_{i=0}^{n-2} [(2i) a\gamma + 1]}{a^{n-2} \gamma^{n-1} \prod_{i=0}^{n-2} [(2i+1) c\alpha + 1]}, \qquad x_{4n-6} = \frac{d^{n-1} \beta^{n-1} \prod_{i=0}^{n-2} [(2i) b\omega + 1]}{b^{n-2} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+1) d\beta + 1]},$$

$$x_{4n-5} = \frac{c^n \alpha^{n-1} \prod_{i=0}^{n-2} [(2i+1) a\gamma + 1]}{a^{n-1} \gamma^{n-1} \prod_{i=0}^{n-2} [(2i+2) c\alpha + 1]}, \qquad x_{4n-4} = \frac{d^n \beta^{n-1} \prod_{i=0}^{n-2} [(2i+1) b\omega + 1]}{b^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i+2) d\beta + 1]}.$$

$$y_{4n-7} = \frac{a^{n-1} \gamma^{n-1} \prod_{i=0}^{n-2} [(2i) c\alpha + 1]}{c^{n-1} \alpha^{n-2} \prod_{i=0}^{n-2} [(2i+1) a\gamma + 1]}, \qquad y_{4n-6} = \frac{b^{n-1} \omega^{n-1} \prod_{i=0}^{n-2} [(2i) d\beta + 1]}{d^{n-1} \beta^{n-2} \prod_{i=0}^{n-2} [(2i+1) b\omega + 1]},$$

$$y_{4n-5} = \frac{a^{n-1} \gamma^n \prod_{i=0}^{n-2} [(2i+1) c\alpha + 1]}{c^{n-1} \alpha^{n-1} \prod_{i=0}^{n-2} [(2i+2) a\gamma + 1]}, \qquad y_{4n-4} = \frac{b^{n-1} \omega^n \prod_{i=0}^{n-2} [(2i+1) d\beta + 1]}{d^{n-1} \beta^{n-1} \prod_{i=0}^{n-2} [(2i+2) b\omega + 1]}.$$

Now, it can be easily observed from system (1) that

$$x_{4n-3} = \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} (1 + x_{4n-5} y_{4n-7})}$$

$$= \frac{\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}}{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\left[1+\frac{c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}\right]}$$

$$= \frac{\frac{c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}}{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\left[1+\frac{c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\right]}$$

$$= \frac{c\alpha\prod_{i=0}^{n-2}[(2i)\,c\alpha+1]\ \ c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)\,a\gamma+1]}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)\,c\alpha+1]\left[\prod_{i=0}^{n-2}[(2i+2)\,c\alpha+1]+c\alpha\prod_{i=0}^{n-2}[(2i)\,c\alpha+1]\right]}$$

$$= \frac{c^n\alpha^n\prod_{i=0}^{n-2}[(2i+2)\,a\gamma+1]}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-1}[(2i+1)\,c\alpha+1]}.$$

Again, from system (1) we can obtain that

$$y_{4n-3} = \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[1+y_{4n-5}x_{4n-7}\right]}$$

$$= \frac{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\ \frac{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}}{\frac{c^{n}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[1+\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\ \frac{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}\right]}$$

$$= \frac{\frac{a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}}{\frac{c^{n}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma+1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[1+\frac{a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma+1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma+1]}\right]}$$

$$= \frac{a\gamma\prod_{i=0}^{n-2}[(2i)\,a\gamma+1]\ \ a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)\,c\alpha+1]}{c^{n}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)\,a\gamma+1]\left[\prod_{i=0}^{n-2}[(2i+2)\,a\gamma+1]+a\gamma\prod_{i=0}^{n-2}[(2i)\,a\gamma+1]\right]}$$

$$= \frac{a^n\gamma^n\prod_{i=0}^{n-2}[(2i+2)\,c\alpha+1]}{c^{n}\alpha^{n-1}\prod_{i=0}^{n-1}[(2i+1)\,a\gamma+1]}.$$

Hence, the other relations can be similarly proven. The proof is complete.

3 Second System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(-1-y_{n-1}x_{n-3})}$

In this section, we shall investigate and analyze the solutions of the following dynamic system of recursive equations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1 + x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1 - y_{n-1}x_{n-3}\right)}. \tag{2}$$

The initial conditions of system (2) are arbitrary real numbers. The following concrete theorem will introduce some sold solutions to our considered system.

Theorem 2 *Let $\{x_n, y_n\}$ be a solution to system (2) and assume that $x_{-3} = a$, $x_{-2} = b$, $x_{-1} = c$, $x_0 = d$, $y_{-3} = \alpha$, $y_{-2} = \beta$, $y_{-1} = \gamma$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ \ldots$ we have*

$$x_{4n-3} = \frac{c^n\alpha^n}{a^{n-1}\gamma^n \prod_{i=0}^{n-1}\left[(2i+1)\,c\alpha + 1\right]}, \qquad x_{4n-2} = \frac{d^n\beta^n}{b^{n-1}\omega^n \prod_{i=0}^{n-1}\left[(2i+1)\,d\beta + 1\right]},$$

$$x_{4n-1} = \frac{(-1)^n c^{n+1}\alpha^n (a\gamma+1)^n}{a^n\gamma^n \prod_{i=0}^{n-1}\left[(2i+2)\,c\alpha + 1\right]}, \qquad x_{4n} = \frac{(-1)^n d^{n+1}\beta^n (b\omega+1)^n}{b^n\omega^n \prod_{i=0}^{n-1}\left[(2i+2)\,d\beta + 1\right]},$$

$$y_{4n-3} = \frac{(-1)^n a^n\gamma^n \prod_{i=0}^{n-1}\left[(2i)\,c\alpha + 1\right]}{c^n\alpha^{n-1}(a\gamma+1)^n}, \qquad y_{4n-2} = \frac{(-1)^n b^n\omega^n \prod_{i=0}^{n-1}\left[(2i)\,d\beta + 1\right]}{d^n\beta^{n-1}(b\omega+1)^n},$$

$$y_{4n-1} = \frac{a^n\gamma^{n+1} \prod_{i=0}^{n-1}\left[(2i+1)\,c\alpha + 1\right]}{c^n\alpha^n}, \qquad y_{4n} = \frac{b^n\omega^{n+1} \prod_{i=0}^{n-1}\left[(2i+1)\,d\beta + 1\right]}{d^n\beta^n}.$$

Proof. For $n = 0$ the relations hold. Now, we suppose that $n > 1$ and assume that the relations hold for $n - 1$. That is

$$x_{4n-7} = \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1} \prod_{i=0}^{n-2}\left[(2i+1)\,c\alpha + 1\right]}, \qquad x_{4n-6} = \frac{d^{n-1}\beta^{n-1}}{b^{n-2}\omega^{n-1} \prod_{i=0}^{n-2}\left[(2i+1)\,d\beta + 1\right]},$$

$$x_{4n-5} = \frac{(-1)^{n-1} c^n\alpha^{n-1} (a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1} \prod_{i=0}^{n-2}\left[(2i+2)\,c\alpha + 1\right]}, \qquad x_{4n-4} = \frac{(-1)^{n-1} d^n\beta^{n-1} (b\omega+1)^{n-1}}{b^{n-1}\omega^{n-1} \prod_{i=0}^{n-2}\left[(2i+2)\,d\beta + 1\right]},$$

$$y_{4n-7} = \frac{(-1)^{n-1} a^{n-1}\gamma^{n-1} \prod_{i=0}^{n-2}\left[(2i)\,c\alpha + 1\right]}{c^{n-1}\alpha^{n-2}(a\gamma+1)^{n-1}}, \qquad y_{4n-6} = \frac{(-1)^{n-1} b^{n-1}\omega^{n-1} \prod_{i=0}^{n-2}\left[(2i)\,d\beta + 1\right]}{d^{n-1}\beta^{n-2}(b\omega+1)^{n-1}},$$

$$y_{4n-5} = \frac{a^{n-1}\gamma^n \prod_{i=0}^{n-2}\left[(2i+1)\,c\alpha + 1\right]}{c^{n-1}\alpha^{n-1}}, \qquad y_{4n-4} = \frac{b^{n-1}\omega^n \prod_{i=0}^{n-2}\left[(2i+1)\,d\beta + 1\right]}{d^{n-1}\beta^{n-1}}.$$

Next, one can obtain from system (2) that

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left[1+x_{4n-5}y_{4n-7}\right]} \\
&= \frac{\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}(a\gamma+1)^{n-1}}}{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\left[1+\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}(a\gamma+1)^{n-1}}\right]} \\
&= \frac{\frac{(-1)^{2n-2}c\alpha\ \prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}}{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\left[1+\frac{(-1)^{2n-2}c\alpha\ \prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\right]} \\
&= \frac{c\alpha\prod_{i=0}^{n-2}\left[(2i)\,c\alpha+1\right]\ c^{n-1}\alpha^{n-1}}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}\left[(2i+1)\,c\alpha+1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\,c\alpha+1\right]+c\alpha\prod_{i=0}^{n-2}\left[(2i)\,c\alpha+1\right]\right]} \\
&= \frac{c^n\alpha^n}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-1}\left[(2i+1)\,c\alpha+1\right]}.
\end{aligned}
$$

Similarly, system (2) gives

$$
\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[-1-y_{4n-5}x_{4n-7}\right]} \\
&= -\frac{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\ \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}}{\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[1+\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\ \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}\right]} \\
&= -\frac{a\gamma}{\frac{(-1)^{n-1}c^n\alpha^{n-1}(a\gamma+1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-1}[(2i)c\alpha+1]}\left[1+a\gamma\right]} = (-1)^n\,\frac{a^n\gamma^n\prod_{i=0}^{n-1}\left[(2i)\,c\alpha+1\right]}{c^n\alpha^{n-1}\left(a\gamma+1\right)^n}.
\end{aligned}
$$

Accordingly, the remaining unassertive relations can be simply proven in a similar way. Thus, the proof has been achieved.

4 Third System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(1+x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

This section is devoted to highlight and formulate the forms of solutions of the following dynamic systems:

$$
x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1+x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)}, \tag{3}
$$

where the initial conditions are as described above. The formulas of the solution are clearly expressed in the following theorem.

Theorem 3 *Let $\{x_n, y_n\}$ be a solution to system (3) and suppose that $x_{-3} = a,\ x_{-2} = b,\ x_{-1} = c,\ x_0 = d,\ y_{-3} = \alpha,\ y_{-2} = \beta,\ y_{-1} = \gamma$ and $y_0 = \omega$. Then, for $n = 0,\ 1,\ ...$ we have*

$$x_{4n-3} = \frac{c^n\alpha^n}{a^{n-1}\gamma^n \prod_{i=0}^{n-1}[(2i+1)c\alpha+1]}, \qquad x_{4n-2} = \frac{d^n\beta^n}{b^{n-1}\omega^n \prod_{i=0}^{n-1}[(2i+1)d\beta+1]},$$

$$x_{4n-1} = \frac{c^{n+1}\alpha^n(a\gamma-1)^n}{a^n\gamma^n \prod_{i=0}^{n-1}[(2i+2)c\alpha+1]}, \qquad x_{4n} = \frac{d^{n+1}\beta^n(b\omega-1)^n}{b^n\omega^n \prod_{i=0}^{n-1}[(2i+2)d\beta+1]},$$

$$y_{4n-3} = \frac{a^n\gamma^n \prod_{i=0}^{n-1}[(2i)c\alpha+1]}{c^n\alpha^{n-1}(a\gamma-1)^n}, \qquad y_{4n-2} = \frac{b^n\omega^n \prod_{i=0}^{n-1}[(2i)d\beta+1]}{d^n\beta^{n-1}(b\omega-1)^n},$$

$$y_{4n-1} = \frac{a^n\gamma^{n+1} \prod_{i=0}^{n-1}[(2i+1)c\alpha+1]}{c^n\alpha^n}, \qquad y_{4n} = \frac{b^n\omega^{n+1} \prod_{i=0}^{n-1}[(2i+1)d\beta+1]}{d^n\beta^n}.$$

Proof. At $n = 0$ our relations hold. Following this, we suppose that $n > 1$ and assume that the formulae hold for $n-1$. That is

$$x_{4n-7} = \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1} \prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}, \qquad x_{4n-6} = \frac{d^{n-1}\beta^{n-1}}{b^{n-2}\omega^{n-1} \prod_{i=0}^{n-2}[(2i+1)d\beta+1]},$$

$$x_{4n-5} = \frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1} \prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}, \qquad x_{4n-4} = \frac{d^n\beta^{n-1}(b\omega-1)^{n-1}}{b^{n-1}\omega^{n-1} \prod_{i=0}^{n-2}[(2i+2)d\beta+1]},$$

$$y_{4n-7} = \frac{a^{n-1}\gamma^{n-1} \prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}(a\gamma-1)^{n-1}}, \qquad y_{4n-6} = \frac{b^{n-1}\omega^{n-1} \prod_{i=0}^{n-2}[(2i)d\beta+1]}{d^{n-1}\beta^{n-2}(b\omega-1)^{n-1}},$$

$$y_{4n-5} = \frac{a^{n-1}\gamma^n \prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}, \qquad y_{4n-4} = \frac{b^{n-1}\omega^n \prod_{i=0}^{n-2}[(2i+1)d\beta+1]}{d^{n-1}\beta^{n-1}}.$$

Next, it can be seen from system (3) that

$$x_{4n-3} = \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}[1+x_{4n-5}y_{4n-7}]}$$

$$= \frac{\frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]} \frac{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}(a\gamma-1)^{n-1}}}{\frac{a^{n-1}\gamma^n\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\left[1+\frac{c^n\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]} \frac{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}(a\gamma-1)^{n-1}}\right]}$$

$$= \frac{\frac{c\alpha \prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}}{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\left[1+\frac{c\alpha \prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\right]}$$

$$= \frac{c\alpha \prod_{i=0}^{n-2}\left[(2i)\,c\alpha+1\right]\ c^{n-1}\alpha^{n-1}}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}\left[(2i+1)\,c\alpha+1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\,c\alpha+1\right]+c\alpha\prod_{i=0}^{n-2}\left[(2i)\,c\alpha+1\right]\right]}$$

$$= \frac{c^{n}\alpha^{n}}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-1}\left[(2i+1)\,c\alpha+1\right]}.$$

We now turn to prove an extra result. System (3) leads to

$$y_{4n-3} = \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[-1+y_{4n-5}x_{4n-7}\right]}$$

$$= \frac{\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\ \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}}{\frac{c^{n}\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[-1+\frac{a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}}\ \frac{c^{n-1}\alpha^{n-1}}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}\right]}$$

$$= \frac{a\gamma}{\frac{c^{n}\alpha^{n-1}(a\gamma-1)^{n-1}}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-1}[(2i)c\alpha+1]}\left[-1+a\gamma\right]}$$

$$= \frac{a^{n}\gamma^{n}\prod_{i=0}^{n-1}\left[(2i)\,c\alpha+1\right]}{c^{n}\alpha^{n-1}\left(a\gamma-1\right)^{n}}.$$

The other formulae can be demonstrated in a similar way.

5 Fourth System $x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}(1+x_{n-1}y_{n-3})}$, $y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}$

This part is allocated to determine and figure out formulae for solutions of the following nonlinear system of difference relations:

$$x_{n+1} = \frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1+x_{n-1}y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1}x_{n-3}}{x_{n-1}\left(1-y_{n-1}x_{n-3}\right)}, \tag{4}$$

where the initial conditions are real numbers. We now turn to establish an intrinsic theorem to point out the solutions of system (4).

Theorem 4 *Assume that* $\{x_n, y_n\}$ *is a solution to system (4) and let* $x_{-3} = a,\ x_{-2} =$

$b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=\alpha,\ y_{-2}=\beta,\ y_{-1}=\gamma$ *and* $y_0=\omega$. *Then, for* $n=0,\ 1,\ \ldots$ *we have*

$$x_{4n-3}=\frac{(-1)^n c^n\alpha^n\prod_{i=0}^{n-1}[(2i)a\gamma-1]}{a^{n-1}\gamma^n\prod_{i=0}^{n-1}[(2i+1)c\alpha+1]},\qquad x_{4n-2}=\frac{(-1)^n d^n\beta^n\prod_{i=0}^{n-1}[(2i)b\omega-1]}{b^{n-1}\omega^n\prod_{i=0}^{n-1}[(2i+1)d\beta+1]},$$

$$x_{4n-1}=\frac{(-1)^n c^{n+1}\alpha^n\prod_{i=0}^{n-1}[(2i+1)a\gamma-1]}{a^n\gamma^n\prod_{i=0}^{n-1}[(2i+2)c\alpha+1]},\qquad x_{4n}=\frac{(-1)^n d^{n+1}\beta^n\prod_{i=0}^{n-1}[(2i+1)b\omega-1]}{b^n\omega^n\prod_{i=0}^{n-1}[(2i+2)d\beta+1]},$$

$$y_{4n-3}=\frac{(-1)^n a^n\gamma^n\prod_{i=0}^{n-1}[(2i)c\alpha+1]}{c^n\alpha^{n-1}\prod_{i=0}^{n-1}[(2i+1)a\gamma-1]},\qquad y_{4n-2}=\frac{(-1)^n b^n\omega^n\prod_{i=0}^{n-1}[(2i)d\beta+1]}{d^n\beta^{n-1}\prod_{i=0}^{n-1}[(2i+1)b\omega-1]},$$

$$y_{4n-1}=\frac{(-1)^n a^n\gamma^{n+1}\prod_{i=0}^{n-1}[(2i+1)c\alpha+1]}{c^n\alpha^n\prod_{i=0}^{n-1}[(2i+2)a\gamma-1]},\qquad y_{4n}=\frac{(-1)^n b^n\omega^{n+1}\prod_{i=0}^{n-1}[(2i+1)d\beta+1]}{d^n\beta^n\prod_{i=0}^{n-1}[(2i+2)b\omega-1]}.$$

Proof. It is obvious that the results hold for $n=0$. Now, we assume that $n>1$ and suppose that the relations hold for $n-1$. That is

$$x_{4n-7}=\frac{(-1)^{n-1} c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]},\qquad x_{4n-6}=\frac{(-1)^{n-1} d^{n-1}\beta^{n-1}\prod_{i=0}^{n-2}[(2i)b\omega-1]}{b^{n-2}\omega^{n-1}\prod_{i=0}^{n-2}[(2i+1)d\beta+1]},$$

$$x_{4n-5}=\frac{(-1)^{n-1} c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]},\qquad x_{4n-4}=\frac{(-1)^{n-1} d^n\beta^{n-1}\prod_{i=0}^{n-2}[(2i+1)b\omega-1]}{b^{n-1}\omega^{n-1}\prod_{i=0}^{n-2}[(2i+2)d\beta+1]},$$

$$y_{4n-7}=\frac{(-1)^{n-1} a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]},\qquad y_{4n-6}=\frac{(-1)^{n-1} b^{n-1}\omega^{n-1}\prod_{i=0}^{n-2}[(2i)d\beta+1]}{d^{n-1}\beta^{n-2}\prod_{i=0}^{n-2}[(2i+1)b\omega-1]},$$

$$y_{4n-5}=\frac{(-1)^{n-1} a^{n-1}\gamma^n\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]},\qquad y_{4n-4}=\frac{(-1)^{n-1} b^{n-1}\omega^n\prod_{i=0}^{n-2}[(2i+1)d\beta+1]}{d^{n-1}\beta^{n-1}\prod_{i=0}^{n-2}[(2i+2)b\omega-1]}.$$

Next, one can obtain from system (4) that

$$x_{4n-3}=\frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(1+x_{4n-5}y_{4n-7}\right)}$$

$$= \frac{\frac{(-1)^{n-1}c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}}{\frac{(-1)^{n-1}a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\left[1+\frac{(-1)^{n-1}c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\ \frac{(-1)^{n-1}a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{c^{n-1}\alpha^{n-2}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}\right]}$$

$$= \frac{\frac{(-1)^{2n-2}c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}}{\frac{(-1)^{n-1}a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\left[1+\frac{(-1)^{2n-2}c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]}{\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\right]}$$

$$= \frac{(-1)^{2n-2}c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]\,c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}{(-1)^{n-1}a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]\left[\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]+c\alpha\prod_{i=0}^{n-2}[(2i)c\alpha+1]\right]}$$

$$= \frac{(-1)^{n}c^n\alpha^n\prod_{i=0}^{n-1}[(2i)a\gamma-1]}{a^{n-1}\gamma^{n}\prod_{i=0}^{n-1}[(2i+1)c\alpha+1]}.$$

Similarly, one can find from system (4) that

$$y_{4n-3} = \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}(1-y_{4n-5}x_{4n-7})}$$

$$= \frac{\frac{(-1)^{n-1}a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\ \frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}}{\frac{(-1)^{n-1}c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[1-\frac{(-1)^{n-1}a^{n-1}\gamma^{n}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}{c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\ \frac{(-1)^{n-1}c^{n-1}\alpha^{n-1}\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{a^{n-2}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+1)c\alpha+1]}\right]}$$

$$= \frac{\frac{(-1)^{2n-2}a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}}{\frac{(-1)^{n-1}c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]}{a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}\left[1-\frac{(-1)^{2n-2}a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma-1]}{\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]}\right]}$$

$$= \frac{(-1)^{2n-2}a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma-1]\,a^{n-1}\gamma^{n-1}\prod_{i=0}^{n-2}[(2i+2)c\alpha+1]}{(-1)^{n-1}c^n\alpha^{n-1}\prod_{i=0}^{n-2}[(2i+1)a\gamma-1]\left[\prod_{i=0}^{n-2}[(2i+2)a\gamma-1]-(-1)^{2n-2}a\gamma\prod_{i=0}^{n-2}[(2i)a\gamma-1]\right]}$$

$$= \frac{(-1)^{n}a^n\gamma^n\prod_{i=0}^{n-1}[(2i)c\alpha+1]}{c^{n}\alpha^{n-1}\prod_{i=0}^{n-1}[(2i+1)a\gamma-1]}.$$

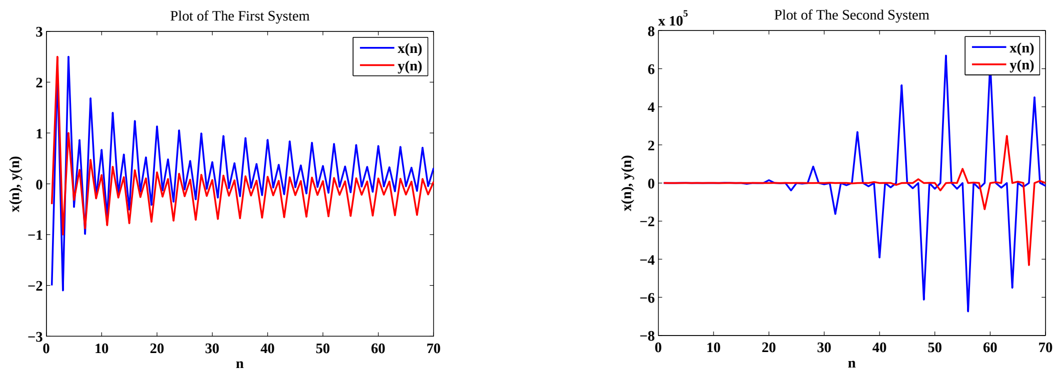
Other results can be likewise shown. Hence, this completes the proof.

6 Numerical Examples

We now turn to give a powerful confirmation and verification on our theoretical discussion. This confirmation is embodied in presenting some numerical examples and illustrative explanations.

Example 1. This example shows the behaviour of the solutions of system (1). Here, we take $x_{-3} = 2.5,\ x_{-2} = -2.1,\ x_{-1} = 2.2,\ x_0 = -2,\ y_{-3} = 1,\ y_{-2} = -1,\ y_{-1} = 2.5$ and $y_0 = -0.4$, as described in Figure (1a).

Example 2. In this example, the graph of system (2) is depicted under the following initial conditions: $x_{-3} = 10,\ x_{-2} = -0.2,\ x_{-1} = -0.1,\ x_0 = -0.01,\ y_{-3} = -0.4,\ y_{-2} = -2.6,\ y_{-1} = -0.5$ and $y_0 = 0.051$. See Figure (1b).

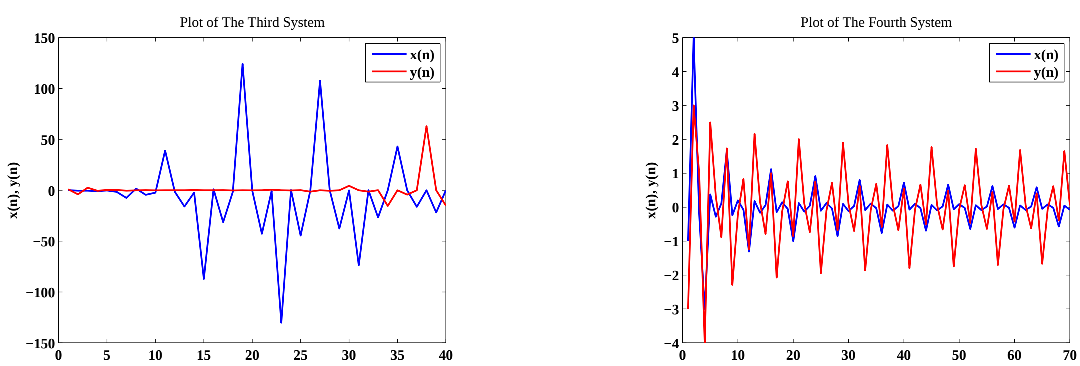


(a) Plot of system 1. (b) Plot of system 2.

Figure 1: Behaviour of solutions of system 1 and system 2.

Example 3. Figure (2a) presents the behaviour of solutions of system (3) when we let $x_{-3} = 10,\ x_{-2} = 0.2,\ x_{-1} = -5,\ x_0 = -0.1,\ y_{-3} = -0.5,\ y_{-2} = 1,\ y_{-1} = 0.6$ and $y_0 = 3$.

Example 4. The solutions of system (4) are plotted in Figure (2b). The considered initial data here are $x_{-3} = -3,\ x_{-2} = -0.2,\ x_{-1} = 5,\ x_0 = -1,\ y_{-3} = -4,\ y_{-2} = 1,\ y_{-1} = 3,\ y_0 = -3$.



(a) Plot of system 3. (b) Plot of system 4.

Figure 2: Behaviour of solutions of system 3 and system 4.

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