The Basic Concepts On Distribution of Decision Power Between The Players and Manipulation in Weighted Voting Games

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Abstract

It is known that voting is a widely used method in social choice theory. In the present paper we consider some concepts of distribution of voting powers between the player and the process of manipulation in weighted voting games. The aim is to show some basic problems in social choice theory by studying the decision powers of players and the three processes of manipulation in weighted voting games: by merging of two players into a single player, by players splitting into a number of smaller units, and by annexation of a part or all of the voting weights of another player.

Indexing terms/Keywords: weighted voting game, decision power, manipulation, merge, split, annex

Introduction

The modern notion of a simple game was introduced by John von Neumann and Oscar Morgenstern in their monumental book "Theory of Games and Economic Behavior" in 1944 [25]. Previous works on this problem were fragmentary and did not attract much attention. The book of Von Neumann and Morgenstern provided some new important developments such as the consideration of information sets and the introduction of formal definitions and decision rules. According to this book a simple game is a conflict in which the only objective is winning and the only rule is an algorithm to decide which coalitions of players are winning.

It is known that voting is a widely used method for social decision making. In particular, voting power of the players and manipulation have been studied intensely in social choice theory and theory of games, starting with the classical works of Gibbard [9], Satterthwaite [17], and Shapley and Shubik [18]. The problem of coalitional manipulation was first explicitly introduced by Conitzer, Sandholm and Lang in [6] where the authors initiated its analysis from a computational perspective.

We start our study with a consideration of key terms, definitions, and notations. Let N be a nonempty finite set of players (individuals or agents) in weighted voting game (or committee) G and every subset $S \subset N$ is referred to as a coalition. The set N is called the grand coalition and $\varnothing$ is called the empty coalition. We denote the collection of all coalitions by 2^N and the number of players of coalition $S \in 2^N$ by $|S|$. Let us label the players by $1, 2, \ldots, n$, $n = |N| \geq 2$.

The aim of this paper is to discuss two basic problems in weighted voting games: (i) the distributions of decision power of players and (ii) the three processes of manipulation in weighted voting games, that is, by merging of two players into a single player, by players splitting into a number of smaller units, and by annexation of a part or all of the voting weights of another player.

A simple game in characteristic-function form is a pair $G = (N, v)$ *where* $N = \{1, 2, \ldots, n\}$ *is a set of players and* $v : 2^N \to \{0,1\}$ *is the characteristic function which satisfies the following three conditions:*

(1) $v(\varnothing) = 0$.

(2) $v(N) = 1$.

(3) v is monotonic, i.e. if $S \subset T \subset N$, then $v(S) \le v(T)$.

Thus, we formalize the idea of coalition decision making. It follows that the characteristic function v for a coalition $S \subset N$ indicates the value of S. This means that for each coalition $S \subset N$ we have either $v(S) = 0$ or $v(S) = 1$.

Two simple games $G_1 = (N_1, v_1)$ and $G_2 = (N_2, v_2)$ in characteristic-function forms are called equal when $N_1 = N_2$ and $v_1 = v_2$.

In this paper we will consider a special class of simple games called weighted voting games with dichotomous voting rule - acceptance ("yes") or rejection ("no"). These games have been found to be well-suited to model economic or political bodies that exercise some kind of control. A weighted voting game is one type of simple cooperative game and it is a formalization model of coalition decision making in which decisions are made by vote [16].

The basic formal framework of this study is as follows. A weighted voting game (N, v) is described by $G = [q; w_1, w_2, ..., w_n]$ where q is positive and $w_1, w_2, ..., w_n$ are nonnegative integer numbers such that $q \le \sum_{k=1}^{n} w_k = \tau$. By convention, we take $w_i \ge w_j$ when $i < j$. For more information see [15] and [23]. This game has the following properties:

(1) $1 \le q \le \tau$.

(2) $n = |N| \ge 2$ is the number of players.

(3) $w_i \ge 0$ is the number of votes of player $i \in N$ and $w_1 \ge 1$.

(4) $w_1 \ge w_2 \ge ... \ge w_n$.

(5) q is the needed quota so that a coalition can win.

(6) the symbol $[q; w_1, w_2, ..., w_n]$ represents the weighted voting game G defined by

$$v(S) = \begin{cases} 1, & \sum_{k \in S} w_k \ge q \\ 0, & \sum_{k \in S} w_k < q \end{cases}, \text{ where } S \subset N.$$

Of course, if $w_i = 0$ for $i \in N$, then player i is powerless, i.e. it is a dummy or null player.

Two weighted voting games $G_1 = [q_1; w_1^1, w_2^1, ..., w_m^1]$ and $G_2 = [q_2; w_1^2, w_2^2, ..., w_n^2]$ are equal when $m = n$, $q_1 = q_2$ and $w_i^1 = w_i^2$ for all players $i \in N$.

For any weighted voting game G, the form $[q; w_1, w_2, ..., w_n]$ is often called a weighted representation of game G. Obviously, one weighted voting game has many representations. For example, the following two weighted voting games $G_1 = [51; 49, 49, 2]$ and $G_2 = [2; 1, 1, 1]$ represent the same voting rule, i.e. they have the same characteristic function and each coalition of two or three players is winning.

2. Preliminaries

We continue our study with a consideration of three basic types of coalitions - winning, losing and blocking.

For any coalition $S \subset N$ in game G, S is winning if $v(S)=1$, S is losing if $v(S)=0$, and S is blocking if S and $N \setminus S$ are both losing coalitions. The collections of all winning, all losing and all blocking coalitions in game G are denoted by $W(G)$, $L(G)$ and $B(G)$, respectively. If game G is known, we simply write W, L and B.

Of course, any simple game has winning and losing coalitions and this game is determined by the set of all winning (or losing) coalitions. We also get that $N \in W$ and $\varnothing \in L$; therefore, W and L are nonempty, $W \bigcap L = \varnothing$, $W \bigcup L = 2^N$, $B \subset L$ and $W \bigcap B = \varnothing$. Observe that a coalition having a winning sub-coalition is also winning, a sub-coalition of a losing coalition is also losing, and the complement of a blocking coalition is also blocking. It is easy to show that B can be either empty or nonempty. From $\varnothing \in L$, $\varnothing \notin B$ and $B \subset L$ it follows that $L \setminus B$ is nonempty. Sometimes, coalitions of $L \setminus B$ are called strictly losing.

First, for any player $i \in N$, the collection of all winning coalitions including i is denoted by W_+^i and the collection of all winning coalitions excluding i is denoted by W_-^i. Clearly, if $S \in W_-^i$, then $S \bigcup \{i\} \in W_+^i$; therefore, we obtain the inequality $\left|W_-^i\right| \le \left|W_+^i\right|$. We also have that $W_+^i \bigcap W_-^i = \varnothing$, $W_+^i \bigcup W_-^i = W$ and $\left|W_-^i\right| \le \frac{1}{2}\left|W\right| \le \left|W_+^i\right|$.

Next, for any player $i \in N$, the collection of all losing coalitions including i is denoted by L_+^i and the collection of all losing coalitions excluding i is denoted by L_-^i, $L_+^i \bigcap L_-^i = \varnothing$ and $L_+^i \bigcup L_-^i = L$. From $S \in L_+^i$ it follows that $S \setminus \{i\} \in L_-^i$; therefore, we get that $\left|L_+^i\right| \le \frac{1}{2}\left|L\right| \le \left|L_-^i\right|$.

Finally, for any player $i \in N$, the collection of all blocking coalitions including i is denoted by B_+^i and the collection of all blocking coalitions excluding i is denoted by B_-^i. In this case we obtain $B_+^i \bigcap B_-^i = \varnothing$ and $B_+^i \bigcup B_-^i = B$.

For any coalition $S \in W$, S is called a minimal winning coalition if $S \setminus \{i\}$ is not winning for all $i \in S$. The collection of all minimal winning coalitions is denoted by MW for a known game or $MW(G)$ for any game G. For any player $i \in N$, the collection of all minimal winning coalitions including i is denoted by MW_+^i and the collection of all minimal winning coalitions excluding i is denoted by MW_-^i.

It is easy to prove that MW and W are finite sets, $MW \subset W$ and MW is nonempty. Clearly, we have that $MW_+^i \bigcap MW_-^i = \varnothing$, $MW_+^i \bigcup MW_-^i = MW$, $MW_+^i \subset W_+^i$ and $MW_-^i \subset W_-^i$ for all $i \in N$.

Thus, a simple game (N,v) can alternatively be defined in winning-set form as (N,W) or (N,MW). There are the extensive winning or the extensive minimal winning form, respectively.

For any coalition $S \in L$, S is called a maximal losing coalition if $S \bigcup \{i\}$ is not losing for all $i \in N \setminus S$. The collection of all maximal losing coalitions is denoted by ML. For any player $i \in N$, the collection of all maximal losing coalitions including i is denoted by ML_+^i and the collection of all maximal losing coalitions excluding i is denoted by ML_-^i.

By analogy, ML and L are finite sets, $ML \subset L$ and ML is nonempty, and $ML_+^i \bigcap ML_-^i = \varnothing$, $ML_+^i \bigcup ML_-^i = ML$, $ML_+^i \subset L_+^i$ and $ML_-^i \subset L_-^i$ for all $i \in N$.

The set of minimal winning coalitions determines a simple game uniquely. When $MW(G_1) = MW(G_2)$ we call that games G_1 and G_2 are equivalent. The same applies to the set of maximal losing coalitions.

A player who does not belong to any minimal winning coalition is called a dummy, i.e. player $i \in N$ is a dummy if $i \notin S$ for all $S \in MW$. A player who belongs to all minimal winning coalitions is called a veto player or vetoer, i.e. player $i \in N$ has the capacity to veto if $i \in S$ for all $S \in MW$. A player $i \in N$ is a dictator if $\{i\}$ is a winning coalition.

In voting power theory, a dummy player has no decision power, a veto player can block every decision and a dictator has all of the decision power. Formally, for any player $i \in N$, i being a dictator is equivalent to $\{i\} \in MW$, i being a veto player is equivalent to $i \in \bigcap_{S \in MW} S$ (or $i \in \bigcap_{S \in W} S$) and i being a dummy is equivalent to $i \notin \bigcup_{S \in MW} S$.

For any player $i \in N$, it is easy to show that $MW_+^i = \varnothing$ is equivalent to player i being a dummy and $MW_+^i = MW$ (or $W_+^i = W$) is equivalent to player i being a veto player.

Now we will consider three examples.

Example 1. The voting method of the Security Council of the United Nations, formed by 5 permanent (USA, UK, France, Russia and China) and 10 temporary members, is a game in which each one of the permanent member has 7 votes and each one of the temporary member has only one vote, the established quota is 39 votes, and there are 45 total votes. We observe that any coalition which does not include all of the 5 permanent members has at most $4 \times 7 + 10 = 38$ votes, which is an inferior number to the fixed quota. As a result this coalition is not winning. Hence, each one of the permanent members has the capacity to veto any proposal. For more information see [13] and [24].

Example 2. The Bulgarian Parliament with 240 seats uses two different rules: a simple majority by quota 121 (more than $\frac{1}{2}$) and a qualified majority by quota 161 (more than $\frac{2}{3}$). The Finish Parliament with 200 seats uses three different rules: a simple majority by quota 101 (more than $\frac{1}{2}$), a qualified majority by quota 134 (more than $\frac{2}{3}$), and in some special cases by quota 167 (more than $\frac{5}{6}$) [13].

Example 3. The U. S. Congress has a nonvoting delegate who represents the District of Columbia; therefore, this delegate is a dummy. Note that in principle a player can be assigned weight zero, but in practice this player would be silly, because it would be a dummy. However, a player having positive weight can also be a dummy. In fact, this was the case with Luxembourg as a member of the European Union Council of Ministers during the period 1958-1972, when its weight was one but it was a dummy [19].

In the following example we illustrate that there exists a simple game that is not a weighted voting game [10].

Example 4. Consider simple game G given by $N=\{1,2,3,4,5\}$ and $MW=\{\{1,2,3\},\{4,5\}\}$. Let us assume that there exist quota q and weights w_1, w_2, w_3, w_4, w_5 such that G is a weighted voting game. This means that $w_1+w_2+w_3 \geq q$ and $w_4+w_5 \geq q$. As a result we obtain $q > w_1+w_2+w_4 \geq q-w_3+w_4$; therefore, we have that $w_3 > w_4$. Thus, we find that $\{3,5\} \in W$, but $\{3\},\{5\} \in L$. It follows that $\{3,5\} \in MW$. This leads to a contradiction; therefore, G is not a weighted voting game.

Now, let us analyze the sum $v(S)+v(N \setminus S)$ for $S \subset N$. Clearly, $0 \leq v(S) \leq 1$ imply inequalities $0 \leq v(S)+v(N \setminus S) \leq 2$.

A weighted voting game G is called proper if $v(S)+v(N \setminus S) \leq 1$ for all $S \subset N$.

Note that a weighted voting game being proper is equivalent to the complement of a winning coalition is not winning. This means that in a proper game both coalitions S and $N \setminus S$ cannot be winning. In this context, if S is winning, then $N \setminus S$ is losing, but the converse statement is not always true.

Clearly, the following statements are true:

(1) A proper game may have only one dictator and if there is a dictator, then player 1 is the only dictator and the only veto player, all other players are dummies and there is no blocking coalition. Note that a proper game with two or more veto players does not have a dictator.

(2) An improper game may have at least one pair of non-intersecting winning coalitions. In particular, an improper game may have more than one dictator.

In what follows, we will only study proper games.

A proper game G is called decisive (or strong) if $v(S)+v(N \setminus S)=1$ for all $S \subset N$.

It is easy to prove that a proper game being decisive (or strong) is equivalent to it having no blocking coalition. For any coalition $S \subset N$ in a decisive game, S being winning is equivalent to $N \setminus S$ being losing. If a proper game has a dictator, then this game is decisive, but the converse statement is not true.

Note that the set of all winning coalitions and the set of all minimal winning coalitions in weighted voting game $[q;\, w_1,\, w_2,\ldots, w_n]$ are the same as the set of all winning coalitions and the set of all minimal winning coalitions in weighted voting game $[\lambda q;\, \lambda w_1,\, \lambda w_2,\ldots, \lambda w_n]$ for every positive integer number λ. As a result we obtain that weighted voting game $G=[q;\, w_1,\, w_2,\ldots, w_n]$ is equivalent to game $G_\lambda=[\lambda q;\, \lambda w_1,\, \lambda w_2,\ldots, \lambda w_n]$. For integer number $\lambda > 1$, the two distinct representations $[q;\, w_1,\, w_2,\ldots, w_n]$ and $[\lambda q;\, \lambda w_1,\, \lambda w_2,\ldots, \lambda w_n]$ are equivalent. It follows that the number of representations of a weighted voting game is infinitive. This means that G and G_λ are equal as two simple games.

For any proper game G, a pair of players $i, j \in N$ is called symmetric (or i and j are symmetric) if $v(S \bigcup \{i\}) = v(S \bigcup \{j\})$ for all coalitions $S \in N \setminus \{i, j\}$. Proper game G is symmetric if every pair of players is symmetric.

Of course, if $w_i = w_j$, then players i and j are symmetric, but the converse statement is not true. For example, see game $G=[30;\, 14,\, 13,\, 11,\, 10]$. The players of G have different weights but every pair of players are symmetric. Hence, symmetric does not imply that all players have equal weights, but symmetric implies that all players are granted equal impact on collective decisions.

It is interesting to note that game $[30; 14, 13, 11, 10]$ is equivalent to game $[3; 1, 1, 1, 1]$ and game $[3; 1, 1, 1, 1]$ is symmetric. It is easy to prove that if G_1 and G_2 are equivalent games and G_1 is symmetric, then G_2 is also symmetric because the set of all minimal winning coalitions of game G_1 is equal to the set of all minimal winning coalitions of game G_2.

Consider proper game $G = [q; w_1, w_2, w_3]$ and let us assume that $\{2,3\} \in W$. For example, let us consider game $G = [9; 7, 6, 4]$. It is easy to prove that $MW = \{\{1,2\},\{1,3\},\{2,3\}\}$. Game G has representation $[2; 1, 1, 1]$; therefore, game G is symmetric and all players have equal decision power.

Consider a pair of symmetric players $i, j \in N$. It follows that if $i \in S \subset N$ and $j \notin S$, then $v(S) = v(S \bigcup \{j\} \setminus \{i\})$. Now, it is easy to see that $|W_+^i| = |W_+^j|$, $|L_+^i| = |L_+^j|$ and $|B_+^i| = |B_+^j|$.

It is easy to see that the player's ability to influence the outcome of a weighted voting game is related to the player's weight, but it is not always directly proportional to the decision power of this player. For example, let us consider a game with 4 players, player 1 has 58 votes, player 2 has 25 votes, player 3 has 10 votes, player 4 has 7 votes and the quota is 55. We may think that player 1 has 58 % of the decision power, players 2, 3 and 4 have 25 %, 10% and 7 %, respectively. But this is not true because player 1 has 100 % of the power, and players 2, 3 and 4 are powerless or they are dummies. Sometimes, a player may want the voting body not to make a decision, i.e. he/she wants to block decision-making. In our example, if the quota is 80, then player 2 can block every decision, i.e. player 2 has capacity to veto. Note that players 3 and 4 do not have the capacity to veto. But, if the quota is 85, then a coalition of players 3 and 4 may block decision-making; therefore, this coalition is blocking.

For $i \in N$ and $S \in W_+^i$, player i is called a negative swing member of S (critical or pivotal) if $S \setminus \{i\}$ is not winning. For any player $i \in N$, the collection of all winning coalitions including i as a negative swing number is denoted by W_s^i. For $i \in N$ and $S \in L_-^i$, player i is called a positive swing member of S (critical or pivotal) if $S \bigcup \{i\}$ is not losing. For any player $i \in N$, the collection of all losing coalitions including i as a positive swing number is denoted by L_s^i.

It is often said that $|W_s^i|$ and $|L_s^i|$ are the number of swings of player $i \in N$.

Note that each member of a minimal winning coalition is a negative swing player, each member of a maximal losing coalition is a positive swing player, a winning coalition may have a negative swing member and a losing coalition may have a positive swing member.

It is easy to show that each positive swing for player $i \in N$ corresponds to a pair of coalitions $(S, S \bigcup \{i\}) \in L_-^i \times W_+^i$ such that S is losing and $S \bigcup \{i\}$ is winning, and each negative swing for player $i \in N$ corresponds to a pair of coalitions $(S \setminus \{i\}, S) \in L_-^i \times W_+^i$ such that $S \setminus \{i\}$ is losing and S is winning. In the first case we say that i is a swing player for the pair $(S, S \bigcup \{i\})$, but in the second case we say that that i is a swing player for the pair $(S \setminus \{i\}, S)$.

It is easy to show that if a weighted voting game has a dictator, then he/she is the only swing player in this game.

Theorem 1 [4]. For any proper game $|W_s^i| = |L_s^i|$ for all $i \in N$.

It is important to note that in the proof of the above statement the authors construct a one-to-one mapping $m_i : W_s^i \to L_s^i$ such that coalition $S \in W_s^i$ only corresponds to coalition $S \setminus \{i\} \in L_s^i$ and conversely, coalition $S \setminus \{i\} \in L_s^i$ only corresponds to coalition $S \in W_s^i$.

3. Two Voting Paradoxes

Weighted voting games are mathematical abstractions of real voting systems. In this section we will describe two paradoxes in real voting systems. In principle a player can be assigned weight zero, but in practice this player would be silly, because it would be a dummy. However, a player having a positive weight can also be a dummy. Note that the possibility of dummy players in a real voting systems is a big problem from a democratic point of view.

3.1. The Voting Paradox of Luxembourg

In this subsection we will see that Luxembourg was a powerless country as a member of the European Union Council of Ministers during the period 1958-1973.

Let us consider the Council of Ministers during the above period from a mathematical point of view. The decision rule is a weighted voting game $L58 = [12; 4, 4, 4, 2, 2, 1]$. In this game the players are Germany, France, Italy, Belgium, Netherlands and Luxembourg, respectively.

Note that the total sum of the weights is 17 and the quota is 12, i.e. $n = 6$, $\tau = 17$ and $q = 12$. It is easy to show that this game is proper, see also [13], [20] and [24].

The number of winning coalitions is 14 and they are: {1,2,3,4,5,6}, {1,2,3,4,5}, {1,2,3,4,6}, {1,2,3,5,6}, {1,2,3,4}, {1,2,3,5}, {1,2,3,6}, {1,2,4,5,6}, {1,3,4,5,6}, {2,3,4,5,6}, {1,2,3}, {1,2,4,5}, {1,3,4,5} and {2,3,4,5}. The number of minimal winning coalitions is 4 and they are: {1,2,3}, {1,2,4,5}, {1,3,4,5} and {2,3,4,5}. As a result we decide that player 6 is a dummy. So player 6 or Luxembourg formally was never able to make any difference in the voting process and was a dummy during the period 1958-1973 [19].

Obviously, game $L58 = [12; 4, 4, 4, 2, 2, 1]$ has no dictator, has no veto player and has a dummy player.

We also get that this game has at least one blocking coalition. Hence, this game is not decisive. For example, coalition $\{1,4\}$ is blocking and coalition $\{4,5,6\}$ is losing but not blocking.

Clearly, players 1, 2 and 3 are symmetric, and players 4 and 5 are symmetric because they have equal weights, respectively.

These notes allow us to discuss the case when a player has positive weight and is a dummy.

Let us consider a proper game $[q; w_1, w_2, ..., w_{n-1}, w_n]$ when $n \geq 3$ and $w_n > 0$. Then, $w_i > 0$ for all $i \in N$.

Theorem 2. Let $i, j \in N$ and $i \neq j$. The following statements are true.

(a) If player i is not a dummy and player j is a dummy, then $w_i > w_j$.

(b) If player i is a dummy and $w_i \geq w_j$, then player j is also a dummy.

Proof. (a) Let us denote $MW_k = \{S \in MW : k \in S\}$ for $k \in N$. It is easy to show that MW_i is not empty and MW_j is empty.

If $S \in MW_i \subset MW$, then $\sum_{k \in S} w_k \geq q$. For $T = S \setminus \{i\}$ it follows that $\sum_{k \in T} w_k + w_i \geq q$ and $\sum_{k \in T} w_k < q$, i.e. T is a losing coalition. It is known that player j is a dummy; therefore, $j \notin S$ and $j \notin T$. Let us now consider the coalition $P = T \bigcup \{j\}$. There are two cases:

<u>Case 1.</u> Let P be a losing coalition. In this case we have $\sum_{k \in T} w_k + w_j < q$.

<u>Case 2.</u> Let P be a winning coalition. From the condition j is a dummy it follows that $T = P \setminus \{j\}$ is a winning coalition too. This leads to a contradiction.

Finally, we obtain $\sum_{k \in T} w_k + w_j < q$.

From the inequalities $\sum_{k \in T} w_k + w_i \geq q$ and $\sum_{k \in T} w_k + w_j < q$ we get $w_i > w_j$.

(b) Let us assume that player j is not a dummy. From player i being a dummy and part (a) it follows that $w_i < w_j$. This leads to a contradiction with the condition $w_i \geq w_j$; therefore, player j is a dummy.

The theorem is proven.

3.2. The Voting Paradox of Nassau Country

In this subsection we will study Nassau Country, New York, which is a region on Long Island. The government of Nassau Country took the form of a Board of Supervisors in 1958 for the period 1958-1964 and in 1964 for the period 1964-1970, one representative for each of various municipalities, who casts a block of votes [14] [20].

There are two special weighted voting games used at various times by Nassau Country. We will discuss these two voting games.

3.2.1. The First Voting Game

The first decision rule in 1958 was a weighted voting game $N58 = [16; 9, 9, 7, 3, 1, 1]$. In this game the players were Hempstead 1, Hempstead 2, North Hempstead, Oyster Bay, Long Beach and Glen Cove, respectively. In game $N58$, the total sum of the weights was 30 and the quota was 16, i.e. $n = 6$, $\tau = 30$ and $q = 16$. As a result we find that game $N58$ is proper. It is easy to show that this game has no blocking coalition; therefore, it is decisive.

The number of minimal winning coalitions is 3 and they are: {1,2}, {1,3} and {2,3}. As a result we see that Oyster Bay, Long Beach and Glen Cove (players 4, 5 and 6) are dummies. See Theorem 2.

Note that the number of all winning coalitions is $\left(\binom{3}{2} + \binom{3}{3}\right) \times \left(\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}\right) = 4 \times 8 = 32$, i.e. $|W| = 32$ and $|MW| = 3$.

In this game players 1, 2 and 3 are symmetric, but they do not have equal weights. Players 4, 5 and 6 do not have equal weights and they are symmetric too.

3.2.2. The Second Voting Game

The second decision rule in 1964 was a weighted voting game $N64 = [58; 31, 31, 28, 21, 2, 2]$. In this game the players were Hempstead 1, Hempstead 2, Oyster Bay, North Hempstead, Long Beach and Glen Cove, respectively. In game $N64$, the total sum of the weights was 115 and the quota was 58, i.e. $n = 6$, $\tau = 115$ and $q = 58$. It is easy to show that this game is also decisive.

The number of all winning coalitions is also 32. The number of minimal winning coalitions is also 3 and they are: {1,2}, {1,3} and {2,3}. As a result we see that North Hempstead, Long Beach and Glen Cove (players 4, 5 and 6) are dummies. See also Theorem 2.

As in game $N58$, we see in game $N64$ that players 1, 2 and 3 are symmetric, they do not have equal weights, players 4, 5 and 6 are also symmetric and they do not have equal weights.

4. Distribution of Decision Power between the Players

The concept of decision power of the players in weighted voting games is well-known. For example, let us consider a game $G = [51; 62, 27, 11]$. We may be tempted to say that player 1 has 62 % of the decision power, players 2 and 3 have 27 % and 11 %, respectively. But this is not true because player 1 has 100 % of the power, and players 2 and 3 are powerless, i.e. player 1 is a dictator and players 2 and 3 are dummies. Weighted voting games use mathematical models to analyze the distribution of decision power of the players. This distribution of decision power is central in economics and political science.

The proportional index $\alpha(\alpha_1, \alpha_2, ..., \alpha_n)$ is the trivial power index given by $\alpha_i = \dfrac{w_i}{\tau}$ for player $i \in N$. This measure is popularly known as Gamson's Law or Gamson's Index [8].

There are two most widely used measures of voting power in the weighted voting games - the Shapley-Shubik power index and the Banzhaf power index. The Shapley-Shubik power index in a voting situation depends on the number of orderings in which each player can affect a positive swing. The Banzhaf power index depends on the number of ways in which each player can affect a negative swing.

The Shapley-Shubik power index was introduced by the mathematician Lloyd Shapley and the economist Martin Shubik in 1954 [18]. For player $i \in N$ this index is defined by

$$\phi_i = \sum_{S \notin L, S \bigcup \{i\} \in W} \frac{s!(n-s-1)!}{n!},$$

where $s = |S|$. If we assume that all $n!$ orderings are equiprobable, then ϕ_i is the probability of player i being a positive swing member in a winning coalition, that is, $S \bigcup \{i\}$ is a winning and S is a losing coalition.

In classical theory, a negative swing for player $i \in N$ corresponds to a pair of coalitions $(S \bigcup \{i\}, S)$ such that $S \bigcup \{i\}$ is winning and S is losing, i.e. $i \notin S$, $v(S \bigcup \{i\}) = 1$ and $v(S) = 0$. It is easy to show that $v(S) - v(S \setminus \{i\})$ is always either zero or one for all $S \subset N$ and all $i \in N$. If $S \subset N$ and $i \notin S$, then $v(S) - v(S \setminus \{i\}) = 0$. If $S \subset N$ and $i \in S$, then $v(S) - v(S \setminus \{i\}) = 1$ (when S is wining and $S \setminus \{i\}$ is losing) or $v(S) - v(S \setminus \{i\}) = 0$ (when S and $S \setminus \{i\}$ are wining or S and $S \setminus \{i\}$ are losing).

It follows two equivalent forms of the Shapley-Shubik index and they are defined as

$$\phi_i = \sum_{S \subset N} \frac{(s-1)!(n-s)!}{n!}\big(v(S) - v(S \setminus \{i\})\big) = \sum_{S \subset N, i \in S} \frac{(s-1)!(n-s)!}{n!}\big(v(S) - v(S \setminus \{i\})\big).$$

The Shapley-Shubik index is the vector $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)$ and it has the normalization property, i.e. $\sum_{i=1}^{n} \phi_i = 1$.

The Banzhaf power index was introduced by the American jurist and law professor John Banzhaf III in 1965 [1]. The absolute Banzhaf index concerns the number of times each player $i \in N$ could change a coalition from losing to winning and it requires that we know the number of negative swings for each player i. For each player $i \in N$, the absolute Banzhaf index is denoted by η_i and it equals the number of negative swings for this player, i.e. $\eta_i = \sum_{S \subset N} \big(v(S) - v(S \setminus \{i\})\big) = \sum_{S \subset N, i \in S} \big(v(S) - v(S \setminus \{i\})\big) = \left|W_s^i\right|$ for all $i \in N$.

The normalized Banzhaf power index is the vector $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, given by

$$\beta_i = \frac{\eta_i}{\sum_{k=1}^{n} \eta_k} \text{ for } i = 1, 2, \ldots, n.$$

The Banzhaf index is similar of the Penrose-Banzhaf (or Banzhaf-Coleman) index which is defined by $b_i = \sum_{S \subset N} \frac{v(S) - v(S \setminus \{i\})}{2^{n-1}} = \frac{\left|W_s^i\right|}{2^{n-1}} = \frac{\eta_i}{2^{n-1}}$ for $i \in N$. The Banzhaf index was originally created in 1946 by Leonel Penrose, but was reintroduced by John Banzhaf in 1965.

In [4] and [11] it is proven that for any player $i \in N$, $\eta_i = \left|W_s^i\right| = \left|L_s^i\right| = \left|W_+^i\right| - \left|W_-^i\right|$.

Example 5. Consider a weighted voting game $[7; 3, 3, 1, 1]$. The collections of all wining and all minimal wining coalitions are $W = \{\{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}\}$ and $MW = \{\{1,2,3\}, \{1,2,4\}\}$, respectively. The swings of player 1 correspond to three pairs of coalitions $(\{2,3\}, \{1,2,3\})$, $(\{2,4\}, \{1,2,4\})$ and $(\{2,3,4\}, \{1,2,3,4\})$, i.e. we get that $\eta_1 = 3$. Similarly, we obtain $\eta_2 = 3$, $\eta_3 = 1$ and $\eta_4 = 1$.

It is important to note that the Shapley-Shubik power index and the Banzhaf power index are monotonic with respect to the weights when we are evaluating the power, i.e. for $i, j \in N$, $\phi_i = \phi_j$ and $\eta_i = \eta_j$ when $w_i = w_j$, and $\phi_i \geq \phi_j$ and $\eta_i \geq \eta_j$ when $w_i > w_j$. We also get that for any proper game, player $i \in N$ being a dummy is equivalent to $\eta_i = \phi_i = 0$, see also [19]. Thus it follows that player $i \in N$ being a dummy is equivalent to i not being a swing player.

Coleman considered two different power indices of the players in a game, see also [2] and [5]. For any player $i \in N$, they are defined as follows.

(a) The preventive power index $P_i = \dfrac{\left|W_s^i\right|}{|W|}$.

(b) The initiative power index $I_i = \dfrac{\left|L_s^i\right|}{|L|}$.

It is easy to show that $0 \le P_i \le 1$ and $0 \le I_i \le 1$ for all $i \in N$.

For any player $i \in N$, both indices P_i and I_i achieve their lower bound of 0 if and only if player i is a dummy; index P_i achieves its upper bound of 1 if and only if i is a veto player; and index I_i achieves its upper bound of 1 if and only if player i is a dictator [2] [3].

It is natural to postulate that player i being a dummy is equivalent to $|W_s^i| = |L_s^i| = 0$ (or player i is never a swing player), that player i being a vetoer is equivalent to $|W_s^i| = |W|$, and player i being a dictator is equivalent to $|L_s^i| = |L|$.

In [7], the authors prove that for any non-dummy player $i \in N$ the Penrose-Banzhaf index b_i is the harmonic mean of the two Coleman indices P_i and I_i, i.e. $\frac{2}{b_i} = \frac{1}{P_i} + \frac{1}{I_i}$. It is important to point out that the Coleman power indexes are monotonic with respect to the weights when we are evaluating the power, i.e. for two different players $i, j \in N$, $P_i = P_j$ and $I_i = I_j$ when $w_i = w_j$ and $P_i \ge P_j$ and $I_i \ge I_j$ when $w_i > w_j$.

5. Methods of Manipulation

Weighted voting games are cooperative games; therefore, analyses of manipulation are natural. The study of methods of manipulation also has practical applications. Manipulation is a change or influence on the weighted voting game that changes the decision power of the players within the legal rules. It particular, decision rules in voting games can be manipulated by coalitions merging into single players and players splitting into a number of smaller units [12]. Decision rules can also be manipulated by annexation of a part or all of the weights of other players.

Now we will study three cases of manipulations – merging, splitting and annexation.

First, we will focus our attention on manipulation by merging, that is, two or more different players merge into a single player. Consider a proper weighted voting game $G = [q; w_1, w_2, ..., w_n]$. We construct a new game $G' = [q; w'_1, w'_2, ..., w'_{n'}]$ such that $n > n'$, each players i' in game G' is a fixed coalition of one or more players in game G and its weight $w'_{i'}$ is the sum of the weights of this fixed coalition. The quota and the total sum of the weights remain the same. We denote the set of all players in games G and G' by N and N', respectively. Game G' is called a derivative game of the original game G. For more information see [21].

Example 6. Let $G = [5; 4, 3, 1, 1]$ be an original game. For the derivative game G', let the coalition of players 2 and 4 in game G be a new player in game G' (players 2 and 4 merge into a single player and it is player $2'$) and the other players remain the same. Thus, we get the derivative game $G' = [5; 4, 4, 1]$.

Example 7. Let $G = [17; 8, 7, 4, 4, 2, 1, 1]$ be an original game where $n = 7$, $q = 17$ and $\tau = 27$. For the derivative game G', let players 2, 3 and 5 in game G as a coalition be a new player in G' and the other players be the same. So we get $G' = [17; 13, 8, 4, 1, 1]$ where $m = 5$, $q = 17$, $\tau = 27$, $1' = \{2,3,5\}$, $2' = \{1\}$, $3' = \{4\}$, $4' = \{6\}$ and $5' = \{7\}$. The sum of the Banzhaf power indices of players 2, 3 and 5 in game G is $\beta_2 + \beta_3 + \beta_5 = 0{,}4851$ but this index of player $1'$ in game G' is $\beta'_{1'} = 0{,}6000$. As a result we obtain $\beta_2 + \beta_3 + \beta_5 < \beta'_{1'}$.

Example 8. Let $G=[30;9,8,5,5,4,3,1]$ be an original game where $n=7$, $q=30$ and $\tau=35$. For the derivative game G', let players 1, 2 and 3 in game G as a coalition be a new player in G' and the other players be the same. In this case we get $G'=[30;22,5,4,3,1]$ where $m=5$, $q=30$, $\tau=35$, $1'=\{1,2,3\}$, $2'=\{4\}$, $3'=\{5\}$, $4'=\{6\}$ and $5'=\{7\}$. Here the sum of the Banzhaf power indices of players 1, 2 and 3 in game G is $\beta_1+\beta_2+\beta_3=0{,}5789$ but this index of player $1'$ in game G' is $\beta'_{1'}=0{,}3684$. Now we obtain $\beta_1+\beta_2+\beta_3>\beta'_{1'}$.

Theorem 3 [22]. Transform game G to game G' by merging of two different players i and j into player i', and the other players remain the same. The following statements are true.

(a) $\eta_i+\eta_j=2\eta'_{i'}$, $|W_s^i(G)|+|W_s^j(G)|=2|W_s^{i'}(G')|$ and $|L_s^i(G)|+|L_s^j(G)|=2|L_s^{i'}(G')|$.

(b) If player j is a dummy in game G, then $\eta_i=2\eta'_{i'}$, $|W_s^i(G)|=2|W_s^{i'}(G')|$ and $|L_s^i(G)|=2|L_s^{i'}(G')|$

(c) Players i and j being dummies in game G is equivalent to player i' being a dummy in game G'.

(d) If $w_i=w_j$, then $2w_i=w'_{i'}$, $\eta_i=\eta'_{i'}$, $|W_s^i(G)|=|W_s^{i'}(G')|$ and $|L_s^i(G)|=|L_s^{i'}(G')|$.

(e) If $k\neq i,j$ and player k in game G transforms to player k' in game G', then $\eta_k\geq\eta'_{k'}$, $|W_s^k(G)|\geq|W_s^{k'}(G')|$ and $|L_s^k(G)|\geq|L_s^{k'}(G')|$.

(f) If player i' is a dummy in game G', then $\beta_i+\beta_j=\beta'_{i'}=0$.

(g) If player i' is not a dummy in game G', then $\beta_i+\beta_j<2\beta'_{i'}$.

(h) If $w_j\leq w_i$ and player i' is not a dummy in game G', then $\beta_j<\beta'_{i'}$.

(i) If player i is a dictator in game G, then player i' is a dictator in game G' and $\beta_i+\beta_j=\beta'_{i'}=1$.

Proof. (a) Let $S\in L_s^{i'}(G')$, $w_i+w_j=w'_{i'}$ and let us assume that $w_j\leq w_i$. This means that $0<q-\sum_{h\in S}w_h\leq w'_{i'}=w_j+w_i$ and $S\bigcup\{i'\}\in W(G')$.

There are three cases for positive swings of each player i, j or i'.

Case 1. If $0<q-\sum_{h\in S}w_h\leq w_j$, then $q\leq\sum_{h\in S}w_h+w_j$ and $q\leq\sum_{h\in S}w_h+w_i$. As a result we see that player i' is a swing member of pair $(S,S\bigcup\{i'\})$ in game G', and players j and i are swing members of pair $(S,S\bigcup\{j\})$ and pair $(S,S\bigcup\{i\})$ in game G, respectively. Hence, in this case we have $\eta_i+\eta_j=2\eta'_{i'}$.

Case 2. If $w_j<q-\sum_{h\in S}w_h\leq w_i$, then $\sum_{h\in S}w_h+w_j<q\leq\sum_{h\in S}w_h+w_i$. Here we get that player i' is a swing member of pair $(S,S\bigcup\{i'\})$ in game G', and players j and i are swing members of pair $((S\bigcup\{i\})\setminus\{j\},S\bigcup\{i\})$ and pair $(S,S\bigcup\{i\})$ in game G, respectively. We also obtain $\eta_i+\eta_j=2\eta'_{i'}$.

Case 3. If $w_i < q - \sum_{h \in S} w_h \le w'_{i'}$, then $\sum_{h \in S} w_h + w_i < q \le \sum_{h \in S} w_h + w_j + w_i$. So we have that player i' is a swing member of pair $(S, S \cup \{i'\})$ in game G', and players j and i are swing members of pair $((S \cup \{i\}) \setminus \{j\}, S \cup \{i\})$ and pair $((S \cup \{j\} \setminus \{i\}), S \cup \{j\})$ in game G, respectively. Here we get $\eta_i + \eta_j = 2\eta'_{i'}$ too.

In summary, we obtain $\eta_i + \eta_j = 2\eta'_{i'}$. From $\eta_i = |W_s^i(G)| = |L_s^i(G)|$, $\eta_j = |W_s^j(G)| = |L_s^j(G)|$ and $\eta'_i = |W_s^{i'}(G')| = |L_s^{i'}(G')|$ it follows that $|W_s^i(G)| + |W_s^j(G)| = 2|W_s^{i'}(G')|$ and $|L_s^i(G)| + |L_s^j(G)| = 2|L_s^{i'}(G')|$.

(b) If player j is a dummy, then $W_s^j(G)$ is empty. According to (a) we get that $\eta_i = 2\eta'_{i'}$, $|W_s^i(G)| = 2|W_s^{i'}(G')|$ and $|L_s^i(G)| = 2|L_s^{i'}(G')|$.

(c) This is immediate from (a).

(d) The proof follows from (a).

(e) Let us consider a player $k' \in N'$ in game G' such that $k' \ne i'$ and coalition $S \in L_s^{k'}(G')$. This means that $S \in L_s^k(G)$; therefore, we obtain $|L_s^k(G)| \ge |L_s^{k'}(G')|$. It also follows that $\eta_k \ge \eta'_{k'}$ and $|W_s^k(G)| \ge |W_s^{k'}(G')|$.

(f) If player i' is a dummy, then $W_s^{i'}(G')$ is empty. This means that $|W_s^{i'}(G')| = 0$, i.e. $\beta'_{i'} = 0$. Thus we find that $\beta_i = \beta_j = 0$; therefore, $\beta_i + \beta_j = \beta'_{i'} = 0$.

(g) If player i' is not a dummy, then $\eta'_{i'} > 0$. Clearly, we have that $\eta_i + \eta_j = 2\eta'_{i'} > 0$, $2\eta'_{i'} > \eta'_{i'}$ and $\eta_i > 0$ or $\eta_j > 0$, see (a) and (f).

Applying now (a) and (e) we calculate that

$$\beta_i + \beta_j = \frac{\eta_i + \eta_j}{\sum_{h \in N} \eta_h} = \frac{\eta_i + \eta_j}{\eta_i + \eta_j + \sum_{h \in N \setminus \{i.j\}} \eta_h} \le \frac{2\eta'_{i'}}{2\eta'_{i'} + \sum_{h \in N \setminus \{i'\}} \eta'_h} < \frac{2\eta'_{i'}}{\eta'_{i'} + \sum_{h \in N'} \eta'_h} < 2\beta'_{i'}.$$

Finally, we obtain $\beta_i + \beta_j < 2\beta'_{i'}$.

(h) We have that $\beta_j \le \beta_i$; therefore, we find that $2\beta_j \le \beta_i + \beta_j < 2\beta'_{i'}$, i.e. $\beta_j < \beta'_{i'}$.

(i) If player i is a dictator in game G, then the other players in game G are dummies. These players are dummies in game G' too, see (a), (e) and (f). Hence, player i' is a dictator in game G'. As a result we obtain $\beta_j = 0$, $\beta_i = \beta'_{i'} = 1$ and $\beta_i + \beta_j = \beta'_{i'} = 1$.

The theorem is proven.

Now we will consider manipulation by splitting, that is, a player splits into a number of smaller different players. Consider a proper weighted voting game $G=[q; w_1, w_2,..., w_n]$. We construct a new game $G'=[q; w'_1, w'_2,..., w'_{n'}]$ such that $n<n'$, each player i in game G splits to one or more players in game G' and the weight w_i of player i in game G is the sum of the weights of the split players in game G'. The quota and the total sum of the weights remain the same. Game G' is called a derivative game of the original game G. In the other words, game G transforms into game G' by the splitting of player i into players i' and j' while the other players remain same, and $w_i = w'_{i'} + w'_{j'}$.

Example 9. Let $G=[5; 4, 4, 1]$ be an original game. For the derivative game G', let player 2 in game G split into two players with weights 3 and 1 while the other players remain the same. Thus, we get the derivative game $G'=[5; 4, 3, 1, 1]$.

Example 10. Let $G=[5; 2, 2, 2]$ be an original game where $n=3$, $q=5$ and $\tau=6$. For the derivative game G', let player 3 with weight 2 split up into two players with weight 1. So in the new game G' players $3'$ and $4'$ will have weight 1. This means that $G'=[5; 2, 2, 1, 1]$ where $m=4$, $q=5$ and $\tau=6$. The Banzhaf power index of player 3 in game G is $\beta_3 = 1/3$ and this index of players $3'$ and $4'$ in game G' is $\beta'_{3'} = \beta'_{4'} = 1/8$. As a result we find that $\beta_3 > \beta'_{3'} + \beta'_{4'}$.

Example 11. Let $G=[4; 2, 2, 2]$ be an original game where $n=3$, $q=4$ and $\tau=6$. For the derivative game G', let player 3 with weight 2 split up into two players with weight 1. So in the new game G' players $3'$ and $4'$ will have weight 1. This means that $G'=[4; 2, 2, 1, 1]$ where $m=4$, $q=4$ and $\tau=6$. The Banzhaf power index of player 3 in game G is $\beta_3 = 1/3$ and this index of players $3'$ and $4'$ in game G' is $\beta'_{3'} = \beta'_{4'} = 1/6$. As a result we find that $\beta_3 = \beta'_{3'} + \beta'_{4'}$.

Example 12. Let $G=[6; 2, 2, 2]$ be an original game where $n=3$, $q=6$ and $\tau=6$. For the derivative game G', let player 3 with weight 2 split up into two players with weight 1. So in the new game G' players $3'$ and $4'$ will have weight 1. This means that $G'=[6; 2, 2, 1, 1]$ where $m=4$, $q=6$ and $\tau=6$. The Banzhaf power index of player 3 in game G is $\beta_3 = 1/3$ and this index of players $3'$ and $4'$ in game G' is $\beta'_{3'} = \beta'_{4'} = 1/4$. As a result we find that $\beta_3 < \beta'_{3'} + \beta'_{4'}$.

It is necessary to note that the converse process of manipulation by merging is the process of manipulation by splitting.

In the end, we will discuss manipulation by annexation, that is, a player annexes a part or all of the voting weights of other players. Consider a proper weighted voting game $G=[q; w_1, w_2,..., w_n]$. We construct a new game $G'=[q; w'_1, w'_2,..., w'_{n'}]$ such that $n=n'$, fix two different players i and j in game G, player i annexes a part or all of the voting weight of player j, i.e. the new weight of player i is $w'_i = w_i + t$ and the new weight of player j is $w'_j = w_j - t$ for $w_j \geq t > 0$. In the other words, player i takes t votes from player j and the other players remain the same. Game G' is called a derivative game of the original game G.

Example 13. Let $G=[6; 4, 3, 2, 1]$ be an original game. For the derivative game G', let player 1 with weight 4 in game G annex one vote of player 3 with weight 2. As a result we get that player 1 has weight 5, player 3 has weight 1 and the other players remain the same. Thus, we get the derivative game $G'=[6; 5, 3, 1, 1]$.

Example 14. Let $G=[18;8,5,4,4,3,3,2]$ be an original game where $n=7$, $q=18$ and $\tau=29$. For the derivative game G', let player 2 with weight 5 annex 4 votes of player 5. In the new game G' player $1'$ will have weight $5+4=9$ and player $7'$ will have weight $4-4=0$. We also get that $m=7$, $q=18$, $\tau=29$, $2'=\{1\}$, $3'=\{3\}$, $4'=\{5\}$, $5'=\{6\}$ and $6'=\{7\}$, i.e. $G'=[18;9,8,4,3,3,2,0]$. The Banzhaf power index of player 2 in game G is $\beta_2=0{,}1712$ and this index of player $1'$ in game G' is $\beta'_{1'}=0{,}3400$. As a result we obtain $\beta_2<\beta'_{1'}$ and $\beta'_{7'}=0$.

Example 15. Let $G=[29;9,9,9,8,7,5,3]$ be an original game where $n=7$, $q=29$ and $\tau=50$. For the derivative game G', let player 4 with weight 8 annex 3 votes of player 6. In the new game G' player $1'$ will have weight $8+3=11$ and player $7'$ will have weight $3-3=0$. We also get that $m=7$, $q=29$, $\tau=50$, $2'=\{1\}$, $3'=\{2\}$, $4'=\{3\}$, $5'=\{5\}$ and $6'=\{6\}$, i.e. $G'=[29;11,9,9,9,7,5,0]$. The Banzhaf power index of player 4 in game G is $\beta_4=0{,}1774$ and this index of player $1'$ in game G' is $\beta'_{1'}=0{,}2167$. As a result we obtain $\beta_4<\beta'_{1'}$ and $\beta'_{7'}=0$.

Example 16. Let $G=[9;3,3,2,1,1,1]$ be an original game where $n=6$, $q=9$ and $\tau=11$. It is known that from $w_2>w_3$ it follows that $\beta_2\geq\beta_3$. We will consider two cases of annexation.

Case 1. Player 1 annexes all votes of player 2. So we obtain a new game $G'=[9;6,2,1,1,1,0]$, the Banzhaf power index of player $1'$ in game G' is $\beta'_{1'}=0{,}4000$ and this index of player $6'$ is $\beta'_{6'}=0$.

Case 2. Player 1 annexes all votes of player 3. In this case we obtain a new game $G''=[9;6,2,1,1,1,0]$, the Banzhaf power index of player $1''$ in game G'' is $\beta''_{1''}=0{,}4118$ and this index of player $6''$ is $\beta''_{6''}=0$.

Finally, we see that it is possible to have $\beta'_{1'}<\beta''_{1''}$ when $\beta_2\geq\beta_3$.

Theorem 4 [22]. Transform game G to game G' such that player i annex $t=\{1,2,\ldots,w_j\}$ voters of player j and the other players remain the same. The following statements are true.

(a) $\eta_i(G')\geq\eta_i(G)$ and $\eta_j(G')\leq\eta_j(G)$.

(b) $\eta_i(G')-\eta_i(G)=\eta_j(G)-\eta_j(G')$.

(c) If player j is a dummy in game G and $w_j>0$, then player j is a dummy in game G' and $\eta_i(G')=\eta_i(G)$.

(d) Players i and j being dummies in game G is equivalent to these two players i and j being dummies in game G'.

(e) If player i is a vetoer in game G, then player i is a vetoer in game G'.

(f) If player i is a dictator in game G, then player i is a dictator in game G'.

Proof. (a) Let $S\in L_s^i(G)$, i.e. $S\in L_-^i(G)$ and $S\bigcup\{i\}\in W_+^i(G)$. This means that $0<q-w(S)\leq w_i$. From $w'_i=w_i+t>w_i$ it follows $0<q-w(S)\leq w'_i$; therefore, $S\in L_-^i(G')$ and $S\bigcup\{i\}\in W_+^i(G')$. As a result we obtain $S\in L_s^i(G')$ and $\eta_i(G')\geq\eta_i(G)$.

On the analogy of the above, let $S \in L_s^j(G')$, i.e. $S \in L_-^j(G')$ and $S \bigcup \{j\} \in W_+^j(G')$. So, $0 < q - w(S) \le w_j' < w_j$ implies $S \in L_-^j(G)$ and $S \bigcup \{j\} \in W_+^j(G)$. In this case we find that $\eta_j(G') \le \eta_j(G)$.

(b) Let transform game G to game $\widetilde{G}_1$ by the merging of players i and j into player i' and the other players be the same. Now, if we transform game G' to game $\widetilde{G}_2$ by the merging of players i and j into player i' and the other players remain the same, then we obtain $\widetilde{G}_1 = \widetilde{G}_2$. According to Theorem 3(a) we obtain $\eta_i(G) + \eta_j(G) = 2\eta_{i'}(\widetilde{G}_1) = 2\eta_{i'}(\widetilde{G}_2) = \eta_{i'}(G') + \eta_j(G')$. Finally, we get that $\eta_i(G') - \eta_i(G) = \eta_j(G) - \eta_j(G')$.

(c) Let player j be a dummy in game G and $w_j > 0$. From (a) and (b) it follows $\eta_i(G') - \eta_i(G) = \eta_j(G) - \eta_j(G') \ge 0$. This means that player j is a dummy in game G' and $\eta_i(G') = \eta_i(G)$.

(d) The proof follows from (c).

(e) If player i is a vetoer in game G, then $\tau - q < w_i$. From $w_i' = w_i + t$ and $t > 0$ it follows that player i is a vetoer in game G'.

(f) If player i is a dictator in game G, then $q \le w_i$. Analogy, $w_i' = w_i + t$ and $t > 0$ imply player i is a dictator in game G'.

The theorem is proven.

Note that one player can consecutively annex a part or all of the voting weights of several players.

6. Conflicts of Interest

The author states that there are no relevant interests that could be perceived as conflicting.

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