A Semi-Analytical Method for The Solution of Linear And Nonlinear Newell-Whitehead-Segel Equations

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Abstract

This work presents a semi-analytical technique, namely the Reduced Differential Transform Method (RDTM) for the solution of linear and nonlinear Newell-Whitehead-Segel Equations (NWSE). The method does not require linearization, transformation, discretization, perturbation or restrictive assumptions. To determine the accuracy of solutions and the performance measure of RDTM, two illustrative examples were considered. The results obtained via RDTM was compared with that of the exact solution. The results show that RDTM is very effective, offers solutions with easily computable components as convergent series and quite accurate to systems of linear and nonlinear equations. Hence, RDTM is a good alternative approach for the solution of partial differential equations that overcomes the shortcoming of complex calculations of differential transform method.

Keywords: Exact solution, Linear and nonlinear equation, Newell-Whitehead-Segel equation, Reduced differential transform, Semi-analytical method

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Introduction

Differential equations play a vital role in various fields such as engineering, mathematics, astronomy, chemistry, physics and biology. Partial Differential Equations (PDEs) are mathematical models that are used to describe complex phenomena emanating from the world around us. In the recent years, many researchers had paid attention to the solutions of partial differential equations by means of the various methods, such as [1], [2], [4], [5], [6], [9], [12], [13], just to mention a few. The RDTM was first proposed by [7] and successfully employed to solve many types of nonlinear PDEs. Also Keskin and Oturanç [8] used RDTM to obtain the analytical solution of linear and nonlinear wave equations. For more details on RDTM see; [3], [10] and [11], just to mention a few.

In this work, a powerful technique, the “Reduced Differential Transform Method (RDTM)” for the solution of the Newell-Whitehead-Segel Equation (NWSE) is presented. The rest of the paper is organized as follows: Section Two is the brief review of RDTM. In Section Three, the analysis of the RDTM is presented. Section Four presents two illustrative examples and the concluding remarks.

Brief review of the reduced differential transform

The reduced differential transform of $u(x,t)$ at $t=0$ is defined as

$$U_k(x)=\frac{1}{k!}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} \qquad (1)$$

where $u(x,t)$ is analytic and continuously differentiable with respect to time t and space x in the domain of interest and $U_k(x)$ is the transformed function. Conversely, the reduced differential inverse transform of $U_k(x)$ is defined as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^k \tag{2}$$

Combining (1) and (2) yields

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\partial^k u(x,t)}{\partial t^k}\right)_{t=0} t^k \tag{3}$$

The properties of the reduced differential transform are given below.

Property 1.

If $a(x,t)=u(x,t)\pm v(x,t)$ then $A_k(x)=U_k(x)\pm V_k(x)$

Property 2.

If $a(x,t)=\alpha u(x,t)$ then $A_k(x)=\alpha U_k(x)$

Property 3.

If $a(x,t)=\frac{\partial u(x,t)}{\partial x}$ then $A_k(x)=\frac{\partial U_k(x)}{\partial x}$

Property 4.

If $a(x,t)=x^m t^n$ then $A_k(x)=x^m\delta(k-n)$,

where the Kronecker delta is given by $\delta(k-n)=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}$

Property 5.

If $a(x,t)=x^m t^n u(x,t)$ then $A_k(x)=x^m U_{k-n}(x)$

Property 6.

If $a(x,t)=\frac{\partial^r u(x,t)}{\partial x^r}$ then $A_k(x)=(k+1)(k+2)...(k+r)U_{k+r}(x)=\frac{(k+r)!}{k!}U_{k+r}(x)$

Property 7.

If $a(x,t)=u(x,t)v(x,t)$ then $A_k(x)=\sum_{r=0}^{k}U_r(x)V_{k-r}(x)$

Property 8.

If $a(x,t)=\left(u(x,t)\right)^{m}$ then $A_k(x)=\begin{cases} U_0(x), & k\geq 0 \\ \sum_{n=1}^{k}\frac{(m+1)_{n-k}}{kU_0(x)}U_n(x)W_{k-n}(x), & k\geq 1 \end{cases}$

Application of the reduced differential transform to the Newell-Whitehead-Segel equation

Consider the Newell-White-Segel equation of the form

$$v_t(x,t)=a_1v_{xx}(x,t)+a_2v(x,t)-a_3v^q(x,t) \tag{4}$$

Subject to the initial condition

$$v(x,0)=h(x) \tag{5}$$

where a_1,a_2,a_3 are real numbers and q is a positive integer. By means of the definition and the fundamental properties of RDTM, (4) and (5) yield

$$(k+1)V_{k+1}(x)=a_1\frac{\partial^2V_k(x)}{\partial x^2}+a_2V_k(x)-a_3H_k(x) \tag{6}$$

with

$$H_k(x)=\begin{cases} V_0(x), & k=0 \\ \sum_{n=1}^{k}\frac{(q+1)_{n-k}}{kV_0(x)}V_n(x)H_{k-n}(x), & k\geq 1 \end{cases} \tag{7}$$

and

$$V_0(x)=h(x)\delta(k) \tag{8}$$

respectively. The terms $V_k(x)$ and $H_k(x)$ are the transformed values of $v(x,t)$ and $v^q(x,t)$ respectively, $h(x)\delta(k)$ is the transformed value of $h(x)$. By iterative calculations on (6) and (8), the following values of $U_k(x)$ are obtained as

$$V_1(x)=\alpha_1(x),V_2(x)=\alpha_2(x),V_3(x)=\alpha_3(x),\ldots,V_n(x)=\alpha_n(x),\ldots \tag{9}$$

Combining (8) and (9) and by means of the reduced differential inverse transform, the solution of (4) yields

$$v(x,t)=h(x)\delta(k)+t\alpha_1(x)+t^2\alpha_2(x)+t^3\alpha_3(x)+\ldots+t^n\alpha_n(x)+\ldots \tag{10}$$

Two Illustrative Examples and Results

This section consists of two illustrative examples to discuss the performance measure and accuracy of RDTM. The results generated are also presented.

Example 1: The Linear Newell-Whitehead-Segel equation

Consider the equation of the form

$$v_t(x,t) - v_{xx}(x,t) + 2v(x,t) = 0 \tag{11}$$

Subject to the initial condition

$$v(x,0) = e^x \tag{12}$$

whose exact solution is obtained as

$$v(x,t) = e^{x-t} \tag{13}$$

By applying the RDTM to (11) and (12) yields

$$(k+1)V_{k+1}(x) = \frac{\partial^2 V_k(x)}{\partial x^2} - 2V_k(x) \tag{14}$$

and

$$V_0(x) = e^x \tag{15}$$

respectively.

The iterative calculations of (14) and (15) are obtained as follows:

$$V_1(x) = -e^x, V_2(x) = \frac{e^x}{2}, V_3(x) = \frac{-e^x}{6}, V_4(x) = \frac{e^x}{24}, V_5(x) = \frac{-e^x}{120}, \ldots, V_n(x) = (-1)^n \frac{e^x}{n!}, \ldots \tag{16}$$

Using (15), (16) and the definition of the reduced differential inverse transform, the analytical approximate solution is obtained as

$$u(x,t) = e^x \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots + (-1)^n \frac{t^n}{n!} + \ldots \right) = e^{x-t} \tag{17}$$

The results generated by means of the RDTM and exact solution for different values of $x = 0.0(0.1)0.5$ and $t = 0.0(0.1)0.5$ are shown in Figures 1 and 2 below respectively.

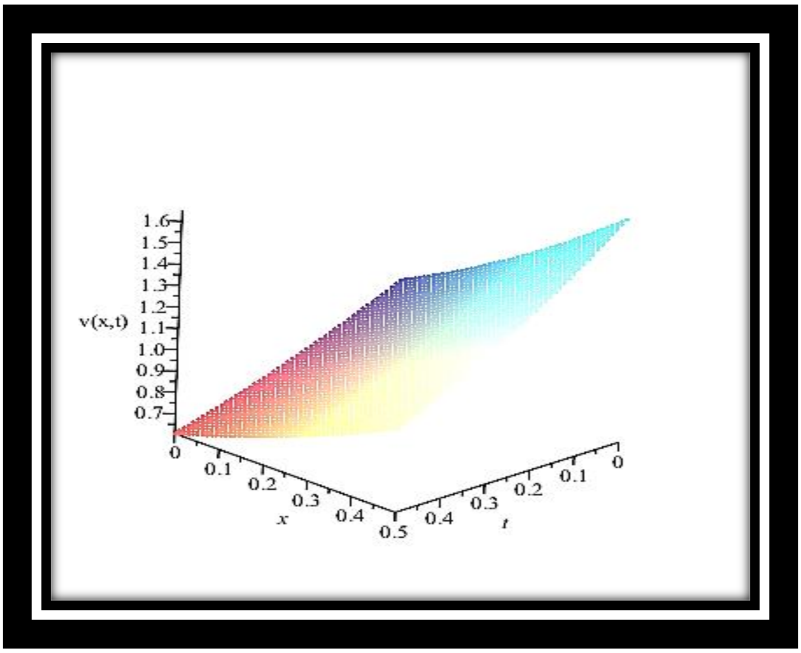

Figure 1: The RDTM solution

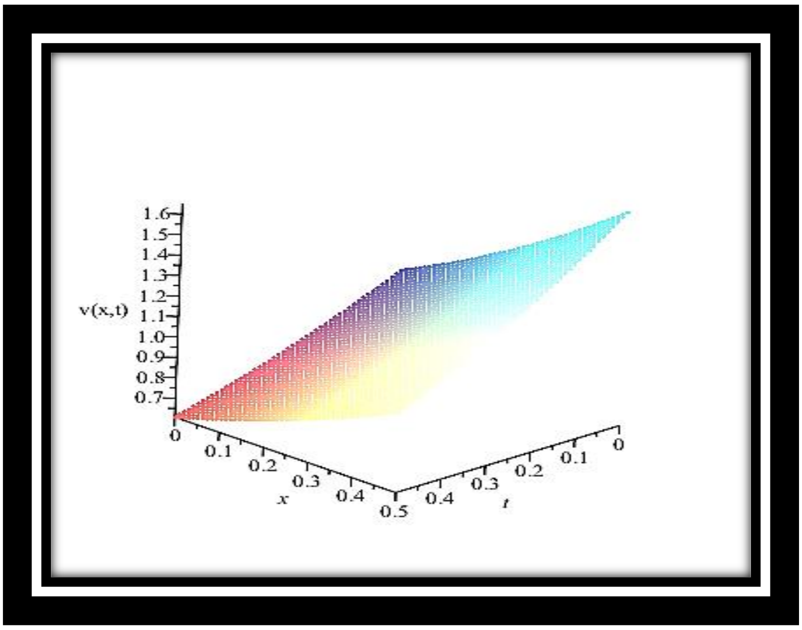

Figure 2: The Exact solution

Example 2: The Non-linear Newell-Whitehead-Segel equation

Consider the equation of the form

$$v_t(x,t) - v_{xx}(x,t) - 2v(x,t) + 3v^2(x,t) = 0 \tag{18}$$

Subject to initial condition

$$v(x,0) = \lambda \tag{19}$$

The exact solution of the problem is obtained as

$$v(x,t)=\left(\frac{2\lambda e^{2t}}{2-3\lambda+3\lambda e^{2t}}\right) \tag{20}$$

Taking the reduced differential transform of (18) and (19), and by means of the fundamental theorems of RDTM, one gets

$$(k+1)V_{k+1}(x)=\frac{\partial^2 V_k(x)}{\partial x^2}+2V_k(x)-3\sum_{i=0}^{k}V_i(x)V_{k-i}(x)=0 \tag{21}$$

and

$$V_0(x)=\lambda \tag{22}$$

Following the same procedures for the solution of the linear case, the iteration calculations on (21) are obtained as follows

$$V_1(x)=2\lambda-3\lambda^2, V_2(x)=\frac{2\lambda(2-3\lambda)(1-3\lambda)}{2}, V_3(x)=\frac{2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)}{6},\ldots \tag{23}$$

From (2), the reduced differential inverse transform is given by

$$v(x,t)=\sum_{k=0}^{\infty}V_k(x)t^k \tag{24}$$

Substituting (22) and (23) into (24) yields

$$\begin{aligned}v(x,t)&=\lambda+(2\lambda-3\lambda^2)t+\left(\frac{2\lambda(2-3\lambda)(1-3\lambda)}{2}\right)t^2+\left(\frac{2\lambda(2-3\lambda)(27\lambda^2-18\lambda+2)}{6}\right)t^3+\ldots\\&=\left(\frac{2\lambda e^{2t}}{2-3\lambda+3\lambda e^{2t}}\right)\end{aligned} \tag{25}$$

Equation (25) is the required approximate solution to (18) subject to (19).

By varying λ, the results generated by means of the RDTM and exact solution for different values of $t=0.2(0.1)1.0$ are shown in Figures 3 and 4 below respectively.

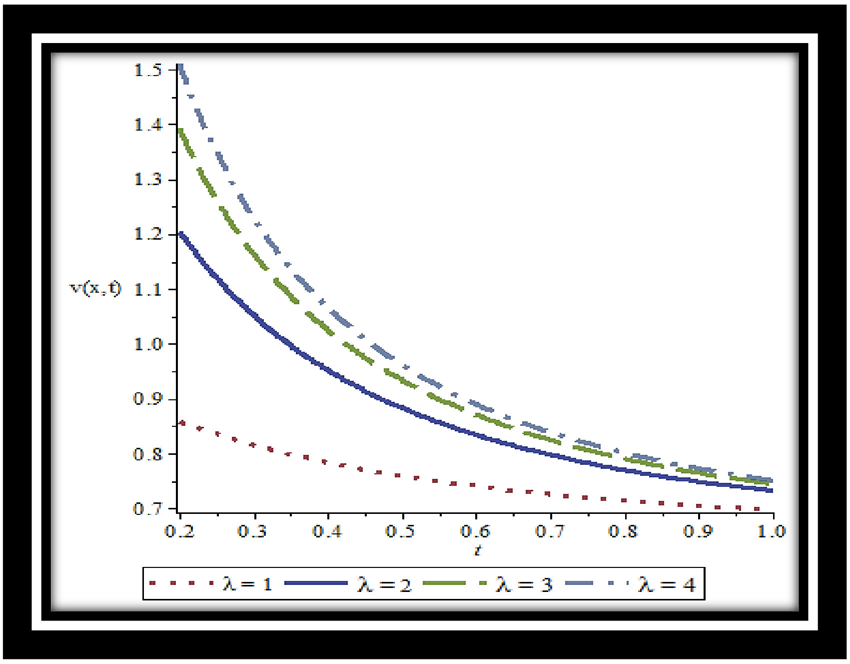

Figure 3: The RDTM solution

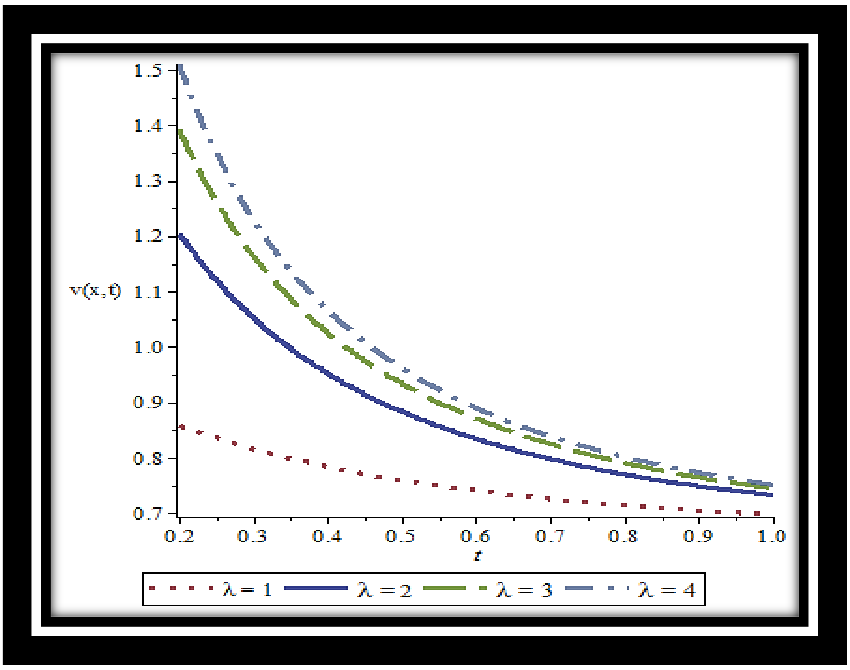

Figure 4: The Exact solution

Conclusions

The RDTM is applied to both linear and nonlinear Newell-Whitehead-Segel equations. RDTM does not require linearization, perturbation or restrictive assumptions and offers solutions with easily computable components as convergent series. The Figures 1, 2, 3 and 4 above are drawn for RDTM as well as the exact solution. It is observed from Figures 1 and 2 that the value of the function $v(x,t)$ increases for different values of $x = 0.0(0.1)0.5$ and $t = 0.0(0.1)0.5$. From Figures 3 and 4, it is observed that as λ increases, the function $v(x,t)$ increases for different values $t = 0.2(0.1)1.0$. It also observed from Figures 1, 2, 3 and 4 that the results obtained via RDTM agree with the results of the exact solution. It is concluded that RDTM is efficient and a powerful semi-analytical method for solving both linear and nonlinear PDEs emanating from science and engineering.

Conflicts of Interest

Author declares that there is no conflict of interest

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