Some Characterizations of The Exponential Family

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Abstract

This paper introduces some characterizations concerning the exponential family. Recurrence relation between two consecutive conditional moments of $h(Z)$ given $x < Z < y$ is presented. In addition, an expression of $V[h(Z)|x < Z < y]$ as well as a closed form of $E[h^r(Z)|x < Z < y]$ in terms of the failure rate and the reversed failure rate is deduced. Finally, the left r^{th}truncated moment of $h(Y_k)$ (where Y_k is the k^{th}order statistic) is expressed in terms of a polynomial, h $(\cdot)$, of degree r. Some results concerning the exponentiated Pareto, exponentiated Weibull, the Modified Weibull, Weibull, generalized exponential, Linear failure rate, 1^{st} type Pearsonian distributions, Burr, power and the uniform distributions are obtained as special cases.

Mathematics Subject Classifications: 62E10

Keywords. Characterization, left , right and double truncated moments, conditional variance, order statistics, recurrence relations, exponentiated Pareto, exponentiated Weibll, Modified Weibull, generalized exponential, Linear failure rate,1^{st}type Pearsonian distributions , Burr, Power, beta, uniform distributions.

1-Introduction.

Characterization theorems play a vital role in many fields such as mathematical statistics, reliability and actuarial science. They form an essential tool of statistical inference as they provide characteristic properties of distributions that enable researchers to identify the particular models. Some excellent references are, e.g., Azlarov and Volodin [6], Galambos and Kotz [13], Kagan, Linnik, and Rao [18], and Patel, Kapadia and Owen [26],among others. Characterization

Different methods have been used to identify many types of distributions. Afify et al. [1],Gupta [13], Ouyang [24], Talwalker [29] and Elbatal et al. [10], among others have used the concept of right truncated moments to identify different probability distributions like Weibull, exponential, Pareto and power distributions. In fact characterizations by right truncated moments are very important in practical since, e.g., in reliability studies some measuring devices may be unable to record values greater than time t. Actually there are some measuring devices that can't be able to record values smaller than time t. This has motivated several authors to deal with the problem of characterizing distributions using left truncated moments, see , e. g., Ahmed [2],Dimaki and Xekalaki [7], Navaro et. al.[20],Elbatal [9] and Gupta [14]. On the other hand, characterizations of some particular distributions based on the concept of conditional variance have been considered by several authors such as, Fakhry[11],El-Arishi [8] and A-Rahman [5].In addition, the concept of order statistics has been used to characterize several types of probability distributions, see, e.g., Ahsanullah [3], Ahanullah and Nevzorov [4], Gupta and Ahsanullah [15], Hamedani et al. [16]. Nassar [20], Obretenov [23] and Wu and Ouyang [30], among others.

Recently, the concept of double truncated moments has been used to identify some probability distributions. Ruiz and Navaro [27] have discussed the necessary and sufficient conditions for a bivariate real valued function $m(x, y)$ to be a conditional expectation $E(Z|x < Z < y)$ of a continuous random variable Z. Khorashadizadeh et al. [18] have studied some of the reliability properties of the variance residual life based on doubly truncated data. Nofal [22] has used this concept to identify some distributions like exponential, geometric, Pareto.

Let Z be a continuous random variable with distribution function $F(z)$ defined by:

$$F(z) = 1 - \exp\left\{-\left[\frac{h(z)-h(\alpha)}{c}\right]\right\} \quad , \qquad z \in (\alpha,\beta) \qquad (1)$$

Such that:

(I) c is a positive constant.

(II) $h(z)$ is a real valued differentiable function defined on (α , β) with:

(a) $\lim_{z\to\alpha^+} h(z) = h(\alpha)$ (b) $\lim_{z\to\beta^-} h(z) = \infty$

(c) $\grave{h}(z) > 0$ for any $z \in (\alpha,\beta)$. (d) $E\left(h^i(Z)\right)$ exists and finite.

(e) $\lim_{y\to\beta^-} h^i(y)G(y) = 0$ for $i \in \{1,2\}$, where $G(\cdot)$ is the survival function of Z.

Ouyang [25] has proposed the above family and claimed that several well-known distributions arise from this family by suitable choices for the function $h(\cdot)$ and the domain (α,β).

We, in this paper, use the concept of double truncated moments as well as double truncated variance to characterize the family (1) in terms of the failure rate and reversed failure rate and hence generalize the results of Ouyang [25] and Fakhry [11]. In addition, we express the r^{th} left truncated moment of some function of the k^{th}order statistic,$h(\cdot)$, as a polynomial of order r. Some well-known results follow as special cases from our results.

2 - Main Results.

The following Theorem identifies the distribution defined by (1) through some equivalent statements.

Theorem 2.1: Let Z be an absolutely continuous random variable with cumulative distribution function $F(\cdot)$,survival function $G(\cdot)$,density function f(·),failure rate λ(·) and reversed failure rate $\tau(\cdot)$ such that $F(\alpha)$ =0 and $F(\beta)$ = 1 and F(·) has continuous first order derivative on (α,β) with $\grave{F}(z)$ > 0 for all z . Then under the conditions (I-II), the following statements are equivalent:

1- $$F(z) = 1 - exp\left\{-\left[\frac{h(z)-h(\alpha)}{c}\right]\right\}, \qquad z \in (\alpha,\beta) .$$

2- For any natural number r and real numbers x, $y \in (\alpha,\beta)$, the following recurrence relation is satisfied:

$$\mu_r(x,y) = E(h^r(Z)|x < Z < y) = \left[\frac{h^r(x)\tau(x)(\lambda(y)+\tau(y)) - h^r(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\,\tau(x) - \lambda(x)\,\tau(y)}\right] + rc\mu_{r-1}(x,y). \qquad (2)$$

3- $$V\big(h(Z|x < Z < y)\big) = \frac{h^2(x)\tau(x)(\lambda(y)+\tau(y)) - h^2(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\,\tau(x) - \lambda(x)\tau(y)} - \left[\frac{h(x)\tau(x)(\lambda(y)+\tau(y)) - h(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x) - \lambda(x)\tau(y)}\right]^2 + c^2. \qquad (3)$$

4- $$V(h(Z)|Z > x) = c^2. \qquad (4)$$

Proof . $1 \Rightarrow 2$

By definition

$$\mu_r(x,y) = E(h^r(Z)|x < Z < y) = \frac{\int_x^y h^r(z)d\,F(z)}{F(y)-F(x)} .$$

Integrating by parts, one gets:

$$\mu_r(x,y) = \frac{h^r(y)F(y)-h^r(x)F(x)}{F(y)-F(x)} - \frac{r\int_x^y h^{r-1}(z)h'(z)F(z)dz}{F(y)-F(x)}$$

Using equation (1) to eliminate $F(z)$, one gets:

$$\mu_r(x,y) = \frac{h^r(y)F(y)-h^r(x)F(x)}{F(y)-F(x)} - \frac{r}{F(y)-F(x)}\int_x^y h^{r-1}(z)\, h'(z)\, \left[1-exp\left\{-\left[\frac{h(z)-h(\alpha)}{c}\right]\right\}\right] dz$$

$$= \frac{h^r(x)G(x)-h^r(y)G(y)}{F(y)-F(x)} + \frac{r}{F(y)-F(x)}\int_x^y h^{r-1}(z) \exp\left\{-\left[\frac{h(z)-h(\alpha)}{c}\right]\right\} dh(z)$$

It is easy to see that:

$$h'(z) = \frac{c\,f(z)}{1-F(z)} = \frac{c\,f(z)}{\exp\left\{-\left[\frac{h(z)-h(\alpha)}{c}\right]\right\}} \quad . \qquad (5)$$

Therefore,

$$\mu_r(x,y) = \frac{h^r(x)G(x)-h^r(y)G(y)}{F(y)-F(x)} + \frac{rc}{F(y)-F(x)}\int_x^y h^{r-1}(z) f(z)dz$$

Recalling that: $\lambda(y) = \frac{f(y)}{1-F(y)}$, $\tau(y) = \frac{f(y)}{F(y)}$ and making use of the definition of $\mu_r(x,y)$, one gets:

$$\mu_r(x,y) = \frac{h^r(x)\tau(x)(\lambda(y)+\tau(y))-h^r(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + rc\,\mu_{r-1}(x,y)$$

$2 \Longrightarrow 3$

Using equation (2) with $r = 2$ and $r = 1$, recalling that $\mu_0(x,y) = 1$, one gets:

$$\mu_2(x,y) = \frac{h^2(x)\tau(x)(\lambda(y)+\tau(y))-h^2(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + 2c\frac{h(x)\tau(x)(\lambda(y)+\tau(y))-h(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + 2c^2$$

and

$$\mu_1(x,y) = \frac{h(x)\tau(x)(\lambda(y)+\tau(y))-h(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + c \quad .$$

Therefore,

$$V(h(Z)|x<Z<y) = \mu_2(x,y) - [\mu_1(x,y)]^2$$

$$= \frac{h^2(x)\tau(x)(\lambda(y)+\tau(y))-h^2(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} - \left[\frac{h(x)\tau(x)(\lambda(y)+\tau(y))-h(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)}\right]^2 + c^2 \ .$$

$3 \Longrightarrow 4$

Writing equation (3) in terms of the survival and distribution functions of the variable y as follows:

$$V(h(Z)|x<Z<y) = \frac{h^2(x)\tau(x)-h^2(y)G(y)(\lambda(x)+\tau(x))}{\tau(x)F(y)-\lambda(x)G(y)} - \left[\frac{h(x)\tau(x)-h(y)G(y)(\lambda(x)+\tau(x))}{\tau(x)F(y)-\lambda(x)G(y)}\right]^2 + c^2.$$

Taking the limit of both sides as $y \to \beta^-$, recalling that : $\lim_{y\to\beta^-} h^i(y)G(y) = 0$ for $i\epsilon\{1,2\}$ and $\lim_{y\to\beta^-} G(y) = 0$ one gets:

$$V(h(Z)|Z > x) = c^2.$$

$4 \Rightarrow 1.$

Writing equation (4) in integral form as follows:

$$G(x)\int_x^\infty h^2(z)dF(z) - \left[\int_x^\infty h(z)dF(z)\right]^2 = c^2G^2(x).$$

Differentiating last equation with respect to x then dividing the result by $\grave{G}(x)$,one gets:

$$G(x)h^2(x) + \int_x^\infty h^2(z)dF(z) - 2h(x)\int_x^\infty h(z)dF(z) = 2c^2G(x).$$

Differentiating with respect to x then cancelling out "$\grave{G}(x)h^2(x)$" then dividing the result by $2\grave{h}(x)$,one gets:

$$G(x)h(x) - \int_x^\infty h(z)dF(z) = c^2\, \grave{G}(x)/\grave{h}(x).$$

Differentiating again with respect to x then cancelling the term " $\grave{G}(x)h(x)$", one gets:

$$c^2\frac{d}{dx}\left(\frac{\grave{G}(x)}{\grave{h}(x)}\right) = G(x)\grave{h}(x)\ . \qquad (6)$$

Now, it is easy to see that: $\frac{\grave{G}(x)}{\grave{h}(x)} = \frac{dG(x)}{dh(x)}$, therefore we have:

$$\frac{d}{dx}\left(\frac{\grave{G}(x)}{\grave{h}(x)}\right) = \frac{d}{dx}\left(\frac{dG(x)}{dh(x)}\right) = \grave{h}(x)\frac{d^2G(x)}{dh^2(x)}.$$

Hence equation (6) becomes: $\frac{d^2G(x)}{dh^2(x)} = \frac{G(x)}{c^2}$.

This is a second order differential equation with constant coefficients, its solution is known to be:

$$G(x) = ae^{-\frac{h(x)}{c}} + be^{\frac{h(x)}{c}}$$, for some constants a and b.

Taking the limit as $x \to \beta^-$, we find that $b = 0$, on the other hand, taking the limit as $x \to \alpha^+$,we conclude that $a = e^{\frac{h(\alpha)}{c}}$.

Hence, $G(x) = e^{-(h(x)-h(\alpha))/c}$.

Remarks 2.1:

(1) Theorem 2.1 can be used to generalize Fakhrey's result [11]. To this end, rewriting equation (2) as follows:

$$\mu_r(x,y) = \frac{h^r(x)\tau(x) - h^r(y)G(y)(\lambda(x)+\tau(x))}{\tau(x)F(y)-\lambda(x)G(y)} + rc\mu_{r-1}(x,y).$$

Taking the limit of both sides as $y \to \beta^-$, recalling that:

(1) $\lim\limits_{y\to\beta^-} F(y)=1$ (2) $\lim\limits_{y\to\beta^-} h^r(y)G(y)=0$ (3) $\lim\limits_{y\to\beta^-} \int_x^y \frac{h^r(z)f(z)dz}{F(y)-F(x)} = \mu_r(x)$

One gets:

$$\mu_r(x) = h^r(x) + rc\mu_{r-1}(x).$$

It is easy to see that, if we set $\emptyset(Z) = h(Z) - h(\alpha), r = 1$ and recalling that: $\mu_0 = 1$, the result reduces to that of Hamdan [16]

(2) Theorem 2.1 generalizes the result of A-Rahman [5]. To see this, rewriting equation (2) as follows:

$$\mu_r(x,y) = \frac{h^r(x)G(x)(\lambda(y)+\tau(y)) - h^r(y)\tau(y)}{\lambda(y)G(x)-\tau(y)F(x)} + rc\mu_{r-1}(x,y).$$

Taking the limit of both sides as $x \to \alpha^+$, recalling that: $\lim\limits_{x\to\alpha^+} F(x)=0$ and $\lim\limits_{x\to\alpha^+} \int_x^y \frac{h^r(z)f(z)dz}{F(y)-F(x)} = \mu_r(y)$,

one gets:

$$\mu_r(y) = -\frac{\tau(y)}{\lambda(y)}h^r(y) + rc\mu_{r-1}(y) + \frac{\lambda(y)+\tau(y)}{\lambda(y)}h^r(\alpha).$$

It is easy to see that for $r = 1$, recalling that $\frac{\lambda(y)+\tau(y)}{\lambda(y)} = \frac{1}{F(y)}$ and setting $\varphi(Z) = h(Z) - h(\alpha)$ the result reduces to that of Talwalker [29].On the other hand, if we set $r = 1$, $h(Z) = Z$, $\alpha = 0$ and $\beta = \infty$ the result coincides with that of Elbatal et.al.[10].

The following Corollary gives a closed form for the double truncated r^{th}moments of the random variable Z in terms of the failure rate, the reversed failure rate and a polynomial $h(\cdot)$ of degree r.

Corollary 2.1: Let Z be a continuous random variable with distribution function $F(\cdot)$, survival function $G(\cdot)$, density function $f(\cdot)$, reversed failure rate $\tau(\cdot)$ and failure rate $\lambda(\cdot)$ such that $F(\alpha)=0$ and $F(\beta) = 1$,then Z has the distribution defined by (1) iff for any natural number ,r, the following equation is satisfied:

$$\mu_r(x,y) = E\big((h^r(Z)|x<Z<y)\big) = \frac{h^r(x)\tau(x)[\lambda(y)+\tau(y)] - h^r(y)\tau(y)[\lambda(x)+\tau(x)]}{\lambda(y)\tau(x)-\lambda(x)\tau\lambda(y)} +$$

$$\sum_{i=1}^{r} c^i \frac{h^{r-i}(x)\tau(x)\big(\lambda(y)+\tau(y)\big) - h^{r-i}(y)\tau(y)\big(\lambda(x)+\tau(x)\big)}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} \prod_{j=1}^{i}\big(r-j+1\big) \qquad (7)$$

Proof. By induction

At $r = 1$, equation (7) will be:

$$\mu_1(x,y) = \frac{h(x)\tau(x)(\lambda(y)+\tau(y)) - h(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + c.$$

Comparing this result with equation (2), we conclude that the relation is true at $r = 1$. Assume that the result is true at $r = k$, i.e.,

$$\mu_k(x,y) = \frac{h^k(x)\tau(x)(\lambda(y)+\tau(y)) - h^k(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} +$$

$$\sum_{i=1}^{k} c^i \frac{h^{k-i}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k-i}(y)\tau(y)(\lambda(x)+\tau(y))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} \prod_{j=1}^{i}(k-j+1). \tag{8}$$

We have to prove that this will be true at $r = k + 1$. From equation (2), we get:

$$\mu_{k+1}(x,y) = \frac{h^{k+1}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k+1}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + (k+1)c\mu_k(x,y)\,. \tag{9}$$

Eliminating $\mu_k(x,y)$ using equation (8), one gets:

$$\mu_{k+1}(x,y) =$$

$$\frac{h^{k+1}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k+1}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} + (k+1)c\frac{h^{k}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} +$$

$$(k+1)c\sum_{i=1}^{k} c^i \frac{h^{k-i}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k-i}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} \prod_{j=1}^{i}(k-j+1).$$

It is easy to see that the second and third term can be combined in a sum term so that we have:

$$\mu_{k,+1}(x,y) = \frac{h^{k+1}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k+1}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} +$$

$$\sum_{i=1}^{k+1} c^i \frac{h^{k-i+1}(x)\tau(x)(\lambda(y)+\tau(y)) - h^{k-i+1}(y)\tau(y)(\lambda(x)+\tau(x))}{\lambda(y)\tau(x)-\lambda(x)\tau(y)} \prod_{j=1}^{i}(k+1-j+1).$$

So equation (7) is true at r=1 and its validity at r=k implies its validity at r=k+1. So by the principal of mathematical induction (7) is true for every natural number n.

Remarks 2.2:

(1) If we set $h(Z) = Z, c = 1/\lambda\,, \alpha = 0$ and $\beta=\infty$, the result coincides with that of Nofal [22].

(2) If we take The limit as $y \to \beta^-$of both sides of equation (7) we obtain a generalization of Fakhry's result [11],namely, A random variable Z has the distribution defined by (1) iff:

$$u_r(x) = E(h^r(Z)|Z > x) = h^r(x) + \sum_{i=1}^{r} c^i h^{r-i}(x) \prod_{j=1}^{i}(r-j+1).$$

(3) If we set $r = 1$ and $\emptyset(Z) = h(Z) - h(\alpha)$, the result coincides with Hamdan's result [16].

(4) If we take the limit as $x \to \alpha^+$of both sides of equation (7), we obtain a generalization of A-Rahman's result [5], namely, A random variable Z has the distribution defined by (1) iff

$$\mu_r(y) = E(h^r(Z)|Z < y) =$$

$$\frac{-\tau(y)}{\lambda(y)}\left[h^r(y)+\sum_{i=1}^{r}c^i h^{r-i}(y)\prod_{j=1}^{i}(r-j+1)\right]+\frac{\lambda(y)+\tau(y)}{\lambda(y)}\left[h^r(\alpha)+\sum_{i=1}^{r}c^i h^{r-i}(\alpha)\prod_{j=1}^{i}(r-j+1)\right].$$

For $r = 1$, recalling that $\frac{\lambda(y)+\tau(y)}{\lambda(y)} = \frac{1}{F(y)}$ and setting $\emptyset(Z) = h(Z) - h(\alpha)$, the result reduces to that of Talwalker [29].

A-Rahman [5] has used the concept of recurrence relations through order statistics to characterize the family (1) and hence generalizes Ouyang's result [25]. The following theorem gives a closed form of the r^{th} truncated moment of some function of the k^{th}order statistic,Y_k . Some important results follow as special cases of this result.

Theorem 2.2: Let X be an absolutely continuous random variable with cumulative distribution function F(·), survival function G(·),and density function f(·). Let X_1, X_2, …, X_n be a random sample from F(·). Denote by Y_1<Y_2< …< Y_n the corresponding ordered sample. Then under the conditions (I-II), the random variable X has the distribution defined by equation (1) iff for any natural number, r, the following recurrence relation is satisfied :

$$\mu_r(t) = E(h^r(Y_k)|\, Y_k > t) = h^r(t) + \sum_{i=1}^{r} c^i \frac{h^{r-i}(t)}{(n-k+1)^i}\prod_{j+1}^{i}(r-j+1)\,,\ k = 1,2,\dots n. \quad (10)$$

Proof. A-Rahman [5] has characterized the family defined by equation (1) using the following recurrence relation:

$$\mu_r(t) = E(h^r(Y_k)|Y_k > t) = h^r(t) + \frac{rc}{n-k+1}\mu_{r-1}(t) .$$

Writing similar expression for $\mu_{r-1}(t)$ and $\mu_{r-2}(t)$, one gets:

$$\mu_{r-1}(t) = E(h^{r-1}(Y_k)|Y_k > t) = h^{r-1}(t) + \frac{(r-1)c}{n-k+1}\mu_{r-2}(t)\,,$$

$$\mu_{r-2}(t) = E(h^{r-2}(Y_k)|Y_k > t) = h^{r-2}(t) + \frac{(r-2)c}{n-k+1}\mu_{r-3}(t)\,.$$

Substituting these results in $\mu_t(t)$, one gets:

$$\mu_r(t) = E(h^r(Y_k)|Y_k > t) = h^r(t) + \frac{rc}{n-k+1}h^{r-1}(t)+\frac{r(r-1)c^2}{(n-k+1)^2}h^{r-2}(t)+\frac{r(r-1)(r-2)c^3}{(n-k+1)^3}\mu_{r-3}(t)\,.$$

Continuing in this manner we conclude that:

$$\mu_r(t) = E(h^r(Y_k)|Y_k > t) = h^r(t) + \sum_{i=1}^{r}\frac{h^{r-i}(t)}{(n-k+1)^i}\prod_{j=1}^{i}(r-j+1)\,.$$

Remarks 2.3:

(1) If we put k=n in Theorem (2.2), we obtain a characterization for the maximum. Moreover, A-Rahman [5], has shown that:

E($h^r(Y_n)$|Y_n> t)= E(h^r(X)|X> y),

(2) Theorem (2.2) generalizes the result of Fakhry[11]. To see this, put k = n.

(3) Theorem (2.2) generalizes the result of Ouyang [25].To see this, put r =1 and k = n.

(4) Theorem (2.2) generalizes the result of Hamdan [16]. To see this put r = 1, k = n and $\emptyset(x) = h(x) - h(\alpha)$

(5) Theorem (2.2) generalizes the result of Shanbhag [28]. To see this set r =1, k = n and $h(x) = x$

(6) If we put k=1, we obtain a recurrence relation for the minimum.

(7) If we put n=2r+1 and k= r+1, we obtain a recurrence relation for the median.

(8) If we set $h(Z) = Z^{\gamma}, \alpha = 0$ and β=∞, we have characterizations concerning Weibull distribution with positive parameter γ.For $r = 1$, the result coincides with that of Ahsanullah [3]. For $r = k = 1$, the result coincides with that of Hamedani et al. [17].

3- General Comments .

In all of the previous theorems, several results can be picked out for some well-known distributions by suitable choices for the function h(X),the values of the parameters d and c and the domain (α, β) as follows:

(1) If we set $= h(Z) = aZ + bZ^{\lambda}$, where λ > 0 and $a, b \geq 0$, such that $a + b > 0$, c=1, α=0 and β = ∞ , we obtain characterizations concerning modified Weibull distribution with non-negative parameters a and b and positive parameter λ . For λ= 2, we have characterizations concerning the linear failure rate distribution with positive parameters a and b . For $a = 0$, we have characterizations concerning Weibull distribution with positive parameters b and λ . For b = 0, we have characterizations concerning the exponential distribution with parameter $a > 0$. For $a = 0$ and λ = 2, we have characterizations concerning Rayleigh distribution with parameter b > 0.

(2) If we set $h(Z) = -ln[1 - (1 - (\alpha/Z)^{a})^{\theta}], c = 1$ and β=∞, we obtain the characterizations concerning the exponentiated Pareto of the 1st type with positive parameters α,θ and a. For θ=1, the results reduce to those of the 1st type Pareto with positive parameters α and a.

(3) If we set $h(Z) = -ln[1 - (1 - (1 + Z)^{-a})^{\theta}], c = 1\ \alpha = 0\ and\ \beta = \infty$, we obtain characterizations concerning Burr distribution of type XII with positive parameters θ and a. For $a = 1$, the result reduces to the 2nd type Pareto distribution.

(4) If we set $h(Z) = -ln[1 - (1 - exp - (Z/b)^{a})^{\theta}], c = 1, \alpha = 0\ and\ \beta = \infty$, we obtain characterizations concerning the exponentiated Weibull with positive parameters θ,a and b. For θ=1, the results reduce to Weibull distribution with positive parameters a and b. For $a = 1$, the results reduce to the generalized exponential distribution with positive parameters θ and b.For $\theta = a = 1$, the results reduce to the exponential distribution with positive parameter b.

(5) If we set $h(Z) = -ln\left[1 - \left(exp - \frac{\gamma}{Z}\right)^{\theta}\right], c = 1, \alpha = 0\ and\ \beta = \infty$, we obtain characterizations concerning the inverse Weibull distribution with positive parameters θ and γ.For θ=2, the results coincide with those of inverse Rayleigh with positive parameter γ.

(6) If we put $h(Z) = -\ln\left[\frac{\beta - Z}{\beta - \alpha}\right]$, $c = 1/\theta$, we obtain characterizations concerning the first type Pearson distribution with parameters β, α and θ. For θ=1, the results reduce to the uniform distribution with parameters β and α.

(7) If we set $h(Z) = -ln[1 - ((Z - \alpha)/(\beta - \alpha))^{\theta}], c = 1$, we obtain characterizations concerning Ferguson's distribution [12] of the first type with parameters α, β and θ.

(8) If we set $h(Z) = -ln[1 - ((r - \beta)/(r - Z))^{\theta}]$, $Z < \beta$, $c = 1, \theta > 0$ and $\alpha = -\infty$, we obtain characterizations concerning Ferguson's distribution [12] of the third type with parameters r, β and θ.

(9) If we set $h(Z)$ = - $\ln(1-Z)$, α = 0 and β = 1, we obtain characterizations concerning beta distribution with parameters 1 , $\frac{1}{c}$.

(10) If we Set $h(Z) = \ln(1+Z^r)$, r > 0, ,α =0 and β =∞ ,we obtain results concerning the 2^{nd}type Burr distribution with parameters r, $\frac{1}{c}$.

(11) If we Set $h(Z) = -\ln(1- Z^{\theta})$, θ > 0, c = 1, α =0 and β = 1, we obtain results concerning the power distribution with parameter θ. For θ= 1, we have results concerning the uniform distribution.

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