On The Gould's Formula for Stirling Numbers of The Second Kind

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Abstract

We present an alternative deduction of the Gould's relation for Stirling numbers of the second kind. Our approach is based in the Nörlund polynomials and in the duality property between the Stirling numbers.

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1. Introduction

In Sec. 2 we show the following identity for Stirling numbers of the first kind:

$$S_n^{(n-k)} = \sum_{j=0}^{k} \binom{n}{k+j} \binom{2k-n}{k-j} S_{k+j}^{(j)}, \qquad 1 \le k \le n, \tag{1}$$

and in Sec. 3 we exhibit that (1) and the duality property [1-5]:

$$S_{-n}^{(-N)} = (-1)^{N+n} \, S_N^{[n]}, \tag{2}$$

imply an expression for Stirling numbers of the second kind:

$$S_n^{[n-k]} = \sum_{j=0}^{k} (-1)^j \binom{n+j-1}{n-k-1} \binom{n+k}{k-j} S_{k+j}^{(j)}, \tag{3}$$

which coincides with the Gould's formula [4, 6]; the relation (3) is the inverse of the Schläfli's identity [7, 8].

2. An identity for Stirling numbers of the first kind

In [9, 10] we have the expression:

$$S_n^{(n-k)} = \binom{n-1}{k} B_k^{(n)}, \tag{4}$$

where $B_n^{(z)}$ are the Nörlund polynomials [11-13] defined via the following generating function:

$$\sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!} = \left(\frac{x}{e^x-1}\right)^z; \tag{5}$$

then from (4):

$$S_{k+j}^{(j)} = \binom{k+j-1}{k} B_k^{(k+j)}. \tag{6}$$

On the other hand, in [6, 9] we find the relation:

$$(-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=0}^{k} \binom{k+j-1}{k} \binom{k-z}{k+j} \binom{k+z}{k-j} B_k^{(k+j)}, \tag{7}$$

that is:

$$(-1)^k \binom{k-n}{k} B_k^{(n)} \overset{(6)}{=} \sum_{j=0}^{k} \binom{n}{k+j} \binom{2k-n}{k-j} S_{k+j}^{(j)}, \tag{8}$$

whose application in (4) implies (1), q.e.d.

3. Gould's formula

In [1] we make the transformation $n \to -n$ to obtain:

$$S_{-n}^{(-(n+k))} = \sum_{j=0}^{k} \binom{-n}{k+j} \binom{2k+n}{k-j} S_{k+j}^{(j)}, \tag{9}$$

where we can use (2) and the relation [4, 14]:

$$\binom{-n}{k+j} = (-1)^{k+j}\binom{n+k+j-1}{k+j}, \tag{10}$$

to deduce:

$$S_{n+k}^{[n]} = \sum_{j=1}^{k} (-1)^j \binom{n+k+j-1}{n-1}\binom{n+2k}{k-j} S_{k+j}^{(j)}\,, \tag{11}$$

and the change $n \to n-k$ gives (3), q.e.d.

It is simple to prove that (3) is equivalent to the expression of Gould [4, 6]:

$$S_n^{[n-k]} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_{k+j}^{(j)}\,, \tag{12}$$

because:

$$\binom{k-n}{k+j} = (-1)^{k+j}\binom{n+j-1}{k+j} = (-1)^{k+j}\binom{n+j-1}{n-k-1}. \tag{13}$$

Our procedure shows the usefulness of (2), (10) and (13), that is, of Stirling numbers and binomial coefficients with negative indices [5].

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