Strong Insertion of a Contra-Continuous Function Between Two Comparable Contra-B−Continuous Functions

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Abstract

Enough condition in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable contra-b−continuous real-valued functions on such topological spaces that kernel of sets is open.

Indexing terms/Keywords: Weak insertion, Strong binary relation, Contra-b-continuous function, kernel-sets, Lower cut set.

Subject Classification: MSC (2010): Primary 54C08, 54C10, 54C50; Secondary 26A15, 54C30.

Supporting Agencies: University of Isfahan and Centre of Excellence for Mathematics(University of Isfahan).

Date of Publication : 30-08-2018

Volume : 1 Issue : 2

Journal : MathLAB Journal

website : https://purkh.com

1 Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [5]. A subset A of a topological space (X, τ) is called preopen or locally dense or nearly open if $A \subseteq Int(Cl(A))$. A set A is called preclosed if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [22], while the concept of a , locally dense, set was introduced by H. H. Corson and E. Michael [5].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [19]. A subset A of a topological space (X, τ) is called semi- open [19] if $A \subseteq Cl(Int(A))$. A set A is called semi-closed if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

D. Andrijevic introduced a new class of generalized open sets in a topo- logical space, so called b−open sets [2]. This type of sets discussed by A. A. El-Atik under the name of γ−open sets [11]. This class is closed under arbitrary union. The class of b−open sets contains all semi-open sets and preopen sets. The class of b−open sets generates the same topology as the class of preopen sets. D. Andrijevic studied several fundamental and inter- esting properties of b−open sets. A subset A of a topological space (X, τ) is called b−open if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ [1]. A set A is called b−closed if its complement is b−open or equivalently if $Cl(Int(A)) \cap Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [21]. He investigated the sets that can be represented as union of closed sets and called them $\vee$−sets. Complements of $\vee$−sets, i.e., sets that are intersection of open sets are called Λ−sets [21].

Recall that a real-valued function f defined on a topological space X is called A−continuous [26] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to [6, 13].

In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [7] introduced a new class of mappings called contra- continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 9, 10, 12, 14, 15, 25].

Hence, a real-valued function f defined on a topological space X is called contra-continuous (resp. contra-b−continuous) if the preimage of every open subset of R is closed (resp. b−closed) in X [7].

Results of Katětov [16, 17] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that Λ−sets or kernel of sets are open [21]. If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [18].

A property P defined relative to a real-valued function on a topological space is a cc−property provided that any constant function has property P and provided that the sum of a function with property P and any contra- continuous function also has property P. If P_1 and P_2 are cc−properties, the following terminology is used:(i) A space X has the weak cc−insertion property for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a contra- continuous function h such that $g \leq h \leq f$.(ii) A space X has the strong cc−insertion property for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f$, g has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose Λ−sets or kernel of sets are open, is given a sufficient condition for the weak cc−insertion property. Also for a space with the weak cc−insertion property, we give a sufficient conditions for the space to have the strong cc−insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the weak insertion of a contra-Baire-1 (Baire-.5) function has also recently considered by the author in [23].

2 The Main Result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated. The abbreviations cc and cbc are used for contra-continuous and contra-b−continuous, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:

$A^{\Lambda} = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^{V} = \cup\{F : F \subseteq A, F^{c} \in (X, \tau)\}$.

In [8, 20, 24], A^{Λ} is called the kernel of A.

The family of all b−open and b−closed will be denoted by $bO(X, \tau)$ and $bC(X, \tau)$, respectively. We define the subsets $b(A^{\Lambda})$ and $b(A^{V})$ as follows:

$b(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in bO(X, \tau)\}$ and $b(A^{V}) = \cup\{F : F \subseteq A, F \in bC(X, \tau)\}$. $b(A^{\Lambda})$ is called the b − kernel of A.

Proposition 2.1. (D. Andrijevic [2]) (i) The union of any family of b−open sets is a b−open set.

(ii) The intersection of an open and a b−open is a b−open set. The following first two definitions are modifications of conditions consid- ered in [16, 17].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \bar{\rho} y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a strong binary relation in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \rho C$ and $C \rho B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \bar{\rho} B$.

3) If $A \rho B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < i\} \subseteq A(f, i) \subseteq \{x \in X : f(x) \leq i\}$ for a real number i, then $A(f, i)$ is called a lower indefinite cut set in the domain of f at the level i.

We now give the following main result: **Theorem 2.1.** Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1)\ \rho\ A(g, t_2)$.

Define functions F and G mapping the rational numbers Q into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of Q with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2)$, $G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [17] it follows that there exists a function H mapping Q into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2)$, $H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t^0)$ for any $t^0 > t$; since x is in $G(t^0) = A(g, t^0)$ implies that $g(x) \leq t^0$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t^0)$ for any $t^0 < t$; since x is not in $F(t^0) = A(f, t^0)$ implies that $f(x) > t^0$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = H(t_2)^{\vee} \backslash H(t_1)^{\wedge}$. Hence $h^{-1}(t_1, t_2)$ is closed in X, i.e., h is a contra-continuous function on X. The above proof used the technique of theorem 1 in [16].

3 Applications

Before stating the consequences of theorems 2.1, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint b−open sets G_1, G_2 of X, there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the weak cc−insertion property for (cbc, cbc).

Proof. Let g and f be real-valued functions defined on X, such that f and g are cbc, and $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $b(A^{\wedge}) \subseteq b(B^{\vee})$, then by hypothesis ρ is a strong binary relation in the power set of X. If $t1$ and $t2$ are any elements of Q with $t1 < t2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a b−open set and since $\{x \in X : g(x) < t_2\}$ is a b−closed set, it follows that $b(A(f, t_1)^{\Lambda}) \subseteq b(A(g, t_2)^{V})$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2.1.

Corollary 3.2. If for each pair of disjoint b−open sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then every contra-b−continuous function is contra-continuous.

Proof. Let f be a real-valued contra-b−continuous function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$.

Corollary 3.3. If for each pair of disjoint b−open sets G_1, G_2 of X, there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$ then X has the strong cc−insertion property for (cbc, cbc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are cbc, and $g \leq f$. Set $h = (f + g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.

Acknowledgement

This research was partially supported by Centre of Excellence for Math- ematics (University of Isfahan).

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