Operational representations for the quadruple hypergeometric function $F_{30}^{(4)}$

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Abstract

Based upon the classical derivative and integral operators we introduce a new symbolic operational representations for the hypergeometric function of four variables $F_{30}^{(4)}$. By means of these symbolic operational representations number of generating functions involving the hypergeometric function $F_{30}^{(4)}$ are then found. Some special cases of the main result here are also considered.

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1.Introduction

Operational representations and relations involving one and more variables hypergeometric series have been given considerable in the literature, see for example, Chen and Srivastava [1], Goyal, Jain and Gaur ([2], [3]) Kalla ([4], [5]), Kalla and Saxena ([6] and [7]), Kant and Koul [8]. In [9], Exton introduced twenty one complete quadruple hypergeometric series, which he denoted by symbols $K_1, K_2, \ldots, K_{21}$. In [10], eighty three complete quadruple hypergeometric series given by $F_1^{(4)}, F_2^{(4)}, \ldots, F_{83}^{(4)}$ were defined by Sharma and Parihar. Very recently, Bin-Saad et al. [11] introduced five new quadruple hypergeometric functions different from the Exton's list of 21 hypergeometric functions of four variables and the Sharma and Parihar's list of 83 hypergeometric functions of four variables whose names are $X_6^{(4)}$, $X_7^{(4)}$, $X_8^{(4)}$, $X_9^{(4)}$, $X_{10}^{(4)}$.

Each quadruple hypergeometric function is of the form

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Lambda(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$

where $\Lambda(m,n,p,q)$ is a certain sequence of complex parameters and there are twelve parameters in each function $X^{(4)}(.)$. Here, for an example, we choose the Sharma and Parihar function $F_{30}^{(4)}$ among their eighty three functions

$$F_{30}^{(4)}(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+n}(a_4)_p(a_5)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol given by

$(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\ldots(a+n-1)\ (n \in N := \{1,2,3,\ldots\})$ and $(a)_0 = 1$.

In this work we will deal with operational definitions ruled by the operators D_x and D_x^{-1} where D_x denoted the derivative operator and D_x^{-1} defines the inverse of the derivative. It is evident that D_x^{-1} is essentially an integral operator and the lower limit has assumed to be zero. The following two formulas are well-known consequences of the derivative operator D_x and the integral operator D_x^{-1} (see, Ross [12]):

$$D_x^m x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-m+1)} x^{\lambda-m}, \tag{1.2}$$

$$D_x^{-m} x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m+1)} x^{\lambda+m}, \tag{1.3}$$

$$m \in N \cup \{0\},\ \lambda \in C - \{-1,-2,\ldots\}.$$

Based on the operational relations (1.2) and (1.3) we have

$$D_{\alpha}^{m}\ D_{\beta}^{-m}\left\{\alpha^{a+m-1}\ \beta^{b-1}\right\}=\alpha^{a-1}\ \beta^{b+m-1}\ \frac{(a)_m}{(b)_m}. \tag{1.4}$$

We have organized the rest of this paper in the following way: Section 2 establish symbolic representations for the quadruple series $F_{30}^{(4)}$. Section 3 deals with special cases of the results of Section 2. The aim of Section 4 is to use the symbolic operational representations obtained in Section 2 to derive a number of generating functions for the Sharma and Parihar function $F_{30}^{(4)}$.

2. Operational Representations

Theorem 2.1. *Let* $\mathrm{Re}(a_i)>0,\ (i=1,2,3,4,5)$ and $\mathrm{Re}(c_i)>0,\ (i=1,2,3)$, then

$$\begin{aligned}&\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]-y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)^{-a_1}\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)^{-a_2}\\&\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\times F_{30}^{(4)}(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u),\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]-y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_1\right]\right)^{-a_1}\left(1-z\left[D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2\right]\right)^{-a_4}\\&\times\left(1-u\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\right)^{-a_5}\times\left\{\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\times F_{30}^{(4)}(a_1,a_1,a_2,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u).\end{aligned} \tag{2.2}$$

Proof. Denote, for convenience, the left-hand side of assertion (2.1) by Ω. Then as a consequence of the binomial theorem, it is easily seen that :

$$\begin{aligned}\Omega=&\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{p+q}\ x^m y^n z^p u^q\ \beta_1^{-(m+p)}\beta_2^{-n}\beta_3^{-q}}{m!\,n!\,p!\,q!}D_{\alpha_3}^{m+n}D_{\alpha_4}^{p}D_{\alpha_5}^{q}\\&\times D_{\beta_1}^{-(m+p)}D_{\beta_2}^{-n}D_{\beta_3}^{-q}\\&\times\left\{\alpha_3^{a_3+m+n-1}\alpha_4^{a_4+p-1}\alpha_5^{a_5+q-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}.\end{aligned}$$

Upon using (1.4) and considering the definition (1.1), we are led finally to right-hand side the assertion (2.1). The proof of the assertion (2.2) runs parallel to that of the assertion (2.1) then we skip the details.

3. Special Cases

Here we consider some special cases of our results in previous section.

Substituting $x=0$ in (2.1), we have

$$\begin{aligned}&\left(1-y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)^{-a_1}\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)^{-a_2}\\&\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\times{}_2F_1(a_1,a_3;c_2;y)\times F_2(a_2,a_4,a_5;c_1,c_3;z,u),\end{aligned}\tag{3.1}$$

Which, $z=u=0$ yields the symbolic operational representation for the Gaussian hypergeometric series ${}_2F_1$ [13]. Another interesting special case of this last result (3.1) when $y=0$ yields the operational representation for Appell's series F_2 (see [14, p. 41]).

Again, if we take $y=0$ in (2.2) and $u=0$ in (2.1), and simplifying, we obtain the operational representations for Lauricella's hypergeometric series [14].

Formula (2.2), with $z=u=0$, yields the operational representation for Appell's series F_4 (see [14, p. 41]).

4. Generating Functions

Theorem 4.1. *The following generating functions for $F_{30}^{(4)}$ in (1.1) holds true:*

$$\begin{aligned}&\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)+v\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)\right)\\&\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k,r=0}^{\infty}\frac{t^k\,v^r}{k!\,r!}F_{30}^{(4)}(-k,-k,-r,-r,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u),\end{aligned}\tag{4.1}$$

$$\begin{aligned}&\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)\right)\\&\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\sum_{k=0}^{\infty}\frac{t^k}{k!}F_{30}^{(4)}(-k,-k,-k,-k,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u),\end{aligned}\tag{4.2}$$

$$\begin{aligned}&\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)+v\left(1-z\left[D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2\right]\right)+w\left(1-u\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\right)\right)\\&\times\left\{\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}\\&=\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\\&\times\sum_{k,r,s=0}^{\infty}\frac{t^k\,v^r\,w^s}{k!\,r!\,s!}F_{30}^{(4)}(-k,-k,a_2,a_2,a_3,a_3,-r,-s;c_1,c_2,c_1,c_3;x,y,z,u),\end{aligned}\tag{4.3}$$

$$\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\left(1-z\left[D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2\right]\right)+v\left(1-u\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\right)\right)$$
$$\times\left\{\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1} \tag{4.4}$$
$$\times\sum_{k,r=0}^{\infty}\frac{t^k\ v^r}{k\,!\,r\,!}F_{30}^{(4)}(-k,-k,a_2,a_2,a_3,a_3,-k,-r;c_1,c_2,c_1,c_3;x,y,z,u),$$

$$\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)+v\left(1-z\left[D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2\right]\right)\left(1-u\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\right)\right)$$
$$\times\left\{\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1} \tag{4.5}$$
$$\times\sum_{k,r=0}^{\infty}\frac{t^k\ v^r}{k\,!\,r\,!}F_{30}^{(4)}(-k,-k,a_2,a_2,a_3,a_3,-r,-r;c_1,c_2,c_1,c_3;x,y,z,u),$$

$$\exp\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]+y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\left(1-z\left[D_{\alpha_2}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_2\right]\right)\left(1-u\left[D_{\alpha_2}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_2\right]\right)\right)$$
$$\times\left\{\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_2^{a_2-1}\alpha_3^{a_3-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1} \tag{4.6}$$
$$\times\sum_{k=0}^{\infty}\frac{t^k}{k\,!}F_{30}^{(4)}(-k,-k,a_2,a_2,a_3,a_3,-k,-k;c_1,c_2,c_1,c_3;x,y,z,u).$$

Proof of (4.1). In (2.1), let $a_1=-k$ and $a_2=-r$, by multiplying both the sides by $\frac{t^k}{k!}\frac{v^r}{r!}$, it is easily seen that

$$\frac{\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]-y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\right)^k}{k\,!}\times\frac{\left(v\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)\right)^r}{r\,!}$$
$$\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\times\frac{t^k\ v^r}{k\,!\,r\,!}F_{30}^{(4)}(-k,-k,-r,-r_1,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u),$$

and then taking the double sum of both sides we get

$$\sum_{k=0}^{\infty}\frac{\left(t\left(1-x\left[D_{\alpha_3}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_3\right]-y\left[D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\alpha_3\right]\right)\right)^k}{k\,!}\times\sum_{r=0}^{\infty}\frac{\left(v\left(1-z\left[D_{\alpha_4}\beta_1^{-1}D_{\beta_1}^{-1}\alpha_4\right]-u\left[D_{\alpha_5}\beta_3^{-1}D_{\beta_3}^{-1}\alpha_5\right]\right)\right)^r}{r\,!}$$
$$\times\left\{\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\right\}$$
$$=\alpha_3^{a_3-1}\alpha_4^{a_4-1}\alpha_5^{a_5-1}\beta_1^{c_1-1}\beta_2^{c_2-1}\beta_3^{c_3-1}\times\sum_{k,r=0}^{\infty}\frac{t^k\ v^r}{k\,!\,r\,!}F_{30}^{(4)}(-k,-k,-r,-r_1,a_3,a_3,a_4,a_5;c_1,c_2,c_1,c_3;x,y,z,u).$$

Using the following exponential expansion:

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k\,!},$$

then, after some simplification, we obtain relation (4.1). A similar argument will establish the other identities (4.2) to (4.6). For their details, we leave to the interested reader.

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