Some Integral Representations for Certain Quadruple Hypergeometric Functions

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Abstract

we aim in this work at establishing new integral representations of Euler type for the Exton hypergeometric functions of four variables $K_6, K_7, K_8, K_9, K_{10}$, the Sharma and Parihar hypergeometric functions of four variables $F_7^{(4)}, F_{41}^{(4)}, F_{61}^{(4)}$ and the Lauricella function of four variables $F_A^{(4)}$, whose kernels include the quadruple hypergeometric functions $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$. Some particular cases and consequences of our main results are also considered.

Keywords: integrals of Euler-type, quadruple hypergeometric functions.

Subject Classification: 33C20, 33C65.

Type (Method/Approach): Series Manuplition ; Integral Transformation;

Language: English

Date of Submission: 2018-02-12

Date of Acceptance: 2018-04-10

Date of Publication: 2018-04-30

Volume: 01 Issue: 01

Journal: *MATLAB Journal*

Website: https://purkh.com

Introduction

In [1], Exton introduced twenty one complete quadruple hypergeometric functions, which he denoted by symbols $K_1, K_2, \ldots, K_{21}$. In [2], eighty three complete quadruple hypergeometric functions given by $F_1^{(4)}, F_2^{(4)}, \ldots, F_{83}^{(4)}$ were defined by Sharma and Parihar. Very recently, Bin-Saad et al. [3] introduced five new quadruple hypergeometric functions whose names are $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$ to investigate their five Laplace integral representations which include the confluent hypergeometric function ${}_0F_1$, ${}_1F_1$, a Humbert functions Φ_2, Φ_3 and Ψ_2 in their kernels, we recall these quadruple hypergeometric functions are defined by

$$X_6^{(4)}(a_1,a_1,a_2,a_1,a_1,a_2,a_2,a_2;c_1,c_1,c_2,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+q}(a_2)_{q+n+2p}}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.1}$$

$$X_7^{(4)}(a_1,a_1,a_1,a_1,a_1,a_2,a_3,a_1;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.2}$$

$$X_8^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_1,c_1,c_2,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{m+n}(c_2)_p(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.3}$$

$$X_9^{(4)}(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_2,c_1,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2q+n+p}(a_2)_n(a_3)_p}{(c_1)_{n+p}(c_2)_m(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{1.4}$$

$$X_{10}^{(4)}(a_1,a_1,a_1,a_2,a_1,a_2,a_3,a_3;c_1,c_2,c_3,c_4;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+q}(a_3)_{p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{1.5}$$

Here, in this paper, we aim at establishing new integral representations of Euler type for the Exton hypergeometric functions of four variables $K_6, K_7, K_8, K_9, K_{10}$, the Sharma and Parihar hypergeometric functions of four variables $F_7^{(4)}, F_{41}^{(4)}, F_{61}^{(4)}$ and the Lauricella function of four variables $F_A^{(4)}$ in terms of the quadruple hypergeometric functions $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$.

2. Integral Representations of Euler-type

For our purpose, we begin by recalling Exton's hypergeometric functions of four variables K_6, K_7, K_8, K_9 and K_{10} defined by

$$K_6(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_2,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_{n+p+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2.1}$$

$$K_7(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_{m+p}(c_2)_{n+q}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2.2}$$

$$K_8(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,y,z,u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2.3}$$

$$K_9(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_3;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \tag{2.4}$$

and

$$K_{10}(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p+q}(a_2)_{m+n}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_4}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2.5}$$

respectively (see[1]). the Sharma's and Parihar's quadruple functions $F_7^{(4)}, F_{41}^{(4)}$ and $F_{61}^{(4)}$ (see[2]) are given by as follows:

$$F_7^{(4)}(a_1,a_1,a_2,a_2,a_3,a_4,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{p+q}(a_3)_{m+p}(a_4)_{n+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{2.6}$$

$$F_{41}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_2;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(a_3)_{m+n+q}(a_4)_p}{(c_1)_{m+p}(c_2)_{n+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \tag{2.7}$$

and

$$F_{61}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_1;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(a_3)_{m+n+q}(a_4)_p}{(c_1)_{m+p+q}(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{2.8}$$

Lauricella hypergeometric function of four variables $F_A^{(4)}$ is as below (see [4])

$$F_A^{(4)}(a,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\sum_{m,n,p,q=0}^{\infty}\frac{(a)_{m+n+p+q}(a_1)_m(a_2)_n(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{2.9}$$

Now, by means of the quadruple functions $X_6^{(4)}, X_7^{(4)}, X_8^{(4)}, X_9^{(4)}, X_{10}^{(4)}$, we investigate some further integral representations of Euler-type for $K_6, K_7, K_8, K_9, K_{10}, F_7^{(4)}, F_{41}^{(4)}, F_{61}^{(4)}$ and $F_A^{(4)}$ as follows:

$$\begin{aligned}&K_6(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_2,c_2;x,y,z,u)=\frac{\Gamma(a_1+a_2)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(c_2-a)}\\&\times\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a-1}(1-\beta)^{c_2-a-1}\\&\times X_8^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;a,a,c_2-a,c_1;\ \alpha\beta(1-\alpha)y,\alpha\beta z,\ \alpha(1-\beta)u,\ \alpha(1-\alpha)x)\\&\times d\alpha\, d\beta\\&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_2-a)>0);\end{aligned} \tag{2.10}$$

$$\begin{aligned}&K_6(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_2,c_2;x,y,z,u)=\frac{4\Gamma(a_1+a_2)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(c_2-a)}\\&\times\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(\sin^2\alpha)^{a_1-\frac12}(\cos^2\alpha)^{a_2-\frac12}(\sin^2\beta)^{a-\frac12}(\cos^2\beta)^{c_2-a-\frac12}\\&\times X_9^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,a,a,c_2-a;\ \tfrac14 x\sin^2 2\alpha,\ z\sin^2\alpha\sin^2\beta,\ u\sin^2\alpha\sin^2\beta,\ \tfrac14 y\sin^2 2\alpha\cos^2\beta)\\&\times d\alpha\, d\beta\\&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_2-a)>0);\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
&K_7(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_2;x,y,z,u)=\frac{\Gamma(a_1+a_2)\,\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a)\,\Gamma(b)\,\Gamma(c_1-a)\,\Gamma(c_2-b)}\\
&\times\int_0^1\int_0^1\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_2-1}\beta^{a-1}(1-\beta)^{c_1-a-1}\,\gamma^{b-1}(1-\gamma)^{c_2-b-1}\\
&\times X_7^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;a,c_1-a,c_2-b,b;\,\alpha\beta\,(1-\alpha)\,x,\alpha(1-\beta)\,z\,,\,\alpha\,(1-\gamma)\,u,\,\alpha\gamma(1-\alpha)\,y)\\
&\times d\alpha\;d\beta\;d\gamma\\
&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c_2-b)>0);
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
&K_7(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_2\;;x,y,z,u)=\frac{\Gamma(a_1+a_2)\,\Gamma(c_2)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a)\,\Gamma(c_2-a)}\\
&\times\int_0^\infty\int_0^\infty \frac{\alpha^{a_1-1}}{(1+\alpha)^{a_1+a_2}}\,\frac{\beta^{a-1}}{(1+\beta)^{c_2}}\\
&\times X_8^{(4)}\left(a_1+a_2,\,a_1+a_2,\,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_1,c_2-a\,,a\,;\,\frac{\alpha\,x}{(1+\alpha)^2}\,,\,\frac{\alpha\,z}{(1+\alpha)}\,,\frac{\alpha\,u}{(1+\alpha)\,(1+\beta)}\,,\frac{\alpha\beta\,u}{(1+\alpha)^2\,(1+\beta)}\right)\\
&\times d\alpha\;d\beta\\
&(\mathrm{Re}(a_1)>0\,,\mathrm{Re}(a_2)>0\,,\ \mathrm{Re}(a)>0\,,\mathrm{Re}(c_2-a)>0);
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&K_8(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\,\Gamma(a_2)}\\
&\times\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_2-1}\\
&\times X_8^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_1,c_3,c_2;\alpha(1-\alpha)\,x,\alpha\,z,\alpha\,u,\alpha(1-\alpha)\,y)\,d\alpha\\
&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0);
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
&K_8(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_1,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\,\Gamma(a_2)}\\
&\times\int_0^\infty \left(e^{-\alpha}\right)^{a_1}\,\left(1-e^{-\alpha}\right)^{a_2-1}\\
&\times X_8^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_1,c_3,c_2;e^{-\alpha}\left(1-e^{-\alpha}\right)x,e^{-\alpha}\ z,e^{-\alpha}\ u,e^{-\alpha}\left(1-e^{-\alpha}\right)y)\\
&\times d\alpha\\
&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0);
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&K_9(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\,\Gamma(a_2)}\\
&\times\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_2-1}\\
&\times X_9^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_3,c_3,c_2;\alpha(1-\alpha)\,x,\alpha\,z,\alpha\,u,\alpha(1-\alpha)\,y)\,d\alpha\\
&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0);
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&K_9(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_3;x,y,z,u)=\frac{\Gamma(a_1+a_2)\,(S-T)^{a_1}\,(R-T)^{a_2}}{\Gamma(a_1)\,\Gamma(a_2)\,(S-R)^{a_1+a_2-1}}\\
&\times\int_R^S \frac{(\alpha-R)^{a_1-1}\,(S-\alpha)^{a_2-1}}{(\alpha-T)^{a_1+a_2}}\\
&\times X_9^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;\,c_1,\,c_3,c_3,c_2;\,\lambda_1 x,\,\lambda_2 z,\,\lambda_2 u,\lambda_1 y\,)\,d\alpha\\
&\left(\lambda_1=\frac{(S-T)(R-T)(\alpha-R)(S-\alpha)}{(S-R)^2\,(\alpha-T)^2}\,,\lambda_2=\frac{(S-T)(\alpha-R)}{(S-R)(\alpha-T)}\right),\\
&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ T<R<S);
\end{aligned}
\tag{2.17}
$$

$$K_{10}(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_2-1}$$
$$\times X_7^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_3,c_4,c_2;\alpha(1-\alpha)x,\alpha z,\alpha u,\alpha(1-\alpha)y)\,d\alpha$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0); \tag{2.18}$$

$$K_{10}(a_1,a_1,a_1,a_1,a_2,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{\Gamma(a_1+a_2)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^1 \frac{\alpha^{a_1-1}(1-\alpha)^{a_2-1}}{(1+M\alpha)^{a_1+a_2}}$$
$$\times X_7^{(4)}(a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;c_1,c_3,c_4,c_2;\lambda_1 x,\lambda_2 z,\lambda_2 u,\lambda_1 y)\,d\alpha$$
$$\left(\lambda_1=\frac{(1+M)\alpha(1-\alpha)}{(1+M\alpha)^2},\ \lambda_2=\frac{(1+M)\alpha}{(1+M\alpha)}\right),$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ M>-1); \tag{2.19}$$

$$F_7^{(4)}(a_1,a_1,a_2,a_2,a_3,a_4,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)}$$
$$\times\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_3-1}$$
$$\times X_{10}^{(4)}(a_1+a_3,a_1+a_3,a_1+a_3,a_4,a_1+a_3,a_4,a_2,a_2;c_1,c_2,c_3,c_4;\alpha(1-\alpha)x,\alpha y,(1-\alpha)z,u)\,d\alpha$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_3)>0); \tag{2.20}$$

$$F_7^{(4)}(a_1,a_1,a_2,a_2,a_3,a_4,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{2\,\Gamma(a_1+a_3)(1+M)^{a_1}}{\Gamma(a_1)\Gamma(a_3)}$$
$$\times\int_0^{\frac{\pi}{2}} \frac{(\sin^2\alpha)^{a_1-\frac{1}{2}}(\cos^2\alpha)^{a_3-\frac{1}{2}}}{(1+M\sin\alpha)^{a_1+a_3}}$$
$$\times X_{10}^{(4)}(a_1+a_3,a_1+a_3,a_1+a_3,a_4,a_1+a_3,a_4,a_2,a_2;c_1,c_2,c_3,c_4;\lambda_1 x,\lambda_2 y,\lambda_3 z,u)\,d\alpha$$
$$\left(\lambda_1=\frac{\frac{1}{4}(1+M)\sin^2 2\alpha}{(1+M\sin^2\alpha)^2},\ \lambda_2=\frac{(1+M)\sin^2\alpha}{(1+M\sin^2\alpha)},\ \lambda_3=\frac{\cos^2\alpha}{(1+M\sin^2\alpha)}\right),$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_3)>0,\ M>-1); \tag{2.21}$$

$$F_{41}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_2;x,y,z,u)=\frac{\Gamma(a_1+a_4)\,\Gamma(a_2+a_3)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(a_4)}$$
$$\times\int_0^1\int_0^1 \alpha^{a_1-1}(1-\alpha)^{a_4-1}\beta^{a_2-1}(1-\beta)^{a_3-1}$$
$$\times X_6^{(4)}(a_1+a_4,a_1+a_4,a_2+a_3,a_1+a_4,a_1+a_4,a_2+a_3,a_2+a_3,a_2+a_3;c_1,c_1,c_2,c_2;\ \alpha(1-\alpha)z,\alpha(1-\beta)x,\ \beta(1-\beta)u,\ \alpha(1-\beta)y)$$
$$\times d\alpha\, d\beta$$
$$(\mathrm{Re}(a_i)>0,\ (i=1,2,3,4)); \tag{2.22}$$

$$F_{41}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_2;x,y,z,u)=\frac{\Gamma(a_1+a_4)\,\Gamma(a_2+a_3)}{\Gamma(a_1)\,\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(a_4)}$$
$$\times\int_0^\infty\int_0^\infty (e^{-\alpha})^{a_1}(1-e^{-\alpha})^{a_4-1}(e^{-\beta})^{a_2}(1-e^{-\beta})^{a_3-1}$$
$$\times X_6^{(4)}(a_1+a_4,a_1+a_4,a_2+a_3,a_1+a_4,a_1+a_4,a_2+a_3,a_2+a_3,a_2+a_3;c_1,c_1,c_2,c_2;\ (e^{-\alpha}-e^{-2\alpha})z,\lambda x,\ (e^{-\beta}-e^{-2\beta})u,\ \lambda y)$$
$$\times d\alpha\, d\beta$$
$$\left(\lambda=\left(e^{-\alpha}-e^{-(\alpha+\beta)}\right)\right)$$
$$(\mathrm{Re}(a_i)>0,\ (i=1,2,3,4)); \tag{2.23}$$

$$\begin{aligned}&F_{61}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_1;x,y,z,u)=\frac{\Gamma(a_1+a_3)\,\Gamma(c_1)}{\Gamma(a_1)\,\Gamma(a_3)\,\Gamma(a)\,\Gamma(c_1-a)}\\&\times\int_0^1\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_3-1}\beta^{a-1}(1-\beta)^{c_1-a-1}\\&\times X_8^{(4)}(a_1+a_3,a_1+a_3,a_1+a_3,a_1+a_3,a_1+a_3,a_4,a_2,a_1+a_3;a,a,c_1-a,c_2;\ \alpha\beta(1-\alpha)\,x,\alpha\beta\, z\ ,\ (1-\alpha)\,(1-\beta)\,u,\ \beta(1-\beta)y)\\&\times d\alpha\ d\beta\\&(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_3)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0);\end{aligned}\tag{2.24}$$

$$\begin{aligned}&F_{61}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_3;c_1,c_2,c_1,c_1\,;x,y,z,u)=\frac{\Gamma(a_1+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}\\&\times\int_0^\infty\int_0^\infty \frac{\alpha^{a_1-1}}{(1+\alpha)^{a_1+a_3}}\,\frac{\beta^{a-1}}{(1+\beta)^{c_1}}\\&\times X_9^{(4)}\left(a_1+a_2,\,a_1+a_2,\,a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_4,a_1+a_2;a,c_1-a,c_1-a\ ,c_2\ ;\frac{\alpha\beta\, x}{(1+\alpha)^2(1+\beta)},\frac{\alpha\ z}{(1+\alpha)(1+\beta)},\frac{u}{(1+\alpha)(1+\beta)},\frac{\alpha\ y}{(1+\alpha)^2}\right)\\&\times d\alpha\ d\beta\\&(\mathrm{Re}(a_1)>0\,,\mathrm{Re}(a_3)>0\,,\mathrm{Re}(a)>0\,,\mathrm{Re}(c_1-a)>0);\end{aligned}\tag{2.25}$$

$$\begin{aligned}&F_A^{(4)}(a,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{\Gamma(a+a_1+a_4)}{\Gamma(a)\Gamma(a_1)\Gamma(a_4)}\\&\times\int_0^1\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_4-1}\beta^{a-1}(1-\beta)^{a_1+a_4-1}\\&\times X_7^{(4)}(a+a_1+a_4,a+a_1+a_4,a+a_1+a_4,a+a_1+a_4,a+a_1+a_4,a_2,a_3,a+a_1+a_4;c_1,c_2,c_3,c_4;\ \alpha\beta\,(1-\beta)\,x,\beta\ y\ ,\ \beta\ z,\ \beta(1-\alpha)(1-\beta)\,u)\\&\times d\alpha\ d\beta\\&(\mathrm{Re}(a)>0,\ \mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_4)>0);\end{aligned}\tag{2.26}$$

$$\begin{aligned}&F_A^{(4)}(a,a_1,a_2,a_3,a_4;c_1,c_2,c_3,c_4;x,y,z,u)=\frac{\Gamma(a+a_1+a_2)}{\Gamma(a)\,\Gamma(a_1)\,\Gamma(a_2)}\\&\times\int_0^1\int_0^1 \alpha^{a_1-1}\,(1-\alpha)^{a_2-1}\beta^{a_1+a_2-1}(1-\beta)^{a-1}\\&\times X_7^{(4)}(a+a_1+a_2,a+a_1+a_2,a+a_1+a_2,a+a_1+a_2,a+a_1+a_2,a_3,a_4,a+a_1+a_2;c_1,c_3,c_4,c_2;\ \alpha\beta(1-\beta)x,(1-\beta)z,(1-\beta)u,\ \beta(1-\alpha)\,(1-\beta)y)\\&\times d\alpha\ d\beta\\&(\mathrm{Re}(a)>0,\ \mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0).\end{aligned}\tag{2.27}$$

Proof of the integral representations

Once substituting the series definition of the special function in each integrand and then, changing the order of the integral and the summation, and finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals (see, for example, [5, p. 9–11], [6, 7, Section 1.1] and [8, p. 26 and p. 86, Problem 1]), we derive each of the integral representations from (2.10) to (2.27).

$$B(a,b)=\begin{cases}\displaystyle\int_0^1 t^{a-1}\,(1-t)^{b-1}\,dt\,,\ (\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0),\\[2mm] \displaystyle\frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\qquad\qquad (a,b\in C\setminus Z_0^-),\end{cases}\tag{2.28}$$

$$\begin{aligned}&B(a,b)=\int_0^1 \alpha^{a-1}\,(1-\alpha)^{b-1}\,d\alpha=\int_0^\infty \left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{b-1}d\alpha\\&(\mathrm{Re}(a)>0,\,\mathrm{Re}(b)>0),\end{aligned}\tag{2.29}$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(\sin\alpha)^{2a-1}(\cos\alpha)^{2b-1}\,d\alpha=\int_0^{\infty}\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\,d\alpha \tag{2.30}$$
$$(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0),$$

$$B(a,b)=\frac{(S-T)^a(R-T)^b}{(S-R)^{a+b-1}}\int_R^S\frac{(\alpha-R)^{a-1}(S-\alpha)^{b-1}}{(\alpha-T)^{a+b}}\,d\alpha\quad(T<R<S)$$
$$=(1+M)^a\int_0^1\frac{\alpha^{a-1}(1-\alpha)^{b-1}}{(1+M\alpha)^{a+b}}\,d\alpha\qquad(M>-1) \tag{2.31}$$
$$(\mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0).$$

3. Special Cases

It is easy to observe that the main results (2.10) to (2.27) gave a number integral representations of Euler-type. In the present section, we will mention only some special cases.

Let $x=0$ in (2.10), (2.12), (2.14), (2.20), (2.22) and (2.24), we get the following integral representations involving triple hypergeometric functions:

$$F_D^{(3)}(a_1,a_2,a_3,a_4;c_2;y,z,u)=\frac{\Gamma(a_1+a_2)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(c_2-a)}\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a-1}(1-\beta)^{c_2-a-1}$$
$$\times X_6(a_1+a_2,a_3,a_4;a,c_2-a;\alpha\beta(1-\alpha)y,\alpha\beta z,\alpha(1-\beta)u)\,d\alpha\,d\beta \tag{3.1}$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_2-a)>0);$$

$$F_G(a_1,a_1,a_1,a_3,a_2,a_4;c_1,c_2,c_2;z,y,u)=\frac{\Gamma(a_1+a_2)\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a)\Gamma(b)\Gamma(c_1-a)\Gamma(c_2-b)}$$
$$\times\int_0^1\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}\beta^{a-1}(1-\beta)^{c_1-a-1}\gamma^{b-1}(1-\gamma)^{c_2-b-1} \tag{3.2}$$
$$\times X_8(a_1+a_2,a_3,a_4;b,c_1-a,c_2-b;\alpha\gamma(1-\alpha)y,\alpha(1-\beta)z,\alpha(1-\gamma)u)\,d\alpha\,d\beta\,d\gamma$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0,\ \mathrm{Re}(b)>0,\ \mathrm{Re}(c_2-b)>0);$$

$$F_A^{(4)}(a_1,a_2,a_3,a_4;c_2,c_1,c_3;y,z,u)=\frac{\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_2-1}$$
$$\times X_8(a_1+a_2,a_3,a_4;c_2,c_1,c_3;\alpha(1-\alpha)y,\alpha z,\alpha u)\,d\alpha \tag{3.3}$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_2)>0);$$

$$F_K(a_1,a_2,a_2,a_4,a_3,a_4;c_2,c_3,c_4;y,z,u)=\frac{\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)}\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_3-1}$$
$$\times H_B(a_1+a_3,a_4,a_2;c_2,c_3,c_4;\alpha y,(1-\alpha)z,u)\,d\alpha \tag{3.4}$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_3)>0);$$

$$F_M(a_4,a_3,a_3,a_1,a_1,a_2;c_1,c_2,c_2;z,y,u)=\frac{\Gamma(a_1+a_4)\Gamma(a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_4-1}\beta^{a_2-1}(1-\beta)^{a_3-1}$$
$$\times X_{10}(a_2+a_3,a_1+a_4;c_1,c_2;\beta(1-\beta)u,\alpha(1-\beta)y,\alpha(1-\alpha)z)\,d\alpha\,d\beta \tag{3.5}$$
$$(\mathrm{Re}(a_i)>0,\ (i=1,2,3,4));$$

$$F_P(a_1,a_2,a_1,a_3,a_3,a_4;c_2,c_1,c_1;y,u,z)=\frac{\Gamma(a_1+a_3)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_3)\Gamma(a)\Gamma(c_1-a)}\int_0^1\int_0^1\alpha^{a_1-1}(1-\alpha)^{a_3-1}\beta^{a-1}(1-\beta)^{c_1-a-1}$$
$$\times X_8(a_1+a_3,a_4,a_2;c_2,c_1-a,a;\beta(1-\beta)y,(1-\alpha)(1-\beta)u,\alpha\beta z)\,d\alpha\,d\beta \tag{3.6}$$
$$(\mathrm{Re}(a_1)>0,\ \mathrm{Re}(a_3)>0,\ \mathrm{Re}(a)>0,\ \mathrm{Re}(c_1-a)>0),$$

where X_6, X_8, X_{10} are the Exton functions of three variables (see [9]), H_B are Srivastava's function (see, for details [10]), $F_A^{(3)}, F_D^{(3)}, F_G, F_K, F_M, F_P$ are three variables Lauricella functions (cf. [10,Sections 1.4 and 1.5]) .

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