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Initial value problems between Taylor and Volterra integral equations

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Abstract. In this work, we solve initial value problem for a second order numerically using Euler and Taylor series. On the other hand, we show that after the transformation of the (IVP) into Volterra integral equation of the second kind, its numerical solution using trapezoidal method and modified Simpson method are better than the two first ones Euler and Tayor methods.

1. Introduction

Initial value problems and in particular initial value problem for second-order are at the heart of many mathematical descriptions of physical systems, as used by engineers, physicists and applied mathematicians [**1, 2, 10**]. As we know, most of those differential equations cannot be solved analytically by a simple formula. So we present in this study a development of a class of methods that solve an initial value problem numerically, using a computer algorithm [**3, 5, 7**]. In industry and science, differential equations are almost always solved numerically because most real-world problems are too complicated to solve analytically and even if the differential equations can be solved analytically often the solution is in the form of a complicated integral

Many solutions have been obtained in [5,6] for further specific second-order ordinary differential equations

2. Initial value problem for second-order

Given a differential equation of the second order

$$y'' = f(x, y, y'). \tag{2.1}$$

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An initial value problem (abbreviated IVP) for a second-order differential equation is the problem of finding a solution $y(x)$ to equation (2.1) that satisfies an initial conditions $y(x_0)$ and $y'(x_0)$, where x_0 is some fixed value of point and $y(x_0)$ and $y'(x_0)$ are a fixed states. We write the IVP concisely as

$$IVP \quad \begin{cases} y'' = f(x, y, y'), \\ y(x_0) = y_0;\ y'(x_0) = y_1. \end{cases} \tag{2.2}$$

The initial conditions usually picks out a specific value of the arbitrary constants C_1 and C_2 that appears in the general solution of the equation. So, it selects one of the many possible states that satisfy the differential equation.

2.1. Numerical Methods of IVP for a second-order. Suppose we want to solve the following initial value problem on the interval $a \leq x \leq b$

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0;\ y'(x_0) = y_1. \tag{2.3}$$

or again

$$y'' + p(x)y' + q(x)y = f(x).$$

If the functions p, q and g are continuous on the interval $I : a \leq x \leq b$ containing the point $x = x_0$. Then there exists a unique solution $y(x)$ of the

problem, and that this solution exists throughout the interval I. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $p(t), q(t)$, or $g(t)$.

Rather than seek a continuous solution defined at each point x, we develop a strategy of discretization the problem to determine an approximation at discrete points in the interval of interest. Therefore, the plan is to replace the continuous function model (2.3) with an approximate discrete function model that is amenable to computer solution.

We divide the interval $a \leq x \leq b$ into N segments of constant length h, called the stepsize. Thus the stepsize is $h = \dfrac{b-a}{N}$. This defines a set of equally spaced discrete points $a = x_0 \leq x_2 \leq x_3 ....., x_N = b$, where $x_{i+1} = x_i + h$, $i = 0, 1, 2, ..., N$. Now, suppose we know the solution $y(x_i)$ and the first derivative $y'(x_i)$ of the initial value problem at the point x_i. How could we estimate the solution at the point x_{i+1}?.

$$Y(x_0) = \begin{pmatrix} y(x_0) \\ z(x_0) \end{pmatrix} = \begin{pmatrix} y(x_0) \\ y'(x_0) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$Y'(x) = \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ f(x, y, z) \end{pmatrix} = \begin{pmatrix} z(x) \\ z'(x) \end{pmatrix}$$

$$Y''(x) = \begin{pmatrix} y''(x) \\ z''(x) \end{pmatrix} = \begin{pmatrix} z'(x) \\ z''(x) \end{pmatrix} = \begin{pmatrix} f(x, y, z) \\ z''(x) \end{pmatrix}$$

- **Euler method**

$$Y(x_{i+1}) = Y(x_i) + hY'(x_i) + \frac{h^2}{2}Y''(\omega_i); \quad x_i < \omega_i < x_{i+1}. \tag{2.4}$$

For the series in (2.4) the integration order one. The truncation error, due to the terms omitted from the series, is

$$E = \frac{h^2}{2}Y''(\omega_i); \quad x_i < \omega_i < x_{i+1}.$$

- **Taylor series Method of order two**

$$Y(x_{i+1}) = Y(x_i) + hY'(x_i) + \frac{h^2}{2}Y''(x_i) + \frac{h^3}{3!}Y'''(\omega_i); \quad x_i < \omega_i < x_{i+1}. \tag{2.5}$$

For the series in (2.5) the integration order two. The truncation error, due to the terms omitted from the series, is

$$E = \frac{h^3}{3!}Y'''(\omega_i); \quad x_i < \omega_i < x_{i+1}.$$

- **Volterra integral equation method**

$$\begin{aligned}
&y'' = f(x, y, y'), \qquad y(x_0) = y_0; \ y'(x_0) = y_1, \\
&\int_{x_0}^{x} y''(x)dx = \int_{x_0}^{x} f(x, y, y')dx, \\
&y'(x) = y_1 + \int_{x_0}^{x} f(x, y, y')dx \\
&\int_{x_0}^{x} y'(x)dx = y_1(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{x} f(z, y, y')dzdx
\end{aligned}$$

$$y(x) = y_0 + y_1(x - x_0) + \int_{x_0}^{x} (x - z)f(z, y, y')dz \tag{2.6}$$

For the series in (2.6) The error of the composite trapezoidal rule is the difference between the value of the integral and the numerical result

$$E = \frac{h^2}{12}M_2$$

where $M_2 = \sup_{a \le x \le b} (x - x_0)^3 |f''(x)|$ and h is the "step length", given by $h = \dfrac{b - a}{N}$.

For the series in (2.6), the error committed by the composite Simpson's rule is bounded (in absolute value) by

$$E = \frac{h^4}{180}M_4$$

where $M_4 = \sup_{a\leq x\leq b} (x-x_0)^4 \left|f^{(4)}(x)\right|$ and h is the "step length", given by $h = \dfrac{b-a}{N}$.

Denoting by y_i the approximation of the solution $y(x_i)$ at $x = x_i$. By algorithm and the given value $y_0 = y(x_0) = \alpha$ and $y_1 = y'(x_0) = \beta$ of the solution and its derivative at the point x_0 we can calculate the approximations solutions $y_i,\ i = 1, 2, .., N$.

Example 1. *solve the initial value problem* [**4**]

$$y'' = 0.1y' - x; \quad y(0) = 0, \quad y'(0) = 1 \tag{2.7}$$

from $x = 0$ to 1 with the Euler, Taylor series of order two, Trapezoidal and modified Simpson methods using $N = 10$, where the analytical solution of the IVP is

$$y(x) = 100x - 5x^2 + 990\left(\exp\left(-0.1x\right) - 1\right)$$

- **Euler Method of IVP for second order**

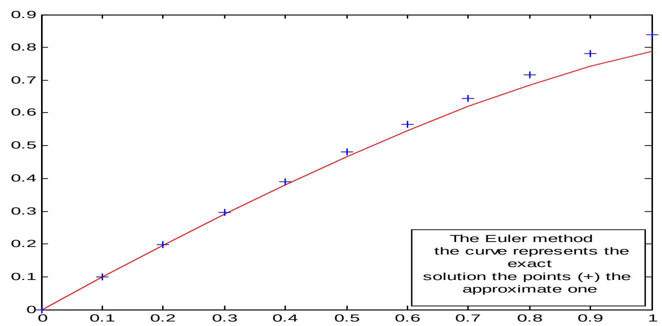
Solution 1. *For $y = y$ and $z = y'$ the equivalent first-order equations and initial conditions are*

$$\begin{aligned}
Y(x) &= \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix};\ Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
Y'(x) &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} z(x) \\ z'(x) \end{pmatrix}\begin{pmatrix} z \\ -0.1z - x \end{pmatrix} \\
\begin{pmatrix} y(x_{i+1}) \\ z(x_{i+1}) \end{pmatrix} &\simeq \begin{pmatrix} y(x_i) \\ z(x_i) \end{pmatrix} + h\begin{pmatrix} z_i \\ -0.1z_i - x_i \end{pmatrix}
\end{aligned}$$

which is computed

values of x	Exact solution	Approx solution	Error
0.000000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000000e-001	1.966866e-001	1.990000e-001	2.313426e-003
4.000000e-001	3.815448e-001	3.900499e-001	8.505139e-003
6.000000e-001	5.468882e-001	5.653479e-001	1.845966e-002
8.000000e-001	6.851829e-001	7.172475e-001	3.206456e-002
1.000000e+000	7.890439e-001	8.382543e-001	4.921040e-002

The method of Euler series $y''=-0.1y'-x;$ $y(0)=0,$ $y('0)=1$



- **Taylor series Method of order two of IVP for second order**

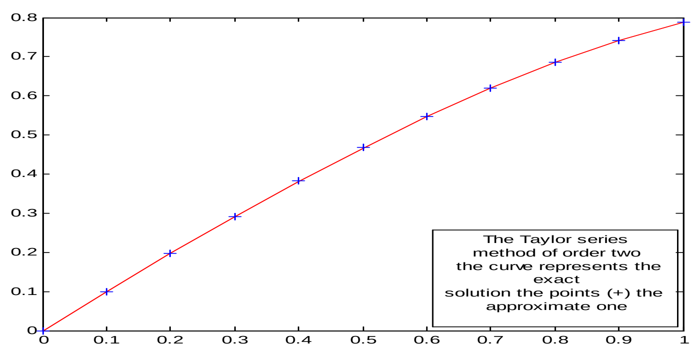
$$
\begin{aligned}
Y(x) &= \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix};\ Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
Y' &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} z \\ -0.1z - x \end{pmatrix} \\
Y'' &= \begin{pmatrix} y''(x) \\ z''(x) \end{pmatrix} \begin{pmatrix} z'(x) \\ z''(x) \end{pmatrix} = \begin{pmatrix} -0.1z - x \\ z''(x) \end{pmatrix} \\
&= \begin{pmatrix} -0.1z - x \\ -0.01z + 0.1x - 1 \end{pmatrix} \\
\begin{pmatrix} y(x_{i+1}) \\ z(x_{i+1}) \end{pmatrix} &\simeq \begin{pmatrix} y(x_i) \\ z(x_i) \end{pmatrix} + h \begin{pmatrix} z_i \\ -0.1z_i - x_i \end{pmatrix} + \\
& \frac{h^2}{2} \begin{pmatrix} -0.1z_i - x_i \\ -0.01z_i + 0.1x_i - 1 \end{pmatrix}
\end{aligned}
$$

which is computed

values of x	Exact solution	Approx solution	Error
0.000000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000000e-001	1.966866e-001	1.970025e-001	3.159513e-004
4.000000e-001	3.815448e-001	3.820853e-001	5.405862e-004
6.000000e-001	5.468882e-001	5.474872e-001	5.990007e-004
8.000000e-001	6.851829e-001	6.856022e-001	4.193082e-004
1.000000e+000	7.890439e-001	7.889764e-001	6.745093e-005

$$y'' = -0.1y' - x; \quad y(\ 0) = 0, \quad y('0) = 1$$

The method of Taylor 2 series



- **Volterra integral equation for IVP second order**

On the other hand, the transformation of the initial value problem (2.7) into Volterra integral equation gives the best approximation solution using the trapezoidal method

$$y'' = 0.1y' - x; \quad y(0) = 0, \quad y'(0) = 1,$$

integral equation

$$\int_0^x y''(x)dx = \int_0^x -0.1y\prime(x)dx - \int_0^x xdx;$$

$$y'(x) - y'(0) = -0.1y(x) - \frac{1}{2}x^2$$

$$y'(x) = -0.1y(x) - \frac{1}{2}x^2 + 1$$

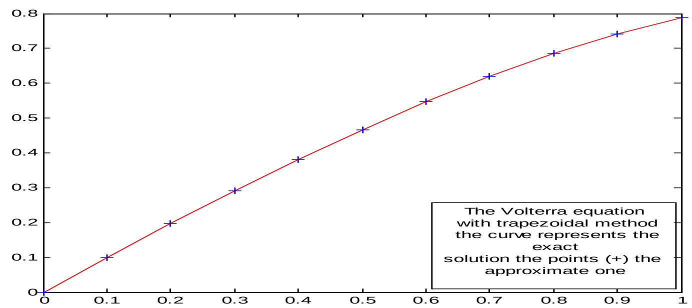
$$\int_0^x y'(x)dx = \int_0^x -0.1y(x)dx + \int_0^x \left(-\frac{1}{2}x^2 + 1\right) dx$$

$$y(x) - y(0) = \int_0^x -0.1y(x)dx - \frac{1}{6}x^3 + x$$

$$y(x) = \int_0^x -0.1y(x)dx - \frac{1}{6}x^3 + x$$

values of x	Exact solution	Approx solution	Error
0.000000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000000e-001	1.966866e-001	1.966899e-001	3.277224e-006
4.000000e-001	3.815448e-001	3.815545e-001	9.692125e-006
6.000000e-001	5.468882e-001	5.469074e-001	1.911916e-005
8.000000e-001	6.851829e-001	6.852144e-001	3.143652e-005
1.000000e+000	7.890439e-001	7.890904e-001	4.652604e-005

The trapezoidal method $y'' = -0.1y' + x^2; \quad y(0) = 0, \; y'(0) = 1$
transformed into VIE $y(x) - \int_0^x -0.1y(x)dx = x - \frac{1}{6}x^3$



EXAMPLE 2. *solve the initial value problem* [**4**]

$$y'' = -4xy' - (3 + 4x^2)y; \quad y(0) = 0, y'(0) = 1, \tag{2.8}$$

from $x = 0$ to 1 with the Euler, Taylor series of order two, Trapezoidal and modified Simpson methods using $N = 10$, where the analytical solution of the IVP is

$$y(x) = \sin x \exp(-x^2);$$

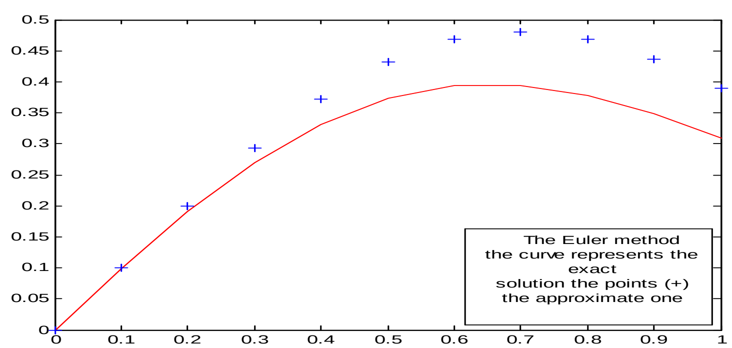
- **Euler Method of IVP for second order**

Solution For $y = y$ and $z = y'$ the equivalent first-order equations and initial conditions are

$$\begin{aligned} Y(x) &= \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}; \; Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ Y'(x) &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} z(x) \\ -4xz - (3 + 4x^2)y \end{pmatrix} \end{aligned}$$

values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	2.000000e-001	9.120605e-003
4.000e-001	3.318404e-001	3.721632e-001	4.032278e-002
6.000e-001	3.939377e-001	4.687503e-001	7.481264e-002
8.000e-001	3.782564e-001	4.692420e-001	9.098559e-002
1.000e+000	3.095599e-001	3.890364e-001	7.947656e-002

The method of Euler series $y''=-4xy'-(3+4x^2)y; \; y(0)=0, y'(0)=1$



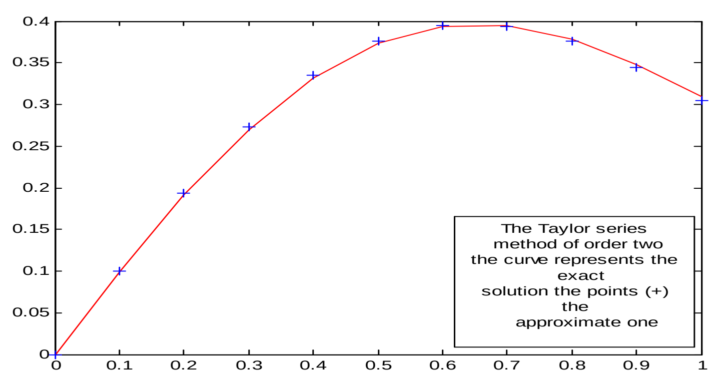
- **Taylor series Method of order two of IVP for second order**

$$\begin{aligned}
Y(x) &= \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}; \; Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
Y'(x) &= \begin{pmatrix} y'(x) \\ z'(x) \end{pmatrix} = \begin{pmatrix} z(x) \\ -4xz - (3+4x^2)y \end{pmatrix} \\
Y''(x) &= \begin{pmatrix} z'(x) \\ -4z - 4xz' - 8xy - (3+4x^2)z \end{pmatrix} \\
&= \begin{pmatrix} -4xz - (3+4x^2)y \\ -4xz' - (7+4x^2)z - 8xy \end{pmatrix}
\end{aligned}$$

which is computed

values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	1.930500e-001	2.170605e-003
4.000e-001	3.318404e-001	3.346036e-001	2.763185e-003
6.000e-001	3.939377e-001	3.948634e-001	9.256742e-004
8.000e-001	3.782564e-001	3.759754e-001	2.281027e-003
1.000e+000	3.095599e-001	3.046570e-001	4.902875e-003

The method of Taylor 2 series $y''=-4xy'-(3+4x^2)y$; $y(0)=0, y'(0)=1$



- **Volterra integral equation of IVP for second order**

On the other hand, the transformation of the initial value problem (2.8) into Volterra integral equation gives the best approximation solution using the trapezoidal method and the modified Simpson method

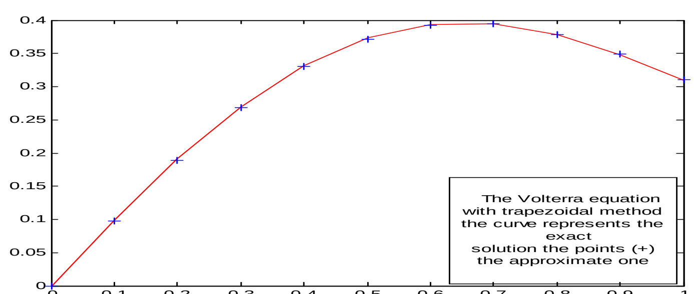
$$y'' = -4xy' - (3+4x^2)y; \quad y(0) = 0, y'(0) = 1,$$

integral equation

$$\int_0^x y''(x)dx = \int_0^x -4xy'dx - \int_0^x (3+4x^2)ydx;$$
$$y'(x) - y'(0) = -4xy(x) + \int_0^x 4ydx; \; - \int_0^x (3+4x^2)ydx;$$
$$y'(x) = \int_0^x (1-4x^2)y(x)dx - 4xy(x) + 1$$
$$\int_0^x y'(x)dx = \int_0^x \int_0^x (1-4x^2)y(x)dx - \int_0^x 4xy(x)dx + x$$
$$y(x) - y(0) = \int_0^x \left((1-4z^2)(x-z) - 4z\right) y(z)dz + x$$
$$y(x) = \int_0^x \left((1-4z^2)(x-z) - 4z\right) y(z)dz + x$$

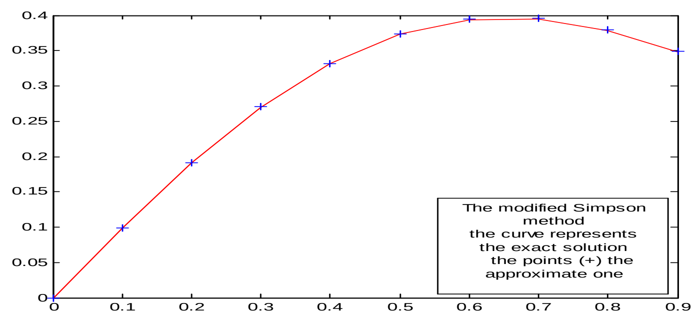
values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	1.894419e-001	1.437464e-003
4.000e-001	3.318404e-001	3.300467e-001	1.793676e-003
6.000e-001	3.939377e-001	3.929823e-001	9.553452e-004
8.000e-001	3.782564e-001	3.784867e-001	2.302680e-004
1.000e+000	3.095599e-001	3.103894e-001	8.295589e-004

The trapezoidal method $y'' = -4xy' - (3+4x^2)y; \quad y(0) = 0, y'(0) = 1$ transformed into VIE $y(x) - \int_0^x \left(\left(1-4z^2\right)(x-z) - 4z\right) y(z)dz = x$



values of x	Exact solution	Approx solution	Error
0.000e+000	0.000000e+000	0.000000e+000	0.000000e+000
2.000e-001	1.908794e-001	1.909270e-001	4.764010e-005
4.000e-001	3.318404e-001	3.321603e-001	3.198637e-004
6.000e-001	3.939377e-001	3.946794e-001	7.417302e-004
8.000e-001	3.782564e-001	3.792457e-001	9.892349e-004
1.000e+000	3.095599e-001	3.103804e-001	8.205444e-004

The modified Simpson method $y''= -4xy' - (3+4x^2)y;\quad y(0)=0, y'(0)=1$ transformed into VIE $y(x) - \int_0^x \left(\left(1-4z^2\right)(x-z) - 4z\right) y(z)dz = x$



2.2. Conclusion. we remark that all initial value problem can be transformed into Volterra integral equation of the second kind, where the numerical solution of those equations using the modified Simpson method ([**8**]) is better than the trapezoidal method and in the other hand, these two methods are better than the Euler ([**7**]) and Taylor methods [**6**].

References

[1] K. Al-Khaled, M. Naim Anwar, Numerical comparison of methods for solving second-order ordinary initial value problems, in Applied Mathematical Modelling 31 (2007) 292–301

[2] L. Bougoffa, On the exact solutions for initial value problems of second-order differential equations in Applied Mathematics Letters 22 (2009) 1248 1251

[3] K. O. Friedrichs, Advanced ordinary differential equations, science Publishers, Inc 1967

[4] J. Kiusalaas, Numerical Methods in Engineering with MATLAB, in Cambridge University Press, New York 2010

[5] J. D. Logan, A First Course in Differential Equations, Springer Science 2006

[6] K. Maleknejad, N. Aghazadeh, Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method, Appl. Math. Comput. 161 (2005) 915–922.

[7] M. Nadir, S. Guechi, Integral Equations and their Relationship to Differential Equations with Initial Conditions, in General Letters in Mathematics. 1(1) , (2016) 1-9.

[8] M. Nadir, A. Rahmoune, Modified method for solving linear Volterra integral equations of second kind. IJMM (2007) 141-146

[9] M. Nadir, Cours sur les équations intégrales. University of Msila, Algeria, 2008.
[10] M. Sezer, A. Akyuz-Dascioglu, Taylor polynomial solutions of general linear differential-difference equations with variable coefficients , Appl.Math.Comp.,174(2), (2006),1526-1538.
[11] M. Sezer, A. Akyuz-Dascioglu, A Taylor method for numerical solution of generalized pantograph equations with linear functional argument, J.Comp.Appl. Math.,200(2007),217-225.
[12] S. Yüzbasi, A Collocation Approach To Solve The Riccati-Type Differential Equation Systems, in Int. J. Comput. Math vol.64, pp.589-603, 2012
[13] S. Yüzbasi, A Laguerre Approach for the Solutions of Singular Perturbated Differential Equations, in Int. J. Comput. Methods 14, 1750034 (2017)

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