Weak Insertion of a Contra-Baire-1 (Baire-.5) Function

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Abstract

A sufficient condition in terms of lower cut sets are given for the weak insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that Fσ−kernel of sets are Fσ−sets.

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J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 19].

Results of Katˇetov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition

1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [16]. He investigated the sets that can be represented as union of closed sets and called them V −sets. Complements of V −sets, i.e., sets that are intersection of open sets are called Λ−sets [16].

Recall that a real-valued function f defined on a topological space X is called A−continuous [20] if the preimage of every open subset of R belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A−continuity. However, for unknown concepts the reader may refer to [4, 10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity. for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that Fσ−kernel of sets are Fσ−sets.

A real-valued function f defined on a topological space X is called contraBaire-1 (Baire-.5) if the preimage of every open subset of R is a Gδ−set in X [21].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [15].

A property P defined relative to a real-valued function on a topological space is a B − .5−property provided that any constant function has property P and provided that the sum of a function with property P and any Baire.5 function also has property P . If P1 and P2 are B − .5−properties, the following terminology is used: A space X has the weak B − .5−insertion property for (P1,P2) iff for any functions g and f on X such that $g \leq f$, g has property P1 and f has property P2, then there exists a Baire-.5 function h such that $g \leq h \leq f$.

In this paper, for a topological space that Fσ−kernel of sets are Fσ−sets, is given a sufficient condition for the weak B − .5−insertion property. Also several insertion theorems are obtained as corollaries of these results.

2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets $A^\wedge$ and $A^\vee$ as follows:

$A^\wedge = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^\vee = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.

In [6, 17, 18], $A^\wedge$ is called the kernel of A.

We define the subsets Gδ(A) and Fσ(A) as follows:

Gδ(A)= ∪{O : O ⊆ A, OisGδ − set} and

Fσ(A)= ∩{F : F ⊇ A, F isFσ − set}.

Fσ(A) is called the Fσ − kernel of A.

The following first two definitions are modifications of conditions considered in [13, 14].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\bar{\rho}y$ if and only if yρv implies xρv and uρx implies uρy for any u and v in S.

Definition 2.3. A binary relation ρ in the power set P (X) of a topological space X is called a strong binary relation in P (X) in case ρ satisfies each of the following conditions:

1) If $A_i \rho B_j$ for any $i \in\{1,...,m\}$ and for any $j \in\{1,...,n\}$, then there exists a set C in P (X) such that $A_i \rho C$ and $C \rho B_j$ for any $i \in\{1,...,m\}$and any $j \in\{1,...,n\}$.

2) If $A \subseteq B$, then $A \bar{\rho} B$.

3) If AρB, then Fσ(A) ⊆ B and A ⊆ Gδ(B).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < l\} \subseteq A(f, l) \subseteq \{x \in X : f(x) \leq l\}$ for a real number l, then A(f, l) is a lower indefinite cut set in the domain of f at the level l.

We now give the following main results:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, that $F\sigma-$kernel of sets in X are $F\sigma-$ sets , with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t1 < t2$ then A(f, t1) ρA(g, t2), then there exists a Baire-.5 function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets A(f, t) and A(g, t) in the domain of f and g at the level t for each rational number t such that if $t1 < t2$ then A(f, t1) ρA(g, t2).

Define functions F and G mapping the rational numbers Qinto the power set of X by F (t)= A(f, t) and G(t)= A(g, t). If t1 and t2 are any elements of Q with $t1 < t2$, then F (t1) $\bar{\rho}$ F (t2),G(t1) ρG(t2), and F (t1) ρG(t2). By Lemmas 1 and 2 of [14] it follows that there exists a function H mapping Q into the power set of X such that if t1 and t2 are any rational numbers with

$t1 < t2$, then F (t1) ρH(t2),H(t1) ρH(t2) and H(t1) ρG(t2).

For any x in X, let $h(x) = \inf\{t \in Q : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in H(t) then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in H(t), then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t1 and t2 with $t1 < t2$, we have $h^{-1}(t1,t2) = G\delta(H(t2)) \setminus F\sigma(H(t1))$. Hence $h^{-1}(t1,t2)$ is a $G\delta-$set in X, i.e., h is a Baire-.5 function on X. •

The above proof used the technique of theorem 1 of [13].

3 Applications

Definition 3.1. A real-valued function f defined on a space X is called contra-upper semi-Baire-.5 (resp. contra-lower semi-Baire-.5) if $f^{-1}(-\infty,t)$ (resp. $f^{-1}(t, +\infty)$) is a $G\delta-$set for any real number t.

The abbreviations usc, lsc, cusB.5 and clsB.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 1. [13, 14]. A space X has the weak c−insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of theorem 2.1, we suppose that X is a topological space that $F\sigma-$kernel of sets are $F\sigma-$sets.

Corollary 3.1. For each pair of disjoint $F\sigma-$sets F1,F2, there are two $G\delta-$sets G1 and G2 such that $F1 \subseteq G1$, $F2 \subseteq G2$ and $G1 \cap G2 = \emptyset$ if and only if X has the weak $B - .5-$insertion property for $(cusB - .5, clsB - .5)$.

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB1,g is usB1, and $g \leq f$.If a binary relation ρ is defined by AρB in case $F\sigma(A) \subseteq G\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with $t1 < t2$, then

$$A(f, t1) \subseteq \{x \in X : f(x) \leq t1\} \subseteq \{x \in X : g(x) < t2\} \subseteq A(g, t2);$$

since $\{x \in X : f(x) \leq t1\}$ is a $F\sigma-$set and since $\{x \in X : g(x) < t2\}$ is a $G\delta-$set, it follows that $F\sigma(A(f, t1)) \subseteq G\delta(A(g, t2))$. Hence $t1 < t2$ implies that A(f, t1) ρA(g, t2). The proof follows from Theorem 2. 1.

On the other hand, let F1 and F2 are disjoint $F\sigma-$sets. Set $f = \chi F1\ c$ and

$g = \chi F_2$, then f is $clsB - .5$,g is $cusB - .5$, and $g \leq f$. Thus there exists Baire-.5 function h such that $g \leq h \leq f$. Set

$G1 = \{x \in X : h(x) < 1/2\}$ and

$G2 = \{x \in X : h(x) > 1/2\}$, then G1 and G2 are disjoint Gδ−sets such that F1 ⊆ G1 and F2 ⊆ G2. •

Remark 2. [22]. A space X has the weak c−insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. For every G of Gδ−set, Fσ(G) is a Gδ−set if and only if X has the weak B − .5−insertion property for (clsB − .5, cusB − .5).

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For every G of Gδ−set we have Fσ(G) is a Gδ−set.

(ii) For each pair of disjoint Gδ−sets as G1 and G2 we have Fσ(G1) ∩ Fσ(G2)= ∅.

The proof of lemma 3.1 is a direct consequence of the definition Fσ−kernel sets. We now give the proof of corollary 3.2.

Proof. Let g and f be real-valued functions defined on the X, such that f is clsB − .5,g is cusB − .5, and f ≤ g.If a binary relation ρ is defined by AρB in case Fσ(A) ⊆ G ⊆ Fσ(G) ⊆ Gδ(B) for some Gδ−set g in X, then by hypothesis and lemma 3.1 ρ is a strong binary relation in the power set of X. If t1 and t2 are any elements of Q with t1 <t2, then

$$A(g, t1)= \{x \in X : g(x) < t1\} \subseteq \{x \in X : f(x) \leq t2\};$$

$$= A(f, t2);$$

since $\{x \in X : g(x) < t1\}$ is a Gδ−set and since $\{x \in X : f(x) \leq t2\}$ is a Fσ−set, by hypothesis it follows that A(g, t1) ρA(f, t2). The proof follows from Theorem 2.1.

On the other hand, Let G1 and G2 are disjoint Gδ−sets. Set f = χG2

and g = χG1 c , then f is clsB − .5,g is cusB − .5, and f ≤ g.

Thus there exists Baire-.5 function h such that f ≤ h ≤ g. Set $F1 = \{x \in X : h(x) \leq 1/3\}$ and $F2 = \{x \in X : h(x) \geq 2/3\}$ then F1 and F2 are disjoint Fσ−sets such that G1 ⊆ F1 and G2 ⊆ F2. Hence Fσ(F1)∩Fσ(F2)= ∅. •

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