ISSN 2347-1921

# Skew Injective Modules Relative to Torsion Theories

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## ABSTRACT

The purpose of this paper is to extend results about skew injective modules to a torsion theoretic setting. Given a hereditary torsion theory $\tau$, a module $M$ is called $\tau$-skew injective if all endomorphisms of $\tau$-dense submodules of $M$ can be extended to endomorphismsof $M$. A characterization of $\tau$-skew injectivity using split short exact sequences is given.

## Indexing terms/Keywords

Torsion theory;$\tau$-torsion module;$\tau$-dense submodule;skew injective module.

## Academic Discipline And Sub-Disciplines

Mathematics; Algebra.

## Mathematics Subject Classification

16D50.

Council for Innovative Research

Peer Review Research Publishing System

**Journal:**Journal of Advances in Mathematics

Vol 9, No 2

editor@cirjam.org

www.cirjam.com, www.cirworld.com

ISSN 2347-1921

## 1 Preliminaries

All modules considered will be right unital$R$-modules, where $R$is some associative ring with a nonzero identity.By $\tau=(\mathcal{T},\mathcal{F})$we denote a hereditary torsion theory on the category mod-$R$ of $R$-modules, where $\mathcal{T}$ (resp. $\mathcal{F}$) denotes the class of all $\tau$-torsion (resp. $\tau$-torsion free) $R$-modules.

A submodule$N$ of a module $M$ is said to be *$\tau$-dense* in $M$ (denoted $N\leq^{\tau d} M$) if $M/N$ is $\tau$-torsion, and $M$ is $\tau$-torsion if and only if all its elements are annihilated by $\tau$-dense right ideals of $R$. A submodule $N$ of $M$ is called *$\tau$-essential* in $M$ (denoted $N\leq^{\tau e} M$) if $N$ is both $\tau$-dense and essential in $M$. In this case, $M$ is called a *$\tau$-essential extension*of $N$. The intersection of any finite number of $\tau$-dense (resp. $\tau$-essential) submodules is again $\tau$-dense (resp. $\tau$-essential). If $N$ and $K$ are submodules of a module $M$ such that $N\leq^{\tau e} M$ then $N\cap K\leq^{\tau e} K$. Any submodule that contains a $\tau$-dense (resp. $\tau$-essential) submodule is itself $\tau$-dense (resp. $\tau$-essential). An $R$-module is called *$\tau$-injective* if it is injective with respect to every short exact sequence having a $\tau$-torsion cokernel. Every $R$-module $M$ admits a *$\tau$-injective envelope*$E=E_\tau(M)$, i.e. a $\tau$-injective $R$-module $E$ containing $M$ as a $\tau$-essential submodule. A module $M$ is called *$\tau$-quasi injective*if homomorphisms from $\tau$-dense submodules of $M$ into $M$ are extendable to endomorphisms of $M$. For preliminaries about torsion theories, we refer to [2].

Charalambides [3] introduced the concept of *$\tau$-essentially closed*submodules. A submodule$N$ of a module $M$ is called $\tau$-essentially closed in $M$ (denoted $N\leq^{\tau c} M$) if $N$ has no proper $\tau$-essential extensions in $M$.

A module $M$ is called *skew injective* [4] if whenever $N$ is a submodule of $M$, any $f$ in $\mathrm{End}(N)$ can be extended to $g\in\mathrm{End}(M)$. Note that in [5] skew injective modules are called semiinjective. In this paper, we generalize this concept to torsion theoretic setting.

## 2 $\tau$-Skew Injective Modules

**Definition.** A module $M$ is called *$\tau$-skew injective* if whenever $N$ is a $\tau$-dense submodule of $M$, any $f$ in $\mathrm{End}(N)$ can be extended to $g\in\mathrm{End}(M)$.

**Remarks.**

a) Every skew injective module is $\tau$-skew injective.
b) Every $\tau$-quasi injective module (and hence every $\tau$-injective) module is $\tau$-skew injective.
c) If $M$ is a $\tau$-torsion $\tau$-skew injective module, then it is skew injective.
d) If $\tau$ is the torsion theory in which every $R$-module is $\tau$-torsion, then a module is $\tau$-skew injective if and only if it is skew injective.

**Proposition 1:** A module $M$ is $\tau$-skew injective if and only if for every $\tau$-essential submodule $N$of $M$, any endomorphism of $N$ can be extended to an endomorphism of $M$.

*Proof.* Let $N$ be a $\tau$-dense submodule of $M$and $f\in\mathrm{End}(N)$. Let $N'$ be a relative complement of $N$in $M$. Then $N\oplus N'$ is a $\tau$-essential submodule of $M$. Moreover, $f$ can be extended to an $R$-endomorphism $g$ of $N\oplus N'$ by putting $g(N')=0$. By the given condition, there is an $R$-homomorphism $h$ of $M$ which extends $g$hence $f$. The other direction is trivial. □

Given a submodule$M$ ofa module $E$ and an endomorphism $f$of $E$,we call $f$an *$M$-$\tau$-essential* endomorphism if $f(N)\subseteq N$ for some $\tau$-essential submodule $N$ of $M$.

**Theorem 2:** If $E$ is the $\tau$-injective envelope of a module $M$, then the following statements are equivalent:

(a) $M$is$\tau$-skew injective and $f(M)\subseteq M$ for any endomorphism $f$ of $E$ having a $\tau$-essential kernel.
(b) $g(M)\subseteq M$for any $M$-$\tau$-essential endomorphism $g$ of $E$.

*Proof.* (a)$\Rightarrow$(b) Let $g$ be an $M$-$\tau$-essential endomorphism of $E$. Then there is a $\tau$-essential submodule $N$ of $M$ such that $g(N)\subseteq N$. By $\tau$-skew injectivityof $M$, there exists $h\in\mathrm{End}(M)$ that extends $g$. Again $\tau$-injectivity of $E$ gives existence of a $k$ in $\mathrm{End}(E)$ such that $k|_M=h$. So $(g-k)(N)=0$. Hence $N\subseteq\ker(g-k)$. So $\ker(g-k)\leq^{\tau e} E$. Then by hypothesis $(g-k)(M)\subseteq M$. Therefore, for any $x$ in $M$ we have $(g-k)(x)=m\in M$, hence $g(x)=m+k(x)\in M$, i.e. $g(M)\subseteq M$.

(b)$\Rightarrow$(a) Since any endomorphism of $E$ having a $\tau$-essential kernel is necessarily an $M$-$\tau$-essential homomorphism, we need only prove that $M$ is $\tau$-skew injective. By Proposition 1, let $N$ be a $\tau$-essential submodule of $M$and $f\in\mathrm{End}(N)$. By $\tau$-injectivity of $E$ we have a $g\in\mathrm{End}(E)$ such that $g(N)=f(N)\subseteq N$. Hence, $g$ is an $M$-$\tau$-essential homomorphism, so $g(M)\subseteq M$. Then, $g|_M\in\mathrm{End}(M)$ is an extension of $f$. □

The following Theorem generalizes Lemma 6 of [5].

**Theorem 3:** Let $M$ be a $\tau$-skew injective module, $E$ a $\tau$-essential extension of $M$ and $f$ an $M$-$\tau$-essential endomorphism of $E$. If for each $x\in M$ there exists a positive integer $n$ such that $f^{n+1}(x)=f^n(x)$, then $f(M)\subseteq M$.

*Proof.* Let $N$ be the sum of all $\tau$-dense submodules $N'$ of $M$ such that $f(N')\subseteq N'$. Therefore$f(N)\subseteq N$ andby hypothesis, $N$ is a $\tau$-essential submodule of $M$. We see that$f^n(N)\subseteq N$ for all $n\geq 1$, henceby$\tau$-skew injectivity of $M$, there exist endomorphisms$g_1,g_2,\ldots$ of $M$ such that $(f^n-g_n)(N)=0$ for all $n\geq 1$. So we have well-defined homomorphisms $h_n$ from $M/N$into $E$:$h_n(m+N)=(f^n-g_n)(m)$ for all $m\in M$, $n\geq 1$. Let $\bar{A}_n=h_n^{-1}(N\cap\operatorname{Im}h_n)$ for all $n$. Hence $\bar{A}_n$ are $\tau$-essential submodulesof $\bar{M}=M/N$, for $\bar{A}_n$ is the inverse image under the homomorphism $h_n$ of the essential submodule $N\cap\operatorname{Im}h_n$ of

$\operatorname{Im} h_n$, this gives essentiality of $\bar{A}_n$ in $\bar{M}$. Moreover,$\bar{M} = M/N$ is a $\tau$-torsion module. This means that $\bar{A}_n$ is $\tau$-dense in$\bar{M}$ for all $n$. If $\bar{M} = \bar{0}$, then $f(M) \subseteq M$ and everything is proved. Assume $b \in M \setminus N$ so that $\overline{N + bR}$ is a non-zero submodule of $\bar{M}$ and choose a natural number $n$ such that $f^{n+1}(b) = f^n(b)$. Put $\bar{A} = \bar{A}_1 \cap \cdots \cap \bar{A}_n$, hence $\bar{A}$ is a $\tau$-essential submodule of $M$ since $\bar{A}$ is the intersection of a finite number of $\tau$-essential submodules of $\bar{M}$. Now $\bar{A} \cap \overline{N + bR} \neq 0$. Therefore, there exists an element $r \in R$ such that $br \in (N + bR) \setminus N$and $br + N \in \bar{A}_1 \cap \cdots \cap \bar{A}_n$. It follows from the definition of the modules $\bar{A}_i$ that $h_i(br) \in M$for $i = 1, \cdots, n$. If $b_1 = br$, then $g_m(b_1) \in M$ for all $m$. From the definition of the homomorphisms$h_i$we see that $f^i(b_1) \in M$for $i = 1, \cdots, n$. But then $f^m(b_1) \in M$ for all $m$, since $f^{m+1}(b_1) = \left(f^{m+1}(b)\right)r = f^m(b_1)$. Now put $N_1 = N + \sum_{i=0}^{\infty} f^i(b_1 R)$. Hence $N_1$ is a $\tau$-densesubmodule of $M$ with$f(N_1) \subseteq N_1$and$N_1 \nsubseteq N$, in contradiction with the choice of the module $N$. □

Charalambides [3] defines a module $M$ to be $\tau$-*quasi continuous* if it is invariant under idempotentsof $\operatorname{End}(E_\tau(M))$. From the above Theorem,we see that if $f$ is an idempotent in $\operatorname{End}(E_\tau(M))$ and $M$ is $\tau$-skew injective, then $f^2(x) = f(x)$ for all $x \in M$, hence $f(M) \subseteq M$. So we have the following corollary.

**Corollary 1:** Every $\tau$-skew injective module is $\tau$-quasi continuous. □

In [3], a module $M$ is defined to be $\tau$-*CS* if every ($\tau$-essentially) closed $\tau$-dense submodule of $M$ is a direct summand. There, it is proved that $\tau$-quasi continuous modules are $\tau$-CS. Hence we have:

**Corollary 2:** Any $\tau$-skew injective module is $\tau$-CS. □

$\tau$-skew injectivity is preserved by taking direct summands:

**Proposition 4:** A direct summand of a $\tau$-skew injective module is $\tau$-skew injective.

*Proof.* Let $M$ be $\tau$-skew injective such that $M = N \oplus N'$. Let $K$ be a $\tau$-dense submodule of $N$. Then $N/K \cong (N \oplus N')/(K \oplus N')$ is $\tau$-torsion, which means that $K \oplus N'$ is $\tau$-dense in $M$. Any homomorphism $f: K \to K$ can be extended to a homomorphism $f': K \oplus N' \to K \oplus N'$ by putting $f'(k + n') = f(k)$ for all $k + n' \in K \oplus N'$. Now $\tau$-skew injectivity of $M$ gives a homomorphism$g \in \operatorname{End}(M)$ that extends $f'$. Hence $h = pgi_N$extends $f$, where $p$ is the projection map of $M$ onto $N$. □

We end this section with a characterization of $\tau$-skew injective modules using split short exact sequences:

**Theorem 5:** For a module $A$, the following statements are equivalent:

(1) $A$is$\tau$-skew injective.

(2) Any short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B$ splits whenever there exists $\beta \in \operatorname{Hom}(A, B)$ such that:
  (a) $\alpha(A) + \beta(A) = B$,
  (b) $\alpha(A) \cap \beta(A) \subseteq \alpha\left(\beta^{-1}(\alpha(A))\right)$ and
  (c) $\beta^{-1}(\alpha(A)) \leq^{\tau-d} A$.

*Proof.* (1)⇒(2) Let $N = \beta^{-1}(\alpha(A))$. By hypothesis, $\alpha(A) \cap \beta(A) = \beta\left(\beta^{-1}(\alpha(A))\right) = \beta(N) \subseteq N$. Hence by $\tau$-skew injectivityof $A$, the homomorphism $\beta: N \to A$ extends to a homomorphism $\gamma: A \to A$. By assumption, $B = \alpha(A) + \beta(A)$, i.e. for each $b \in B$ there exist $a, a'$ in $A$ such that $b = \alpha(a) + \beta(a')$. Define $\delta: B \to A$by $\delta(b) = \alpha(a) + \gamma(a')$. It is an easy matter to verify that $\delta$ is an $R$-homomorphism. Moreover, for $a \in A$, $\delta(\alpha(a)) = \delta(\alpha(a) + \beta(0)) = \alpha(a) + \gamma(0) = \alpha(a)$ so that $\alpha(A)$ is a direct summand of $B$.

(2) ⇒ (1) Let $N$ be a $\tau$-dense submodule of $A$ and $g \in \operatorname{End}(N)$. Form the pushout diagram:

$$\begin{array}{ccc} N & \xrightarrow{i} & A \\ g\downarrow & & \downarrow\beta \\ 0 \to A & \xrightarrow{\alpha} & B \end{array}$$

where $B = (A \oplus A)/W$, with $W = \{(n, -g(n)), n \in N\}$. If the second row splits then we get a homomorphism $\alpha': B \to A$ such that $\alpha'\alpha = 1_A$. Hence $\alpha'\beta i$ is an extension of $g$. So we need to show that conditions in (2) hold. But it is easy to see that conditions (a) and (b) hold. Moreover, $\beta^{-1}(\alpha(A)) = N$ which is $\tau$-dense in $A$. Hence the lower sequence splits by (2). □

## 3 Direct sum of $\tau$-skew injective modules

In Theorem 2 of the last section, we proved that a module $M$ is (a) $\tau$-skew injective and (b) $f(M) \subseteq M$ for any $f \in \operatorname{End}(E_\tau(M))$ having a $\tau$-essential kernel if and only if $g(M) \subseteq M$ for any $M$-$\tau$-essential endomorphism $g$of $E$. In this section, we seek conditions on the module $M$ and/or the ring $R$ so that condition (b) above is already satisfied.

Let us see first what happens if we take direct sum or summands of modules satisfying condition (b):

**Proposition 6.**A module $M$ satisfies condition (b) if and only if any direct summand of $M$ satisfies condition (b).

*Proof.* Let $M = M_1 \oplus M_2$ so that $E_\tau(M) = E_\tau(M_1) \oplus E_\tau(M_2)$. Let $f$ be an endomorphism of $E_\tau(M_1)$ having a $\tau$-essential kernel. Now $f$ can be easily extended to an endomorphism of $E_\tau(M_1) \oplus E_\tau(M_2)$ by $(x, y) \mapsto (f(x), 0)$ whose kernel is now equal to $\ker f \oplus E_\tau(M_2)$ which is clearly a $\tau$-essential submodule of $E_\tau(M_1) \oplus E_\tau(M_2)$. By assumption, the image of $M$under this map is contained in $M$. So $f(M_1) \subseteq M_1$. Conversely, Let $f$ be an endomorphism of $E_\tau(M)$ having a $\tau$-essential kernel.

Now for all $i$, $M_i \cap \ker f$ is $\tau$-essential in $M_i$. But $M_i \cap \ker f$ is the kernel of $p_i \circ (f|M_i)$, where $p_i$ is the projection map of $M$ onto $M_i$. Hence $p_i \circ (f|_{M_i})(M_i) \subseteq M_i$ for all $i$. So $f(M) \subseteq M$. □

Recall that a module $M$ is called *τ-nonsingular* [1] if $Z_\tau(M) = 0$where $Z_\tau(M) = \{m \in M | \operatorname{ann}_R(m) \leq^{\tau e} R\}$. The following proposition shows that if we assume $\tau$-nonsingularity of the module $M$, then we can remove condition (b) above from Theorem 2:

**Proposition 7.** Let $M$ be a $\tau$-nonsingular module. Then $M$ is $\tau$-skew injective if and only if $M$ is invariant over endomorphisms $g \in \operatorname{End}(\mathrm{E}_\tau(M))$ having $M$-$\tau$-essential kernels.

*Proof.* We will show that the only endomorphism of $M$ that has a $\tau$-essential kernel is the zero homomorphism. But this implies that $M$ is invariant under such homomorphisms and hence by Theorem 2, the result follows. To this end, let $f \in \operatorname{End}(M)$with $\ker f \leq^{\tau-e} M$, and let $g = 1_M - f \in \operatorname{End}(M)$. We will show that $g = 1_M$ and hence $f = 0$. For each $x \in M$, there exists a non-zero element $r \in R$ such that $0 \neq xr \in \ker f$, so $f(xr) = 0$ hence $g(x)r = g(xr) = xr - f(xr) = xr$, then $(g(x) - x)r = 0$ and $\operatorname{ann}_R(g(x) - x)$ is a non-zero ideal of $R$. But $(\ker f : x) \leq^{\tau e} R$ and hence $\operatorname{ann}_R(g(x) - x) \leq^{\tau e} R$, i.e. $g(x) - x \in Z_\tau(M) = 0$ and therefore$f(x) = 0$. □

In the next result, if the $\tau$-injective envelope of $M$ satisfies some ascending chain condition, then we can get rid of condition (b).

**Proposition 8.** Let $M$ be a $\tau$-skew injective module. If $E = \mathrm{E}_\tau(M)$ satisfies the ascending chain condition on $\tau$-essential submodules, then $f(M) \subseteq M$ for every $M$-$\tau$-essential $f \in \operatorname{End}(E)$.

*Proof.* Consider the ascending chain $M \cap \ker f \subseteq M \cap \ker f^2 \subseteq \cdots \subseteq M$. It is clear that $M \cap \ker f^k \leq^{\tau e} M$ for each $k \geq 1$, so by assumption there is a positive integer $n_\circ$ such that $M \cap \ker f^n = M \cap \ker f^{n+1}$ for all $n \geq n_\circ$. We claim that $\operatorname{Im}(f^n) \cap \ker fn \cap M=0$. To see this, let $x \in \operatorname{Im} fn \cap \ker fn \cap M$. So there is $y \in E$ such that $x=fny$and $0=fnx=fnfny=f2ny$. Hence $y \in \ker f^{2n} = \ker f^n$. So $x = f^n(y) = 0$. But $\ker f^n \leq^{\tau-e} E$ implies that $\operatorname{Im} f^n = 0$. Now for $x \in M$, $x - f(x) \in E$ and $0 = f^n\big(x - f(x)\big) = f^n(x) - f^{n+1}(x)$ or $f^n(x) = f^{n+1}(x)$ for $n \geq n_\circ$. So by Theorem 3, we have $f(M) \subseteq M$. □

Now, combining the above propositions with Theorem 2, we get:

**Corollary.** If $M$ is a $\tau$-nonsingular module or $\mathrm{E}_\tau(M)$ satisfies the ascending chain condition on $\tau$-essential submodules, then $M$ is $\tau$-skew injectiveif and only if it is invariant under all $M$-$\tau$-essential endomorphisms of $\mathrm{E}_\tau(M)$. □

Now, we put conditions on the ring $R$ to help us remove condition (b). For this we give a concept that generalizes both noetherian and weakly noetherian modules in [5].

**Definition.** A module $M$ is said to be *τ-weakly noetherian* if for every ascending chain $L_1 \subseteq L_2 \subseteq \cdots$ of submodules of $M$ with $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for all $i$, there is a positive integer $k$ such that $L_{n+1} = L_n$ for all $n \geq k$. A ring $R$ is called $\tau$-weakly noetherian if it is $\tau$-weakly noetherian as an $R$-module.

**Remarks.**

(1) Every module with ascending chain condition on $\tau$-essential submodules is $\tau$-weakly noetherian.
(2) If $M$ is $\tau$-weakly noetherian then so is any homomorohic image of $M$.
(3) Every cyclic module over a $\tau$-weakly noetherian ring is $\tau$-weakly noetherian.

*Proof.*(1) Let $L_1 \subseteq L_2 \subseteq \cdots$ be an ascending chain of submodules of a $\tau$-weakly noetherian module $M$ with $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for all $i$. For each $i$, under the natural map $M \to M/L_i$, we have $L_{i+1}$ is the preimage of $L_{i+1}/L_i$. So it must be essential in in $M$. Moreover, $M/L_{i+1} \cong (M/L_i)/(L_{i+1}/L_i)$ is $\tau$-torsion, hence $L_{i+1} \leq^{\tau e} M$. Thus the ascending chain $L_2 \subseteq L_3 \subseteq \cdots$ (and hence the ascending chain $L_1 \subseteq L_2 \subseteq \cdots$) terminates.

(2) Let $N$ be a submodule of $M$. We want to show that $M/N$ is $\tau$-weakly noetherian. Let $L_1/N \subseteq L_2/N \subseteq \cdots$ be an ascending chain of submodules of $M/N$ such that $(L_{i+1}/N)/(L_i/N) \leq^{\tau e} (M/N)/(L_i/N)$ for each $i$. Hence $L_{i+1}/L_i \leq^{\tau e} M/L_i$ for each $i$. Now for every $i$, $(M/L_i)/(L_{i+1}/L_i) \cong \frac{(M/N)/(L_i/N)}{(L_{i+1}/N)/(L_i/N)}$, so that $L_{i+1}/L_i \leq^{\tau e} M/L_i$. So by assumption there is a positive integer $k$ such that $L_{n+1} = L_n$ for all $n \geq k$or $L_{n+1}/N = L_n/N$.

(3) Let $M$ be a cyclic module over a $\tau$-weakly noetherian ring $R$. This means that $M \cong R/\operatorname{ann}_R(m)$ for some $m \in M$. By (2) it follows that $M$ is $\tau$-weakly noetherian. □

**Theorem 9.** Let $M$ be a module over a $\tau$-weakly noetherian ring, then for each endomorphism $f$ of $\operatorname{End}(\mathrm{E}_\tau(M))$ that has a $\tau$-essential kernel, there is a positive integer $n$ such that $f^n(x) = 0$ for every $x \in \mathrm{E}_\tau(M)$.

*Proof.* Put $E = \mathrm{E}_\tau(M)$ and let $K_0 = 0$, $K_1 = \ker f$, … , $K_{n+1} = f^{-1}\big(K_n \cap f(E)\big)$. Hence $K_1 \subseteq K_2 \subseteq \cdots$ is an ascending chain of submodules of $E$. Now $\ker f = K_1 \leq^{\tau d} E$. This implies that $K_n \leq^{\tau d} E$ for each $n$ and hence $E/K_{n+1} \cong (E/K_n)/(K_{n+1}/K_n)$ which is $\tau$-torsion, gives that $K_{n+1}/K_n \leq^{\tau d} E/K_n$. So $K_{n+1}/K_n \leq^{\tau e} E/K_n$ for each $n$ since $K_{n+1}/K_n \leq^{e} E/K_n$ for each $n$. For each $x \in E$, let $A = xR$ which by remark (3) is $\tau$-weakly noetherian. Put $A_0 = 0$, $A_1 = A \cap \ker f$, … , $A_n = A \cap K_n$, … which gives an ascending chain $A_0 \subseteq A_1 \subseteq \cdots$ of submodules of $A$. Since each $K_n$ is $\tau$-essentialin $E$, $A_n = A \cap K_n$ is $\tau$-essential in $A$ [3], and hence $A_{n+1}/A_n \leq^{\tau d} A/A_n$ for all $n$. But $A$ is $\tau$-weakly noetherian. So there is a positive integer $k$ such that $A_{n+1} = A_n$for $n \geq k$. But $A_{n+1}/A_n$ is an essential submodule of $E/A_n$ for all $n \geq k$. This is equivalent to saying that $A_n = A$

for all $n \geq k$. Hence $A = A_n = A \cap K_n$. So $A \subseteq K_n$, but $x \in A$ which implies that $f(x) \in K_{n-1} = f^{-1}(K_{n-2} \cap f(E))$. Thus $f^2(x) \in K_{n-2}$ and so on, we have $f^n(x) = 0$. □

Now we can remove condition (b) provided $R$ is $\tau$-weakly noetherian:

**Theorem 10.** Let $M$ be a module over a $\tau$-weakly noetherian ring. Then $M$ is $\tau$-skew injective if and only if it is invariant under $M$-$\tau$-essential endomorophisms of $\mathrm{E}_\tau(M)$.

*Proof.* By Theorem 2 it is enough to show that $M$ is invariant under all endomorphisms of $E = \mathrm{E}_\tau(M)$ that have $\tau$-essential kernels. Let $f$ be such an endomorphism, thus by Theorem 9, there is a positive integer $n$ such that $f^n(x) = 0$ for all $x \in E$. In particular, for every $m \in M$ we have $f^n(m - f(m)) = 0$. Thus $f^{n+1}(m) = f^n(m)$. Using Theorem 3, we have $f(M) \subseteq M$. □

So far, examples of modules satisfying condition (b) are:

1. $\tau$-nonsingular modules,
2. Modules whose $\tau$-injective envelopes satisfy the ascending chain condition on $\tau$-essential submodules, and
3. Modules over $\tau$-weakly noetherian rings.

Now we are ready to study direct sums of $\tau$-skew injective modules. Here we give necessary and sufficient conditions for a direct sum of $\tau$-skew injective modules to be $\tau$-skew injective.

**Theorem 11.** Let $M = M_1 \oplus \cdots \oplus M_n$ be an $R$-module satisfying condition (b). Then $M$ is $\tau$-skew injectiveif and only if $K_{ij} M_i \subseteq M_j$ for each $i, j = 1,2, \ldots, n$,

where $K_{ij} = \{f \in \mathrm{Hom}_R(\mathrm{E}_\tau(M_i), \mathrm{E}_\tau(M_j)) \mid f(N_i) \subseteq N_j$ for some $\tau$-essential submodules $N_i$ of $M_i$ and $N_j$ of $M_j\}$.

*Proof.* Put $E = \mathrm{E}_\tau(M)$and $E_i = \mathrm{E}_\tau(M_i)$, $i = 1,2, \ldots, n$. Then $E = E_1 \oplus \cdots \oplus E_n$. Suppose that $M$ is $\tau$-skew injective and let $f_{ij} \in K_{ij}$, i.e $f_{ij}: E_i \to E_j$ is a map with $f_{ij}(N_i) \subseteq N_j$ for some $\tau$-essential submodules $N_i$ of $M_i$ and $N_j$ of $M_j$. Consider the direct sum $N = \oplus N_k'$, where $N_k' = N_k$ if $k = i$ or $k = j$ and otherwise $N_k' = E_k$. Now $N$ is clearly $\tau$-essential in $E$ and hence $N \cap M$ is $\tau$-essential in $M$. But $f_{ij}$ can easily be extended to a map $f: E \to E_j$, $(x_1, \ldots, x_n) \mapsto f_{ij}(x_i)$. So that $f(N) \subseteq N_j$. Hence $f(N \cap M) \subseteq N_j = N_j \cap M \subseteq N \cap M$. This means that $f$ is an $M$-$\tau$-essential endomorphism of $E$. Hence $f(M) \subseteq M$ since $M$ is $\tau$-skew injective satisfying condition (b). So $f_{ij}(M_i) \subseteq M_j$. Conversely, let $f$ be an $M$-$\tau$-essential endomorphism of $E$, so that $f(N) \subseteq N$ for some $\tau$-essential submodule $N$ of $M$. Now for each $i$and $j$, we have $N \cap M_i \leq^{\tau-e} M_i$ and $p_j(N \cap M_i) \subseteq N \cap M_j$, where $p_j$ is the projection of $M$ onto $M_j$. So if we compose $p_j$ with the restriction of $f$ on $M_i$ we get a map $f_{ij} \in K_{ij}$, i.e. $f_{ij}(M_i) \subseteq M_j$ for all $i$ and $j$. Now it easy to see that $f(M) \subseteq M$. □

**Corollary.**If $M$ is a $\tau$-skew injective$R$-module satisfying condition (b) then $M^n$ is also $\tau$-skew injective. □

**Proposition 12.** Let $M = M_1 \oplus M_2$ be a $\tau$-skew injective$R$-module satisfying condition (b). Then $\mathrm{E}_\tau(M_1) \cong \mathrm{E}_\tau(M_2)$ if and only if $M_1 \cong M_2$.

*Proof.* Let $f: \mathrm{E}_\tau(M_1) \to \mathrm{E}_\tau(M_2)$ be an isomorphism. Then $f$ extends to an endomorphism $F$ of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$by $(x, y) \mapsto (0, f(x))$. If we prove that $F$ is $M$-$\tau$-essential then, by Theorem 11 we must have $F(M_1) \subseteq M_2$. Now since $M_2$ is essential in $\mathrm{E}_\tau(M_2)$ we have $f^{-1}(M_2)$ is essential in $\mathrm{E}_\tau(M_1)$, and since $M_1$ is $\tau$-essential in $\mathrm{E}_\tau(M_1)$ we must have $M_1 \cap f^{-1}(M_2)$ is $\tau$-essential in $M_1$ and hence $(M_1 \cap f^{-1}(M_2)) \oplus M_2$ is $\tau$-essential in $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$. Now $F\left((M_1 \cap f^{-1}(M_2)) \oplus M_2\right) = f(M_1 \cap f^{-1}(M_2)) \subseteq f(M_1) \cap M_2 \subseteq M_2 \subseteq (M_1 \cap f^{-1}(M_2)) \oplus M_2$. So $F$ is $M$-$\tau$-essential and $F(M_1) \subseteq M_2$. Similarly we get an $M$-$\tau$-essential endomorphism $G$ of $\mathrm{E}_\tau(M_1) \oplus \mathrm{E}_\tau(M_2)$ so that $G(M_2) \subseteq M_1$. Now for every $m_1 \in M_1$ we have $G \circ F(m_1) = G(F(m_1)) = G(f(m_1)) = m_1$. So $G \circ F = 1_{M_1}$. And similarly we have $F \circ G = 1_{M_2}$. The other direction is obvious. □

**Corollary.** Let $M$ be a $\tau$-skew injective module satisfying condition (b) and $E = \mathrm{E}_\tau(M)$. Then $M \oplus E$ is $\tau$-skew injective if and only if $M = E$.

*Proof.*It is obvious that $E$ satisfies condition (b), and by Proposition 6 so does $M \oplus E$. So we can apply Theorem 11 on $M \oplus E$. Clearly$1_{E \oplus E}$ is an $M \oplus M$-$\tau$-essential endomorphism of $E \oplus E$. So if $M \oplus E$ is $\tau$-skew injective then $1_E(E) \subseteq M$ by Theorem 11. But this means that $M = E$. The other direction is trivial. □

**Proposition 13.** The following statements are equivalent for any ring $R$:

(1) The direct sum of any two $\tau$-skew injective $R$-modules satisfying condition (b) is $\tau$-skew injective.
(2) Every $\tau$-skew injective$R$-module satisfying condition (b) is $\tau$-injective.

*Proof.*(1) ⇒ (2) Let $M$ be $\tau$-skew injective satisfying condition (b). Then $M \oplus E$ is $\tau$-skew injective by 1. So by the above corollary, $M = E$. (2) ⇒ (1) is trivial. □

**Corollary.**The following statements are equivalent for a $\tau$-weakly noetherian ring:

(1) The direct sum of any two $\tau$-skew injective $R$-modules is $\tau$-skew injective.
(2) Every $\tau$-skew injective $R$-module is $\tau$-injective. □

## ACKNOWLEDGMENTS

The authors would like to express their gratitude to Professor Edgar Enochs who has read the manuscript and provided them with his valuable remarks.

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