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A PRIORI ESTIMATE FOR THE SOLUTION OF SYLVESTER EQUATION

Abdelouahab Mansour,
Camille Jordan Institute, Claude Bernard University-Lyon1, France.
amansour@math.univ-lyon1.fr
L. Hariz Bekkar
Mathematics departement, Eloued University, Algeria.
harizlnd@yahoo.fr
H. Gaaya
Camille Jordan Institute, Claude Bernard University-Lyon1, France.
gaaya@math.univ-lyon1.fr

ABSTRACT

For A, B and Y operators in $B(H)$ it's well known the importance of sylvester equation $AX - XB = Y$ in control theory and its applications. In this paper -using integral calculus- we were able to give a priori estimate of the solution of famous sylvester equation when A and B are selfadjoint operators, some other results are also given.

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1 INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space, and $B(\mathrm{H})$ denote the algebra of all bounded linear operators on H. In general we define the generalized derivation on $B(\mathrm{H})$ by $\delta_{A,B}X = AX - XB$.

It's well known the importance of sylvester equation

$$\delta_{A,B}X = AX - XB = Y \qquad (1)$$

in control theory and its applications in quantum mechanics. The study of these operator equations generated many work, and several problems with these equations are still unanswered. The mathematical key is the elementary operator $\delta_{A,B}$, that is find an operator $X \in B(H)$ satisfying the operator equation. Solving operator equations of this kind returns to some tens of years, but giving the best priori estimate of the solution with simple integral calculation, easily manipulated by hand in our knowledge is not exist. In our present work we used the selfadjoiness of operators A, B (their spectra are real) and some technics of integral calculation. Also, our work has its roots in physics, the Heisenberg uncertainly principle may be mathematically formulated as saying that there exists a pair A, X of linear transformations and a non-zero scalar λ for which $AX - XA = \lambda I$. In $[7]$ W.E.Roth has shown for finite matrices A and B over a field that $AX - XB = C$ and $AX - YB = C$ are solvable if and only if the matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} and \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ are similar.

In $[6]$ Rosenblum showed that the result remains true when A and B are bounded selfadjoint operators in $B(H)$. In general if A is an invertible operator in $B(H)$ and $A^{-1}Y$ is contraction we can easly see that $\|X\| \le \frac{1}{|\|A^{-1}B\| - 1|}$, in this paper we give the best priori estimate of the solution of operator equation (1) under some suitable conditions, and some other results are also given.

2 Preliminaries

Theorem 2.1 For A and B operators in $B(H)$ such that the spectrum of A, B given by $\sigma(B) = \{z, |z| < \rho\}$, $\sigma(A) = \{z, |z| > \rho\}$, where ρ is real number, the solution of sylvester equation (1) is given by $X = \sum_{n=0}^{+\infty} A^{-n-1}YB^{n}$.

Lemma 2.2 For A and B operators in $B(H)$ such that $\sigma(B) = \{z, |z| < 0\}$, $\sigma(A) = \{z, |z| > 0\}$, the integral $\int_0^{+\infty} e^{-tA}Ye^{tB}dt$ is convergent where $Y \in B(H)$.

Theorem 2.3 (BHATIA[2]) For A and B operators in $B(H)$ such that $\sigma(B) = \{z, |z| < 0\}$, $\sigma(A) = \{z, |z| > 0\}$, the solution of the sylvester equation (1) is given by $X = \int_0^{+\infty} e^{-tA}Ye^{tB}dt$.

3 Main results

Proposition 3.1 *Let be A a selfadjoint operator in $B(H)$ such that $\sigma(A) \subset [\lambda_m, \lambda_M]$, where $\lambda_m = \inf_{\lambda \in \sigma(A)}(\lambda)$, $\lambda_M = \sup_{\lambda \in \sigma(A)}(\lambda)$, for any z in the circle $C(\frac{\lambda_m + \lambda_M}{2}, r)$ with the radius $r > \frac{\lambda_M - \lambda_m}{2}$, and $\phi = \frac{\lambda_M - \lambda_m}{2r}$ we have the following estimation*

$$\int_{-\pi}^{\pi} (A - zI)^{-1} d\theta \le \frac{2}{r}[ln\frac{1+\phi}{1-\phi} + 2\frac{Arccos\phi}{1-\phi}]$$

Lemma 3.2 Let be A a normal operator in $B(H)$, then $(A - zI)^{-1}$ is normal for any complex number z and we have for any $z \in \rho(A)$ the resolvent of A, $\|(A - zI)^{-1}\| = \frac{1}{d(z, \sigma(A))}$, where

$d(z,\sigma(A)) = \inf_{\lambda\in\sigma(A)} (|z-\lambda|)$.

Proof. It's clear that if A is normal, then $A-zI$ is normal and consequently $(A-zI)^{-1}$ is normal. from this we can write $\|(A-zI)^{-1}\| = \rho((A-zI)^{-1})$ the spectrum radius.

Then $\|(A-zI)^{-1}\| = \sup_{\lambda\in\sigma(A)} (|z-\lambda|)$ that's $\|(A-zI)^{-1}\| = \dfrac{1}{d(z,\sigma(A))}$.

Proof (of proposition (3.1)). Let be A a selfadjoint operator in $B(H)$, by lemma (3.2) we have $\|(A-zI)^{-1}\| = \dfrac{1}{d(z,\sigma(A))}$, for any $z\in\rho(A)$.

If C is the circle with center $\dfrac{\lambda_m+\lambda_M}{2}$ and radius $r > \dfrac{\lambda_M-\lambda_m}{2}$, we consider the parametrization $z = \dfrac{\lambda_m+\lambda_M}{2} + re^{i\theta}$ and $\cos\psi = \dfrac{\lambda_M-\lambda_m}{2r} = \phi$.

We will estimate $d(z,\sigma(A))$ the distance between the point z of the circle C and $\sigma(A)$ the spectrum of A, we have three different cases:

If $|\theta|<\psi$: $d(z,\sigma(A)) = |z-\lambda_M|$.

If $\pi-\psi<|\theta|<\pi$: $d(z,\sigma(A)) = |z-\lambda_m|$.

If $\psi<|\theta|<\pi-\psi$: $d(z,\sigma(A)) \geq r|\sin\theta|$. Then

$$\int_{-\pi}^{\pi}(A-zI)^{-1}d\theta = \int_{|\theta|<\psi}(A-zI)^{-1}d\theta + \int_{\pi-\psi<|\theta|<\pi}(A-zI)^{-1}d\theta + \int_{\psi<|\theta|<\pi-\psi}(A-zI)^{-1}d\theta$$

we will estimate the three integrals in the right side of equality

$$I = \int_{|\theta|<\psi}(A-zI)^{-1}d\theta = \int_{|\theta|<\psi}\frac{d\theta}{|z-\lambda_M|} = \int_{|\theta|<\psi}\frac{d\theta}{\left|\frac{\lambda_m+\lambda_M}{2}+re^{i\theta}-\lambda_M\right|} = \int_{|\theta|<\psi}\frac{d\theta}{\left|\frac{\lambda_m-\lambda_M}{2}+re^{i\theta}\right|}$$

since $\left|\dfrac{\lambda_m-\lambda_M}{2}+re^{i\theta}\right| \geq r-\dfrac{\lambda_M-\lambda_m}{2}$, we get

$$I = \int_{|\theta|<\psi}(A-zI)^{-1}d\theta \leq \frac{2\psi}{r-\frac{\lambda_M-\lambda_m}{2}}$$

$$J = \int_{\pi-\psi<|\theta|<\pi}\frac{d\theta}{\left|\frac{\lambda_m+\lambda_M}{2}+re^{i\theta}-\lambda_m\right|} = \int_{\pi-\psi<|\theta|<\pi}\frac{d\theta}{\left|\frac{\lambda_M-\lambda_m}{2}+re^{i\theta}\right|}$$

it is easy to prove that the integral J is equal to I as follows

$$J = \int_{\pi-\psi<\theta<\pi}\frac{d\theta}{\left|\frac{\lambda_M-\lambda_m}{2}+re^{i\theta}\right|} + \int_{-\pi<\theta<\psi-\pi}\frac{d\theta}{\left|\frac{\lambda_M-\lambda_m}{2}+re^{i\theta}\right|}.$$

With change of variables $\theta' = \theta-\pi$ we get

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$$\int_{\pi-\psi<\theta<\pi}\frac{d\theta}{|\frac{\lambda_M-\lambda_m}{2}+re^{i\theta}|}=\int_{-\psi<\theta'<0}\frac{d\theta'}{|\frac{\lambda_m-\lambda_M}{2}+re^{i\theta'}|}.$$

With change of variables $\theta'=\theta+\pi$ we get

$$\int_{-\pi<\theta<\psi-\pi}\frac{d\theta}{|\frac{\lambda_M-\lambda_m}{2}+re^{i\theta}|}=\int_{0<\theta'<\psi}\frac{d\theta'}{|\frac{\lambda_m-\lambda_M}{2}+re^{i\theta'}|},$$

by summation the last two equalities term by term we get $J=I$. Then

$$J=\int_{\pi-\psi\le|\theta|\le\pi}\|(A-zI)^{-1}\|d\theta\le\frac{2\psi}{r-\frac{\lambda_M-\lambda_m}{2}}.$$

On the interval $\psi\le|\theta|\le\pi-\psi$ we have $d(z,\sigma(A))\ge r|sin\theta|$

$$\int_{\psi\le|\theta|\le\pi-\psi}\|(A-zI)^{-1}\|d\theta=\int_{\psi\le\theta\le\pi-\psi}\frac{d\theta}{r.sin\theta}-\int_{\psi-\pi\le\theta\le-\psi}\frac{d\theta}{r.sin\theta},$$

then

$$\int_{\psi\le|\theta|\le\pi-\psi}\|(A-zI)^{-1}\|d\theta=2\int_{\psi\le\theta\le\pi-\psi}\frac{d\theta}{r.sin\theta}.$$

A simple calculation gives us

$$\int_{\psi\le\theta\le\pi-\psi}\frac{d\theta}{r.sin\theta}=-\frac{1}{r}ln(tan^2\frac{\psi}{2})$$

And since $tan^2\frac{\psi}{2}=\frac{1-cos\psi}{1+cos\psi}$, then $\int_{\psi\le\theta\le\pi-\psi}\frac{d\theta}{r.sin\theta}=-\frac{1}{r}ln\frac{1-\frac{\lambda_M-\lambda_m}{2r}}{1+\frac{\lambda_M-\lambda_m}{2r}}$ with $\phi=\frac{\lambda_M-\lambda_m}{2r}$

i.e. $\int_{\psi\le\theta\le\pi-\psi}\frac{d\theta}{r.sin\theta}=-\frac{1}{r}ln\frac{1-\phi}{1+\phi}$. Then

$$\int_{\psi\le|\theta|\le\pi-\psi}\|(A-zI)^{-1}\|d\theta=-\frac{2}{r}ln\frac{1-\phi}{1+\phi}$$

Therefore

$$\int_{-\pi\le\theta\le\pi}\|(A-zI)^{-1}\|d\theta\le2(\frac{2\psi}{r-\frac{\lambda_M-\lambda_m}{2}})-\frac{2}{r}ln\frac{1-\phi}{1+\phi}=\frac{2}{r}[\frac{2Arccos\phi}{1-\phi}+ln\frac{1+\phi}{1-\phi}]$$

Remark 3.3 Let be

$$\varphi(r)=\frac{2}{r}[\frac{2Arccos\phi}{1-\phi}+ln\frac{1+\phi}{1-\phi}]$$

where $\phi=\frac{\lambda_M-\lambda_m}{2r}$ and $r>\frac{\lambda_M-\lambda_m}{2}$. By proposition (3.1)

$$\int_{-\pi\le\theta\le\pi}\|(A-zI)^{-1}\|d\theta\le\varphi(r)$$

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We can examine the rate of this estimate as the values â€‹â€‹of r which will be best when r tends to infinity.

Theorem 3.4 If A and B are selfadjoint operators in $B(H)$ such that $\sigma(B)=\{z,|z|<b\}$, $\sigma(A)=\{z,|z|>a\}$ where a and b are real numbers and $a>0$ and $b<0$, then we have the estimations $\text{P}e^{-tA}\text{P}\le C_1e^{-ta}$ and $\text{P}e^{tB}\text{P}\le C_2e^{tb}$ where C_1 and C_2 are positif constants.

Proof. It well known that if f is an analytic function on a neighborhood of $\sigma(T)$ the spectrum of T, for any $\lambda\in\rho(T)$ the resolvent of T:

$$f(T)=\frac{1}{2\pi i}\int_\Gamma f(\lambda)R_\lambda(T)d\lambda$$

where $R_\lambda(T)$ is the mapping resolvent of T for λ, Γ is closed cantour of Cauchy. for $f(T)=e^{-tA}$ we have

$$e^{-tA}=\frac{1}{2\pi i}\int_\Gamma f(\lambda)R_\lambda(T)d\lambda$$

Theorem 3.5 *Let be A and B self adjoint operators in $B(H)$, $\sigma(A)$ and $\sigma(B)$ their spectrum respectively such that $\sigma(A)\subseteq[\lambda_m,\lambda_M]$ and $\sigma(B)\subseteq[\mu_m,\mu_M]$, where*

$$\lambda_m=\inf_{\lambda\in\sigma(A)}\lambda,\lambda_M=\sup_{\lambda\in\sigma(A)}\lambda,\mu_m=\inf_{\mu\in\sigma(B)}\mu,\mu_M=\sup_{\mu\in\sigma(B)}\mu,$$

with $\mu_M<\lambda_m$, then we can estimate the solution of Sylvester equation (1) by

$$\text{P}X\text{P}\le\frac{1}{2a.\pi^2}[\frac{2arccos\phi}{1-\phi}+ln\frac{1+\phi}{1-\phi}][\frac{2arccos\phi'}{1-\phi'}+ln\frac{1+\phi'}{1-\phi'}]$$

where $\phi=\dfrac{\lambda_M-\lambda_m}{2r}$ and $\phi'=\dfrac{\mu_M-\mu_m}{2r'}$

Proof. Since the conditions of Bhatia theorem are satisfying then the solution of the equation (1) is given by

$$X=\int_0^{+\infty}e^{-tA}Ye^{tB}dt.$$

Then $\text{P}X\text{P}\le\text{P}Y\text{P}\int_0^{+\infty}\text{P}e^{-tA}\text{PP}e^{tB}\text{P}dt.$ Using theorem (3.4) and proposition (3.1) we get

$$\text{P}X\text{P}\le\frac{\text{P}Y\text{P}}{4\pi^2}\frac{4}{r.r'}[\frac{2arccos\phi}{1-\phi}+ln\frac{1+\phi}{1-\phi}][\frac{2arccos\phi'}{1-\phi'}+ln\frac{1+\phi'}{1-\phi'}].K.L.\int_0^{+\infty}e^{-2ta}dt.$$

with $K=(\dfrac{\lambda_M+\lambda_m}{2}-a)$ and $L=(-a-\dfrac{\mu_M+\mu_m}{2})$.

Since the spectra of A and B are isolated, we can always put $r=m-a$ and $r'=-a-m$, then

$$\text{P}X\text{P}\le\frac{\text{P}Y\text{P}}{2a\pi^2}[\frac{2arccos\phi}{1-\phi}+ln\frac{1+\phi}{1-\phi}][\frac{2arccos\phi'}{1-\phi'}+ln\frac{1+\phi'}{1-\phi'}]$$

with $\phi=\dfrac{\lambda_M-\lambda_m}{2r}$ and $\phi'=\dfrac{\mu_M-\mu_m}{2r'}$ and $a>0$.

Corollary 3.6 If A and B self adjoint operators in $B(H)$ with spectrum reduced to one point i.e. $\sigma(A)=\lambda$, $\sigma(B)=\mu$ and $\mu<\lambda$ then the estimation in thorem (3.5) comes $\text{P}X\text{P}\le\dfrac{1}{2a}\text{P}Y\text{P}$

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