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Dual strongly Rickart modules

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ABSTRACT

In this paper we introduce and study the concept of dual strongly Rickart modules as a stronger than of dual Rickart modules [8] and a dual concept of strongly Rickart modules. A module M is said to be dual strongly Rickart if the image of each single element in $S = End_R(M)$ is generated by a left semicentral idempotent in S. If M is a dual strongly Rickart module, then every direct summand of M is a dual strongly Rickart. We give a counter example to show that direct sum of dual strongly Rickart module not necessary dual strongly Rickart. A ring R is dual strongly Rickart if and only if R is a strongly regular ring. The endomorphism ring of d-strongly Rickart module is strongly Rickart. Every d-strongly Rickart ring is strongly Rickart. Properties, results, characterizations are studied.

Indexing terms/Keywords

strongly Rickart rings, strongly Rickart modules, Rickart modules, dual Rickart modules; strongly regular rings.

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INTRODUCTION

Throughout this paper R is an associative ring with identity and all modules will be unitary right R-modules. A module M is Rickart if the right annihilator in M of any single element of S = $End_R(M)$ is generated by an idempotent of S[7]. Recently, the authors in [4] introduced the concept of strongly Rickart rings as stronger concepts of Rickart rings. A ring R is strongly Rickart if the right annihilator of each single element in R is generated by left semicentral idempotent of R. A module M is strongly Rickart if the right (resp. left) annihilator in M of any single element of S is generated by an left (resp. right) semicentral idempotent of S[5]. Following [8], a module M is dual Rickart if the image in M of any single element of S is generated by an idempotent of S. In this paper we introduce a dual concept of strongly Rickart modules as a strong concept of dual Rickart modules and a dual concept of strongly Rickart modules. A module M is dual strongly Rickart if the image in M of any single element of S is generated by a left semicentral idempotent of S.

Recall that a submodule N of a module M is stable (resp. fully invariant) if for each α : N→ M (resp. α : M→ M), α(N) ≤ N [1](resp. [10]). A module M is weak duo if every direct summand of M is fully invariant for each α ∈ S= $End_R(M)$ [ ]. A module M is said to be abelian if for each f ∈S, e^2 = e ∈ S, m ∈ M, fem = efm [10]. A module M is an abelian if and only if S = $End_R(M)$ is an abelian ring [10]. An idempotent e ∈S is called left (resp. right) semicentral if fe = efe (resp. ef = efe), for all f ∈S. An idempotent e ∈ S= $End_R(M)$ is called central if it commute with each g∈S. A monomorphism α : N→ M is a strongly splits if α(N) is a stable direct summand of M (i.e fully invariant direct summand)for every direct summand N of M [3, Definition (2.3.39)]. A module M is strongly direct injective, if for every direct summand N of M, every monomorphism α : N→ M is strongly splits [3, Definition (2.3.40)].

Notations. R is a ring and S is the endomorphism ring of a module M. For a ring S and α∈ S, the set $r_M(\alpha)$ = {m ∈M: αm = 0} (resp. $\ell_M(\alpha)$ ={m∈M : mα = 0}) is said to be the right (resp. left) annihilator in M of α in S. The sets $S_\ell(S)$, $S_r(S)$ and B(S) are the set of all left semicentral, right semicentral and central idempotent of S respectively. The samples$\leq$, $\trianglelefteq$, $\leq^{\oplus}$, $\trianglelefteq^{\oplus}$, $\leq^{e}$and ◘ refer to submodule, fully invariant submodule, direct summand, fully invariant direct summand, essential submodule and end the proof.

2. ON DUAL STRONGLY RICKART MODULES

Definition 2.1. A module M is said to be dual strongly Rickart (shortly, d-strongly Rickart) if the image of any single element of S = $End_R(M)$ is generated by a left semicentral idempotent element of S. A ring R is d-strongly Rickart if and only if R_R is d-strongly Rickart as right R-module.

Remarks and examples 2.2.

1. A module M is d-strongly Rickart if and only if Imα is a fully invariant direct summand of M.

Proof . Since for any e^2 = e ∈ S, eM $\trianglelefteq$ M if and only if e^2 = e ∈ $S_\ell(S)$ [6, Lemma 1.9], then the proof is obvious. ◘

2. A module M is d-strongly Rickart if and only if Imα is stable direct summand of M.

Proof. From the fact: every fully invariant direct summands of a module M is stable [3, Lemma 2.1.6]. ◘

3. Let R = $\begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$ and I = $\begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$ be an ideal in R. From [3, Remarks and examples 2.2.2(6)], $End_R(I)$ ≅ $\begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 2\mathbb{Z}_4 & \mathbb{Z}_4 \end{pmatrix}$. One can takes α ∈ $End_R(I)$ such that Imα = $\begin{pmatrix} 0 & \mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$, Imα is a right direct summand of I_R[3]. But Imα is not fully invariant in I. Let g ∈ $End_R(I)$ defined by g(β) = $\begin{pmatrix} a & c \\ 2b & d \end{pmatrix}$β for all β ∈ $End_R(I)$ and for some a, b, c, d ∈ $\mathbb{Z}_4$.

   So g(Imα) = { $\begin{pmatrix} 0 & ax \\ 0 & 2bx \end{pmatrix}$ | x ∈ $\mathbb{Z}_4$ } $\nleq$ Imα. Therefore, I is not d-strongly Rickart.

4. A module M is d-strongly Rickart if and only if the short exact sequence

$$0 \to Im\alpha \xrightarrow{i} M \xrightarrow{\alpha} \frac{M}{Im\alpha} \to 0$$

is a strongly split for any α ∈ S = $End_R(M)$.

Proof. Obvious, since Imα $\trianglelefteq^{\oplus}$M if and only if i(Imα) is stable direct summand of M. ◘

5. Every d-strongly Rickart module is strongly direct injective.

Proof. Let N $\leq^{\oplus}$M and α : N → M any monomorphism. There exist β = α ⊕ $0|_L$ since M = N ⊕ L for some L ≤ M. By hypothesis, M is a d-strongly Rickart module, then by (1), Imβ $\trianglelefteq^{\oplus}$M. But Imα = Imβ, so Imα $\trianglelefteq^{\oplus}$M. Therefore, M is a strongly direct injective. ◘

6. Every d-strongly Rickart module is d-Rickart. The converse is not true in general. In fact, the Z-module M = $Z_2 \oplus Z_2$ is d-Rickart [8, Example 4.6], which is not d-strongly Rickart. If one takes α : M → M defined by $\alpha(\bar{x}, \bar{y}) = (\bar{x}, \bar{0})$ for all $\bar{x} \in Z_2$, then Imα = $Z_2 \oplus \{\bar{0}\}$ is a direct summand of M. Now, let g ∈ S = $End_R(M)$ defined by $g(\bar{x}, \bar{y}) = (\bar{y}, \bar{x})$. So, g(α(M)) = g($Z_2 \oplus \{\bar{0}\}$) = $\{\bar{0}\} \oplus Z_2 \nleq Z_2 \oplus \{\bar{0}\}$. Hence Imα is not fully invariant submodule of M. Therefore, M = $Z_2 \oplus Z_2$ is not d-strongly Rickart Z-module.
7. Following (4), (5) and (6), a module M is d-strongly Rickart if and only if M is strongly direct injective and d-Rickart module.
8. A module M is a d-strongly Rickart if and only if M is a d-Rickart and weak duo (and hence abelian) module.

Proof. Let N $\leq^{\oplus}$ M and α ∈ S = End_R (M) such that Imα = N. By hypothesis, N = Imα $\trianglelefteq^{\oplus}$ M. Hence M is a weak duo module. Following (6), M is a d-Rickart. The converse is an obvious. ◘

9. A module M is d-strongly Rickart if and only if Imα is generated by a central idempotent element of S for each α ∈S = End_R(M).

Proof. An immediately consequence from(8).

10. Every d- strongly Rickart module has strictly SIP and strictly SSP.

Proposition 2.3. A module M is d-strongly Rickart if and only if $\sum_{\alpha \in I} \mathrm{Im}\,\alpha$ is generated by left semicentral idempotent element in S = End_R(M), for any finite generated ideal I of S.

Proof. ⇒) Let I be any nonzero left ideal of S with finite generators α_1, ..., α_n. Since M is a d-strongly Rickart module, then $\mathrm{Im}\alpha_i = e_iM$ for some left semicentral idempotent $e_i^2 = e_i \in S_\ell(S)$, i = 1, …,n. So $\sum_{i=1}^{n} \mathrm{Im}\alpha_i = \sum_{i=1}^{n} e_iM$. But M is satisfies the strictly SSP (Remarks and examples (2.2(10))). Now, each of e_iM is a direct summand of M and $e_i^2 = e_i \in S_\ell(S)$ for each i = 1,2,…,n so there is a $e^2 = e = \sum_{i=1}^{n} e_i$- $e_1 e_2 \dots e_n \in S_\ell(S)$ such that $\sum_{i=1}^{n} \mathrm{Im}\alpha_i$ = eM.

(2)⟹ (1) Let μ ∈ S and I = Sμ be a principle left ideal of S. By hypothesis, Imα = eM for $e^2 = e^2 \in S_\ell(S)$. Hence M is d-strongly Rickart module.

Proposition 2.4. For a module M and S = End_R(M), the following conditions hold:

1. If M is d-strongly Rickart with D_2-condition, then M is strongly Rickart.
2. If M is strongly Rickart with C_2-condition, then M is d- strongly Rickart.
3. If M is projective morphic, then M is a strongly Rickart if and only if M is a d-strongly Rickart.
4. If M is Rickart with SC_2-condition, then M is d-strongly Rickart.
5. A module M is d-strongly Rickart satisfies the D_2-condition if and only if M is strongly Rickart satisfies the C_2-condition.

Proof. 1. Let α ∈ S, then Imα $\trianglelefteq^{\oplus}$ M. But Imα ≅ $\frac{M}{\ker\alpha}$, so kerα $\leq^{\oplus}$ M. Since M is weak duo module, so kerα $\trianglelefteq^{\oplus}$M. Thus M is strongly Rickart module.

2. Suppose that M is strongly Rickart module and α ∈S. Then kerα $\trianglelefteq^{\oplus}$M. Hence M = kerα⊕K for some K ≤ M. Then Imα ≅ $\frac{M}{\ker\alpha}$ ≅ K $\leq^{\oplus}$M. By C_2-Condition, Imα $\leq^{\oplus}$ M. But M is a weak duo module(Remarks and examples(2.2(6)), hence Imα $\trianglelefteq^{\oplus}$M.

3. Suppose that M is a d-strongly Rickart module. Since, for each α ∈ S, $\frac{M}{\ker\alpha}$ ≅ Imα and by hypothesis, Imα $\trianglelefteq^{\oplus}$M. Then, by D_2-condition, kerα $\leq^{\oplus}$M. Then kerα $\trianglelefteq^{\oplus}$M (Remarks and examples(2.2(6)) and hence M is strongly Rickart. Conversely, suppose that M is a strongly Rickart module and α ∈ S then kerα $\trianglelefteq^{\oplus}$M. Since M is morphic ($\frac{M}{\mathrm{Im}\alpha}$ ≅ kerα) and satisfies the D_2, hence Imα $\leq^{\oplus}$M. Therefore, M is a d-strongly Rickart module (Remarks and examples(2.2(6)).

4. Since the SC_2-condition implies the C_2-condition and weak duo module, so from (2) the proof obvious.

5. Obvious. ◘

Examples 2.5.

1. The Z-module Z is projective (and hence satisfies the D_2-condition) module which is not morphic. From[5], Z is strongly Rickart. If $\alpha \in S = End_R(Z)$, such that $\alpha(n) = 2n$ for each $n \in Z$, so $Im\alpha \not\trianglelefteq^{\oplus} Z$. Hence Z is not d-strongly Rickart Z-module.

2. The Z-module Z_{p^∞} is morphic (also satisfies the C_2) which is not projective module. Since every endomorphism of Z_{p^∞} is an epimorphism so Z_{p^∞} is d-strongly Rickart (see Proposition 3.9). From[5], Z_{p^∞} is not strongly Rickart Z-module [5].

A submodule of a d-strongly Rickart module may be not d-strongly Rickart. In fact, $End_Z(Q) \cong Q$ and every endomorphism of Q is either isomorphism or zero. Hence Q is d-strongly Rickart while the submodule Z_Z is not, where there is $\alpha: Z \to 2Z$ have $Im\alpha = 2Z \not\trianglelefteq^{\oplus} Z$.

Proposition 2.6. If M is a d-strongly Rickart module, then every direct summand of M is a d-strongly Rickart.

Proof. Let $M = N \oplus L$ and $\alpha \in H = End_R(N)$. So α can be extended to $\beta \in S = End_R(M)$. i.e $\beta = \alpha \oplus 0|_L$. Since M is d-strongly Rickart module, then $Im\beta \trianglelefteq^{\oplus} M$. But $Im\beta = \alpha N$. So $Im\alpha \trianglelefteq^{\oplus} M$. Thus $Im\alpha \leq^{\oplus} N$ since $Im\alpha \leq N$. Now, let $g \in H$, consider the following sequence $M \xrightarrow{\rho} Im\alpha \xrightarrow{j_1} N \xrightarrow{g} N \xrightarrow{j_2} M$, where ρ is the projection epimorphism and j_1, j_2 are the injection monomorphism. So, $Im\alpha \geq j_2 g j_1 \rho(Im\alpha) = g(Im\alpha)$. Hence, $Im\alpha \trianglelefteq^{\oplus} N$. Therefore, N is a d-strongly Rickart module. ◘

Corollary 2.7. If R is a d-strongly Rickart ring, then, so is eR for each $e^2 = e \in R$ as an R-module.

Recall that a module M is an epi-retractable if every submodule of M is a homomorphic image of M [11].

Corollary 2.8. Let M be an epi-retractable module. If M is a d-strongly Rickart module, then so is every submodule of M.

Proof. Let N be any submodule of a d-strongly Rickart module M. By hypothesis, there exists an epimorphism $\alpha : M \to M$ such that $N = \alpha(M)$. So $N = Im\alpha \trianglelefteq^{\oplus} M$, since M is a d-strongly Rickart module. Therefore, N is a d-strongly Rickart module (Proposition 2.6).

Examples 2.9

1. The Z-module Q is not epi-retractable module[11]. From Example (2.5), the submodule Z is not d-strongly Rickart although the Z-module Q is d-strongly Rickart.

2. The Z-module Z_4 is not strongly Rickart while the submodule $2Z_4 \cong Z_2$ is d-strongly Rickart module.

In general, d-strongly Rickart property is not closed under direct sum, see Remarks and example (2.2(6)), although it closed under direct summand. The following proposition gives the necessary condition to a direct sum of d-strongly Rickart.

Proposition 2.10. Let $M = M_1 \oplus M_2$. Then M is d-strongly Rickart if and only if M_i is d-strongly Rickart module ($i \in \{1, 2\}$) and $M_i \trianglelefteq M$, $i \in \{1, 2\}$.

Proof. $\Leftarrow$) Suppose that M_i, $i \in \{1, 2\}$, is d-strongly Rickart modules and $S_i = End_R(M_i)$. Since $M_i \trianglelefteq M$, So $S = End_R(M) = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$. Let $\alpha \in S$, then $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$, where $\alpha_i \in S_i$. But M_i is d-strongly Rickart module, hence $Im\alpha_i = e_i M_i$ for $e_i^2 = e_i \in S_\ell(S_i)$. We claim that $Im\alpha = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$ and $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in S_\ell(S)$. Firstly, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}^2 = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ and $\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} x_1 e_1 & 0 \\ 0 & x_2 e_2 \end{pmatrix} = \begin{pmatrix} e_1 x_1 e_1 & 0 \\ 0 & e_2 x_2 e_2 \end{pmatrix}$ for all $\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \in S$. Thus $e = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \end{pmatrix}$ is a left semicentral idempotent of S. Now, let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in Im\alpha$ where $m_i \in Im\alpha_i = e_i M_i$. So $m_i = e_i m_i$. Hence $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in eM$. Clearly that, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \leq Im\alpha$. Thus $Im\alpha = eM$ for $e^2 = e \in S_\ell(S)$. Therefore M is d-strongly Rickart.

$\Rightarrow$) The proof is a consequence immediately from (Proposition (2.6)) and (Remarks and examples (2.2(8))) respectively. ◘

3. ENDOMORPHISM RING OF d-STRONGLY RICKART MODULES.

As well as d-Rickart [8], the following proposition proves that the endomorphism ring of d-strongly Rickart module is strongly Rickart ring. Following [4], every strongly Rickart ring is a left-right symmetric.

Proposition 3.1. The endomorphism ring of d-strongly Rickart module is strongly Rickart.

Proof. Let M be a d-strongly Rickart module and $\alpha \in S = End_R(M)$. Then $Im\alpha = eM$ for some $e^2 = e \in S_\ell(S)$. Hence $\ell_S(\alpha) = \ell_S(\alpha M) = \ell_S(eM) = S(1-e)$. Since $(1-e)^2 = 1-e \in S_r(S)$, therefore S is a strongly Rickart ring. ◘

It's well known that the endomorphism ring of Z_Z is an isomorphic to Z, this example shows that the converse of Proposition (3.1) is not true in general.

Corollary 3.2. Let M be a retractable module. Then every d-strongly Rickart module is a strongly Rickart.

Proof. Following (Proposition (3.1)), $S = End_R(M)$ is a strongly Rickart ring. By Proposition [5, Proposition (2.3)], M is strongly Rickart module.

Corollary 3.3. Every d-strongly Rickart ring is strongly Rickart ring.

Corollary 3.4. If R is a d-strongly Rickart ring, then for each $e^2 = e \in R$, eRe is strongly Rickart ring.

Proof. Since for each $e^2 = e \in R$, $eRe = End_R(eR)$[12, 7.8, p.60]. Then, by Proposition (2.7) and Proposition (3.1) the proof is complete. ◘

Recall that a module is called self-cogenerator if it cogenerates all its factor modules [12, Exercises17.15, p.147]. It's easy to prove that if a module M is d-strongly Rickart and $f \in S = End_R(M)$. Then Sf is projective left S-module.

Proposition 3.5. If a module M is self-cogenerater and S is a strongly Rickart ring, then M is d-strongly Rickart modules.

Proof. Suppose that $S = End_R(M)$ is strongly Rickart ring and $\alpha \in S$. Since M is self-cogenerater, by [12, 39.11, p.335], $Im\alpha \leq^{\oplus} M$. But S is an abelian ring, so $Im\alpha \trianglelefteq^{\oplus} M$. ◘

Recall that a homomorphic image of projective module over semihereditary ring is projective.

Proposition 3.6. Let M be a finitely generated projective R-module satisfies the SC_2-condition over a right (semi)hereditary ring R. Then M is a d-strongly Rickart module. Furthermore, $S = End_R(M)$ is a strongly regular ring.

Proof. Let M be a finitely generated projective over right semihereditary ring. Then for each $\alpha \in S$, $Im\alpha$ is a projective module. So $M = ker\alpha \oplus N$. But $\frac{M}{ker\alpha} \cong Im\alpha \cong N \leq^{\oplus} M$. Since M satisfies SC_2-condition, then $Im\alpha \trianglelefteq^{\oplus} M$ and so M is d-strongly Rickart. Thus, $Im\alpha$ and $ker\alpha$ are fully invariant direct summand in M. Hence, S is strongly regular ring. ◘

Corollary 3.7. Every right semihereditary right SC_2- ring R is d-strongly Rickart as right R-module and strongly regular ring.

Remark 3.8 . It's well known that the ring Z is a left semihereditary ring (since its hereditary) which is not satisfies the SC_2-condition, and then Z is not d-strongly Rickart ring.

Proposition 3.9. A module M is d-strongly Rickart and $S = End_R(M)$ is a domain if and only if every nonzero element of S is an epimorphism.

Proof. ⇐) A module M is d-strongly Rickart since $Im\alpha = M$ for each nonzero endomorphism α of M. Now, if $\beta\alpha = 0$ and $\alpha \neq 0$, then $\alpha(M) = M$. Hence $\beta\alpha(M) = \beta(M) = 0$. Thus $\beta = 0$. So S is domain.

⇒) Suppose that M is a d-strongly Rickart and $0 \neq \alpha \in S$, then $Im\alpha = eM$. Since S is a domain and $\alpha \neq 0$, then e =1 and hence $Im\alpha = M$. This implies that α is an epimorphism. ◘

Recall that a module M is an indecomposable strongly Rickart if and only if each nonzero element of S is a monomorphism[5]. The following result is the dual of this fact can be proved in the following proposition.

Proposition 3.10. A module M is indecomposable d-strongly Rickart if and only if each nonzero element of S is an epimorphism.

Proof .⇒) Let $\alpha \in S$. Since M is d-strongly Rickart, then $\alpha(M) = eM$ for some $e^2 = e \in S_\ell(S)$. But M is indecomposable module then either e = 1and then α is an epimorphism or e = 0 and so α is zero.

⇐) By hypothesis, if $(0 \neq) e^2 = e \in S$, then, e =1. Hence M is an indecomposable. In the same way, for any $\alpha \in S$, either $\alpha = 0$ and so $\alpha(M) = 0 \trianglelefteq^{\oplus} M$ or α is an epimorphism and hence $\alpha(M) = M \trianglelefteq^{\oplus} M$. Then M is d-strongly Rickart module. ◘

Proposition 3.11. Let M be a module and $S = End_R(M)$. Then the following conditions are equivalent

1. M is d-strongly Rickart

2. S is a strongly Rickart ring and $\alpha(M) = r_M(\ell_S(\alpha M))$, for all $\alpha \in S$.

Proof . (1⇒2) By Proposition (3.1), S is a strongly Rickart ring. Let $\alpha \in S$, then $Im\alpha = eM$ for some $e^2 = e \in S_\ell(S)$. Then $\ell_S(\alpha M) = S(1-e)$ and hence $r_M(\ell_S(\alpha(M)) = eM = \alpha(M)$.

(2⇒1) Suppose that S is strongly Rickart ring. Let $\alpha \in S$, then $\ell_S(\alpha) = Se$ for some $e^2=e \in S_r(S)$. Since $\alpha(M) = r_M(\ell_S(\alpha(M)))$, then $\alpha(M) = (1-e)M$ for $(1-e) \in S_\ell(S)$. Therefore M is d-strongly Rickart module. ◘

Corollary 3.12. For a module M and $S = End_R(M)$, the following conditions are equivalent:

1. M is a d-strongly Rickart module.

2. $\alpha(M) = r_M(\ell_S(\alpha(M))) \trianglelefteq^{\oplus} M$ for all $\alpha \in S$.

Proof. Obvious. ◘

Proposition 3.10. For a module M and $S = End_R(M)$, the following conditions are equivalent:

1. M is d-strongly Rickart module;
2. M is satisfies SC_2-condition and $Im\alpha$ is isomorphic to a direct summand of M for all $\alpha \in S$.

Proof. 1⇒2) Let N be a submodule of M such that $N \cong L \leq^{\oplus} M$. Hence $N = i\alpha\rho(M)$, where $\rho : M \rightarrow L$ be projection, $\alpha : L \rightarrow N$ be an isomorphism and $i: N \rightarrow M$ be injection. Since M is d-strongly Rickart module, so $N = Im(i\alpha\rho) \trianglelefteq^{\oplus} M$. the second condition is an obvious.

2⇒1) Let $\alpha \in S = End_R(M)$. Then by hypothesis, $Im\alpha$ is an isomorphic to a direct summand of M. Hence by SC_2-condition $Im\alpha \trianglelefteq^{\oplus} M$. ◘

Proposition 3.11. A module M is d-strongly Rickart satisfies the D_2-condition if and only if $S = End_R(M)$ is a strongly regular ring.

Proof. ⇐) Following [5], $Im\alpha \trianglelefteq^{\oplus} M$ for each $\alpha \in S$.

⇒) Let $\alpha \in S$. So $Im\alpha \trianglelefteq^{\oplus} M$, since M is d-strongly Rickart. Indeed, $\frac{M}{ker\alpha} \cong Im\alpha$ and by D_2-condition, $ker\alpha \leq^{\oplus} M$. But M is an abelian module, so $ker\alpha \trianglelefteq M$. Therefore, S is strongly regular ring. ◘

We can summarize the previous propositions in the following theorem

Theorem 3.12. For a module M and $S = End_R(M)$, the following conditions are equivalent:

1. S is a strongly regular ring;
2. M is d-strongly Rickart module satisfies the D_2-condition;
3. M is satisfies D_2-condition and SC_2-condition, and $Im\alpha$ is isomorphic to a direct summand of M for all $\alpha \in S$;
4. M is an abelian module and $S = End_R(M)$ is a von Neumann regular ring.

Proposition 3.13. A ring R is d-strongly Rickart if and only if R is a strongly regular ring.

Proof . ⇒) Let aR be a principle right ideal in R for $a \in R$. There is $\alpha : R \rightarrow aR$ such that $\alpha(r) = ar$ for each $r \in R$. It's clear that α is an endomorphism of R and $Im\alpha = aR$. By hypothesis, $aR = Im\alpha = eR$ for $e^2 = e \in B(R)$. Therefore, R is a strongly regular ring [12, 3.11, p.21].

⇐) Since $S = End_R(R) \cong R$, by (Theorem 3.12), the proof holds. ◘

A quotient $\frac{M}{N}$ of quasi-projective is quasi-projective module M, if a submodule N is fully invariant of M [12, 18.2(4), p.149].

Proposition 3.15. Let M be a quasi-projective module. If M is a d-strongly Rickart, then so is $\frac{M}{L}$ for each fully invariant submodule L of M.

Proof. Let $\beta \in End_R(\frac{M}{L})$ and $S = End_R(M)$. Since M is a quasi-projective module, so there is an epimorphism $\mu: S \rightarrow End_R(\frac{M}{L})$ defined by: $\mu(\alpha) = \beta$. It's easy to show that μ is a well define and ring homomorphism. So $End_R(\frac{M}{L}) \cong \frac{S}{ker\mu}$ Furthermore, M is d-strongly Rickart module satisfies the D_2-condition (since M is quasi-projective), hence S is a strongly regular ring (Proposition 3.11). So $\frac{S}{ker\mu}$ and hence $End_R(\frac{M}{L})$ is strongly regular ring. Therefore by Proposition (3.11), $\frac{M}{N}$ is a d-strongly Rickart module. ◘

Corollary 3.16. If a module M is d-strongly Rickart and quasi-projective then $\frac{M}{Im\alpha}$ is a d-strongly Rickart and quasi-projective module for all $\alpha \in S = End_R(M)$.

Recall that Soc M = ∩ {L≤ M｜ L$\leq^{e}$M} is fully invariant in M [12, 21.1, p. 174] and Rad M = ∩ {K ≤ M｜ K is a maximal submodule of M} is fully invariant in M [12, 21.5, P.176]

Corollary 3.17. If M is a quasi-projective and d-strongly Rickart, then $\frac{M}{Rad(M)}$ and $\frac{M}{Soc(M)}$ are d-strongly Rickart.

4. RELATIVE d-STRONGLY RICKART MODULES

Definition 4.1. Let M and N be modules. Then M is called N-d-strongly Rickart (relative d-strongly Rickart to N) if for all α: M → N, Imα $\trianglelefteq^{\oplus}$ N.

Remarks and examples 4.2.

1. A module M is d-strongly Rickart if and only if M is M-d-strongly Rickart.

2. For each semisimple abelian module N, M is N-d-strongly Rickart for each module M.

3. Let M and N are modules such that $Hom_R(M, N) = 0$. Then M is N-d-strongly Rickart. In fact, Let N = Z_p and M = Z_{p^∞}. It's well known that $Hom_Z(M, N) = 0$. Then M is N-d-strongly Rickart. Furthermore N is not M-d-strongly Rickart. In fact, if α ∈ $Hom_Z(N, M)$ since N is simple module, then either α is zero or monomorphism. If α is monomorphism then Imα is not direct summand in M, since M is an indecomposable.

Proposition 4.3. For a module M and N⊕L$\leq^{\oplus}$M if M satisfies the strictly SSP then N is L-d-strongly Rickart.

Proof. By the strictly SSP, every direct summand of M is a fully invariant. Then $Hom_R(N, L)$= $Hom_R(L, N) = 0$. ◘

Proposition 4.4. If M⊕M satisfies the strictly SSP then M is d-strongly Rickart module.

Proof. Since M⊕M satisfies the SSP, so M is a d-Rickart module[8, Corollary 2.17]. But M satisfies the strictly SSP, hence M is d-strongly Rickart module

Proposition 4.5. Let M and N be modules. Then M is N-d-strongly Rickart if and only if for any A $\leq^{\oplus}$M and B ≤ N, A is B-d-strongly Rickart.

Proof. Let A $\leq^{\oplus}$M, B ≤ N and α : A→ B be any homomorphism. Then α can be extended to β = iαρ: M → N where ρ: M → A is projection and i: B → N is injection. Since M is N-d-strongly Rickart, so Imβ = α(A) $\trianglelefteq^{\oplus}$N. But Imα ≤ B, so Imα $\leq^{\oplus}$ B. Now, let g ∈ $End_R(B)$, then g(α(A)) = igα(A) ≤ α(A), where i is the inclusion homomorphism from B → N. Therefore Imα $\trianglelefteq^{\oplus}$ B and hence A is B-d-strongly Rickart.

For the converse, put M = A and N = B. ◘

Corollary 4.6. For modules M, N, and a direct summand A of M, if M is N-d-strongly Rickart then A is N-d-strongly Rickart.

Corollary 4.7. A modules M is d-strongly Rickart if and only if for any submodule L of M and a direct summand A of M, A is L-d-strongly Rickart.

Corollary 4.8. Let N satisfies the strictly SSP and M = $\oplus_{i=1}^{n}M_i$, then $\oplus_{i=1}^{n}M_i$ is N-d-strongly Rickart if and only if M_i is N-d-strongly Rickart for each i = 1,…,n.

Proof. From Proposition (4.5), if M= $\oplus_{i=1}^{n}M_i$ is N-d-strongly Rickart, then M_i is N-d-strongly Rickart for each i = 1, ..., n. Conversely, let α ∈ $Hom_R(\oplus_{i=1}^{n}M_i, N)$. Then α =$(\alpha)_{i=1}^{n}$ where α_i ∈ $Hom_R(M_i, N)$ for each i = 1,…, n. Since each M_i is N-d-strongly Rickart, then $Im\alpha_i \trianglelefteq^{\oplus}$N. But N satisfies the strictly SSP and Imα = $\sum_{i=1}^{n} Im\alpha_i \trianglelefteq^{\oplus}$ N. Therefore $\oplus_{i=1}^{n}M_i$ is N-d-strongly Rickart module. ◘

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