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The global attractors and dimensions estimation for the Kirchhoff type wave equations with nonlinear strongly damped terms [1]

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This paper studies the long time behavior of the solution to the initial boundary value problems for a class of nonlinear strongly damped Kirchhoff type wave equations:

$$u_{tt}-\varepsilon_1\Delta u_t+\alpha|u_t|^{p-1}u_t+\beta|u|^{q-1}u-\phi(\|\nabla u\|^2)\Delta u=f(x).$$

Firstly, we prove the existence and uniqueness of the solution by priori estimate and the Galerkin method. Then we obtain to the existence of the global attractor. Finally, we consider that the estimation of the upper bounds of Hausdorff and fractal dimensions for the global attractor is obtained.

Key words: The nonlinear strong damping; Kirchhoff wave equation; The existence and uniqueness; Global attractor; Hausdorff dimension; Ftactal dimension

1 Introduction

It is well known that the dynamical systems that arise in physics, chemistry or biology, are often generated by a partial differential equation or a functional differential equation and thus the underlying state space is infinite-dimensional. The long time behavior of many dynamical systems generated by evolution equations can be described naturally in term of attractors of corresponding semigroups. The attractor is a basic concept in the study of the asymptotic behavior of solutions for the nonlinear evolution equations with various dissipation.

In recent years, many scholars have made useful researches for the existence of global attractors and their dimensions estimation about some the infinite dimension dynamic systems[1,2,3,4,5].

In this paper, we are concerned with the Kirchhoff type wave equations with nonlinear strongly damped terms referred to as follows:

$$u_{tt}-\varepsilon_1\Delta u_t+\alpha|u_t|^{p-1}u_t+\beta|u|^{q-1}u-\phi(\|\nabla u\|^2)\Delta u=f(x)\quad in\quad \Omega\times\mathbb{R}^+, \tag{1}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x),\quad x\in\Omega, \tag{2}$$

$$u(x,t)|_{\partial\Omega}=0,\Delta u(x,t)|_{\partial\Omega}=0,\quad x\in\Omega. \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\varepsilon_1,\alpha,\beta$ are positive constants, and the assumptions on $\phi(\|\Delta u\|^2)$ will be specified later.

In [6], G. Kirchhoff firstly proposed the so called Kirchhoff string model in the study nonlinear vibration of an elastic string:

$$\rho h u_{tt}+\delta u_t=p_0+\frac{Eh}{2L}\left(\int_0^L|u_x|^2dx\right)u_{xx}+f(x),0<x<L,t>0, \tag{4}$$

where $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t, E is the Young modules, h is the cross-section area, ρ is the mass density, L is the length, p_0 is the initial axial tension, δ is the resistance modules and f is the external force.

Yang Zhijian, Ding Pengyan and Liu Zhiming [7] studied the Global attractor for the Kirchhoff type equations with strong nonlinear damping and supercritical nonlinearity:

$$u_{tt}-\sigma(\|\Delta u\|^2)\Delta u_t-\phi(\|\Delta u\|^2)\Delta u+f(u)=h(x)\quad in\quad \Omega\times\mathbb{R}^+, \tag{5}$$

$$u(x,t)|_{\partial\Omega}=0, u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\Omega. \tag{6}$$

where Ω is a bounded domain in R^N with the smooth boundary $\partial\Omega$, $\sigma(s)$, $\phi(s)$ and $f(s)$ are nonlinear functions, and $h(x)$ is an external force term.

Yang Zhijian, Wang Yunqing [8] also studied the global attractor for the Kirchhoff type equation with a strong dissipation:

$$u_{tt}-M(\|\Delta u\|^2)\Delta u-\Delta u_t+h(u_t)+g(u)=f(x) \quad in \quad \Omega\times R^+, \tag{7}$$

$$u(x,t)|_{\partial\Omega}=0, t>0, u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\Omega. \tag{8}$$

where $M(s)=1+s^{\frac{m}{2}}, 1\le m\le \frac{4}{(N-2)}$, Ω is a bounded domain in R^N, with smooth boundary $\partial\Omega$, $h(s)$ and $g(s)$ are nonlinear functions, and $f(x)$ is an external force term.

Recently, Meixia Wang, Cuicui Tian, Guoguang Lin [9] studied the global attractor and dimension estimation for a 2D generalized Anisotropy Kuramoto-Sivashinsky equation:

$$u_t+\alpha\Delta^2 u+\gamma u+(\varphi(u))_{xx}+(g(u))_{yy}=f(x), (x,y)\in\Omega\subset R^2, \tag{9}$$

$$u(x,y,t)|_{t=0}=u_0(x,y), (x,y)\in\Omega\subset R^2, \tag{10}$$

$$u(x,y,t)|_{\partial\Omega=0}=0, \Delta u(x,y,t)|_{\partial\Omega}=0, (x,y)\in\Omega\subset R^2. \tag{11}$$

where $\Omega\subset R^2$ is bounded set; $\partial\Omega$ is the bound of Ω; $\varphi(u)$ and $g(u)$ are considered as smooth functions of $u(x,y,t)$.

There have been many researches on the long-time behavior of solutions to the nonlinear damped wave equations with delays. For more related results we refer the reader to [10]-[13]. In order to make these equations more normal, in section 2 and in section 3, some assumptions, notations and the main results are stated. Under these assumptions, we prove the existence and uniqueness of solution, then we obtain the global attractors for the problems (1.1)-(1.3). According to [9]-[13], in section 4, we consider that the global attractor of the above mentioned problems (1.1)-(1.3) has finite Hausdorff dimensions and fractal dimensions.

2 Statement of main results

For convenience, we denote the norm and scalar product in $L^2(\Omega)$ by $\|.\|$ and $(.,.)$;

$$f=f(x),\ L^p=L^p(\Omega),\ H^k=H^k(\Omega),\ H_0^k=H_0^k(\Omega),\ \|\cdot\|=\|\cdot\|_{L^2},\ \|\cdot\|_p=\|\cdot\|_{L^p}.$$

In this section, we present some assumptions and notations needed in the proof of our results. For this reason, we assume that

(G_1) $\phi(\|\nabla u\|^2): R^+\to R^+$ is a differentiable function;

(G_2) There exist constant $\varepsilon_1>0, \varepsilon>0, \gamma_1>0, \gamma_2>0, K\ge 0$, such that $K-2\varepsilon\ge 0$, $\varepsilon_1\varepsilon\le\phi(\|\nabla u\|^2)\le\frac{\gamma_1}{K-2\varepsilon}(1+\gamma_2 e^{-(K-2\varepsilon)t})$.

Lemma 1. Assume $(G_1),(G_2)$ hold, and $(u_0,u_1)\in(L^{q+1}(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$, $f\in L^2(\Omega), v=u_t+\varepsilon u$, let

$$\begin{cases} p\ge 2, & n=1,2; \\ 2<p<\dfrac{n+4}{n}, & n\ge 3. \end{cases}$$

$$\begin{cases} q \ge 2, & n = 1,2; \\ 2 < q < \dfrac{n+2}{n-2}, & n \ge 3. \end{cases}$$

Then the solution (u,v) of the problems (1.1)-(1.3) satisfies $(u,v) \in (L^{q+1}(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega), \mathrm{H}_1 := L^{q+1}(\Omega) \cap H_0^1(\Omega)$, and

$$\|(u,v)\|^2_{\mathrm{H}_1 \times L^2} = \|\nabla u\|^2 + \|v\|^2 \le \frac{W(0)}{N} e^{-\alpha_1 t} + \frac{C}{N\alpha_1}(1 - e^{-\alpha_1 t}). \tag{1}$$

where $v = u_t + \varepsilon u$, $0 < N < min\{1, \phi(\|\nabla u\|^2) - \varepsilon_1 \varepsilon\}$, and $W(0) = \|v_0\|^2 + (\phi(\|\nabla u_0\|^2) - \varepsilon_1 \varepsilon)\|\nabla u_0\|^2$, $v_0 = u_1 + \varepsilon u_0$, thus there exist R_0 and $t_1 = t_1(\Omega) > 0$, such that

$$\|(u,v)\|^2_{\mathrm{H}_1 \times L^2} = \|\nabla u\|^2 + \|v\|^2 \le R_0 (t > t_1). \tag{2}$$

Proof. Let $v = u_t + \varepsilon u$, We multiply v with both sides of equation (1.1) and obtain

$$(u_{tt} - \varepsilon_1 \Delta u_t + \alpha |u_t|^{p-1} u_t + \beta |u|^{q-1} u - \phi(\|\nabla u\|^2)\Delta u, v) = (f(x), v). \tag{3}$$

By using Hölder's inequality, Young's inequality and Poincare''s inequality. one by one as follows:

$$(u_{tt}, v) = \frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon(v - \varepsilon u, v) \ge \frac{1}{2}\frac{d}{dt}\|v\|^2 - \varepsilon\|v\|^2 - \frac{\varepsilon^2}{2\lambda_1}\|\nabla u\|^2 - \frac{\varepsilon^2}{2}\|v\|^2. \tag{4}$$

$$(-\varepsilon_1 \Delta u_t, v) = -\varepsilon_1(\Delta(v - \varepsilon u), v) \ge \varepsilon_1 \lambda_2 \|v\|^2 - \frac{\varepsilon_1 \varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2 - \varepsilon_1 \varepsilon^2 \|\nabla u\|^2. \tag{5}$$

$$\begin{aligned} &(\alpha |u_t|^{p-1} u_t, v) \\ &= \alpha(|u_t|^{p-1} u_t, u_t + \varepsilon u) \\ &= \alpha \|u_t\|^{p+1}_{p+1} + \alpha\varepsilon \int_\Omega |u_t|^{p-1} u_t \cdot u dx, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &\alpha\varepsilon \int_\Omega |u_t|^{p-1} u_t \cdot u dx \\ &\le \alpha\varepsilon \int_\Omega |u_t|^p \cdot |u| dx \\ &\le \alpha\varepsilon \left(\int_\Omega |u_t|^{p+1} dx\right)^{\frac{p}{p+1}} \cdot \left(\int_\Omega |u|^{p+1} dx\right)^{\frac{1}{p+1}} \\ &= \alpha\varepsilon \|u_t\|^p_{p+1} \cdot \|u\|_{p+1} \\ &\le \frac{\alpha p}{p+1}\|u_t\|^{p+1}_{p+1} + \frac{\alpha\varepsilon^{p+1}}{p+1}\|u\|^{p+1}_{p+1}, \end{aligned} \tag{7}$$

And using Interpolation Theorem, we have

$$\begin{aligned} &\frac{\alpha p}{p+1}\|u_t\|^{p+1}_{p+1} + \frac{\alpha\varepsilon^{p+1}}{p+1}\|u\|^{p+1}_{p+1} \\ &\le \frac{\alpha p}{p+1}\|u_t\|^{p+1}_{p+1} + C_0(\alpha, \varepsilon, p, \|u\|)\|\nabla u\|^{\frac{n(p-1)}{2}}, \end{aligned} \tag{8}$$

then from $2 < p < \dfrac{n+4}{n}, n \ge 3$, according to Embedding Theorem

$$\begin{aligned}&\frac{\alpha p}{p+1}\|u_t\|_{p+1}^{p+1}+C_0(\alpha,\varepsilon,p,\|u\|)\|\nabla u\|^{\frac{n(p-1)}{2}}\\&\le\frac{\alpha p}{p+1}\|u_t\|_{p+1}^{p+1}+\frac{\varepsilon_1\varepsilon^2}{2}\|\nabla u\|^2+C_1(\alpha,\varepsilon,p,\|u\|,\varepsilon_1).\end{aligned}\tag{9}$$

and

$$(\beta|u|^{q-1}u,v)\quad=\frac{\beta}{(q+1)}\frac{d}{dt}\|u\|_{q+1}^{q+1}+\beta\varepsilon\|u\|_{q+1}^{q+1}.\tag{10}$$

$$\begin{aligned}&(-\phi(\|\nabla u\|^2)\Delta u,v)\\&=\phi(\|\nabla u\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2+\varepsilon\phi(\|\nabla u\|^2)\|\nabla u\|^2\\&=\frac{d}{dt}[\frac{1}{2}\phi(\|\nabla u\|^2)\|\nabla u\|^2]-\frac{1}{2}\|\nabla u\|^2\frac{d}{dt}(\phi(\|\nabla u\|^2))+\varepsilon\phi(\|\nabla u\|^2)\|\nabla u\|^2.\end{aligned}\tag{11}$$

$$(f(x),v)\quad\le\|f\|\|v\|\le\frac{\varepsilon^2}{2}\|v\|^2+\frac{1}{2\varepsilon^2}\|f\|^2.\tag{12}$$

From the above, we have

$$\begin{aligned}&\frac{d}{dt}[\|v\|^2+(\phi(\|\nabla u\|^2)-\varepsilon_1\varepsilon)\|\nabla u\|^2+\frac{2\beta}{q+1}\|u\|_{q+1}^{q+1}]+(2\varepsilon_1\lambda_2-2\varepsilon-2\varepsilon^2)\|v\|^2\\&+[2\varepsilon\phi(\|\nabla u\|^2)-\frac{d}{dt}(\phi(\|\nabla u\|^2))-\frac{\varepsilon^2}{\lambda_1}-3\varepsilon_1\varepsilon^2]\|\nabla u\|^2+2\beta\varepsilon\|u\|_{q+1}^{q+1}+\\&\frac{2\alpha}{p+1}\|u_t\|_{p+1}^{p+1}\le\frac{1}{\varepsilon^2}\|f\|^2+2C_1(\alpha,\varepsilon,p,\|u\|,\varepsilon_1):=C.\end{aligned}\tag{13}$$

Next, we take proper $\varepsilon,\varepsilon_1$, such that: $\varepsilon_1\varepsilon\le\phi(\|\nabla u\|^2)$,

let constant K, such that $K-2\varepsilon\ge 0$,

$$0\le K(\phi(\|\nabla u\|^2)-\varepsilon_1\varepsilon)\le 2\varepsilon\phi(\|\nabla u\|^2)-\frac{d}{dt}(\phi(\|\nabla u\|^2))-\frac{\varepsilon^2}{\lambda_1}-3\varepsilon_1\varepsilon^2,\tag{14}$$

Where $C_3=C_3(\varepsilon,\lambda_1,\varepsilon_1)=\frac{\varepsilon^2}{\lambda_1}+3\varepsilon_1\varepsilon^2$ £¬ such that

$$0\le K(\phi(\|\nabla u\|^2)-\varepsilon_1\varepsilon)\le 2\varepsilon\phi(\|\nabla u\|^2)-\frac{d}{dt}(\phi(\|\nabla u\|^2))-C_3,\tag{15}$$

$$(K-2\varepsilon)\phi(\|\nabla u\|^2)+\frac{d}{dt}(\phi(\|\nabla u\|^2))\le K\varepsilon_1\varepsilon-C_3:=\gamma_1.\tag{16}$$

Multiplying(2.16) by $e^{(K-2\varepsilon)t}$, then

$$\phi(\|\nabla u\|^2)\frac{d}{dt}(e^{(K-2\varepsilon)t})+e^{(K-2\varepsilon)t}\frac{d}{dt}(\phi(\|\nabla u\|^2))\le\gamma_1 e^{(K-2\varepsilon)t},\tag{17}$$

We integrate (2.17)with respect to time t and get that

$$\phi(\|\nabla u\|^2)\le\frac{\gamma_1}{K-2\varepsilon}(1+\gamma_2 e^{-(K-2\varepsilon)t}).\tag{18}$$

Where $\varepsilon, \gamma_1, \gamma_2$ is constant. From the above, we obtain

$$\varepsilon_1 \varepsilon \le \phi(\|\nabla u\|^2) \le \frac{\gamma_1}{K-2\varepsilon}(1+\gamma_2 e^{-(K-2\varepsilon)t}). \tag{19}$$

At last, we take proper $\varepsilon, \varepsilon_1, \beta$, such that:

$$\begin{cases} a_1 = 2\varepsilon_1\lambda_2 - 2\varepsilon - 2\varepsilon^2 \ge 0 \\ a_2 = 2\varepsilon\phi(\|\nabla u\|^2) - \dfrac{d}{dt}(\phi(\|\nabla u\|^2)) - \dfrac{\varepsilon^2}{\lambda_1} - 3\varepsilon_1\varepsilon^2 \ge 0. \end{cases}$$

Taking $\alpha_1 = min\{a_1, \dfrac{a_2}{\phi(\|\nabla u\|^2) - \varepsilon_1\varepsilon}, \varepsilon(q+1)\}$, then

$$\frac{d}{dt}W(t) + \alpha_1 W(t) \le C. \tag{20}$$

Where $W(t) = \|v\|^2 + (\phi(\|\nabla u\|^2) - \varepsilon_1\varepsilon)\|\nabla u\|^2 + \dfrac{2\beta}{q+1}\|u\|_{q+1}^{q+1}$, by using Gronwall's inequality, we obtain:

$$W(t) \le W_0 e^{-\alpha_1 t} + \frac{C}{\alpha_1}(1-\mathrm{e}^{-\alpha_1 t}). \tag{21}$$

From $2 < q < \dfrac{n+2}{n-2}, n \ge 3$, according to Embedding Theorem $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$, let $0 < N = min\{1, \phi(\|\nabla u\|^2) - \varepsilon_1\varepsilon\}$, such that:

$$\|(u,v)\|^2_{H_1 \times L^2} = \|\nabla u\|^2 + \|v\|^2 \le \frac{W(0)}{N}\mathrm{e}^{-\alpha_1 t} + \frac{C}{N\alpha_1}(1-\mathrm{e}^{-\alpha_1 t}). \tag{22}$$

where $v = u_t + \varepsilon u$, and $W(0) = \|v_0\|^2 + (\phi(\|\nabla u_0\|^2) - \varepsilon_1\varepsilon)\|\nabla u_0\|^2 + \dfrac{\beta}{q+1}\|u_0\|_{q+1}^{q+1}, v_0 = u_1 + \varepsilon u_0$, then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H_1 \times L^2} \le \frac{C}{N\alpha_1}, \tag{23}$$

thus there exist R_0 and $t_1 = t_1(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H_1 \times L^2} = \|\nabla u\|^2 + \|v\|^2 \le R_0 (t > t_1). \tag{24}$$

Lemma 2. In addition to the assumptions of Lemma 1, $(G_1), (G_2)$ hold.

If $(G_3): f \in H_0^1(\Omega)$, let

$$\begin{cases} p \ge 2, & n = 1,2; \\ 2 < p < \dfrac{n+4}{n}, & n \ge 3. \end{cases}$$

$$\begin{cases} q \ge 2, & n = 1,2; \\ 2 < q < \dfrac{n+4}{n}, & n \ge 3. \end{cases}$$

and $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega), v=u_t+\varepsilon u$, Then the solution (u,v) of the problems (1.1)-(1.3) satisfies $(u,v)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega), \mathrm{H}_2:=H^2(\Omega)\cap H_0^1(\Omega)$, and

$$\|(u,v)\|^2_{\mathrm{H}_2\times H_0^1}=\|\Delta u\|^2+\|\nabla v\|^2\le\frac{U(0)}{M}\mathrm{e}^{-\alpha_2 t}+\frac{C_1}{M\alpha_2}(1-\mathrm{e}^{-\alpha_2 t}). \quad (25)$$

where $v=u_t+\varepsilon u$, $0<M<min\{1,\phi(\|\nabla u\|^2)+\varepsilon_1\varepsilon\}$, and $U(0)=\|\nabla v_0\|^2+(\phi(\|\nabla u_0\|^2)+\varepsilon_1\varepsilon)\|\Delta u_0\|^2$, thus there exist R_1 and $t_2=t_2(\Omega)>0$, such that

$$\|(u,v)\|^2_{\mathrm{H}_2\times H_0^1}=\|\Delta u\|^2+\|\nabla v\|^2\le R_1(t>t_2). \quad (26)$$

Proof. Let $-\Delta v=-\Delta u_t-\varepsilon\Delta u$, We multiply $-\Delta v$ with both sides of equation (1.1) and obtain

$$(u_{tt}-\varepsilon_1\Delta u_t+\alpha|u_t|^{p-1}u_t+\beta|u|^{q-1}u-\phi(\|\nabla u\|^2)\Delta u,-\Delta v)=(f(x),-\Delta v). \quad (27)$$

By using H$\ddot{o}$lder's inequality, Young's inequality and Poincare''s inequality, we deal with the terms in (2.27), one by one as follows:

$$(u_{tt},-\Delta v)\quad\ge\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2-\varepsilon\|\nabla v\|^2-\frac{\varepsilon^2}{2\mu_1}\|\Delta u\|^2-\frac{\varepsilon^2}{2}\|\nabla v\|^2. \quad (28)$$

$$\begin{aligned}&(-\varepsilon_1\Delta u_t,-\Delta v)\\&=(-\varepsilon_1\Delta u_t,-\Delta u_t-\varepsilon\Delta u)\\&=\varepsilon_1\|\Delta u_t\|^2+\varepsilon_1\|\Delta v\|^2-\varepsilon_1(\Delta v,\Delta u_t)-\varepsilon_1\varepsilon(\Delta u,\Delta v)+\varepsilon_1\varepsilon(\Delta u,\Delta u_t)\\&\ge\frac{\varepsilon_1-\varepsilon_1\varepsilon}{2}\|\Delta v\|^2+\frac{\varepsilon_1}{2}\|\Delta u_t\|^2+\frac{\varepsilon_1\varepsilon}{2}\frac{d}{dt}\|\Delta u\|^2-\frac{\varepsilon_1\varepsilon}{2}\|\Delta u\|^2.\end{aligned} \quad (29)$$

$$(\alpha|u_t|^{p-1}u_t,-\Delta v)\quad=\alpha(|u_t|^{p-1}u_t,-\Delta u_t)+\alpha(|u_t|^{p-1}u_t,-\varepsilon\Delta u). \quad (30)$$

Where

$$\begin{aligned}&\alpha(|u_t|^{p-1}u_t,-\Delta u_t)\\&=\alpha(\nabla((u_t^2)^{\frac{p-1}{2}}u_t),\nabla u_t)\\&=\alpha(\frac{p-1}{2}(u_t^2)^{\frac{p-3}{2}}2u_t^2\nabla u_t,\nabla u_t)+\alpha(|u_t|^{p-1}\nabla u_t,\nabla u_t)\\&=\alpha(p-1)(|u_t|^{p-1}\nabla u_t,\nabla u_t)+\alpha(|u_t|^{p-1}\nabla u_t,\nabla u_t)\\&=\alpha p\int_\Omega|u_t|^{p-1}|\nabla u_t|^2\,dx>0,\end{aligned} \quad (31)$$

$$\begin{aligned}\alpha(|u_t|^{p-1}u_t,-\varepsilon\Delta u)\quad&\le\alpha\varepsilon\int_\Omega|u_t|^p|\Delta u|\,dx\\&\le\alpha\varepsilon(\int_\Omega|u_t|^{2p}\,dx)^{\frac{1}{2}}(\int_\Omega|\Delta u|^2\,dx)^{\frac{1}{2}}\\&=\alpha\varepsilon\|u_t\|^p_{2p}\|\Delta u\|,\end{aligned} \quad (32)$$

By using Interpolation Theorem and Embedding Theorem:

$$\begin{aligned}\alpha\varepsilon\|u_t\|_{2p}^{p}\|\Delta u\| &\le \alpha\varepsilon C_0(\|u_t\|)\|\Delta u_t\|^{\frac{n(p-1)}{4}}\|\Delta u\| \\ &\le \frac{\varepsilon^2}{2\mu_1}\|\Delta u\|^2+\frac{\mu_1\alpha^2C_0^2(\|u_t\|)}{2}\|\Delta u_t\|^{\frac{n(p-1)}{2}} \\ &\le \frac{\varepsilon^2}{2\mu_1}\|\Delta u\|^2+\frac{\varepsilon_1}{4}\|\Delta u_t\|^2+C_1(\alpha,C_0,\mu_1,\varepsilon_1),\end{aligned} \tag{33}$$

Substituting (2.31),(2.32),(2.33) into (2.30), we receive

$$\begin{aligned}&(\alpha|u_t|^{p-1}u_t,-\Delta v) \\ &\ge \alpha p\int_\Omega |u_t|^{p-1}|\nabla u_t|^2\,dx-\frac{\varepsilon^2}{2\mu_1}\|\Delta u\|^2-\frac{\varepsilon_1}{4}\|\Delta u_t\|^2-C_1.\end{aligned} \tag{34}$$

By using Interpolation Theorem and Embedding Theorem:

$$\begin{aligned}&(\beta|u|^{q-1}u,-\Delta v) \\ &\le \beta\|u\|_{2q}^{q}\|\Delta v\| \\ &\le \beta C_2(\|u\|)\|\Delta u\|^{\frac{n(q-1)}{4}}\|\Delta v\| \\ &\le \frac{\varepsilon_1-\varepsilon_1\varepsilon}{4}\|\Delta v\|^2+\frac{\beta^2C_2^2(\|u\|)}{\varepsilon_1-\varepsilon_1\varepsilon}\|\Delta u\|^{\frac{n(q-1)}{2}} \\ &\le \frac{\varepsilon_1-\varepsilon_1\varepsilon}{4}\|\Delta v\|^2+\frac{\varepsilon_1\varepsilon}{2}\|\Delta u\|^2+C_3(\varepsilon,\beta,\varepsilon_1,\|u\|).\end{aligned} \tag{35}$$

$$\begin{aligned}&(-\phi(\|\nabla u\|^2)\Delta u,-\Delta v) \\ &=\phi(\|\nabla u\|^2)(\Delta u,\Delta u_t+\varepsilon\Delta u) \\ &=\frac{1}{2}\phi(\|\nabla u\|^2)\frac{d}{dt}\|\Delta u\|^2+\varepsilon\phi(\|\nabla u\|^2)\|\Delta u\|^2 \\ &=\frac{d}{dt}[\frac{1}{2}\phi(\|\nabla u\|^2)\|\Delta u\|^2]-\frac{1}{2}\|\Delta u\|^2\frac{d}{dt}(\phi(\|\nabla u\|^2))+\varepsilon\phi(\|\nabla u\|^2)\|\Delta u\|^2.\end{aligned} \tag{36}$$

$$(f,-\Delta v)\le\|\nabla f\|\|\nabla v\|\le\frac{\varepsilon_1-\varepsilon_1\varepsilon}{8\mu_2}\|\nabla v\|^2+\frac{2\mu_2}{\varepsilon_1-\varepsilon_1\varepsilon}\|\nabla f\|^2. \tag{37}$$

From the above, we have

$$\begin{aligned}&\frac{d}{dt}[\|\nabla v\|^2+(\varepsilon_1\varepsilon+\phi(\|\nabla u\|^2))\|\Delta u\|^2]+(\frac{\varepsilon_1-\varepsilon_1\varepsilon-8\varepsilon\mu_2-4\mu_2\varepsilon^2}{4\mu_2})\|\nabla v\|^2 \\ &+[2\varepsilon\phi(\|\nabla u\|^2)-\frac{d}{dt}(\phi(\|\nabla u\|^2))-\frac{2\varepsilon^2}{\mu_1}-2\varepsilon_1\varepsilon]\|\Delta u\|^2\le C.\end{aligned} \tag{38}$$

Similar to lemma 1.1 formulas (2.14)-(2.19) we can obtain that

$$\varepsilon_1\varepsilon\le\phi(\|\nabla u\|^2)\le\frac{\kappa_1}{M-2\varepsilon}(1+\kappa_2e^{-(M-2\varepsilon)t}). \tag{39}$$

where $\varepsilon_1,\varepsilon,M,\kappa_1,\kappa_2$ is constants.

At last, we take proper $\varepsilon,\varepsilon_1$, such that:

$$\begin{cases} b_1 = \dfrac{\varepsilon_1 - \varepsilon_1\varepsilon - 8\varepsilon\mu_2 - 4\mu_2\varepsilon^2}{4\mu_2} \geq 0 \\ b_2 = 2\varepsilon\phi(\|\nabla u\|^2) - \dfrac{d}{dt}(\phi(\|\nabla u\|^2)) - \dfrac{2\varepsilon^2}{\mu_1} - 2\varepsilon_1\varepsilon \geq 0. \end{cases}$$

Taking $\alpha_2 = min\{b_1, \dfrac{b_2}{\varepsilon_1\varepsilon + \phi(\|\nabla u\|^2)}\}$, then

$$\frac{d}{dt}U(t) + \alpha_2 U(t) \leq C. \tag{40}$$

Where $U(t) = \|\nabla v\|^2 + (\varepsilon_1\varepsilon + \phi(\|\nabla u\|^2))\|\Delta u\|^2$, by using Gronwall's inequality, we obtain:

$$U(t) \leq U(0)e^{-\alpha_2 t} + \frac{C}{\alpha_2}(1 - e^{-\alpha_2 t}). \tag{41}$$

Let $0 < L = min\{1, \varepsilon_1\varepsilon + \phi(\|\nabla u\|^2)\}$, so we get

$$\|(u,v)\|^2_{H_2 \times H_0^1} = \|\Delta u\|^2 + \|\nabla v\|^2 \leq \frac{U(0)}{L}e^{-\alpha_2 t} + \frac{C}{\alpha_2 L}(1 - e^{-\alpha_2 t}). \tag{42}$$

where $v = u_t + \varepsilon u$, and $U(0) = \|\nabla v_0\|^2 + (\varepsilon_1\varepsilon + \phi(\|\nabla u_0\|^2))\|\Delta u_0\|^2$, $v_0 = u_1 + \varepsilon u_0$, then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H_2 \times H_0^1} \leq \frac{C}{\alpha_2 L}. \tag{43}$$

So, there exist R_1 and $t_2 = t_2(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H_2 \times H_0^1} = \|\Delta u\|^2 + \|\nabla v\|^2 \leq R_1 (t > t_2). \tag{44}$$

3 Global attractor

3.1 The existence and uniqueness of solution

Theorem 3.1 Assume $(G_1),(G_2)$ *hold, let*

$$\begin{cases} p \geq 2, & n = 1,2; \\ 2 < p < \dfrac{n+4}{n}, & n \geq 3. \end{cases}$$

$$\begin{cases} q \geq 2, & n = 1,2; \\ 2 < q < min\{\dfrac{n+4}{n}, \dfrac{n+2}{n-2}\}, & n \geq 3. \end{cases}$$

and $H_2(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$, $(u_0, u_1) \in H_2(\Omega) \times H_0^1(\Omega)$, $f \in H_0^1(\Omega)$, $v = u_t + \varepsilon u$, so Equation (1.1) exists a unique smooth solution

$$(u,v) \in L^\infty([0,+\infty), H_2(\Omega) \times H_0^1(\Omega)). \tag{1}$$

Proof. By the Galerkin method, Lemma 1 and Lemma 2, we can easily obtain the existence of Solutions. Next, we prove the uniqueness of Solutions in detail.

Assume u, v are two solutions of the problems (1.1)-(1.3), let $w = u - v$, then

$w(x,0) = w_0(x) = 0, w_t(x,0) = w_1(x) = 0$ and the two equations subtract and obtain

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$$w_{tt}-\varepsilon_1\Delta w_t+\alpha(|u_t|^{p-1}u_t-|v_t|^{p-1}v_t)+\beta(|u|^{q-1}u-|v|^{q-1}v)+$$
$$(\phi(\|\nabla v\|^2)\Delta v-\phi(\|\nabla u\|^2)\Delta u)=0. \quad (2)$$

By multiplying (3.2) by w_t , we get

$$(w_{tt}-\varepsilon_1\Delta w_t+\alpha(|u_t|^{p-1}u_t-|v_t|^{p-1}v_t)+\beta(|u|^{q-1}u-|v|^{q-1}v)+$$
$$(\phi(\|\nabla v\|^2)\Delta v-\phi(\|\nabla u\|^2)\Delta u),w_t)=0. \quad (3)$$

By using H$\ddot{o}$lder's inequality, Young's inequality and Poincare''s inequality.

One by one as follows:

$$(w_{tt},w_t)=\frac{1}{2}\frac{d}{dt}\|w_t\|^2. \quad (4)$$

$$(-\varepsilon_1\Delta w_t,w_t)=\varepsilon_1\|\nabla w_t\|^2\geq\varepsilon_1\lambda_1\|w_t\|^2. \quad (5)$$

$$\begin{aligned}&(\alpha(|u_t|^{p-1}u_t-|v_t|^{p-1}v_t),w_t)\\&=\alpha\int_\Omega(|u_t|^{p-1}u_t-|v_t|^{p-1}v_t)w_tdx\\&\leq\alpha p\int_\Omega(|u_t|^{p-1}+|v_t|^{p-1})|w_t||w_t|dx.\end{aligned} \quad (6)$$

According to Lemma 1, So, there exists $C_0>0$, such that

$$(\alpha(|u_t|^{p-1}u_t-|v_t|^{p-1}v_t),w_t)\leq\alpha pC_0\|w_t\|^2. \quad (7)$$

$$\begin{aligned}&(\beta(|u|^{q-1}u-|v|^{q-1}v),w_t)\\&=\beta\int_\Omega(|u|^{q-1}u-|v|^{q-1}v)w_tdx\\&\leq\beta q\int_\Omega(|u|^{q-1}+|v|^{q-1})|w||w_t|dx.\end{aligned} \quad (8)$$

According to Lemma 1, So, there exists $C_1>0$, such that

$$\begin{aligned}&(\beta(|u|^{q-1}u-|v|^{q-1}v),w_t)\\&\leq\beta qC_1\|w\|\|w_t\|\\&\leq\frac{\varepsilon_1\lambda_1}{2}\|w_t\|^2+\frac{\beta^2q^2C_1^2}{2\varepsilon_1\lambda_1}\|w\|^2.\end{aligned} \quad (9)$$

$$\begin{aligned}&(\phi(\|\nabla v\|^2)\Delta v-\phi(\|\nabla u\|^2)\Delta u,w_t)\\&=\phi(\|\nabla v\|^2)\Delta v-\phi(\|\nabla v\|^2)\Delta u+\phi(\|\nabla v\|^2)\Delta u-\phi(\|\nabla u\|^2)\Delta u,w_t)\\&=\phi(\|\nabla v\|^2)(-\Delta w,w_t)+[\phi(\|\nabla v\|^2)-\phi(\|\nabla u\|^2)](\Delta u,w_t)\\&=\phi(\|\nabla v\|^2)\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\phi'(\xi)(\|\nabla v\|+\|\nabla u\|)(\|\nabla v\|-\|\nabla u\|)(\Delta u,w_t).\end{aligned} \quad (10)$$

where

$$\begin{aligned}&\phi'(\xi)(\|\nabla v\|+\|\nabla u\|)(\|\nabla v\|-\|\nabla u\|)(\Delta u,w_t)\\&\leq|\phi'(\xi)|(\|\nabla v\|+\|\nabla u\|)(\|\nabla v-\nabla u\|)\|\Delta u\|\|w_t\|\\&\leq\|\phi'(\xi)\|_\infty(\|\nabla v\|+\|\nabla u\|)\|\Delta u\|\|\nabla w\|\|w_t\|.\end{aligned} \quad (11)$$

According to Lemma 1 and Lemma 2, So, there exists $C_2 > 0$, such that

$$\begin{aligned} &\phi'(\xi)(\|\nabla v\| + \|\nabla u\|)(\|\nabla v\| - \|\nabla u\|)(\Delta u, w_t) \\ &\le C_2 \|\nabla w\| \|w_t\| \\ &\le \frac{\varepsilon_1 \lambda_1}{2} \|w_t\|^2 + \frac{C_2^2}{2\varepsilon_1 \lambda_1} \|\nabla w\|^2. \end{aligned} \tag{12}$$

Next, according to the basic assumption of Lemma 1 and Lemma 2:

$$\varepsilon_1 \varepsilon \le \phi(\|\nabla u\|^2) \le min\{\frac{\gamma_1}{K-2\varepsilon}(1+\gamma_2 e^{-(K-2\varepsilon)t}), \frac{\kappa_1}{M-2\varepsilon}(1+\kappa_2 e^{-(M-2\varepsilon)t})\}.$$

Then, we have

$$\begin{aligned} &(\phi(\|\nabla v\|^2)\Delta v - \phi(\|\nabla u\|^2)\Delta u, w_t) \\ &\ge \frac{\varepsilon_1 \varepsilon}{2} \frac{d}{dt} \|\nabla w\|^2 - \frac{\varepsilon_1 \lambda_1}{2} \|w_t\|^2 - \frac{C_2^2}{2\varepsilon_1 \lambda_1} \|\nabla w\|^2. \end{aligned} \tag{13}$$

From the above, we have

$$\begin{aligned} &\frac{d}{dt}[\|w_t\|^2 + \varepsilon_1 \varepsilon \|\nabla w\|^2] \\ &\le 2\alpha p C_0 \|w_t\|^2 + \frac{C_2^2}{\varepsilon_1 \lambda_1} \|\nabla w\|^2 + \frac{\beta^2 q^2 C_1^2}{\varepsilon_1 \lambda_1} \|w\|^2 \\ &\le 2\alpha p C_0 \|w_t\|^2 + \frac{C_2^2}{\varepsilon_1 \lambda_1} \|\nabla w\|^2 + \frac{\beta^2 q^2 C_1^2}{\varepsilon_1 \lambda_1 \mu_1} \|\nabla w\|^2 \\ &\le 2\alpha p C_0 \|w_t\|^2 + \frac{C_2^2 \mu_1 + \beta^2 q^2 C_1^2}{\varepsilon_1 \lambda_1 \mu_1} \|\nabla w\|^2. \end{aligned} \tag{14}$$

Taking M=max $\{2\alpha p C_0, \frac{C_2^2 \mu_1 + \beta^2 q^2 C_1^2}{\varepsilon_1^2 \varepsilon \lambda_1 \mu_1}\}$, then

$$\frac{d}{dt}(\|w_t\|^2 + \varepsilon_1 \varepsilon \|\nabla w\|^2) \le M(\|w_t\|^2 + \varepsilon_1 \varepsilon \|\nabla w\|^2). \tag{15}$$

By using Gronwall's inequality, we obtain

$$\|w_t\|^2 + \varepsilon_1 \varepsilon \|\nabla w\|^2 \le (\|w_t(0)\|^2 + \varepsilon_1 \varepsilon \|\nabla w(0)\|^2) e^{Mt}. \tag{16}$$

So , we can get $\|w_t\|^2 + \varepsilon_1 \varepsilon \|\nabla w\|^2 \le 0$, because of $w_0(x) = 0, w_1(x) = 0$.

That shows that

$$\|w_t\|^2 = 0, \quad \|\nabla w\|^2 = 0.$$

That is

$$w(x,t) = 0.$$

Therefore

$$u = v.$$

So we get the uniqueness of the solution.

Remark 3.1. Under the assumptions of Lemma 1, Lemma 2 and Theorem 3.1, We claim that $S(t)(u_0, u_1) = (u(t), u_t(t))$. Then $S(t)$ composes a continuous semigroup in $H_2(\Omega) \times H_0^1(\Omega)$.

3.2 Global attractor

Theorem 3.2 [1,2] Let E be a Banach space, and $\{S(t)\}(t \geq 0)$ are the semigroup operator on E. $S(t): E \to E, S(t+\tau) = S(t)S(\tau)(\forall t, \tau \geq 0), S(0) = I$, where I is a unit operator. The semigroup operator $S(t)$ satisfies the following conditions.

1) $S(t)$ is uniformly bounded, namely $\forall R > 0, \|u\|_E \leq R$, it exists a constant $C(R)$, so that

$$\|S(t)u\|_E \leq C(R)(t \in [0, +\infty));$$

2) It exists a bounded absorbing set $B_0 \subset E$, namely, $\forall B \subset E$, it exists a constant t_0, so that

$$S(t)B \subset B_0 (t \geq t_0);$$

where B_0 and B are bounded sets.

3) When $t > 0, S(t)$ is a completely continuous operator A.

Therefore, the semigroup operator S(t) exists a compact global attractor.

Theorem 3.3 *Under the assume of Lemma 1, Lemma 2 and Theorem 3.1, equations have global attractor*

$$\mathcal{A} = \omega(B_0) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t) B_0},$$

where $B_0 = \{(u,v) \in \mathcal{H}_2(\Omega) \times H_0^1(\Omega) : \|(u,v)\|^2_{\mathcal{H}_2 \times H_0^1} = \|u\|^2_{\mathcal{H}_2} + \|v\|^2_{H_0^1} \leq R_0 + R_1\}, B_0$ is the bounded absorbing set of $\mathcal{H}_2 \times H_0^1$ and satisfies

1) $S(t)\mathcal{A} = \mathcal{A}, t > 0$;

2) $\lim_{t \to \infty} dist(S(t)B, \mathcal{A}) = 0$, here $B \subset \mathcal{H}_2 \times H_0^1$ and it is a bounded set,

$$dist(S(t)B, \mathcal{A}) = \sup_{x \in B} (\inf_{y \in \mathcal{A}} \|S(t)x - y\|_{\mathcal{H}_2 \times H_0^1}) \to 0, t \to \infty.$$

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup S(t), $S(t): \mathcal{H}_2 \times H_0^1 \to \mathcal{H}_2 \times H_0^1$, here $E = \mathcal{H}_2(\Omega) \times H_0^1(\Omega)$.

(1) From Lemma 1 to Lemma 2, we can get that $\forall B \subset \mathcal{H}_2(\Omega) \times H_0^1(\Omega)$ is a bounded set that includes in the ball $\{\|(u,v)\|_{\mathcal{H}_2 \times H_0^1} \leq R\}$,

$$\|S(t)(u_0, v_0)\|^2_{\mathcal{H}_2 \times H_0^1} = \|u\|^2_{\mathcal{H}_2} + \|v\|^2_{H_0^1} \leq \|u_0\|^2_{\mathcal{H}_2} + \|v_0\|^2_{H_0^1} + C \leq R^2 + C,$$

$$(t \geq 0, (u_0, v_0) \in B).$$

This shows that $S(t)(t \geq 0)$ is uniformly bounded in $\mathcal{H}_2(\Omega) \times H_0^1(\Omega)$.

(2) Furthermore, for any $(u_0, v_0) \in \mathcal{H}_2(\Omega) \times H_0^1(\Omega)$, when $t \geq max\{t_1, t_2\}$, we have

$$\|S(t)(u_0, v_0)\|^2_{\mathcal{H}_2 \times H_0^1} = \|u\|^2_{\mathcal{H}_2} + \|v\|^2_{H_0^1} \leq R_0 + R_1.$$

So we get B_0 is the bounded absorbing set.

(3) Since $E_1 := \mathcal{H}_2(\Omega) \times H_0^1(\Omega) \hookrightarrow E_0 := \mathcal{H}_2(\Omega) \times L^2(\Omega)$ is compact embedded, which means that the

bounded set in E_1 is the compact set in E_0, so the semigroup operator S(t) exists a compact global attractor A. Furthermore we can know, the global attractor A is ω-limited set of the absorptive set B_0, $A = \omega(B_0) = \bigcap_{\tau \ge 0} \overline{\bigcup_{t \ge \tau} S(t)B_0}$.

4 The estimates of the upper bounds of Hausdorff and fractal dimensions for the global attractor

In order to obtain an estimate of the upper bounds of Hausdorff and fractal dimensions for the global attractor A of the problems (1.1)-(1.3). we rewrite the problems (1.1)-(1.3):

$$u_{tt} + \varepsilon_1 Au_t + \phi(\|A^{\frac{1}{2}}u\|^2)Au + h(u) = f(x) \quad in \quad \Omega \times R^+, \tag{17}$$

$$u(x,0) = u_0(x); u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{18}$$

$$u(x,t)|_{\partial\Omega} = 0, Au(x,t)|_{\partial\Omega} = 0, \quad x \in \Omega. \tag{19}$$

Let Au=-Δu, $h(u) = \alpha |u_t|^{p-1} u_t + \beta |u|^{q-1} u$, where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, and $\varepsilon_1, \alpha, \beta$ are positive constants. We consider the abstract linearized equations of the above equations as follows:

$$U_t + AU = FU, \tag{20}$$

$$U_0 = \xi, U_t(0) = \zeta. \tag{21}$$

Let $U_0 \in H_0^1(\Omega), U(t)$ is the solution of the problems (4.20)-(4.21). We can prove that the problems (4.20)-(4.21) have a unique solution $U \in L^\infty(0,T,H_0^1(\Omega)), U_t \in L^\infty(0,T,L^2(\Omega))$. The equation (4.20) is the linearized equation by the equation (4.17).

Define the mapping $Ls(t)_{u_0} : Ls(t)_{u_0}\zeta = U(t)$.

Let $u(t) = s(t)u_0$, $\varphi_0 = (u_0, u_1), \overline{\varphi_0} = \varphi_0 + \{\xi, \zeta\} = \{u_0 + \xi, u_1 + \zeta\}$; $\|\varphi_0\|_{E_0} \le R_1, \|\overline{\varphi_0}\|_{E_0} \le R_2, E_0 = V \times H, V := H_0^1(\Omega), H := L^2(\Omega); S(t)\varphi_0 = \varphi(t) = \{u(t), u_t(t)\}, S(t)\overline{\varphi_0} = \{\overline{\varphi(t)}, \overline{\varphi_t}(t)\}$.

Lemma 4.1[11] Assume H is a Hilbert space, E_0 is a compact set of H.

$S(t): E_0 \to H$ is a continuous mapping,satisfy the follow conditions.

1) $S(t)E_0 = E_0, t > 0$;

2) If $S(t)$ is Fre'chet differentiable, it exists a bounded linear differential operator $L(t,\varphi_0) \in C(R^+; L(E_0, E_0)), \forall t > 0$, that is

$$\frac{\|S(t)\overline{\varphi_0} - S(t)\varphi_0 - L(t,\varphi_0)(u,v)\|_{E_0}^2}{\|\{\xi,\zeta\}\|_{E_0}^2} \to 0, \{\xi,\zeta\} \to 0.$$

where $L(t,\varphi_0): \{\xi,\zeta\} \to \{U(t), U_t(t)\}$. $U(t)$ is the solution of problems (4.20)-(4.21).

The proof of lemma 4.1 see ref. [11], is omitted here.

According to Lemma 4.1, we can get the following theorem :

Theorem 4.1 [11,12] Let A is the global attractor that we obtain in section 3. In that case, A has finite Hausdorff dimensions and fractal dimensions in $(H_2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, that is $d_H(A) \le n, d_F(A) \le \frac{8n}{7}$.

Proof. Firstly, we rewrite the equations $(4.17),(4.18)$ into the first order abstract evolution equations in E_0.

Let $\Psi = R_\varepsilon \varphi = \{u, u_t + \varepsilon u\}$, $R_\varepsilon : \{u, u_t\} \to \{u, u_t + \varepsilon u\}$, is an isomorphic mapping. So let A is the global attractor of $\{S(t)\}$, then R_εA is also the global attractor of $\{S_\varepsilon(t)\}$, and they have the same dimensions. $0 < \varepsilon \le \varepsilon_0, \varepsilon_0 = min\{\frac{\varepsilon_1}{4}, \frac{\lambda_1}{2\varepsilon_1}\}$, then Ψ satisfies as follows:

$$\Psi_t + \Lambda_\varepsilon \Psi + \bar{h}(\Psi) = \bar{f}, \tag{22}$$

$$\Psi(0) = \{u_0, u_1 + \varepsilon u_0\}^T. \tag{23}$$

where $\Psi = \{u, u_t + \varepsilon u\}^T, \bar{h}(\Psi) = \{0, h(u)\}^T, \bar{f} = \{0, f(x)\}^T,$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \phi(\|A^{\frac{1}{2}}\|^2)A - \varepsilon_1 \varepsilon A & \varepsilon_1 A \\ & \end{pmatrix} \tag{24}$$

$$\Psi_t := F(\Psi) = \bar{f} - \Lambda_\varepsilon \Psi - \bar{h}(\Psi). \tag{25}$$

$$P_t = F_t(\Psi). \tag{26}$$

$$P_t + \Lambda_\varepsilon P + \bar{h}_t(\Psi)P = 0. \tag{27}$$

where $P = \{U, U_t + \varepsilon U\}^T, \bar{h}_t(\Psi)P = \{0, h_t(u)U\}^T$. The initial condition (4.21) can be written in the following form:

$$P(0) = \omega, \omega = \{\xi, \zeta\} \in E_0. \tag{28}$$

We take $n \in N$, then consider the corresponding n solutions: $(P = P_1, P_2, ..., P_n.\ \ P_j \in E_0)$ of the initial values: $(\omega = \omega_1, \omega_2, ..., \omega_n.\ \ \omega_j \in E_0)$ in the equations (4.26)-(4.28) . So there is

$$|P_1(t) \wedge P_2(t) \wedge ... \wedge P_n(t)|_{\wedge^n E_0} = |\omega_1 \wedge \omega_2 \wedge ... \wedge \omega_n|_{\wedge^n E_0} \cdot e^{\int_0^t TrF_t(S_\varepsilon(\tau)\Psi_0) \cdot Q_n(\tau) d\tau}.$$

from $\psi(\tau) = S_\varepsilon(\tau)\Psi_0$, we get $S_\varepsilon(\tau): \{u_0, v_1 = u_1 + \varepsilon u_0\} \to \{u(\tau), v(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$, $\psi(\tau) = \{u(\tau), v_t(\tau) = u_t(\tau) + \varepsilon u(\tau)\}$.

Here u is the solution of the problems $(4.17)-(4.19)$; $\wedge$ represents the outer product, Tr represents the trace, $Q_n(\tau) = Q_n(\tau, \Psi_0; \omega_1, \omega_2, ..., \omega_n)$ is an orthogonal projection from the space $E_0 = V \times H$ to the subspace spanned by $\{P_1(\tau), P_2(\tau), ..., P_n(\tau)\}$.

For a given time τ, let $\phi_j(\tau) = \{\xi_j(\tau), \zeta_j(\tau)\}, j = 1,2,...,n. \{\phi_j(\tau)\}_{j=1,2,...,n}$ is the standard orthogonal basis of the space $Q_n(\tau)_{E_0} = span[P_1(\tau), P_2(\tau), ..., P_n(\tau)]$.

From the above, we have

$$\begin{aligned} TrF_t(\Psi(\tau)) \cdot Q_n(\tau) &= \sum_{j=1}^{\infty} F_t(\Psi(\tau)) \cdot Q_n(\tau)\phi_j(\tau), \phi_j(\tau))_{E_0}. \\ &= \sum_{j=1}^{n} F_t(\Psi(\tau))\phi_j(\tau), \phi_j(\tau))_{E_0}. \end{aligned} \tag{29}$$

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where $(\cdot,\cdot)_{E_0}$ is the inner product in E_0. Then

$(\{\xi,\zeta\},\{\bar{\xi},\bar{\zeta}\})_{E_0} = (\xi,\bar{\xi}) + (\zeta,\bar{\zeta}); (F_t(\Psi)\phi_j,\phi_j)_{E_0} = -(\Lambda_\varepsilon\phi_j,\phi_j)_{E_0} - (h_t(u)\xi_j,\xi_j)$;

Let $A\zeta_j = \lambda_j\zeta_j$, here λ_j is the eigenvalue of characteristic vector ζ_j about A. Using the method similar to the priori estimates in Lemma 1 and Lemma 2 to obtain:

$$\begin{aligned}(\Lambda_\varepsilon\phi_j,\phi_j)_{E_0} &= \varepsilon \mathrm{P}\xi_j\mathrm{P}^2 + (\phi-\varepsilon_1\varepsilon)(A\zeta_j,\xi_j) - (\xi_j,\zeta_j) + \varepsilon_1(A\zeta_j,\zeta_j). \\ &= \varepsilon \mathrm{P}\xi_j\mathrm{P}^2 + (\phi-\varepsilon_1\varepsilon)\lambda_j(\zeta_j,\xi_j) - (\xi_j,\zeta_j) + \varepsilon_1\lambda_j \mathrm{P}\zeta_j\mathrm{P}^2. \\ &\geq a(\mathrm{P}\xi_j\mathrm{P}^2 + \mathrm{P}\zeta_j\mathrm{P}^2).\end{aligned} \tag{30}$$

where $a := min\{\frac{2\varepsilon - l\lambda_j - 1}{2}, \frac{2\varepsilon_j - l\varepsilon_j - 1}{2}\}$, let $l = \phi - \varepsilon_1\varepsilon$.

Now, suppose that $\{u_0,u_1\} \in \mathrm{A}$, according to theorem 3.3, A is a bounded absorbing set in E_1.

$\Psi(t) = \{u(t), u_t(t) + \varepsilon u(t)\} \in E_1, u(t) \in D(A); D(A) = \{u \in \mathrm{V}, Au \in \mathrm{H}\}$.

Then there is a $s \in [0,1]$ to make the mapping $h_t : D(A) \to \rho(\mathrm{V}_s, \mathrm{H})$. At the same time, there are the following results:

$$\begin{aligned} &R_A = \sup_{\{\xi,\zeta\}\in \mathrm{A}} |A\xi| < \infty; \\ &\sup_{\substack{u\in D(A)\\ |Au|<R_A}} |h_t(u)|_{\rho(v_s,\mathrm{H})} \leq r < \infty. \end{aligned} \tag{31}$$

where $\mathrm{P}h_t(u)\xi_j,\zeta_j\mathrm{P}$ meets: $\mathrm{P}h_t(u)\xi_j,\zeta_j\mathrm{P} \leq r\mathrm{P}\xi_j\mathrm{P}_s\mathrm{P}\zeta_j\mathrm{P}$.

Comprehensive above can be obtained:

$$\begin{aligned}(F_t(\Psi)\phi_j,\phi_j)_{E_0} &\leq -a(\mathrm{P}\xi_j\mathrm{P}^2 + \mathrm{P}\zeta_j\mathrm{P}^2) + r\mathrm{P}\xi_j\mathrm{P}_s\mathrm{P}\zeta_j\mathrm{P}. \\ &\leq -\frac{a}{2}(\mathrm{P}\xi_j\mathrm{P}^2 + \mathrm{P}\zeta_j\mathrm{P}^2) + \frac{r^2}{2a}\mathrm{P}\xi_j\mathrm{P}_s^2.\end{aligned} \tag{32}$$

$\mathrm{P}\xi_j\mathrm{P}^2 + \mathrm{P}\zeta_j\mathrm{P}^2 = \mathrm{P}\phi_j\mathrm{P}_{E_0}^2 = 1$, due to $\{\phi_j(\tau)\}_{j=1,2,\ldots,n}$ is a standard orthogonal basis in $Q_n(\tau)_{E_0}$. So

$$\sum_{j=1}^{n} F_t(\Psi(\tau))\phi_j(\tau),\phi_j(\tau))_{E_0} \leq -\frac{na}{2} + \frac{r^2}{2a}\mathrm{P}\xi_j\mathrm{P}_s^2. \tag{33}$$

Almost to all t, making

$$\sum_{j=1}^{n}\mathrm{P}\xi_j\mathrm{P}_s^2 \leq \sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{34}$$

So

$$TrF_t(\Psi(\tau))\cdot Q_n(\tau) \leq -\frac{na}{2} + \frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{35}$$

Let us assume that $\{u_0,u_1\} \in \mathrm{A}$, is equivalent to $\Psi_0 = \{u_0, u_1 + \varepsilon u_0\} \in R_\varepsilon \mathrm{A}$.

Then

$$q_n(t)=\sup_{\Psi_0\in R_\varepsilon \mathrm{A}}\ \sup_{\substack{\omega\in E_0\\ \|\omega\|_{E_0}\le 1}}\left(\frac{\int_0^t TrF_t(S_\varepsilon(\tau)\Psi_0)\cdot Q_n(\tau)d\tau}{t_0}\right).j=1,2,\ldots,n.$$

$$q_n=\limsup_{t\to\infty} q_n(t). \tag{36}$$

According to (4.35),(4.36), so

$$q_n(t)\le -\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1},$$
$$q_n\le -\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1}. \tag{37}$$

Therefore, the Lyapunov exponent of $\mathrm{A}(or R_\varepsilon \mathrm{A})$ is uniformly bounded.

$$\mu_1+\mu_2+\cdots+\mu_n\le -\frac{na}{2}+\frac{r^2}{2a}\sum_{j=1}^{n}\lambda_j^{s-1}. \tag{38}$$

From (4.36), when $n\to\infty, q_n\to 0$. By the compactness of the operator A^{-1}, we can get further to :When $j\to\infty, \lambda_i\to\infty$. So,when $n\to\infty$,such that

$$\frac{1}{n}\sum_{j=1}^{n}\lambda_j^{s-1}\to 0. \tag{39}$$

From what has been discussed above, it exists $n\ge 1$,a,r are constants,

then

$$\frac{1}{n}\sum_{j=1}^{n}\lambda_j^{s-1}\le \frac{a^2}{8r^2}. \tag{40}$$

$$q_n\le -\frac{na}{2}\left(1-\frac{r^2}{a^2}\sum_{j=1}^{n}\lambda_j^{s-1}\right)\le -\frac{7na}{16}. \tag{41}$$

$$(q_j)_+\le \frac{r^2}{2a}\sum_{i=1}^{j}\lambda_i^{s-1}\le \frac{r^2}{2a}\sum_{i=1}^{j}\lambda_i^{s-1}\le \frac{na}{16},\ j=1,2,\ldots,n. \tag{42}$$

So we finally obtained the following conclusions:

$$\max_{1\le j\le n-1}\frac{(q_j)_+}{|q_n|}\le \frac{1}{7}. \tag{43}$$

According to the reference [11,12], we immediately to the Hausdorff dimension and fractal dimension are respectively $d_H(\mathrm{A})\le n, d_F(\mathrm{A})\le \frac{8n}{7}$.

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