About some robustness and complexity properties of G-graphs networks

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Abstract

Given a finite group G and a set $S \subset G$, we consider the different cosets of each cyclic group $\langle s \rangle$ with $s \in S$. Then the *G-graph* $\Phi(G, S)$ associated with G and S can be defined as the intersection graph of all these cosets. These graphs were introduced in [4] as an alternative to Cayley graphs: they still have strong regular properties but a more flexible structure. We investigate here some of their *robustness* properties (connectivity and vertex/edge-transitivity) recognized as important issues in the domain of network design. In particular, we exhibit some cases where G-graphs are optimally connected, i.e. their edge and vertex-connectivity are both equal to the minimum degree. Our main result concerns the case of a G-graph associated with an abelian group and its canonical base $\widetilde{S}$, which is shown to be optimally connected. We also provide a combinatorial characterization for this class as clique graphs of Cartesian products of complete graphs and we show that it can be recognized in polynomial time. These results motivate future researches in two main directions: revealing new classes of optimally connected G-graphs and investigating the complexity of their recognition.

Key words: Graphs and groups, G-graphs, orbit graphs, optimal connectivity, vertex and edge-transitivity, robustness, network, Hamming graphs, clique graphs

1 Introduction

Designers of communication infrastructure must assume that networks will continue to function despite failures (up to some level of damage); in other terms, they must assume that the communication network has the highest possible degree of robustness. Robustness of a network is given by the existence of alternate paths, enabling the communication even if the network is damaged [11]. Then there are two aspects in studying a network robustness, each one corresponding to the two main threats: inside threats (link failures) and outside threats (node destruction). In the topological model of a communications network as an (undirected) graph, the two aspects of network robustness correspond to two types of connectivity of the associated graph: vertex-connectivity and edge-connectivity.

Regarding network robustness, a particular interesting property is given by *vertex (resp. edge)-transitivity* which represents that every node (resp. link) is similar to others. In this kind of network, any damage produced does not depend on its location. Graphs generated from finite groups generally satisfy such property as they have a high regularity due to their underlying algebraic structure [10]. The most famous examples are Cayley graphs which have highly symmetric properties. In [11] it is shown that they are vertex-transitive and thus regular and optimally connected (notions defined later). Still not all networks are regular and symmetric. For example, they can also be semi-symmetric (edge-transitive but not vertex-transitive). To overcome this limit, another family of graphs constructed from groups was introduced and first studied in [4], [5] and [6]. They are called *G-graphs* (or also Orbit graphs [22]). These graphs still have high-regular and symmetric properties but with more flexibility than Cayley graphs. In particular, they can be semi-symmetric. With the help of G-graphs, many symmetric and semi-symmetric graphs have been computed in [5] up to 800 vertices.

In this paper, [2] we motivate further studies on this class of graphs by investigating some of its connectivity properties and revealing good robustness properties in some particular cases. For possible applications complexity results are also investigated. A natural question is the complexity of recognizing these graphs. We give a first answer in this direction by exhibiting a polynomial case.

The paper is organized as follows: in Section 2, we first give the main definitions

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needed from group and graph theories and present the class of G-graphs with some of its basic properties. In Section 3, we discuss some important notions for the study of network architectures: vertex-transitivity, edge-transitivity and optimal connectivity. We present some optimally connected families of G-graphs, in particular when the group has a symmetric presentation and in the bipartite case. The main result of the section is that G-graphs associated with abelian (commutative) finite groups (called *canonical abelian G-graphs* - defined later) are optimally connected. Finally, in Section 4 we investigate the structure of canonical abelian G-graphs and show that they are clique graphs (intersection graph of maximal cliques) of Cartesian products of complete graphs (also called *Generalized Hamming Graphs*). For these purposes, we revisit some results of [7] and [22] and precise the links between some G-graphs and some Cayley graphs. We deduce a polynomial time algorithm for the recognition of this class.

2 Definitions and preliminary results

In this section we give the main definitions and notations from group and graph theories needed in the sequel. For further definitions not given here, the reader is referred to [20] and [24].

2.1 Finite groups

Groups and Cyclic Groups: In this paper, we only consider finite groups, usually denoted by G. We adopt a multiplicative notation: "$\cdot$" denoting a canonical operation (sometimes omitted in expressions, $g \cdot h$ being denoted by gh). g^k denotes $\underbrace{g \cdot \ldots \cdot g}_{k}$ and g^{-1} the inverse of $g \in G$. e is the neutral element.

For every g in G, the order of g, denoted by $o(g)$, is the smallest positive integer k such that $g^k = e$. The set $\{e, g, g^2, \ldots, g^{o(g)-1}\}$ forms a subgroup of G, called the *cyclic group* generated by g and denoted by $\langle g \rangle$. S_n will denote the permutation group on n elements.

Generating Set: A set $S \subset G$ is a *generating set* of a group G, if every element g in G can be written as a finite combination of the elements of S, $g = s_1^{a_1} s_2^{a_2} \cdots s_p^{a_p}$ with $s_i \in S$ not necessary distinct and $a_1, \ldots, a_p \in \mathbb{Z}$. Here we can assume that $s_i \neq s_{i+1}, i = 1, \ldots, p-1$. If G is finite we can assume that $a_1, \ldots, a_p \in \mathbb{N}$. Such an expression is called a word on S.

Group Presentation: Another way for describing a group G is by defining a generating set S and a set of relations $R = \{r_1, r_2, \ldots, r_m\}$, called *relators.* $r_1, \ldots, r_m$ are words on S, each equal to e. Moreover every word whose value in G is equal to e can be written as $g_1 r_1^{b_1} g_1^{-1} g_2 r_2^{b_2} g_2^{-1} \cdots g_z r_z^{b_z} g_z^{-1}$ with $g_i \in G, r_i \in R, b_i \in \mathbb{Z}, i = 1, \ldots, z$ and $r_i \neq r_{i+1}, i = 1, \ldots, z-1$. This kind of description is called a *presentation of* G, denoted by $G = \langle S|R \rangle$. Every group has a presentation and it is not unique (see [20] for a formal definition and more details about groups presentations). For instance, the cyclic group of order n has $C_n = \langle a|a^n \rangle$ as one of its presentations.

Symmetric Presentation: Let G be a group with the presentation $\langle S|R \rangle$ and $|S| = n$. For a permutation $\pi \in S_n$ and a relator $r = s_1^{a_1} s_2^{a_2} \cdots s_p^{a_p}$ with $s_i \in S$, we set $\pi(r) = s_{\pi(1)}^{a_1} s_{\pi(2)}^{a_2} \cdots s_{\pi(p)}^{a_p}$. We say that R is *invariant under* π if for every $r \in R$ we have $\pi(r) \in R$. If R is invariant under the action of the full symmetric permutation group S_n, then the presentation $\langle S|R \rangle$ is called a *symmetric presentation* [1]. A group with a symmetric presentation $\langle S|R \rangle$ is called *symmetrically generated* by S. One example is the Cartesian product of cyclic groups of the same order $G = \langle s_1 \rangle \times \cdots \times \langle s_k \rangle$ with $o(s_i) = \ell$, $i = 1, \ldots, k$. Then $G = \langle s_1, \ldots, s_k | s_1^\ell, \ldots, s_k^\ell \rangle$. Also the Alternating group A_4 of permutations on $\{1, 2, 3, 4\}$ with signature 1 has the symmetric presentation $A_4 = \langle a, b | a^3, b^3, (ab)^2, (ba)^2 \rangle$. Here, a and b correspond for instance to the 3-cycles $a = (1, 2, 3)$ and $b = (2, 3, 4)$. Other examples can be found in [19] and [23].

Independence: A set $S \subset G$ is *independent*, if for any $\{s_1, \ldots, s_p\} \subseteq S$, $a_i \in \mathbb{Z}, i = 1, \ldots, p$ such that $s_1^{a_1} \cdots s_p^{a_p} = e$ we have $s_1^{a_1} = \cdots = s_p^{a_p} = e$, which is equivalent to $a_i \equiv 0 \bmod o(s_i), i = 1, \ldots, p$. Elements $s_1, ..., s_k$ are *pairwise independent* in G if any pair $\{s_i, s_j\}$ is independent. Similarly we can define independence by triples. Independence implies independence by triples that implies pairwise independence. Note that in this definition elements $s_1, \ldots, s_p$ are supposed to be distinct. In the abelian (commutative) case this notion corresponds to the usual linear independence [20].

Group Isomorphism: Given two groups G and G', we denote by $G \simeq G'$ that G is isomorphic to G'. $Aut(G)$ denotes the group of automorphisms of G under composition law.

Finite Abelian Groups Fundamental Theorem: The fundamental theorem of finite abelian groups [20] states that in a finite abelian group G, we have $G \simeq \langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_k \rangle$, where $S = \{s_1, \ldots, s_k\}$ is an independent set of elements of G. An important consequence is that S induces for every element $g \in G$ a unique decomposition $g = s_1^{a_1} s_2^{a_2} \cdots s_k^{a_k}$ with $a_i \in \mathbb{Z}/o(g_i)\mathbb{Z}$, $i = 1, \ldots, k)$. Hence the set

$$\widetilde{S} = \{(\underbrace{s_1, e, \ldots, e}_{k}), (\underbrace{e, s_2, e \ldots, e}_{k}), \ldots, (\underbrace{e, \ldots, e, s_k}_{k})\}$$

in $\langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_k \rangle$ is an independent generating set, called the *canonical base*.

Subgroups and Cosets: We will use the notation $H \leq G$ if H is a subgroup of G. If H is a fixed subgroup of a group G and $x \in G$, the subset $Hx = \{hx | h \in H\} \subset G$ is called *right coset of H containing x.* The key property of cosets is that, for any $x, y \in G$, either $Hx = Hy$ or $Hx \cap Hy = \emptyset$. Thus the collection of all cosets of H yields a partition of G.

2.2 Graphs

We limit ourselves to simple graphs (undirected, with no multi-edge and no loop), denoted by $\Gamma = (V, E)$. $N(v)$ will denote the neighborhood of the vertex v and $\delta(\Gamma)$ the minimum vertex degree in Γ.

Connectivity: For a connected graph Γ, the *vertex-connectivity* $\kappa(\Gamma)$ is the smallest number of vertices whose removal induces a disconnected or single-vertex graph. Similarly *edge-connectivity* $\lambda(\Gamma)$ is the smallest number of edges whose removal induces a disconnected graph. Γ is *k-(vertex)-connected* if $\kappa(\Gamma) \geq k$ and *k-edge-connected* if $\lambda(\Gamma) \geq k$.

Disjoint Paths: We remind that two paths in a graph are *internally disjoint* or *internally vertex-disjoint* if they have no common internal vertex. Two paths are called *edge-disjoint* if they have no common edge. It is well known *(Menger's Theorem)* [25] that a graph is k-(vertex)-connected (resp. k-edge-connected) if and only if any two vertices u, v are linked by k internally vertex (resp. edge)-disjoint paths.

Theorem 1 *[25] For any graph Γ we have*

$$\kappa(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma)$$

Optimal Connectivity: Consequently $\delta(\Gamma)$ corresponds to the best possible (highest) connectivity. A graph Γ satisfying $\lambda(\Gamma) = \delta(\Gamma)$ is called *optimally edge-connected* and a graph with $\kappa(\Gamma) = \delta(\Gamma)$ is called *optimally connected.*

Graph Isomorphism: Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs, we denote by $\Gamma_1 \simeq \Gamma_2$ that Γ_1 and Γ_2 are isomorphic and by $Aut(\Gamma)$ the group of automorphisms of Γ under composition law.

Transitivity: A graph $\Gamma = (V, E)$ is *vertex-transitive* if for any two vertices $u, v \in V$ there exists an automorphism $h \in Aut(\Gamma)$ such that $h(u) = v$. It is *edge-transitive* if for any two edges $uv, u'v' \in E$, there exists an automorphism $h \in Aut(\Gamma)$ such that $h(u)h(v) = u'v'$. Using usual group terminology [20] vertex (resp. edge)-transitivity means that the group of automorphisms $Aut(\Gamma)$ acts transitively upon vertices (resp. edges).

Orbit in a Graph: For a graph $\Gamma = (V, E)$, let $H \leq Aut(\Gamma)$ be a subgroup of automorphisms. We define an equivalence relation on V regarding H as follows: for any u and v in V, u is in relation with v if and only if there exists $h \in H$ such as $h(u) = v$. An *orbit* is an equivalence class. The *orbit partition* of V regarding H is the partition of V associated with this relation.

Cartesian Product of Graphs: The Cartesian product of k graphs $\Gamma_i = (V_i, E_i)$, $i = 1, \dots, k$ is defined by: $\Gamma_1 \times \dots \times \Gamma_k = (V_1 \times \dots \times V_k, E)$ where $(v_1, \dots, v_k)$ $(v'_1, \dots, v'_k) \in E \Leftrightarrow [\exists i \in \{1, \dots, k\}, (v_i v'_i \in E_i) \wedge (v_j = v'_j, \forall j \neq i)]$

Generalized Hamming Graphs: A complete graph (or clique) on p vertices is denoted by K_p. We will call *Generalized Hamming Graph* the Cartesian product of complete graphs $K_{n_1} \times \dots \times K_{n_k}$. We can equivalently define this graph by its edge set using the *Hamming distance* d_H: $(v_1, \dots, v_k)$ and $(v'_1, \dots, v'_k)$ are linked if $d_H((v_1, \dots, v_k), (v'_1, \dots, v'_k)) = 1 = |\{i, u_i \neq v_i\}| = 1$. In some references, in particular in [14], these graphs are called Hamming Graphs. However we chose to keep the terminology *Generalized Hamming* since in most cases *Hamming Graphs* correspond to the case $n_1 = \dots = n_k$.

Cayley Graphs: For a group G and $S \subseteq G$ a set we denote $C(G, S)$ the Cayley graph associated with G and S.

2.3 A short introduction to G-graphs

G-graphs have been introduced in [4] as an alternative way to associate a graph to a group.

Definition 1 *Consider G a finite group. Let $S = \{s_1, s_2, \dots, s_k\}$ be a nonempty subset of G. The* (right) G-graph $\Phi(G, S)$*, is the intersection graph of the right cosets of cyclic groups $\langle s_i \rangle$ for all $i \in \{1, 2, \dots, k\}$.*

In other way:

(1) The vertices of $\Phi(G, S)$ are $V = \bigcup_{s_i \in S} V_{s_i}$ where $V_{s_i} = \{\langle s_i \rangle x | x \in G\}$.
(2) For $\langle s_i \rangle x, \langle s_j \rangle y \in V$ $(i \neq j)$, there exists an edge between $\langle s_i \rangle x$ and $\langle s_j \rangle y$ if and only if $|\langle s_i \rangle x \cap \langle s_j \rangle y| \geq 1$.

Left G-graphs are similarly defined. As left and right G-graphs are isomorphic [21], we will only consider right G-graphs and call it simply G-graphs.

Remark 2 *In [4], G-graphs are defined as multi-graphs with one edge between*

$\langle s_i\rangle x$ and $\langle s_j\rangle y$ for every element of $\langle s_i\rangle x \cap \langle s_j\rangle y$. The definition we choose replaces a multi-edge by a single edge. This gives a natural labeling of edges by associating the edge $\langle s_i\rangle x\langle s_j\rangle y$ with $\langle s_i\rangle x \cap \langle s_j\rangle y$. It also gives a natural weight system by associating the edge $\langle s\rangle x\langle t\rangle y$ with the weight $|\langle s\rangle x \cap \langle t\rangle y|$. Thus the degree of the vertex $\langle s_i\rangle x$ in the multi-graph is equal to the weighted degree $d_w(\langle s_i\rangle x)$, defined as the sum of weights of edges incident to $\langle s_i\rangle x$.

Let us give a few examples illustrating the definition.

For $G = \mathbb{Z}/12\mathbb{Z}$ and $S = \{1, 4, 6\}$, the associated G-graph is given in Figure 1.

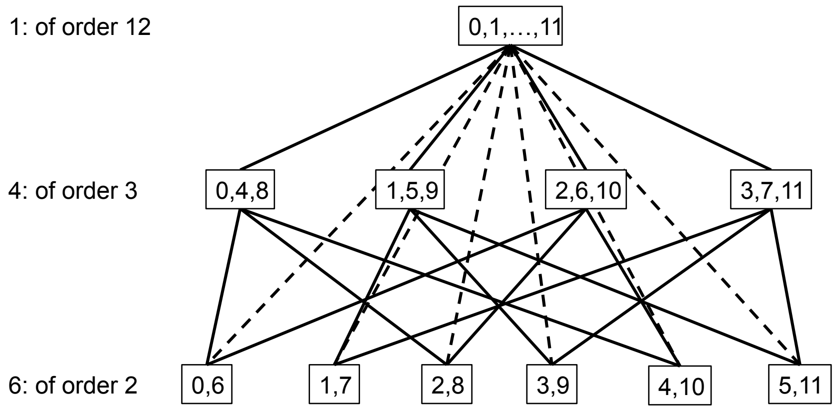

Fig. 1. The G-graph $\Phi(\mathbb{Z}/12\mathbb{Z}, \{1, 4, 6\})$.

A complete bipartite graph $K_{p,q}$ with parts of size p and q can be seen as a G-graph by considering $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ and $S = \{(1, 0), (0, 1)\}$.

Consider now A_4 with the symmetric presentation $\langle a, b|a^3, b^3, (ab)^2, (ba)^2\rangle$ and $S = \{a, b\}$. The associated G-graph is a cube [5]. Similarly a hypercube is a G-graph and more generally, in [8] it is shown that Hamming graphs can be identified as G-graphs.

Remark 3 *If s_i and s_j are independent, we have $|\langle s_i\rangle x \cap \langle s_j\rangle y| \le 1$. So, if elements of S are pairwise independent, then the related G-graph $\Phi(G, S)$, defined as a multi-graph, is a simple graph. For example in Fig.1 the edges connecting the top vertices to medium and bottom rows have weight* 3 *and* 2 *respectively, while edges between medium and bottom row have weight* 1.

Lemma 4 *[6] $\Phi(G, S)$ is connected if and only if S is a generating set of G.*

Lemma 5 *[6,21] $\Phi(G, S)$ is a $|S|$-partite graph with V_{s_i} $s_i \in S$ as parts (independent sets). Every vertex of V_{s_i}has the weighted degree $(|S| - 1)o(s_i)$.*

The partition $V = \bigcup_{i=1}^{k} V_{s_i}$ is called the *canonical partition* of the G-graph. V_{s_i} corresponds to the collection of all cosets associated with the cyclic groups $\langle s_i \rangle x$ with $x \in G$. These cosets induce a partition of the group G (see Section 2) and consequently every element of G is associated with exactly one vertex in V_{s_i} for every $s_i \in S$. Moreover V_{s_i} is a stable set and for every element $x \in G$, the set of all cosets containing x induces a clique of size k in the graph. Consequently $\Phi(G, S)$ is of chromatic number $k = |S|$ and the canonical partition is an optimal coloring.

Remark 6 *The notion of* equitable partition *of a graph is introduced in [13]. Given a graph $\Gamma = (V, E)$ a partition P of its vertex set $P = \bigcup_{1 \leq i \leq r} V_i$ is equitable if for all $1 \leq i, j \leq r$ (not necessarily distinct) there exists b_{ij}, such that each vertex $v \in V_i$ has exactly b_{ij} neighbors in V_j, regardless of the choice of v. Such a partition describes a nice regularity of the graph, especially when the number of parts is limited. G-graphs give a good example of such partition. Indeed, Lemma 5 states that in the G-graph, seen as a multi-graph (see Remark 2), the canonical partition is an equitable partition into stable sets. However it is easy to see that for all $x, y \in G$ and $\forall i, j \in \{1, \ldots, k\}, i \neq j, |\langle s_i \rangle x \cap \langle s_j \rangle y| = b_{ij}$. As a consequence, the canonical partition into stable sets is also an equitable partition in the G-graph, seen as a simple graph.*

Given several partitions of a fixed finite set, the related *partition intersection graph* [18] is defined as the intersection graph of all parts of these partitions. Each partition is associated with a stable set and each element with a clique intersecting all these stable sets. G-graphs correspond to the particular case where the considered partitions are induced by cosets of cyclic groups in a given finite group G.

Let $\Phi(G, S) = (V, E)$ be a G-graph. To any $g \in G$ we associate the mapping $\delta_g : V \longrightarrow V$, defined by $\delta_g(\langle s \rangle x) = \langle s \rangle xg$. Using this notion we have:

Theorem 7 *[21] Let $\Phi(G, S) = (V, E)$ be a G-graph.*

(1) $\delta_g \in Aut(\Phi(G, S))$.
(2) $\delta_G = \{\delta_g, g \in G\}$ forms a group under the composition law, and V_{s_i} for an s_i fixed is a fixed orbit regarding to δ_G.
(3) Let s_i and s_j be two distinct elements of S. Then, for every $u, u' \in V_{s_i}$, $v, v' \in V_{s_j}$ with $uv, u'v' \in E$, there exists $g \in G$ such that $\delta_g(u) = u'$ and $\delta_g(v) = v'$.

In this paper we will assume that S is a generating set, thus dealing only with connected graphs. In some cases we will also restrict ourselves to the case where S is independent, independent by triples or by pairs. We also study the G-graph $\Phi(G, \widetilde{S})$, called *canonical abelian G-graph*, issued from an abelian group

$G = \langle s_1 \rangle \times \cdots \times \langle s_k \rangle$, with $k \geq 1$ and $o(s_i) \geq 2$ for every $i \in \{1, 2, \ldots, k\}$ and its canonical base

$$\widetilde{S} = \{(\underbrace{s_1, e, \ldots, e}_{k}), (\underbrace{e, s_2, e \ldots, e}_{k}), \ldots, (\underbrace{e, \ldots, e, s_k}_{k})\}$$

3 G-graphs Network Robustness

As mentioned in Section 1, vertex and edge-connectivity of graphs are two key aspects of network robustness; the best case corresponds to optimally connected networks for which $\kappa(\Gamma) = \delta(\Gamma)$. We also have mentioned vertex and edge-transitivity as desirable properties for networks. These notions are also interesting for their close link with connectivity illustrated by the following result.

Theorem 8 *[11] A connected vertex-transitive graph is* optimally edge-connected*. Moreover if it is also edge-transitive, then it is* optimally connected*.*

In this section we first investigate some transitivity and connectivity properties of G-graphs (Subsection 3.1) and several optimally connected cases (Subsection 3.2) with a particular focus on the abelian case. These first results motivate to further study the class of G-graphs and to obtain more general optimal connected classes of graphs that can be easily computed from a group.

3.1 Vertex and edge-transitivity

We exhibit a close link between vertex and edge-transitivity of G-graphs and the group presentation. The following lemma will be used for proving Theorem 10.

Lemma 9 *Let $\langle S|R \rangle$ be a presentation of group G and π a permutation on the set S ($k = |S|$). If R is invariant under π, then π can be associated with a group automorphism $f_\pi \in Aut(G)$, with $f_\pi(s_i) = s_{\pi(i)}$ for every $s_i \in S$.*

PROOF.

It is shown in [15] that given a presentation $G = \langle S|R \rangle$ of a group G, a group H and a mapping $\theta : S \longrightarrow H$, θ extends to a homomorphism $\tilde{\theta} : G \longrightarrow H$ if and only if for all $r \in R$ the result of substituting in r every $s \in S$ with $\theta(s)$ yields the identity of H.

We directly apply this theorem to π and $G = H$ since R is invariant by π and we get an extended homomorphism f_π. Note that, since G is finite, R is also invariant by π^{-1}. Hence we can similarly define $f_{\pi^{-1}}$. We have $f_\pi \circ f_{\pi^{-1}} = f_{\pi^{-1}} \circ f_\pi = id$, where id denotes the identity automorphism. So, f_π and $f_{\pi^{-1}}$ are automorphisms of G. $\square$

Under the hypothesis of Lemma 9 we associate to π a graph automorphism $\tau_\pi \in Aut(\Phi(G,S))$ defined by:

$$\tau_\pi(\langle s_i \rangle x) = \langle f_\pi(s_i) \rangle f_\pi(x) = \langle s_{\pi(i)} \rangle f_\pi(x) \tag{1}$$

We similarly define $\tau_{\pi^{-1}}$. Since f_π and $f_{\pi^{-1}}$ are group automorphisms of G, we have $\tau_\pi, \tau_{\pi^{-1}} \in Aut(\Phi(G,S))$ and are inverse one of the other.

We are now ready to state the main result of this section:

Theorem 10 *If G is a finite group with symmetric presentation $\langle S|R \rangle$, then $\Phi(G,S)$ is vertex-transitive and edge-transitive.*

PROOF.

Vertex-transitivity:

Let us consider two vertices $u, v \in \bigcup_{i=1}^{k} V_{s_i}$. We prove that there is an (graph) automorphism $h \in Aut(\Phi(G,S))$ such that $h(u) = v$.

Theorem 7-(1) allows to conclude that, given two vertices $u = \langle s_i \rangle x, v = \langle s_i \rangle y$ in V_{s_i} for i fixed, $\delta_{x^{-1}y}$ is the required automorphism. Suppose now that $u \in V_{s_i}$ and $v \in V_{s_j}$, $i \neq j$ and consider the transposition $\pi = (i,j)$. By Lemma 9, there exists a group automorphism $f_\pi \in Aut(G)$ and a graph automorphism $\tau_\pi \in Aut(\Phi(G,S))$ such that $f_\pi(s_i) = s_{\pi(i)} = s_j$ and $\tau_\pi(u) = v' \in V_{s_j}$. From above, there exists $\delta_{g'} \in Aut(\Phi(G,S))$ such that $\delta_{g'}(v') = v$. By composition we obtain $\delta_{g'}(\tau_\pi(u)) = v$. Hence the automorphism $h = \delta_{g'} \circ \tau_\pi$ satisfies $h(u) = v$. This concludes vertex-transitivity.

Edge-transitivity:
Considering two edges uv and $u'v'$ of $\Phi(G,S)$ we show that there is a (graph) automorphism $h \in Aut(\Phi(G,S))$ such that $h(uv) = (u'v')$.

We distinguish two cases:

Case 1. Let $u, u' \in V_{s_i}$ and $v, v' \in V_{s_j}$, for i, j fixed, the result immediately follows from Theorem 7-(3).

Case 2. In the general case, suppose we have i_1, i_2, i_3, i_4 with $i_1 \neq i_2$, and let $u \in V_{s_{i_1}}$, $u' \in V_{s_{i_2}}$, $v \in V_{s_{i_3}}$, $v' \in V_{s_{i_4}}$. Let π be a permutation on the set S such that $\pi(s_{i_1}) = s_{i_2}$ and $\pi(s_{i_3}) = s_{i_4}$. Hence, as the hypothesis of Lemma 9 is verified, we can consider the graph automorphism $\tau_\pi \in Aut(\Phi(G,S))$ given by Expression 1. Then we have $\tau_\pi(u) = u'' \in V_{s_{i_2}}$ and $\tau_\pi(v) = v'' \in V_{s_{i_4}}$ with $u''v'' \in E$. From *Case 1* there exists an automorphism $\delta_{g''}$ such that $\delta_{g''}(u'') = u'$ and $\delta_{g''}(v'') = v'$. By composition the automorphism $h = \delta_{g''} \circ \tau_\pi$ satisfies $h(u) = u'$ and $h(v) = v'$, which concludes the proof. $\Box$

The two following remarks are immediately deduced from the proof:

Remark 11 *If R is invariant under the transposition $\pi = (i,j)$, then for all $u \in V_{s_i}$ and $v \in V_{s_j}$ there is an automorphism $h_\pi \in Aut(\Phi(G,S))$ such that $h_\pi(u) = v$.*

Remark 12 *The proof of Theorem 10 reveals in fact that, for any two edges uv and $u'v'$, there is an automorphism h with $h \in Aut(\Phi(G,S))$ such that $h(u) = u'$ and $h(v) = v'$. A graph with this property is sometimes called symmetric.*

3.2 Optimal connectivity

Using the previous results, we exhibit some interesting connectivity properties for G-graphs.

Theorem 13 *If G is a finite group with symmetric presentation $\langle S|R\rangle$, then $\Phi(G,S)$ is optimally connected.*

PROOF. By Theorem 10, $\Phi(G,S)$ is vertex-transitive and edge-transitive. Also, by Lemma 4, $\Phi(G,S)$ is connected. Using Theorem 8, the result follows. $\Box$

Another interesting case is the bipartite case, useful in particular for interconnecting two separated networks. For this case, a sufficient condition is already known in terms of orbits:

Theorem 14 *[17] If $\Gamma = (V,E)$ is a connected bipartite graph with two orbits regarding $Aut(\Gamma)$, then $\kappa(\Gamma) = \delta(\Gamma)$.*

Lemma 15 *Let G be a finite group and $S = \{s_1, s_2\} \subseteq G$. Then $\Phi(G,S)$ is edge-transitive.*

PROOF. The result follows from Theorem 7. $\square$

Theorem 16 *Any connected bipartite G-graph is optimally connected.*

PROOF. Let $\Phi(G, S) = (V, E)$ where $S = \{s_1, s_2\}$ is a generating set and $V = V_{s_1} \cup V_{s_2}$. $\Phi(G, S)$ is connected (Lemma 4), bipartite (Lemma 5) and edge-transitive (Lemma 15). From Theorem 7-(2), $\Phi(G, S)$ has at most two orbits regarding $Aut(\Phi(G, S))$. If $\Phi(G, S)$ has a single orbit regarding $Aut(\Phi(G, S))$, then $\Phi(G, S)$ is vertex-transitive and $\Phi(G, S)$ is optimally connected by Theorem 8. If $\Phi(G, S)$ has two orbits regarding $Aut(\Phi(G, S))$, then $\Phi(G, S)$ is optimally connected (Theorem 14). $\square$

The sequel of this subsection is dedicated to canonical abelian G-graphs: let $G = \langle s_1 \rangle \times \langle s_2 \rangle \times \ldots \times \langle s_k \rangle$, with $k \geq 1$ and $o(s_i) \geq 2$ for every $i \in \{1, 2, \ldots, k\}$ and let $\widetilde{S} \subseteq G$ denote the canonical base.

The order of an element $(\underbrace{e, \ldots, e, s_i, e, \ldots, e}_{k}) \in \widetilde{S}$ is $o(s_i)$, for $1 \leq i \leq k$. Since $\widetilde{S}$ is independent, $\Phi(G, \widetilde{S})$ is a simple graph.

The following intermediary results are useful for proving Theorem 19.

Lemma 17 (Expansion Lemma) [24] *If Γ is a k-connected graph and Γ' is obtained from Γ by adding a new vertex y, with at least k neighbors in Γ, then Γ' is k-connected.*

We immediately deduce:

Lemma 18 *Let Γ be k-connected graph and $S, T \subseteq V(\Gamma)$ two subsets of vertices such that $|S| = |T| = p \leq k$. Then there exist p disjoint paths from S to T.*

PROOF. Γ is k-connected and since $p \leq k$, Γ is also p-connected. Let Γ' be the graph obtained from Γ by adding a vertex x adjacent to all vertices of S and a vertex y adjacent to all vertices of T. Using Expansion Lemma 17 for Γ', the result follows. $\square$

Theorem 19 *Canonical abelian G-graphs $\Phi(G, \widetilde{S})$ are optimally connected.*

PROOF.

We can assume w.l.o.g. that $2 \leq o(s_1) \leq o(s_2) \leq \cdots \leq o(s_k)$, since the product of groups is commutative. Denote $p = o(s_1) \geq 2$ and $q = o(s_k) \geq p$. Based on the results before we consider two cases:

Case 1. $p = q$, hence, $o(s_1) = o(s_2) = \cdots = o(s_k) = p$. Then G has a symmetric presentation. By Theorem 13, $\Phi(G, S)$ is optimally connected.

Case 2. $p < q$. As the elements of $\widetilde{S}$ are pairwise independent, we can easily see from Lemma 5 that the minimum degree of $\Phi(G, S)$ is $p(k-1)$ and the degree of any vertex $\langle s_i \rangle x$ is $o(s_i)(k-1)$.

We prove by induction that $\kappa(\Phi(G, \widetilde{S})) = p(k-1)$, which corresponds to optimal connectivity.

For $k = 1$, $\Phi(G, \widetilde{S})$ is a graph with one vertex. For $k = 2$, Theorem 16 allows to conclude.

Assume that the affirmation is true for a $k \geq 2$. Consider the group $G = \langle s_1 \rangle \times \cdots \times \langle s_{k+1} \rangle$ and its canonical base $\widetilde{S} = \{(\underbrace{s_1, e, \ldots, e}_{k+1}), \ldots, (\underbrace{e, \ldots, e, s_{k+1}}_{k+1})\}$. We prove that $\Phi(G, \widetilde{S})$ is pk-connected.

Let $\overline{G} = \langle s_1 \rangle \times \langle s_2 \rangle \times \cdots \times \langle s_k \rangle$. We denote by $\overline{x}$ an element of $\overline{G}$ ($\overline{e}$ being the neutral element) and $\widetilde{\overline{S}} = \{\overline{s_1}, \ldots, \overline{s_k}\}$ its canonical base. We have $G \simeq \overline{G} \times \langle s_{k+1} \rangle$. Hence, any element of G can be written as $(\overline{x}, s_{k+1}^t)$, with $0 \leq t \leq q-1$, for example, $(\underbrace{e, ..., e, s_i, e, \ldots, e}_{k+1}) = (\overline{s_i}, e)$.

Note that, for any fixed t, $0 \leq t \leq q-1$, the subgraph Φ_t of $\Phi(G, \widetilde{S})$ induced by the set of vertices $V_t = \{\langle(\overline{s_i}, e)\rangle(\overline{x}, s_{k+1}^t), 1 \leq i \leq k, \overline{x} \in \overline{G}\}$ is isomorphic to $\Phi(\overline{G}, \widetilde{\overline{S}})$. Note also that for $t \neq u$, Φ_t and Φ_u are not linked by any edge in $\Phi(G, \widetilde{S})$, since all elements of G belonging to cosets in V_t have s_{k+1}^t as last component. More formally, $\Phi(G, \widetilde{S})$ has the following structure (see Figure 2):

- q vertex-disjoint copies of $\Phi(\overline{G}, \widetilde{\overline{S}})$, Φ_t, $0 \leq t \leq q-1$, where $V(\Phi_t) = \bigcup_{i=1}^{k} V_{s_i}(\Phi_t)$ with $V_{s_i}(\Phi_t) = \{\langle(\overline{s_i}, e)\rangle(\overline{x}, s_{k+1}^t) | \overline{x} \in \overline{G}\}$
- a set of vertices $V_{s_{k+1}} = \{\langle(\overline{e}, s_{k+1})\rangle(\overline{x}, e) | \overline{x} \in \overline{G}\}$ which is a stable set in $\Phi(G, \widetilde{S})$.
- each vertex $\langle(\overline{e}, s_{k+1})\rangle(\overline{x}, e) \in V_{s_{k+1}}$ is adjacent to a clique of k vertices in each copy Φ_t: $\langle(\overline{s_i}, e)\rangle(\overline{x}, s_{k+1}^t) \in V_{s_i}(\Phi_t)$, with $1 \leq i \leq k$.

Note also that $\forall a \in V_{s_i}(\Phi_t)$, $|N(a) \cap V_{s_{k+1}}| = o(s_i)$.

By induction hypothesis, each subgraph Φ_t is $p(k-1)$-connected. We prove below that for any pair a, b of vertices in $\Phi(G, \widetilde{S})$, there exist pk internally

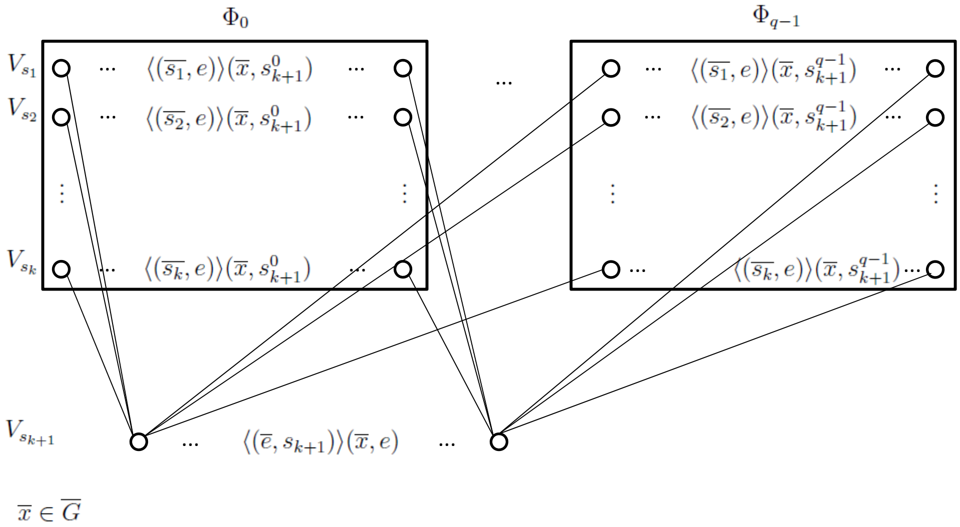

Fig. 2. The structure of $\Phi(G,\widetilde{S})$

disjoint paths between a and b.

We consider four cases:

Case 2.1. a and b are in the same subgraph Φ_t with $t \in \{0,\dots,q-1\}$. Assume w.l.o.g. that $t=0$. Hence, $a = \langle(\overline{s_i},e)\rangle(\overline{x},e)$, $b = \langle(\overline{s_j},e)\rangle(\overline{y},e)$, with $\overline{x},\overline{y} \in \overline{G}$ and $1 \le i,j \le k$. Since Φ_0 is $p(k-1)$-connected, there are $p(k-1)$ internally disjoint paths from a to b in Φ_0.

Consider now $A_{k+1} = \{\langle(\overline{e},s_{k+1})\rangle(\overline{s_i}^h\overline{x},e) | 0 \le h \le p-1\} \subseteq N(a) \cap V_{s_{k+1}}$ a selection of p neighbors of a in $V_{s_{k+1}}$ (recall that $p \le o(s_i)$). We similarly consider $B_{k+1} = \{((\overline{e},s_{k+1}))(\overline{s_j}^h\overline{y},e) | 0 \le h \le p-1\} \subseteq N(b) \cap V_{s_{k+1}}$. We have $|A_{k+1}| = |B_{k+1}| = p$.

Let $C = A_{k+1} \cap B_{k+1}$. Each vertex $c \in C$ corresponds to a path from a to b, having only c as internal vertex. There are $|C|$ such paths from a to b. We then associate to each vertex in $A_{k+1} \setminus C$ a neighbor in a distinct subgraph Φ_h, with $1 \le h \le p-|C| \le q-1$, as follows. For each $a_h = \langle(\overline{e},s_{k+1})\rangle(\overline{s_i}^{(h-1)}\overline{x},e) \in A_{k+1} \setminus C$, we consider the neighbor $a'_h = \langle(\overline{s_i},e)\rangle(\overline{x},s_{k+1}^h) \in V(\Phi_h)$ with $1 \le h \le p$. Similarly, consider the set $B = \{b'_h = \langle(\overline{s_j},e)\rangle(\overline{y},s_{k+1}^h) | 1 \le h \le p\}$ of distinct neighbors of vertices in $B_{k+1} \setminus C$ in different Φ_h. Since Φ_h is connected, for every $1 \le h \le p$, there exists a path from a'_h to b'_h in Φ_h. Moreover as all Φ_h are disjoint, we can extend these paths in $p-|C|$ disjoint paths from $A_{k+1} \setminus C$ to $B_{k+1} \setminus C$, leading to $p-|C|$ internally disjoint paths from a to b. Since these paths do not contain any vertex of $C \cup V(\Phi_0) \setminus \{a,b\}$, all the obtained paths from a to b are internally disjoint. There are $p(k-1)+|C|+p-|C| = pk$ such paths, which concludes the Case 2.1.

Case 2.2. a and b are both in $V_{s_{k+1}}$.

Let $a = \langle(\overline{e}, s_{k+1})\rangle(\overline{x}, e)$ and $b = \langle(\overline{e}, s_{k+1})\rangle(\overline{y}, e)$, with $\overline{x}, \overline{y} \in \overline{G}$ and $a \neq b$. We will prove that for every Φ_t, $0 \leq t \leq q-1$, there exist k internally disjoint paths from a to b having internal vertices only in $V(\Phi_t)$. It will follow that $\Phi(G, \widetilde{S})$ contains at least $qk > pk$ internally disjoint paths from a to b.

Consider, w.l.o.g. that $t = 0$. As already mentioned (see Figure 2) $|N(a) \cap V(\Phi_0)| = k$ and $|N(b) \cap V(\Phi_0)| = k$. Since Φ_0 is $p(k-1)$-connected and $k \leq p(k-1)$, by Lemma 18, there exist, in Φ_0, k disjoint paths from $N(a) \cap V(\Phi_0)$ to $N(b) \cap V(\Phi_0)$, corresponding to k internally disjoint paths from a to b with internal vertices in $V(\Phi_0)$. This concludes the Case 2.2.

Case 2.3. $a \in V(\Phi_t)$ and $b \in V_{s_{k+1}}$.

Assume for sake of simplicity that $a \in V_{s_1}(\Phi_0)$, the other cases being similar. Let $a = \langle(\overline{s_1}, e)\rangle(\overline{x}, e)$ and $b = \langle(\overline{e}, s_{k+1})\rangle(\overline{y}, e)$. For every $l \in \{0, \ldots, p-1\}$, we denote by $N_l(a) = \{a_{li} = \langle(\overline{s_i}, e)\rangle(\overline{s_1}^l\overline{x}, e) | 2 \leq i \leq k\}$. We have $N(a) \cap V(\Phi_0) = \cup_{l=1,\ldots,p} N_l(a)$. Since $|N_l(a)| = k-1$ and $\forall l, t \in \{0, \ldots, p-1\}, l \neq t, N_l(a) \cap N_t(a) = \emptyset$, we have $|N(a) \cap V(\Phi_0)| = p(k-1)$.

We then define an application $p^a_{k+1} : N(a) \cap V(\Phi_0) \longrightarrow V_{s_{k+1}}$ as follows: for every vertex $a_{li} = \langle(\overline{s_i}, e)\rangle(\overline{s_1}^l\overline{x}, e) \in N(a) \cap V(\Phi_0)$ we set $p^a_{k+1}(a_{li}) = \langle(\overline{e}, s_{k+1})\rangle(\overline{s_i}\overline{s_1}^l\overline{x}, e)$. Since $\widetilde{S}$ is independent, p^a_{k+1} is injective. Moreover $\forall a_{li} \in N(a) \cap V(\Phi_0)$, we have $p^a_{k+1}(a_{li}) \notin N(a)$ and $p^a_{k+1}(a_{li}) \in N(a_{li})$. Similarly, for any vertex $c \in V(\Phi_t), t = 0, \ldots, q-1$, we can define $p^c_{k+1} : N(c) \cap V(\Phi_t) \longrightarrow V_{s_{k+1}}$.

Using the same notations as in Case 2.1., we set $A_{k+1} = (N(a) \cap V_{s_{k+1}})$ with $a \in V_{s_1}(\Phi_0)$. We also denote $A'_{k+1} = p^a_{k+1}(N(a) \cap V(\Phi_0))$, where $p^a_{k+1}(U) = \cup_{u \in U}\{p^a_{k+1}(u)\}$. Let $A''_{k+1} = A_{k+1} \cup A'_{k+1}$, we have $|A''_{k+1}| = |N(a) \cap V(\Phi_0))| + |N(a) \cap V_{s_{k+1}}| = pk$.

Suppose first that $b \notin A''_{k+1}$. We partition A''_{k+1} in p sets $A^1_{k+1}, \ldots, A^p_{k+1}$, each of size k. For every $t \in \{1, \ldots, p\}$ (recall that $p \leq q-1$), we denote $A^t_{k+1} = \{a^t_{k+1,1}, \ldots, a^t_{k+1,k}\}$ and we define, for every $i \in \{1, \ldots, k\}$, a^t_i as the neighbor of $a^t_{k+1,i}$ in $V_{s_i}(\Phi_t)$. We set $A_t = \{a^t_1, \ldots, a^t_k\} \subset V(\Phi_t)$. We similarly define B_t as the k distinct neighbors of b in $V(\Phi_t)$, $t = 1, \ldots, p$. Each Φ_t, $t = 1, \ldots, p$, is $p(k-1)$-connected and $p(k-1) \geq k$, since $p \geq 2$ and $k \geq 2$. Hence, by Lemma 18, there are k disjoint paths in Φ_t between A_t and B_t.

Adding to these paths edges from b to $B_t, t = 1, \ldots, p$, edges $a^t_i a^t_{k+1,i}$ with $t = 1, \ldots, p, i = 1, \ldots, k$, edges $aa_j, a_j \in N(a) \cap V_{s_{k+1}}$ and paths $(a, a_{li}, p^a_{k+1}(a_{li}))$ with $a_{li} \in N(a) \cap V(\Phi_0)$, we get pk internally disjoint paths between a and b.

Suppose now that $b \in A''_{k+1}$. Using the same method we construct $pk-1$ paths passing through vertices in $A''_{k+1} \setminus \{b\}$. We add either the edge ab if $b \in N(a)$ or the path (a, a_{li}, b) with $b = p^a_{k+1}(a_{li})$ to devise the last path. This concludes

Case 2.3.

Case 2.4. $a \in V(\Phi_{t_1})$, $b \in V(\Phi_{t_2})$ with $t_1 \neq t_2 \in \{0, 1, \ldots, q-1\}$.

Assume w.l.o.g. $a \in V(\Phi_0)$, $b \in V(\Phi_1)$ with $a = \langle(\overline{s_{i_0}}, e)\rangle(\overline{x}, e)$ and $b = \langle(\overline{s_{j_0}}, e)\rangle(\overline{y}, s_{k+1})$.

Using same notations as previously, we first consider $A'_{k+1} = p^a_{k+1}(N(a) \cap V(\Phi_0))$ with $|A'_{k+1}| \geq p(k-1)$. For every element $a_{li} = \langle(\overline{s_i}, e)\rangle(\overline{s_{i_0}}^l\overline{x}, e)$, $i \neq i_0, 1 \leq i \leq k$, $0 \leq l \leq o(s_{i_0})$, we set $p^a_{k+1}(a_{li}) = \langle(\overline{e}, s_{k+1})\rangle(\overline{s_i s_{i_0}}^l\overline{x}, e)$ in A'_{k+1}. We then consider $a^i_l = \langle(\overline{s_i}, e)\rangle(\overline{s_{i_0}}^l\overline{x}, s_{k+1}) \in V(\Phi_1)$. By independence, all these elements are distinct; we denote by $A_1 \subset V(\Phi_1)$ this set. Moreover A'_{k+1} and A_1 are linked by a matching. Φ_1 being $p(k-1)$-connected and $|A_1| \geq p(k-1)$, we have $p(k-1)$ paths in Φ_1 from b to $p(k-1)$ vertices in A_1 with only b in common. Eventually one of these paths could be reduced to b. Adding some edges of the matching between A'_{k+1} and A_1 and some paths $(a, a_{li}, p^a_{k+1}(a_{li}))$, we get $p(k-1)$ internally disjoint paths between a and b visiting only vertices in $V(\Phi_1) \cup A'_{k+1} \cup (N(a) \cap V(\Phi_0))$. Moreover we can assume that these paths contain exactly $p(k-1)$ neighbors of b in $V(\Phi_1) \cup V_{s_{k+1}}$ (else we could reduce some of them). b has at least p other neighbors in $V(\Phi_1) \cup V_{k+1}$, all distinct from these $p(k-1)$ paths. Using the function p^b_{k+1} we get p additional paths from b to a set $\bar{B}_{k+1} \subset V_{k+1}$, with only b in common regarding the other constructed paths. We consider $A_{k+1} \subset (N(a) \cap V_{k+1})$ of size p, as in Case 2.1. $A_{k+1} \cap \bar{B}_{k+1}$ leads to paths of two edges between a and b and using similar arguments as previously. We can find two sets A_2 and B_2 in $V(\Phi_2)$, $|A_2| = |B_2| = p - |A_{k+1} \cap \bar{B}_{k+1}|$ $(q-1 \geq p \geq 2)$ with a matching between $A_{k+1} \setminus A_{k+1} \cap \bar{B}_{k+1}$ and A_2 and a matching between $\bar{B}_{k+1} \setminus A_{k+1} \cap \bar{B}_{k+1}$ and B_2. Since $p \leq p(k-1)$ and using that Φ_2 is $p(k-1)$-connected, we can find $p - |A_{k+1} \cap \bar{B}_{k+1}|$ internally disjoint additional paths between a and b in Φ_2 with only a and b in common regarding the other constructed paths. This concludes the proof of the last case. $\square$

4 Structure and recognition of abelian G-graphs

Generally, graphs issued from groups have high regular properties. As for Cayley graphs, within G-graphs case the graph can be derived efficiently from the group table. Indeed, given the group table, for each $s \in S$, the cosets associated with s can be computed in linear time with respect to the number of elements in the group. Moreover the intersections between two cosets can be also checked in linear time. However an important question deals with complexity issues within graphs defined from groups in general and G-graphs in particular. As for Cayley graphs (see [2]), the complexity of deciding whether a given graph is a G-graph is still unknown. This motivates considering this question even for restricted classes of G-graphs. In this section, we give a first complexity result for the case of canonical abelian G-graphs. We first give a combinatorial characterization of this class and show that it can be recognized in $O(|E|^2)$. The combinatorial characterization uses close links between some G-graphs and some Cayley graphs that have been shown in [7] and [22].

We have already mentioned in Remark 2 that any simple G-graph $\Phi(G, S)$, with elements in S pairwise independent, has a natural edge-labeling with labels in G. Similarly a Cayley graph $C(G, S)$, with $s \in S \Leftrightarrow s^{-1} \in S$, has also a natural labeling, associating to an edge uv the label $\{uv^{-1}; vu^{-1}\} \in S \times S$. We denote both labeling *natural labeling* of a simple G-graph and a Cayley graph, respectively. Using such labelings, a *color-clique* (see [22]) in a G-graph is a maximal clique containing edges of the same label. A similar notion is defined for a particular family of Cayley graphs $C(G, S^*)$, where $S^* = \cup_{s \in S} \langle s \rangle \setminus \{e\}$ (if $s \in S^*$, then for any integer $p \geq 1$, $s^p \in S^*$). In such Cayley graphs, using the same definition as for G-graphs (see [22]), a color-clique contains edges with labels of the form $\{s^p, s^{-p}\}$ with $s^p \in \langle s \rangle \setminus \{e\}$ for a fixed $s \in S^*$. Note that in a G-graph, there is a single clique for a given label while, in the Cayley case, color-cliques associated to $s \in S^*$ are exactly the cosets of the form $\langle s \rangle x$.

Theorem 20 *Consider a group G and $S \subset G$ a set of pairwise independent elements. We denote by $S^* = \cup_{s \in S} \langle s \rangle \setminus \{e\}$.*

(1) [7] The intersection graph of color-cliques of $\Phi(G, S)$ is isomorphic to $C(G, S^*)$.
(2) [22] Conversely if $e \notin S$, The intersection graph of color-cliques of $C(G, S^*)$ is $\Phi(G, S)$.

Note that for item 2, pairwise independence allows to define S from S^* without ambiguity.

This correspondence allows in particular to link the recognition of some G-graphs to the recognition of some Cayley graphs. In general, a difficulty will be to identify color-cliques while labels are not known. In the case of canonical abelian G-graphs, as stated below, this problem will not hold since all maximal cliques are color-cliques.

Clique-helly: A graph Γ is called *clique-helly* if its set of maximal cliques satisfies the Helly property, i.e. any set of pairwise intersecting maximal cliques has a common vertex. It is *hereditary clique-helly* if Γ and all its subgraphs are clique-helly. Given a graph Γ, we denote by $c(\Gamma)$ the *clique graph* associated to Γ, defined as the intersection graph of all maximal cliques. A graph is *diamond-free* if it has no induced diamond, where a *diamond* is a K_4 minus one edge.

A vertex v in a graph is called *simplicial* if $N(v)$ is a clique: v belongs to a single maximal clique.

The following results are well-known:

Proposition 21 *If* Γ *is diamond-free, then:*

(1) for any edge uv*, there is a unique maximal clique containing* u *and* v*.*
(2) $c(\Gamma)$ *is diamond-free [9].*
(3) Γ *is hereditary clique-helly (see e.g. [3]).*
(4) If furthermore Γ *has no simplicial vertex, then* $c(c(\Gamma)) \simeq \Gamma$*.*

PROOF.

Item 1 is immediately deduced from the definition, so we only need to prove item 4. For any vertex v of Γ, we denote by $u_1, \ldots, u_{d(v)}$ the neighbors of v, where $d(v)$ is the degree of v. For any $i \in \{1, \ldots, d(v)\}$ we denote by K^i the unique maximal clique containing v and u_i (see Item 1). The maximal cliques $K^1, \ldots, K^{d(v)}$ of Γ are not all distinct but there are at least two distinct maximal cliques among them since v is not simplicial. Hence, using the Helly property, no other maximal clique can intersect all of them. Consequently these cliques constitute a maximal clique $K_v^{c(\Gamma)}$ in $c(\Gamma)$, hence correspond to a vertex in $c(c(\Gamma))$. Conversely a maximal clique in $c(\Gamma)$, denoted by $K^{c(\Gamma)}$, corresponds to a bundle of cliques in Γ sharing a single vertex $v(K^{c(\Gamma)})$. The composition of both transformations leads to the identity, which means that the application $v \mapsto K_v^{c(\Gamma)}$ is a bijection from $V(\Gamma)$ to $V(c(c(\Gamma)))$. Note that $K_v^{c(\Gamma)}$ and $K_u^{c(\Gamma)}$ are linked in $c(c(\Gamma))$ if and only if the two related bundles of cliques in $c(\Gamma)$ share a common clique that contains both u and v. This is equivalent to u and v are linked in Γ. So, this bijection is also an isomorphism from Γ to $c(c(\Gamma))$. $\square$

Note that a single clique can be a G-graph and does not satisfy item 4. However if $e \notin S$ and $|S| \geq 2$, then $\Phi(G, S)$ does not have any simplicial vertex. On the other hand, if $|S| = 1$, then $\Phi(G, S)$ is a stable set and satisfies all items of Proposition 21. So we have:

Proposition 22 *Let* $\Gamma = (V, E)$ *be a* G*-graph* $\Phi(G, S)$*, where* S *is independent by triples and* $e \notin S$*, then:*

(1) Γ *is diamond-free and all maximal cliques are color-cliques and maximum cliques.*
(2) Γ *is hereditary clique-helly.*
(3) $c(\Gamma) \simeq C(G, S^*)$
(4) $\Gamma \simeq c(C(G, S^*))$

This holds in particular for canonical abelian G-graphs.

PROOF. Let $\Gamma = (V, E)$ be a G-graph $\Phi(G, S)$, where S is independent by triples. We first show that every triangle in Γ is constituted by edges of the same label. Let three vertices $\langle s_i \rangle x$, $\langle s_j \rangle y$ and $\langle s_l \rangle z$ of Γ inducing a triangle. Hence i, j, l are pairwise distinct. As there is an edge between any two of these vertices, there exist integers a, a', b, b', c, c' such that $\alpha = s_i^a x = s_j^{a'} y$, $\beta = s_j^b y = s_l^{b'} z$ and $\gamma = s_l^c z = s_i^{c'} x$. Taking into account the structure of a G-graph, we have the following identities:

$$s_i^a x = s_i^{a-c'} s_i^{c'} x = s_i^{a-c'} s_l^c z = s_i^{a-c'} s_l^{c-b'} s_j^{b'} z = s_i^{a-c'} s_l^{c-b'} s_j^{b-a'} s_i^a x. \tag{2}$$

Hence $s_i^{a-c'} s_l^{c-b'} s_j^{b-a'} = e$. Based on the fact that S is independent by triples, we deduce $a \equiv c' \bmod o(s_i)$, $c \equiv b' \bmod o(s_l)$, $b \equiv a' \bmod o(s_j)$, thus $\alpha = \beta = \gamma$. So, every edge $uv \in E$ belongs to a unique maximal clique obtained by $\{u, v\} \cup (N(u) \cap N(v))$. Since any edge also belongs to a color-clique, maximal cliques are color-cliques and are all of size k, this concludes the proof of Item 1. Items 2, 3, and 4 immediately follow from Propositions 21 and Theorem 20 (independence by triples implies pairwise independence). $\square$

As a consequence, for the case where S is independent, $c(\Gamma)$ can be computed in polynomial time and the recognition of $\Phi(G, S)$ is polynomially equivalent to the recognition of $C(G, S^*)$. Unfortunately, to our knowledge this last problem remains open in general (see [2]).

We immediately deduce the following combinatorial characterization of canonical abelian G-graphs:

Theorem 23 *Given a diamond-free graph Γ, the following statements are equivalent:*

(1) Γ is an abelian canonical G-graph $\Phi(\langle s_1 \rangle \times \cdots \times \langle s_k \rangle, \widetilde{S})$.
(2) $c(\Gamma) \simeq K_{o(s_1)} \times \cdots \times K_{o(s_k)}$ (Generalized Hamming).
(3) $\Gamma \simeq c(K_{o(s_1)} \times \cdots \times K_{o(s_k)})$ (clique graphs of a generalized Hamming).

PROOF. This immediately follows from Proposition 22, more precisely $C(\langle s_1 \rangle \times \cdots \times \langle s_k \rangle, S^*) \simeq K_{o(s_1)} \times \cdots \times K_{o(s_k)}$. $\square$

Remark 24 *Note that the Generalized Hamming Graph $K_{n_1} \times \cdots \times K_{n_k}$ is also diamond-free. We immediately deduce from Proposition 21 that maximal cliques in $c(K_{n_1} \times \cdots \times K_{n_k})$ correspond to bundles of maximal cliques sharing a single vertex $(x_1, \ldots, x_k)$ in $K_{n_1} \times \cdots \times K_{n_k}$. There are exactly k such cliques,* *which shows that all maximal cliques in $c(K_{n_1} \times \cdots \times K_{n_k})$ are of the same size.*

Remark 25 *In [8], it is shown that Hamming graphs are (diamond-free) G-graphs and consequently regular canonical abelian G-graphs $\Phi(\langle s_1 \rangle \times \cdots \times \langle s_k \rangle, \widetilde{S})$, with $o(s_1) = \cdots = o(s_k)$. Using Theorem 20 we can see that they are also Cayley graphs. This does not hold for general canonical abelian G-graphs that are not necessary regular.*

In [14], a $O(|E|)$ algorithm is proposed for deciding whether a given graph $\Gamma = (V, E)$ is isomorphic to a Cartesian product of k complete graphs (Generalized Hamming). Moreover the algorithm computes a Hamming labeling associating to any vertex its representation as a k-vector. We can immediately use it to recognize canonical abelian G-graphs.

Theorem 26 *It can be decided in $O(|E|^2)$ whether a given graph is a canonical abelian G-graph.*

Algorithm 1 Deciding whether a fixed graph is canonical abelian

Require: A graph $\Gamma = (V, E)$ with $|V| = n$ and $|E| = m$.
if Γ is diamond-free **then**
    Compute $c(\Gamma)$
    if $c(\Gamma) \simeq K_{n_1} \times \cdots \times K_{n_k}$ **then**
        Γ is a canonical abelian G-graph
    else
        Γ is not a canonical abelian G-graph
    end if
else
    Γ is not a canonical abelian G-graph
end if

PROOF.

Using Theorem 23, Algorithm 1 decides whether a given graph $\Gamma = (V, E)$ is a canonical abelian G-graph. From [16], we know that it can be decided in $O(|V|^{2.376} + |E|^{1.5})$ whether Γ is diamond-free. Moreover the number of maximal cliques in Γ is $O(|E|)$ and the graph $c(\Gamma)$ can be computed in $O(|E|^2)$. Then using the algorithm of [14] one can decide whether $c(\Gamma)$ is a Generalized Hamming Graph.

Note that if the graph is a canonical abelian G-graph, then we can use the isomorphism of Theorem 23 (Item 3) to compute the labeling associating $\langle s_i \rangle g$ with any vertex in Γ. $\square$

5 Concluding remarks

We already emphasized in Section 1 the importance of graphs issued from groups, like Cayley graphs, for generating networks with regular topologies and good connectivity properties. In this paper, we studied the connectivity properties of one of this type of graphs, the G-graphs. These graphs constitute a more flexible class than Cayley graphs and in particular they may be not regular. However we have shown that, in some cases, they still have interesting transitivity and connectivity properties. In particular, we showed that clique graphs of Cartesian products of cliques (e.g. canonical abelian G-graphs) are optimally connected. The originality of our proof consists in using the structure of the G-graph. It gives some ideas how we could handle more general classes of optimally connected G-graphs. Moreover it also illustrates whether it would be possible to deduce the connectivity of the clique graph $c(\Gamma)$ from the connectivity of the initial graph Γ.

Another major issue is the complexity of recognizing G-graphs. To our knowledge, this has not been done before. Moreover their close links with Cayley graphs make this question close to the recognition of Cayley graphs, which is still open. The polynomial case studied here motivates further works in this direction.

As a last remark in this direction, let us show that recognizing partition intersection graphs (see Section 2.3), a class containing G-graphs, is NP-complete. The following characterization is given in [18]:

Theorem 27 *[18] A graph $\Gamma = (V, E)$ is a partition intersection graph if and only if:*

- *It has no isolated vertex.*
- *E can be covered by cliques of size $\chi(\Gamma)$, where $\chi(\Gamma)$ is the chromatic number of Γ.*

Note that if Γ is a partition intersection graph, then $\chi(\Gamma)$ is exactly the number k of partitions used to construct it. Theorem 27 allows immediately to show that recognizing a partition intersection graph is NP-complete, even if the number of partitions is fixed to 3:

Proposition 28 *The following problems are NP-complete:*

(1) Deciding whether a graph Γ is a partition intersection graph.
(2) For any fixed $k \geq 3$, deciding whether a fixed graph Γ is a partition intersection graph over k partitions.

PROOF. Both problems are clearly in NP. Item 1 can be easily shown by reducing 3-colorability, the NP-complete problem of deciding whether a fixed graph is 3-colorable [12]. Let Γ be a graph instance of 3-colorability, without loss of generality we can suppose that $\chi(\Gamma) \geq 3$. Then for every edge uv we add a vertex x_{uv} and edges $x_{uv}u, x_{uv}v$. Let us denote by Γ' the new graph. Since $\chi(\Gamma) \geq 3$ and the added vertices are all of degree 2 in Γ', we have $\chi(\Gamma') = \chi(\Gamma)$ and moreover edges of $\chi(\Gamma')$ can be covered by triangles. So, using Theorem 27, Γ' is a partition intersection graph if and only if Γ is 3-colorable, which concludes the proof of Item 1. Item 2, for $k = 3$ and $k > 3$ can be shown in a very similar way by using a reduction from k-colorability and adding, for any edge uv, a clique K_{k-2} completely connected to u and v. $\square$

An interesting further work could be to narrow the boundary line between hard and easy cases for recognizing partition intersection graphs and investigating the complexity of recognizing G-graphs.

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