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THE SPECIFIC FORCED RADIATION WAVE IMPEDANCE OF A FINITE RECTANGULAR PANEL EXCITED BY A PLANE SOUND WAVE

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The average specific forced radiation wave impedance of a finite rectangular panel, excited by a plane sound wave incident at a particular angle of incidence, is of importance for the prediction of both sound insulation and sound absorption. In 1982, Thomasson published numerical calculations of the average specific forced radiation wave impedance of a square of side length 2*e* for wave number *k* for values of *ke* in half octave steps from 0.25 to 64. Thomasson also published approximate formulae for values of *ke* above and below the published results. This paper combines Thomasson's high and low frequency formulae and compares this combined formula with Thomasson's numerical calculations. The real part of the approximate formula is between 0.7 dB higher and -1 dB lower than the numerical calculations. The imaginary part of the approximate formula is between 2.3 dB higher and -2.6 dB lower than the numerical calculations. These extreme differences occur in the region of *ke* equals 2. However the imaginary part also has a difference of 1.7 dB for *ke* equals 64 at an angle of incidence of the forcing wave of 60 degrees relative to the normal to the square panel. An approximate formula for the real part developed previously by Davy is between 0.4 dB higher and -0.6 dB lower than Thomasson's numerical calculations. Thus it is in slightly better agreement with Thomasson's numerical calculations than the formula obtained by combining Thomasson's high and low frequency approximations. The imaginary specific radiation impedance of a rigid disk of equal area to a square and vibrating in phase gives a good approximation, up to *ke* equals 5, to the imaginary part of the specific forced radiation impedance of a square which has been excited by a plane wave incident normally to the square.

1. Introduction

This paper presents the derivation of a combined method for calculating the average specific radiation wave impedance of a finite rectangular panel. The equations derived by Thomasson are combined to derive an approximate formula for the average specific radiation impedance.

In this paper, the sinusoidal variation with time is assumed to be proportional to $e^{j\omega t}$, where ω is the angular frequency, t is the time, j is the square root of -1 and e is Euler's number. e is also used to define half the typical distance across the panel, but this should not create any confusion. It should be noted that the assumption of $e^{-j\omega t}$ for the sinusoidal variation with time gives the opposite sign for the imaginary part of the impedance. The impedances in this paper are normalised by dividing by the characteristic impedance of the fluid medium Z_c, which is equal to the product of the ambient density of the fluid medium ρ_0 and the speed of sound in the fluid medium c.

An infinite one dimensional (either forced or unforced) sinusoidal bending wave with bending wave number k_b travelling in an infinite panel immersed in a fluid medium with freely propagating wave number k has a one sided normalised specific radiation wave impedance z given by[1]

$$z = \begin{cases} 1/\sqrt{1-\left(k_b/k\right)^2} = 1/\cos\left(\theta\right) & \text{if } k_b < k \\ \infty & \text{if } k_b = k \\ j/\sqrt{\left(k_b/k\right)^2 - 1} & \text{if } k_b > k \end{cases} \tag{1}$$

where

$$\theta = \arccos\left(\sqrt{1-\left(k_b/k\right)^2}\right) \tag{2}$$

is the angle of incidence in radians of an incident plane wave. This is defined as the angle between the normal of the panel and the direction of travel of the incident infinite plane wave with wave number k in the fluid medium. This incident plane wave produces a forced bending wave of wave number k_b in the panel.

The first line of Eq. (1) suggests for a bending wave, forced by an incident plane wave in the fluid medium, on a finite panel whose dimensions are large compared to the wavelength of sound in the fluid medium and which is mounted in an infinite baffle, that the real part of the averaged normalised specific wave impedance will be approximately $1/\cos\left(\theta\right)$ and that the imaginary part will be close to zero, except for values of the incident angle which are close to grazing incidence ($\pi/2$ radians or 90°). This suggestion is correct.

The third line of Eq. (1) correctly suggests that the real part of the normalised specific radiation wave impedance of a freely propagating bending wave on a finite panel below the critical frequency of the panel in the fluid medium is close to zero and that the imaginary part is a mass like loading.

The normalised specific radiation impedance of a uniformly sinusoidally vibrating sphere of radius r is[1]

$$z = \frac{\left(kr\right)^2}{1+\left(kr\right)^2} + j\frac{kr}{1+\left(kr\right)^2}. \tag{3}$$

By symmetry, Eq. (3) also applies for a uniformly sinusoidally vibrating hemisphere of radius r whose base is on an infinite rigid baffle. The real part of Eq. (3) also applies to panels or openings which are small compared to the wavelength of sound, are mounted in an infinite baffle and are vibrating uniformly (the angle of incidence of the forcing wave in the fluid medium θ is zero) if the area of the hemisphere is equal to the area of the panel or opening. Applying the same approach to the imaginary part produces the correct qualitative behaviour, but the constant derived from Eq. (3) by applying this method needs to be modified to produce the correct quantitative behaviour.

The first line of Eq. (1) and the real part of Eq. (3) also correctly suggest that the normalised specific acoustic wave impedance for a uniformly sinusoidally vibrating panel or opening mounted in an infinite baffle tends to 1 as ke tends to infinity. The uniform vibration means that $k_b = 0$ and $\theta = 0$.

Eqs. (1) - (3) give a semi-quantitative understanding of the average specific forced radiation wave impedance of a finite size rectangular panel mounted in an infinite rigid baffle, which is excited by an infinite plane sound wave incident at a particular angle of incidence to the normal to the panel.

There are a number of ways of numerically calculating this impedance. The best way appears to be that given by Eq. (64) of Brunskog[2].

$$z = \frac{jk}{2\pi ab}\int_0^{2a}\int_0^{2b}\cos\left(k\mu_x\kappa\right)\cos\left(k\mu_y\tau\right)\frac{e^{-jk\sqrt{\kappa^2+\tau^2}}}{\sqrt{\kappa^2+\tau^2}}\left(2a-\kappa\right)\left(2b-\tau\right)d\kappa d\tau \tag{4}$$

where

$$\mu_x = \sin\left(\theta\right)\cos\left(\phi\right) \text{ and } \mu_y = \sin\left(\theta\right)\sin\left(\phi\right) \tag{5}$$

and the sides of the rectangle are $2a$ and $2b$. Note that other authors often assume that the sides of the rectangle are a and b. The azimuthal angle of the incident plane wave is ϕ. A version of this equation is given in Appendix 12.A of Allard and Atalla[3] but some errors need to be corrected.

2. Approximations

Thomasson[4] showed that the dependence on the azimuthal angle ϕ is small and can be ignored to a first approximation. This means that Eq. (4) can be averaged over the azimuthal angle ϕ for most practical purposes. He also showed that the results for a rectangle, when the ratio a/b was not too different to 1, could be approximated by the results for a square of side $2e$ where

$$2e = 4S/U = 4ab/(a+b). \tag{6}$$

Note that Thomasson's e is equal to the $2e$ used in this paper. S is the area of the rectangle and U is the perimeter of the rectangle. Thomasson[4] gave approximations for $ke \le 0.25$ and for $ke \ge 64$. For $0.25 \le ke \le 64$ he gave numerically calculated values for a square at half octave intervals.

Thomasson[4]'s approximations are as follows.

$$z = 1/\sqrt{\cos^2\left(\theta\right) - j2\beta\sin\left(\theta\right)/ke + \left[\beta/\left(ke\right)\right]^2} \text{ if } ke \ge 64 \text{ where } \beta = 0.956, \tag{7}$$

$$z = 2k^2ab/\pi + j2k\left[bH(a/b) + aH(b/a)\right]/\pi \text{ if } ke \le 0.25 \tag{8}$$

where

$$H(q) = \ln\left(\sqrt{1+q^2}+q\right) - \left(\sqrt{1+q^2}-1\right)/\left(3q\right). \tag{9}$$

For a square Eq. (8) becomes

$$z = 2\left(ke\right)^2/\pi + 0.946\,jke \text{ if } ke \le 0.25. \tag{10}$$

Equating the area of the square to the area of the hemisphere, $4e^2 = 2\pi r^2$ and $r = e\sqrt{2/\pi}$. Putting these values into Eq. (3) and assuming that $(kr)^2 << 1$ gives

$$z = 2(ke)^2/\pi + \sqrt{2/\pi}\, jke = 2(ke)^2/\pi + 0.798\, jke \text{ if } (kr)^2 << 1. \tag{11}$$

This approach gives the correct real part of Eq. (10), but the constant in the imaginary part is slightly in error. For $ke >> 1$, Eq. (7) shows the $1/\cos(\theta)$ behaviour predicted by Eq. (1). For $\theta = \pi/2$ radians (90°) and $(ke)^2 >> 1$, the real and imaginary parts of Eq. (7) are both positive and approximately equal. For θ not near $\pi/2$ radians (90°) and $ke >> 1$, the imaginary part of Eq. (7) is very much less than the real part.

3. Combined formulae

Davy[5] combined high and low frequency approximations for the real part of the averaged normalised specific forced radiation wave impedance which is also equal to the radiation efficiency. The aim of this paper is to combine Thomaason[4]'s low and high frequency approximations in order to cover the whole frequency range. This would give a formula for the imaginary part which is not provided by Davy[5].

Following Davy[5], the low x_L and x_H approximations are combined using the following formula

$$x = \frac{1}{\sqrt[n]{\dfrac{1}{x_L^n} + \dfrac{1}{x_H^n}}}. \tag{12}$$

Table 1. Difference in decibels between the combined approximate method developed in this paper and Thomasson[4]'s numerical calculations for the real part of the specific forced radiation wave impedance of a square panel of side length $2e$.

ke	0°	15°	30°	45°	60°	70°	75°	80°	85°	90°
0.25	-0.1	-0.1	-0.1	-0.1	-0.1		-0.1			-0.1
0.35	-0.1	-0.1	-0.1	-0.1	-0.2		0.4			0.4
0.50	0.0	0.0	0.0	0.3	0.3		0.2			0.2
0.71	0.0	0.0	0.1	0.2	0.3		0.3			0.5
1.00	-0.3	-0.2	-0.1	0.2	0.4		0.5			0.5
1.41	-0.7	-0.6	-0.2	0.2	0.4		0.6			0.6
2.00	-1.0	-0.8	-0.4	0.2	0.5		0.7			0.7
2.83	-0.4	-0.5	-0.5	0.0	0.4		0.5			0.5
4.00	0.1	-0.1	-0.5	-0.3	0.2		0.4			0.4
5.66	-0.2	-0.1	-0.1	-0.5	-0.1		0.3			0.3
8.00	-0.1	0.0	0.0	-0.3	-0.3		0.2			0.2
11.31	-0.1	0.0	0.0	0.0	-0.5		0.1			0.2
16.00	0.0	0.0	0.0	-0.1	-0.4		0.0			0.1
22.63	0.0	0.0	0.0	0.0	-0.1		-0.2			0.1
32.00	0.0	0.0	0.0	0.0	-0.1	-0.4	-0.4	0.0	0.1	0.1
45.25	0.0	0.0	0.0	0.0	0.0		-0.5			0.0
64.00	0.0	0.0	0.0	0.0	0.0		-0.4	-0.3	0.1	0.0

The real part of the specific forced radiation wave impedance is given by the x in Eq. (12) when x_L and x_H are the real parts of Eqs. (8) and (7) respectively and $n=2$.

The imaginary part is more complicated. For a normally incident exciting wave ($\theta=0$), Eq. (7) gives zero imaginary part. Although it is small for large values of ke, the imaginary part is not completely zero. A straight line of best fit was applied in the log-log domain to Thomasson[4]'s numerical calculations for the imaginary part for a normally incident exciting wave ($\theta=0$) versus ke for values of ke from 1.41 to 64. This produced the following equation.

$$\mathrm{Im}\left[z\left(\theta=0\right)\right]=\frac{0.67}{ke} \text{ if } ke\geq 1.41. \tag{13}$$

Table 2. Difference in decibels between the combined approximate method developed in this paper and Thomasson[4]'s numerical calculations for the imaginary part of the specific forced radiation wave impedance of a square panel of side length $2e$.

ke	0°	15°	30°	45°	60°	70°	75°	80°	85°	90°
0.25	0.1	0.1	0.1	0.1	0.1		0.1			0.1
0.35	0.2	0.2	0.2	0.2	0.3		0.3			0.3
0.50	0.4	0.4	0.4	0.4	0.5		0.5			0.5
0.71	0.3	0.3	0.4	0.5	0.6		0.6			0.6
1.00	-0.4	-0.4	-0.3	-0.2	-0.2		-0.1			-0.1
1.41	-1.4	-1.4	-1.5	-1.6	-1.6		-1.6			-1.5
2.00	-0.9	-1.4	-2.3	-2.6	-1.7		-1.1			-1.0
2.83	2.3	0.3	-2.6	-2.5	-1.5		-0.9			-0.7
4.00	-0.5	0.2	-1.7	-2.2	-1.4		-0.7			-0.5
5.66	1.2	-0.7	-1.0	-1.4	-1.4		-0.6			-0.4
8.00	-0.3	0.2	-2.0	0.3	-1.4		-0.6			-0.2
11.31	-0.1	-0.1	-1.0	0.3	-0.8		-0.6			-0.2
16.00	0.2	0.2	-1.2	-0.1	0.9		-0.7			-0.1
22.63	-0.1	-0.1	-0.9	0.2	1.6		-0.9			-0.1
32.00	0.2	0.2	-1.2	0.7	0.5	0.6	-0.8	-0.5	-0.1	-0.1
45.25	-1.3	-1.3	-0.9	0.2	1.2		-0.2			0.0
64.00	0.2	0.2	0.6	0.0	1.7		1.4	-0.8	-0.1	0.0

Note that apart from the 0.67 scaling factor, this is in agreement with the high frequency asymptotic behaviour of the imaginary part of Eq. (3). The imaginary part of the specific forced radiation wave impedance for a normally incident exciting wave ($\theta=0$) for all values of ke is obtained by using Eq. (12) where $n=3$, x_L is the imaginary part of Eq. (8) for the $\theta=0$ case and x_H is given by Eq. (13). The value of the imaginary part for any angle of incidence is calculated as the maximum of the imaginary part for $\theta=0$ case calculated as described in this paragraph and the imaginary part of Eq. (7).

4. Comparison with numerically calculated values

Table 1 and Table 2 show the amounts in decibels by which the real and imaginary parts of the combined approximate method developed in this paper were greater than Thomasson[4]'s numerical calculations for the real and imaginary parts respectively of the specific forced radiation wave impedance of a square panel of side length $2e$. For the real part, the differences are between 0.7 and -1.0 dB. For the imaginary part, the differences are between 2.3 and -2.6 dB. These extreme differences occur in region of ke equals 2. However the imaginary part also has a difference of 1.7

dB for *ke* equals 64 at an angle of incidence of the forcing wave of 60 degrees relative to the normal to the square panel. This difference occurs where the imaginary part is increasing very rapidly from a very low value for angles of incidence close to normal to a very large value at grazing angles of incidence.

Table 3 shows the amount in decibels that Davy[5]'s method was greater than Thomasson[4]'s numerical calculations for the real part of the specific forced radiation wave impedance of a square panel of side length $2e$. The differences are between 0.4 and -0.6 dB. Thus Davy[5]'s method is in slightly better agreement with Thomasson[4]'s numerical calculations for the real part than the formula obtained in this paper by combining Thomasson[4]'s high and low frequency approximations. Davy[5]'s method does not predict the imaginary part of the impedance.

Davy[5] compared his method with the numerical calculations of Sato[6] for the real part of the impedance. A comparison was made between Sato[6]'s and Thomasson[4]'s numerical results for the real part of the impedance across those values of ke and incident excitation angle θ for which they had both calculated results. Thomasson[4]'s values were between 0.2 dB greater and -0.1 dB less than Sato[6]'s values for the real part of the impedance.

Table 3. Difference in decibels between Davy[5]'s method and Thomasson[4]'s numerical calculations for the real part of the specific forced radiation wave impedance of a square panel of side length $2e$.

ke	0°	15°	30°	45°	60°	70°	75°	80°	85°	90°
0.25	0.0	0.0	0.0	0.0	-0.1		-0.1			-0.2
0.35	0.0	0.0	-0.1	-0.1	-0.2		0.3			0.2
0.50	0.2	0.2	0.2	0.4	0.2		0.1			-0.2
0.71	0.3	0.3	0.4	0.4	0.3		-0.1			-0.2
1.00	0.2	0.2	0.2	0.3	0.2		-0.1			-0.4
1.41	-0.2	-0.2	0.0	0.1	0.1		0.0			-0.4
2.00	-0.6	-0.6	-0.4	-0.2	0.0		0.1			-0.2
2.83	-0.2	-0.2	-0.6	-0.6	-0.3		-0.1			-0.2
4.00	0.2	0.1	-0.2	-0.4	-0.3		-0.1			-0.2
5.66	-0.1	0.0	0.0	-0.1	-0.3		-0.1			-0.1
8.00	0.0	0.0	0.1	-0.1	-0.2		-0.1			-0.1
11.31	0.0	0.0	0.0	0.1	0.0		-0.1			-0.1
16.00	0.0	0.0	0.0	0.0	-0.1		-0.2			0.0
22.63	0.0	0.0	0.0	0.0	0.1		-0.2			0.0
32.00	0.0	0.0	0.0	0.0	0.0	-0.1	0.0	-0.1	0.1	0.0
45.25	0.0	0.0	0.0	0.0	0.0		0.0			0.0
64.00	0.0	0.0	0.0	0.0	0.0		-0.1	0.0	0.0	0.1

5. The normally incident excited case

The 2.3 dB difference at $ke = 2.83$ and $\theta = 0$ in Table 2 prompted a further investigation of the normally incident excited case. For the normally incident excited case $\theta = 0$ for a disc of radius r, the specific forced radiation wave impedance is[4]

$$z = 1 - \frac{\mathrm{J}_1(2kr)}{kr} + j\frac{\mathrm{H}_1(2kr)}{kr} \tag{14}$$

where J_1 is the Bessel function of order 1 and H_1 is the Struve function of order 1. The Struve function of order 1 is not available in Excel or Matlab but can be calculated using the approximations given by Newman[6]. Equation (14) was used to estimate the specific forced radiation wave impedance of a square of side $2e$ by equating the area of the disc to the area of the square.

$$r = \frac{2e}{\sqrt{\pi}} \quad (15)$$

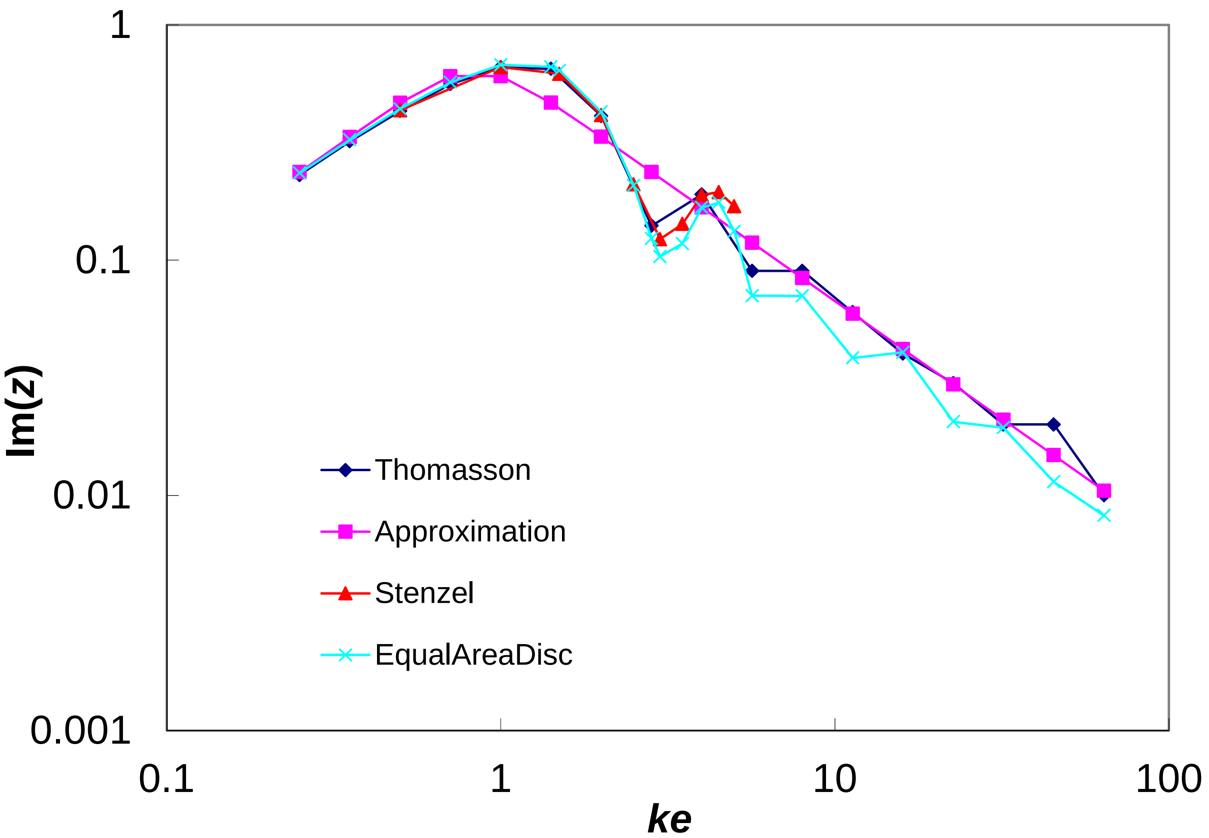

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Figure 1. The imaginary part of the normalised specific forced radiation wave impedance of a square panel of side length $2e$ for a normally incident exciting wave. Thomasson[4]'s and Stenzel[8]'s numerical calculations are compared to the formula for a disc of equal area and the combined approximation method developed in this paper.

The imaginary part of the impedance for normally incidence excitation was calculated using Eqs. (14) and (15). This was compared to Thomasson[4]'s numerical calculations, Stenzel[8]'s numerical calculations and the combined approximation method developed in this paper in Fig. 1. The equal area disc approximation is a better predictor than the combined approximation method up to about $ke = 5$. Above $ke = 5$, the equal area disc approximation predicts too low a value and the combined approximation method is more accurate. It was initially hoped that the equal area disc approximation for normal incident excitation could be used to improve the combined approximation method's prediction of the imaginary part. Unfortunately, the occurrences of differences of 2.3, -2.6 and -2.5 dB for angles of excitation of 0, 30 and 45 degrees when $ke = 2.83$ in Table 2 meant that this approach produced worse agreement.

For the normally incident excited case, Stenzel[8] calculated the normalised specific forced radiation wave impedance when the ratio of the lengths of the sides of the rectangle a/b had the values of 0.1, 0.2, 0.5 and 1. These values together with Thomasson[4]'s values for a square were graphed against ke when e was calculated in several different ways. The idea was to see which way of calculating e made the curves overlap the most.

e was calculated using Eq. (6) and also using the equal area approach

$$2e = \sqrt{S}\ . \quad (16)$$

For the imaginary part e was also calculated as the square bracketed term in Eq. (8) divided by π.

$$2e = 2\left[bH(a/b) + aH(b/a)\right]/\pi . \qquad (17)$$

For the real part, the curves all asymptote to one at high frequencies. At mid frequencies, the use of Eq. (6) causes the curves to overlap, while at low frequencies the use of Eq. (16) causes the curves to overlap. For the imaginary part, the use of Eqs. (6) or (17) causes the curves to overlap at low frequencies while the use of Eq. (16) does not.

6. Conclusion

A combined approximation method for calculating both the real and the imaginary parts of the single sided averaged normalized specific forced radiation wave impedance of a finite rectangular panel, excited by a plane sound wave incident at a particular angle of incidence has been derived. For the real part, the approximate method is between 0.7 dB higher and -1 dB lower than numerical calculations. For the imaginary part, the approximate method is between 2.3 dB higher and -2.6 dB lower than numerical calculations. The method for the real part is not quite as good as the approximate method for the real part developed previously by Davy[5] which is between 0.4 dB higher and -0.6 dB lower than the numerical calculations. However, unlike Davy[5]'s method, the method developed in this paper can also calculate the imaginary part.

For the normal incidence excited case, the equal area disc approximation is a better predictor than the combined approximation method up to about $ke = 5$. Above $ke = 5$, the equal area disc approximation predicts too low a value and the combined approximation method is more accurate. In the normal incidence excited case, several different ways of calculating the side of the square whose impedance is closest to that of a given rectangle have been investigated. The best method depends on the part of the complex impedance and the frequency range.

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