How long is a curved beam?

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The manuscript was received on 5 May 2004 and was accepted after revision for publication on 15 October 2004.

DOI: 10.1243/030932405X7719

Abstract: For a straight or thin curved beam, the expression for strain energy due to bending is $U = M^2L/(2EI)$; for this to be applicable to a thick curved beam, the requisite length is slightly greater than the centre-line length.

Keywords: strain energy, thick curved beam

1 INTRODUCTION

The strain energy U of a uniform straight beam, or a thin curved beam, having length L, second moment of area I, Young's modulus E, and subjected to a pure bending moment M, is well known to be given by the expression $U = M^2L/(2EI)$; the thin curved beam is often chosen as an example in undergraduate texts to illustrate Castigliano's theorems (see for example reference [**1**]), and the elemental length is then of the form $\mathrm{d}s = R\,\mathrm{d}\theta$ if the moment varies along the beam. For the uniform thick curved beam shown in Fig. 1, again subjected to pure bending, it must also be possible to express strain energy in the form $U = M^2L_{\text{eff}}/(2EI)$, where L_{eff} is some effective length of the beam. The question is: what is this effective length? This problem arose in the determination of the equivalent continuum beam properties, such as the second moment of area, cross-sectional area, and shear coefficient, of a pin-jointed curved repetitive beam-like structure [**2**], by demanding equality of strain energy of the single repeating cell and the equivalent continuum. The two immediate candidates are centre-line length, and length of the neutral surface. In turn, an effective radius R_{eff} where $L_{\text{eff}} = R_{\text{eff}}\phi$ can be defined when the above possibilities are equivalent to $R_{\text{cl}} = (a+b)/2$, where a and b are the inner and outer radii respectively and $R_{\text{na}} = R_{\text{cl}} - e$, where e represents the amount by which the neutral axis is closer to the centre of curvature. Employing the exact linear elasticity plane stress solution [**3**, article 29], it transpires that neither of the above is true; rather, the effective radius is greater than the centre-line radius, although the latter must be very small before the difference is significant.

2 THEORY

Within a Cartesian coordinate system the strain energy density is [**3**, article 90]

$$U_0 = \frac{\sigma_x^2 + \sigma_y^2}{2E} - \frac{\nu\sigma_x\sigma_y}{E} + \frac{\tau_{xy}^2}{2G} \tag{1}$$

The equivalent expression within polar coordinates is

$$U_0 = \frac{\sigma_r^2 + \sigma_\theta^2}{2E} - \frac{\nu\sigma_r\sigma_\theta}{E} + \frac{\tau_{r\theta}^2}{2G} \tag{2}$$

Again from reference [**3**, article 29], the stress components are

$$\sigma_r = -\frac{4M}{N}\left[\frac{a^2b^2}{r^2}\log\left(\frac{b}{a}\right) + b^2\log\left(\frac{r}{b}\right) + a^2\log\left(\frac{a}{r}\right)\right] \tag{3a}$$

$$\sigma_\theta = -\frac{4M}{N}\left[-\frac{a^2b^2}{r^2}\log\left(\frac{b}{a}\right) + b^2\log\left(\frac{r}{b}\right) + a^2\log\left(\frac{a}{r}\right) + b^2 - a^2\right] \tag{3b}$$

$$\tau_{r\theta} = 0 \tag{3c}$$

where

$$N = (b^2 - a^2)^2 - 4a^2b^2\left[\log\left(\frac{b}{a}\right)\right]^2 \tag{4}$$

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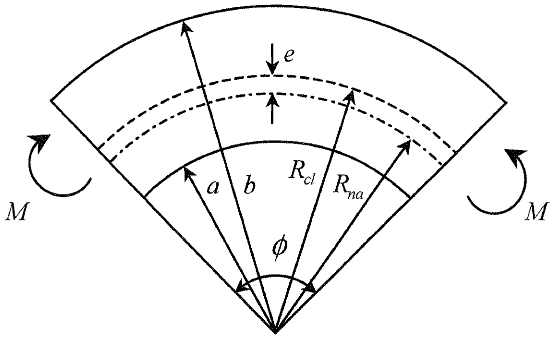

Fig. 1 Thick curved beam subject to pure bending

Equation (2) immediately simplifies as the shearing stress is zero, and the strain energy for the complete beam becomes

$$U = \int_a^b \int_0^\phi \left(\frac{\sigma_r^2 + \sigma_\theta^2}{2E} - \frac{\nu}{E} \sigma_r \sigma_\theta \right) r \, \mathrm{d}r \, \mathrm{d}\theta \tag{5}$$

Further, the strain energy per radian may be expressed as

$$U_{\text{per radian}} = \int_a^b \left(\frac{\sigma_r^2 + \sigma_\theta^2}{2E} - \frac{\nu}{E} \sigma_r \sigma_\theta \right) r \, \mathrm{d}r \tag{6}$$

Employing the computer algebra program MAPLE to perform the integration, initially it is found that $\int_a^b \sigma_r \sigma_\theta r \, \mathrm{d}r = 0$, and so the strain energy is independent of Poisson's ratio, as would be expected, and the expression

$$U_{\text{per radian}} = \frac{M^2 (b-a)^4 (a+b)}{3EIN} = \frac{M^2 (b-a)^4 (a+b)}{3EI\{(b^2-a^2)^2 - 4a^2b^2[\log(b/a)]^2\}} \tag{7}$$

with $I = (b-a)^3/12$ for a beam of unit width. This can now be equated to the expression $U_{\text{per radian}} = M^2 R_{\text{eff}}/(2EI)$, to give the effective radius as

$$R_{\text{eff}} = \frac{2(b-a)^4(a+b)}{3N} \tag{8}$$

For the case when $b = 2a$, as in reference [**3**, article 29], it is found that $R_{\text{eff}} = 1.5235a$, which is greater than the centre-line radius $R_{\text{cl}} = 1.5a$ by some 1.6 per cent and, as will be seen, some 5.5 per cent greater than the radius of the neutral axis. It can be argued that this is a consequence of the fact that the greatest circumferential stress occurs at the inner radius. For the straight beam, a linear stress variation across the depth implies that regions in tension and compression contribute equally to the strain energy. For the curved beam shown in Fig. 1, the smaller region below the neutral axis has a tensile stress σ_θ whose magnitude is greater than that of the compressive stress σ_θ acting over the larger region above the neutral axis; thus the larger region contributes more of a lesser stress to the total strain energy. The greater effective length in the expression for the strain energy is the compensation for this lesser contribution.

Table 1 Effective and neutral axis radii, and percentage deviations from the centre-line radius

Centre-line radius R_{cl}/d	Radius of neutral axis R_{na}/d	Effective radius R_{eff}/d
0.75	0.6368 (−15.1%)	0.8076 (+7.87%)
1	0.9154 (−8.47%)	1.0381 (+3.82%)
1.5	1.4440 (−3.73%)	1.5235 (+1.57%)
2.5	2.4666 (−1.34%)	2.5136 (+0.54%)
5	4.9833 (−0.33%)	5.0067 (+0.13%)
12.5	12.4933 (−0.05%)	12.5027 (+0.02%)
25	24.9967 (−0.013%)	25.0013 (+0.005%)
50	49.9983 (−0.003%)	50.0007 (+0.001%)

The neutral axis is located by setting the expression for stress σ_θ equal to zero, and solving for r; MAPLE provides an analytical solution in terms of the Lambert W function [**4**]; so it is more useful to proceed numerically and, for $b = 2a$, it is found that $R_{\text{na}} = 1.4440a$. This is very close to the value given in Fig. 43 of reference [**3**], which is $1.443a$; note that R_{na} is 3.73 per cent less than the centre-line radius. Both the effective and the neutral axis radii are shown in Table 1 for a curved beam over a range of centre-line radii, normalized with respect to the beam depth $d = b - a$; the example considered above is then equivalent to $b = 2d$, $a = d$, and $R_{\text{cl}} = 3d/2$. As would be expected, both radii converge on to the centre-line radius as the latter becomes large, i.e. as the straight beam is approached.

Over the range considered in Table 1, the neutral axis is located towards the centre of curvature by twice to three times as much as the effective radius is located outwards. The implication for the original problem described in reference [**2**], where an effective radius is not known, is that maximum accuracy is achieved if the centre-line length is employed; moreover, since the strain energy is linearly dependent on radius, the potential error is indicated in the third column of Table 1.

REFERENCES

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3 **Timoshenko, S. P.** and **Goodier, J. N.** *Theory of Elasticity*, 3rd edition, 1982 (McGraw-Hill, New York).
4 **Weisstein, E. W.** Lambert W-function, http://mathworld.wolfram.com/LambertW-Function.html.

APPENDIX

Notation

a, b	inner and outer radii respectively of the curved beam
d	beam depth $= b - a$
e	neutral axis offset from centre-line
E	Young's modulus of elasticity
G	shear modulus of elasticity
I	second moment of area
L	length
M	bending moment
N	see equation (4)
r, θ	polar coordinates
R	radius
U	strain energy
x, y	Cartesian coordinates
ν	Poisson's ratio
σ	direct stress
τ	shearing stress