Alma Mater Studiorum Università di Bologna
Laurea Magistrale in Fisica del Sistema Terra
Corso: Oceanografia Costiera

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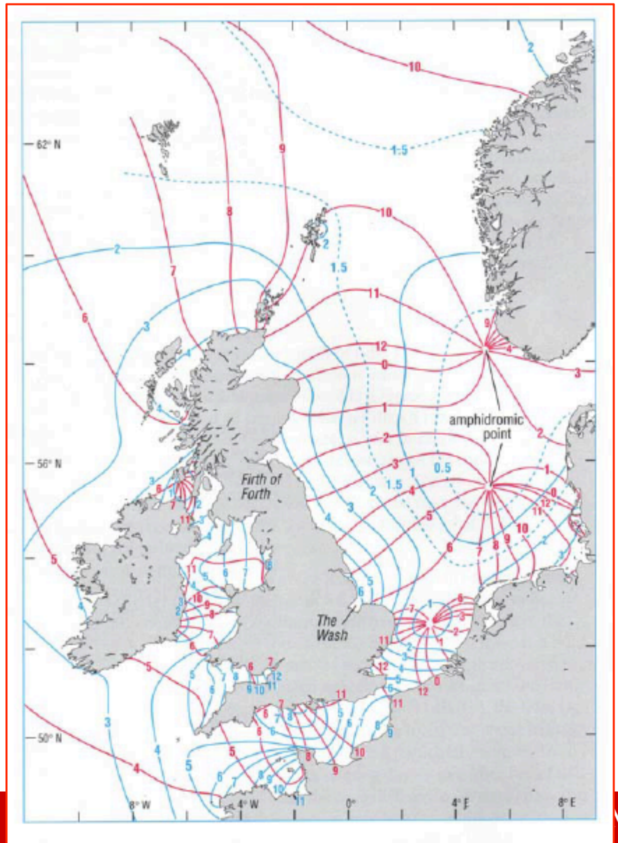
Tides (Part 2)



Equilibrium tide vs. :"real" tide"

Real tides do not exactly behave as equilibrium tides because:

Tidal waves are shallow water waves, whose propagation velocity is $c = \sqrt{gH}$

For $H=4000$ m (average depth of the global ocean) the propagation speed is

$$c = \sqrt{gH} \cong 198 ms^{-1}$$

Assuming the moon over the equator, the linear velocity of the earth surface relative to the moon is given by:

$$v_e = \frac{2\pi R^2}{d_m} = \frac{2\pi\left(6370\cdot 10^3\right)^2}{89400} \cong 449 ms^{-1}$$

R (radius of the earth) =6370km

d_m (lunar day)=24h50min=89400 s.

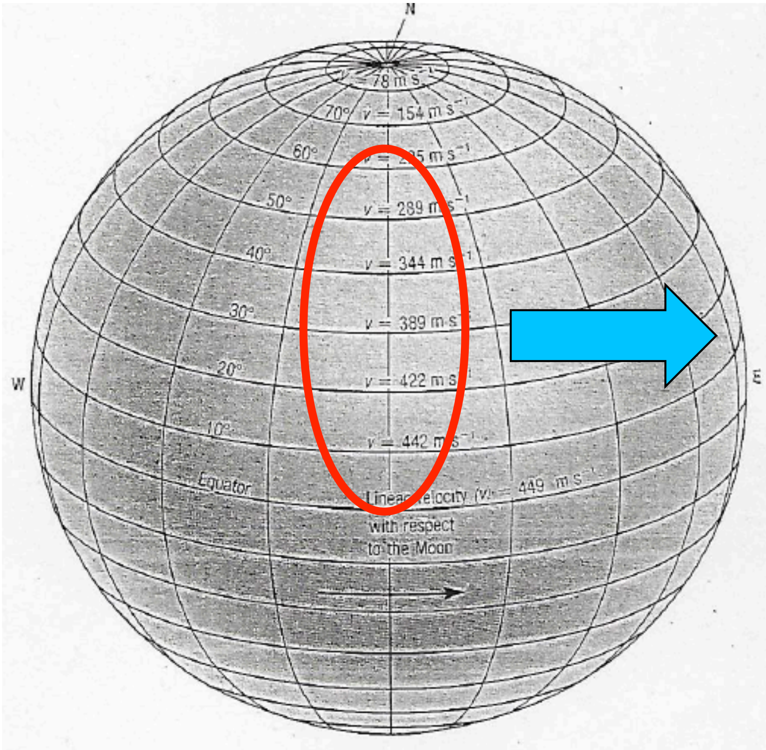
At such speed the "bulge" of the equilibrium tide would remain constantly under the moon, at the sublunar point (and at its antipode), **BUT**,In order to have a shallow water wave propagating at such speed, the ocean depth should be

$$H = \frac{c^2}{g} = \frac{(448)^2}{9.81} \cong 20.5 km$$

Clearly an equilibrium tide at the equator is not possible.

Equilibrium tide vs. :"real" tide"

Real tides do not exactly behave as equilibrium tides because:



Equilibrium tide impossible

The linear velocity of the earth surface decrease with latitude and becomes comparable with the shallow water wave propagation speed (computed at h=4000m) only above 60°N or S.

Then an equilibrium wave is not possible because of depth or because earth linear velocity (with respect to moon) over a very large part of the ocean.

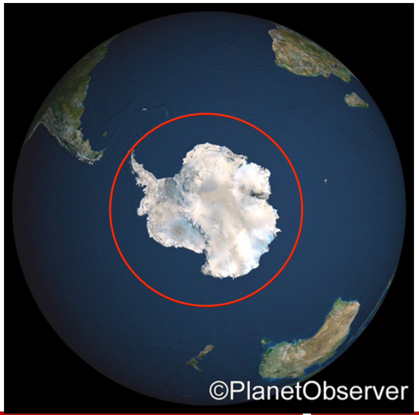
Equilibrium tide vs. :"real" tide"

Real tides do not exactly behave as equilibrium tides because:

Earth rotate on its axis too rapidly for either the inertia of water mass, or the frictional forces at the seabed to be overcome fast enough to allow for an equilibrium tide.

There is an inevitable time lag in the ocean response to the TGF
Usually HT arrive some hours after the passage of the moon.
The tidal lag decrease with latitude :
6 hr at the equator
0 at 65° N/S

The presence of land masses prevent tidal bulges to circumnavigate the planet (possible only in the circumpolar current region around antarctica.



Last but not least….

The Coriolis force deflects water movements including tidal currrents.

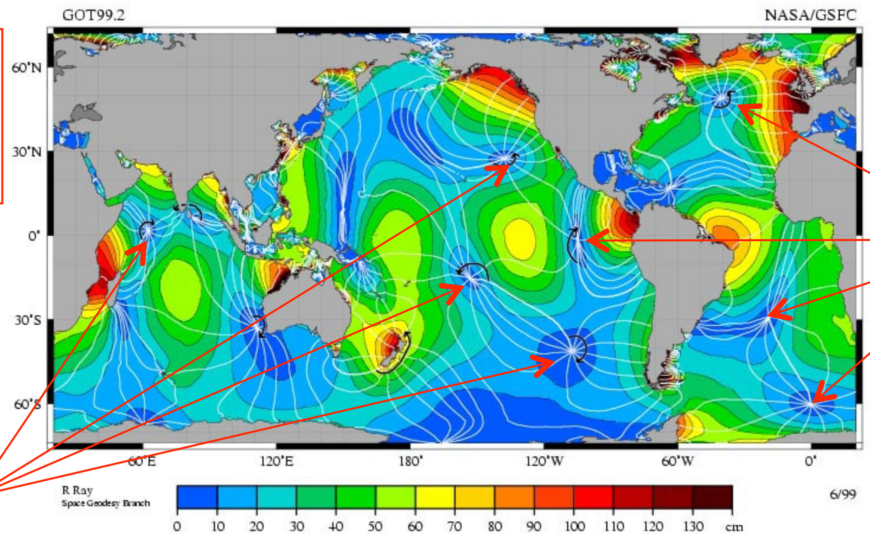
Dynamic theory of tides

Understanding tides by considering how depth basins shape, rotation inertia and friction influence the behaviour of fluids under the action of the moon/sun TGF.

Complex theory with complicated solutions of the equations. Currently relying strongly on numerical computation.

The combined constraint of ocean basin geometry and the influence of the Coriolis force results in the development of **amphidromic systems**, where the crest of a tidal wave circulate around the amphidromic point once over a tidal period. At the amphidromic point the tidal range is zero.

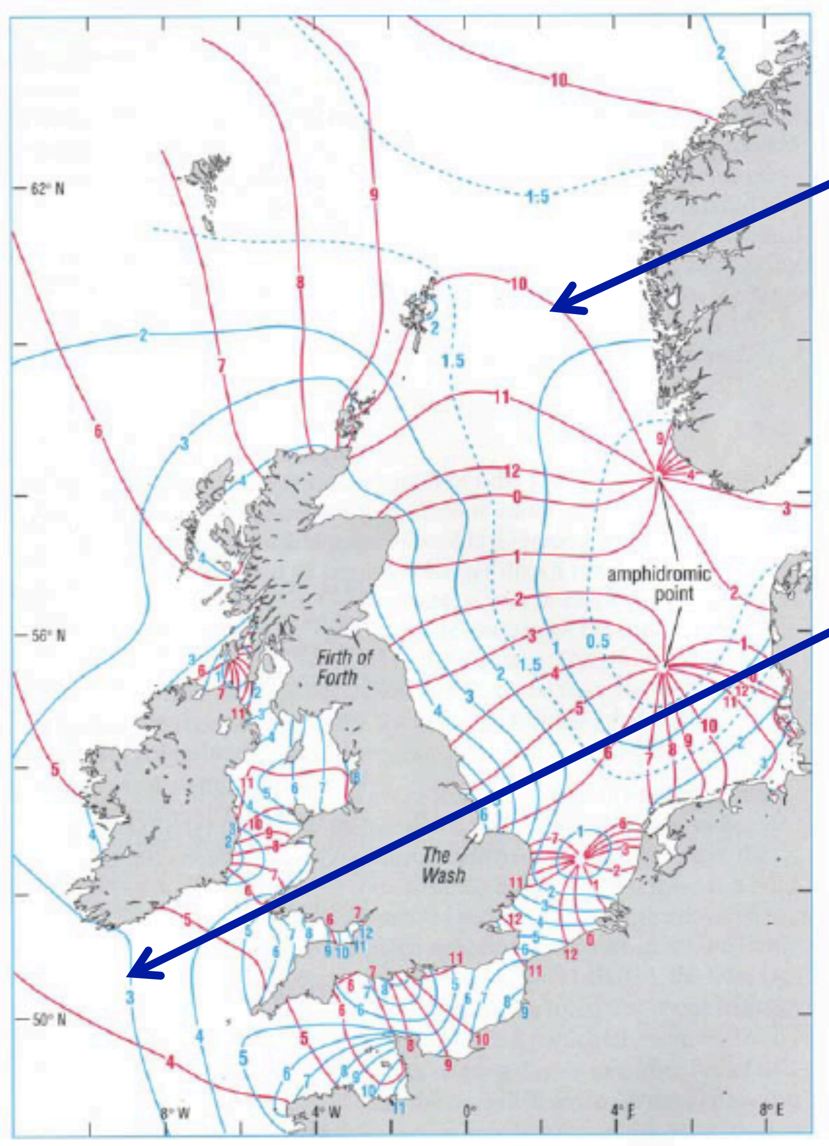
M2
Amphidromic systems
In the global ocean



M2
Amphidromic points

M2
Amphidromic points

Dynamic theory of tides



For each amphidromic systems (of a specific tidal constituent) the theory define
“co-tidal lines” (red lines)
radiating from the amphidromic point, indicating the timing of HT in hours after the passage of the moon over the Greenwich meridian, and marking the rotation of the tidal wave around the amphidromic point.

The
“co-range” lines” (blue lines)
cutting the co-tidal lines at (almost) 90° indicates the location characterised by the same tidal range. More or less concentric circles around the amphidromic point.

Equation of motion for tidal analysis

The equation of motion under the hydrostatic and Boussinesq approximation can be written as:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + F_x$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + F_y$$

$$\frac{\partial p}{\partial z} = -\rho g \qquad\qquad p = p_a + \rho g(\eta - z)$$

p_a=atmospheric pressure

F_x and F_y indicate the components of force (per unit mass) other that the pressure force.
To the above equations must be (obviously) added the continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Equation of motion for tidal analysis

Under constant p_a and under barotropic conditions:

$$\frac{du}{dt} - fv = -g\frac{\partial \eta}{\partial x} + F_x$$

$$\frac{dv}{dt} + fu = -g\frac{\partial \eta}{\partial y} + F_y$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Equation of motion for tidal analysis

$$\frac{du}{dt} - fv = -g\frac{\partial \eta}{\partial x} + F_x$$

$$\frac{dv}{dt} + fu = -g\frac{\partial \eta}{\partial y} + F_y$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

F_x and F_y include (as usual) the vertical frictional stresses in the water (horizontal stress are disregarded)

$$\frac{1}{\rho_0}\frac{\partial \tau_x}{\partial x} \qquad \frac{1}{\rho_0}\frac{\partial \tau_y}{\partial y}$$

but now also the the TGF

$$-\frac{\partial \Omega}{\partial x} \qquad -\frac{\partial \Omega}{\partial y}$$

since $\Omega = -g\eta_T$

we have:

$$F_x = g\frac{\partial \eta_T}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z} \qquad F_y = g\frac{\partial \eta_T}{\partial y} + \frac{1}{\rho_0}\frac{\partial \tau_y}{\partial z}$$

Where (from now on) η_T indicates the elevation in the equilibrium tide

Equation of motion for tidal analysis

The equation system therefore becomes:

$$\frac{du}{dt} - fv = -g\frac{\partial}{\partial x}\left(\eta - \eta_T\right) + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}$$

$$\frac{dv}{dt} + fu = -g\frac{\partial}{\partial y}\left(\eta - \eta_T\right) + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Equation of motion for tidal analysis

And after linearisation………

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial}{\partial x}\left(\eta - \eta_T\right) + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial}{\partial y}\left(\eta - \eta_T\right) + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Equation of motion for tidal analysis

Posing:

$$\overline{u} = \frac{1}{H+\eta}\int_{-H}^{\eta} u\,dz \qquad \overline{v} = \frac{1}{H+\eta}\int_{-H}^{\eta} v\,dz$$

So that $\overline{u}$ and $\overline{v}$ are the vertically averaged components of the velocity field.

Defining also

$$\frac{1}{H+\eta}\int_{-H}^{\eta}\left(\eta-\eta_T\right)dz = \left(\eta-\eta_T\right) \qquad \frac{1}{H+\eta}\int_{-H}^{\eta}\left(\tau_x,\tau_y\right)dz = \frac{\left[\left(\tau_w^{(x)},\tau_W^{(y)}\right)-\left(\tau_B^{(x)},\tau_B^{(y)}\right)\right]}{\left(H+\eta\right)}$$

The equation for the vertically averaged flow equations are obtained (NOT transport equations)

$$\frac{\partial\overline{u}}{\partial t} - f\overline{v} = -g\frac{\partial}{\partial x}\left(\eta-\eta_T\right) + \frac{\tau_w^{(x)}-\tau_B^{(x)}}{\rho_0\left(H+\eta\right)}$$

$$\frac{\partial\overline{v}}{\partial t} + f\overline{u} = -g\frac{\partial}{\partial y}\left(\eta-\eta_T\right) + \frac{\tau_w^{(y)}-\tau_B^{(y)}}{\rho_0\left(H+\eta\right)}$$

Equation of motion for tidal analysis

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial}{\partial x}(\eta - \eta_T) + \frac{\tau_w^{(x)} - \tau_B^{(x)}}{\rho_0 (H + \eta)}$$

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = -g\frac{\partial}{\partial y}(\eta - \eta_T) + \frac{\tau_w^{(y)} - \tau_B^{(y)}}{\rho_0 (H + \eta)}$$

By the same procedure the continuity equation becomes:

$$\frac{\partial}{\partial x}\left[(H + \eta)\bar{u}\right] + \frac{\partial}{\partial y}\left[(H + \eta)\bar{v}\right] + \frac{\partial \eta}{\partial t} = 0$$

In the following the system is not forced by the wind stress $\tau_w^{(x)} = \tau_w^{(y)} = 0$ Then:

$$\frac{\partial \bar{u}}{\partial t} - f\bar{v} = -g\frac{\partial}{\partial x}(\eta - \eta_T) - \frac{\tau_B^{(x)}}{\rho_0 (H + \eta)}$$

$$\frac{\partial \bar{v}}{\partial t} + f\bar{u} = -g\frac{\partial}{\partial y}(\eta - \eta_T) - \frac{\tau_B^{(y)}}{\rho_0 (H + \eta)}$$

Equation of motion for tidal analysis

$$\frac{\partial \overline{u}}{\partial t} - f\overline{v} = -g\frac{\partial}{\partial x}\left(\eta - \eta_T\right) - \frac{\tau_B^{(x)}}{\rho_0\left(H+\eta\right)}$$

$$\frac{\partial \overline{v}}{\partial t} + f\overline{u} = -g\frac{\partial}{\partial y}\left(\eta - \eta_T\right) - \frac{\tau_B^{(y)}}{\rho_0\left(H+\eta\right)}$$

The bottom stress arise form a linear drag law $\left(\tau_b^{(x)},\tau_b^{(y)}\right) = c_d\rho_0\overline{U}\left(\overline{u},\overline{v}\right) = C\rho_0\left(\overline{u},\overline{v}\right)$

with $\overline{U} = \left(\overline{u}^2 + \overline{v}^2\right)^{1/2}$ amplitude of the depth averaged current

Assuming also $H << \eta$ the equations above become:

$$\frac{\partial \overline{u}}{\partial t} - f\overline{v} = -g\frac{\partial}{\partial x}\left(\eta - \eta_T\right) - \frac{C\overline{u}}{H}$$

$$\frac{\partial \overline{v}}{\partial t} + f\overline{u} = -g\frac{\partial}{\partial y}\left(\eta - \eta_T\right) - \frac{C\overline{v}}{H}$$

Equation of motion for tidal analysis

$$\frac{\partial \overline{u}}{\partial t} - f\overline{v} = -g\frac{\partial}{\partial x}\left(\eta - \eta_T\right) - \frac{C\overline{u}}{H} \qquad \frac{\partial \overline{v}}{\partial t} + f\overline{u} = -g\frac{\partial}{\partial y}\left(\eta - \eta_T\right) - \frac{C\overline{v}}{H}$$

The linearised equation may be solved for any particular harmonic tidal constituent (given the appropriate boundary conditions. The solution for each constituent may be superposed to give the resultant tide.

The time varying resultant elevation at a given place can be written: $\eta = \sum_{i=1}^{N} H_i \cos\left(\sigma_i t + \alpha_i - \gamma_i\right)$

i refers to the to the i^{th} harmonic constituents considered

N: total number of harmonic constituents

H_i: amplitude of the constituent

ω_i: angular speed

α_i : phase of the constituent at $t=0$ in the equilibrium tide

γ_i: phase lag of the constituent In the actual tide behind that of the same constitueni in the equilibrium tide

N.B.: from now on the overbars $\left(\overline{u}, \overline{v}\right)$ will be omitted.

It is useful in the study of the tides to regards them as being due to the superposition of waves of various type generated by the TGF. Such waves are forced waves, but their amplitude is increased if there is a tendency to resonance between tidal forces and free waves.

Useful insight into tidal movements may be obtained by considering free waves in the ocean

Progressive waves

The linearised equations for free waves travelling in an ocean of constant depth, without friction With $\eta_T=0$ are:

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial \eta}{\partial t}$$

A particular solution may be obtained for a progressive wave travelling in the x direction ($v=0$):

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

$$fu = -g\frac{\partial \eta}{\partial y}$$

$$H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

Progressive waves

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad fu = -g\frac{\partial \eta}{\partial y} \qquad H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

u can be eliminated by differentiating:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

with respect to x and

$$H\frac{\partial u}{\partial x} = -\frac{\partial \eta}{\partial t}$$

with respect to t, to obtain:

$$\frac{\partial^2 \eta}{\partial t^2} = gH\frac{\partial^2 \eta}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 \eta}{\partial t^2} = c^2\frac{\partial^2 \eta}{\partial x^2}$$

N.B.: compare with the 2D equation obtained when dealing with the wind setup

By analogous procedure a similar equation can be obtained for u:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Progressive waves

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \frac{\partial^2 \eta}{\partial x^2} \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

A wave of general form travelling in the x direction satisfying the above equations is:

$$\eta = Y(y)F(x-ct) \qquad u = \frac{c}{H}Y(y)F(x-ct)$$

From $fu = -g\frac{\partial \eta}{\partial y}$ it can be found: $Y(y) = Ae^{-fy/c} = Ae^{-y/R}$

with: $A=constant$ and $\boxed{R = \frac{C}{f} = \frac{\sqrt{gH}}{f}}$ the barotropic deformation radius.

The factor $e^{-y/R} = e^{-fy/c}$ indicates that the wave amplitude decrease exponentially in the y direction.

This type of motion is known as the **Kelvin wave.**

Kelvin Wave

As a special case consider a wave of simple harmonic form of wavelength λ by setting:

$$F(x-ct)=\cos\frac{2\pi}{\lambda}(x-ct)$$

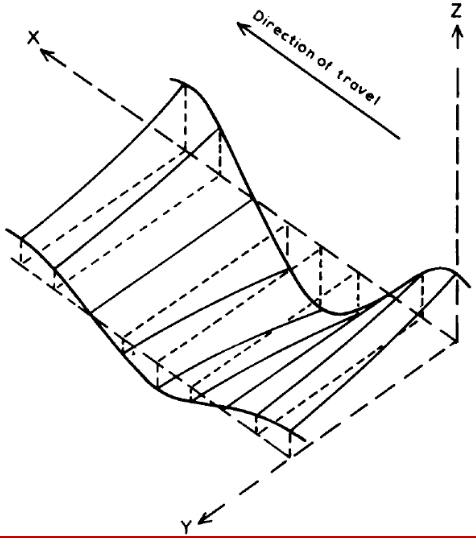
The period is given by $T=\lambda/c$ .Setting $\varkappa=2\pi/\lambda$ and $\sigma=2\pi/T$

The solution is

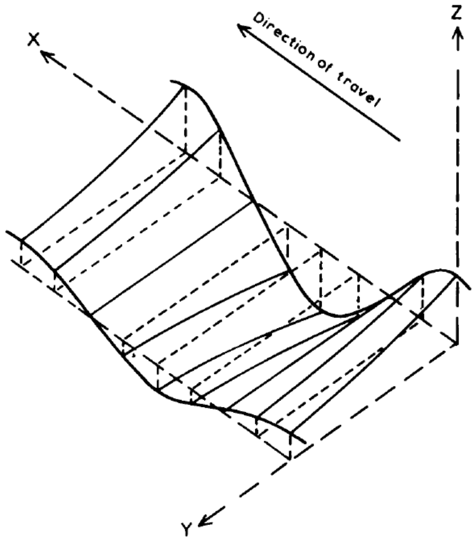
$$\eta = Ae^{-y/R}\cos(\kappa x-\sigma t) \qquad u=\frac{c}{H}\eta$$

These equations represents a Kelvin wave travelling in the x direction.

Amplitude increase exponentially in the negative y direction if $f>0$ Amplitude increase to the right of the propafation direction (n. Hemisphere



Kelvin Wave



Having: $H=4000\ m$ we get $c=98\ m\ s^{-1}$

For a M2 tidal wave ($T=12h25min$) the relation $\lambda=ct$ gives $\lambda=8850$ km .

The rate of change of amplitude across the wave front depend on latitude:

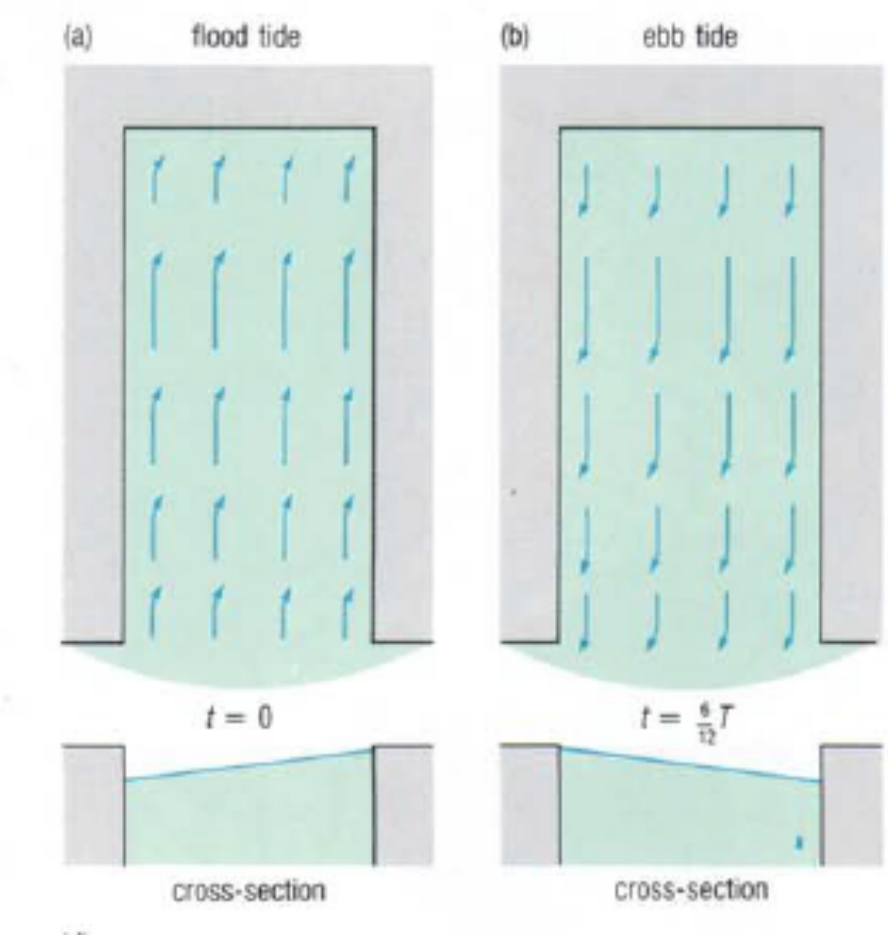
At 30°N $f=7.29\ 10\text{-}5\ s^{-1}$. then $R=2720\ km$. The amplitude would then increase by a factor e in a distance of 2720 (to the right along the wavefront).

Open ocean tidal waves amplitude (A) are at most $0.5\ m\ s^{-1}$. then $(cA)/H=2.5\ cm\ s^{-1}$ a typical value for tidal current in the ocean.

However, a wave with an indefinite growth in amplitude is not realistic.

In fact a Kelvin wave is found travelling parallel to the coast with the coast at its right (N. Hemisphere), or along the equator ($f=0$)

Amphidromic system development



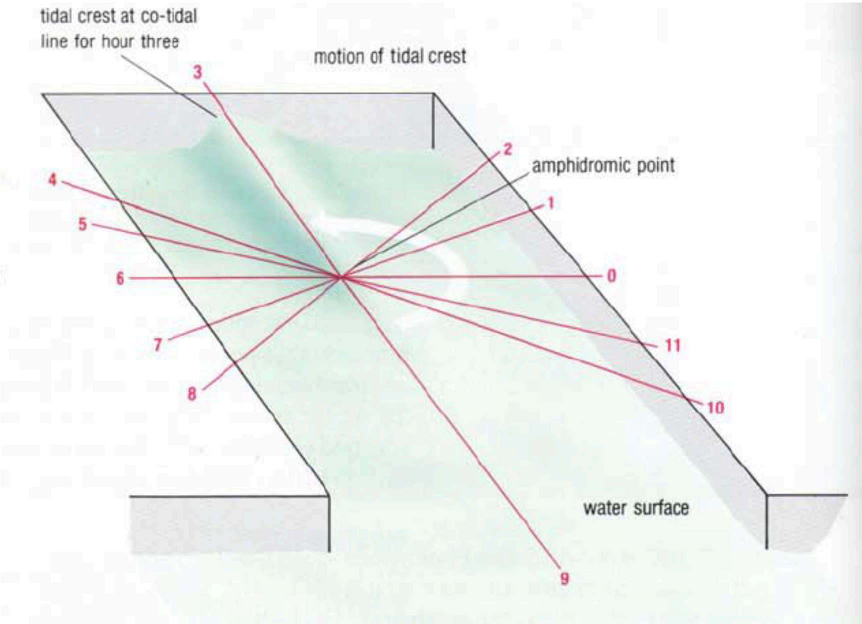
Consider the enclosed basin below
The bent arrows show how water is moving in a flood and ebb tide and how Coriolis force deflects The currents (toward the “right”/”left” coast at flood/ebb) .

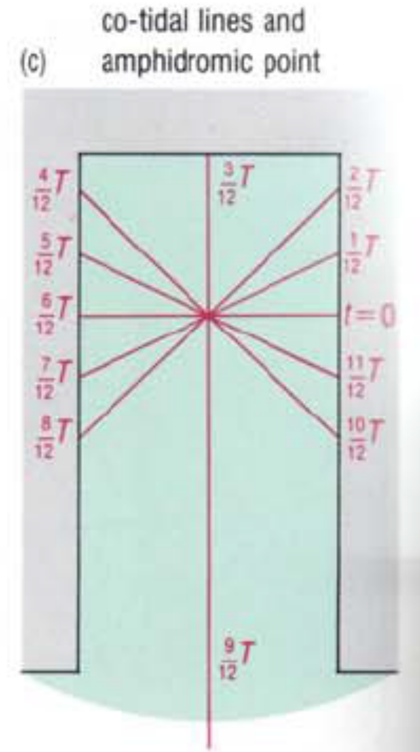
The deflection is constrained by the two coasts

Water is then periodically piled up against the two coasts.

The coastal constraints cause the tidal wave to behave as a Kelvin wave, rotating along the coast of the basin.

Amphidromic system development





The coastal constraints cause the tidal wave to behave as a Kelvin wave, rotating along the coast of the basin.

An amphidromic system is then generated, With a kelvin wave running parallel to the coast around the amphidromic point (location where the Kelvin wave amplitude is Zero)

Here co-tidal lines are "time-labelled" in twelwfth of the tidal period (for M2 12h25min)