Section 1.1: Definition of Functions

Definition of a function

A *function* f from a set A to a set B ($f : A \to B$) is a rule of correspondence that assigns to each element x in the set A **exactly** one element y in the set B. The set A is called the *domain* of the function f. The *range* or *codomain* of the function is the set of elements in B that are in correspondence with elements in A.
In the case of functions described as equations in two variables, the variable x is the *independent* variable and the variable y is the *dependent* variable. In general a function is denoted as $f(x)$ (read *f of x*), where f is the name of the function, x is the domain value and $f(x)$ is the range value y for a given x. The process of finding the value of $f(x)$ for a given value of x is called *evaluating a function*.

Ex.1

- Demand function: $Q_d = f(P) = 15 - 2P$.
- Supply function: $Q_s = g(P) = 1 + 5P$.
- Cobb-Douglas production function: $Q(K, L) = K^\alpha L^\beta$.
- Cobb-Douglas utility function: $U(X, Y) = a \log(X) + (1 - a) \log(Y)$.

Ex.2
Constant functions are functions that assign every object in the domain to the same object in the target. For example, $f(x) = 3$ is a constant function. The *identity function* is the function that assigns every object in the domain to itself, that is $f(x) = x$ for every x in the domain.

Ex.3
Let $f(x) = x^2$. Find the domain and the range of $f(x)$. Compute

- $f(3)$
- $f(2)$
- $f(-2)$
- $f(3 + h)$

Ex.4
Find the domain and the range of $f(x) = 1/x$.

Ex.5
Find the domain and the range of $f(x) = 5 - \sqrt{x - 1}$.

Graph of a function

Let $f(x)$ be a function. The *graph of the function f* consists of those points (x, y) such that $y = f(x)$. Not every curve is the graph of a function. The reason is that a function assigns to a given input a single number as the output. A line parallel to the y axis therefore meets the graph of a function in at most one point. Hence, if some line parallel to the y axis meets the curve more than once, then the curve is NOT the graph of a function.

Ex.6
Graph the function $f(x) = x^2$.

Ex.7
Graph the function $f(x) = 1/x$.

Zoo of function

In mathematics there are many kinds of functions. Here is a short list of some of them:

- Polynomial functions: linear (ex. $f(x) = 2x - 1$), quadratic (ex. $f(x) = -x^2$), cubic (ex. $f(x) = 4x^3 - 3x^2 + 5$)
- Rational functions (ex. $f(x) = \frac{2}{x-3}$)
- Irrational functions (ex. $f(x) = \sqrt{2-x}$)
- Absolute value functions (ex. $f(x) = |x-9|$)
- Exponential functions (ex. $f(x) = 2^x$)
- Logarithmic functions (ex. $f(x) = \log_2(x)$)
- Trigonometric functions (ex. $f(x) = \sin(x)$, $f(x) = \cos(x)$, $f(x) = \tan(x)$)

Transformations of functions

Let $f(x)$ be a function, then

- $y = f(x) + C$
  - $C > 0$ moves it up
  - $C < 0$ moves it down
- $y = f(x + C)$
  - $C > 0$ moves it left
  - $C < 0$ moves it right
- $y = Cf(x)$
  - $C > 1$ stretches it in the y-direction
  - $0 < C < 1$ compresses it
- $y = f(Cx)$
  - $C > 1$ compresses it in the x-direction
  - $0 < C < 1$ stretches it
- $y = -f(x)$ reflects it about x-axis
- $y = f(-x)$ reflects it about y-axis

Ex.8
Graph the function $f(x) = \frac{1}{x^2+1}$.

Definition of composition of two functions

The *composition* of the functions f and g is given by

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $(f \circ g)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Ex.9
Let $f(x) = 1 + 2x$ and $g(x) = x^2$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

Ex.10
Let $f(x) = \sqrt{x}$ and $g(x) = x^3 - 1$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

Ex.11
Let $f(x) = \sqrt{x}$ and $g(x) = x^2$. Compute $(f \circ g)(x)$ and $(g \circ f)(x)$.

Even and odd functions
A function $f(x)$ such that
$$f(-x) = f(x)$$
is called an *even function*.
A function $f(x)$ such that
$$f(-x) = -f(x)$$
is called an *odd function*.

Ex.12
The function $f(x) = x^4$ is an even function. The function $g(x) = x^3$ is an odd function.

Section 1.2: Bijective Functions and Inverse Functions

Bijective functions
A function $f : X \to Y$ that assigns distinct outputs to distinct inputs is called an *injective or one-to-one function*. Hence, a function is injective if for every $a, b \in X$ such that $f(a) = f(b)$, then $a = b$. The graph of a one-to-one function has the property that every horizontal line meets it in at most one point and if each horizontal line meets the graph of a function in at most one point, then the function is one-to-one.
The function is *surjective* or *onto* if every element of the codomain is mapped to by at least one element of the domain. Hence, a function $f : X \to Y$ is surjective if the range of f is Y.
A function is *bijective* if it is BOTH injective and surjective.

Monotonic functions
If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is an *increasing function*. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is a *decreasing function*. These two types of functions are called *monotonic*.

Inverse functions
Let $f(x)$ be a one-to-one function. The function $g(x)$ that assigns to each output of f the corresponding unique input is called the *inverse* of f. The symbol f^{-1} denotes the inverse function.

Ex.1
Determine the inverse of the following functions and then graph them.
- $f(x) = 2x$
- $f(x) = x^3$
- $f(x) = 3x + 2$

Section 1.3: Limits

Definition of limit 1

The *limit* of $f(x)$ as x approaches x_0 is the number L if given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about x_0 such that for all x,

$$0 < |x - x_0| < \delta$$

implies $|f(x) - L| < \varepsilon$. In other words, if the values of a function $f(x)$ approach the value L as x approaches x_0, we say that f has limit L as x approaches x_0 and we write

$$\lim_{x \to x_0} f(x) = L.$$

The limit of $f(x)$ as x approaches x_0 *from the right* is the number L if given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about x_0 such that for all x,

$$x_0 < x < x_0 + \delta$$

implies $|f(x) - L| < \varepsilon$. We write

$$\lim_{x \to x_0^+} f(x) = L.$$

The limit of $f(x)$ as x approaches x_0 *from the left* is the number L if given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about x_0 such that for all x,

$$x_0 - \delta < x < x_0$$

implies $|f(x) - L| < \varepsilon$. We write

$$\lim_{x \to x_0^-} f(x) = L.$$

A function has a limit as x approaches x_0 if and only if the right-hand and left-hand limits at x_0 exist and are equal.

Ex.1
Find

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Ex.2
Let

$$f(x) = \begin{cases} 2 & \text{if} \quad x \geq 3 \\ -1 & \text{if} \quad x < 3 \end{cases}$$

Find $\lim_{x \to 3} f(x)$.

Ex.3
Find

$$\lim_{x \to 5} \frac{3}{x - 5}$$

Ex.4
Show that the function $y = \sin(1/x)$ has no limit as x approaches zero from either side.

Proof: As $x \to 0$, its reciprocal $\frac{1}{x}$ becomes infinite and the value of $\sin(1/x)$ cycles repeatedly from -1 to 1. Thus there is no single number L such that the function's values get close to a single value when $x \to 0$. This is true even if we restrict x to positive values or to negative values, therefore the function has neither a right-hand limit nor a left-hand limit as x approaches zero. In conclusion, the function $y = \sin(1/x)$ has no limit from either side as $x \to 0$.

Properties of limits

If $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, then

- Sum rule:
$$\lim_{x\to x_0} [f(x) + g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = L_1 + L_2$$
- Difference rule:
$$\lim_{x\to x_0} [f(x) - g(x)] = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x) = L_1 - L_2$$
- Product rule:
$$\lim_{x\to x_0} [f(x) \cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = L_1 \cdot L_2$$
- Constant multiple rule:
$$\lim_{x\to x_0} [k \cdot g(x)] = k \cdot \lim_{x\to x_0} g(x) = k \cdot L_2$$
for any number k.
- Quotient rule:
$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2}$$
if $L_2 \neq 0$.

Ex.5
Prove:

- If $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$, then
$$\lim_{x\to x_0} [f(x) + g(x)] = L_1 + L_2$$
- $\lim_{x\to 2} x + 5 = 7$
- $\lim_{x\to 5} \sqrt{x-1} = 2$

Ex.6
Compute the following limits:
- $\lim_{x\to 3} x^2(2-x)$
- $\lim_{x\to 2} \frac{x^2+2x+4}{x+2}$
- $\lim_{x\to -5} \frac{x^2-5}{3(x+5)}$
- $\lim_{t\to \pi} \frac{3+\sin(t)}{1-\cos(t)}$
- $\lim_{t\to 3} \frac{\sqrt{3t+7}-\sqrt{7}}{2}$
- $\lim_{x\to -5} \frac{x^2-25}{2(x+5)}$
- $\lim_{x\to -2}(x^3+3x^2-2x-17)$
- $\lim_{x\to -1^+} \frac{x+3}{x^3+3x+1}$
- $\lim_{x\to 2} \frac{x+3}{x+6}$
- $\lim_{y\to -3} \frac{y^2}{3-y}$
- $\lim_{x\to 6} 8(x-5)(x-7)$
- $\lim_{x\to -3} \sqrt{x+7}$
- $\lim_{x\to 0} \frac{5}{\sqrt{5x+4}+2}$
- $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$
- $\lim_{v\to 2} \frac{v^3-8}{v^4-16}$

Definition of limit 2

Let $f(x)$ be a function defined on an interval that contains x_0, except possibly at x_0. Then we say that

$$\lim_{x\to x_0} f(x) = +\infty$$

if for every $M > 0$ there is some number $\delta > 0$ such that $f(x) > M$ for all x such that $0 < |x - x_0| < \delta$.
Let $f(x)$ be a function defined on an interval that contains x_0, except possibly at x_0. Then we say that

$$\lim_{x\to x_0} f(x) = -\infty$$

if for every $N < 0$ there is some number $\delta > 0$ such that $f(x) < N$ for all x such that $0 < |x - x_0| < \delta$.

Ex.7
Compute the following limits:
- $\lim_{x\to 0} \frac{1}{x^2}$
- $\lim_{x\to 0^+} \frac{1}{x}$
- $\lim_{x\to 0^-} \frac{1}{x}$

Definition of limit 3

Let $f(x)$ be a function defined on $x > K$ for some K. Then we say that

$$\lim_{x\to +\infty} f(x) = L$$

if for every $\varepsilon > 0$ there is some number $M > 0$ such that $|f(x) - L| < \varepsilon$ for all x such that $x > M$.
Let $f(x)$ be a function defined on $x < K$ for some K. Then we say that

$$\lim_{x\to -\infty} f(x) = L$$

if for every $\varepsilon > 0$ there is some number $N < 0$ such that $|f(x) - L| < \varepsilon$ for all x such that $x < N$.

<u>Definition of limit 4</u>
Let $f(x)$ be a function defined on $x > K$ for some K. Then we say that

$$\lim_{x\to+\infty} f(x) = +\infty$$

if for every $N > 0$ there is some number $M > 0$ such that $f(x) > N$ for all x such that $x > M$.
Let $f(x)$ be a function defined on $x < K$ for some K. Then we say that

$$\lim_{x\to-\infty} f(x) = +\infty$$

if for every $N > 0$ there is some number $M < 0$ such that $f(x) > N$ for all x such that $x < M$.
In a similar way we can define $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

Ex.8
Compute the following limits:
- $\lim_{x\to+\infty} \frac{1}{x+3}$
- $\lim_{x\to-\infty} 7 - \frac{1}{x}$
- $\lim_{x\to-\infty} \frac{11x+2}{2x^3-1}$
- $\lim_{x\to+\infty} \frac{2x^2-3}{5x+4}$
- $\lim_{x\to2^+} \frac{7-x}{x-2}$
- $\lim_{x\to1^+} \frac{x^2+5}{3-3x}$
- $\lim_{x\to2^+} \frac{5-3x}{x^2-6x+8}$
- $\lim_{x\to0} \frac{\sqrt{x+9}-3}{4x}$
- $\lim_{x\to-\infty} \frac{\sqrt{8x^{14}-3x^5+x^3-x^2+2}}{5x^4-x^2+x-5}$
- $\lim_{x\to+\infty} \frac{5x^2-4x-2\sqrt{x}+3}{2x^2-x+\sqrt{x}}$
- $\lim_{x\to1} \frac{\sqrt{x}-1}{x-1}$

<u>Sandwich Theorem</u>
Suppose that $g(x) \le f(x) \le h(x)$ for all $x \neq x_0$ in some open interval about x_0 and that

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L.$$

Then

$$\lim_{x\to x_0} f(x) = L.$$

Ex.9
Compute the following limits:
- $\lim_{x\to0} \sin(x)$
- $\lim_{x\to0} \cos(x)$
- $\lim_{x\to0} \tan(x)$

<u>Theorem</u>
If θ is measured in radians, then

$$\lim_{\theta\to0} \frac{\sin(\theta)}{\theta} = 1.$$

Ex.10
Compute the following limits:

- $\lim_{x\to 0} \frac{\sin(7x)}{7x}$
- $\lim_{x\to 0} \frac{\sin(x/2)}{x/2}$
- $\lim_{x\to 0} \frac{\sin(6x)}{x}$
- $\lim_{x\to 0} \frac{\tan(2x)}{5x}$
- $\lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)}$
- $\lim_{x\to 0} x\sin(1/x)$

<u>Standard Limits</u>
Limits to remember:

- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$
- $\lim_{x\to 0} \frac{\tan(x)}{x} = 1$
- $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\ln(a)}$, $(a > 0)$
- $\lim_{x\to 0} \frac{e^x-1)}{x} = 1$
- $\lim_{x\to 0} \frac{a^x-1}{x} = \ln(a)$, $(a > 0)$
- $\lim_{x\to \pm\infty} (1 + \frac{1}{x})^x = e$
- $\lim_{x\to 0} \frac{(1+x)^c-1}{x} = c$, $(c \in \mathbb{R})$

Ex.11
Compute the following limits:

- $$\lim_{x\to 0} \frac{\log_3(1+3x)}{e^{2x}-1}$$
- $$\lim_{x\to 0} \frac{\sin(x)}{\ln(1+x)}$$
- $$\lim_{x\to +\infty} \left(1 + \frac{1}{2x}\right)^{3x}$$
- $$\lim_{x\to 0} \frac{\tan(2x)}{x}$$
- $$\lim_{x\to 0} \frac{(1+x)^4-1}{x}$$

Section 1.4: Continuous Functions

Definition of continuity

A function $f(x)$ is *continuous at* x_0 if and only if it meets all three of the following conditions:

- $f(x_0)$ exists;
- $\lim_{x \to x_0} f(x)$ exists;
- $\lim_{x \to x_0} f(x) = f(x_0)$.

Continuity at an endpoint:

A function is continuos at a left endpoint a of its domain if $\lim_{x \to a^+} f(x) = f(a)$.

A function is continuos at a right endpoint b of its domain if $\lim_{x \to b^-} f(x) = f(b)$.

A function is *continuous* if it is continuous at each point of its domain. If a function f is not continuous at a point c, we say that f is *discontinuous* at c and call c a point of discontinuity of f.

Ex.1

Sine and Cosine are continuous at $x = 0$.

Properties of continuous functions

If f and g are continuous functions at $x = c$, then

- Sum: $f + g$
- Difference: $f - g$
- Product: $f \cdot g$
- Constant multiple: $k \cdot f$, for any number k.
- Quotient: f/g, provided $g(c) \neq 0$.

are continuous functions at $x = c$.

Moreover, if f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Ex.2

The following functions

- $f(x) = 3x - 5x^2 + \frac{1}{x^2+2}$
- $f(x) = 4x\cos(x)$
- $f(x) = \tan(x)$

are continuous.

Removable and non-removable discontinuities

One single type of discontinuity, called a *removable discontinuity*, occurs whenever $\lim_{x \to c} f(x) \neq f(c)$. We remove the discontinuity by defining $f(c)$ to have the same value as $\lim_{x \to c} f(x) \neq f(c)$.

The removability of a discontinuity of a function at a point $x = c$ requires the existence of $\lim_{x \to c} f(x) = f(c)$. Without it, there is no way to fulfill the conditions of the continuity test, and the discontinuity is *non-removable*.

Ex.3

The function

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}$$

is not defined at $x = 2$. Is $x = 2$ a removable discontinuity? If so, how can you extend the function to make it continuous at $x = 2$?

Ex.4
Solve the following problems:

- Compute
$$\lim_{x\to 3}\frac{x^2-7x+12}{x-3}$$
- Compute
$$\lim_{x\to 4}\frac{x^2+x-20}{x-4}$$
- Compute
$$\lim_{t\to 1}\frac{t^2-3t+2}{t-1}$$
- Let
$$f(x)=\begin{cases} 1+x^2 & \text{if} \quad x<2 \\ x^3 & \text{if} \quad x\geq 2 \end{cases}$$
Find $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$. Does $\lim_{x\to 2} f(x)$ exist?
- Let
$$f(x)=\begin{cases} -5x+7 & \text{if} \quad x<3 \\ x^2-16 & \text{if} \quad x\geq 3 \end{cases}$$
Does $\lim_{x\to 3} f(x)$ exist?
- Let
$$f(t)=\begin{cases} -t & \text{if} \quad t<1 \\ t^2 & \text{if} \quad t\geq 1 \end{cases}$$
Does $\lim_{t\to 1} f(t)$ exist?
- Suppose the total cost $C(Q)$ of producing a quantity Q of a product equals a fixed cost of $\$1000$ plus $\$3$ times the quantity produced.
  (1) Write $C(Q)$.
  (2) Find the average cost per unit quantity $A(Q)$.
  (3) Compute
$$\lim_{Q\to 0^+} A(Q)$$
.

Section 1.5: The Intermediate Value Theorem for Continuous Functions

Intermediate Value Theorem
A function $y = f(x)$ that is continuous on a closed interval $I = [a, b]$ takes on every value between $f(a)$ and $f(b)$.

Connectivity
Suppose we want to graph a function $y = f(x)$ that is continuous throughout some interval I on the x-axis. The Intermediate Value Theorem tells us that the graph of f over I will never move from one y-value to another without taking on the y-values in between. The graph of f over I will be connected, that is it will consist of a single, unbroken curve.

Root finding
Suppose that $f(x)$ is continuous at every point of a closed interval $[a, b]$ and that $f(a)$ and $f(b)$ differ in sign. Then zero lies between $f(a)$ and $f(b)$ differ in sign, so there is at least one number c between a and b where $f(c) = 0$. In other words, if $f(x)$ is continuous and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b). A point c where $f(c) = 0$ is called a *zero* or *root* of f. Hence, the zeros of f are the points where the graph of f intersects the x-axis.

Ex.1
Is any real number exactly 1 less than its cube?

Ex.2
Show that $x^3 - x - 1 = 0$ has a root somewhere in the interval $[-1, 2]$.

Section 1.6: Extreme Value Theorem

Maxima and Minima
Suppose that f is a function which is continuous on the closed interval $[a, b]$. Then there exist real numbers c and d in $[a, b]$ such that

- We say that $f(x)$ has an *absolute (or global) maximum* at $x = c$ if for every x in the domain we are working on we have $f(x) \leq f(c)$.
- We say that $f(x)$ has a *relative (or local) maximum* at $x = c$ if for every x in some open interval around $x = c$, $f(x) \leq f(c)$.
- We say that $f(x)$ has an *absolute (or global) minimum* at $x = d$ if for every x in the domain we are working on we have $f(x) \geq f(d)$.
- We say that $f(x)$ has a *relative (or local) minimum* at $x = d$ if for every x in some open interval around $x = d$, $f(x) \geq f(d)$.

A function f defined on X is called *bounded*, if there exists a real number M such that $|f(x)| \leq M$ for all x in X. A function that is not bounded is said to be *unbounded*. If $f(x) \leq A$ for all x in X, then the function is said to be *bounded above* by A. If $f(x) \geq B$ for all x in X, then the function is said to be *bounded below* by B.

Extreme Value Theorem
Suppose that f is a function which is continuous on the closed interval $[a, b]$. Then there exist real numbers c and d in $[a, b]$ such that

- f has a maximum value at $x = c$ and
- f has a minimum value at $x = d$.

Section 1.7: Piecewise and Uniform Continuous Functions

Piecewise continuity and uniform continuity
A function or curve is *piecewise continuous* if it is continuous on all but a finite number of points at which certain matching conditions are sometimes required.
A function f is *uniformly continuous* if it is possible to guarantee that $f(x)$ and $f(y)$ are as close to each other as we please by requiring only that x and y are sufficiently close to each other: for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in I$ with $|y - x| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
Every uniformly continuous function is continuous, but the converse does not hold. Consider for instance the function $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$. Given an arbitrarily small positive real number ε, uniform continuity requires the existence of a positive number δ such that for all x_1, x_2 with $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \varepsilon$. But

$$f(x + \delta) - f(x) = 2x\delta + \delta^2 = \delta(2x + \delta),$$

and for all sufficiently large x this quantity is greater than ε.

Ex.1
The function

$$f(x) = \begin{cases} x + 4 & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 < x < 5 \\ 7 & \text{if} \quad x \geq 5 \end{cases}$$

is piecewise continuous.

Ex.2
The function $f(x) = 4x - 1$ is uniformly continuous.

Section 1.8: Economic Applications of Continuous and Discontinuous Functions

<u>Introduction</u>
There are many natural examples of discontinuities from economics. In fact, economists often adopt continuous functions to represent economic relationships (that is, they build a continuous model) when the use of discontinuous functions would be a more literal interpretation of reality. It is important to know when the simplifying assumption of continuity can be safely made for the sake of convenience and when it is likely to distort the true relationship between economic variables too much.

Ex.1
We illustrate a class of situations in which it is usual to use a continuous model even if this is a distortion of reality. The first step in modeling the decisions of a firm is usually the analysis of the available technology. This relationship between inputs used and outputs generated is generally presumed to be represented by some production function: $y = f(x)$. What does it mean to say that this function is continuous on some domain (usually $x \geq 0$)? To assume that $f(x)$ is continuous at a point $x = c$ implies that $f(x)$ is defined on some open interval of real numbers containing c. This means x must be *infinitely divisible*: one can choose x to be a value that deviates even by infinitesimal amounts from $x = c$.
An example of input that would not be infinitely divisible would be bolts used in the production of a car. Since one would not use a fraction like a half of a bolt, it would only make literal sense to treat bolts as integer valued. Therefore, it does not make sense to contemplate an open interval of points including some value $x = c$ bolts. However, if we denote by x the number of bolts used and by y the number of cars produced, we have

$$y = \frac{x}{1,050}$$

If one uses this continuous function in the process of solving some value of x that is not a multiple of $1,050$, then using the closest value that is a multiple of $1,050$ would probably be reasonably accurate. Thus, even if a commodity is not infinitely divisible, we may often assume that it is, without distorting realty very much. Draw the graph of this liner function considering the domain of real numbers $x \geq 0$.

Ex.2
Suppose that a salesperson receives a salary according to a contract that establishes a relationship between pay and the level of sales made by the salesperson. In particular, suppose that the contract stipulates that the salesperson's monthly salary will be composed by three parts:

- a basic amount of $\$800$,
- a commission of $\%10$, and
- a lump-sum bonus of $\$500$, if the salesperson's sales for the month reach or exceed $\$20,000$.

Let S represent sales per month and P represent the salesperson's pay for the month, it follows that the function describing her salary-sales relationship is

$$P = \begin{cases} 800 + (0.1)S & \text{if} \quad S < 20,000 \\ 1,300 + (0.1)S & \text{if} \quad S \geq 20,000 \end{cases}$$

Draw the graph of the function. Is the function discontinuous at $S = 20,000$? Can we remove the discontinuity?

Ex.3
Many welfare programs or income support programs offer individuals who are not employed a fixed or lump-sum monthly payment that is made only if the individual does not earn any income. Once an individual earns any income whatsoever, the payment is stopped. For example, suppose a single parent of two preschool-aged children can collect a monthly welfare payment of $750 provided she doesn't work. However, once she earns any positive amount of income, the welfare payment stops. Assume she could earn $15 per hour at some job for which the number hours worked per month is entirely flexible.
The income of this person, Y, as a function of hours worked, h, is given by the function:

$$Y(h) = \begin{cases} 750 & \text{if} \quad h = 0 \\ 15h & \text{if} \quad h > 0 \end{cases}$$

Sketch the graph of the function. It is discontinuous at $h = 0$. This type of discontinuity, which is a property of many *all-or-nothing* income support programs, has been the subject of a great deal of debate. One can see that a person in such a program would have to work 50 hours per month just to match the income earned from the support payments. Since the person would face childcare and other costs of working, the all-or-nothing property of this program presents a serious deterrent to the incentive to work.
An alternative scheme would be to allow a person in this situation to keep a certain fraction of income earned in addition to the $750 monthly payment. For example, suppose that the person was allowed to retain 50% of any earnings, with the other 50% representing a payback of the income support up to the level where the entire $750 is paid back. A person facing a wage rate of $15 per hour will have paid back the full $750 only after working 100 hours or more per month. The 50% of 15 is 7.5.
Hence, the new function is

$$Y(h) = \begin{cases} 750 + 7.5h & \text{if} \quad 0 \le h \le 100 \\ 15h & \text{if} \quad h > 100 \end{cases}$$

Draw the graph of this function. Prove that it is continuous at $h = 100$.
Many economists prefer this second placeboes it avoids the discontinuity of the first plan. In the first plan there is effectively a large penalty for working at all, since income drops from $750 per month to almost zero if the individual chooses only a few hours of work. Under the second plan the person always earns more income by choosing to work more. The result is that the person will be more likely to choose some positive hours of work under the second plan making himself/herself better off and also reducing the cost of the program to the government.

Section 1.9: Economic Applications of the Intermediate Value Theorem

Introduction

The Intermediate Value Theorem is a very powerful tool in the study of equilibrium, which is one of the most important concepts in economics.

Theorem

Let Q_D and Q_S be the demand and the supply function respectively, and let $z(P) = Q_D - Q_S$. If the demand and supply functions are continuous and the following two conditions are satisfied:

- at zero price, demand exceeds supply, $Q_D(0) > Q_S(0)$, meaning that $z(0) > 0$,
- there exists some price $\hat{P}$ at which supply exceeds demand, $Q_S(\hat{P}) > Q_D(\hat{P})$, meaning that $z(\hat{P}) < 0$,

then there exists a positive equilibrium price in the market.

Ex.1

Consider the following market demand and supply functions:

$$Q_D = 100 - 2P$$

$$Q_S = 3P$$

Graph the demand and supply functions on one diagram and $z = Q_D - Q_S$ on another. Find the equilibrium price and quantity for this market and illustrate on both graphs. Show that these demand and supply functions satisfy the conditions for the existence of a positive equilibrium price (look at the previous theorem!).

QUESTIONS AND PROBLEMS

(1) What is a function?
(2) What is a continuous function?
(3) List the properties of continuous functions.
(4) What is the graph of a function?
(5) What is an even function?
(6) What is an odd function?
(7) What is an injective function?
(8) What is a surjective function?
(9) What is a bijective function?
(10) What is a monotonic function? Give an example of monotonic functions.
(11) Define $\lim_{x\to x_0} f(x) = L$.
(12) Prove $\lim_{x\to 1} 2x+3 = 5$.
(13) State and prove the Intermediate Value Theorem.
(14) State and prove the Extreme Value Theorem.
(15) Define a uniformly continuous function.
(16) Find the domain and the range of the following functions and sketch their graphs.
  (a) $f(x) = 4x$
  (b) $f(x) = 5x^2$
  (c) $f(x) = -2x-3$
  (d) $f(x) = x^2 + x$
  (e) $f(x) = \sqrt{x}$
  (f) $f(x) = \sqrt{x+1}$
  (g) $f(x) = \frac{1}{x-2}$
  (h) $f(x) = \frac{1}{x^2-1}$
(17) Graph $f(x) = x^2 - x$. For which values of x is $f(x) = 0$?
(18) Which of the following functions are even? odd? neither?
  (a) $f(x) = 3x^2 + 6x^4$
  (b) $f(x) = |x|$
  (c) $f(x) = |x+2|$
  (d) $f(x) = x^3 - 2x$
  (e) $f(x) = 5x^2 - 9$
  (f) $f(x) = 3x^2 + 4x$
  (g) $f(x) = 1 + x^7$
  (h) $f(x) = \frac{x-1}{x^2}$
  (i) $f(x) = \frac{x^6}{x^2+1}$
  (j) $f(x) = \sin(x)$
  (k) $f(x) = \cos(x)$
  (l) $f(x) = \cos(x) + \sin(x)$
(19) Let $f(x) = 2x^2 - 1$ and $g(x) = 4x + 2$ Find $f \circ g$ and $g \circ f$.
(20) Determine whether each given function is bijective on the given domain. If it is, obtain a formula for its inverse.
  (a) $f(x) = x^4$ on [-1,1]
  (b) $f(x) = x^4$ on [0,2]
  (c) $f(x) = (x-1)^2$ on [1,3]
  (d) $f(x) = (x-1)^2$ on [0,2]
  (e) $f(x) = 1 + x^5$ on [0,1]
(21) Is the function $f(x) = x^3 + 2$ one-to-one?

(22) Let $A = \{1, 2\}$ and $B = \{a, b, c\}$.
  (a) How many functions are there from A to B?
  (b) How many of them are one-to-one?
(23) Find domain and range of the following functions and sketch their graph.
  (a) $f(x) = 5^x - 2$
  (b) $f(x) = \sqrt{x-9} + 4$
  (c) $f(x) = x^{\frac{1}{3}} - 8$
  (d) $f(x) = \log_3(x-1) - 3$
  (e) $f(x) = (x-1)^3$
  (f) $f(x) = x^3 - 1$
  (g) $f(x) = 5x^2$
  (h) $f(x) = \frac{1}{5}x^2$
  (i) $f(x) = (x+5)^2$
  (j) $f(x) = (x-2)^2 + 4$
  (k) $f(x) = \sin(x) - 3$
  (l) $f(x) = \sin(x-3)$
  (m) $f(x) = \frac{1}{2}\cos(x)$
  (n) $f(x) = 2\cos(x)$
(24) Compute the following limits:
  (a)
$$\lim_{x \to 1} 5x - 4$$
  (b)
$$\lim_{x \to -3} |7x + 10|$$
  (c)
$$\lim_{x \to 3} (x+2)(x-5)$$
  (d)
$$\lim_{x \to -0.5} \sqrt{\frac{x+2}{x+1}}$$
  (e)
$$\lim_{x \to 3} \frac{x-9}{x^2}$$
  (f)
$$\lim_{x \to 3} \frac{5x^2 - 8x - 13}{x^2 - 5}$$
  (g)
$$\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 - 4}$$
  (h)
$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$
  (i)
$$\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x^3 + 8}$$
  (j)
$$\lim_{x \to +\infty} \frac{x^3 - 7x}{x^3}$$
  (k)
$$\lim_{x \to 0} \frac{x^3 - 7x}{x^3}$$

(l)
$$\lim_{x\to 1^-} \frac{x^2+x+1}{x-1}$$
(m)
$$\lim_{x\to -\infty} \frac{2^x}{x}$$
(n)
$$\lim_{x\to 3^+} \frac{4}{x^2-9}$$
(o)
$$\lim_{x\to 2^-} \frac{x-3}{\sqrt{2}-\sqrt{x}}$$
(p)
$$\lim_{x\to \pi} \frac{\tan(x)-3}{\tan(x)-1}$$
(q)
$$\lim_{x\to 0^-} \frac{-5+7x}{e^x-1}$$
(r)
$$\lim_{x\to +\infty} \frac{1}{5x+2}$$
(s)
$$\lim_{x\to 0^+} \frac{e^{\frac{1}{x}}}{x^3+\sqrt{x}}$$
(t)
$$\lim_{x\to +\infty} 2+\left(1+\frac{3}{x}\right)^{-x}$$
(u)
$$\lim_{x\to \pi^+} \frac{\cos(x)+2}{\sin(x)}$$
(v)
$$\lim_{x\to 0} \frac{e^3+e^{-x}}{3}$$
(w)
$$\lim_{x\to +\infty} (\sqrt{2x^3-1}-\sqrt{2x^3+4})$$
(x)
$$\lim_{x\to +\infty} \frac{3x^2-4x^4+2x^5-1}{14-3x^5+3x}$$
(y)
$$\lim_{x\to 4^-} \frac{x^3-64}{x^2-8x+16}$$
(z)
$$\lim_{x\to 3} \frac{x^2+2x-15}{x-3}$$

(25) Compute the following limits:

(a)
$$\lim_{x\to 0} \frac{3(1-\cos(x))}{4x^2+\sin(x^2)}$$

(b)

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{x \sin(x)}$$

(c)

$$\lim_{x \to +\infty} \left(1 + \frac{1}{3x+2}\right)^{20x}$$

(d)

$$\lim_{x \to +\infty} \frac{5x^{20} + 3e^x}{6x - e^x}$$

(26) Consider the function

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x < -1 \\ 2 & \text{if} \quad -1 \leq x < 1 \\ 3 & \text{if} \quad x = 1 \\ x+1 & \text{if} \quad 1 < x \leq 2 \\ \frac{-1}{(x-2)^2} & \text{if} \quad x > 2 \end{cases}$$

(a) Sketch the graph of f.
(b) Determine the following limits:
(i) $\lim_{x \to -1^+} f(x)$
(ii) $\lim_{x \to -1^-} f(x)$
(iii) $\lim_{x \to -1} f(x)$
(iv) $\lim_{x \to 1^+} f(x)$
(v) $\lim_{x \to 1^-} f(x)$
(vi) $\lim_{x \to 1} f(x)$
(vii) $\lim_{x \to 2^+} f(x)$
(viii) $\lim_{x \to 2^-} f(x)$
(ix) $\lim_{x \to 2} f(x)$
(x) $\lim_{x \to -3} f(x)$
(xi) $\lim_{x \to 5} f(x)$

(27) Is there any value of k that will make the function

$$f(x) = \begin{cases} \frac{x^2-2x-8}{x+2} & \text{if} \quad x \neq -2 \\ k & \text{if} \quad x = -2 \end{cases}$$

continuous at $x = -2$? If so, what is it? Give reasons for your answer.

(28) Is there any value of k that will make the function

$$f(x) = \begin{cases} \frac{x^2+2x-15}{x-3} & \text{if} \quad x \neq 3 \\ k & \text{if} \quad x = 3 \end{cases}$$

continuous at $x = 3$? If so, what is it? Give reasons for your answer.

(29) Show that $x^2 - 6x + 5 =$ has a root somewhere in the interval $[0, 2]$.

(30) Prove that the function $f(x) = 2x + 5$ is uniformly continuous.

(31) Consider the following market demand and supply functions:

$$Q_D = 50 - 2P$$

$$Q_S = -10 + P$$

Graph the demand and supply functions on one diagram and $z = Q_D - Q_S$ on another. Find the equilibrium price and quantity for this market and illustrate it on both graphs. Show that these demand and supply functions satisfy the conditions for the existence of a positive equilibrium price.