Escrow Account and Moral Hazard

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Diego

Abstract

This paper investigates the optimal strategies for traders whom invest in the market on behalf of banks introducing the definition of the escrow account and its consequences on traders' strategies by solving the Exponential Utility Maximization Problem with Hamilton-Jacobi-Belmann equation. Setting the model on the usual filtered probabilistic space in continuous time with a risky asset driven by an exponential Brownian Motion. This paper considers as well the option to trading on a Fraud Asset, which is a jump process, in order to maximize their expected utility function, i.e., their satisfaction given by their earnings. This aforementioned Fraud Asset could mean their dismissal from the bank. We find that there exists an equilibrium between their strategies in which each trader decides or not to invest in such an asset and will keep that strategy afterwards.

Introduction

On 19 June 2015, the Council agreed its negotiating stance on structural measures to improve the resilience of EU credit institutions in order to avoid bankruptcy from potentially risky trading activities as their magnitude could have devastating consequences for the community. This potential risk is included in the definition of *Operational Risk*. The Basel Committee for Banking Supervision (BIS) defines Operational Risk as *"the risk of losses resulting from inadequate or failed internal processes, people and systems or from external events"*. Banks have many reasons to have this risk into account. For instance, in the past, banks hired traders to invest on their behalf. Banks would pay the traders by giving them bonuses based on profit they make, and assume the losses they incur. This led to an aggressive strategy with some traders becoming rogue. They could earn a higher amount of money but if they lost the investment, they could just quit and leave the bank with all the losses. An excellent example for the case is the role of Nick Leeson in the collapse of Barings Bank in 1995, hiding poor speculative investments until the accumulated debt was over $\pounds 800M$.

Power (2005), [15], explains that Basel 2, the latest regulation for the banking sector introduces operational account measures under the label of Operational Risk. Under this label there are risks coming from IT, business disruption, fraud and legal liabilities among others. However there are still many critics to these attempts of regulating Operational Risks. For example, Gillet, Hübner & Plunus (2010), [2], propose that Operational Risk should account *"the difference between the market value loss and the announced loss amount of the firm. This adjustment allows us to isolate the pure reputational effect of the operational loss event on market returns"*.

Banks are not the only financial institutions caring about Operational Risks, the EU introduced Solvency II, known as the Basel of the insurance sector. Mittnik (2011), [12], Vukovic (2015), [16], and Wang (2013), [17], says that Solvency II focuses in three pillars and one of them is the Operational Risk and focuses on the Solvency Capital Requirement, SCR, which represents the amount of own funds that would potentially be consumed by unexpected loss events and should be the sum of the Basic Solvency Capital Requirement, BSCR, the adjustment for the risk absorbing effect of technical provisions and deferred taxes, Adj, and finally the Capital Requirement of Operational Risk, SCR_{op}. Therefore, it is clear that after the crisis of 2009 all the financial sectors are devoting increased effort and resources to managing Operational Risk.

One of the most important reasons to introduce Operational Risk is the existence of rogue traders. For example, the famous incident in which Jerome Kerviel, a junior trader, had lost over $7 billion on unauthorized trades in equities and equity index futures, becoming the largest rogue trader in history in January 2008. Krawiec (2000), [8], defines a rogue trader as

"a market professional who engages in unauthorized purchases or sales of securities, commodities or derivatives, often for a financial institution's proprietary trading account".

Krawiec (2009), [9], argues whether self-regulatory controls will successfully manage Operational Risk, and Wexler (2010), [18], attempts to explain why rogue traders exist arguing that

"financial edgework, social psychological study of voluntary risk, to speculative trading, demonstrates how why rogue trading is a result of the security industry's pursuit of and desire to capitalize upon yet not publicize an occupational culture stressing a "risk-and-win" ethos".

Brown, Goetzmann, Liang & Schwarz (2008), [1], considers that within the Mandatory Disclosures (regulatory tool intended to allow market participants to assess Operational Risk) some market participants, such as equity fund investors and prime brokers extending credit, are able to distinguish problem from non-problem funds, i.e., traders are aware of their actions.

Moodie (2009), [13], suggests additional measures to control rogue traders. Likewise, banks have recently been restricting trader's profits to minimize the risk on the strategies of rogue traders. But still one trader or even a group traders could find an equilibrium among themselves in which everyone would be willing to invest in fraudulent assets to satisfy their eagerness of maximizing their returns even though they risk their jobs. So, rather than paying bonuses as before, they hold the trader's proportion of the wealth earned or lost (for example, the day traders of a company) placing it in an escrow account (an account in which the trader can take money to trade but cannot take it for her own consumption) and letting the trader take only a proportion of the money held for her own consumption at a time. As an immediate outcome, this time investors would lose liquidity if they lost the investment and also they would have limited access to liquidity.

However, in this scenario, multiple traders could secretly invest in fraudulent assets to maximize their earnings in order to deal with this new handicap imposed by the banks. Investing in fraudulent assets gives them a much higher returns increasing their wealth in the escrow account and hence they receive greater wealth for their consumption even though this behaviour could mean their dismissal.

Considering a simpler scenario, if just one trader is hired with the possibility to trade one risk-free asset and one risky asset driven by a drifted geometric Brownian Motion, i.e., a stochastic asset which price fluctuates randomly up and down, then the real earnings that the trader can make for herself are a proportion of the wealth held in the escrow account as mentioned above. Thus, the trader would like to maximize her expected Utility Function, i.e., maximize her own satisfaction given by trading in this scenario. The best strategy she could have is not affected by the proportion fixed by the bank, i.e., she knows the money will come so she will be patient and consistent with this. In the real world, traders would react to this parameter as the cash flow, which goes from the escrow account to her private account, decreases. Hereafter, we consider the exponential Utility Function due to its indifference of the amount of wealth which reflects somehow the freedom of a trader using bank's wealth and not being affected. CARA utilities have been widely use for last decades. For instance, they are used in Frei and Schewizer (2008) for indifference pricing of contingent claims. However, exponential utility function leads to a feasible negative consumption s it is exposed in Muraviev and Wüthrich (2012) who says:

" *An essential drawback challenging the classical paradigm of exponential utility is the fact that negative consumption policies are included in the set of feasible controls. [...] Economically, this is a vague assumption which cannot be reasonably justified, but is rather made to allow a simplified framework".*

Another example is explained in Caballero (1990) :

"This paper specializes to this type of preferences (exponential) in spite of some of its ***unpleasant*** *features like the possibility of* ***negative consumption****".*

As a conclusion, we could set that consumption should be non-negative for $t \geq 0$ but this would lead to the impossibility of finding tractable solutions. Therefore, let us acknowledge this fact, accepting negative consumption and define the utility maximization problem for this special case.

Suppose that the trader has a new asset, called the Fraud Asset, which works as follows: the more an investor trades on it, the more likely a jump occurs ("picking up pennies in front of a steam roller"). The economist John Kay describes this behaviour as the "Taleb Distribution" - a returns profile that appears at times deceptively low-risk with steady returns, but experiences periodically catastrophic drawdowns. He gives many examples like Hedge Funds in [7]:

"We find Taleb distributions not only on the road and in financial institutions. Hedge funds make them accessible to a general public. Business people learnt centuries ago that you can water the milk again and again and again. Until Taleb strikes back: people notice and take their custom elsewhere. Marks and Spencer, the UK retailer, pushed to the limit of what its customers could stand until it discovered that it had crossed that limit".

Going back to our case, the aforementioned Fraud Asset is represented by a jump process driven by a Poisson Process. If the jump occurs then it would be reflected on the escrow account and therefore the bank would realize that the trader is cheating. With the Fraud Asset, the trader has the possibility to make more profit so that she could earn more but this asset would risk her job, i.e., she could be fired. Guegan & Hassani (2009). [4], explains that there two kinds of distributions to model the events that are associated to Operational Risks: the discrete distribution functions such as Poisson, Binomial or Negative Binomial distributions for modelling the frequency of rare events and their intensity and the continuous distributions such as the Brownian Motion which model the non-rogue trading.

In a scenario such as this one, the trader could think that they could earn so much from the Fraud Asset that even if she is fired, she could live better from the money taken until the jump happens. If the trader chooses her optimal strategy she would avoid trading in such an asset and, as a result, would retain her position in the long term, as opposed to making money in the short term but lose her job. And again, the proportion fixed by the bank is not relevant at all.

Consider the case in which the bank hire more than one trader with different initial wealth and proportions given by the bank. Both have the possibility to either trade the Fraud Asset or not trade at all. If they both trade the Fraud Asset the intensity of their investment would combine to affect the frequency of the jump (again the Taleb Distribution with more participants), and therefore, the bank would quickly recognize their behaviour. One strategy could depend on the others' strategy. For example, if one trader had little money with respect to the other, i.e. the former would be close to bankruptcy, one logical reaction would be that the former would invest more than the latter as she has nothing to lose and would incur more risk to gain wealth quicker.

In addition, traders could invest the money from the investment pool, i.e., if the jump happens, this jump would be reflected on the pool leading bankruptcy and therefore everyone would be fired.

In this scenario, one trader would invest depending on her initial wealth and the wealth of the other trader. For example, if one trader has less initial wealth than the other, she would be willing to trade more on the fraud asset than the other and viceversa.

While the jump has not occurred yet and one trader invests more than other, then the former would earn more than the latter and therefore, the difference between initial wealth decreases. This new initial wealth would presumably make them want to change their optimal strategy, i.e., one trader decides to invest in the fraud asset, since this would modify the initial conditions for the second trader, the latter could decide to change her previous decision and invest in the fraud asset differently, suited accordingly to the new scenario. This new change of the settings would modify the initial conditions of the former trader and she could change again depending on how much she could invest and so on.

If they continue this process, they achieve an optimal equilibrium for any initial conditions they could start from, i.e., there is a moment when both traders are happy with their decisions given the other trader's decision. This would satisfy what is called a Nash equilibrium for a mixed strategy game. Therefore, in this scenario, the traders could agree to trade on the Fraud Asset taking risk together increasing the possibility of being fired.

Chapter 1

Single Trader and Moral Hazard

1.1 The general framework

Let us define the model under the probabilistic space $(\Omega, F, \{F_t)\}_{t\geq 0}, \mathbb{P})$ within an infinite time horizon in which we assume that the filtration satisfies the usual conditions. Let us define a Standard Brownian Motion which will represent our risky asset. A Standard Brownian Motion is a stochastic process, $W = \{W_t \in \mathbb{R} : t \in (0, \infty)\}$ with the following properties:

- $W_0 = 0$, $\mathbb{P}$ a.s.
- W has Gaussian increments: $W_{t+s} - W_s$ is normally distributed with null mean and variance t, $N(0, t)$.
- W has independent increments: for $0 \leq t < s \leq u < v$, $W_t - W_s$ and $W_v - W_u$ are independent.
- W has continuous paths $\mathbb{P}$ a.s.

This aforementioned risky asset will be based on the Black-Scholes model, i.e., a market where prices follow a geometric Brownian Motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1.1}$$

where μ (returns) and σ (volatility) are constants ($\sigma \neq 0$).

Let us include another risky asset which will be used below in order to simulate the repercussion of a fraudulent asset. This fraudulent asset will express the following, if an investor desires to invest in such an asset, she will increase her income by taking the risk of being caught and get fired. Moreover, if this investor decided to invest a higher amount of wealth, the consequence will be that the probability of being caught will increase as well. To represent this phenomena, this fraudulent asset is driven by a Poisson Process, N_t, with intensity $\varphi_t \geq 0$, i.e., a stochastic control which reflects the investor eagerness on the fraud asset. Then,

$$\frac{dF_t}{F_t} = \varphi_t dt - dN_t. \tag{1.2}$$

Thus, we should consider the definition of a "stopping time", τ, which specifies the moment in which the jump happens. A "stopping time" is a random variable, $\tau : \Omega \to \mathbb{R}^+$, if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in F_t$ for all $t \geq 0$. The probability that jump does not happen is

$$\mathbb{P}[\tau > t] = e^{-\int_0^t \varphi_s ds}; \tag{1.3}$$

Let us emphasize that the aforementioned consequence of increasing the intensity of investment on the fraud asset as well as increasing the probability of being fired is reflected on the definition of stopping time. We could also argue whether the size of the jump is relevant or not for firing the trader. The model could include another different scenario: if the loss is smaller than the reserve in the pool, traders could keep investing but with less margin; if losses are far greater than wealth reserved, then the bank would fire the trader. Nevertheless, as we consider this jump process as a vehicle of moral hazard, let us assume that in case of a jump, regardless the size, our trader will be fired.

Suppose now that a bank hires a trader. This trader will have access to the bank pool for investments in which the investor's transactions will take place. On the other hand, her salary will be composed by a fixed periodic amount and performance bonuses given by a proportion of the wealth generated. In other words, this trader gains or loses a proportion of the wealth invested at any time t. Hereafter this percentage will be represented by $\delta \in [0, 1]$. This percentage of earnings will be reflected in the pool so that the investor could use it to trade. Hence, supposing the bank has only one trader with access to the pool, $(X_t)_{t\geq 0}$, defined by the following dynamics:

$$dX_t = \theta_t X_t \frac{dS_t}{S_t} - \delta X_t dt + rX_t dt + \varphi_t X_t dt - X_t dN_t. \tag{1.4}$$

where $\theta_t \in [0, 1]$ represents the pool's wealth reserved for trading on the risky asset; and $r \in [0, 1]$ is the interest rate affecting the market. It is noted that the pool could be negative. The peculiarity of the model resides in the following, instead of giving the bonuses directly to her, the bank puts it in an escrow account, $(E_t)_{t\geq 0}$, i.e., the trader can take a proportion, $\lambda \in [0, 1]$, of the savings from the account for her own savings and consumption, i.e., the variable bonuses aforementioned. But to consume earnings, trader should wait this new filter, λ, as earnings are first going to her escrow account and thereafter to her actual savings account. In case of being fired, there are two different points of views. For the bank's perspective, it will depend on the difference between the intensity of the jump, φ_t and the wealth held in the pool. Whereas the trader will lose the access to the pool as well as to the escrow account. Therefore, all earned savings accumulated in the escrow account would be lost. We would like to know whether traders' behavior and strategy would change by this control on her winnings:

$$dE_t = \delta X_t dt + rE_t dt - \lambda E_t dt. \tag{1.5}$$

Finally, let $(Y_t)_{t\geq 0}$ be the trader's private wealth, i.e., the trader's profit (or savings) which satisfies the following diffusion:

$$dY_t = (Y_0 + \lambda E_t - c_t + rY_t)dt \tag{1.6}$$

where Y_0 is the fixed periodic salary and $c_t \geq 0$ is her consumption. i.e., her savings are composed of money from the escrow account, her fixed salary the trader's consumption and the interest rate.

The aim of the problem is to understand if a trader would have any moral hazard in order to maximize her profit under these handicaps imposed by banks. Mathematically speaking, this objective is translated into an utility maximization problem for consumption of the trader's savings with respect to the triple stochastic optimal control $u = (c, \theta, \varphi)$, i.e.,

$$V(E_t, Y_t) := \max_{u \in A_t} \mathbb{E}\left[\int_t^{T \wedge \tau} e^{-\beta s} U(c_s) ds + W_{t' \geq T \wedge \tau} | F_t\right], \tag{1.7}$$

where $U(\cdot)$ is the utility function in which $\beta > 0$ is the impatience parameter, i.e., how important the time value of money is for the trader; and $W(\cdot)$ is the value function after the jump occurs and the investor is fired. In other words, when the investor is fired, she can only live with her savings thereafter.

Let us recall the definition of a **control process**, $u_t(\omega) : [t, T] \times \Omega \to \mathbb{R}^m$ adapted to the filtration $\{F_t)\}_{t \geq 0}$ and the **set of admissible controls** is $A_{t,T}$.

1.2 One trader and no moral hazard

Due to the complexity of the model, let us consider a simpler scenario in which there is a complete market with no possible option for a moral hazard. The only new handicap that trader faces is the escrow account. For sake of simplicity, let us consider an escrow account which has a dual function. In other words, in this simpler scenario, we have bank's pool and escrow account merged following just one dynamic:

$$dX_t = \delta \theta_t X_t \frac{dS_t}{S_t} - \lambda X_t dt + r X_t dt \tag{1.8}$$

or equivalently,

$$X_t = X_0 e^{(r-\lambda)t + \delta \int_0^t \theta_s \left(\mu - \frac{1}{2}\sigma^2 \delta \theta_s\right) ds + \sigma \delta \int_0^t \theta_s dS_s}. \tag{1.9}$$

This new assumption will let us simplify computations, as there will be just one term X_t in all components as opposed of (1.5). The peculiarity of this diffusion with respect to (1.5) is that appears $\delta \theta_t X_t dS_t$ which is the wealth corresponding to the trader and the rest, $(1 - \delta)\theta_t X_t dS_t$, belongs to the bank. Finally, let Y_t be the trader's private wealth remains mostly the same, i,e., for simplicity consider that the salary of the trader is only composed by bonuses. Hereafter, Y_0 will be the initial private wealth of the trader. Therefore:

$$dY_t = \lambda X_t dt - c_t dt + r Y_t dt. \tag{1.10}$$

or equivalently,

$$Y_t = e^{rt}\left(\int_0^t e^{-rs}(\lambda X_s - c_s) ds + Y_0\right) \tag{1.11}$$

where $U(x) = -\frac{1}{\alpha}e^{-\alpha x}$, and the Value Function:

$$V(X_t, Y_t) := \max_{u=(\theta,c)\in A_{t,T}} \mathbb{E}\left[\int_t^\infty e^{-\beta s}U(c_s)ds|F_t\right], \tag{1.12}$$

We assume as well that the strategy for consumption is admissible. Let as recall that an y_0-admissible strategy is a process $(c_t)_{t\geq 0}$ such that the stochastic integral $Z_t = \int_0^t c_s dS_s$ is defined, and satisfies $Z_t \geq -y_0$ a.s. for all $t > 0$ where Z_t is the portfolio process. For this special case c_t is assumed deterministic. Therefore:

$$\int_0^\infty e^{-rt}c_t dt \geq y_0. \tag{1.13}$$

The intuition behind is that we are not interested in what happens whether the trader gets credit for her consumption or not but what is her behavior towards the company and the market in order to increase her wealth. Hence, we assume the trader cannot consume more than what she has and then she does not have such a possibility, so we introduce this constraint.

1.2.1 Optimal Solution

Theorem 1. *Let consider the exponential expected utility maximization problem defined by (1.12). The optimal value function is given by:*

$$V(X_t, Y_t) = \frac{-1}{\alpha r}e^{-(X_t+Y_t)\alpha r - \frac{1}{r}(\frac{\mu^2}{2\sigma^2}+\beta-r)}, \tag{1.14}$$

where the optimal consumption and investment are:

$$c_t = (X_t + Y_t)r + \frac{1}{\alpha r}\left(\frac{\mu^2}{2\sigma^2} + \beta - r\right) \tag{1.15}$$

$$\pi_t = \frac{\mu}{\alpha r\delta\sigma^2 X_t} \tag{1.16}$$

Proof. The proof is composed by two steps:

1. Solving the PDE given by the HJB equation for this particular value function done below.

2. Proving that the solution which solves the HJB equation satisfies the hypothesis of the Verification Theorem and therefore, the solution is optimal. We will not prove the Verification Theorem as our goal is to find a more generic environment where traders have response to λ.

The HJB equation for an infinite horizon and impatience β is given by

$$\sup_{c,\theta}\left[U(c_t) + L^u V(X_t, Y_t)\right] = 0. \tag{1.17}$$

where L^u is the operator with respect to the stochastic control $u = (c, \theta)$ such that $L^u\cdot = -\beta\cdot + \frac{\mathbb{E}[d(\cdot)]}{dt}$. The value function depends on the following diffusions

$$dY_t = (\lambda X_t - c_t + rY_t)dt \tag{1.18}$$
$$dX_t = \delta\theta_t X_t \frac{dS_t}{S_t} - \lambda X_t dt + rX_t dt \tag{1.19}$$
$$= (\mu\delta\theta_t X_t + (r-\lambda)X_t)dt + \sigma\delta\theta_t X_t dW_t. \tag{1.20}$$

On the other hand, $V(X_t, Y_t)$ is stochastic (the term X_t is stochastic as it is driven by the Brownian Motion S_t) so by the Itô's formula its derivative is the following :

$$dV = V_x dX_t + V_y dY + \frac{1}{2}(V_{xx} d\langle X\rangle_t + V_{yy} d\langle Y\rangle_t + 2V_{xy} d\langle X, Y\rangle_t) \tag{1.21}$$
$$= V_x((\mu\delta\theta_t X_t + (r-\lambda)X_t)dt + \sigma\delta\theta_t X_t dW_t) + V_y(\lambda X_t - c_t + rY_t)dt \tag{1.22}$$
$$+ \frac{1}{2}V_{xx}(\sigma\delta\theta_t X_t)^2 dt \tag{1.23}$$

where $d\langle Y\rangle_t = d\langle X, Y\rangle_t = 0$ as Y is deterministic, i.e., has not a dW term. dV is still stochastic so we need its expected value:

$$\frac{\mathbb{E}\left[dV\right]}{dt} = V_x(\mu\delta\theta_t X_t + (r-\lambda)X_t) + V_y(\lambda X_t - c_t + rY_t) \tag{1.24}$$
$$+ \frac{1}{2}V_{xx}(\sigma\delta\theta_t X_t)^2. \tag{1.25}$$

Hence, the HJB equation is

$$\sup_c \left(-c_t V_y - \frac{1}{\alpha}e^{-\alpha c_t}\right) + \sup_\theta \left(V_x \mu\delta\theta_t X_t + \frac{1}{2}V_{xx}(\sigma\delta\theta_t X_t)^2\right) \tag{1.26}$$
$$+(V_y - V_x)\lambda X_t + r(V_y Y_t + V_x X_t) - \beta V = 0 \tag{1.27}$$

We shall find out when the HJB equation reaches the maximum with respect to the consumption and the trading strategy, so we differentiate with respect to c_t and θ_t and set the derivative equal to 0. The expressions of c_t and θ which maximize the PDE are

$$c_t = \frac{-\log V_y}{\alpha} \quad \text{and} \quad \theta_t = -\frac{V_x \mu}{V_{xx}\delta\sigma^2 X_t}$$

Plugging c_t and θ_t, the PDE is:

$$\frac{1}{\alpha}V_y \log V_y - \frac{1}{\alpha}e^{\alpha \log V_y/\alpha} - V_x \delta\mu X_t \frac{V_x \mu}{V_{xx}\delta\sigma^2 X_t} + \frac{1}{2}V_{xx}\delta^2\sigma^2 X_t^2 (\frac{V_x \mu}{V_{xx}\delta\sigma^2 X_t})^2 \tag{1.28}$$
$$+V_t + (V_y - V_x)\lambda X_t + r(V_y Y_t + V_x X_t) - \beta V = 0 \tag{1.29}$$

Equivalently,

$$\frac{1}{\alpha}\left(V_y \log V_y - V_y\right) + V_y \lambda X_t + \frac{V_x^2}{\sigma^2 V_{xx}}\left(-\frac{1}{2}\mu^2\right) - V_x \lambda X_t + r(V_y Y_t + V_x X_t) - \beta V = 0 \tag{1.30}$$

Thus, the final PDE to be solved is

$$\frac{1}{\alpha}V_y\left(\log V_y - 1\right) - \frac{V_x^2\mu^2}{2\sigma^2 V_{xx}} + (Vy - V_x)\lambda X_t + r(V_x X_t + V_y Y_t) - \beta V = 0 \quad (1.31)$$

1. Before tackling the general problem let consider $\lambda = 0$ and $r = 0$. There are two different solutions of V as is showing below, to find a solution of V, the introduction of an interest rate is needed, i.e., $r \neq 0$.

   In this case, dY_t does not depend on X_t any longer and the terms V_x and V_{xx} vanish. Moreover, there will be no trading dependence, θ_t, in the value function. Then, the previous diffusion becomes:

   $$dY_t = -c_t dt \quad (1.32)$$

   and with the same optimized c_t, the HJB equation is:

   $$\frac{1}{\alpha}V_y\left(\log V_y - 1\right) - \beta V = 0 \quad (1.33)$$

   with solutions

   $$V = \frac{e^{\mp\sqrt{1+2\alpha\beta(x+y)+2C_1}}\left(-1 \mp \sqrt{1+2\alpha\beta(x+y)+2C_1}\right)}{\alpha\beta} \quad (1.34)$$

   which become just one (explained below):

   $$V(y) := \frac{-1}{\alpha r}e^{-\alpha r y - \frac{\beta - r}{r}}. \quad (1.35)$$

   In this scenario, the constant C_1 aforementioned would presumably cancel out the square root outside the exponential.

2. Consider $r \neq 0$. To solve that problem, let us recall the assumption that trader cannot consume more than all the wealth generated plus the initial wealth available, i.e., the trader has to follow an admissible strategy for consumption. Again, as $\lambda = 0$, X_t does not have any affect on Y_t so that it is a deterministic problem. There is only one term more in Y_t dynamics:

   $$dY_t = rY_t dt - c_t dt. \quad (1.36)$$

   Let us consider a different approach and define the Lagrange multipliers problem for this case subject to the constraint (1.13):

   $$V(y) = \max_{c_t,\eta}\left\{\int_0^\infty e^{-\beta t}U(c_t)dt - \eta\left(\int_0^\infty e^{-rt}c_t dt - y_0\right)\right\}. \quad (1.37)$$

Finding the derivative with respect to c_t and setting it equal to 0:

$$\frac{\partial V}{\partial c_t} = \int_0^\infty e^{-\beta t - \alpha c_t + rt} - \eta dt = 0 \tag{1.38}$$

$$e^{-\beta t - \alpha c_t + rt} - \eta = 0 \tag{1.39}$$

$$c_t = -\frac{1}{\alpha}\left((\beta - r)t + \ln \eta\right). \tag{1.40}$$

We shall find η so that the constrain is satisfied

$$\int_0^\infty e^{-rt} c_t dt = y_0 \tag{1.41}$$

$$\int_0^\infty e^{-rt}\frac{1}{\alpha}\left((\beta - r)t + \ln \eta\right) dt = -y_0 \tag{1.42}$$

$$(\beta - r)\int_0^\infty e^{-rt} t dt + \ln \eta \int_0^\infty e^{-rt} dt = -\alpha y_0 \tag{1.43}$$

$$(\beta - r)\frac{1}{r^2} + \ln \eta \frac{1}{r} = -\alpha y_0 \tag{1.44}$$

$$\ln \eta = \alpha r\left(-y_0 - (\beta - r)\frac{1}{r^2}\right) \tag{1.45}$$

$$\eta = e^{-y_0 \alpha r - \frac{\alpha}{r}(\beta - r)}. \tag{1.46}$$

Then

$$c_t = -\frac{1}{\alpha}\left((\beta - r)t + \alpha r\left(-y_0 - (\beta - r)\frac{1}{r^2}\right)\right) \tag{1.47}$$

$$= r y_0 - \frac{1}{\alpha}\left((\beta - r)(t - \frac{\alpha}{r})\right) \tag{1.48}$$

and finally the value function

$$V = -\frac{1}{\alpha}\int_0^\infty e^{-\beta t - \alpha\left(r y_0 - \frac{1}{\alpha}\left((\beta - r)(t - \frac{\alpha}{r})\right)\right)} dt \tag{1.49}$$

$$V = -\frac{1}{\alpha} e^{-\alpha r y_0 - \frac{(\beta - r)\alpha}{r}}\left(e^{-rt}\right)_0^\infty \tag{1.50}$$

$$V = \frac{-1}{\alpha r} e^{-\alpha r y_0 - \frac{(\beta - r)\alpha}{r}}. \tag{1.51}$$

3. If we consider $\mu = 0$, we assume no expected returns from the market, so no one would want to trade in such a market. Likewise, if $\delta = 0$, even if there is an expected return, the trader will not see any change in the escrow account. In both cases, trader is able to consume both X_0 and Y_0. Thus, the diffusions become

$$dX_t = \delta \theta_t \sigma S_t dW_t + (r - \lambda) X_t dt \quad \text{and} \quad dY_t = (\lambda X_t - c_t + r Y_t) dt \tag{1.52}$$

$$dX_t = (r - \lambda) X_t dt \quad \text{and} \quad dY_t = (\lambda X_t - c_t + r Y_t) dt \tag{1.53}$$

with the very same PDE

$$\frac{1}{\alpha} V_y \left(\log V_y - 1\right) + \lambda X_t (V_y - V_x) + r(V_y Y_t + V_x X_t) - \beta V = 0 \tag{1.54}$$

and $c_t = -\log V_y/\alpha$ and $\theta_t = 0$ (since $V_{xx}\delta^2\sigma^2 S_t^2 \theta_t = 0$) for $\forall t \geq 0$.

Let $V(x, y) = V(x + y)$ such that $V_y = V_x = V'(z)$ with $z = x + y$. Subbing the new variable, we end up with the same PDE as before

$$\frac{1}{\alpha} V'(z) \left(\log(V'(z)) - 1\right) + rV'(z)(x + y) - \beta V = 0 \tag{1.55}$$

$$\frac{1}{\alpha} V'(z) \left(\log(V'(z)) + \alpha r z - 1\right) - \beta V = 0. \tag{1.56}$$

Therefore, the solution is

$$V(z) = \frac{-1}{\alpha r} e^{-z\alpha r - \frac{(\beta - r)\alpha}{r}} \tag{1.57}$$

$$V(x, y) = \frac{-1}{\alpha r} e^{-(x+y)\alpha r - \frac{(\beta - r)\alpha}{r}} \quad \text{and} \tag{1.58}$$

$$c_t = -\frac{1}{\alpha} \left((x + y)\alpha r - \frac{(\beta - r)\alpha}{r} \right) \tag{1.59}$$

$$= (x + y)r + \frac{\beta - r}{r} \tag{1.60}$$

General Case

We shall go back to the general case, i.e.

$$\frac{1}{\alpha} V_y \left(\log V_y - 1\right) - \frac{V_x^2 \mu^2}{2\sigma^2 V_{xx}} + (Vy - V_x)\lambda X_t + r(V_x X_t + V_y Y_t) - \beta V = 0 \tag{1.61}$$

We shall Plug the solution (1.58) found above times a constant, i.e:

$$V(x, y) = \frac{-A}{\alpha r} e^{-(x+y)\alpha r - \frac{\alpha}{r}(\beta - r)} \tag{1.62}$$

where

$$V'(x) = V'(y) = V'(z) = -\alpha r A V(z) \tag{1.63}$$

$$V''(z) = (\alpha r)^2 A V(z) \tag{1.64}$$

Then

$$\frac{1}{\alpha}(-\alpha r)AV(z)\left(\log A - z\alpha r - \frac{\alpha}{r}(\beta - r) - 1\right) - \frac{(AV(z)(-\alpha r))^2\mu^2}{2\sigma^2(\alpha r)^2 AV(z)} \tag{1.65}$$

$$-rzAV(z)\alpha r - \beta AV(z) = 0 \tag{1.66}$$

$$-r\left(\log A - \frac{\alpha}{r}(\beta - r) - 1\right) - \frac{\mu^2}{2\sigma^2} - \beta = 0 \tag{1.67}$$

$$\frac{1}{r}\left(-\frac{\mu^2}{2\sigma^2} + \alpha(\beta - r) + r - \beta\right) = \log A \tag{1.68}$$

$$\tag{1.69}$$

then the constant $A = e^{\frac{1}{r}(-\frac{\mu^2}{2\sigma^2}+(\alpha-1)(\beta-r))}$ and finally the solution is

$$V(x,y) = \frac{-1}{\alpha r}e^{-(x+y)\alpha r - \frac{1}{r}(\frac{\mu^2}{2\sigma^2}+\beta-r)} \tag{1.70}$$

$$c_t = (x+y)r + \frac{1}{\alpha r}\left(\frac{\mu^2}{2\sigma^2} + \beta - r\right) \tag{1.71}$$

$$\theta_t = \frac{\mu}{\alpha r\delta\sigma^2 X_t S_t} \tag{1.72}$$

This is the usual Merton's problem, hence λ does not interfere into the trader's behavior which seems that traders are not affected by this retention of the money in the escrow account. They would just wait for the money to come. The question now is if traders will have any change of behaviour under a more general model, i.e., if there is another asset to trade which would put their perpetuity in the bank under risk.

1.3 Single Trader with Moral Hazard

With the same setting as the previous subsection, suppose now that the investor has the possibility to trade on other jump asset, F_t, driven by a Poisson Process, N_t, with intensity φ_t -the more the investor trades on the asset, the higher probability to be fired is. Then,

$$\frac{dF_t}{F_t} = \varphi_t dt - dN_t \tag{1.73}$$

She should not trade at this risky asset, called the Fraud Asset, but this investor has the possibility to trade this asset as long as the jump does not happen. When it does happen, the bank will acknowledge immediately the investor's behavior as there will be a "crash" in the pool, or in this simple setting, in the escrow account, and fire her consequently. Meanwhile, the trader can take a proportion, λ, of the money from the account for her own savings and consumption as in the previous setting. When she is fired, they will have no access to the escrow account any more and has to live with her savings gained up to the moment of the dismissal.

Consider that the trader decides to invest $\pi_t S_t$ at a time t , then their profit will be $\pi_t dS_t$ (number of shares times the change of the price). If $\pi_t dS_t > 0$ the bank puts in the escrow account $\pi_t dS_t$. If not, the bank takes $\pi_t dS_t$ out from the escrow account. Simultaneously, this investor trades dF_t which as long as there is no jump will give cash flow return to her account.

Thus, the cash in that escrow account should be

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t} + \frac{dF_t}{F_t} - \lambda dt + (1-\pi) r dt \tag{1.74}$$

$$\frac{dX_t}{X_t} = rdt + \pi\mu dt + \pi\sigma dB_t + \varphi_t dt - dN_t - \lambda dt \tag{1.75}$$

The trader's private wealth, Y_t, remains equal. When $dF_T = 0$ for some T, then $\lambda dX_{T+t} = 0$ for all $t \geq 0$.

The problem is again to maximize the long horizon expected exponential utility for consumption of the traders savings where τ is the stopping time when the jump happens:

$$V(X_t, Y_t) := \max_{u_t \in A_{t,T}} \mathbb{E}\left[\int_t^{T\wedge\tau} e^{-\beta s} U(c_s) ds + W(Y_{t'>T\wedge\tau})\right], \quad U(x) = \frac{e^{-\alpha x}}{-\alpha} \tag{1.76}$$

where $W(Y_t)$ is the value function of the trader once she has no access to the escrow account. It is clear the value of this function as it has been found within the proof of Theorem 1 for the case $\mu = 0$ but with $X_t = 0$ for all $t > \tau$., i.e., $W(Y_t) = -\frac{1}{\alpha r} e^{-\alpha r Y_t + \frac{\beta - r}{r}}$.

In conclusion,

$$X_t = X_0 e^{rt + \int_0^t (\mu\pi_s - \frac{\sigma^2}{2}\pi_s^2 + \varphi_t) ds + \int_0^t \sigma\pi_s dB_s} \mathbb{1}_{\{t<\tau\}} \tag{1.77}$$

where the probability that jump does not happen is

$$\mathbb{P}[\tau > t] = e^{-\int_0^t \varphi_s ds} \tag{1.78}$$

which makes $F_t = e^{\int_0^t \varphi_t ds} \mathbb{1}_{\{t<\tau\}}$ a martingale (i.e. $E[F_t] = 1$).

Theorem 2. *The optimal expected utility function is the same as Theorem 1:*

$$V(x,y) = \frac{-1}{\alpha r} e^{-(x+y)\alpha r - \frac{1}{r}(\frac{\mu^2}{2\sigma^2} + \beta - r)}, \tag{1.79}$$

where the optimal consumption and investment are the same:

$$c_t = (x+y)r + \frac{1}{\alpha r}\left(\frac{\mu^2}{2\sigma^2} + \beta - r\right) \tag{1.80}$$

$$\theta_t = \frac{\mu}{\alpha r \delta \sigma^2 S_t} \tag{1.81}$$

and the optimal amount of investment in the Fraud Asset is $\varphi_t = 0$ for $\forall t \geq 0$.

Proof. Now we shall use the HJB equation again but this time we need to introduce the discontinuity forced by the jump at time τ like ([10]). We shall prove that this new HJB will give us the same PDE as before. For the moment we know that PDE should be the same as before up to time τ and thereafter the final function, W, by the intensity, i.e.

$$\sup_{c,\pi,\varphi} \left(U(c_t) + \boldsymbol{L}^u V(X_t, Y_t)\right) = 0 \tag{1.82}$$

where

$$\boldsymbol{L}^u = (\varphi_t + \beta)\cdot + \varphi_t W + \frac{E[d(\cdot)]}{dt} \tag{1.83}$$

$$U(c_t) = -\frac{1}{\alpha} e^{-\alpha c_t} \tag{1.84}$$

$$\frac{E[dV]}{dt} = V_x X_t(r + \pi_t\mu + \varphi_t - \lambda) + V_y(\lambda X_t - c_t + rY_t) + V_{xx}(\pi\sigma X_t)^2 \tag{1.85}$$

Then

$$\sup_c \left(\frac{-e^{-\alpha x}}{\alpha} - c_t V_y\right) + \sup_\pi \left(V_x \pi_t \mu X_t + \frac{1}{2} V_{xx}(\pi_t\sigma X_t)^2\right) \tag{1.86}$$

$$+ \sup_\varphi \left(X_t V_x \varphi_t - \varphi_t V + \varphi_t W\right) - \beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) = 0 \tag{1.87}$$

Differentiating wrt c_t, θ_t and φ_t and setting them equal to zero.

$$c_t = -\frac{1}{\alpha}\ln(V_y) \tag{1.88}$$

$$\pi_t = -\frac{\mu}{\sigma^2 X_t}\frac{V_x}{V_{xx}} \tag{1.89}$$

$$0 = X_t V_x - V + W \tag{1.90}$$

For φ_t is needed to consider the cases:

1. If $X_tV_x - V + W \geq 0$ then $\sup_\varphi (X_t V_x \varphi_t - \varphi_t V + \varphi_t W) = \infty$. Therefore, $\varphi_t = \infty$.
2. If $X_tV_x - V + W \leq 0$ then $\sup_\varphi (X_t V_x \varphi_t - \varphi_t V + \varphi_t W) = 0$. Therefore, $\varphi_t = 0$.

But as V is concave $X_tV_x - V + W \leq 0$ is always satisfied. Hence $\varphi_t = 0$ for $\forall t \geq 0$.

Hence, the HJB equation becomes:

$$\frac{1}{\alpha}V_y(\ln V_y - 1) - \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - \beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) = 0 \tag{1.91}$$

Plugging the change of variable $V(x,y) = W(y)u(x)$ where $W(y) = -\frac{1}{\alpha r}e^{-\alpha r y}$ then

$$-ru(\ln u - 1) - \frac{\mu^2}{2\sigma^2}\frac{u'^2}{u''} - \beta u + u'(r-\lambda)x - \alpha r\lambda u x = 0 \tag{1.92}$$

The solution is

$$u(x) = e^{-\alpha r x - \frac{1}{r}(\frac{\mu^2}{2\sigma^2}+\beta-r)}.$$

Then

$$V(x,y) = -\frac{1}{\alpha r}e^{-\alpha r(x+y)-\frac{1}{r}(\frac{\mu^2}{2\sigma^2}+\beta-r)} \tag{1.93}$$

which again comes back to the previous result, i.e., the trader avoids the jump and performs a similar strategy. Again traders are not affected by the proportion λ.

Consider that the trader is obliged to invest regularly on the Fraud Asset, i.e., $\varphi_t = \bar{\varphi} \neq 0$ fixed. Therefore the HJB equation becomes:

$$\sup_c \left(\frac{-e^{-\alpha x}}{\alpha} - c_t V_y\right) + \sup_\pi \left(V_x \pi_t \mu X_t + \frac{1}{2}V_{xx}(\pi_t \sigma X_t)^2\right) + \bar{\varphi}\left(X_t V_x - V + W\right) \tag{1.94}$$

$$-\beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) = 0 \tag{1.95}$$

Plugging the same c_t and π_t

$$\frac{1}{\alpha}V_y(\ln V_y - 1) - \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - \beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) + \bar{\varphi}\left(X_t V_x - V + W\right) = 0 \tag{1.96}$$

Substituting by $V(x,y) = W(y)u(x)$ where $W(y) = -\frac{1}{\alpha r}e^{-\alpha r y}$ then

$$-ru(\ln u - 1) - \frac{\mu^2}{2\sigma^2}\frac{u'^2}{u''} - \beta u + u'(r-\lambda)x - \alpha r \lambda u x + \bar{\varphi}\left(xu' - u + 1\right) = 0 \tag{1.97}$$

equivalently

$$-ru(\ln u - 1) - \frac{\mu^2}{2\sigma^2}\frac{u'^2}{u''} - (\beta + \bar{\varphi})u + u'(r - \lambda + \bar{\varphi})x - \alpha r \lambda u x + \bar{\varphi} = 0 \tag{1.98}$$

Which we could not find a explicit solution for this ODE. Instead, we tried with the logarithmic utility function, $U(x) = \log(x)$.

$$\sup_c \left(\log(c_t) - c_t V_y\right) + \sup_\pi \left(V_x \pi_t \mu X_t + \frac{1}{2}V_{xx}(\pi_t \sigma X_t)^2\right) + \bar{\varphi}\left(X_t V_x - V + W\right) \tag{1.99}$$

$$-\beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) = 0 \tag{1.100}$$

Then we have

$$c_t = \frac{1}{V_y} \tag{1.101}$$

$$\pi_t = -\frac{\mu}{\sigma^2 X_t}\frac{V_x}{V_{xx}} \tag{1.102}$$

$$-\ln V_y - 1 - \frac{\mu^2}{2\sigma^2}\frac{V_x^2}{V_{xx}} - \beta V + V_x(r-\lambda)X_t + V_y(\lambda X_t + rY_t) + \bar{\varphi}\left(X_tV_x - V + W\right) = 0 \tag{1.103}$$

And again we have not found a solution yet.

Simpler Attempt

As the previous problem seems tedious, let try to solve the simpler version without consumption and the risky asset as these are the troublesome. Let just consider the following setting:

$$dX_t = \varphi X_t dt - \lambda X_t dt - dN_t \tag{1.104}$$
$$dY_t = \lambda X_t dt \tag{1.105}$$

where φ is the intensity of the Poisson and λ is the proportion of wealth which goes to the saving account. Therefore the log-value function maximization problem, i.e.

$$V(x,y) = E_y[U(Y_t)] \tag{1.106}$$

where $U(y) = \log(y)$.

Consider what is

$$V(kx,ky) = E_y[U(kY_t)] = E_y[U\log(kY_t)] \tag{1.107}$$
$$= E_y[U\log(Y_t)] + E_y[U\log(k)] \tag{1.108}$$
$$= V(x,y) + \log(k) \tag{1.109}$$

Set $k = \frac{1}{y}$

$$V(\frac{1}{y}x, \frac{1}{y}y) = V(x,y) + \log(\frac{1}{y}) \tag{1.110}$$
$$v(z) = V(z,1) = V(x,y) - \log(y) \tag{1.111}$$

Where $z = \frac{x}{y}$. When $x = 0$, $v(0) = V(x,y) - \log(y) = \log(y) - \log(y) = 0$. The HJB equation starts being:

$$-\varphi V + \varphi\log(y) + xV_x(\varphi - \lambda) + xV_y\lambda = 0 \tag{1.112}$$

and becomes

$$-\varphi v + zv'(\varphi - \lambda) - z^2v'\lambda + \lambda z = 0 \tag{1.113}$$

1. Case 1: $\lambda = 0$. Then $dY_t = 0$. Therefore V depends on just $Y_T = y$ Therefore

$$V(y) = E_y[U(Y_t)] = E_y[U(y)] = \log(y) \tag{1.114}$$

2. Case 2: $\varphi = 0$.

$$+zv'(-\lambda) - z^2v'\lambda + \lambda z = 0 \tag{1.115}$$
$$v(z) = \log|1+z| \tag{1.116}$$

3. Case 3: $\lambda = \varphi$

$$-\lambda v - z^2 v'\lambda + \lambda z = 0 \tag{1.117}$$

for $\lambda > \varphi > 0$:

$$v(z) = e^{\frac{1}{z}} ExpIntegralEi[-(1/z)] = e^{\frac{1}{z}} \int_{\frac{1}{z}}^{\infty} \frac{e^{-t}}{t} dt \tag{1.118}$$

4. and the general case: for $\lambda > \varphi > 0$:

$$v(z) = -\left(\frac{-z\lambda}{\lambda + z\lambda - \varphi}\right)^{\frac{-\varphi}{\lambda-\varphi}} \int_0^{\frac{z\lambda}{\varphi-\lambda}} t^{\frac{\varphi}{\lambda-\varphi}} (1-t)^{\frac{-\lambda}{\lambda-\varphi}} dt \tag{1.119}$$

$$V(x,y) = -\left(\frac{-\frac{x}{y}\lambda}{\lambda + \frac{x}{y}\lambda - \varphi}\right)^{\frac{-\varphi}{\lambda-\varphi}} \int_0^{\frac{\frac{x}{y}\lambda}{\varphi-\lambda}} t^{\frac{\varphi}{\lambda-\varphi}} (1-t)^{\frac{-\lambda}{\lambda-\varphi}} dt + \log(y) \tag{1.120}$$

or equivalently

$$V(x,y) = -\left(\frac{-\frac{x}{y}\lambda}{\lambda + \frac{x}{y}\lambda - \varphi}\right)^{\frac{-\varphi}{\lambda-\varphi}} Beta\left[\frac{\frac{x}{y}\lambda}{-\lambda+\varphi}, \frac{\lambda}{\lambda-\varphi}, -\frac{\varphi}{\lambda-\varphi}\right] + \log(y) \tag{1.121}$$

These are the limits

$$\lim_{\varphi \to 0} V(x,y) = \log(x+y) \tag{1.122}$$

$$\lim_{\varphi \to \infty} V(x,y) = \log(y) \tag{1.123}$$

$$\lim_{\lambda \to 0} V(x,y) = \log(y) \tag{1.124}$$

$$\lim_{\lambda \to \infty} V(x,y) = \log(x+y) \tag{1.125}$$

If we define $\kappa = \frac{\varphi}{\lambda}$

$$v(z) = -\left(\frac{-z}{1+z-\kappa}\right)^{\frac{-\kappa}{1-\kappa}} \int_0^{-\frac{z}{1-\kappa}} t^{\frac{\kappa}{1-\kappa}} (1-t)^{\frac{-1}{1-\kappa}} dt \tag{1.126}$$

As an illustration of how $v(z)$ varies on κ. we plug $v(z)$ for $z > 0$ with respect different κ ($\kappa = 0$ equivalent as $v(z) = \log|1+z|$, $\kappa = \frac{1}{2}$ and $\kappa = 1$ equivalent to $v(z) = e^{\frac{1}{z}} \int_{\frac{1}{z}}^{\infty} \frac{e^{-t}}{t} dt$)

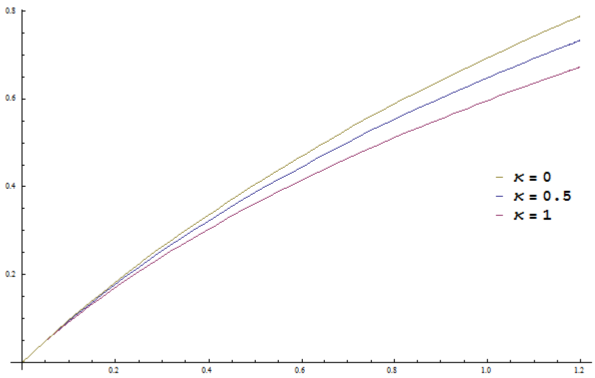

Figure 1.1: $v(z)$, $z > 0$

As a conclusion, it seems that the best choice for the trader to maximize her utility function is not to trade with the fraud asset as the maximum value achieved is at $\varphi = 0$ because λ cannot be greater than 1, i.e., λ cannot go to infinity.

Risky asset back

We introduce back the asset:

$$\frac{dS_t}{S_t} = (r + \mu)dt + \sigma dB_t, \tag{1.127}$$

So therefore

$$\frac{dX_t}{X_t} = rdt + \pi\mu dt + \pi\sigma dB_t + \varphi_t dt - dN_t - \lambda dt \tag{1.128}$$

$$dY_t = (\lambda X_t + rY_t)dt \tag{1.129}$$

Our initial value is just the same $V(0, y) = \log(y)$. The HJB becomes

$$\sup_{\pi} \left(V_x \pi_t \mu X_t + \frac{1}{2} V_{xx} (\pi_t \sigma X_t)^2 \right) + \bar{\varphi} \left(X_t V_x - V + \log(y) \right) \tag{1.130}$$

$$-\beta V + V_x(r - \lambda)X_t + V_y(\lambda X_t + rY_t) = 0 \tag{1.131}$$

$$-\frac{\mu^2}{2\sigma^2} \frac{V_x^2}{V_{xx}} - \beta V + V_x(r - \lambda)X_t + V_y(\lambda X_t + rY_t) + \bar{\varphi} \left(X_t V_x - V + \log(y) \right) = 0 \tag{1.132}$$

1. For $\lambda = \beta = 0$ the HJB becomes

$$V_y dY_t + \bar{\varphi}(\log(y) - V) = 0 \tag{1.133}$$

With solution $V(y) = \log(y) + \frac{r}{\varphi}$.

2. For $r = \beta = 0$ and $\varphi = \lambda$

$$-\frac{\mu^2}{2\sigma^2} \frac{V_x^2}{V_{xx}} + V_y \lambda X_t + \bar{\varphi} \left(-V + \log(y) \right) = 0 \tag{1.134}$$

And then using the same transformation $V(x, y) = v(z) + \log(y)$

$$-\frac{\mu^2}{2\lambda\sigma^2}\frac{v'^2}{v''} - v'z^2 + z - v = 0 \tag{1.135}$$

Using dual method as ([6]) $u(\xi) = \sup_z (v(z) - z\xi)$

$$\frac{\mu^2}{2\lambda\sigma^2}u''\xi^2 - \xi u'^2 - u' - u + u'\xi = 0 \tag{1.136}$$

Again, we could not find an explicit solution.

Nevertheless, the optimal strategy is to not trade on the fraud asset at all, so what if the bank hires a second agent whom could trade as the former trader with the only difference is in the initial wealth and the proportion which she can take the wealth gained to her private account? What if both traders would be able to invest in the fraud asset increasing the probability that the bank acknowledge their portfolio? Would they invest now? Would the poorer trader take advantage of the richer to earn quicker the money?

Chapter 2

Multiple Traders and Moral Hazard

2.1 The new general Framework

Before we were considering just one trader using the pool to invest. However, this is unrealistic. In the Nick Lesson's case, it is clear that the consequences for the rest of the traders in Barings Bank were devastating. Therefore, we could assume that one trader's strategy could significantly affect the others' strategies. Even more, it could mean their jobs.

Suppose that Trader 1 is investing fraudulently or just acting amorally and suppose as well that Trader 2 does not know this and this person only can see the good performance of Trader 1. Would this person trade on the Fraud Asset to achieve a similar results? If Trader 2 knows what Trader 1 is doing, and assuming that Trader 2 warn the back, then Trader 2 knows the existence of the risk of getting fired due to bankruptcy. Would Trader 2 start trading on the Fraud Asset and get benefit or would he keep avoiding such an investment?
Let us considering the same general framework as before, but this time, including n traders. Both traders have the same opportunities which means that they could trade on the Fraud Asset. As a consquence, we could expect that the intensity of the jump process would increase corresponding to the joined amount of wealth that both traders use. Therefore, let us redefine the Fraud Asset:

$$\frac{dF_t}{F_t} = \sum_{i=1}^{n} \varphi_{i,t} dt - dN_t. \tag{2.1}$$

We can define now the diffusion of the pool:

$$dX_t = \sum_{i=1}^{n} \theta_{i,t} X_t \frac{dS_t}{S_t} - \sum_{i=1}^{n} \delta_i X_t dt + rX_t dt + \sum_{i=1}^{n} \varphi_{i,t} X_t dt - X_t dN_t. \tag{2.2}$$

Each trader uses the money of the pool to trader and from their investments and they receive a proportion of the winnings or losses in their respective escrow account:

$$dE_{i,t} = \delta_i X_t dt + rE_{i,t} dt - \lambda_i E_{i,t} dt, \tag{2.3}$$

and they both can consum with their earnings hold in their private account:

$$dY_{i,t} = (Y_{i,0} + \lambda_i E_{i,t} - c_{i,t} + rY_{i,t})dt. \tag{2.4}$$

The goal is to maximize their respective value function as before.

2.2 Two Traders with Moral Hazard

Using the same approach as before, instead of having a pool and two escrow accounts, we assume there is only two escrow accounts that jointly will act as the pool. We still want to keep the condition that if the jump occurs everyone gets fired as a consequence of bankruptcy. One possible way to do this is allowing both traders invest everyone's money and if the jump occurs, all money is lost, i.e., escrow accounts have no money left to trade with. Given this assumption, the diffusions of the escrow accounts should be

$$dX_t^1 = \pi_t^1(X_t^1 + X_t^2)\frac{dS_t}{S_t} + (1-\pi_t^1)rX_t^1 + \varphi_t^1(X_t^1 + X_t^2)dt - \lambda_1 X_t^1 dt - (X_t^1 + X_t^2)dN_t \tag{2.5}$$

$$dX_t^2 = rX^2dt + \pi_t^2(X_t^1 + X_t^2)\frac{dS_t}{S_t} + \varphi_t^2(X_t^1 + X_t^2)dt - \lambda_2 X_t^2 dt - (X_t^1 + X_t^2)dN_t \tag{2.6}$$

where λ_1 and λ_2 are proportions of cash that traders are allowed to take at a time as in the previous settings; r is a fixed interest rate so that current money increases with time at the aforementioned rate; and the size of the jump, dN_t, is equal to 1 when multiplied by $X_t^1 + X_t^2$, traders lose all their accumulated wealth.

Finally, the traders' private wealth (or savings) are

$$dY_t^1 = \lambda_1 X_t^1 dt - c_t^1 dt + rY_t^1 dt \tag{2.7}$$
$$dY_t^2 = \lambda_2 X_t^2 dt - c_t^2 dt + rY_t^2 dt \tag{2.8}$$

where c_t^i is their consumption. i.e., savings is composed of money from the escrow account, the trader's consumption and the interest rate. When $dF_T = 0$ for some T, then $\lambda dX_{T+t} = 0$ for all $t \geq 0$.

We want to maximize their respective long horizon expected exponential utility for consumption of the traders savings as before. For the simpler case, where there is not a risky asset $S_t = 0$; nor interest rate either, r_t, nor consumption, $c_t = 0$, the maximized utility functions are

$$V_1(X_1, Y_1) = e^{Y_1}\left(1 + C(X_1 + X_2)\right) \tag{2.9}$$
$$V_2(X_2, Y_2) = e^{Y_2}\left(1 + C(X_1 + X_2)\right) \tag{2.10}$$

For the general case there is not an explicit solution.

Proof. If $S_t = r_t = c_t = 0$ the previous diffusions become

$$dX_t^1 = \varphi_t^1(X_t^1 + X_t^2)dt - \lambda_1 X_t^1 dt \tag{2.11}$$
$$dX_t^2 = \varphi_t^2(X_t^1 + X_t^2)dt - \lambda_2 X_t^2 dt \tag{2.12}$$
$$dY_t^1 = \lambda_1 X_t^1 dt \tag{2.13}$$
$$dY_t^2 = \lambda_2 X_t^2 dt \tag{2.14}$$

Now we have two HJB equations. We want to maximize over φ_t^1 for trader 1 and over φ_t^2 for trader 2:

$$\sup_{\varphi}\left(-(\varphi_t^1 + \varphi_t^2)V^1 + (\varphi_t^1 + \varphi_t^2)W^1 + \frac{E[dV^1]}{dt}\right) \tag{2.15}$$
$$\sup_{\varphi}\left(-(\varphi_t^1 + \varphi_t^2)V^2 + (\varphi_t^1 + \varphi_t^2)W^2 + \frac{E[dV^2]}{dt}\right) \tag{2.16}$$

W^i is the function after the jump and the utility functions vanish as there is no consumption. The derived PDEs for the first trader is

$$-\varphi_t^2 V^1 + \varphi_t^2 W^1 - V_{x_1}^1 \lambda_1 X_t^1 + V_y^1 \lambda_1 X_t^1 = 0 \tag{2.17}$$
$$-V^1 + W^1 + V_{x_1}^1 (X_t^1 + X_t^2) = 0 \tag{2.18}$$

Let substitute $V_i(\vec{x}, y) = u(\vec{x})W_i(y)$ where $W'(y) = W(y)$ and $u(\vec{x}) = u(x_1, x_2)$. The four ODEs for both traders are

$$-\varphi_t^2 u + \varphi_t^2 - u_{x_1}\lambda_1 X_t^1 + u\lambda_1 X_t^1 = 0 \tag{2.19}$$
$$-u + 1 + u_{x_1}(X_t^1 + X_t^2) = 0 \tag{2.20}$$
$$-\varphi_t^1 u + \varphi_t^1 - u_{x_2}\lambda_2 X_t^2 + u\lambda_2 X_t^2 = 0 \tag{2.21}$$
$$-u + 1 + u_{x_2}(X_t^1 + X_t^2) = 0 \tag{2.22}$$

From the HJB ODEs, we can find the values for φ^1 and φ^2

$$\varphi_1 = \frac{\lambda_2 X_2 (u_{x_2} - u)}{1 - u} \tag{2.23}$$
$$\varphi_2 = \frac{\lambda_1 X_1 (u_{x_1} - u)}{1 - u} \tag{2.24}$$
$$\tag{2.25}$$

and the first order conditions

$$-u + 1 + u_{x_1}(X_t^1 + X_t^2) = 0 \tag{2.26}$$
$$-u + 1 + u_{x_2}(X_t^1 + X_t^2) = 0 \tag{2.27}$$

which give the function joint

$$u(X_1, X_2) = 1 + C(X_1 + X_2)$$

where C must be equal for both ODEs. Therefore

$$\varphi_1 = \frac{\lambda_2 X_2}{X_1 + X_2}\frac{1-C}{C} + \lambda_1 X_2 \tag{2.28}$$
$$\varphi_2 = \frac{\lambda_1 X_1}{X_1 + X_2}\frac{1-C}{C} + \lambda_2 X_1 \tag{2.29}$$

and the value functions are

$$V_1(X_1, Y_1) = e^{Y_1}\left(1 + C(X_1 + X_2)\right) \tag{2.30}$$
$$V_2(X_2, Y_2) = e^{Y_2}\left(1 + C(X_1 + X_2)\right) \tag{2.31}$$

Here we see that φ_i depends on the amount λ_i. If we introduce back the risky asset, the HJB equation will not have an explicit solution since $u(x)$ would need to be exponential, and the first order condition with respect to φ_i needs $u(x)$ to be linear. Now this result tells us that a trader's behavior towards the fraud asset depends on the other trader's strategy and the proportion which they get their wealth from their escrow account. So we wonder if there is an equilibrium between both, i.e., if they would adapt their strategy until both are satisfied. To study this special case, we set a simpler model but this time in the Discrete Time.

2.3 Two Traders in Discrete Time

2.3.1 Definition of the model

Suppose now that a bank gives a sum of money, $x_{1,0}$ and $x_{2,0}$, to two traders. Instead of giving the money directly to them, the bank puts it in an escrow account for each other and the end of the 1-period, T. Both investors have the possibility to trade only on a jump asset, F_t, driven by a Poisson process, N_t, with a combined intensity $\varphi_{1,t}+\varphi_{2,t}$ (the first is the intensity which trader 1 invests in the fraud asset and the second for the second trader) -the more investors trades on the asset, the higher probability to be both fired is. Then,

$$\frac{dF_t}{F_t} = (\varphi_1 + \varphi_2)dt - dN_t \tag{2.32}$$

They should not trade at this risky asset, but these investor will trade this asset as long as the jump does not happen. When it does happen, the bank will acknowledge the investors' behavior and consequently fire them.

Thus, the cash in that escrow account should be

$$\dot{x}_{1,t} = \varphi_1(x_{1,t} + x_{2,t}) \tag{2.33}$$

$$\dot{x}_{2,t} = \varphi_2(x_{1,t} + x_{2,t}) \tag{2.34}$$

and solving the ODEs one can obtain the solutions:

$$x_{1,t} = x_{1,0} + \frac{\varphi_1}{\varphi_1 + \varphi_2}(x_{1,0} + x_{2,0})(e^{(\varphi_1+\varphi_2)t} - 1) \tag{2.35}$$

$$x_{2,t} = x_{2,0} + \frac{\varphi_2}{\varphi_1 + \varphi_2}(x_{1,0} + x_{2,0})(e^{(\varphi_1+\varphi_2)t} - 1) \tag{2.36}$$

Finally, the goal is to maximize their both Expected Utility, i.e.

$$V_i = E(U(x_{i,t})) = U(x_{i,t})e^{-(\varphi_1+\varphi_2)t} + U(0)(1 - e^{-(\varphi_1+\varphi_2)t}) \tag{2.37}$$

where $e^{-(\varphi_1+\varphi_2)t}$ is the probability that the jump does not happen.

Let define an algorithm such that iterating the function defined by

$$P(\varphi_1, \varphi_2) := (\varphi_1, \varphi_2) \to (\hat{\varphi}_1, \hat{\varphi}_2) \tag{2.38}$$

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) \tag{2.39}$$

$$\hat{\varphi}_2 = argmax_{\varphi_2} V_2(\varphi_1, \varphi_2) \tag{2.40}$$

where there exists an equilibrium so that both traders are happy with their respective φ. To find such equilibrium there should be a contraction at first. For the trader one, an optimal φ_1 should satisfy

$$\frac{\partial}{\partial \varphi_1} V_1(\hat{\varphi}_1(\varphi_2), \varphi_2) = 0 \tag{2.41}$$

Using implicit differentiation, differentiate with respect to φ_2

$$\frac{\partial^2}{\partial \varphi_1^2} V_1(\hat{\varphi}_1(\varphi_2), \varphi_2) \frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial \varphi_2} + \frac{\partial^2}{\partial \varphi_1 \varphi_2} V_1(\hat{\varphi}_1(\varphi_2), \varphi_2) = 0 \tag{2.42}$$

Solve for $\frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial \varphi_2}$:

$$\frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial \varphi_2} = -\frac{\frac{\partial^2}{\partial \varphi_1 \varphi_2} V_1(\hat{\varphi}_1(\varphi_2), \varphi_2)}{\frac{\partial^2}{\partial \varphi_1^2} V_1(\hat{\varphi}_1(\varphi_2), \varphi_2)} \tag{2.43}$$

$$= -\frac{U''(x)\frac{\partial x}{\partial \varphi_1}\frac{\partial x}{\partial \varphi_2} + U'(x)\left(\frac{\partial^2 x}{\partial \varphi_1 \varphi_2} - (\frac{\partial x}{\partial \varphi_1} + \frac{\partial x}{\partial \varphi_2})\right) + U(x) - U(0)}{U''(x)(\frac{\partial x}{\partial \varphi_1})^2 + U'(x)\left(\frac{\partial^2 x}{\partial \varphi_1^2} - 2\frac{\partial x}{\partial \varphi_1}\right) + U(x) - U(0)} \tag{2.44}$$

This derivative means that given any change of φ_2 the change that φ_1 will have to be smaller than the one for φ_2. Sometimes during the proof of the theorem mentioned below, instead of considering α, x_{10} and x_{20}, let consider $\alpha x_{10} = k_i$ and $x_{20} = q x_{10}$ to reduce one variable:

$$V_i = -\frac{1}{\alpha} e^{-\alpha(x_{i,0} + \frac{\varphi_1}{\varphi_1+\varphi_2}(x_{1,0}+x_{2,0})(e^{(\varphi_1+\varphi_2)t}-1))} e^{-(\varphi_1+\varphi_2)t} - \frac{1}{\alpha}(1 - e^{-(\varphi_1+\varphi_2)t}) \tag{2.45}$$

$$= -\frac{1}{\alpha} e^{-k(1 + \frac{\varphi_1}{\varphi_1+\varphi_2}(1+q^{-1^{i+1}})(e^{(\varphi_1+\varphi_2)t}-1))} e^{-(\varphi_1+\varphi_2)t} - \frac{1}{\alpha}(1 - e^{-(\varphi_1+\varphi_2)t}) \tag{2.46}$$

2.3.2 Theorem

Theorem 3. *Consider $x_{10}, x_{20}, \alpha_1, \alpha_2 > 0$ and the value functions:*

$$V_1 = U(x_1)e^{-\varphi_1-\varphi_2} + U(0)\left(1 - e^{-\varphi_1-\varphi_2}\right) \tag{2.47}$$
$$V_2 = U(x_2)e^{-\varphi_1-\varphi_2} + U(0)\left(1 - e^{-\varphi_1-\varphi_2}\right) \tag{2.48}$$

1. *Given a pair $(\varphi_1, \varphi_2) \in \mathbb{R}^{+^2}$ then there exist a unique new pair, $(\hat{\varphi}_1, \hat{\varphi}_2)$, such that*

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) \tag{2.49}$$
$$\hat{\varphi}_2 = argmax_{\varphi_2} V_2(\varphi_1, \varphi_2) \tag{2.50}$$

2. *If $(\varphi_1, \varphi_2) \in (\mathbb{R}^+)^2$, then*

$$\left|\frac{d(\varphi_i(\varphi_j))}{d(\varphi_j)}\right| < 1. \tag{2.51}$$

Proof

1. (a) Firstly, we want to check when the derivative is positive, negative or equal to zero:

$$dV_1 = U'(x_1)\frac{\partial x_1}{\partial \varphi_1}e^{-(\varphi_1+\varphi_2)} - U(x_1)e^{-(\varphi_1+\varphi_2)} + U(0)e^{-(\varphi_1+\varphi_2)} > 0 \tag{2.52}$$
$$U'(x_1)\frac{\partial x_1}{\partial \varphi_1} - U(x_1) + U(0) > 0 \tag{2.53}$$
$$\tag{2.54}$$

Hence, the derivative is null when

$$\frac{\partial x_1}{\partial \varphi_1} = \frac{U(x_1) - U(0)}{U'(x_1)} \tag{2.55}$$

where $U'(x_1)$ is positive since is an utility function. At the same time:

$$\frac{\partial x_1}{\partial \varphi_1} = (x_{1,0} + x_{2,0})\left(\frac{\varphi_2}{(\varphi_1+\varphi_2)^2}e^{(\varphi_1+\varphi_2)} + \frac{\varphi_1}{\varphi_1+\varphi_2}(e^{(\varphi_1+\varphi_2)} - 1)\right) \tag{2.56}$$
$$\frac{U(x_1) - U(0)}{U'(x_1)} = \frac{U(x_1) - U(0)}{-\alpha_1 U(x_1)} = \frac{-1 + U^{-1}(x_1)}{\alpha_1} = \frac{-1 + e^{\alpha_1 x_1}}{\alpha_1} \tag{2.57}$$

Plugging the aforementioned results on the derivative:

$$(x_{1,0}+x_{2,0})\left(\frac{\varphi_2}{(\varphi_1+\varphi_2)^2}e^{(\varphi_1+\varphi_2)}+\frac{\varphi_1}{\varphi_1+\varphi_2}(e^{(\varphi_1+\varphi_2)t}-1)\right)=\frac{U(x_1)-U(0)}{U'(x_1)} \tag{2.58}$$

$$(x_{1,0}+x_{2,0})\left(\frac{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))}{(\varphi_1+\varphi_2)^2}\right)=\frac{U(x_1)-U(0)}{U'(x_1)} \tag{2.59}$$

$$(x_{1,0}+x_{2,0})=\frac{U(x_1)-U(0)}{U'(x_1)}\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.60}$$

$$(x_{1,0}+x_{2,0})=\frac{-1+e^{\alpha_1 x_1}}{\alpha_1}\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.61}$$

$$(x_{1,0}+x_{2,0})\frac{\alpha_1}{-1+e^{\alpha_1 x_1}}=\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.62}$$

$$\tag{2.63}$$

If we call $k_1:=\alpha_1 x_{1,0}$ and $q:=\frac{x_{2,0}}{x_{1,0}}$, then:

$$(1+q)\frac{k_1}{-1+e^{k_1(1+\frac{\varphi_1}{\varphi_1+\varphi_2}(1+q)(e^{(\varphi_1+\varphi_2)t}-1))}}=\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.64}$$

If the derivative is positive,

$$(1+q)\frac{k_1}{-1+e^{k_1(1+\frac{\varphi_1}{\varphi_1+\varphi_2}(1+q)(e^{(\varphi_1+\varphi_2)t}-1))}}>\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.65}$$

and if it is negative,

$$(1+q)\frac{k_1}{-1+e^{k_1(1+\frac{\varphi_1}{\varphi_1+\varphi_2}(1+q)(e^{(\varphi_1+\varphi_2)t}-1))}}<\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.66}$$

To prove that the new pair $(\hat{\varphi}_1,\hat{\varphi}_2)$ is unique the steps are the following:

i. Let

$$M(\varphi_1):=(1+q)\frac{k_1}{-1+e^{k_1(1+\frac{\varphi_1}{\varphi_1+\varphi_2}(1+q)(e^{(\varphi_1+\varphi_2)t}-1))}} \tag{2.67}$$

and

$$m(\varphi_1):=\frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{(\varphi_1+\varphi_2)}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.68}$$

are continuous and always decreasing with respect to φ_1, respectively, which is true since they are a composition of continuous functions and their derivatives are always negative.

ii. Bounded by above at $\varphi_1 = 0$ by $(1+q)\frac{k_1}{-1+e^{k_1}}$ and $\frac{\varphi_2}{-1+e^{\varphi_2}}$, respectively. In the worse case scenario when $k \to 0$ and $\varphi_2 \to 0$ the limit is $(1+q)$ and 1. As q is finite then both are bounded by above. This is only for proposition (4) using a finite starting point at $\varphi_1 = 0$.

iii. We apply proposition (4).
Therefore these functions can intersect at most once.

iv. Consider the four different scenarios.

A. (2.67) is always greater than (2.68).
Then $dV > 0$ for every φ_1. But this scenario is not possible because in such a scenario, V should increase, but $V \geq -\frac{1}{\alpha}$ with $\lim_{\varphi_i} V_i = -\frac{1}{\alpha}$, so V has to decrease after some point.

B. (2.67) is greater than (2.68) at first.
There is a maximum at φ_1^*

C. (2.68) is always greater than (2.67).
Then $dV < 0$ for every φ_1. Therefore the maximum value is at $\varphi_1^* = 0$

D. (2.68) is greater than (2.67) at first.
There is a minimum at φ_1^*. But this is again not possible because, after φ_1^*, V should increase, but $V \geq -\frac{1}{\alpha}$ with $\lim_{\varphi_i} V_i = -\frac{1}{\alpha}$, so V has to decrease after some point.

Therefore there is a unique $\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2)$. For $\hat{\varphi}_2$ the proof is analogous.

2. We want to prove that there exists a contraction. The steps are the following:

(a) We want to prove that the denominator of the contraction defined below is negative:

$$\frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial\varphi_2} = -\frac{\frac{\partial^2}{\partial\varphi_1\varphi_2}V_1(\hat{\varphi}_1(\varphi_2), \varphi_2)}{\frac{\partial^2}{\partial\varphi_1^2}V_1(\hat{\varphi}_1(\varphi_2), \varphi_2)} \tag{2.69}$$

$$= -\frac{U''(x)\frac{\partial x}{\partial\varphi_1}\frac{\partial x}{\partial\varphi_2} + U'(x)\left(\frac{\partial^2 x}{\partial\varphi_1\varphi_2} - (\frac{\partial x}{\partial\varphi_1} + \frac{\partial x}{\partial\varphi_2})\right) + U(x) - U(0)}{U''(x)(\frac{\partial x}{\partial\varphi_1})^2 + U'(x)\left(\frac{\partial^2 x}{\partial\varphi_1^2} - 2\frac{\partial x}{\partial\varphi_1}\right) + U(x) - U(0)} \tag{2.70}$$

(b) If so then if we call

$$Num = -\left(U''(x)\frac{\partial x}{\partial\varphi_1}\frac{\partial x}{\partial\varphi_2} + U'(x)\left(\frac{\partial^2 x}{\partial\varphi_1\varphi_2} - (\frac{\partial x}{\partial\varphi_1} + \frac{\partial x}{\partial\varphi_2})\right)\right) \tag{2.71}$$

and

$$Dem = -\left(U''(x)(\frac{\partial x}{\partial \varphi_1})^2 + U'(x)\left(\frac{\partial^2 x}{\partial \varphi_1^2} - 2\frac{\partial x}{\partial \varphi_1}\right)\right), \tag{2.72}$$

then we have

$$\frac{-Num + U(x) - U(0)}{-Dem + U(x) - U(0)} < 1$$

if $Dem > Num$ and we are done.

(a) The denominator is equal to

$$\frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha(x_{10}+(x_{10}+x_{20})(-1+e^{\varphi_1+\varphi_2})\frac{\varphi_1}{\varphi_1+\varphi_2})} \tag{2.73}$$

$$-\frac{1}{\alpha}e^{-\alpha(x_{10}+(x_{10}+x_{20})(-1+e^{\varphi_1+\varphi_2})\frac{\varphi_1}{\varphi_1+\varphi_2})} \tag{2.74}$$

$$\frac{(x_{10}+x_{20})^2\alpha^2(\varphi_2 - e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2)))^2}{(\varphi_1+\varphi_2)^4} \tag{2.75}$$

$$-\frac{1}{\alpha}e^{-\alpha(x_{10}+(x_{10}+x_{20})(-1+e^{\varphi_1+\varphi_2})\frac{\varphi_1}{\varphi_1+\varphi_2})} \tag{2.76}$$

$$\frac{(x_{10}+x_{20})\alpha(-2\varphi_2(1+\varphi_1+\varphi_2)+e^{\varphi_1+\varphi_2}(2\varphi_2+\varphi_1(\varphi_1+\varphi_2)^2))}{(\varphi_1+\varphi_2)^3} \tag{2.77}$$

Now $1 - e^{-\alpha(x_{10}+(x_{10}+x_{20})(-1+e^{\varphi_1+\varphi_2})\frac{\varphi_1}{\varphi_1+\varphi_2})} < 0$ always. The first fraction is always positive (as every element is to the power of an even number) and then, by the exponential, negative. The second fraction is a bit trickier.

Consider only the numerator without as $(x_{10} + x_{20})\alpha$ is positive and the denominator as well.

$$(-2\varphi_2(1+\varphi_1+\varphi_2)+e^{\varphi_1+\varphi_2}(2\varphi_2+\varphi_1(\varphi_1+\varphi_2)^2)) \tag{2.78}$$

Its first derivative with respect to φ_1 is

$$2\varphi_2(-1+e^{\varphi_1+\varphi_2})+e^{\varphi_1+\varphi_2}((\varphi_1+\varphi_2)(\varphi_2+\varphi_1(3+\varphi_1+\varphi_))) \tag{2.79}$$

which is always positive therefore

$$-2\varphi_2(1+\varphi_1+\varphi_2)+e^{\varphi_1+\varphi_2}(2\varphi_2+\varphi_1(\varphi_1+\varphi_2)^2) > 2\varphi_2(-1+e^{\varphi_2})+e^{\varphi_2}\varphi_2^2 > 0 \tag{2.80}$$

Hence the denominator is always negative.

(b) We want to prove that $Dem > Num$. If we call

$$A := \frac{\partial x}{\partial \varphi_1}\frac{\partial x}{\partial \varphi_2} \tag{2.81}$$

$$A' := (\frac{\partial x}{\partial \varphi_1})^2 \tag{2.82}$$

$$B := \left(\frac{\partial^2 x}{\partial \varphi_1 \varphi_2} - (\frac{\partial x}{\partial \varphi_1} + \frac{\partial x}{\partial \varphi_2})\right) \tag{2.83}$$

$$B' := \left(\frac{\partial^2 x}{\partial \varphi_1^2} - 2\frac{\partial x}{\partial \varphi_1}\right) \tag{2.84}$$

or equivalently,

$$A := (x_{10} + x_{20})^2 \varphi_1 \tag{2.85}$$

$$\frac{(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2))(-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)))}{(\varphi_1 + \varphi_2)^4} \tag{2.86}$$

$$A' := (x_{10} + x_{20})^2 \frac{(-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)))^2}{(\varphi_1 + \varphi_2)^4} \tag{2.87}$$

$$B := (x_{10} + x_{20}) \tag{2.88}$$

$$\frac{(-\varphi_1 - \varphi_1^2 + \varphi_2 + \varphi_2^2 - e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(-1 + \varphi_1 + \varphi_2)(1 + \varphi_1 + \varphi_2)))}{(\varphi_1 + \varphi_2)^3} \tag{2.89}$$

$$B' := (x_{10} + x_{20}) \frac{(2\varphi_2(1 + \varphi_1 + \varphi_2) - e^{\varphi_1+\varphi_2}(2\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)^2))}{(\varphi_1 + \varphi_2)^3} \tag{2.90}$$

and if we prove that $A' \geq A \geq 0$ and $B' \leq B \leq 0$ then $-Num = U''(x)A + U'(x)B$ and $-Dem = U''(x)A' + U'(x)B'$ are negative since we know that the first derivative of the utility function, $U(x)$, has to be positive and the second negative. Therefore $-Dem < -Num$ and we have finished.

i. $A' \geq A \geq 0$: It is clear to see that $A > 0$ and $A' > 0$. Now we want to prove that $A' \geq A \geq 0$, so we divide A by A' to take out what is equal:

$$\frac{A}{A'} = \frac{\varphi_1(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2))}{-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2))} \tag{2.91}$$

and we take away the numerator from the denominator, if the result is positive we are done.

$$-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)) \tag{2.92}$$

$$-\left(\varphi_1(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2))\right) \tag{2.93}$$

$$= (-1 + e^{\varphi_1+\varphi_2})(\varphi_1 + \varphi_2) > 0 \tag{2.94}$$

Therefore $A' \geq A \geq 0$.

ii. $B' \leq B \leq 0$: This part is a bit trickier. To see that B is always negative we take derivatives to simplify the expression until one is always negative and then go back. This works if all the derivatives are always negative. Since $(x_{10} + x_{20})\frac{1}{(\varphi_1+\varphi_2)^3}$ is always positive, we will not consider it. We now call the remaining part as D.

$$
\begin{aligned}
D &:= -\varphi_1 - \varphi_1^2 + \varphi_2 + \varphi_2^2 && (2.95)\\
&\quad - e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(-1+\varphi_1+\varphi_2)(1+\varphi_1+\varphi_2)) && (2.96)\\
D' &= -1 - 2\varphi_1 - e^{\varphi_1+\varphi_2}(-1+\varphi_1(-1+\varphi_1(3+\varphi_1))) && (2.97)\\
&\quad - e^{\varphi_1+\varphi_2}(\varphi_2 + 2\varphi_1(2+\varphi_1)\varphi_2 + (1+\varphi_1)\varphi_2^2) && (2.98)\\
D'' &= -2 - e^{\varphi_1+\varphi_2}(-2+\varphi_1(1+\varphi_1)(5+\varphi_1)) && (2.99)\\
&\quad - e^{\varphi_1+\varphi_2}(5\varphi_2 + 2\varphi_1(4+\varphi_1)\varphi_2 + (2+\varphi_1)\varphi_2^2) && (2.100)\\
D''' &= -e^{\varphi_1+\varphi_2}(3+\varphi_1(17+\varphi_1(9+\varphi_1))) && (2.101)\\
&\quad - e^{\varphi_1+\varphi_2}(13\varphi_2 + 2\varphi_1(6+\varphi_1)\varphi_2 + (3+\varphi_1)\varphi_2^2) < 0 && (2.102)
\end{aligned}
$$

The third derivative is negative everywhere and then the second derivative is less or equal than the second derivative evaluated at $\varphi_1 = 0$, and so on for the rest.

$$
\begin{aligned}
D'' &\leq -2 - e^{\varphi_2}(-2 + 5\varphi_2 + 2\varphi_2^2) \leq 0 && (2.103)\\
D' &\leq -1 - e^{\varphi_2}(-1 + \varphi_2 + \varphi_2^2) \leq 0 && (2.104)\\
D &\leq \varphi_2(1 + \varphi_2 - e^{\varphi_2}) \leq 0 && (2.105)
\end{aligned}
$$

the maximum value is at the origin so $D \leq 0$ and finally $B \leq 0$.
If we take the ratio, $\frac{B}{B'}$, to take out the parts where B and B' are equal as before

$$
\frac{B}{B'} = \frac{-\varphi_1 - \varphi_1^2 + \varphi_2 + \varphi_2^2 - e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(-1+\varphi_1+\varphi_2)(1+\varphi_1+\varphi_2))}{2\varphi_2(1+\varphi_1+\varphi_2) - e^{\varphi_1+\varphi_2}(2\varphi_2 + \varphi_1(\varphi_1+\varphi_2)^2)} \tag{2.106}
$$

and we take away the denominator from the numerator then it should give us $B - B' > 0$ to have $B' \leq B \leq 0$.

$$
\begin{aligned}
&-\varphi_1 - \varphi_1^2 + \varphi_2 + \varphi_2^2 - e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(-1+\varphi_1+\varphi_2)(1+\varphi_1+\varphi_2)) && (2.107)\\
&-\big(2\varphi_2(1+\varphi_1+\varphi_2) - e^{\varphi_1+\varphi_2}(2\varphi_2 + \varphi_1(\varphi_1+\varphi_2)^2)\big) && (2.108)\\
&= (\varphi_1+\varphi_2)(e^{\varphi_1+\varphi_2} - (1+\varphi_1+\varphi_2)) > 0 && (2.109)
\end{aligned}
$$

Thus, the denominator of $\frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial\varphi_2}$ is greater of equal than the numerator and therefore

$$
0 \geq \frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial\varphi_2} \geq -1 \tag{2.110}
$$

Looking for when the contraction is equal to one, we check again the fraction

$$\frac{U''(x)\frac{\partial x}{\partial \varphi_1}\frac{\partial x}{\partial \varphi_2} + U'(x)\left(\frac{\partial^2 x}{\partial \varphi_1 \varphi_2} - (\frac{\partial x}{\partial \varphi_1} + \frac{\partial x}{\partial \varphi_2})\right) + U(x) - U(0)}{U''(x)(\frac{\partial x}{\partial \varphi_1})^2 + U'(x)\left(\frac{\partial^2 x}{\partial \varphi_1^2} - 2\frac{\partial x}{\partial \varphi_1}\right) + U(x) - U(0)} \tag{2.111}$$

and there are two options: the exponential utility function is equal to 0, so that when any of the parameters α_1, x_{10}, x_{20}, φ_1 and φ_2 is infinity; or when the coefficients multiplying the exponential are equal ($A = A'$ and $B = B'$) and this is when $\varphi_1 + \varphi_2 = 0$. Since φ_i cannot be negative then $\varphi_1 = \varphi_2 = 0$. Let consider when $A = A'$:

$$(x_{10} + x_{20})^2 \varphi_1 \tag{2.112}$$

$$\frac{(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2))(-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)))}{(\varphi_1 + \varphi_2)^4} \tag{2.113}$$

$$= (x_{10} + x_{20})^2 \frac{(-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)))^2}{(\varphi_1 + \varphi_2)^4} \tag{2.114}$$

$$\varphi_1(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2))(-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2))) \tag{2.115}$$

$$= (-\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)))^2 \tag{2.116}$$

$$\varphi_1(1 + e^{\varphi_1+\varphi_2}(-1 + \varphi_1 + \varphi_2)) = -\varphi_2 + e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)) \tag{2.117}$$

$$\varphi_1 + \varphi_2 + e^{\varphi_1+\varphi_2}(-\varphi_1 - \varphi_2 + \varphi_1 e^{\varphi_1+\varphi_2}(\varphi_1 + \varphi_2) - \varphi_1 e^{\varphi_1+\varphi_2}(\varphi_1 + \varphi_2)) = 0 \tag{2.118}$$

$$\varphi_1 + \varphi_2 - e^{\varphi_1+\varphi_2}(\varphi_1 + \varphi_2) = 0 \tag{2.119}$$

$$1 - e^{\varphi_1+\varphi_2} = 0 \tag{2.120}$$

$$\varphi_1 + \varphi_2 = 0 \tag{2.121}$$

Now for $B = B'$, as done with A and A', let only consider the numerators

$$-\varphi_1 - \varphi_1^2 + \varphi_2 + \varphi_2^2 - e^{\varphi_1+\varphi_2}(\varphi_2 + \varphi_1(-1 + \varphi_1 + \varphi_2)(1 + \varphi_1 + \varphi_2)) \tag{2.122}$$

$$= 2\varphi_2(1 + \varphi_1 + \varphi_2) - e^{\varphi_1+\varphi_2}(2\varphi_2 + \varphi_1(\varphi_1 + \varphi_2)^2) \tag{2.123}$$

$$-\varphi_1 - \varphi_1^2 - \varphi_2 - \varphi_2^2 - 2\varphi_1\varphi_2 \tag{2.124}$$

$$= -e^{\varphi_1+\varphi_2}(-\varphi_2 - \varphi_1(-1 + (\varphi_1 + \varphi_2)^2) + \varphi_1(\varphi_1 + \varphi_2)^2) \tag{2.125}$$

$$-(\varphi_1 + \varphi_2) - (\varphi_1 + \varphi_2)^2 = -e^{\varphi_1+\varphi_2}(-\varphi_2 - \varphi_1) \tag{2.126}$$

$$1 + (\varphi_1 + \varphi_2) = -e^{\varphi_1+\varphi_2} \tag{2.127}$$

$$1 + (\varphi_1 + \varphi_2) + e^{\varphi_1+\varphi_2} = 0 \tag{2.128}$$

$$\varphi_1 + \varphi_2 = 0 \tag{2.129}$$

Therefore the coefficients are equal only for $\varphi_i = 0$, and finally

$$0 \geq \frac{\partial(\hat{\varphi}_1(\varphi_2))}{\partial \varphi_2} > -1 \tag{2.130}$$

or

$$\left|\frac{d(\varphi_i(\varphi_j))}{d(\varphi_j)}\right| < 1. \tag{2.131}$$

when $(\varphi_1, \varphi_2) \in (\mathbb{R}^+)^2$. The proof for the second trader is analogous.

□

Proposition 4. *The functions (2.67) and (2.68) can intersect at most once.*

Proof. Consider the two possible scenarios:

1. If $\alpha x_{10} \geq k_1$ since (2.67) is decreasing then

$$(1+q)\frac{\alpha x_{10}}{-1+e^{\alpha x_{10}(1+(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2}(e^{(\varphi_1+\varphi_2)}-1))}} \tag{2.132}$$

$$\leq (1+q)\frac{k_1}{-1+e^{k_1(1+\frac{\varphi_1}{\varphi_1+\varphi_2}(1+q)(e^{(\varphi_1+\varphi_2)}-1))}} \tag{2.133}$$

$$= \frac{(\varphi_1+\varphi_2)^2}{-\varphi_2+e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))} \tag{2.134}$$

Therefore, if $\alpha x_{10} \geq k_1$ then $dV \leq 0$.

2. If $\alpha_i x_{i0} < k_i(\varphi_j, q)$ we know that dV cannot be always $dV > 0$ as mentioned above, although for $\varphi_1 = 0$, $dV > 0$. And thus we prove that (2.67) and (2.68) can intersect at most once.

$$\frac{\partial V}{\partial \varphi_1}(0) = \frac{1}{\alpha_1}e^{-\varphi_2}\left(-1+e^{-\alpha_1 x_{10}}\left(\frac{(-1+e^{\varphi_2})(1+q)x_{10}\alpha}{\varphi_2}+1\right)\right) \tag{2.135}$$

we use $q+1 = \frac{\varphi_2}{k}\frac{-1+e^k}{-1+e^{\varphi_2}}$ or $(q+1)\frac{-1+e^{\varphi_2}}{\varphi_2} = \frac{-1+e^k}{k}$

$$\frac{\partial V}{\partial \varphi_1}(0) = \frac{1}{\alpha_1}e^{-\varphi_2}\left(-1+e^{-\alpha_1 x_{10}}((-1+e^k)\frac{x_{10}\alpha}{k}+1)\right) \tag{2.136}$$

$$= \frac{1}{\alpha_1}e^{-\varphi_2}\left(-1+e^{-\alpha_1 x_{10}}(-1+e^k)\frac{x_{10}\alpha}{k}+e^{-\alpha_1 x_{10}}\right) \tag{2.137}$$

where the derivative with respect to k of $(-1+e^k)\frac{x_{10}\alpha}{k}$ is always positive $(-1+e^k)\frac{x_{10}\alpha}{k}$ is greater than the minimal value, i.e., $k = x_{10}\alpha_1$:

$$> \frac{1}{\alpha_1}e^{-\varphi_2}\left(-1+e^{-\alpha_1 x_{10}}(-1+e^{\alpha_1 x_{10}})\frac{x_{10}\alpha}{\alpha_1 x_{10}}+e^{-\alpha_1 x_{10}}\right) \tag{2.138}$$

$$= \frac{1}{\alpha_1}e^{-\varphi_2}\left(-1+(-e^{-\alpha_1 x_{10}}+1)+e^{-\alpha_1 x_{10}}\right) = 0 \tag{2.139}$$

We recall (2.67) and (2.68) and we check $\frac{M(\varphi_1)}{m(\varphi_1)}$ (it is well defined since both are continuous and greater than zero), its first value at $\varphi_1 = 0$ is larger than 1 as $M(0) >$

$m(0)$. Now if we check that $\frac{\partial}{\partial\varphi_1}\left(\frac{M(\varphi_1)}{m(\varphi_1)}\right) < 0$ for every φ_1, therefore only hits 1 once, then we are done.

To do so, we find $\frac{\partial}{\partial\varphi_1}\left(\frac{1}{(1+q)k}\frac{M(\varphi_1)}{m(\varphi_1)}\right)$ equal to

$$=\frac{e^{\varphi_1+\varphi_2}(-1+e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})})(2+\varphi_1)(\varphi_1+\varphi_2)^2}{(-1+e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})})^2(\varphi_1+\varphi_2)^3} \tag{2.140}$$

$$-\frac{\frac{e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})}k(1+q)(\varphi_2-e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))^2)}{(\varphi_1+\varphi_2)}}{(-1+e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})})^2(\varphi_1+\varphi_2)^3} \tag{2.141}$$

$$-\frac{2(-1+e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})})(-\varphi_2+e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))))}{(-1+e^{k(1+(-1+e^{\varphi_1+\varphi_2})(1+q)\frac{\varphi_1}{\varphi_1+\varphi_2})})^2(\varphi_1+\varphi_2)^3} \tag{2.142}$$

We get rid of $(1+q)k$ as is a positive constant like the denominator,i.e, take the numerator and derive by q.

$$-e^{\varphi_1+\varphi_2}(\varphi_1+\varphi_2)(\varphi_1^4+2\varphi_1(-2+\varphi_1^2)\varphi_2+(-2+\varphi_1^2)\varphi_2^2) \tag{2.143}$$
$$-e^{\varphi_1+\varphi_2}(\varphi_1+\varphi_2)2\varphi_2(2\varphi_1+\varphi_2)Cosh[\varphi_1+\varphi_2] \tag{2.144}$$
$$-(-1+e^{\varphi_1+\varphi_2})k(1+q)\varphi_1(\varphi_2-e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2)))^2 \tag{2.145}$$

Its derivative is always less than zero therefore the numerator is less than the numerator evaluated at $q=0$.

$$=\frac{e^{\frac{k(e^{\varphi_1+\varphi_2}\varphi_1+\varphi_2)}{(\varphi_1+\varphi_2)}}}{(\varphi_1+\varphi_2)^2}k(e^{\varphi_1+\varphi_2}(-1+e^{\varphi_1+\varphi_2})\varphi_1(2+\varphi_1)(\varphi_1+\varphi_2)^3) \tag{2.146}$$

$$-\frac{e^{\frac{k(e^{\varphi_1+\varphi_2}\varphi_1+\varphi_2)}{(\varphi_1+\varphi_2)}}}{(\varphi_1+\varphi_2)^2}k^2(-1+e^{\varphi_1+\varphi_2})\varphi_1(\varphi_2-e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2)))^2 \tag{2.147}$$

$$-\frac{e^{\frac{k(e^{\varphi_1+\varphi_2}\varphi_1+\varphi_2)}{(\varphi_1+\varphi_2)}}}{(\varphi_1+\varphi_2)^2}k(\varphi_1+\varphi_2)(\varphi_2-e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2)))^2 \tag{2.148}$$

$$-\frac{e^{\frac{k(e^{\varphi_1+\varphi_2}\varphi_1+\varphi_2)}{(\varphi_1+\varphi_2)}}}{(\varphi_1+\varphi_2)}2k(-1+e^{\varphi_1+\varphi_2})\varphi_1(-\varphi_2+e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2))) \tag{2.149}$$

Everything is multiplied by something positive $\frac{1}{(\varphi_1+\varphi_2)^2}e^{(\frac{k(e^{\varphi_1+\varphi_2}\varphi_1+\varphi_2)}{(\varphi_1+\varphi_2)}}k$, we take the rest and derive by k now.

$$-(-1+e^{\varphi_1+\varphi_2})k\varphi_1(\varphi_2-e^{\varphi_1+\varphi_2}(\varphi_2+\varphi_1(\varphi_1+\varphi_2)))^2 \tag{2.150}$$
$$-e^{\varphi_1+\varphi_2}(\varphi_1+\varphi_2)\left(\varphi_1^4+2\varphi_1(-2+\varphi_1^2)\varphi_2+(-2+\varphi_1^2)\varphi_2^2\right) \tag{2.151}$$
$$-e^{\varphi_1+\varphi_2}2\varphi_2(2\varphi_1+\varphi_2)Cosh[\varphi_1+\varphi_2] \tag{2.152}$$

Again the derivative is always negative therefore the numerator is less than itself evaluated at $k = 0$

$$-\frac{e^{\varphi_1+\varphi_2}(\varphi_1^4 + 2\varphi_1(-2+\varphi_1^2)\varphi_2 + (-2+\varphi_1^2)\varphi_2^2 + 2\varphi_2(2\varphi_1+\varphi_2)Cosh[\varphi_1+\varphi_2])}{\varphi_1+\varphi_2} \tag{2.153}$$

which is less than 0 everywhere. Therefore, $\frac{\partial}{\partial\varphi_1}\left(\frac{M(\varphi_1)}{m(\varphi_1)}\right) < 0$ for every φ_1.

□

As we can see, the proof corresponds with the two possible scenarios aforementioned. The only thing lefy to see is what happens when if $(\varphi_1, \varphi_2) = (0, 0)$? The Value Function for trader i which we would like to maximize is

$$V_i = -\frac{1}{\alpha}e^{-\alpha(x_{i,0}+(x_{1,0}+x_{2,0})(e^{\varphi_i}-1))}e^{-\varphi_i} - \frac{1}{\alpha}(1-e^{-\varphi_i}) \tag{2.154}$$

which derivative

$$V_i' = -\frac{1}{\alpha}e^{-\alpha(x_{i,0}+(x_{1,0}+x_{2,0})(e^{\varphi_i}-1))}e^{-\varphi_i}\left(-\alpha(x_{1,0}+x_{2,0})-1\right) - \frac{1}{\alpha}(e^{-\varphi_i}) \tag{2.155}$$

which is equal to zero when

$$-e^{-\alpha(x_{i,0}+(x_{1,0}+x_{2,0})(e^{\varphi_i}-1))}\left(\alpha(x_{1,0}+x_{2,0})e^{\varphi_i}+1\right)+1=0 \tag{2.156}$$

or equivalently

$$e^{-\alpha(x_{i,0}+(x_{1,0}+x_{2,0})(e^{\varphi_i}-1))}\left(\alpha(x_{1,0}+x_{2,0})e^{\varphi_i}+1\right)=1 \tag{2.157}$$

For any $\varphi_i \neq 0$, we already know that there is a contraction, so consider $\varphi_i = 0$

$$e^{-\alpha x_{i,0}}\left(\alpha(x_{1,0}+x_{2,0})+1\right)=1 \tag{2.158}$$

If we recall $k_i = \alpha x_{i,0}$ and $x_{1,0} = q x_{2,0}$ then

$$e^{-k_i}\left(k_1(1+q)+1\right)=1 \tag{2.159}$$

1. This equation is satisfied at $k_i = 0$ and $q = \frac{e^k}{k} - 1$. Therefore, $dV = 0$.
2. The derivative of this function with respect to q is always positive. Hence, If

$$q > \frac{e^k-1}{k} - 1 \tag{2.160}$$

   then $dV > 0$ at $\varphi_i = 0$ and there will be a maximum afterward, i.e., the maximum is some $\varphi_i^* > 0$ as the derivative dV should be negative at the end as explained above.
3. If $q < \frac{e^k-1}{k} - 1$ then $dV < 0$ everywhere. Therefore the the optimal φ_i is equal to 0.

Therefore, if the initial pair is $(0, 0)$ then there will be a maximum $(\varphi_1^*, \varphi_2^*)$ greater or equal than 0. So the algorithm works for all reals including 0.

2.4 Conclusion

The obtained results can lead to interesting interpretations. Chapter 1 sets up a model including moral hazard in investment and the result tells us that just one trader would not be interested to risk her position trading fraudulently and become rogue while Chapter 2 gives us a glance of the possible dynamics between more traders. Moreover, Chapter 2 tells us that if one trader starts risking his portfolio, others could follow. This also leads to open questions such as how traders could be coordinated to act fraudulently? In case of more possible assets, how would they react? How can we model and therefore anticipate these events? Could banks introduce indicators of such a strategy in advance?

Nevertheless, there is much to be researched on this direction like including constraints to non- negative consumptions; studying the difference on the diffusions depending on size of the Poisson Process or the impact of fixed salaries in moral hazard reactions. With respect to the Discrete time process, one could still research the interactions between traders and formalize it in continuous time.

Chapter 3

Appendix

The algorithm can be written in two different ways. The first one, which is the one presented below, both traders react at the same time to the other trader strategy. The second method, one trader makes her strategy and the other reacts consequently. The former is a bit longer, since both traders reacts at the same time while the latter goes quicker since the second trader is already reacting from the other's strategy.

$*X[\varphi1\_, \varphi2\_, x10\_, x20\_] := x10 + (x10 + x20)\dfrac{\varphi1}{\varphi1 + \varphi2}(e^{\varphi1+\varphi2} - 1)$

$*U[y\_, \alpha\_] := -(\dfrac{1}{\alpha})e^{-\alpha y}$

$*V1[\varphi1\_, \varphi2\_, x10\_, x20\_, \alpha\_] := U[X[\varphi1, \varphi2, x10, x20], \alpha]e^{-(\varphi1+\varphi2)} + U[0, \alpha](1 - e^{-(\varphi1+\varphi2)})$

$*V2[\varphi1\_, \varphi2\_, x10\_, x20\_, \alpha\_] := U[X[\varphi2, \varphi1, x20, x10], \alpha]e^{-(\varphi1+\varphi2)} + U[0, \alpha](1 - e^{-(\varphi1+\varphi2)})$

$*\$Assumptions\{tol = 10^{-16}; \varphi11 = 0; \varphi22 = 0; x10 = 1; x20 = 1; \alpha1 = 1; \alpha2 = 1\};$

$*\Phi1 = If[Sign[\varphi1/.Last[FindMaximum[V1[\varphi1, \varphi22, x10, x20, \alpha1], \varphi1]]] == -1, 0.,$
$\quad \varphi1/.Last[FindMaximum[V1[\varphi1, \varphi22, x10, x20, \alpha1], \varphi1]]];$

$*\Phi2 = If[Sign[\varphi2/.Last[FindMaximum[V2[\varphi11, \varphi2, x10, x20, \alpha2], \varphi2]]] == -1, 0.,$
$\quad \varphi2/.Last[FindMaximum[V2[\varphi11, \varphi2, x10, x20, \alpha2], \varphi2]]];$

$*cnt = 1;$

$*While[Abs[\Phi1 - \varphi11] + Abs[\Phi2 - \varphi22] > tol, cnt++; \varphi11 = \Phi1; \varphi22 = \Phi2;$
$\quad \Phi1 = If[Sign[\varphi1/.Last[FindMaximum[V1[\varphi1, \varphi22, x10, x20, \alpha1], \varphi1]]] == -1, 0.,$
$\qquad \varphi1/.Last[FindMaximum[V1[\varphi1, \varphi22, x10, x20, \alpha1], \varphi1]]];$
$\quad \Phi2 = If[Sign[\varphi2/.Last[FindMaximum[V2[\varphi11, \varphi2, x10, x20, \alpha2], \varphi2]]] == -1, 0.,$
$\qquad \varphi2/.Last[FindMaximum[V2[\varphi11, \varphi2, x10, x20, \alpha2], \varphi2]]];$

$*\Phi1$

$*\Phi2$

$*cnt$

Examples

1. Suppose $x_1 = 1$, $x_2 = 2$, $\alpha_1 = 1$, and $\alpha_2 = 2$. Then if we start with $\varphi_1 = \varphi_2 = 0$:

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.155647$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$

   Again with $\varphi_1 = 0.155647$ and $\varphi_2 = 0$

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.155647$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$

   If we start with $\varphi_1 = \varphi_2 = 1$.

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.172975$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$

   and again:

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.155647$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$

   Here we can see that starting from different initial strategies, the equilibrium is the same at the end. For this particular case we can see that the poorer trader and is less risk averse wants to trade while the wealthy trader does not.

2. Suppose $x_1 = 1$, $x_2 = 2$, $\alpha_1 = 2$, and $\alpha_2 = 1$:

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$

   Now if the trader is poorer but more or equally risk averse does not want to invest either.

3. Suppose $x_1 = 1$, $x_2 = 1$, $\alpha_1 = 2$, and $\alpha_2 = 1$:

$$\hat{\varphi_1} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0$$
$$\hat{\varphi_2} = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.0705485$$

   If both have the same initial wealth, the one who is less risk averse wants to trade.

4. Suppose $x_1 = 1$, $x_2 = 1$, $\alpha_1 = 1$, and $\alpha_2 = 1$:

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.0705485$$
$$\hat{\varphi}_2 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.0705485$$

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.0832432$$
$$\hat{\varphi}_2 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.0832432$$

and so on until

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) \approx 0.0858535$$
$$\hat{\varphi}_2 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) \approx 0.0858535$$

If they are equally wealthy and risk averse, they want to trade equally, i.e, their behavior is equal.

5. Suppose $x_1 = 1$, $x_2 = 2$, $\alpha_1 = 1$, and $\alpha_2 = 1$:

$$\hat{\varphi}_1 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.155647$$
$$\hat{\varphi}_2 = argmax_{\varphi_1} V_1(\varphi_1, \varphi_2) = 0.$$

They both are equally risk averse. The trader who wants to invest would be the poorer.

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