PRICING OF FORWARDS AND OTHER DERIVATIVES IN COINTEGRATED COMMODITY MARKETS

FRED ESPEN BENTH AND STEEN KOEKEBAKKER

ABSTRACT. We analyse cointegration in commodity markets, and propose a parametric class of pricing measures which preserves cointegration for forward prices with fixed time to maturity. We present explicit expressions for the term structure of volatility and correlation in the context of our spot price models based on continuous-time autoregressive moving average dynamics for the stationary components. The term structures have many interesting shapes, and we provide some empirical evidence from refined oil futures prices at NYMEX defending our modelling idea. Motivated from these results, we present a cointegrated forward price dynamics using the Heath-Jarrow-Morton modelling idea. In this setting, the concept of cointegration is extended to what we call *cointegration in the limit*, which is an asymptotic form of the notion. The Margrabe formula for spread option prices is shown to hold, with an explicit plug-in volatility. We present several numerical examples showing that cointegration leads to significantly cheaper spread options compared to the complete market case, where cointegration disappears for the pricing measure.

1. INTRODUCTION

In this paper we investigate pricing of forwards and other derivatives in a multiple commodities framework with cointegration. In financial economics, the standard modeling choice for the joint stochastic dynamics is correlated geometric Brownian motion (see e.g. Merton [30], Margrabe [29] and Stulz [40] for early contributions). Geometric Brownian motion is a non-stationary process and the spread of two correlated geometric Brownian motions will have infinite variance as time approaches infinity. On the other hand, when commodities are cointegrated there exists a linear combination of (log) prices which becomes stationary. The individual commodities may be non-stationary, but there exists a stationary long term linear relationship between them.

There exist two main modeling approaches for contingent claim valuation in commodity markets; spot price models and forward curve models. In a spot price model the starting point is the specification of the stochastic dynamics of the underlying commodity. The unobservable (net) convenience yield plays the same role as a dividend yield for common stocks, since it benefits the spot commodity holder but not the holder of a derivative asset. After an appropriate change of probability measure, forward, futures and (real) option prices can be computed as conditional expectations of the underlying spot price under the pricing measure. Examples of this approach can be found in Brennan and Schwartz [12], Gibson and Schwartz [24] and Schwartz [38]. The main problem with spot price based models is that forward prices are given endogenously from the spot price dynamics. As a result, theoretical forward prices will in general not be consistent with market observed forward prices. As a response to this, a line of research has focused on modeling the evolution of the whole forward curve using only a few stochastic factors taking the initial term structure as given. Examples of this research, building on the modeling framework of Heath, Jarrow and Morton (HJM) [26], are Cortazar and Schwartz [19], Clewlow and Strickland [17], Clewlow and Strickland [16] and Miltersen and Schwartz [31]. Empirical investigations in commodity markets have been conducted by, among others, Cortazar and Schwartz [19], Clewlow and Strickland [18] and Casassus, Liu and Tang [14].

Recently derivatives pricing in cointegrated commodity markets have produced results that, at first, seem inconsistent. For instance, Duan and Thierault [21] consider a cointegrated forward curve approach in the

---

Date: October 27, 2014.

Key words and phrases. Cointegration, risk premium, CARMA processes, commodity markets, spot and forward relationship, Heath-Jarrow-Morton modeling.

We are grateful for discussions with Alvaro Cartea, Rikard Green and Trygve Kastberg Nilssen. Sjur Westgaard is thanked for providing data from NYMEX. Financial support from the projects "Greenshiprisk" (F. E. Benth and S. Koekebakker) and "Managing Weather Risk in Energy Markets" (F. E. Benth), funded by the Norwegian Research Council, are gratefully acknowledged.

market for crude oil and oil products. In their model, cointegration has no effect on cross commodity option valuation. The long term stationary relationship disappears in the transition from the real world probability measure to the pricing measure. On the other hand, Casassus, Liu and Tang [15] develop an equilibrium model where spot prices of crude oil and oil products are cointegrated through linkages in the convenience yields. In their framework cointegration is preserved after changing from the real world to the pricing measure. The two approaches give very different valuation results for (long term) spread options.

The purpose of this paper is to develop a rigorous and coherent modeling framework for cointegrated commodity markets. We do this both in the spot price framework and in the HJM framework. We present some new insights and reconcile results from previous literature in this area.

Starting with a spot price framework, we propose a generalised two-factor model similar to the short-term/long-term commodity model of Schwartz and Smith [37]. The log commodity spot price dynamics consists of two separate processes; a stationary (short term) factor and a non-stationary (long term) factor. We model the stationary factor as a continuous time autoregressive moving average (CARMA) process, and the non-stationary process as an arithmetic Brownian motion. These stochastic processes are analogous to the short term and the long term factors in the Schwartz-Smith model respectively, however, the Ornstein-Uhlenbeck dynamics for the short term factor in the Schwartz-Smith model is replaced by a general CARMA dynamics. The non-stationary process is common to both commodities, while the stationary CARMA processes are specific to each commodity.[1] For our joint dynamics there exist a stationary, linear relationship between the log prices in which the common non-stationary process cancels out. Our spot price model does not explicitly consider stochastic convenience yields.[2] Nevertheless, as shown by Schwartz and Smith [37], if short term dynamics is governed by a Ornstein-Uhlenbeck process (which corresponds to a CARMA(1,0) model in our framework) the model is equivalent to the stochastic convenience yield model developed in Gibson and Schwartz [24]; the state variables in each model can be represented as linear combinations of the state variables in the other.

Schwartz and Smith [37] argue that their model specification of stochastically evolving short-term deviations and long term equilibrium prices seems more natural and intuitive than the stochastic convenience yield set up. Adding to that, the Schwartz-Smith model approach is better suited for modelling non-storable "commodity" markets, like electricity, weather or freight rates, where the convenience of holding inventory makes little sense. Schwartz and Smith [37] also note that these factors are more "orthogonal" in their dynamics, which leads to more transparent analytical results. This is especially true when generalising to multiple cointegrated commodity markets. The long term factor must be common to all commodities, while the the short term factor can be idiosyncratic to each commodity. Note also that this model is suitable for other asset classes as well. For instance, Fama and French [22] proposed a discrete time model for stock price dynamics similar to the Schwartz-Smith model. This means that our framework can also be used for derivatives pricing in cointegrated stock markets as well.

Cointegration is a real world phenomenon and therefore defined under the market (objective) probability measure. For the purpose of derivative valuation, we change the probability measure to the pricing measure. There are two main approaches. The first approach assumes that the commodity itself is a traded asset similar to common stock in the Black-Scholes model (see for instance Brennan and Schwartz [12]). Derivatives can be replicated by dynamic trading in the underlying commodity, and there exist a unique pricing measure for the commodity. In the second approach the spot price plays the role of an underlying state variable upon which contingent claims can be written. In this latter approach the pricing measure can only be identified after an additional assumption regarding the market price of risk. The market price of risk is typically assumed to have a functional form that makes the spot price dynamics qualitatively similar

[1]Paschke and Prokopczuk [34] propose a slightly different cointegrated spot price model than the one we present here. In their model commodity prices are driven by an n-dimensional log price dynamics. The non-stationary factor is an arithmetic Brownian motion, while the $n-1$ stationary components are all correlated Ornstein-Uhlenbeck processes. They estimate the model using the Kalman filter approach on 3 commodities (crude oil, heating oil and gasoline) using a 6 factor model.

[2]Nakajima and Ohashi [32] also consider cointegration through the convenience yield, but they state their model directly under the pricing measure.

under the real world and the pricing measure (see for instance Gibson and Schwartz [24], Schwartz [38], Schwartz and Smith [37] and Casassus and Collin-Dufresne [14]).[3]

This basic market assumption is crucial when it comes to derivatives valuation in cointegrated commodity markets. When assuming perfect tradability in the underlying spot commodity, forward prices can be replicated by a simple buy-hold strategy, and thus, the volatility of forwards is equal to the volatility of spot prices. Both the short term and the long term factor of the spot price has the same effect on all parts of the forward curve, and the mean reverting property of the short term factor is not transfered to forward prices. Cointegration disappears under the pricing measure and the joint spot (and forward) dynamics reduces to correlated geometric Brownian motions. On the other hand, using the state variable approach, the non-stationary long term factor and the mean reverting short term factor both exist after adjusting for the price of risk. This way spot commodity prices remain cointegrated also under the pricing measure. This explains the fundamental difference between the model of Duan and Thierault [21][4] and Casassus, Liu and Tang [15].

In our analysis we proceed with the state variable approach and assume constant market prices of risks.[5] We derive several implications for the forward price dynamics and cross-commodity option pricing when cointegration is preserved under the pricing measure. As is known, for a given forward contract the maturity time is fixed, but time to maturity decreases as we move forward in time. We show that the dynamics of two commodity forward contracts are not cointegrated even though the underlying spot prices are.[6] Rolling a contract forward means closing an initial shorter-term contract and opening a new longer-term contract. Doing so continually provides us with a time series with the time-to-maturity fixed, the so called Musiela parametrization of forward prices. By changing to the Musiela parametrization we find that cointegration is preserved. This means that cointegration under the pricing measure applies to all fixed time to maturities along the commodity term structure. The spot prices are simply a special case with time to maturity being zero. Next we derive the term structure of volatility, covariance and correlation of the logarithmic returns of forward prices. In the short end of the curve, each market is driven by both the stationary and the non-stationary factor. In this way, each market incorporates Samuelson-type maturity-effects, with volatility increasing as time to maturity decreases. In the long end of the curve, the stationary component in both markets have no effect, and prices are driven only by the common non-stationary component. Thus, the term structure of volatility flattens in the long end for both commodities. This also implies that the correlation between forward returns is perfect in the long end. The term structure of correlation increases to unity, as time to maturity tends infinity. The stationary factors with CARMA-dynamics allow for a wide range of volatility and correlation term structures.

Turning our attention to cross commodity derivatives valuation, we focus on spread options, and in particular on the Margrabe exchange option. In the original article by Margrabe [29] the exchange option is written on two assets driven by correlated geometric Brownian motions. In our cointegrated framework the option pricing formula is similar to Margrabe's, except for the variance plug-in that accounts for the volatility and correlation effects noted above. Cointegration under the pricing measure makes long term spread options cheaper than is the case for commodity dynamics represented by correlated geometric Brownian motions.

Finally, we turn to modeling cointegration in the HJM framework. We stay within a two factor model, with marginal forward price dynamics specified as correlated geometric Brownian motions under the market (objective) probability measure for both commodity markets. The "non-stationary" Brownian motion is common to both markets, while there are two "stationary" Brownian motions, one for each individual

---

[3]See Secomandi and Seppi [39] for a nice discussion on risk neutral pricing with different commodity market assumptions. They use the term *dynamically complete market* when the commodity itself is a tradeable asset and the market is perfect in the Black-Scholes sense. They argue that this approach applies to precious metals, like silver and gold, which are investible stores of value over time. They use the term *dynamically incomplete markets* when there is non-traded randomness in the spot price dynamics, and the market price of risk approach applies. This latter approach is relevant for most commodities other than precious metals.

[4]The approach of Duan and Thierault [21] is in fact an application of the model by Duan and Pliska [20] developed for the stock market. They set up a model for cointegrated stock prices allowing for GARCH-type stochastic volatility. They found that although cointegration disappeared under the pricing measure, the joint stock price dynamics still had non-trivial GARCH effects.

[5]For previous studies assuming constant market price of risks see e.g. Gibson and Schwartz [24], Schwartz [38] and Schwartz and Smith [37].

[6]Forward contracts in the long end of the market will move with a fixed distance respective to each other on the log scale. Hence, long term forward contracts in a cointegrated set up will stay close to each other, and in this sense they are asymptotically cointegrated.

market. We define cointegration *in the limit* if there exist a linear combination of forwards, under the Musiela parametrization, with a stationary distribution as time approaches infinity. It turns out we need a structural (consistency) condition on the initial forward curves to obtain cointegration. We further state sufficient conditions for an arbitrage-free dynamics that preserves the correlation structure under the pricing measure.

The rest of this paper is organised as follows: In section 2 we present our cointegrated commodity price model under the market (objective) measure and discuss the change to the pricing measure. In section 3 we derive forward prices for cointegrated spot commodities and compute the term structure of volatility and correlation. With constant market prices of risk forward prices in the Musiela parametrization remain cointegrated under the pricing measure. In section 4 we derive spread option prices. We present the Margrabe-formula for an exchange option when the underlying forwards are cointegrated. In section 5 we set up cointegrated commodity curves in the HJM framework and present sufficient conditions for an arbitrage free dynamics. Section 6 concludes. Some derivations are deferred to the appendix.

2. A PRICING MEASURE PRESERVING COINTEGRATION

On a complete filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ we define three correlated Brownian motions $\widetilde{B}$ and $\widetilde{W}_i$, $i = 1, 2$. The correlation between $\widetilde{W}_1$ and $\widetilde{W}_2$ is denoted $\rho \in (-1, 1)$, whereas the correlation between $\widetilde{B}$ and $\widetilde{W}_i$ is $\rho_i \in (-1, 1)$, $i = 1, 2$. To have a well-defined three-dimensional Brownian motion process we assume that

$$\rho^2 + \rho_1^2 + \rho_2^2 - 2\rho\rho_1\rho_2 \leq 1$$

See the Appendix for the argument behind this condition.

Define the spot prices at time t, $S_i(t), i = 1, 2$ of two commodities to be

$$S_i(t) = \exp\left(c_i X(t) + Y_i(t)\right), i = 1, 2, \tag{2.1}$$

where $c_i, i = 1, 2$ are two constants and X is a drifted Brownian motion (a non-stationary process) with dynamics

$$dX(t) = \mu\, dt + \sigma\, d\widetilde{B}(t). \tag{2.2}$$

The parameters μ and $\sigma > 0$ are constants. On a logarithmic scale we see that

$$\ln S_1(t) - \frac{c_1}{c_2} \ln S_2(t) = Y_1(t) - \frac{c_1}{c_2} Y_2(t).$$

If the processes $Y_i(t)$ are stationary, we say that S_1 and S_2 are *cointegrated* by following the definition given by Duan and Pliska [20]. To reduce notation, we assume from now on that $c_1 = 1$ and $c_2 = c$ for some constant c. This constant can be interpreted as a conversion factor between two commodities, for example oil and gas or power and gas, in order to measure the two commodities on the same scale. It can also be a factor measuring the differences in quality, as for refined oil products.

We remark in passing that we ignore possible seasonal effects in our model for simplicity. We could incorporate that easily by adding to the logarithmic spot price a deterministic seasonality function. In that case we have to modify the cointegration concept slightly as well, as the log-difference between the prices will not be stationary, but stationary after de-seasonalizing the prices.

Usually, the stationary processes $Y_i, i = 1, 2$ are taken as Ornstein-Uhlenbeck processes, being a simple mean-reverting dynamics. In this paper we suppose that $Y_i, i = 1, 2$, follow stationary continuous-time autoregressive moving-average processes, known as CARMA-processes in the literature.

Following Benth and Šaltytė Benth [4], we let $Y_i(t)$ be a CARMA(p_i, q_i) process, for $p_i > q_i, i = 1, 2$ defined as

$$Y_i(t) = \mathbf{b}_i' \mathbf{Z}_i(t), \tag{2.3}$$

where $\mathbf{b}_i'$ is the transpose of the row vector $\mathbf{b}_i \in \mathbb{R}^{p_i}$ with $\mathbf{b}_i' = (b_{0,i}, b_{1,i}, \ldots, b_{q_i,i} = 1, 0, \ldots, 0)$. Furthermore, $\mathbf{Z}_i(t)$ is a p_i-dimensional Ornstein-Uhlenbeck process given by

$$d\mathbf{Z}_i(t) = A_i \mathbf{Z}_i(t)\, dt + \sigma_i \mathbf{e}_{p_i}\, d\widetilde{W}_i(t), \tag{2.4}$$

for σ_i being positive constant, $\mathbf{e}_{p_i}$ the p_ith canonical coordinate vector in $\mathbb{R}^{p_i}$, and

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_{p_i,i} & -\alpha_{p_i-1,i} & -\alpha_{p_i-2,i} & \dots & -\alpha_{1,i} \end{bmatrix}, \tag{2.5}$$

for positive numbers $\alpha_{1,i},\dots,\alpha_{p_i,i}$. We have a stationary CARMA-process $Y_i(t)$ if and only if the eigenvalues of A_i all have negative real part, which will be the class of models we will focus on in this paper. Thus, we assume that $A_i, i = 1,2$ have eigenvalues with negative real part from now on.

The explicit dynamics of $\mathbf{Z}_i(t)$ is know to be (see e.g. Benth and Šaltytė Benth [4])

$$\mathbf{Z}_i(t) = \exp(A_i t)\mathbf{Z}_i(0) + \int_0^t \sigma_i \exp(A_i(t-s))\mathbf{e}_{p_i}\, d\widetilde{W}_i(s)\,. \tag{2.6}$$

From this we see that under P, $(\mathbf{Z}_1,\mathbf{Z_2})$ is p_1+p_2-variate Gaussian process.

We now introduce a general pricing measure Q which accomodates various market situations typical for commodities. For Itô integrable processes ξ,ξ_1,ξ_2 on $[0,T]$ for some $T<\infty$ and vectors $\theta_i' = (\theta_{p_i,i},\theta_{p_i-1,i},\dots,\theta_{1,i})$, $i=1,2$, introduce the processes

$$dB(t) = \sigma^{-1}\xi(t)\,dt + d\widetilde{B}(t) \tag{2.7}$$

$$dW_i(t) = -\sigma_i^{-1}\left(\xi_i(t) + \theta_i'\mathbf{Z}_i(t)\right)\,dt + d\widetilde{W}_i(t)\,, i=1,2\,. \tag{2.8}$$

Suppose that the process $t\mapsto M(t)$ for $t\le T$ defined by

$$M(t) = \mathcal{E}\left(\sigma^{-1}\xi(t), -\sigma_1^{-1}\left(\xi_1(t)+\theta_1'\mathbf{Z}_1(t)\right), -\sigma_2^{-1}\left(\xi_2(t)+\theta_2'\mathbf{Z}_2(t)\right)\right)(t)$$

is a martingale, with $\mathcal{E}(X,Y,Z)$ being the stochastic exponential for the vector-valued Itô integrable stochastic process $t\mapsto(X(t),Y(t),Z(t))$ with respect to the Brownian motions $\widetilde{B},\widetilde{W}_i$, $i=1,2$. It follows by Girsanov's Theorem (see e.g. Øksendal [33]) that there exists a probability measure $Q\sim P$ such that B, W_1 and W_2 are three correlated Q-Brownian motions on $[0,T]$, which have the same correlation structure as $\widetilde{B}$, $\widetilde{W}_1$ and $\widetilde{W}_2$. We have chosen this representation of the measure change deliberately in order to cover interesting special cases in various financial markets.

We assume that

$$\theta_{k_i,i} < \alpha_{k_i,i}\,, \tag{2.9}$$

for $k_i = 1,\dots,p_i$, $i=1,2$. The Q-dynamics of state processes of the spot price become

$$dX(t) = (\mu - \xi(t))\,dt + \sigma\,dB(t) \tag{2.10}$$

$$d\mathbf{Z}_i(t) = \left(\xi_i(t)\mathbf{e}_{p_i} + A_i^\theta\mathbf{Z}_i(t)\right)\,dt + \sigma_i\mathbf{e}_{p_i}\,dW_i(t)\,, i=1,2\,, \tag{2.11}$$

with

$$A_i^\theta = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -(\alpha_{p_i,i}-\theta_{p_i,i}) & -(\alpha_{p_i-1,i}-\theta_{p_i-1,i}) & -(\alpha_{p_i-2,i}-\theta_{p_i-2,i}) & \dots & -(\alpha_{1,i}-\theta_{1,i}) \end{bmatrix}. \tag{2.12}$$

We note that under the condition (2.9) on the $\theta_i, i=1,2$ vectors we have that the elements of the last row of A_i^θ are strictly negative, and therefore $\mathbf{Z}_i$ is a CARMA-dynamics also under Q, but now with a stochastic stationarity level ξ_i. We have assumed that the eigenvalues of A_i all have negative real part to ensure stationarity, but this does not imply that A_i^θ has eigenvalues which have negative real part for arbitrary choices of θ. However, we note that the eigenvalues of a matrix depend continuously on the matrix elements because of the continuous dependency of the roots on the parameters of the characteristic polynomial (see Tyrtyshnikov [41]). The characteristic polynomial of A_i is

$$p_i(\lambda) = \lambda^{p_i} + \alpha_{1,i}\lambda^{p_i-1} + \ldots + \alpha_{p_i,i}$$

with the roots being the eigenvalues of A_i. The corresponding characteristic polynomial of A_i^θ is

$$p_i^\theta(\lambda) = \lambda^{p_i} + (\alpha_{1,i}-\theta_{1,i})\lambda^{p_i-1} + \ldots + (\alpha_{p_i,i}-\theta_{p_i,i})\,.$$

As the roots of a polynomial are continuously depending on the parameters of the polynomial, there exists at least some $\theta_{k,i} < \alpha_{k,i}$ for which the real part of the roots of p_i^θ are negative. As the real parts of the eigenvalues of A_i are strictly negative, there is some "space" left before they reach or cross zero to be positive.

We now investigate some special cases of the measure change (2.10)-(2.11). First, let us assume that we are in a complete market, in which the pricing measure Q turns both assets S_1, S_2 into Q-martingales after discounting. To this end, denoting $r > 0$ the risk-free interest rate and fixing θ, we find by Itô's Formula, for $i = 1, 2$,

$$\begin{aligned} d(\mathrm{e}^{-rt}S_i(t)) = \Big\{\mu - r - c_i\xi(t) + \mathbf{b}_i' A_i^\theta \mathbf{Z}_i(t) + \xi_i(t)\mathbf{b}_i'\mathbf{e}_{p_i} \\ + \frac{1}{2}(c_i^2\sigma^2 + 2\rho_i c_i \sigma\sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i}) + \sigma_i^2(\mathbf{b}_i'\mathbf{e}_{p_i})^2)\Big\}\,\mathrm{e}^{-rt}S_i(t)\,dt \\ + \mathrm{e}^{-rt}S_i(t)\left\{c_i\sigma\,dB(t) + \sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i})\,dW_i(t)\right\}\,. \end{aligned}$$

If we can find $\xi, \xi_i, i = 1, 2$ such that

$$c_i\xi(t) - \xi_i(t)(\mathbf{b}_i'\mathbf{e}_{p_i}) = \mu - r + \mathbf{b}_i' A_i^\theta \mathbf{Z}_i(t) + \frac{1}{2}\left\{c_i^2\sigma^2 + 2\rho_i\sigma\sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i}) + \sigma_i^2(\mathbf{b}_i'\mathbf{e}_{p_i})^2\right\}\,, \tag{2.13}$$

it follows that

$$d(\mathrm{e}^{-rt}S_i(t)) = \mathrm{e}^{-rt}S_i(t)\left\{c_i\sigma\,dB(t) + \sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i})\,dW_i(t)\right\}\,,$$

which is a Q-martingale for $i = 1, 2$. We see that the cointegration between the spot prices has completely disappeared, since both assets under Q will have a dynamics

$$dS_i(t) = rS_i(t)\,dt + S_i(t)\left\{c_i\sigma\,dB(t) + \sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i})\,dW_i(t)\right\}\,. \tag{2.14}$$

This is the situation analysed by Duan and Pliska [20]. For example, in a liquid stock market this would be the relevant situation. In the complete market commodity set up the drift rate is typically the risk free rate minus a constant convenience yield., see e.g. Brennan and Schwartz [12].

If $q_i = p_i - 1$, then $\mathbf{b}_i'\mathbf{e}_{p_i} \neq 0$, otherwise $\mathbf{b}_i'\mathbf{e}_{p_i} = 0$. So, in the case *both* CARMA processes $Y_i(t)$ are such that $q_i < p_i - 1$, $i = 1, 2$, then we cannot find any equivalent martingale measure as $\xi(t)$ must satisfy two linearly independent equations in (2.13). We recall from the theory of CARMA processes that if $q_i < p_i - 1$, then $Y_i(t)$ is a finite variation process (see Brockwell [13]). If at least one of the two CARMA processes satisfy $q_i = p_i - 1$, we find two equations in (2.13) for the three unknowns ξ, ξ_1, ξ_2. Hence, we will have infinitely many solutions, giving infinitely many pricing measures. Although we have infinitely many martingale measures Q, the risk-neutral dynamics of S_i is unique. We see that if only one of the two assets have a CARMA dynamics with $q_i = p_i - 1$, let us say for $i = 1$, then ξ and ξ_1 are uniquely determined while ξ_2 can be chosen arbitrarily. Indeed, we find in this case that

$$c\xi(t) = \mu - r + \mathbf{b}_2' A_2^\theta \mathbf{Z}_2(t) + \frac{1}{2}c_2^2\sigma^2\,,$$

and

$$\begin{aligned} \xi_1(t)(\mathbf{b}_1'\mathbf{e}_{p_1}) = (\frac{1}{c} - 1)(\mu - r) + \frac{1}{c}\mathbf{b}_2' A_2^\theta \mathbf{Z}_2(t) - \mathbf{b}_1' A_1^\theta \mathbf{Z}_1(t) \\ - \frac{1}{2}\left\{\sigma^2 + 2\rho_1\sigma\sigma_1(\mathbf{b}_1'\mathbf{e}_{p_1}) + \sigma_1^2(\mathbf{b}_1'\mathbf{e}_{p_1})^2\right\}\,. \end{aligned}$$

Note that ξ and ξ_1 are linearly dependent on the state vectors $\mathbf{Z}_i(t)$, $i = 1, 2$.

Turning our attention to commodity markets, the situation may become very different. Typically, the spot market may be very illiquid, or not existing as a market where one can trade in the normal sense of the word. For example, in power markets one has an auction-based spot market, where physical transmission of power is committed in return for a fixed price. As power cannot be stored, but has to be used once produced, one cannot for example buy the spot and sell it later for speculative purposes. Markets for weather and freight derivatives are common examples where the underlying "spot" cannot be traded (see Benth and Šaltytė Benth [4] for an extensive analysis of weather markets). In gas and oil spot markets one must have storage facilities and means of transportation to speculate. Therefore, also in these markets it is highly questionable whether one can talk of liquid trading as in a stock market. Metals and agriculture are other examples of commodity markets which are illiquid and cannot be analysed within a complete market framework.

A quite common assumption in commodity markets where the spot is a highly illiquid asset, as discussed above, is that the pricing measure Q is a simple parametrization of the market price of risk. A parametric market price of risk can again be viewed as a parametrization of the risk premium. Typically (see for example Lucia and Schwartz [28], Kolos and Ronn [27] and Benth, Šaltytė Benth and Koekebakker [5]), one introduces a constant level change in one or both factors in the spot price process. Thus, one specifies the pricing measure given by the choices $\theta_i = \mathbf{0}$, $\xi_i(t) = \xi_i$ and $\xi(t) = \xi$, all being constants. Then the drift of X becomes $\mu - \xi$ under Q, while $Y_i(t)$ will have a long-term mean level given by

$$\lim_{t\to\infty} \mathbf{b}_i' \int_0^t \xi_i \exp(A_i(t-s))\mathbf{e}_{p_i}\, ds = -\xi_i(\mathbf{b}_i' A_i^{-1}\mathbf{e}_{p_i})\,.$$

Hence, the drift of the non-stationary component X is either increased or decreased by a constant factor ξ, while a long-term mean level of Y_i is different from zero. Contrary to the complete market case, such a measure change will preserve the cointegration between the two spots. We observe that the cointegration structure will be slightly different under Q compared to P, as the mean-level of the processes Y_i, $i = 1, 2$ are different.

For constant ξ, ξ_i's, we may in addition slow down the speed of mean reversion coefficients of the A-matrices by letting $\theta_i \neq 0$. The associated pricing measure becomes an extension of the class of pricing measures studied by Benth and Ortiz-Latorre [9] to the CARMA case. Note that under the condition (2.9) this measure change will be structure preserving, as the processes Y_i will be CARMA processes under Q as well. We note that we need to impose additional conditions on θ_i for the matrices A_i^θ to have eigenvalues with negative real parts, and thus Y_i being stationary processes for $i = 1, 2$. In this situation the cointegration between the spot price processes is preserved, however, with different stationary processes under Q than under P (due to the change of mean level as well as the mean-reversion coefficients in the A matrices). If, for example, $Y_i(t)$, $i = 1, 2$ are Ornstein-Uhlenbeck processes, then choosing $\theta_i \neq 0$ means to slow down the speed of mean reversion under Q. This can be viewed as the market letting shocks in the stationary processes Y_i last longer under Q than in the actual market, which is a way of adding a risk loading to the short-term spot price variations. Benth, Cartea and Pedraz [6] show empirically that the mean reversion speed is slowed down in energy markets based on spot and forward data analysis. Generally speaking, we have a class of pricing measures for which the cointegration under P is preserved. This class of measures are structurally similar to the measure change in the complete case. As we shall see in the next Section, the preservation of cointegration will have immediate consequences on the forward price dynamics, and in turn on the pricing of options on forwards.

Worth observing is that we have looked at a class of pricing measures Q ranging from complete to incomplete markets. All the different cases of interest discussed above can be represented generically by the Girsanov change,

$$dB(t) = (a + \mathbf{c}'\mathbf{Z}(t))\,dt + d\widetilde{B}(t)\,, \tag{2.15}$$

and

$$dW_i(t) = (a_i + \mathbf{c}_i'\mathbf{Z}(t))\,dt + d\widetilde{W}_i(t)\,, \tag{2.16}$$

for a, a_i being constants and $\mathbf{c}, \mathbf{c_i}$ being vectors in $\mathbb{R}^{p_1+p_2}$, $i = 1, 2$. Here, $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2)$. By following the arguments in the proof of Prop. 5.1 in Benth and Šaltytė Benth [4], one can verify by Girsanov's Theorem that there exists a probability $Q \sim P$ such that (B, W_1, W_2) are Q-Brownian motions, with the same correlation structure as $(\widetilde{B}, \widetilde{W}_1, \widetilde{W}_2)$. Thus, in all the cases of interest to us, we have a valid pricing measure Q and a characterization of the spot price dynamics under this measure.

We end this section with a discussion on when A_i^θ has eigenvalues with negative real part. We consider the cases $p = 2$ and $p = 3$, which seems to be of most practical interest (see e.g. Garcia, Klüppelberg and Müller [23] and Pasche and Prokopczuk [35] for a CARMA(2,1) model for power spot and crude oil futures prices, respectively, and Benth and Šaltytė Benth [4] and Härdle and Lopez-Cabrera [25] for CARMA(3,0) models applied to temperature dynamics.) Let us first consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{bmatrix}\,,$$

which has eigenvalues

$$\lambda_{1,2} = -\frac{1}{2}\alpha_1 \pm \frac{1}{2}\sqrt{\alpha_1^2 - 4\alpha_2}\,.$$

If $\alpha_1^2 < 4\alpha_2$, then the real part of $\lambda_{1,2}$ is $-\alpha_1/2$ being negative. If $\alpha_1^2 > 4\alpha_2$, we have two real eigenvalues. However, since $\alpha_1^2 - 4\alpha_2 < \alpha_1^2$, they are both negative. In conclusion, if $p_i = 2$, then A_i^θ will always have eigenvalues with negative real part as long as $\theta_{k_i,i} < \alpha_{k_i,i}$ for $k_i = 1, 2$. The situation $p = 3$ is far more complex, as we now show. The characteristic polynomial of the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix},$$

is

$$p(\lambda) = \lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3\,.$$

This is a cubic polynomial with positive coefficients, thus having at least one real root. By the conjugate root theorem, if λ is a complex root of $p(\lambda)$, then $\bar{\lambda}$, the complex conjugate is also a root. Hence, the situation is that we may have a) 3 real distinct roots, b) two real distinct roots, one with multiplicity 2, c) 1 real root, with multiplicity 3, or d) one real and two complex roots. Note that $p'(\lambda)$ has roots

$$\widetilde{\lambda}_{1,2} = -\frac{1}{3}\alpha_1 \pm \frac{1}{3}\sqrt{\alpha_1^2 - 3\alpha_2}\,.$$

Now, if $\alpha_1^2 < 3\alpha_2$, $p'(\lambda)$ has no real roots and will therefore be either positive or negative. But $p'(0) = \alpha_2 > 0$, and thus $p(\lambda)$ must be an increasing function. We are hence in case c or d concerning the roots of $p(\lambda)$. But as $p(0) = \alpha_3 > 0$, the real root of $p(\lambda)$ must be negative. We recover case c when $\alpha_2 = \alpha_1^2/3$ and $\alpha_3 = \alpha_1^3/27$, in which case the root is $\lambda = -\alpha_1/3$. As it turns out, the real part of two complex roots may be either positive or negative, depending on the coefficients. We look at the case $\alpha_1^2 > 3\alpha_2$, for which $p'(\lambda)$ has two real roots. This means that $p(\lambda)$ has two extreme points. We have that $p(\lambda)$ is concave for $\lambda < -\alpha_1/3$ and convex for $\lambda > -\alpha_1/3$, and therefore $p(\lambda)$ has a local maximum at λ_1 and local minimum at λ_2, where

$$\lambda_{1,2} = -\frac{1}{3}\alpha_1 \mp \frac{1}{3}\sqrt{\alpha_1^2 - 3\alpha_2}\,,$$

which both are located on the negative part of the real axis. Depending on the coefficients, we may now have that there are three real roots, one to the left of λ_1, one inbetween λ_1 and λ_2, and one to the right of λ_2. As this happens if $p(\lambda_2) < 0$, the root above λ_2 must be negative by the mean value theorem as $p(0) = \alpha_3 > 0$. Hence, all theree roots are negative. Other situations that may occur are $p(\lambda_1) < 0$ or $p(\lambda_2) > 0$, which means we have only one real root and two complex. In both cases the real root will be negative. If either $p(\lambda_1) = 0$ or $p(\lambda_2) = 0$, we will have two real roots, both negative, one with multiplicity 2. Possibly, we may have configurations of coefficients α_1, α_2 and α_3 for which we do not have roots with negative real parts. In conclusion, for $p_i = 3$ we may have that A_i has eigenvalues with negative real part, while A_i^θ may not. From the discussion above we can figure out configurations where this happen, but the criteria are rather technical.

3. FORWARD PRICES AND COINTEGRATION

In this Section we are concerned with the implied forward price dynamics from the cointegrated spot model in the previous Section. We introduced and argued for a pricing measure Q in commodity markets which changed the level of mean and drift, as well as the autoregressive coefficients in the stationary part. We recall our state process dynamics under the pricing measure Q to be

$$dX(t) = \zeta\,dt + \sigma\,dB(t) \tag{3.1}$$

$$d\mathbf{Z}_i(t) = (\xi_i\mathbf{e}_{p_i} + A_i\mathbf{Z}_i(t))\,dt + \sigma_i\mathbf{e}_{p_i}\,dW_i(t)\,, \tag{3.2}$$

for $i = 1, 2$. To reduce the notational burden, we have deleted the superscript θ from the matrix A, but assume that this has eigenvalues with negative real parts also under Q. We also use the notation ζ for the drift $\mu - \xi$.

The forward price $F_i(t,T)$ at time $t \geq 0$ for a contract delivering commodity $i = 1,2$ at time $T \geq t$ is defined as

$$F_i(t,T) = \mathbb{E}_Q\left[S_i(T)\,|\,\mathcal{F}_t\right] . \tag{3.3}$$

We find the following result for the forward price:

Proposition 3.1. *The forward price $F_i(t,T)$ at time $t \leq T$ is*

$$F_i(t,T) = H_i(T-t)\exp\left(c_i X(t) + \mathbf{b}_i' \mathrm{e}^{A_i(T-t)}\mathbf{Z}_i(t)\right) ,$$

for $i = 1,2$, where

$$\ln H_i(x) = c_i\zeta x + \xi_i \mathbf{b}_i' A_i^{-1}(\mathrm{e}^{A_i x} - I_i)\mathbf{e}_{p_i} + \frac{1}{2}c_i^2\sigma^2 x + \rho_i c_i \sigma\sigma_i \int_0^x \mathbf{b}_i'\mathrm{e}^{A_i s}\mathbf{e}_{p_i}\,ds$$
$$+ \frac{1}{2}\sigma_i^2 \int_0^x (\mathbf{b}_i'\mathrm{e}^{A_i s}\mathbf{e}_{p_i})^2\,ds ,$$

for I_i being the $p_i \times p_i$ identity matrix.

Proof. See Appendix. □

Observe that the log-forward price becomes

$$\ln F_i(t,T) = \ln H_i(T-t) + c_i X(t) + \mathbf{b}_i'\mathrm{e}^{A_i(T-t)}\mathbf{Z}_i(t) .$$

Recalling our assumption that $c_1 = 1, c_2 = c$, we find that

$$\ln F_1(t,T) - \frac{1}{c}\ln F_2(t,T) = H_1(T-t) - \frac{1}{c}H_2(T-t) + \mathbf{b}_1'\mathrm{e}^{A_1(T-t)}\mathbf{Z}_1(t) - \frac{1}{c}\mathbf{b}_2'\mathrm{e}^{A_2(T-t)}\mathbf{Z}_2(t) . \tag{3.4}$$

Therefore, the two forward prices are not cointegrated. However, we have the following asymptotic result:

Corollary 3.2. *For fixed t, it holds that*

$$\lim_{T-t\to\infty} \ln F_1(t,T) - \frac{1}{c}\ln F_2(t,T) = -\xi_1\mathbf{b}_1' A_1^{-1}\mathbf{e}_{p_1} + \frac{1}{c}\xi_2\mathbf{b}_2' A_2^{-1}\mathbf{e}_{p_2}$$
$$+ \sigma\int_0^\infty \left(\rho_1\sigma_1\mathbf{b}_1'\mathrm{e}^{A_1 s}\mathbf{e}_{p_1} - \rho_2\sigma_2\mathbf{b}_2'\mathrm{e}^{A_2 s}\mathbf{e}_{p_2}\right)\,ds$$
$$+ \frac{1}{2}\int_0^\infty \left(\sigma_1^2(\mathbf{b}_1'\mathrm{e}^{A_1 s}\mathbf{e}_{p_1})^2 - \frac{1}{c}\sigma_2^2(\mathbf{b}_2'\mathrm{e}^{A_2 s}\mathbf{e}_{p_2})^2\right)\,ds .$$

Proof. We obtain this result after recalling that $A_i, i = 1,2$ have eigenvalues with negative real parts, which implies that $\exp(A_i x)\mathbf{x}$ can be written as a sum of exponentials which converges to zero as x tends to infinity for any vector $\mathbf{x} \in \mathbb{R}^{p_i}$. □

This result tells us that although the forward prices are not cointegrated for fixed maturity times, one may say that they are *asymptotically* cointegrated in the sense that the prices of the two forward contracts in the long end of the market will (as a linear combination on logarithmic scale) move with a fixed distance respective to each other.

By changing to the so-called Musiela parametrization, that is, to consider the forward price dynamics for contracts with fixed time to maturity $x \geq 0$, cointegration is in fact preserved. This is immediate from the following Corollary:

Corollary 3.3. *Define $f_i(t,x) := F_i(t,t+x)$. Then it holds*

$$\ln f_1(t,x) - \frac{1}{c}\ln f_2(t,x) = H_1(x) - \frac{1}{c}H_2(x) + \mathbf{b}_1'\mathrm{e}^{A_1 x}\mathbf{Z}_1(t) - \frac{1}{c}\mathbf{b}_2'\mathrm{e}^{A_2 x}\mathbf{Z}_2(t) .$$

Proof. This is straightforward from (3.4). □

Note that the stochastic processes $\mathbf{Z}_i(t)$ are stationary p_i-variate Gaussian processes, with stationary mean and variance with respect to the pricing probability Q given by

$$\lim_{t\to\infty} \mathbb{E}_Q\left[\mathbf{Z}_i(t)\right] = -\xi_i A_i^{-1}\mathbf{e}_{p_i}$$

and

$$\lim_{t\to\infty} \mathrm{Var}_Q(\mathbf{Z}_i(t)) = \sigma_i^2 \int_0^\infty \mathrm{e}^{A_i s}\mathbf{e}_{p_i}\mathbf{e}_{p_i}' \mathrm{e}^{A_i' s}\, ds\,,$$

for $i = 1, 2$. In view of the discussion on the relationships between the market probability P and the pricing measure Q in Sect. 2, $\mathbf{Z}_i(t)$ will be a stationary process also under P. Hence, f_1 and f_2 are cointegrated processes under P and Q, since the linear combination $f_1(t, x) - c^{-1} f_2(t, x)$ can be expressed as a linear transformation of the stationary processes $\mathbf{Z}_i(t), i = 1, 2$. This means that if we consider rolling forwards, they are cointegrated as long as the spots are cointegrated. In fact, we can have two different times to maturities $x_i, i = 1, 2$ and still have cointegration, that is, $f_1(t, x_1)$ and $f_2(t, x_2)$ are cointegrated. In conclusion, we have that $F_1(t, T)$ and $F_2(t, T)$ *are not* cointegrated, while $f_1(t, x_1)$ and $f_2(t, x_2)$ *are*!

We move on with the forward dynamics of $F_i(t, T)$:

Proposition 3.4. *The dynamics of the forward price is*

$$\frac{dF_i(t,T)}{F_i(t,T)} = c_i\sigma\, dB(t) + \sigma_i \mathbf{b}_i' \mathrm{e}^{A_i(T-t)}\mathbf{e}_{p_i}\, dW_i(t)\,,$$

for $t \leq T$ *and* $i = 1, 2$

Proof. This follows immediately from an application of Itô's Formula on $F_i(t, T)$ as given in Prop. 3.1. □

The forward price dynamics is therefore a two-factor geometric Brownian motion, with a time-dependent volatility. As is evident from the dynamics, we find

$$\frac{dF_i(t,T)}{F_i(t,T)} \approx c_i\sigma\, dB(t)\,, \tag{3.5}$$

for $T - t \to \infty$. Thus, not unexpectedly, $c_i\sigma$ is the long-term (constant) volatility of the forward dynamics, that is, the volatility for contracts which are far from maturity. Moreover, the long-term contracts do not depend on the second factor W_i, and both forwards will have a dynamics given by the same geometric Brownian motion (modulo the constant $c_i, i = 1, 2$). This is simply re-stating Cor. 3.2 on what we called asymptotic cointegration.

We derive the term structure of volatility for $dF_i(t, T)/F_i(t, T)$ and the term structure of covariance between $dF_1(t, T)/F_1(t, T)$ and $dF_2(t, T)/F_2(t, T)$ in the next Proposition:

Proposition 3.5. *For* $t \leq T$ *and* $i = 1, 2$, *it holds that*

$$\mathit{Var}\left(\frac{dF_i(t,T)}{F_i(t,T)}\right) = \left\{c_i^2\sigma^2 + 2\rho_i c_i\sigma\sigma_i \mathbf{b}_i'\mathrm{e}^{A_i(T-t)}\mathbf{e}_{p_i} + \sigma_i^2(\mathbf{b}_i'\mathrm{e}^{A_i(T-t)}\mathbf{e}_{p_i})^2\right\}\, dt\,.$$

Moreover,

$$\begin{aligned}\mathit{Cov}\left(\frac{dF_1(t,T)}{F_1(t,T)}, \frac{dF_2(t,T)}{F_2(t,T)}\right) = \Big\{&c\sigma^2 + c\sigma\rho_1\sigma_1(\mathbf{b}_1'\mathrm{e}^{A_1(T-t)}\mathbf{e}_{p_1}) + \sigma\rho_2\sigma_2(\mathbf{b}_2'\mathrm{e}^{A_2(T-t)}\mathbf{e}_{p_2}) \\ &+ \rho\sigma_1\sigma_2(\mathbf{b}_1'\mathrm{e}^{A_1(T-t)}\mathbf{e}_{p_1})(\mathbf{b}_2'\mathrm{e}^{A_2(T-t)}\mathbf{e}_{p_2})\Big\}\, dt\end{aligned}$$

Proof. To compute the covariance, we find

$$\begin{aligned}&\mathrm{Cov}\left(\frac{dF_1(t,T)}{F_1(t,T)}, \frac{dF_2(t,T)}{F_2(t,T)}\right) \\ &\quad= \mathbb{E}_Q\left[\left(\sigma\, dB(t) + \sigma_1(\mathbf{b}_1'\mathrm{e}^{A_1(T-t)}\mathbf{e}_{p_1})\, dW_1(t)\right)\left(c\sigma\, dB(t) + \sigma_2(\mathbf{b}_2'\mathrm{e}^{A_2(T-t)}\mathbf{e}_{p_2})\, dW_2(t)\right)\right]\,.\end{aligned}$$

The result follows by direct computation recalling the correlation structure between the Brownian motions B, W_1 and W_2. The derivation of the variance is analogous. □

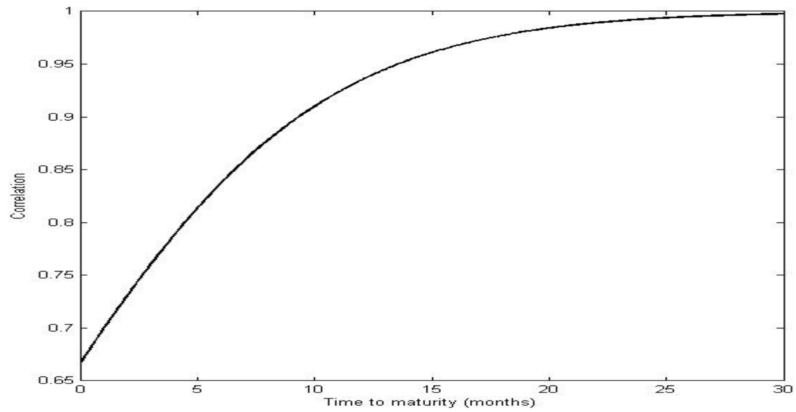

FIGURE 1. The correlation term structure for the case of $Y_i, i = 1, 2$ being Ornstein-Uhlenbeck processes.

We see that the volatility term structure (being the square root of the variance in the Proposition above) incorporates a Samuelson effect, as the volatility tends to a constant $c_i\sigma$ for $T-t$ large, and has a mixture of exponentially behaving functions for general times to maturity $T-t$. Moreover, the forward volatility converges to the volatility of the logarithmic spot price dynamics as $T-t$ converges to zero. However, the shape of the volatility term structure is not simply an exponential one as in the classical Samuelson case (see Samuelson [36]). We refer to Benth, Šaltytė Benth and Koekebakker [5] for more on the modification of the Samuelson effect in CARMA-models.

More interesting in our context is the term structure of correlation between the two forward contracts. It is straightforward to see, after recalling that the eigenvalues of A_i have negative real parts by assumption, that the correlation between $dF_1(t,T)/F_1(t,T)$ and $dF_1(t,T)/F_2(t,T)$ is one when $T-t\to\infty$. This is as expected taking into account the discussion above on the price dynamics in the far end of the forward curve. Thus, forward prices are perfectly correlated for $T-t$ large. However, for general $T-t$ we may achieve many different term structures depending on the choice of CARMA models.

Let us consider a numerical example: suppose that the Brownian motions W_1 and W_2 driving the short term variations are independent, and that the correlation between B and W_i is equal to $\rho_i = 0.5$ for $i = 1, 2$. Moreover, we suppose the stationary dynamics is driven by Ornstein-Uhlenbeck processes (CAR(1), that is), with volatilities being $\sigma_i = 0.015$, and speed of mean-reversion $\alpha_i = 0.1$, for $i = 1, 2$. In Fig. 1 we have plotted the correlation term structure computed from the expressions in Prop. 3.5 as a function of time to maturity $T-t$. As we observe, the correlation term structure is monotonely increasing to one.

For comparison, we have estimated the empirical correlation term structure for three refined oil products traded at NYMEX. We had accessible times series of daily prices for the first 12-15 positions of forward contracts on crude oil, heating oil and gasoline, ranging from October 4 2005 until January 31, 2012. In Fig. 2 we see the estimated correlation term structure for crude oil and heating oil (black line), heating oil and gasoline (broken line) and crude oil and gasoline (dashed line) computed from the time series of logarithmic returns. There is a similar monotone increase in terms of time to maturity, here measured in terms of positions. However, they do not seem to converge to one as in the theoretical case of Ornstein-Uhlenbeck processes.

In Fig. 3 we have plotted the correlation term structure in the case of $Y_i(t)$ being CAR(3) processes, $i = 1, 2$. The volatilites and correlations are the same as in the Ornstein-Uhlenbeck case above, but the matrices A_1 and A_2 are chosen to have the last rows being the vectors

$$(-\alpha_{3,1}, -\alpha_{2,1}, -\alpha_{1,1}) = (-0.187, -1.311, -2.034)$$
$$(-\alpha_{3,2}, -\alpha_{2,2}, -\alpha_{1,2}) = (-0.177, -1.399, -2.043)\,,$$

respectively. These numbers are taken from an empirical estimation of daily temperature data collected in Vilnius (Lithuania) and Stockholm (Sweden), respectively, in order to have reasonable values (see Benth

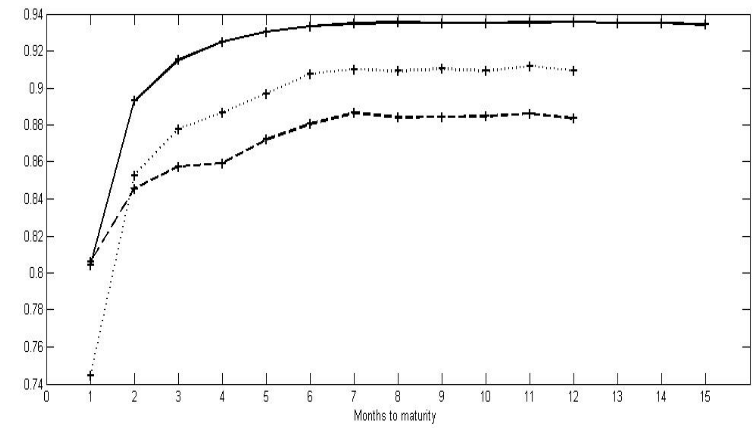

FIGURE 2. The correlation term structure for three refined oil products traded at NYMEX: Crude oil and heating oil (complete line), heating oil and gasoline (broken line), and crude oil and heating oil (dashed line)

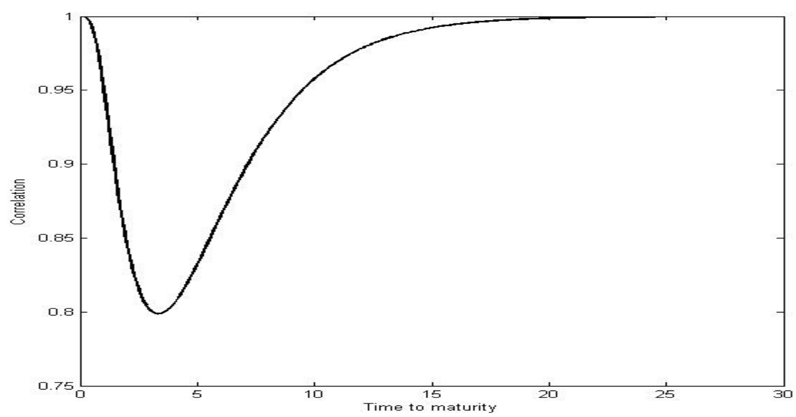

FIGURE 3. The correlation term structure for the case of $Y_i, i = 1, 2$ being CAR(3) processes.

and Šaltytė Benth [4] for more details on the modelling and estimation of temperatures). The correlation term structure will have a much more complex shape than the one obtained for the Ornstein-Uhlenbeck model. It starts at one, decays to a minimum value before it monotonely increases towards one. It would be interesting to see whether such shapes can be detected in any commodity data.

As a final example we consider two CARMA(2,1) processes for $Y_i, i = 1, 2$. Recall that Garcia, Klüppelberg and Müller [23], Benth et al. [7] and Paschke and Prokopczuk [35] showed empirically that power and oil prices follow a CARMA(2,1) dynamics. In Fig. 4 we show the autocorrelation function for parameters in Y_i being $b_1 = [10.29]$, $b_2 = [10.69]$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.091 & -1.49 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -0.23 & -2.33 \end{bmatrix}.$$

These parameters are taken from the estimates of EEX power base and peak load spot prices, resp., in Benth et al. [7]. From the plot, we see that the correlation term structure is similar to the the CAR(3) case, but differs in that the correlation starts at a value lower than 1, and is convex rather than concave for small times to maturity.

As we see in the dynamics of the forward price and the examples above, cointegration in the spot dynamics manifests itself as a non-constant term structure of volatility and correlation. The drift in the

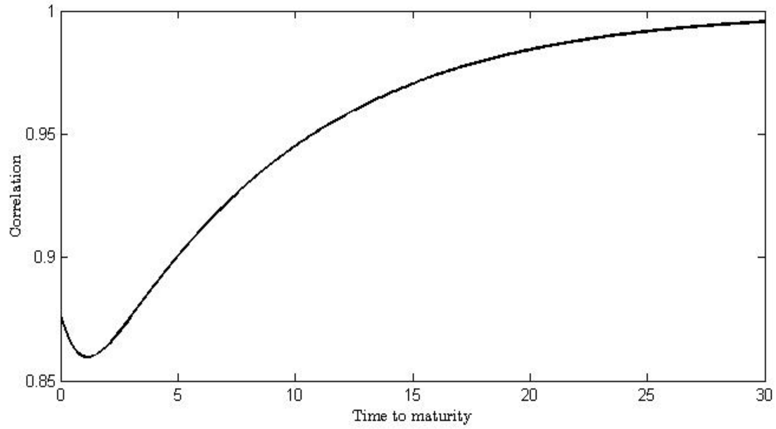

FIGURE 4. The correlation term structure for the case of $Y_i, i = 1, 2$ being CARMA(2,1) processes.

spot depending on the stationary components is inherited in the volatility of the forward, contrary to the complete market case where the Q-dynamics of the spots will be geometric Brownian motions (recall (2.14)) with constant drift being the risk-free interest rate. In the next Section we shall see that this may have dramatic effects when pricing spread options where one applies a pricing measure that preserves the cointegration in the spot.

4. PRICING OF SPREAD OPTIONS

In this Section we consider options on the spread between to the forward prices F_1 and F_2. To this end, consider a call option on the spread $F_1(\tau, T_1) - hF_2(\tau, T_2)$ between the two forwards, where the exercise time of the option is $0 \leq \tau \leq \min(T_1, T_2)$ at a strike zero. Here $h > 0$ is a positive constant signifying some conversion factor between the two commodities (in case they are not denoted in the same currency and/or in the same measurement scale). For example, h could be the heat rate in case we look at the spread between power and gas, say. Note that the maturity times of the forwards is T_1 and T_2, resp., which could possibly be different. The payoff of this option at time τ is

$$\max\left(F_1(\tau, T_1) - hF_2(\tau, T_2), 0\right) . \tag{4.1}$$

In the special case $\tau = T_1 = T_2$, we have a call option written on the spread between the two spots with exercise time τ, that is, an option with payoff

$$\max\left(S_1(\tau) - hS_2(\tau), 0\right) \tag{4.2}$$

This follows from the fact that $F_i(\tau, \tau) = S_i(\tau)$.

Our goal now is to derive the arbitrage-free price of spread options with payoff function (4.1). Recall Prop. 3.4, which tells us that we are in the framework of exponential Brownian-based models, and therefore the spread option price will become a variation of the Margrabe formula (see Margrabe [29]).

It is simple to see from Prop. 3.4 that the forward price $F_i(\tau, T)$ for $F_i(t, T)$ given, with $t \leq \tau \leq T$, is

$$\begin{aligned} F_i(\tau, T) = F_i(t, T) \exp\Bigg(& c_i\sigma(B(\tau) - B(t)) + \sigma_i \int_t^\tau \mathbf{b}_i' \mathrm{e}^{A_i(T-s)} \mathbf{e}_{p_i} \, dW_i(s) \\ & -\frac{1}{2} c_i^2 \sigma^2 (\tau - t) - \rho_i c_i \sigma \sigma_i \int_t^\tau (\mathbf{b}_i' \mathrm{e}^{A_i(T-s)} \mathbf{e}_{p_i}) \, ds - \frac{1}{2}\sigma_i^2 \int_t^\tau (\mathbf{b}_i' \mathrm{e}^{A_i(T-s)} \mathbf{e}_{p_i})^2 \, ds \Bigg) \end{aligned} \tag{4.3}$$

The Margrabe formula for our situation is stated in the next Proposition, with proof reported in the Appendix.

Proposition 4.1. *The price $P(t)$ at time $t \geq 0$ of the spread option with payoff given by* (4.1) *at exercise time $\tau \geq t$ on two forwards with maturities T_1 and T_2, $\tau \leq \min(T_1, T_2)$, is given by*

$$P(t) = \mathrm{e}^{-r(\tau - t)} \left\{ F_1(t, T_1)\Phi(d_1) - hF_2(t, T_2)\Phi(d_2) \right\}$$

where Φ is the cumulative standard normal distribution function, $d_1 = d_2 + \Sigma(t,\tau,T_1,T_2)$,

$$d_2 = \frac{\ln F_1(t,T_1) - \ln F_2(t,T_2) - \ln h - \frac{1}{2}\Sigma^2(t,\tau,T_1,T_2)}{\Sigma(t,\tau,T_1,T_2)}\,,$$

and

$$\begin{aligned}\Sigma^2(t,\tau,T_1,T_2) = \int_t^\tau &(1-c)^2\sigma^2 - 2(1-c)\sigma\sigma_2\rho_2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2}) \\ &+ 2(1-c)\sigma\sigma_1\rho_1(\mathbf{b}_1'\mathrm{e}^{A_1(T_1-s)}\mathbf{e}_{p_1}) - 2\rho\sigma_1\sigma_2(\mathbf{b}_1'\mathrm{e}^{A_1(T_1-s)}\mathbf{e}_{p_1})(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2}) \\ &+ \sigma_2^2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2})^2 + \sigma_1^2(\mathbf{b}_1'\mathrm{e}^{A_1(T_1-s)}\mathbf{e}_{p_1})^2\,ds\,.\end{aligned}$$

Proof. See Appendix. □

We remark that if $c = 1$, the "total volatility" $\Sigma(t,\tau,T_1,T_2)$ reduces to

$$\begin{aligned}\Sigma^2(t,\tau,T_1,T_2) = \int_t^\tau &\sigma_1^2(\mathbf{b}_1'\mathrm{e}^{A_1(T_1-s)}\mathbf{e}_{p_1})^2 - 2\rho\sigma_1\sigma_2(\mathbf{b}_1'\mathrm{e}^{A_1(T_1-s)}\mathbf{e}_{p_1})(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2}) \\ &+ \sigma_2^2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2})^2\,ds\end{aligned}$$

and the spread option price becomes independent of the non-stationary part of the dynamics, namely σ, ρ_1 and ρ_2.

For the case of a spread option on the spots S_i, $i = 1,2$, we find that the option price is expressible in terms of the forward prices $F_i(t,T)$, $i = 1,2$. Note that as the forward prices F_1 and F_2 can be expressed in terms of the factors X and $\mathbf{Z}_i, i = 1,2$, it is not straightforward to derive the spread option price as a function of the spot. However, see Benth and Solanilla Blanco [10] for an analysis of the connection between the spot and the state vector $\mathbf{Z}_i(t)$.

In the remainder of this section we want to analyse the spread option price in the case of cointegration in the spot dynamics under the pricing measure versus the complete market case, where there is no cointegration under the pricing measure. Recall that in the latter case, when the market is complete, we have the risk-neutral spot price dynamics given as in (2.14). It is a simple exercise to show that the forward price dynamics becomes

$$\frac{dF_i(t,T)}{F_i(t,T)} = c_i\sigma\,dB(t) + \sigma_i(\mathbf{b}_i'\mathbf{e}_{p_i})\,dW_i(t)\,.$$

The corresponding spread option price is given by the formula in Prop. 4.1 where the total volatility becomes $\Sigma^2(t,\tau,T_1,T_2) = \Sigma_c^2 \times (\tau - t)$ with

$$\begin{aligned}\Sigma_c^2 = &(1-c)^2\sigma^2 - 2(1-c)\sigma\sigma_2\rho_2(\mathbf{b}_2'\mathbf{e}_{p_2}) + 2(1-c)\sigma\sigma_1\rho_1(\mathbf{b}_1'\mathbf{e}_{p_1}) \\ &- 2\rho\sigma_1\sigma_2(\mathbf{b}_1'\mathbf{e}_{p_1})(\mathbf{b}_2'\mathbf{e}_{p_2}) + \sigma_2^2(\mathbf{b}_2'\mathbf{e}_{p_2})^2 + \sigma_1^2(\mathbf{b}_1'\mathbf{e}_{p_1})^2\,.\end{aligned}$$

We now do a numerical study illustrating the effect of cointegration on spread option prices.

Let us focus on the case of $p_1 = p_2 = 1$, meaning that $Y_i(t), i = 1,2$ are Ornstein-Uhlenbeck processes. Furthermore, we fix $c = 1$ to find

$$\Sigma_c^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2\,.$$

In Fig. 5 we have plotted a panel of four cases comparing the spread option prices in the complete market situation with the cointegrated ones. We have supposed $h = 1$ and fixed the exercise time to be $\tau = 10$ days. Furthermore, we have supposed that the speed of mean reversions as well as the volatilites are equal for $Y_i, i = 1,2$, being chosen as $\alpha_i = 0.05$ and $\sigma_i = 0.015$, resp. This implies an assumption of a half life of approximately 14 days, and an annual volatility of 24%. We separate our comparison into the case of backwardation and contango, and with either strong positive or negative correlation, $\rho = \pm 0.95$. The initial forward curves decays or increases by the rate α_i, having a long term fixed level of 100. In Fig. 5 we have plotted the spread option prices in the complete market situation (thin line) as a function of time to maturity $T_1 = T_2 = T$. The red (thick) line is the corresponding option prices in case cointegration is preserved under the pricing measure. In all four cases the cointegrated prices are less than the complete ones, significantly so for larger maturities. In the top row, we observe that in case of backwardation, the option prices are qualitatively behaving similarily, whereas in the contango case the spread option prices

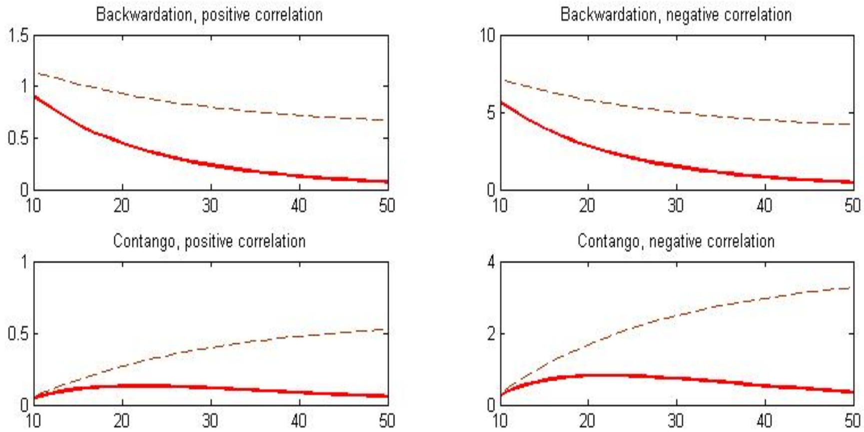

FIGURE 5. Spread option prices for the complete market situation (thin black line) and the cointegrated ones (red line) as a function of time to maturity of the underlying forwards.

in the cointegration model first increase monotonically as a function of time to maturity, before tailing off and converging to a constant. This is due to the stationarity, of course. The difference in prices in both the contango and backwardation cases between the complete market situation and the one with contango is rather dramatic, and shows the importance of cointegration on option prices. Failing to account for cointegration under the pricing measure, will lead to serious pricing errors of spread options.

5. HEATH-JARROW-MORTON MODELING

Motivated from the analysis in Sect. 3, we suggest a Heath-Jarrow-Morton (HJM) modeling framework with cointegration for forwards (see Heath, Jarrow and Morton [26] for the seminal paper on modeling forward rates in fixed income markets, an idea later used in commodity markets by Clewlow and Strickland [18] and Benth and Koekebakker [8]). Recalling our analysis above, the marginal forward price dynamics were geometric Brownian motion with time-dependent volatility, whereas using fixing time to maturity, the forward prices (under the Musiela parametrization) were cointegrated. We use this as starting point for our HJM model.

Suppose that the forward price dynamics under the market (objective) probability P is given by

$$\frac{dF_i(t,T)}{F_i(t,T)} = \alpha_i(t,T)\,dt + c_i\sigma(t)\,d\widetilde{B}(t) + g_i(T-t)\,d\widetilde{W}_i(t) \tag{5.1}$$

for $i = 1,2$, where $\widetilde{B}, \widetilde{W}_1$ and $\widetilde{W}_2$ are correlated Brownian motions as before and $c_1 = 1, c_2 = c$. Furthermore, we assume $t \mapsto \sigma(t)$ is a deterministic, square integrable function (on any subset of interest), and $t \mapsto g_i(T-t)$ for $t \leq T$ is also deterministic and square integrable, for $i = 1,2$. Finally, $t \mapsto \alpha_i(t,T)$ for $t \leq T$ is a square-integrable deterministic function for every $T > 0$, jointly measurable in (t,T), $i = 1,2$. An explicit solution of the dynamics in (5.1) is

$$\begin{aligned} F_i(t,T) = F_i(0,T)\exp\Bigg(c_i\int_0^t \sigma(s)\,d\widetilde{B}(s) + \int_0^t g_i(T-s)\,d\widetilde{W}_i(s) \\ + \int_0^t \alpha_i(s,T) - \frac{1}{2}\left\{c_i^2\sigma(s) + 2\rho_i c_i\sigma(s)g_i(T-s) + g_i^2(T-s)\right\}\,ds\Bigg)\,, \end{aligned} \tag{5.2}$$

for $i = 1,2$.

Obviously, we can add more Brownian motions in the dynamics (5.1) with corresponding volatility term structure parameters. However, we dispense with this generality here and focus on two-factor models. We also remark that we have assumed a time-dependent volatility in the "non-stationary" dB-term. We could have easily extended the non-stationary factor X in the spot-model in Sect. 2 to allow for this generality as well, but decided to keep matters there slightly simpler. A time-dependent volatility σ may incorporate possible seasonal fluctuations, like for example more variability in prices in the winter than in summer as one observes in the Nordic power market. Choosing

$$g_i(x) = \sigma_i \mathbf{b}_i' \exp(A_i x)\mathbf{e}_{p_i}\,, i = 1, 2\,,$$

we recover the forward model based on CARMA processes studied in Sect. 3, derived from the cointegrated spot model in Sect. 2. Here, the definitions of $\mathbf{b}_i$, A_i and $\mathbf{e}_{p_i}$ are as in Sect. 2 and 3.

Let us analyse cointegration in relation to the HJM forward model (5.1). As it turns out, it is convenient to introduce a slightly modified notion of cointegration, namely what we will call *cointegrated in the limit*. After changing to the Musiela parametrization with the notation $f_i(t,x) := F_i(t, t+x)$, $x = T - t$, we define:

Definition 5.1. *For $x \geq 0$, we say that $f_1(t,x)$ and $f_2(t,x)$ are cointegrated in the limit if $f_1(t,x) - \frac{1}{c}f_2(t,x)$ has a stationary distribution when $t \to \infty$.*

We observe that

$$\begin{aligned}\ln f_1(t,x) &- \frac{1}{c}\ln f_2(t,x) \\ &= \ln F_1(0,t+x) - \frac{1}{c}\ln F_2(0,t+x) \\ &\quad + \int_0^t g_1(t-s+x)\,d\widetilde{W}_1(s) - \frac{1}{c}\int_0^t g_2(t-s+x)\,d\widetilde{W}_2(s) \\ &\quad + \int_0^t \alpha_1(s,t+x) - \frac{1}{c}\alpha_2(s,t+x)\,ds \\ &\quad - \frac{1}{2}\int_0^t \sigma^2(s)(1-c) + 2\sigma(s)(\rho_1 g_1(t-s+x) - \rho_2 g_2(t-s+x)) \\ &\qquad + g_1^2(t-s+x) - \frac{1}{c}g_2^2(t-s+x)\,ds\,,\end{aligned} \tag{5.3}$$

where we have that $F_i(0,x) = f_i(0,x)$. From the analysis in Section 3 we recall that the forward prices were cointegrated when considered in the Musiela parametrization. It turns out that we need a structural (consistency) condition on the initial forward curve to obtain cointegration in the HJM setting. We find the following:

Proposition 5.2. *Define the function $H : \mathbb{R}_+^2 \to \mathbb{R}$ such that*

$$\begin{aligned}H(t,x) = \ln F_1(0,t+x) &- \frac{1}{c}\ln F_2(0,t+x) \\ &+ \int_0^t \alpha_1(s,t+x) - \frac{1}{c}\alpha_2(s,t+x)\,ds \\ &- \frac{1}{2}\int_0^t \sigma^2(s)(1-c) + 2\sigma(s)(\rho_1 g_1(t-s+x) - \rho_2 g_2(t-s+x)) \\ &\quad + g_1^2(t-s+x) - \frac{1}{c}g_2^2(t-s+x)\,ds\,.\end{aligned} \tag{5.4}$$

If $\lim_{t\to\infty} H(t,x)$ exists pointwise in x, then the forwards $f_1(t,x)$ and $f_2(t,x)$ for every fixed time-to-maturity $x \geq 0$ are cointegrated in the limit, that is

$$\ln f_1(t,x) - \frac{1}{c}\ln f_2(t,x) = H(t,x) + \int_0^t g_1(t-s+x)\,d\widetilde{W}_1(s) - \frac{1}{c}\int_0^t g_2(t-s+x)\,d\widetilde{W}_2(s)\,, \tag{5.5}$$

has a stationary limit as $t \to \infty$ for every $x \geq 0$.

Proof. With the condition (5.4), we find directly from (5.3) that,

$$\ln f_1(t,x) - \frac{1}{c}\ln f_2(t,x) = H(t,x) + \int_0^t g_1(t-s+x)\,d\widetilde{W}_1(s) - \frac{1}{c}\int_0^t g_2(t-s+x)\,d\widetilde{W}_2(s)\,.$$

Since by assumption the limit of $H(t,x)$ exists for every $x \geq 0$ as $t \to \infty$, we will have that the forwards with fixed time-to-maturity $x \geq 0$ are cointegrated in the limit as long as the stochastic integrals on the right hand side of the equation above converge to stationary Gaussian processes. From the Itô Isometry we find

$$\mathbb{E}_Q\left[\left(\int_0^t g_i(t-s+x)\,d\widetilde{W}_i(s)\right)^2\right]=\int_0^t g_i^2(s+x)\,ds\,.$$

Thus, we have a stationary process if and only if

$$g_i(x+\cdot)\in L^2(\mathbb{R}_+)\,,\ \text{for any } x\geq 0\,.$$

But this is satisfied by the standing condition $g_i\in L^2(\mathbb{R}_+)$. The Proposition follows. □

The limit condition on $H(t,x)$ defined in (5.4) imposes a structural relationship between the drifts α_i, the volatilities σ, g_i and the initial curves F_i. In Sect. 3, where we started with a cointegrated spot model, the resulting function H became time-independent (see Cor. 3.3). If we have $H(t,x)=\widehat{H}(x)$ for some function $\widehat{H}:\mathbb{R}_+\to\mathbb{R}$, then trivially the limit condition holds and we have cointegration in the limit for f_1 and f_2. If α_i, g_i and σ are given, then the case $H(t,x)=\widehat{H}(x)$ imposes a condition on the initial curves $F_i(0,T)$ in the sense that we must select these curves taking the volatility structure into account.

Note that the limit condition on the function $H(t,x)$ in (5.4) is more flexible than imposing marginally that

(5.6)
$$H_i(t,x)=\ln F_i(0,t+x)+\int_0^t \alpha_i(s,t+x)-\frac{1}{2}\left\{\sigma^2(s)+2\sigma(s)\rho_i g_i(t-s+x)+g_i^2(t-s+x)\right\}\,ds$$

has a limit when $t\to\infty$ for $i=1,2$, which would be a sufficient condition for the existence of a limit of $H(t,x)$ in (5.4) since $H(t,x)=H_1(t,x)-\frac{1}{c}H_2(t,x)$.

The implied spot price dynamics is achieved for $T=t$, yielding $S_i(t)=F_i(t,t)$. The spot prices $S_1(t)$ and $S_2(t)$ become cointegrated in the limit under the market probability P as a special case of Prop. 5.2 by setting $x=0$. Hence, as time t goes to infinifty, $S_1(t)-\frac{1}{c}S_2(t)$ has a limiting distribution. Furthermore, we note that the spot price is expressed in terms of $\int_0^t g_i(t-s)\,d\widetilde{W}_i(s)$, which is a so-called Brownian stationary process. Brownian stationary processes constitute a special case of the more general Lévy semistationary processes, which have been extensively studied in connection to power and energy spot markets in Barndorff-Nielsen, Benth and Veraart [1]. Apart from CARMA models, other choices of functions g_i relevant to energy prices were discussed and empirically analysed, including the Bjerksund model $g(x)=a/b+x$ for positive constants a and b. The motivation for this function comes from Bjerksund, Stensland and Rasmussen [11] who suggested this in an HJM forward model for power prices in the Nordic power market to incorporate an extreme form of the Samuelson effect. Barndorff-Nielsen, Benth and Veraart [2] treat ambit field models as forward price dynamics, having the dynamics analysed in this Section as special case. A multivariate framework based on ambit fields is analysed in Barndorff-Nielsen, Benth and Veraart [3], however, not from the viewpoint of cointegration as we do here.

The volatility and correlation term structures of the HJM model (5.1) are easily derived: one can show that

$$\text{Var}\left(\frac{dF_i(t,T)}{F_i(t,T)}\right)=\left\{c_i^2\sigma^2(t)+2c_i\rho_i\sigma(t)g_i(T-t)+g_i^2(T-t)\right\}\,dt\,,$$

for $i=1,2$, and

$$\begin{aligned}\text{Cov}\left(\frac{dF_1(t,T)}{F_1(t,T)},\frac{dF_2(t,T)}{F_2(t,T)}\right)=&\left\{c\sigma^2(t)+\sigma(t)(c\rho_1 g_1(T-t)+\rho_2 g_2(T-t))\right.\\&\left.+\rho g_1(T-t)g_2(T-t)\right\}\,dt\,.\end{aligned}$$

The term structures are very flexible as we can choose appropriate functions g_i rather freely.

In order to have an arbitrage-free model for the forward price $t\mapsto F_i(t,T)$ for $i=1,2$, there must exist a risk neutral probability Q such that the forward dynamics become (local) Q-martingales. The next Proposition states sufficient conditions for an arbitrage-free dynamics.

Proposition 5.3. *Suppose there exist functions $\theta, \theta_i : \mathbb{R}_+ \mapsto \mathbb{R}$, $i = 1, 2$, which are square-integrable on any compact subset of $\mathbb{R}_+$ and satisfying*

$$g_1(T-t)\theta_1(t) - \frac{1}{c} g_2(T-t)\theta_2(t) = \alpha_1(t,T) - \frac{1}{c}\alpha_2(t,T)$$

for $0 \leq t < T < \infty$. Then, $t \mapsto F_i(t,T)$, $t \leq T$ is a Q-martingale with dynamics

$$\frac{dF_i(t,T)}{F_i(t,T)} = c_i\sigma(t)\, dB(t) + g_i(T-t)\, dW_i(t)$$

for $i = 1, 2$, where B, W_1 anbd W_2 are Q-Brownian motions with the same correlation structure as $\widetilde{B}, \widetilde{W}_1$ and $\widetilde{W}_2$.

Proof. First, define

$$dB(t) = \theta(t)\, dt + d\widetilde{B}(t)\,,$$

and

$$dW_i(t) = \theta_i(t)\, dt + d\widetilde{W}_i(t)$$

for $i = 1, 2$. It follows from Girsanov's theorem the existence of a probability $Q \sim P$ such that (B, W_1, W_2) is a trivariate Brownian motion with the same correlation structure as $(\widetilde{B}, \widetilde{W}_1, \widetilde{W}_2)$. We find

$$\begin{aligned}\frac{dF_i(t,T)}{F_i(t,T)} &= \{\alpha_i(t,T) - c_i\sigma(t)\theta(t) - g_i(T-t)\theta_i(t)\}\, dt \\ &\quad + c_i\sigma(t)\, dB(t) + g_i(T-t)\, dW_i(t)\,.\end{aligned}$$

Hence, if θ, θ_i are such that

$$c_i\sigma(t)\theta(t) + g_i(T-t)\theta_i(t) = \alpha_i(t,T)$$

for $t < T$ and $i = 1, 2$, $t \mapsto F_i(t,T)$ become Q-martingales. But the assumption on θ_i implies that this holds true. Note that by Cauchy-Schwartz' inequality, $s \mapsto g_i(T-s)\theta_i(s) \in L^1([0,T])$ for any $T > 0$. □

As we need to have *one* risk-neutral probability for all maturities $T \geq 0$, we cannot allow for T-dependency in the so-called *market prices of risk* functions θ, θ_i, $i = 1, 2$. It is also noteworthy that we can choose the market price of risk for the non-stationary component θ freely. On the other hand, given θ and volatility functions σ and g_i, $i = 1, 2$, we can only allow for drift coefficients α_i which satisfies

$$\alpha_i(t,T) = g_i(T-t)\theta_i(t) - c_i\sigma(t)\theta(t)\,,$$

for $i = 1, 2$.

We recall from Sect. 3 that a cointegrated spot yielded a cointegrated forward dynamics under the Musiela parametrization under both probabilities P and Q. With the limit condition on the function $H(t,x)$ in Prop. 5.2, we recall that we have cointegration in the limit of $f_1(t,x)$ and $f_2(t,x)$ under the market probability P. For the Q-dynamics, we have from Prop. 5.3 (recalling that $c_1 = 1, c_2 = c$),

$$\begin{aligned}\ln f_1(t,x) - \frac{1}{c}\ln f_2(t,x) &= H(x) - \int_0^t \alpha_1(s,t+x) - \frac{1}{c}\alpha_2(s,t+x)\, ds \\ &\quad + \int_0^t g_1(t-s+x)\, dW_1(s) - \frac{1}{c}\int_0^t g_2(t-s+x)\, dW_2(s)\,.\end{aligned}$$

Hence, by the same argument as in Prop. 5.2, it follows that $f_1(t,x) - \frac{1}{c} f_2(t,x)$ has a stationary limiting distribution if the function

$$\widetilde{H}(t,x) = H(t,x) - \int_0^t \alpha_1(s,t+x) - \frac{1}{c}\alpha_2(s,t+x)\, ds$$

has a pointwise (in $x \geq 0$) limit when $t \to \infty$.

Consider next an example where the drift coefficients are *defined* as

$$\alpha_i(t,T) = \theta_i g_i(T-t) + c_i\theta\sigma(t)\,,$$

for constants θ, θ_i, $i = 1, 2$. Then the condition in Prop. 5.3 is satisfied, so that we have an arbitrage-free forward dynamics. Suppose that the function

$$\begin{aligned}\widetilde{H}(t,x) = \ln F_1(0,t+x) - \frac{1}{c}\ln F_2(0,t+x) \\ - \frac{1}{2}\int_0^t \sigma^2(s)(1-c) + 2\sigma(s)(\rho_1 g_1(t-s+x) - \rho_2 g_2(t-s+x)) \\ + g_1^2(t-s+x) - \frac{1}{c}g_2^2(t-s+x)\,ds\,,\end{aligned}$$

has a pointwise limit (in $x \geq 0$) when $t \to \infty$. Then $f_1(t,x)$ and $f_2(t,x)$ are cointegrated in the limit with respect to the risk neutral probability Q. Furthermore, to ensure cointegration in the limit with respect to the market probability P, we must have that

$$\int_0^t \alpha_1(s,t+x) - \frac{1}{c}\alpha_2(s,t+x)\,ds = \theta_1 \int_0^t g_1(s+x)\,ds - \frac{\theta_2}{c}\int_0^t g_2(s+x)\,ds\,,$$

has a pointwise limit (in $x \geq 0$) as $t \to \infty$. However, if we add the assumption that $g_i \in L^1(\mathbb{R}_+)$, then by monotone convergence,

$$\lim_{t\to\infty}\int_0^t g_i(s+x)\,ds = \int_0^\infty g_i(s+x)\,ds\,.$$

Therefore, we have a stationary limit for $\ln f_1(t,x) - c^{-1}\ln f_2(t,x)$ under P as well, implying that $f_1(t,x)$ and $f_2(t,x)$ are cointegrated in the limit both under P and Q. Note that if $\sigma(t) \equiv \sigma$, a constant, and $c = 1$, then the limit condition on $\widetilde{H}(t,x)$ can be reduced to assuming that $\ln F_1(0,t+x) - \ln F_2(0,t+x)$ is independent of x, as long as $g_i(\cdot + x) \in L^1(\mathbb{R}_+)$. This can be seen by the same monotone convergence argument as above.

We remark that the Margrabe formula in Prop. 4.1 is the same for our risk neutral HJM model with the modification of the total volatility given by

$$\begin{aligned}\Sigma^2(t,\tau,T_1,T_2) = \int_t^\tau (1-c)^2\sigma^2(s) - 2(1-c)\sigma(s)\rho_2 g_2(T_2-s) + 2(1-c)\sigma(s)\rho_1 g_1(T_1-s) \\ - 2\rho g_1(T_1-s)g_2(T_2-s) + g_2^2(T_2-s) + g_1^2(T_1-s)\,ds\,.\end{aligned}$$

Here, $\tau \leq \min(T_1,T_2)$.

6. Conclusions and outlook

We have argued that in many illiquid commodity markets, cointegration of the spot price dynamics will be inherited under the pricing measure. This is different from the complete market situation analyzed by Duan and Pliska [20], where the drift in the discounted dynamics of any tradeable asset will be zero (or, in more mathematical words, the discounted dynamics is a martingale). Hence, cointegration will be lost, and does not affect pricing of forwards and derivatives in the market. In commodity markets like power, shipping and weather, to mention some, this is not the case as the spot is highly illiquid (in fact, not a tradeable asset in the financial sense). We have proposed a class of measure changes where in fact cointegration of the spot is preserved. This parametric class of pricing measures encompasses the most used pricing measures in commodity markets, including the risk-neutral probability for the complete case.

It is shown that cointegration in the spot dynamics in these illiquid markets leads to cointegrated forward prices under the Musiela parametrization. This means that rolling forwards, that is, forwards with fixed time to maturity, are cointegrated. Forward prices with given time of maturity are not cointegrated, on the other hand. We derive rather flexible term structures of volatility and correlation under the hypothesis that the stationary dynamics of the spots are modelled as continuous-time autoregressive moving average processes. A simple empirical study reveals that refined oil products traded at NYMEX have similar correlation term structures as provided by our model based on simple mean reverting stationary dynamics.

Indeed, the Margrabe formula for spread options holds in our context, with a plug-in volatility taking the time-dependent correlation terms structure into account. As we demonstrate in several numerical examples, cointegration leads to a significant reduction in the spread option price compared to the complete market situation, where the price spread becomes non-stationary.

In this paper we have proposed a new cointegrated forward price model within the HJM framework. Motivated from our spot-based forward pricing, we specify a forward dynamics with time-dependent parameters. We state a structural condition on the initial forward curve and the parameters of the dynamics to ensure what we call *cointegration in the limit*, meaning an asymptotic concept of cointegration extending the classical one. It is worth pointing out that the cointegration property is under the Musiela parametrization. We discuss relevant cases, showing for example that we can recover the situation of the cointegrated spot.

In future studies we would like estimate our proposed HJM cointegrated forward price model to market data. It is likely that one must extend the models to more factors, in particular including non-Gaussian stationary factors driven by jump processes. This will be technically more demanding, however, will open up for a better and more flexible fit to the marginal forward curves. Thinking about gas and power, spot prices tend to spike, which results in leptokurtic behaviour in the forward price returns. This is prominent, at least in the short-end of the forward curve (see Benth *et al.* [7] for highly leptokurtic forward prices observed in the German power market EEX).

By introducing non-Gaussian stationary factors in the marginal forward dynamics, one also opens up for the possibility to have even more complex term structures of correlation and volatility. It is of interest to see how one could capture empirically observed term structures with cointegrated models. We leave these challenging problems for future research.

Appendix A. Some technical results and Proofs

Suppose that U_1, U_2 and U_3 are three independent Brownian motions, and define, for $\rho, \rho_1, \rho_2 \in (-1, 1)$

$$\begin{bmatrix} d\widetilde{B} \\ d\widetilde{W}_1 \\ d\widetilde{W}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \frac{\rho-\rho_1\rho_2}{\sqrt{1-\rho_1^2}} & \sqrt{1-\rho_2^2-\frac{(\rho-\rho_1\rho_2)^2}{1-\rho_1^2}} \end{bmatrix} \begin{bmatrix} dU_1 \\ dU_2 \\ dU_3 \end{bmatrix}. \tag{A.1}$$

In order to have the elements of the matrix well-defined, we need to impose the condition

$$\rho_2^2 + \frac{(\rho-\rho_1\rho_2)^2}{1-\rho_1^2} \leq 1\,,$$

or, equivalently,

$$\rho^2 + \rho_1^2 + \rho_2^2 - 2\rho\rho_1\rho_2 \leq 1\,.$$

We observe that $\widetilde{B}(t), \widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ all are Gaussian processes with mean zero and variance equal to t. Furthermore, as linear combinations of Brownian motions, they will have stationary and independent increments and thus being Brownian motions. A direct calculation shows that the correlation between $\widetilde{B}$ and $\widetilde{W}_i$ is ρ_i, $i = 1, 2$, and the correlation between $\widetilde{W}_1$ and $\widetilde{W}_2$ is ρ. Variances are equal to 1.

Proof of Prop. 3.1. By an application of the multi-dimensional Itô Formula, we find

$$\mathbf{Z}_i(T) = \exp(A_i(T-t))\mathbf{Z}_i(t) + \xi_i A_i^{-1}(\exp(A_i(T-t)) - I_i)\mathbf{e}_{p_i} + \sigma_i \int_t^T \exp(A_i(T-s))\mathbf{e}_{p_i}\, dW_i(s)\,,$$

where I_i is the $p_i \times p_i$ identity matrix. Trivially, we have

$$X(T) = X(t) + \zeta(T-t) + \sigma(B(T) - B(t))\,.$$

Hence, by appealing to $\mathcal{F}_t$-measurability of $X(t)$ and $\mathbf{Z}_i(t)$, we derive

$$\begin{aligned} F_i(t,T) &= \mathbb{E}_Q\left[\exp\left(c_i X(T) + Y_i(T)\right) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_Q\Big[\exp\Big(c_i X(t) + c_i\zeta(T-t) + c_i\sigma(B(T)-B(t)) \\ &\qquad + \mathbf{b}_i' \mathrm{e}^{A_i(T-t)}\mathbf{Z}_i(t) + \xi_i \mathbf{b}_i' A_i^{-1}(\mathrm{e}^{A_i(T-t)} - I_i)\mathbf{e}_{p_i} + \sigma_i \int_t^T \mathbf{b}_i' \mathrm{e}^{A_i(T-s)}\mathbf{e}_{p_i}\, dW_i(s)\Big) \mid \mathcal{F}_t\Big] \\ &= \exp\left(c_i X(t) + \mathbf{b}_i' \mathrm{e}^{A_i(T-t)}\mathbf{Z}_i(t) + c_i\zeta(T-t) + \xi_i \mathbf{b}_i' A_i^{-1}(\mathrm{e}^{A_i(T-t)} - I_i)\mathbf{e}_{p_i}\right) \\ &\qquad \times \mathbb{E}_Q\left[\exp\left(c_i\sigma(B(T)-B(t)) + \sigma_i \int_t^T \mathbf{b}_i' \mathrm{e}^{A_i(T-s)}\mathbf{e}_{p_i}\, dW_i(s)\right)\right] \end{aligned}$$

where we have used the independent increment property of Brownian motion in the last equality. The random variable in the exponent is normally distributed with zero mean and variance equal to

$$c_i^2\sigma^2(T-t)+2\rho_i\sigma\sigma_i\int_0^{T-t}\mathbf{b}_i'\mathrm{e}^{A_is}\mathbf{e}_{p_i}\,ds+\sigma_i^2\int_0^{T-t}(\mathbf{b}_i'\mathrm{e}^{A_is}\mathbf{e}_{p_i})^2\,ds\,.$$

Hence, the Proposition follows. □

Proof of Prop. 4.1. By $\mathcal{F}_t$-measurability, we find

$$\mathbb{E}_Q\left[(F_1(\tau,T_1)-hF_2(\tau,T_2))^+\mid\mathcal{F}_t\right]=F_2(t,T_2)\mathbb{E}_Q\left[\frac{F_2(\tau,T_2)}{F_2(t,T_2)}\left(\frac{F_1(\tau,T_1)}{F_2(\tau,T_2)}-h\right)^+\mid\mathcal{F}_t\right].$$

But the process $\tau\mapsto F_2(\tau,T_2)/F_2(t,T_2)$, $\tau\geq t$,

$$\begin{aligned}\frac{F_2(\tau,T_2)}{F_2(t,T_2)}=\exp\Bigg(&c\sigma(B(\tau)-B(t))+\sigma_2\int_t^\tau(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2})\,dW_2(s)\\&-\frac{1}{2}c^2\sigma^2(\tau-t)-\rho_2c\sigma\sigma_2\int_t^\tau(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2})\,ds\\&-\frac{1}{2}\sigma_2^2\int_t^\tau(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-s)}\mathbf{e}_{p_2})^2\,ds\Bigg)\end{aligned}$$

is a martingale. Thus, by the Girsanov Theorem, it is the density process of the Radon-Nikodym derivative for the probability $\widetilde{Q}\sim Q$. Furthermore, the measure change turns the processes

$$\begin{aligned}d\widetilde{B}(t)&=-(c\sigma+\rho_2\sigma_2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-t)}\mathbf{e}_{p_2}))\,dt+dB(t)\\d\widetilde{W}_1(t)&=-(c\rho_1\sigma+\rho\sigma_2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-t)}\mathbf{e}_{p_2}))\,dt+dW_1(t)\\d\widetilde{W}_2(t)&=-(c\rho_2\sigma+\sigma_2(\mathbf{b}_2'\mathrm{e}^{A_2(T_2-t)}\mathbf{e}_{p_2}))\,dt+dW_2(t)\,,\end{aligned}$$

into a three dimensional $\widetilde{Q}$-Brownian motion with the same correlation structure as (B,W_1,W_2). Next, substituting these Brownian motions into the expression for $F_1(\tau,T_1)/F_2(\tau,T_2)$, we find after some algebra

$$\left(\frac{F_1(\tau,T_1)}{F_2(\tau,T_2)}-h\right)^+=\left(\frac{F_1(t,T_1)}{F_2(t,T_2)}\exp\left(\Sigma(t,\tau,T_1,T_2)Z-\frac{1}{2}\Sigma^2(t,\tau,T_1,T_2)\right)-h\right)^+,$$

where Z is a standard normally distributed random variable. The equality above is in distribution. Hence, the Proposition follows after a straightforward computation referring to the Black-Scholes formula for a European call option. □

REFERENCES

[1] Barndorff-Nielsen, O. E., Benth, F. E., and Veraart, A. E. D. (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes, *Bernoulli*, **19**(3), pp. 803–845.

[2] Barndorff-Nielsen, O. E., Benth, F. E., and Veraart, A. E. D. (2014). Modelling electricity futures by ambit fields, *Adv. Applied Prob.*, **46**(3), pp. 719–745.

[3] Barndorff-Nielsen, O. E., Benth, F. E., and Veraart, A. E. D. (2014). Cross-commodity modelling by multivariate ambit fields fields multivariate paper. Forthcoming in *Commodities, Energy and Environmental Finance*, Fields Institute Communications Series, R. Aid, M. Ludkovski and R. Sircar (eds.), Springer Verlag.

[4] Benth, F.E., and Šaltytė Benth, J. (2013). *Modeling and Pricing in Financial Markets for Weather Derivatives*, World Scientific, Singapore.

[5] Benth, F.E., Šaltytė Benth, J., and Koekebakker, S. (2008). *Stochastic Modeling of Electricity and Related Markets,* World Scientific, Singapore.

[6] Benth, F. E., Cartea, A., and Pedraz C. G. (2014). *Manuscript in preparation*.

[7] Benth, F. E., Klüppelberg, C., Müller, G., and Vos, L. (2014). Futures pricing in electricity markets based on stable CARMA spot models. *Energy Econ.*, **44**, pp. 392–406.

[8] Benth, F. E., and Koekebakker, S. (2008). Stochastic modeling of financial electricity contracts. *Energy Econ.*, **30**(3), pp. 1116–1157.

[9] Benth, F. E., and Ortiz-Latorre, S. (2013). A pricing measure to explain the risk premium in power markets. *Submitted manuscript*. Avaliable on http://arxiv.org/abs/1308.3378

[10] Benth, F. E., and Solanilla Blanco, S. (2013). Forward prices as functionals of the spot path in commodity markets modeled by Lévy semistationary processes. Forthcoming in *Intern. J. Theor. Appl. Finance*.

[11] Bjerksund, P., Rasmussen, H., and Stensland, G. (2010). Valuation and risk management in the Nordic electricity market. In *Energy, Natural Resources and Environmental Economics*. ed. P. M. P. E. Bjrndal, M. Bjrndal and M. Rønnqvist. Springer Verlag, pp. 167–185.

[12] Brennan, M. J., and Schwartz, E. S. (1985). Evaluating natural resource investments. *J. Business*, **58**(2), pp. 135–157.

[13] Brockwell, P.J. (2001). Lévy-driven CARMA processes. *Ann. Inst. Statist. Math.*, **53**(1), pp. 113–124.

[14] Casassus, J., and Collin-Dufresne, P. (2005). Stochastic convenience yield implied from commodity futures and interest rates. *J. Finance*, **60**(5), pp. 2283–2331.

[15] Casassus, J., Liu, P., and Tang, K. (2013). Economic linkages, relative scarcity, and commodity futures returns. *Rev. Financial Stud.*, **26**(5), pp. 1324–1362.

[16] Clewlow, L., and Strickland, C. (1999). A multifactor model for energy derivatives. *Technical Report*, Quantitative Finance Research Group, University of Technology, Sydney.

[17] Clewlow, L., and Chris Strickland, C. (1999). Valuing energy options in a one factor model fitted to forward prices. *Technical Report*, Quantitative Finance Research Group, University of Technology, Sydney. Available on http://ssrn.com/abstract=160608.

[18] Clewlow, L., and Strickland, C. (2000). *Energy Derivatives: Pricing and Risk Management*, Lacima Publications, London.

[19] Cortazar, G., and Schwartz, E. S. (1994). The valuation of commodity-contingent claims. *J. Deriv.*, **1**(4), pp. 27–39.

[20] Duan, J.-C., and Pliska, S. R. (2004). Option valuation with cointegrated asset prices. *J. Econ. Dynam. & Control*, **28**, pp. 727–754.

[21] Duan, J.-C., and Theriault, A. (2007). Co-integration in crude oil components and the pricing of crack spread options. *Manuscript*.

[22] Fama, E. F. and French, K. R. (1988). Permanent and temporary components of stock prices. *J. Political Economy*, **96**(2), pp. 246–273.

[23] Garcia, I., Klüppelberg, C., Müller, G. (2011). Estimation of stable CARMA models with an application to electricity spot prices. *Statist. Modelling*, **11**(5), pp. 447–470.

[24] Gibson, R., and Schwartz, E. S. (1990). Stochastic convenience yield and the pricing of oil contingent claims. *J. Finance*, **45**(3), pp. 959–976.

[25] Härdle, W., and Lopez Cabrera, B. (2012). The implied market price of weather risk. *Appl. Math. Finance*, **19**(1), pp. 59–95.

[26] Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation. *Econometrica*, **60**, pp. 77–105.

[27] Kolos, S.P., and Ronn, E.I. (2008). Estimating the commodity market price of risk for energy prices. *Energy Economics*, **30**, pp. 621–641.

[28] Lucia, J., and Schwartz, E. S. (2002). Electricity prices and power derivatives: evidence from the Nordic Power Exchange. *Rev. Derivatives Res.*, **5**(1), pp. 5–50.

[29] Margrabe, W. (1978). The value of an option to exchange one asset for another. *J. Finance*, **33**, pp. 177–186.

[30] Merton, R. C (1973). An intertemporal capital asset pricing model. *Econometrica*, **41**(5), pp. 868–887.

[31] Miltersen, K. R., and Schwartz, E. S. (1998). Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. *J. Financial Quant. Anal.*, **33**(1), pp. 33–59.

[32] Nakajima, K., and Ohashi, K. (2012). A cointegrated commodity pricing model. *J. Futures Markets*, **32**(11), pp. 995–1033.

[33] Øksendal, B. (1998). *Stochastic Differential Equations. An Introduction with Applications,* Fifth Ed., Springer Verlag, Berlin-Heidelberg-New York.

[34] Paschke, R., and Prokopczuk, M. (2009). Integrating multiple commodities in a model of stochastic price dynamics. *J. Energy Markets*, **2**(3), pp. 47–82.

[35] Paschke, R., and Prokopczuk, M. (2010). Commodity derivatives valuation with autoregressive and moving average components in the price dynamics. *J. Banking Finance*, **34**(11), pp. 2741–2752 .

[36] Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *Indust. Manag. Rev.*, **6**, pp. 41–44.

[37] Schwartz, E. S., and Smith, J. E. (2000). Short-term variations and long-term dynamics in commodity prices. *Manag. Science*, **46**(7), pp. 893–911.

[38] Schwartz, E. S. (1997). The stochastic behavior of commodity prices: implications for valuation and hedging. *J. Finance*, **52**(3), pp. 923–973.

[39] Secomandi, N., and Seppi, D. J. (2014). Real options and merchant operations of energy and other commodities. *Foundations Trends Tech. Inform. Operations Manag.*, **6**, pp. 161–331.

[40] Stulz, R. (1982). Options on the minimum or the maximum of two risky assets: analysis and applications. *J. Financial Econ.*, **10**, pp. 161–185.

[41] Tyrtyshnikov, E.E. (1997). *A Brief Introduction to Numerical Analysis,* Birkhäuser, Boston.

FRED ESPEN BENTH, CENTRE FOR ADVANCED STUDY, DRAMMENSVEIEN 78, N-0271 OSLO, NORWAY, AND, CENTRE OF MATHEMATICS FOR APPLICATIONS, UNIVERSITY OF OSLO, P.O. BOX 1053, BLINDERN, N–0316 OSLO, NORWAY

E-mail address: fredb@math.uio.no

URL: http://folk.uio.no/fredb/

STEEN KOEKEBAKKER, SCHOOL OF BUSINESS AND LAW, UNIVERSITY OF AGDER, SERVICEBOKS 422, N-4604 KRISTIANSAND, NORWAY

E-mail address: steen.koekebakker@uia.no