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A maximum principle for optimal control of stochastic systems with delay, with applications to finance

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Dedicated to Professor Alain Bensoussan on the occasion of his 60th birthday

Abstract

We consider optimal control problems for systems described by stochastic differential equations with delay. We prove two (sufficient) maximum principles for certain classes of such systems, one for ordinary stochastic delay control and one which also includes singular stochastic delay control. As an application we find explicitly the optimal consumption rate from an economic quantity described by a stochastic delay equation of a certain type. We also solve a Merton type optimal portfolio problem in a market with delay.

1 Introduction

Suppose the state $X(t) = X^\xi(t)$ of a quantity (e.g. in physics, economics or biology) at time $t \geq 0$ is described by an Itô stochastic delay equation of the form

$$(1.1) \qquad \begin{cases} dX(t) = b(t, X(t), Y(t), Z(t), u(t))dt \\ \qquad\qquad +\sigma(t, X(t), Y(t), Z(t), u(t))dB(t) \; ; \quad t \geq 0 \\ X(s) = \xi(s) \; ; \quad -\delta \leq s \leq 0 \end{cases}$$

Here $B(t) = B(t, \omega)$; $t \geq 0$, $\omega \in \Omega$, is 1-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t P)$, $b : \mathbf{R}_+ \times \mathbf{R}^3 \times U \to \mathbf{R}$ and $\sigma : \mathbf{R}_+ \times \mathbf{R}^2 \times U \to \mathbf{R}$ are given continuously differentiable (i.e. C^1) functions, $u(t) = u(t, \omega)$ is an $\mathcal{F}_t$-adapted stochastic process (our *control process*) with values in given closed, convex set $U \subset \mathbf{R}^k$ and

$$(1.2) \qquad Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s) ds \; , \qquad Z(t) = X(t - \delta)$$

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represent given functionals of the path segment $X_t := \{X(t+s); s \in [-\delta, 0]\}$ of X, $\lambda \in \mathbf{R}$ is the given *averaging parameter* and $\delta > 0$ is the given *delay*. The continuous function $\xi : [-\delta, 0] \to \mathbf{R}$ is the *initial path* of X.

Suppose we are given a *performance functional* of the form

$$J(u) = E^\xi\Big[\int_0^T f(t, X(t), Y(t), Z(t), u(t))dt + g(X(T), Y(T))\Big] \tag{1.3}$$

where $f : \mathbf{R} \times \mathbf{R}^3 \times U \to \mathbf{R}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ are given lower bounded C^1 functions and $E^\xi = E$ denotes the expectation given that the initial path of X is $\xi \in C[-\delta, 0]$, the set of continuous functions from $[-\delta, 0]$ into $\mathbf{R}$. Let $\mathcal{A}$ be a given family of *admissible* adapted controls $u(t, \omega) : \mathbf{R}_+ \times \Omega \to U$ with the property that if $u \in \mathcal{A}$ then (1.1) has a unique, strong solution for all $\xi \in C[-\delta, 0]$. We consider the problem of finding $u^* \in \mathcal{A}$ such that

$$J(u^*) = \sup\{J(u); u \in \mathcal{A}\} . \tag{1.4}$$

Such $u^* \in \mathcal{A}$ (if it exists) is called an *optimal control*.

For each given u the system (1.1)–(1.2) is an example of a *stochastic differential equation with delay*. We refer the reader to [M1] and [M2] for the general theory of such systems. The problem (1.1)–(1.4) is an example of a *stochastic control problem for delay equations*. More information about such problems can be found in [KS].

In general stochastic control problems for delay equations are difficult to solve because they are infinite-dimensional. However, in certain cases the problem can be reduced to a finite-dimensional problem and solved explicitly. To the best of our knowledge the first example of such a solvable problem was a linear delay system with a quadratic cost functional [KM] (see also [KS]). Another example, which is more recent, is a singular control problem for delay solved in [EØS]. There sufficient variational inequalities are formulated and applied to solve a problem of optimal harvesting/divident for certain systems with delay. Recently a similar method has been applied to certain impulse control problems with delay [E].

In all the above papers the main idea is to reduce the problem to a finite-dimensional Markovian system and then apply suitable versions of the Hamilton-Jacobi-Bellman equation/inequalities. The purpose of this paper, however, is to present an alternative method, based on the *maximum principle* rather than dynamic programming. Such a maximum principle for delay systems is established in Theorem 2.2 for ordinary control and Theorem 2.3 for singular control. Then in Section 3 the method is illustrated on two stochastic delay control problems in finance.

We refer to [B], [CH], [H], [P] and [YZ] and the references therein for more information on the stochastic maximum principle for systems *without* delay.

2 A maximum principle for delay equations

As before we let $X_t \in C[-\delta, 0]$ be the segment of the path of X from $t - \delta$ to t, i.e.

$$X_t(s) = X(t+s) ; \qquad -\delta \le s \le 0 . \tag{2.1}$$

Define

$$G(t) = F(t, X(t), Y(t)) \tag{2.2}$$

where F is a given function in $C^{1,2,1}(\mathbf{R}^3)$ and

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds \tag{2.3}$$

as in (1.2). Then we have:

Lemma 2.1 (The Itô formula for delay)

$$dG(t) = LF\,dt + \sigma(t,x,y,z,u)\frac{\partial F}{\partial x}dB(t) + [x - \lambda y - e^{-\lambda\delta}z]\frac{\partial F}{\partial y}dt \tag{2.4}$$

where

$$LF = LF(s,x,y,z,u) = \frac{\partial F}{\partial s} + b(s,x,y,z,u)\frac{\partial F}{\partial x} + \tfrac{1}{2}\sigma^2(s,x,y,z,u)\frac{\partial^2 F}{\partial x^2}$$

and $LF(s,x,y,z,u)$ and the other functions appearing in (2.4) are evaluated at

$$s = t, \quad x = X(t), \quad y = Y(t), \quad z = Z(t) = X(t-\delta) \quad \text{and} \quad u = u(t)\ .$$

Proof. This is proved in [KM] and also in [EØS].

We now return to the stochastic control problem (1.1)–(1.4). Define the *Hamiltonian* $H : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times U \times \mathbf{R}^3 \times \mathbf{R}^2 \to \mathbf{R}$ for this problem by

$$\begin{aligned} H(t,x,y,z,u,p,q) &= f(t,x,y,z,u) + b(t,x,y,z,u)\cdot p_1 \\ &\quad + (x - \lambda y - e^{-\lambda\delta}z)\cdot p_2 + \sigma(t,x,y,z,u)q_1\ , \end{aligned} \tag{2.5}$$

where $p = (p_1, p_2, p_3)^T \in \mathbf{R}^3$ and $q = (q_1, q_2) \in \mathbf{R}^2$, $((\cdot)^T$ denotes matrix transposed).

For each $u \in \mathcal{A}$ the associated *adjoint equations* are the following backward stochastic differential equations in the unknown $\mathcal{F}_t$-adapted processes $p(t) = (p_1(t), p_2(t), p_3(t))^T$ and $q(t) = (q_1(t), q_2(t))$:

$$dp_1(t) = -\frac{\partial H}{\partial x}(t, X(t), Y(t), Z(t), u(t), p(t), q(t))dt + q_1(t)dB(t)\ , \quad t \in [0,T] \tag{2.6}$$

$$dp_2(t) = -\frac{\partial H}{\partial y}(t, X(t), Y(t), Z(t), u(t), p(t), q(t))dt + q_2(t)dB(t)\ ; \quad t \in [0,T] \tag{2.7}$$

$$dp_3(t) = -\frac{\partial H}{\partial z}(t, X(t), Y(t), Z(t), u(t), p(t), q(t))dt\ ; \quad t \in [0,T] \tag{2.8}$$

$$p_1(T) = \frac{\partial g}{\partial x}(X(T), Y(T)) \tag{2.9}$$

$$p_2(T) = \frac{\partial g}{\partial y}(X(T), Y(T)) \tag{2.10}$$

$$p_3(T) = 0 \tag{2.11}$$

Here $X(t), Y(t), Z(t)$ is the solution of (1.1)-(1.2) corresponding to u.

The first main result of this paper is the following:

Theorem 2.2 (A maximum principle for stochastic control of delay equations) *Suppose $\bar{u} \in \mathcal{A}$ and let $\bar{X}(t), \bar{Y}(t), \bar{Z}(t)$ and $p(t), q(t)$ be the corresponding solutions of (1.1)–(1.2) and (2.6)–(2.11), respectively. Suppose the following, (2.12)–(2.14), hold:*

(2.12) $$H(t, \cdot, \cdot, \cdot, \cdot, p(t), q(t)) \text{ and } g(\cdot, \cdot) \text{ are concave, for all } t \in [0, T]$$

(2.13) $$H(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t)) = \sup_{v \in U} H(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), v, p(t), q(t)) \quad \text{for all } t \in [0, T]$$

(2.14) $$p_3(t) = 0 \qquad \text{for all } t \in [0, T] .$$

Then $\bar{u}$ is an optimal control for problem (1.4).

Proof. Choose $u \in \mathcal{A}$ and let $X(t), Y(t), Z(t)$ be the corresponding solution of (1.1)–(1.2). To simplify the notation we put

$$\zeta(t) = (X(t), Y(t), Z(t)) \quad \text{and} \quad \bar{\zeta}(t) = (\bar{X}(t), \bar{Y}(t), \bar{Z}(t)) .$$

Let

$$D_1 = E\Big[\int_0^T \{ f(t, \bar{\zeta}(t), \bar{u}(t)) - f(t, \zeta(t), u(t)) \} dt \Big]$$

and

$$D_2 = E[h(\bar{X}(T), \bar{Y}(T)) - h(X(T), Y(T))] .$$

We want to prove that

(2.15) $$J(\bar{u}) - J(u) = D_1 + D_2 \geq 0 .$$

To this end note that by (2.5) we have

$$\begin{aligned} D_1 = {} & E\Big[\int_0^T \{ H(t, \bar{\zeta}(t), \bar{u}(t), p(t), q(t)) - H(t, \zeta(t), u(t), p(t), q(t)) \} dt \\ & - E\Big[\int_0^T \{ b(t, \bar{\zeta}(t), \bar{u}(t)) - b(t, \zeta(t), u(t)) \} p_1(t) dt \Big] \\ & - E\Big[\int_0^T \{ (\bar{X}(t) - \lambda \bar{Y}(t) - e^{-\lambda\delta} \bar{Z}(t)) - (X(t) - \lambda Y(t) - e^{-\lambda\delta} Z(t)) \} p_2(t) dt \Big] \\ & - E\Big[\int_0^T \{ \sigma(t, \bar{\zeta}(t), \bar{u}(t)) - \sigma(t, \zeta(t), u(t)) \} q_1(t) dt \Big] \end{aligned}$$

(2.16) $$=: \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 .$$

Since $(\zeta, u) \to H(\zeta, u) = H(t, \zeta, u, p, q)$ is concave, we have

$$\begin{aligned} H(\zeta, u) - H(\bar{\zeta}, \bar{u}) &\leq H_\zeta(\bar{\zeta}, \bar{u}) \cdot (\zeta - \bar{\zeta}) + H_u(\bar{\zeta}, \bar{u}) \cdot (u - \bar{u}) \\ &\leq H_\zeta(\bar{\zeta}, \bar{u}) \cdot (\zeta - \bar{\zeta}), \quad \text{by (2.13)}, \end{aligned}$$

where $H_\zeta = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right)$. Subsituting this in Δ_1 we get

$$\begin{aligned} \Delta_1 &\geq E\Big[\int_0^T -H_\zeta(t, \bar{\zeta}(t), \bar{u}(t), p(t), q(t)) \cdot (\zeta(t) - \bar{\zeta}(t))dt\Big] \\ &= E\Big[\int_0^T (\zeta(t) - \bar{\zeta}(t)) \cdot dp(t) - \int_0^T (X(t) - \bar{X}(t))q_1(t)dB(t)\Big] \\ &= E\Big[\int_0^T (X(t) - \bar{X}(t))dp_1(t) + \int_0^T (Y(t) - \bar{Y}(t))dp_2(t)\Big], \end{aligned} \tag{2.17}$$

by (2.4).

Next we consider D_2. Since g is concave, we have

$$\begin{aligned} D_2 &= E[g(\bar{X}(T), \bar{Y}(T)) - g(X(t), Y(T))] \\ &\geq -E\Big[\frac{\partial g}{\partial x}(\bar{X}(T), \bar{Y}(T))(X(T) - \bar{X}(T)) + \frac{\partial g}{\partial y}(\bar{X}(T), \bar{Y}(T))(Y(T) - \bar{Y}(T))\Big] \\ &= -E[(X(T) - \bar{X}(T))p_1(T) + (Y(T) - \bar{Y}(T)) \cdot p_2(T)] \\ &= -E\Big[\int_0^T (X(t) - \bar{X}(t))dp_1(t) + \int_0^T p_1(t)d(X(t) - \bar{X}(t)) \\ &\qquad + \int_0^T \{\sigma(t, \zeta(t), u(t)) - \sigma(t, \bar{\zeta}(t), \bar{u}(t))\}q_1(t)dt\Big] \\ &\qquad -E\Big[\int_0^T (Y(t) - \bar{Y}(t))dp_2(t) + \int_0^T p_2(t)d(Y(t) - \bar{Y}(t))\Big]. \end{aligned} \tag{2.18}$$

Combining this with (2.17) and (2.16) we get

$$D_2 \geq -\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4 = -D_1 .$$

Hence $J(\bar{u}) - J(u) = D_1 + D_2 \geq 0$.

Since $u \in \mathcal{A}$ was arbitrary this proves that $\bar{u}$ is optimal. □

A singular control version

It is possible to extend Theorem 2.2 to include *singular* control problems for stochastic systems with delay. We now formulate such a result, which may be regarded as a partial extension of Theorem 4.1 in [CH] to the delay case. We will not try to state the most general result, but settle with a special case which nevertheless is sufficient to cover some interesting applications.

Suppose the state $(X_0(t), X_1(t)) = (X_0(t), X(t)) \in \mathbf{R}^2$ is described by the following equations

$$\begin{aligned} dX_0(t) &= b_0(t,\zeta(t),u(t))dt + \sigma_{00}(t,\zeta(t),u(t))dB_0(t) + \sigma_{01}(t,\zeta(t),u(t))dB_1(t) \\ &\quad + a_{11}(t)d\mathcal{L}(t) + a_{12}(t)d\mathcal{M}(t)\ ; \qquad X_0(0^-) = x_0 \end{aligned} \tag{2.19}$$

$$\begin{aligned} dX(t) &= b_1(t,\zeta(t),u(t))dt + \sigma_{10}(t,\zeta(t),u(t))dB_0(t) + \sigma_{11}(t,\zeta(t),u(t))dB_1(t) \\ &\quad + a_{21}(t)d\mathcal{L}(t) + a_{22}(t)d\mathcal{M}(t)\ ; \qquad X(s) = \xi(s) \quad \text{for } s \in [-\delta, 0) \end{aligned} \tag{2.20}$$

where $B_0(t), B_1(t)$ are independent Brownian motions in $\mathbf{R}$,

$$\zeta(t) = (X_0(t), X(t), Y(t), Z(t)) \tag{2.21}$$

and, as before

$$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds\ , \qquad Z(t) = X(t-\delta)\ . \tag{2.22}$$

Here $b_i : \mathbf{R}_+ \times \mathbf{R}^4 \times U \to \mathbf{R}$, $\sigma_{ij} : \mathbf{R}_+ \times \mathbf{R}^4 \times U \to \mathbf{R}$ are given C^1 functions $(i, j = 0, 1)$ and $a_{ij}(t)$ are given continuous deterministic functions; $1 \le i, j \le 2$. The processes $\mathcal{L}(t)$ and $\mathcal{M}(t)$ are assumed to be predictable, right-continuous, non-decreasing processes with $\mathcal{L}(0^-) = \mathcal{M}(0^-) = 0$. As before we assume that $u(t) \in U$ is an adapted process. We let $\mathcal{A}$ denote the set of such controls $(u, \mathcal{L}, \mathcal{M})$ with the property that the corresponding system (2.19)–(2.20) has a unique strong solution $(X_0(t), X(t))$ for $0 \le t \le T$ and we call such controls *admissible.*

Thus in this model we assume that we have delay in $X_1(t) = X(t)$ only, not in $X_0(t)$.

Suppose we are given a *performance functional* of the form

$$\begin{aligned} J(u,\mathcal{L},\mathcal{M}) = E\Big[&\int_0^T f(t, X_0(t), X(t), Y(t), Z(t), u(t))dt \\ &+ g(X_0(T), X(T), Y(T)) + \int_0^T \theta_1(t)d\mathcal{L}(t) + \theta_2(t)d\mathcal{M}(t)\Big] \end{aligned} \tag{2.23}$$

with f, g lower bounded C^1-functions and $\theta_1(t), \theta_2(t)$ continuous deterministic functions. Then the problem is to find $(u^*, \mathcal{L}^*, \mathcal{M}^*) \in \mathcal{A}$ such that

$$J(u^*, \mathcal{L}^*, \mathcal{M}^*) = \sup\{J(u, \mathcal{L}, \mathcal{M});\ (u, \mathcal{L}, \mathcal{M}) \in \mathcal{A}\}\ . \tag{2.24}$$

For this problem we define the *Hamiltonian* $H : \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R} \times U \times \mathbf{R}^4 \times \mathbf{R}^{3\times 2} \to \mathbf{R}$ by

$$\begin{aligned} H(t, x_0, x, y, z, u, p, q) &= H(t, \zeta, u, p, q) = f(t, \zeta, u) \\ &+ b_0(t,\zeta,u) \cdot p_0 + b_1(t,\zeta,u) \cdot p_1 + (x - \lambda y - e^{-\lambda\delta} z) \cdot p_2 \\ &+ \sigma_{00}(t,\zeta,u)q_{00} + \sigma_{01}(t,\zeta,u)q_{01} + \sigma_{10}(t,\zeta,u)q_{10} + \sigma_{11}(t,\zeta,u)q_{11} \end{aligned} \tag{2.25}$$

where $p = (p_0, p_1, p_2, p_3)^T \in \mathbf{R}^4$, $q = (q_{ij})_{0 \le i \le 2, 0 \le j \le 1} \in \mathbf{R}^{3\times 2}$ and $\zeta = (x_0, x, y, z)$.

The corresponding *adjoint equations* are

$$\begin{cases} dp_0(t) = -\frac{\partial H}{\partial x_0}(t, \zeta(t), u(t), p(t), q(t))dt + q_{00}(t)dB_0(t) \\ \qquad\qquad +q_{01}(t)dB_1(t); \qquad t \in [0,T] \\ p_0(T) = \frac{\partial g}{\partial x_0}(X_0(T), X(T), Y(T)) \end{cases} \tag{2.26}$$

$$\begin{cases} dp_1(t) = -\frac{\partial H}{\partial x}(t, \zeta(t), u(t), p(t), q(t))dt + q_{10}(t)dB_0(t) \\ \qquad\qquad +q_{11}(t)dB_1(t); \qquad t \in [0,T] \\ p_1(T) = \frac{\partial g}{\partial x}(X_0(T), X(T), Y(T)) \end{cases} \tag{2.27}$$

$$\begin{cases} dp_2(t) = -\frac{\partial H}{\partial y}(t, \zeta(t), u(t), p(t), q(t))dt + q_{20}(t)dB_0(t) \\ \qquad\qquad +q_{21}(t)dB_1(t); \qquad t \in [0,T] \\ p_2(T) = \frac{\partial g}{\partial y}(X_0(T), X(T), Y(T)) \end{cases} \tag{2.28}$$

and

$$\begin{cases} dp_3(t) = -\frac{\partial H}{\partial z}(t, \zeta(t), u(t), p(t), q(t))dt\,; \qquad t \in [0,T] \\ p_3(T) = 0 \end{cases} \tag{2.29}$$

We can now state the second main result of this paper:

Theorem 2.3 (A maximum principle for singular control of stochastic systems with delay)
Suppose $(\bar{u}, \bar{\mathcal{L}}, \bar{\mathcal{M}}) \in \mathcal{A}$ and let $\bar{\zeta}(t) = (\bar{X}_0(t), \bar{X}(t), \bar{Y}(t), \bar{Z}(t))$ and $p(t), q(t)$ be the corresponding solutions of (2.19)–(2.20) and (2.26)–(2.29). Suppose the following, (2.30)–(2.33), hold:

$$(\zeta, u) \to H(t, \zeta, u, p(t), q(t)) \quad \textit{and } g(\cdot) \quad \textit{are concave functions, for all } t \in [0,T] \tag{2.30}$$

$$H(t, \bar{\zeta}(t,), \bar{u}(t), p(t), q(t)) = \sup_{v \in U} H(t, \bar{\zeta}(t), v, p(t), q(t)) \qquad \textit{for all } t \in [0,T] \tag{2.31}$$

$$E\Big[\int_0^T \{\theta_1(t) + a_{11}p_0(t) + a_{21}p_1(t)\} \cdot d(\mathcal{L} - \bar{\mathcal{L}})(t) + \int_0^T \{\theta_2(t) + a_{12}(t)p_0(t) + a_{22}(t)p_1(t)\} \cdot d(\mathcal{M} - \bar{\mathcal{M}})(t)\Big] \le 0 \quad \textit{for all } (u, \mathcal{L}, \mathcal{M}) \in \mathcal{A} \tag{2.32}$$

$$p_3(t) = 0 \qquad \textit{for all } t \in [0,T]\,. \tag{2.33}$$

Then $(\bar{u}, \bar{\mathcal{L}}, \bar{\mathcal{M}})$ is an optimal control for problem (2.23).

Proof. The proof is similar to the proof of Theorem 2.2. The main difference is that now we must also consider

$$D_3 := E\Big[\int_0^T \theta_1(t)d(\bar{\mathcal{L}} - \mathcal{L})(t) + \theta_2(t)d(\bar{\mathcal{M}} - \mathcal{M})(t)\Big]$$

and we must take into account the effect of $\mathcal{L}, \mathcal{M}$ and $\bar{\mathcal{L}}, \bar{\mathcal{M}}$ on $\zeta(T), \bar{\zeta}(T)$, respectively, when we compute

$$D_2 := E\big[g(\bar{X}_0(T), \bar{X}(T), \bar{Y}(T)) - g(X_0(T), X(T), Y(T))\big] \ .$$

Doing this we end up with the additional condition (2.32). We omit the details. □

3 Applications

We now give some examples to illustrate how Theorem 2.2 and Theorem 2.3 can be applied.

Example 1 (Optimal consumption)
Suppose that the size $X(t)$ of an economic quantity at time t is given by

$$\text{(3.1)} \qquad \begin{cases} dX(t) = [\mu X(t) + \alpha Y(t) + \beta Z(t) - u(t)]dt \\ \qquad\qquad +\sigma(t, X(t), Y(t), Z(t), u(t))dB(t) \ ; \quad t > 0 \\ X(s) = \xi(s) \ ; \qquad -\delta \le s \le 0 \end{cases}$$

Here $\sigma : \mathbf{R}^5 \to \mathbf{R}$ is a given C^1 function and, as before

$$\text{(3.2)} \qquad Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds \ , \qquad Z(t) = X(t-\delta) \ ,$$

$\theta, \alpha, \beta, \lambda$ and $\delta > 0$ are constants and $\xi \in C[-\delta, 0]$. In this model the mean growth rate of $X(t)$ is a linear combination of the present value plus some average of previous values. We may interprete the control $u(t) \ge 0$ as our consumption rate.

Suppose the performance associated to the consumption rate $u(t)$ is given by

$$\text{(3.3)} \qquad J(u) = E\Big[\int_0^T e^{-\rho t} \frac{u^\gamma(t)}{\gamma} dt + X(T) + \nu Y(T)\Big]$$

where $T > 0$, $\rho > 0$, $\gamma \in (0,1)$ and $\nu \in \mathbf{R}$ are constants ($1-\gamma$ is the relative risk aversion of the consumer). The problem is to find an $\mathcal{F}_t$-adapted $u^*(t, \omega)$ such that

$$\text{(3.4)} \qquad J(u^*) = \sup_u J(u) \ .$$

In this case the Hamiltonian (2.5) gets the form

$$H(t,x,y,z,u,p,q) = e^{-\rho t}\,\frac{u^\gamma}{\gamma} + \big[\mu x + \alpha y + \beta z - u\big]p_1 \qquad (3.5)$$
$$+[x - \lambda y - e^{-\lambda\delta}z]p_2 + \sigma(t,x,y,z,u)q_1 \; .$$

Hence the adjoint equations (2.6)–(2.11)are

$$\begin{cases} dp_1(t) = -\Big[\mu p_1(t) + p_2(t) + \frac{\partial\sigma}{\partial x}(t,X(t),Y(t),Z(t),u(t))q_1(t)\Big]dt + q_1(t)dB(t) \\ p_1(T) = 1 \end{cases} \qquad (3.6)$$

$$\begin{cases} dp_2(t) = -\Big[\alpha p_1(t) - \lambda p_2(t) + \frac{\partial\sigma}{\partial y}(t,X(t),Y(t),Z(t),u(t))q_1(t)\Big]dt + q_2(t)dB(t) \\ p_2(T) = \nu \end{cases} \qquad (3.7)$$

$$\begin{cases} dp_3(t) = -\Big[\beta p_1(t) - e^{-\lambda\delta}p_2(t) + \frac{\partial\sigma}{\partial z}(t,X(t),Y(t),Z(t),u(t))q_1(t)\Big]dt \\ p_3(T) = 0 \end{cases} \qquad (3.8)$$

Since $p(T)$ and the coefficients of $p_1(t), p_2(t)$ are deterministic we can choose $q_1(t) = q_2(t) = 0$. Therefore the condition (2.14) that $p_3(t) = 0$ can be formulated as follows:

$$\beta p_1(t) - e^{-\lambda\delta}p_2(t) = 0 \qquad \text{for all } \; t \in [0,T] \qquad (3.9)$$

where $p_1(t), p_2(t)$ are the solutions of

$$\begin{cases} dp_1(t) = -[\mu p_1(t) + p_2(t)]dt \\ p_1(T) = 1 \end{cases} \qquad (3.10)$$

$$\begin{cases} dp_2(t) = -[\alpha p_1(t) - \lambda p_2(t)]dt \\ p_2(T) = \nu \end{cases} \qquad (3.11)$$

Choosing $t = T$ in (3.9) we see that it is necessary that

$$\nu = \beta e^{\lambda\delta} \qquad \text{and} \quad \beta \neq 0 \; . \qquad (3.12)$$

Put

$$\widetilde{p}_2(t) = \beta^{-1}e^{-\lambda\delta}p_2(t) \; .$$

Then in terms of $p_1, \widetilde{p}_2$ (3.10)–(3.11) gets the form

$$\begin{cases} dp_1(t) = -[\mu p_1(t) + \beta e^{\lambda\delta}\widetilde{p}_2(t)]dt \\ p_1(T) = 1 \end{cases} \qquad (3.13)$$

$$\begin{cases} d\widetilde{p}_2(t) = -[\alpha\beta^{-1}e^{-\lambda\delta}p_1(t) - \lambda\widetilde{p}_2(t)]dt \\ \widetilde{p}_2(T) = 1 \end{cases} \qquad (3.14)$$

The solution of this system is

$$\begin{bmatrix} p_1(t) \\ \widetilde{p}_2(t) \end{bmatrix} = e^{A(T-t)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(T-t)^n}{n!} A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} \mu & \beta e^{\lambda\delta} \\ \alpha\beta^{-1} e^{-\lambda\delta} & -\lambda \end{bmatrix} .$$

From this we see that $p_1(t) = \widetilde{p}_2(t)$ for all t if and only if

$$\mu + \beta e^{\lambda\delta} = \alpha\beta^{-1} e^{-\lambda\delta} - \lambda$$

or

$$\alpha = \beta e^{\lambda\delta}(\mu + \lambda + \beta e^{\lambda\delta}) \; . \tag{3.15}$$

We conclude that $p_3(t) = 0$ for all $t \in [0, T]$ if and only if (3.12) and (3.15) hold.

Then we find $u^*(t)$ by maximizing

$$v \to H(t, X(t), Y(t), Z(t), v, p(t), 0)$$

over all $v \geq 0$. Since

$$\frac{\partial H}{\partial v}(t, X(t), Y(t), Z(t), v, p(t), 0) = e^{-\rho t} v^{\gamma - 1} - p_1(t)$$

we see by concavity that

$$u^*(t) = \left[e^{\rho t} p_1(t) \right]^{\frac{1}{\gamma - 1}} \tag{3.16}$$

where, by (3.9) and (3.10),

$$p_1(t) = \exp\{(\mu + \beta e^{\lambda\delta})(T - t)\} \; . \tag{3.17}$$

We summarize what we have proved as follows:

Theorem 3.1 *Suppose that (3.12) and (3.15) hold. Then the optimal consumption rate $u^*(t)$ for problem (3.1)–(3.4) is given by*

$$u^*(t) = \exp\left\{ \frac{1}{1-\gamma} \big((\mu + \beta e^{\lambda\delta} - \rho)t - (\mu + \beta e^{\lambda\delta})T \big) \right\} . \tag{3.18}$$

Example 2 (Optimal portfolio in a market with delay)
Consider a market with the following two investment possibilities:

a) a *safe* (risk free) investment (e.g. a bond or a bank account), with price dynamics given by

$$dx_0(t) = r\,x_0(t)dt\ ; \qquad x_0(0) = 1$$

where $r > 0$ is a constant

b) a *risky* investment (e.g. a stock) with price dynamics described by a stochastic delay equation of the form

$$dx_1(t) = \Big[\mu x_1(t) + \alpha \int_{-\delta}^{0} e^{\lambda s} x_1(t+s)ds + \beta x_1(t-\delta)\Big]dt$$

$$+\sigma\Big[x_1(t) + \nu \int_{-\delta}^{0} e^{\lambda s} x_1(t+s)ds\Big]dB(t)\ ; \qquad x_1(s) = b(s) \quad \text{for} \ \ s \in [-\delta, 0]\ ,$$

where $\mu, \alpha, \lambda, \beta, \delta > 0$, σ and ν are constants.

Now suppose that an agent is free to transfer money from the safe investment to the risky investment and conversely. Let $\mathcal{L}(t)$ be the total amount transferred *from the safe to the risky* investment up to time $t \geq 0$, and let $\mathcal{M}(t)$ be the total amount transferred *from the risky to the safe* investment up to time t. Then the amounts of money $X_0(t), X(t)$ held in the safe and the risky investment, respectively, at time t are given by

(3.19) $$dX_0(t) = rX_0(t)dt - d\mathcal{L}(t) + d\mathcal{M}(t)\ ; \qquad X_0(0^-) = x_0$$

(3.20) $$dX(t) = \big[\mu X(t) + \alpha Y(t) + \beta Z(t)\big]dt$$
$$+\sigma[X(t) + \nu Y(t)]dB(t) + d\mathcal{L}(t) - d\mathcal{M}(t)\ ; \quad X(s) = \xi(s)\ ;\ s \in [-\delta, 0)\ ,$$

where, as before,

(3.21) $$Y(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s)ds\ , \qquad Z(t) = X(t-\delta)\ .$$

As before we assume that $\mathcal{L}(t)$ and $\mathcal{M}(t)$ are predictable, right-continuous, non-decreasing processes and that $\mathcal{L}(0^-) = \mathcal{M}(0^-) = 0$. Such portfolios $(\mathcal{L}, \mathcal{M})$ are called *admissible.* We let $\mathcal{A}$ denote the set of all admissible portfolios.

We consider the problem of finding a portfolio $(\mathcal{L}^*, \mathcal{M}^*) \in \mathcal{A}$ such that

(3.22) $$J(\mathcal{L}^*, \mathcal{M}^*) = \sup\{J(\mathcal{L}, \mathcal{M})\ ;\ (\mathcal{L}, \mathcal{M}) \in \mathcal{A}\}$$

where, for given constants $\theta \in \mathbf{R}$, $T > 0$ and $\gamma \in (0,1)$,

$$J(\mathcal{L}, \mathcal{M}) = E\Big[\frac{1}{\gamma}(X_0(T) + X(T) + \theta Y(T))^{\gamma}\Big]\ .$$

The quantity J represents the expected utility of a linear combination of the terminal total amount of the accounts, $X_0(T) + X(T)$, and the average $Y(T)$ of previous $X(t)$-values.

This problem may be regarded as a delay generalization of the classical Merton problem of optimal portfolio in a Black-Scholes market [M]. In the no delay case ($\delta = \alpha = \beta = \nu = 0$) it was proved by Merton that it is optimal to choose $\mathcal{L}(t), \mathcal{M}(t)$ such that

$$\frac{X(t)}{X_0(t) + X(t)} = \frac{\mu - r}{\sigma^2(1-\gamma)} \qquad \text{for all } t \in [0, T] \; . \tag{3.23}$$

We will show that – under certain conditions – a similar portfolio is optimal also for the delay generalization. To achive this we apply Theorem 2.3.

In this case the Hamiltonian (2.25) gets the form

$$\begin{aligned} H(t, x_0, x, y, z, p, q) = r\, x_0 p_0 + [\mu x + \alpha y + \beta z] p_1 \\ + [x - \lambda y - e^{-\lambda\delta} z] p_2 + \sigma [x + \nu y] q_1 \; , \end{aligned} \tag{3.24}$$

where

$$p = (p_0, p_1, p_2, p_3)^T \in \mathbf{R}^4, \quad q = (q_0, q_1, q_2) \in \mathbf{R}^3, \qquad x, y, z \in \mathbf{R} \; , \; t \geq 0 \; .$$

Put

$$D = (X_0(T) + X(T) + \theta Y(T))^{\gamma - 1} \; . \tag{3.25}$$

The corresponding adjoint equations are

$$\begin{cases} dp_0(t) = -r\, p_0(t) dt + q_0(t) dB(t) \; ; \qquad t \in [0, T] \\ p_0(T) = D \end{cases} \tag{3.26}$$

$$\begin{cases} dp_1(t) = -[\mu p_1(t) + p_2(t) + \sigma q_1(t)] dt + q_1(t) dB(t) \; ; \qquad t \in [0, T] \\ p_1(T) = D \end{cases} \tag{3.27}$$

$$\begin{cases} dp_2(t) = -[\alpha p_1(t) - \lambda p_2(t) + \sigma \nu q_1(t)] dt + q_2(t) dB(t) \; ; \qquad t \in [0, T] \\ p_2(T) = \theta\, D \end{cases} \tag{3.28}$$

$$\begin{cases} dp_3(t) = -[\beta p_1(t) - e^{-\lambda\delta} p_2(t)] dt \; ; \qquad t \in [0, T] \\ p_3(T) = 0 \; . \end{cases} \tag{3.29}$$

As in Example 1 we get that

$$p_3(t) = 0 \qquad \text{for all } t \in [0, T] \tag{3.30}$$

if and only if

$$\beta\, p_1(t) = e^{-\lambda\delta} p_2(t) \; . \tag{3.31}$$

Motivated by the result obtained there we conjecture that if

$$\theta = \nu = \beta e^{\lambda\delta} \qquad \text{and} \qquad \alpha = \beta e^{\lambda\delta}(\lambda + \mu + \beta e^{\lambda\delta}) \tag{3.32}$$

then (3.31) holds.

To verify this we substitute (3.32) into (3.27) and (3.28) and get

$$\begin{cases} dp_1(t) = -[\mu p_1(t) + p_2(t) + \sigma q_1(t)]dt + q_1(t)dB(t) \ ; \quad t \in [0,T] \\ p_1(T) = D \end{cases} \tag{3.33}$$

$$\begin{cases} dp_2(t) = -[\beta e^{\lambda\delta}(\lambda + \mu + \beta e^{\lambda\delta})p_1(t) - \lambda p_2(t) + \sigma\beta e^{\lambda\delta} q_1(t)]dt \\ \qquad\qquad +q_2(t)dB(t) \ ; \quad t \in [0,T] \\ p_2(T) = \beta e^{\lambda\delta} D \end{cases} \tag{3.34}$$

As in Example 1 let us put

$$\widetilde{p}_2(t) = \beta^{-1} e^{-\lambda\delta} p_2(t) \ . \tag{3.35}$$

Then the equations for $(p_1(t), \widetilde{p}_2(t))$ become

$$\begin{cases} dp_1(t) = -[\mu p_1(t) + \beta e^{\lambda\delta}\widetilde{p}_2(t) + \sigma q_1(t)]dt + q_1(t)dB(t) \\ p_1(T) = D \end{cases} \tag{3.36}$$

$$\begin{cases} d\widetilde{p}_2(t) = -[(\lambda + \mu + \beta e^{\lambda\delta})p_1(t) - \lambda\widetilde{p}_2(t) + \sigma q_1(t)]dt + \beta^{-1}e^{-\lambda\delta}q_2(t)dB(t) \ , \\ \widetilde{p}_2(T) = D \ . \end{cases} \tag{3.37}$$

Define

$$y(t) = \widetilde{p}_2(t) - p_1(t) \ . \tag{3.38}$$

Subtracting (3.36) from (3.37) we get

$$\begin{cases} dy(t) = (\lambda + \beta e^{\lambda\delta})y(t)dt + (\beta^{-1}e^{-\lambda\delta}q_2(t) - q_1(t))dB(t) \\ y(T) = 0 \ . \end{cases} \tag{3.39}$$

The unique solution of this backward stochastic differential equation is

$$y(t) = 0 \ , \qquad \beta^{-1}e^{-\lambda\delta}q_2(t) - q_1(t) = 0 \ . \tag{3.40}$$

This proves that (3.32) implies (3.30), as claimed.

Next we consider the maximum principle condition (2.32): In our case the condition gets the form

$$E\Big[\int_0^T (p_0(t) - p_1(t))(d\mathcal{L}(t) - d\bar{\mathcal{L}}(t)) - (p_0(t) - p_1(t))(d\mathcal{M}(t) - d\bar{\mathcal{M}}(t))\Big] \geq 0 \tag{3.41}$$

for all $(\mathcal{L}, \mathcal{M}) \in \mathcal{A}$.

To have this satisfied it is clearly sufficient to find $(\bar{\mathcal{L}}, \bar{\mathcal{M}}) \in \mathcal{A}$ such that the corresponding $p_0(t), p_1(t)$ satisfy

$$p_0(t) = p_1(t) \qquad \text{for all } \; t \in [0,T] \; . \tag{3.42}$$

To this end, let us try the portfolio $(\bar{\mathcal{L}}, \bar{\mathcal{M}})$ which makes sure that

$$R(t) := \frac{V(t)}{W(t)} = \frac{\mu + \beta e^{\lambda\delta} - r}{\sigma^2(1-\gamma)} \qquad \text{for all } \; t \in [0,T] \; . \tag{3.43}$$

Here

$$V(t) := X(t) + \beta e^{\lambda\delta} Y(t) \tag{3.44}$$

is the *delay-included* wealth held in the risky investment at time t and

$$W(t) := X_0(t) + V(t) \tag{3.45}$$

is the *total* delay-included wealth held by the agent at time t.

This choice is motivated by the solution (3.23) for the no-delay case.

To verify (3.42) for this choice $(\bar{\mathcal{L}}, \bar{\mathcal{M}})$ we first prove that the solution $(p_0(t), q_0(t))$ of (3.26) is given by

$$p_0(t) = e^{\rho(T-t)} W(t)^{\gamma-1} \tag{3.46}$$

$$q_0(t) = (\gamma - 1)\sigma R \, p_0(t) \; , \tag{3.47}$$

where

$$\rho = r\gamma + \frac{(\mu + \beta e^{\lambda\delta} - r)^2 \gamma}{2\sigma^2(1-\gamma)} \; . \tag{3.48}$$

To this end, note that with $V(t)$ as in (3.43) we have, by (3.20) and (3.32),

$$dV(t) = (\mu + \beta e^{\lambda\delta}) V(t) dt + \sigma V(t) dB(t) + d\mathcal{L}(t) - d\mathcal{M}(t) \; . \tag{3.49}$$

Define

$$A(t) = e^{\rho(T-t)} (X_0(t) + V(t))^{\gamma-1} = e^{\rho(T-t)} W^{\gamma-1}(t) \; . \tag{3.50}$$

Then by the Itô formula

$$\begin{aligned} dA(t) = e^{\rho(T-t)} \big[&- \rho W^{\gamma-1}(t) + (\gamma-1) W^{\gamma-2}(t) (r X_0(t) + (\mu + \beta e^{\lambda\delta}) V(t)) \\ &+ \tfrac{1}{2}(\gamma-1)(\gamma-2) W^{\gamma-3}(t) \sigma^2 V^2(t) \big] dt + e^{\rho(T-t)} \big[(\gamma-1) W^{\gamma-2}(t) \sigma V(t) \big] dB(t) \; . \end{aligned}$$

Substituting

$$V(t) = R\,W(t) \; , \qquad X_0(t) = (1-R) W(t) \tag{3.51}$$

we get

$$dA(t) = \big[-\rho + (\gamma-1)\{r(1-R) + (\mu+\beta e^{\lambda\delta})R\} + \tfrac{1}{2}(\gamma-1)(\gamma-2)\sigma^2 R^2\big]A(t)dt + (\gamma-1)\sigma R\, A(t)dB(t)\ . \tag{3.52}$$

Now, by (3.48) and (3.45), with $\bar{\mu} = \mu + \beta e^{\lambda\delta}$,

$$\begin{aligned} -\rho + (\gamma-1)\{r(1-R) + (\mu+\beta e^{\lambda\delta})R\} + \tfrac{1}{2}(\gamma-1)(\gamma-2)\sigma^2 R^2 \\ = -r\gamma - \frac{(\bar{\mu}-r)^2\gamma}{2\sigma^2(1-\gamma)} + (\gamma-1)\Big\{r\Big(1-\frac{\bar{\mu}-r}{\sigma^2(1-\gamma)}\Big) + \frac{\bar{\mu}(\bar{\mu}-r)}{\sigma^2(1-\gamma)}\Big\} \\ +\tfrac{1}{2}(\gamma-1)(\gamma-2)\sigma^2\,\frac{(\bar{\mu}-r)^2}{\sigma^4(1-\gamma)^2} \\ = \frac{(\bar{\mu}-r)^2}{2\sigma^2(1-\gamma)}\big[-\gamma + 2(\gamma-1) - (\gamma-2)\big] - r\,\gamma + (\gamma-1)r = -r\ . \end{aligned} \tag{3.53}$$

Hence, by (3.52),

$$\begin{cases} dA(t) = -r\,A(t) + (\gamma-1)\sigma R\,A(t)dB(t) \\ A(T) = D\ . \end{cases} \tag{3.54}$$

This shows that $p_0(t) = A(t)$, $q_0(t) = (\gamma-1)\sigma R\,A(t)$ solve (3.26) as claimed.

It remains to verify that $(p,q) := (p_0,q_0)$ also solves the equation (3.33) for (p_1,q_1). In view of (3.31) this means that

$$dp_0(t) = -[\mu p_0(t) + \beta e^{\lambda\delta}p_0(t) + \sigma q_0(t)]dt + q_0(t)dB(t)\ . \tag{3.55}$$

For this it suffices to have

$$(\mu+\beta e^{\lambda\delta})p_0(t) + \sigma q_0(t) = r\,p_0(t)\ . \tag{3.56}$$

Since

$$q_0(t) = (\gamma-1)\sigma\,Rp_0(t) = -\frac{\mu+\beta e^{\lambda\delta}-r}{\sigma}p_0(t)\ ,$$

we see that (3.56) holds.

We have verified that if $(\bar{\mathcal{L}},\bar{\mathcal{M}})$ is chosen such that

$$V(t) = \frac{\mu+\beta e^{\lambda\delta}-r}{\sigma^2(1-\gamma)}W(t) \qquad \text{for all } t\in[0,T]$$

then $p_0(t) = p_1(t)$. That completes the proof of the following result:

Theorem 3.2 *Assume that (3.32) holds. Then an optimal portfolio $(\mathcal{L}^*,\mathcal{M}^*)$ for problem (3.22) is the portfolio $(\mathcal{L}^*,\mathcal{M}^*) = (\bar{\mathcal{L}},\bar{\mathcal{M}})$ defined by the property that the corresponding wealth processes*

$$V(t) = X(t) + \beta e^{\lambda\delta}Y(t) \qquad \textit{(delay-included wealth in stocks)}$$
$$W(t) = X_0(t) + V(t) \qquad \textit{(total delay-included wealth)}$$

satisfy

$$V(t) = \frac{\mu+\beta e^{\lambda\delta}-r}{\sigma^2(1-\gamma)}W(t) \qquad \textit{for all } t\in[0,T]\ .$$

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