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Black polarization sandwiches are square roots of zero

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Abstract
In the 2×2 matrices representing retarders and ideal polarizers, the eigenvectors are orthogonal. An example of the opposite case, where eigenvectors collapse onto one, is matrices $\mathbf{M}$ representing crystal plates sandwiched between a crossed polarizer and analyser. For these familiar combinations, $\mathbf{M}^2 = 0$, so black sandwiches can be regarded as square roots of zero. Black sandwiches illustrate physics associated with degeneracies of non-Hermitian matrices.

Keywords: polarization, degeneracy, nilpotence, matrix optics

Tudor [1, 2] has recently remarked that the common classification of polarizing devices into retarders, represented by unitary operators, and ideal polarizers, represented by projection operators, fails to include some commonly used combinations of optical elements. Both retarders and ideal polarizers are represented by 2×2 Jones matrices [3–5] whose eigenvectors are orthogonal; the difference lies in the eigenvalues: for retarders, two complex eigenvalues lie on the unit circle, and for ideal polarizers, one eigenvalue is zero. Tudor gives the example of a linear polarizer followed by a quarter-wave plate, where the eigenvectors are not orthogonal. He also gives an example of the extreme case of nonorthogonal eigenvectors, namely *parallel* eigenvectors: a linear polarizer sandwiched between two identical quarter-wave plates oriented at 45° to the axes of the polarizer; this example falls into a class described by de Lang [6].

Here we extend Tudor's remark in a very simple way, by pointing out that this extreme case of parallel eigenvectors is realized optically by a much wider class of devices including some of the most familiar combinations employed in polarization optics, namely any specimen (e.g. a crystal plate) between a crossed polarizer and analyser—which we call a *black sandwich*, for obvious reasons. The matrix for a black sandwich is

$$\mathbf{M} = \mathbf{P}_+ \mathbf{A} \mathbf{P}_-, \tag{1}$$

where

$$\mathbf{P}_\pm = |\pm\rangle\langle\pm| \tag{2}$$

are the projection matrices corresponding to the ideal polarizer ($-$) and analyser ($+$) that select orthogonal states represented by the column (e.g. Jones) vectors $|\pm\rangle$, and $\mathbf{A}$, representing the specimen, can be any 2×2 matrix.

Because of its unsymmetrical form (polarizer and analyser different), $\mathbf{M}$ is non-Hermitian, and explicit calculation shows that both of its eigenvalues are zero. Therefore the two eigenvectors, that are different for the general case, have here collapsed onto one, namely $|+\rangle$. A polarization $|\psi\rangle$ entering the black sandwich emerges in the state

$$\mathbf{M}|\psi\rangle = (\langle +|\mathbf{A}|-\rangle\langle -|\psi\rangle)|+\rangle. \tag{3}$$

In contrast to the ideal polarizer $\mathbf{P}_+$, from which the emerging light is also in the state $|+\rangle$ but which extinguishes incident light in the orthogonal state $|-\rangle$, the black sandwich extinguishes its own eigenstate. This is obvious when $\mathbf{M}$ is written in the form

$$M = (\langle +|A|-\rangle)|+\rangle\langle -|. \tag{4}$$

It follows immediately that

$$\mathbf{M}^2 = 0, \tag{5}$$

i.e. $\mathbf{M}$ is nilpotent, reflecting the obvious fact that the combination of two black sandwiches extinguishes all light. Therefore black sandwiches (1), incorporating general matrices $\mathbf{A}$, can be regarded as nontrivial square roots of zero. (The trivial square root $\mathbf{M} = 0$ corresponds to $\mathbf{A} = 1$—simply a crossed polarizer and analyser, i.e. a sandwich with no filling.)

A familiar black sandwich consists of a transparent crystal plate between a linear polarizer and analyser. If the polarizer and analyser are

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(polarizer)}, \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(analyser)}, \tag{6}$$

and the orthogonal eigenpolarizations and eigenvalues of the crystal are

$$|\phi_1\rangle = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \text{eigenvalue } \lambda_0 + \lambda,$$
$$|\phi_2\rangle = \begin{pmatrix} v^* \\ -u^* \end{pmatrix}, \qquad \text{eigenvalue } \lambda_0 - \lambda, \tag{7}$$

with $|u|^2 + |v|^2 = 1$, then $\mathbf{A}$ is the unitary matrix

$$\mathbf{A} = \exp(\mathrm{i}\lambda_0)[|\phi_1\rangle\langle\phi_1|\exp(\mathrm{i}\lambda) + |\phi_2\rangle\langle\phi_2|\exp(-\mathrm{i}\lambda)]$$
$$= \exp(\mathrm{i}\lambda_0)\left[\cos\lambda\mathbf{I} + \mathrm{i}\sin\lambda\begin{pmatrix} |u|^2 - |v|^2 & 2uv^* \\ 2u^*v & |v|^2 - |u|^2 \end{pmatrix}\right], \tag{8}$$

and the black sandwich matrix is

$$\mathbf{M} = \langle+|\mathbf{A}|-\rangle|+\rangle\langle-| = 2\mathrm{i}uv^*\exp(\mathrm{i}\lambda_0)\sin\lambda\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

For an anisotropic material, the polarization components u and v depend on direction, and any diffuse light (e.g. the sky, or white paper) viewed through the sandwich will exhibit the familiar conoscopic figures [7, 8]. An easy way to see the conoscopic figures displaying the polarization singularity at the optic axis of a biaxial material is with a 'crystal' consisting of a sheet of overhead-projector transparency film [9], so this 'black plastic sandwich' is a square root of zero. For conoscopic figures corresponding to more general polarization singularities, with $\mathbf{A}$ representing crystals that are gyrotropic and dichroic as well as birefringent, see [10].

The de Lang class [6] of nilpotent devices, mentioned at the end of the first paragraph, is more restricted, being of the form $\mathbf{M} = \mathbf{UPU}$, where $\mathbf{U}$ is a quarter-wave retarder whose eigenpolarizations are directions on the Poincaré sphere perpendicular to that selected by $\mathbf{P}$. This class forms a four-parameter family of devices (including overall phase), whereas the black sandwiches form a ten-parameter family (or, if $\mathbf{A}$ is restricted to the class of retarders, a six-parameter family).

The polarization optics of black sandwiches joins a growing class of physics associated with the collapse of two eigenvectors onto one at a degeneracy of eigenvalues of non-Hermitian matrices. Other examples occur in the diffraction of atoms by 'crystals of light' [11–13], in nuclear physics [14–17], and in the linewidths of unstable lasers [18]. Such degeneracies also occur in the optics of absorbing crystals [8, 10, 19–21], for light travelling along a 'singular axis' (to avoid confusion, we emphasize that in (1) these degeneracies occur in the crystal matrix $\mathbf{A}$, not the black sandwich matrix $\mathbf{M}$).

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