Inhomogeneous Dark Radiation Dynamics on a de Sitter Brane

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Abstract

Assuming spherical symmetry we analyse the dynamics of an inhomogeneous dark radiation vaccum on a Randall and Sundrum 3-brane world. Under certain natural conditions we show that the effective Einstein equations on the brane form a closed system. On a de Sitter brane and for negative dark energy density we determine exact dynamical and inhomogeneous solutions which depend on the brane cosmological constant, on the dark radiation tidal charge and on its initial configuration. We also identify the conditions leading to the formation of a singularity or of regular bounces inside the dark radiation vaccum.

1 Introduction

In the context of the intensive search for extra dimensions the Randall and Sundrum (RS) brane world scenario is specially compelling for its simplicity and depth [1]. In this scenario the observable Universe is a 3-brane boundary of a non-compact Z_2 symmetric 5-dimensional Anti-de Sitter (AdS) space. The matter fields are restricted to the brane but gravity exists in the whole AdS bulk and is bound to the brane by the warp of the infinite fifth dimension.

In recent years numerous studies have been conducted within the RS scenario [see [2] for a recent review and notation]. From an effective 4-dimensional point of view [3, 4, 5] the interactions existing between the brane and the bulk lead to a modification of the Einstein equations by a set of two distinct terms, namely a local high energy embedding term generated by the matter energy-momentum tensor and a non-local term induced by the bulk Weyl tensor. The resulting equations have a complex non-linear dynamics. For example the exterior vaccum of a collapsing distribution of

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matter on the brane is now filled with modes originated by the bulk Weyl curvature and can no longer be a static space [6, 7].

Previous investigations of the RS scenario have been focused on static or homogeneous dynamical solutions. In this proceedings we analyse some aspects of the inhomogeneous dynamics of a RS brane world vaccum [see [8] for more details].

2 Vaccum Einstein Equations on the Brane

In the 4-dimensional effective geometric approach to the RS brane world scenario first introduced by Shiromizu, Maeda and Sasaki [3, 4, 5], the induced Einstein vaccum field equations on the brane are given by

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{1}$$

where Λ is the brane cosmological constant and the tensor $\mathcal{E}_{\mu\nu}$ is the limit on the brane of the projected 5-dimensional Weyl tensor. On one hand the Weyl symmetries ensure it is a symmetric and traceless tensor. On the other the Bianchi identities constrain it to satisfy the conservation equations

$$\nabla_\mu \mathcal{E}^\mu_\nu = 0. \tag{2}$$

The tensor $\mathcal{E}_{\mu\nu}$ can be written in the following general form [5]

$$\mathcal{E}_{\mu\nu} = -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \left[\mathcal{U}\left(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}\right) + \mathcal{P}_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu\right], \tag{3}$$

where u_μ such that $u^\mu u_\mu = -1$ is the 4-velocity field and $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the tensor which projects orthogonaly to u_μ. The forms $\mathcal{U}$, $\mathcal{P}_{\mu\nu}$ and $\mathcal{Q}_\mu$ represent different characteristics of the effects induced on the brane by the free gravitational field in the bulk. Thus, $\mathcal{U}$ is interpreted as an energy density, $\mathcal{P}_{\mu\nu}$ as stress and $\mathcal{Q}_\mu$ as energy flux.

Since the 5-dimensional metric is unknown, in general $\mathcal{E}_{\mu\nu}$ is not fully determined on the brane [3, 4]. As a consequence the effective 4-dimensional theory is not closed and to close it we need simplifying assumptions about the bulk degrees of freedom. For example we may consider a static and spherically symmetric brane vaccum with $\mathcal{Q}_\mu = 0$, $\mathcal{P}_{\mu\nu} \neq 0$ and $\mathcal{U} \neq 0$ to find the Reissner-Nordström black hole solution on the brane [9]. Let us now show that it is possible to take a non-static spherically symmetric brane vaccum with $\mathcal{Q}_\mu = 0$, $\mathcal{U} \neq 0$, $\mathcal{P}_{\mu\nu} \neq 0$ and still close the system of dynamical equations.

Consider the general, spherically symmetric metric in comoving coordinates (t, r, θ, ϕ),

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^\sigma dt^2 + A^2 dr^2 + R^2 d\Omega^2, \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $\sigma = \sigma(t,r)$, $A = A(t,r)$, $R = R(t,r)$ and R is interpreted as the physical spacetime radius. If the stress is isotropic then the traceless $\mathcal{P}_{\mu\nu}$ will have the general form

$$\mathcal{P}_{\mu\nu} = \mathcal{P}\left(r_\mu r_\nu - \frac{1}{3}h_{\mu\nu}\right),$$ (5)

where $\mathcal{P} = \mathcal{P}(t,r)$ and r_μ is the unit radial vector, given in the above metric by $r_\mu = (0, A, 0, 0)$. Then

$$\mathcal{E}_\mu^\nu = \left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \text{diag}\left(\rho, -p_r, -p_T, -p_T\right),$$ (6)

where the energy density and pressures are, respectively, $\rho = \mathcal{U}$, $p_r = (1/3)\left(\mathcal{U} + 2\mathcal{P}\right)$ and $p_T = (1/3)\left(\mathcal{U} - \mathcal{P}\right)$. Substituting in the conservation Eq. (2) we obtain the following expanded system [10]

$$2\frac{\dot{A}}{A}\left(\rho + p_r\right) = -2\dot{\rho} - 4\frac{\dot{R}}{R}\left(\rho + p_T\right),$$

$$\sigma'\left(\rho + p_r\right) = -p_r' + 4\frac{R'}{R}(p_T - p_r),$$ (7)

where the dot and the prime denote, respectively, derivatives with respect to t and r. A synchronous solution is obtained by taking the equation of state $\rho + p_r = 0$, giving $\mathcal{P} = -2\mathcal{U}$ with $\mathcal{U}$ having the dark radiation form

$$\mathcal{U} = \left(\frac{\kappa}{\tilde{\kappa}}\right)^4 \frac{Q}{R^4},$$ (8)

where the dark radiation tidal charge Q is constant. Consequently, we obtain

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \frac{Q}{R^4}\left(u_\mu u_\nu - 2r_\mu r_\nu + h_{\mu\nu}\right),$$ (9)

an exactly solvable closed system which depends on the two free parameters Λ and Q. Indeed, its solutions can be written in the LeMaître-Tolman-Bondi form

$$ds^2 = -dt^2 + \frac{R'^2}{1+f}dr^2 + R^2 d\Omega^2,$$ (10)

where R satisfies

$$\dot{R}^2 = \frac{\Lambda}{3}R^2 - \frac{Q}{R^2} + f.$$ (11)

The function $f = f(r) > -1$ is naturally interpreted as the energy inside a shell labelled by r in the dark radiation vaccum and is fixed by its initial configuration.

3   Inhomogeneous de Sitter Solutions

Let us assume from now on that $Q < 0$ and $\Lambda > 0$ [note that according to the recent supernovae data $\Lambda \sim 10^{-84} \text{GeV}^2$ - see *e.g.* [11]]. Then we have a de Sitter brane with negative dark energy density and the gravitational field is confined to the vicinity of the brane. The marginally bound models corresponding to $f = 0$ are the static solutions

$$ds^2 = -\left(1 + \frac{Q}{R^2} - \frac{\Lambda}{3}R^2\right)dT^2 + \left(1 + \frac{Q}{R^2} - \frac{\Lambda}{3}R^2\right)^{-1}dR^2 + R^2 d\Omega^2 \qquad (12)$$

and were used for the static exterior of a collapsing sphere of homogeneous dark radiation [6, 7]. When $\Lambda = 0$ this corresponds to the zero mass limit of the tidal Reissner-Nordström black hole solution on the brane [9].

It is for $f \neq 0$ that the dark radiation vaccum really becomes dynamical. The solutions are better organized by a redefinition of f, the function $\beta = (3/\Lambda)[3f^2/(4\Lambda) + Q]$. As an explicit example take $\beta > 0$. Then $|f| > 2\sqrt{-Q\Lambda/3}$ and

$$\left|R^2 + \frac{3f}{2\Lambda}\right| = \sqrt{\beta}\cosh\left[\pm 2\sqrt{\frac{\Lambda}{3}}t + \cosh^{-1}\left(\frac{\left|r^2 + \frac{3f}{2\Lambda}\right|}{\sqrt{\beta}}\right)\right], \qquad (13)$$

where $+$ or $-$ refer respectively to expansion or collapse. Since R is a non-factorizable function of t and r these are new non-static, cosmological and intrinsically inhomogeneous exact solutions for negative dark energy dynamics on a spherically symmetric de Sitter brane.

4   Singularities and Rebounces

The dark radiation dynamics defined by Eq. (11) can lead to the formation of shell focusing singularities at $R = 0$ and of regular rebounce points at some $R \neq 0$. To analyse these issues consider

$$R^2\dot{R}^2 = V(R, r) = \frac{\Lambda}{3}R^4 + fR^2 - Q. \qquad (14)$$

If for all $R \geq 0$ we find $V > 0$ then a shell focusing singularity forms at $R = R_s = 0$. On the other hand if there exists $R = R_* \neq 0$ where $V = 0$ then a regular rebounce point forms at $R = R_*$. For the dark radiation vaccum it is clear that no more than two regular rebounce epochs can be found. Since $\Lambda > 0$ there is always a phase of continuous expansion to infinity with ever increasing speed. For $\Lambda > 0$ and $Q < 0$ other phases are possible depending on $f(r)$. To ilustrate consider $\beta > 0 \Leftrightarrow |f| > 2\sqrt{-Q\Lambda/3}$. If $f > 2\sqrt{-Q\Lambda/3}$ then V is positive for all $R \geq 0$. There are no rebounce points and the dark radiation shells may either expand continuously or collapse to a shell focusing singularity at $R_s = 0$.
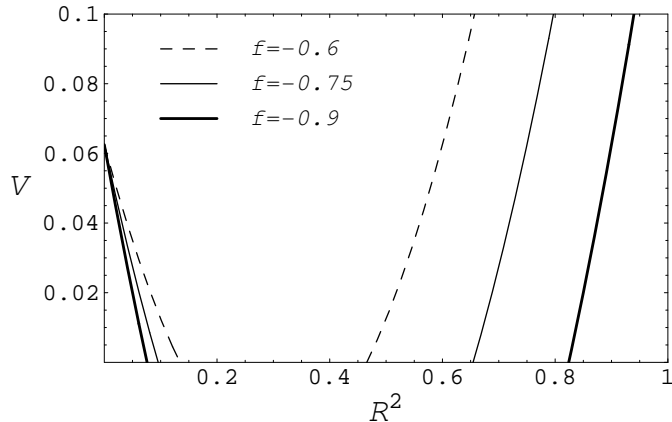
Figure 1: Plot of V for $\beta > 0$. Non-zero values of f belong to the interval $-1 < f < -2\sqrt{-Q\Lambda/3}$ and correspond to shells of constant r.

However for $-1 < f < -2\sqrt{-Q\Lambda/3}$ (see Fig. 1) there are two rebounce epochs at $R = R_{*\pm}$ with $R_{*\pm}^2 = -3f/(2\Lambda) \pm \sqrt{\beta}$. Since $V(0,r) = -Q > 0$ a singularity also forms at $R_s = 0$. The region between the two rebounce points is forbidden because there V is negative. The phase space of allowed dynamics is thus divided in two disconnected regions separated by the forbidden interval $R_{*-} < R < R_{*+}$. For $0 \le R \le R_{*-}$ the dark radiation shells may expand to a maximum radius $R = R_{*-}$, rebounce and then fall to the singularity. If $R \ge R_{*+}$ then there is a collapsing phase to the minimum radius $R = R_{*+}$ followed by reversal and subsequent accelerated continuous expansion. The singularity at $R_s = 0$ does not form and so the solutions are globally regular.

5   Conclusions

In this work we have reported some new results on the gravitational dynamics of inhomogeneous dark radiation on a RS brane. Taking an effective 4-dimensional approach we have shown that under certain simplifying but natural assumptions the Einstein field equations on the brane form a closed, solvable system. We have presented exact dynamical and inhomogeneous solutions for $\Lambda > 0$ and $Q < 0$ showing they further depend on the energy function $f(r)$. We have also described the conditions under which a singularity or a regular bounce develop inside the dark radiation vaccum. Left for future research are for instance the analysis of the confinement of gravity near the brane and the introduction of an inhomogeneous collapsing distribution of matter.

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