Average Consensus and Gossip Algorithms in Networks with Stochastic Asymmetric Communications

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Outline

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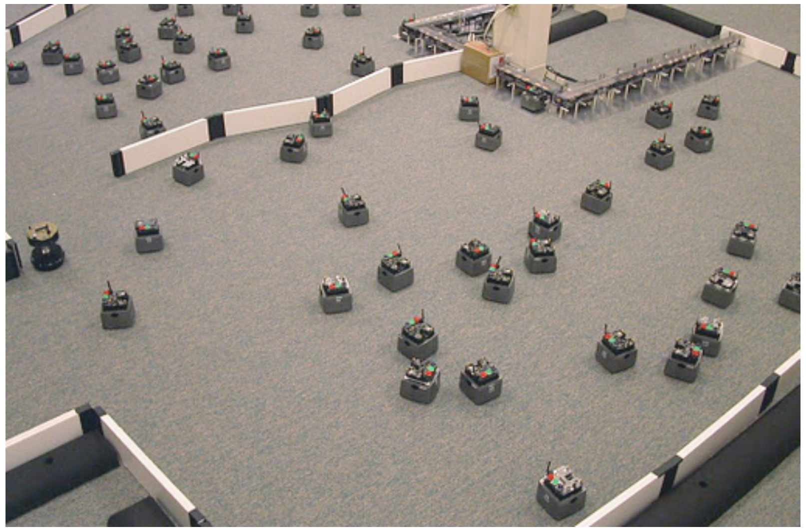
Motivation

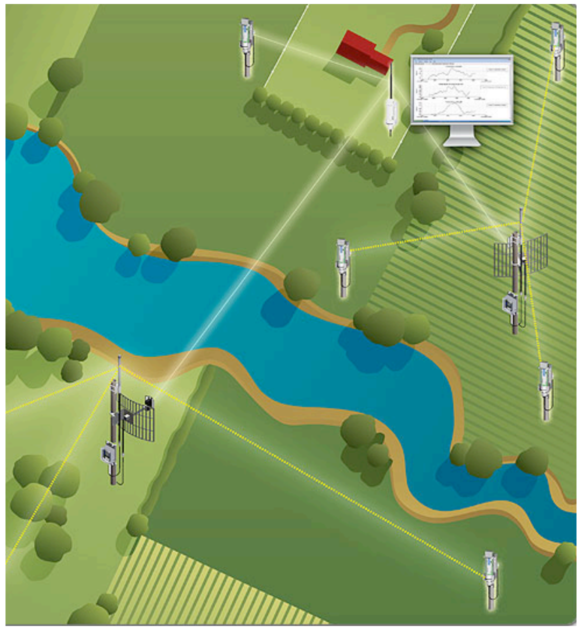
- Average Consensus Problem - M nodes connected according to a connectivity graph wish to agree on the average of their initial values.

- Robot Coordination - Fleet of robots wish to agree on direction/speed or rendezvous point.

- Sensor Networks - Compute the mean of a noise corrupted set of sensor measurements.

- Distributed Optimization - Average Consensus is a subproblem of distributed optimization such as ADMM[1]





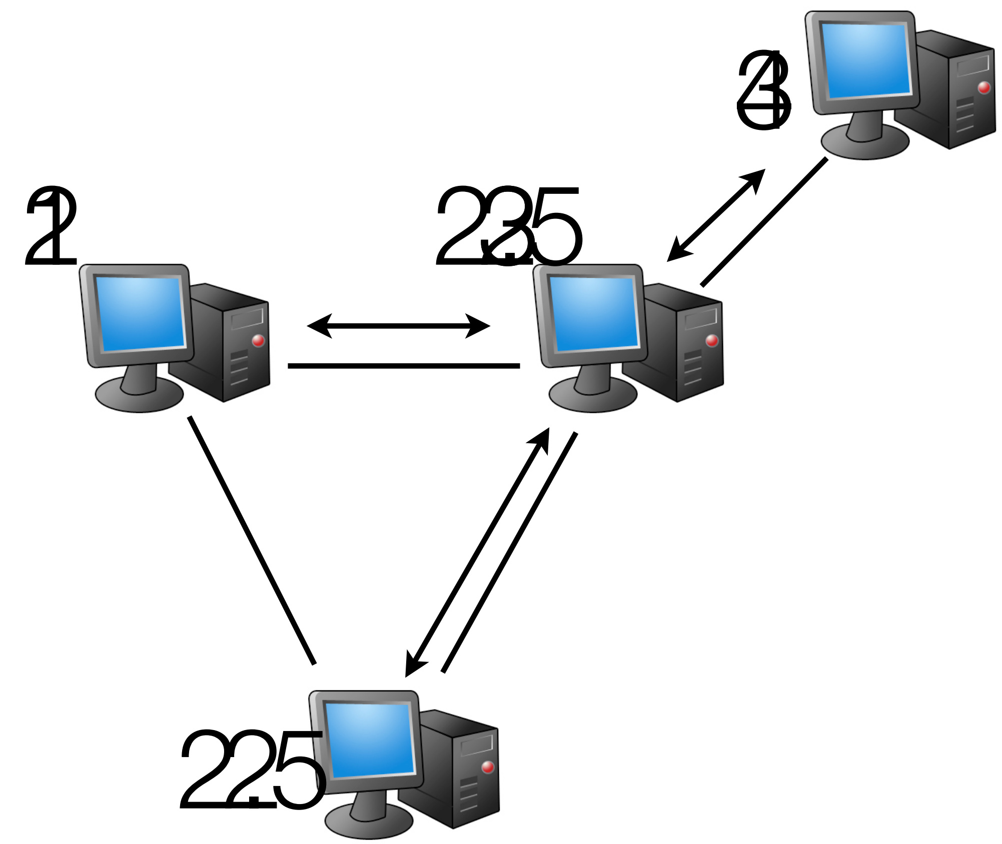
[1] Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato and Jonathan Eckstein (2011) "Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers", *Foundations and Trends® in Machine Learning*: Vol. 3: No 1, pp 1-122

Gossip Algorithm[2]

- Nodes transmit at random times, exponentially distributed, i.e. $t^l_{k+1} - t^l_k \sim \mathrm{Exp}(\lambda)$

- Each node transmits only to one adjacent node at each time.

- **Both** nodes average their values (symmetric communications).



- What if communication is **Asymmetric**? Meaning communication at each transmission time is unidirectional and only the receiver can update its state.

Much more realistic in wireless communication!

[2] S. Boyd, A. Ghosh,B. Prabhakar, D. Shah, "Randomized gossip algorithms," *Information Theory, IEEE Transactions on* , vol.52, no.6, pp. 2508- 2530, June 2006

Motivating Example

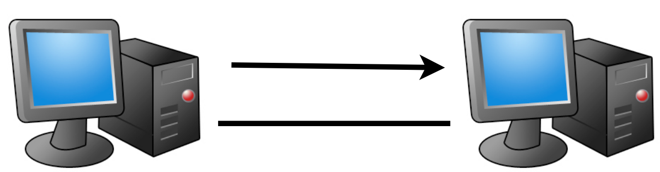
- Let us suppose a simple network of 2 nodes

- To keep the average we need to have

$$x_1(t_{k+1}) + x_2(t_{k+1}) = x_1(t_k) + x_2(t_k)$$

- Simple network:

$x_1(t_k) = a$ / $x_1(t_{k+1}) = a$



$x_2(t_k) = b$ / $x_2(t_{k+1}) = \alpha * a + \beta * b$

$$\implies \alpha = 0$$

So there is no change!

Previous Work

- In [3] a Gossip algorithm with quantization is proposed for asynchronous communication with state augmentation for directed topology graphs.

Cai and Ishii[3]	Present Work
More general - directed graphs	Symmetric graphs
Use of Quantization allows finite time convergence	Asymptotic infinite time convergence
Nonlinear	Linear

In this work we propose a linear distributed algorithm!

Advantages:

- Easy to compute convergence rates
- Easy to implement by nodes with limited computational power

[3] Kai Cai and H. Ishii, "Gossip consensus and averaging algorithms with quantization," *American Control Conference (ACC), 2010* , vol., no., pp.6306-6311, June 30 2010-July 2 2010

Problem Statement

- The network topology is defined by an adjacency matrix:

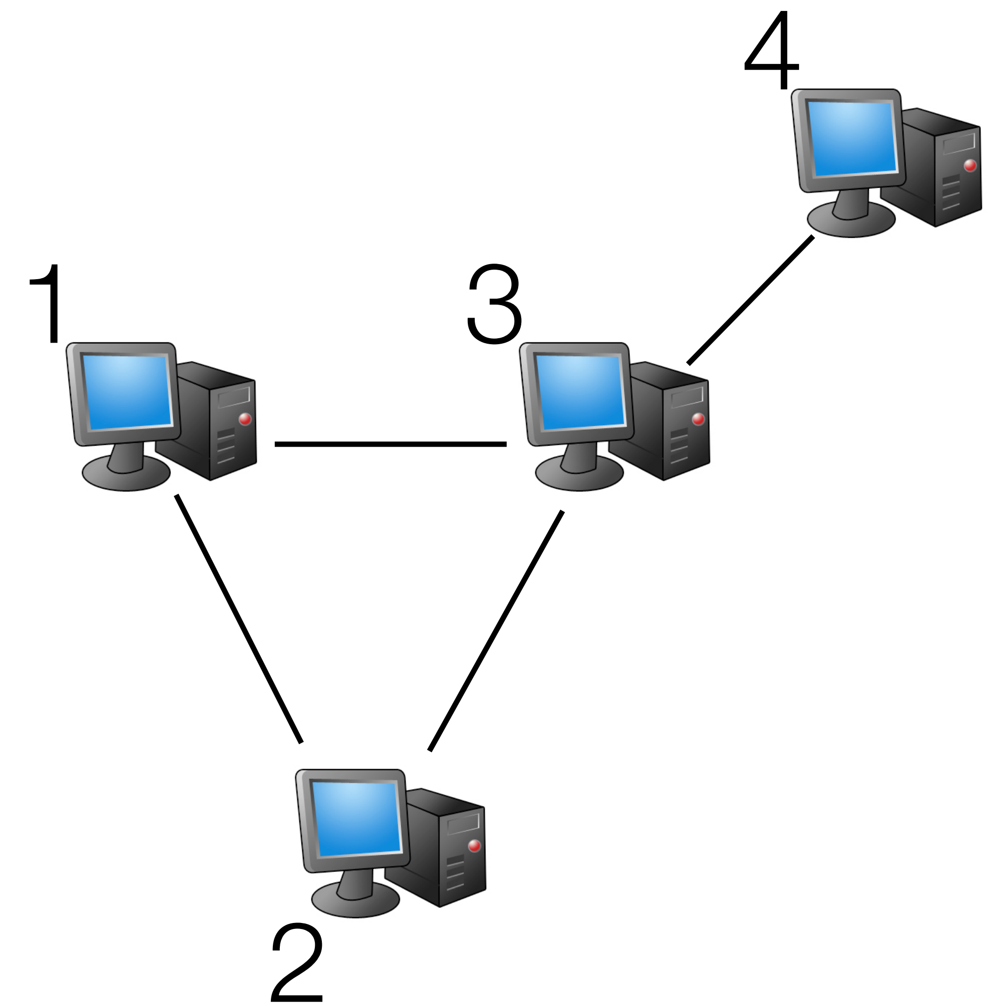
$$N_{ij} := \begin{cases} 1, \text{ if } (i,j) \in E \\ 0, \text{ otherwise} \end{cases} \qquad N = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- The algorithm is defined by the equation:
$x(t_{k+1}) = W_k x(t_k), x \in \mathbb{R}^4 \qquad \{W_k, k \geq 0\}$

- The set of transition matrix W_k are chosen randomly and define what happens to the state if node *i* transmits to node *j*.

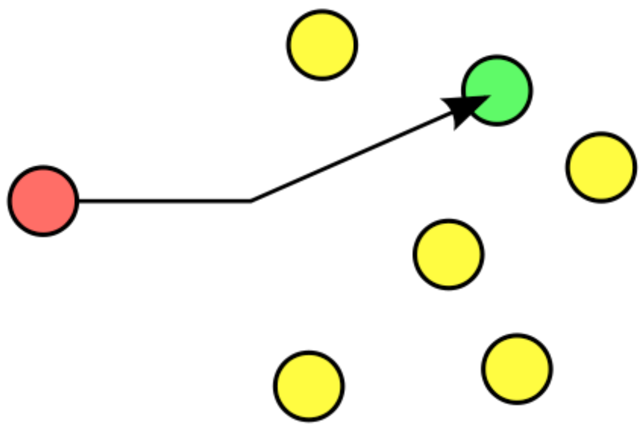
- The algorithm asymptotically achieves consensus means:

$$\lim_{t \to \infty} x_i(t) = x_{\text{av}} := \frac{1}{m} \sum_{i=1}^{m} x_i(0)$$

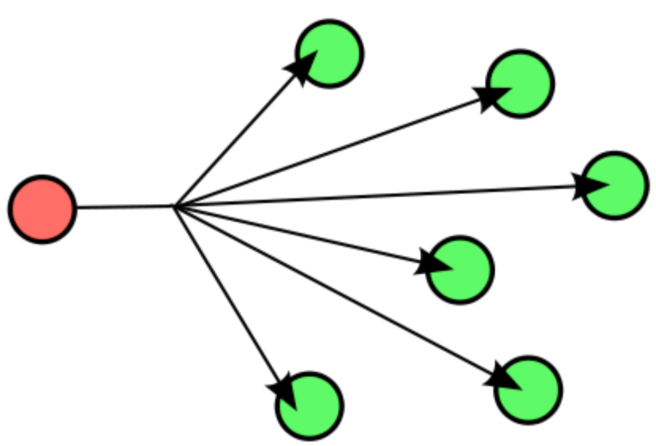


Problem Statement (cont'd)

- Transmission constraints:
- Gossip - node *i* transmits to node *j* either symmetrically or asymmetrically.



- Broadcast - node *i* transmits and all the other nodes receive.



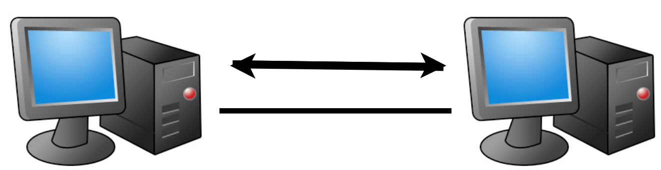
Proposed Method

- The proposed solution is to augment the state with a variable *y*.

- Intuitively *y* is like the "debt" that keeps track of the change in *x*.

$$x_1(0) = 1 \quad x_2(0) = 3$$

$$y_1(0) = 0 \quad y_2(0) = 0$$



- We can generalize and we get the following system:

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \quad z \in \mathbb{R}^{2m} \qquad z(t_{k+1}) = U_k z(t_k)$$

- Where U_k *are built to satisfy the equalities:*

$$x_j(t_{k+1}) = (1-\alpha)x_j(t_k) + \alpha x_i(t_k) + \beta y_j(t_k) + \gamma y_i(t_k)$$

$$y_j(t_{k+1}) = \frac{y_i(t_k)}{n} + y_j(t_k) + x_j(t_k) - x_j(t_{k+1})$$

Convergence Definitions

- An Algorithm is said to converge to consensus:

i. Almost Surely:

$$\text{Prob}[\lim_{t\to\infty} x_i(t) = x_{\text{av}}] = 1$$

ii. In Expectation:

$$\lim_{t\to\infty} \mathbb{E}[x_i(t)] = x_{\text{av}} \quad , \quad \forall_{i\in\{1,\ldots,m\}}$$

iii. In Second Moment:

$$\lim_{t\to\infty} \mathbb{E}[(x_i(t) - x_{\text{av}})^2] \to 0 \quad , \quad \forall_{i\in\{1,\ldots,m\}}$$

Main Result

Theorem 7

For every network with a topology graph that has a symmetric adjacency matrix *N*, there always exists a symmetric doubly stochastic weighted adjacency matrix *P*. Moreover, if the probability of node *i* transmitting to node *j*, $p_{ij} = [P]_{ij}$ and $\alpha = \beta = \gamma = 1/2$ then the Gossip algorithm converges in expectation.

Other Results

Theorem 6

If we have: $Prob[U_k = B_i] = pi, \sum_{i=1}^{n} p_i = 1$

Then the algorithm converges in expectation *if and only if*:

$$\sigma(\sum_{i=1}^{n_p} p_i B_i - \frac{1}{m} \begin{bmatrix} \mathsf{1}_m \\ \mathsf{0}_m \end{bmatrix} \begin{bmatrix} \mathsf{1}_m^\intercal & \mathsf{1}_m^\intercal) \end{bmatrix}) < 1$$

And converges in second moment *if and only if*:

$$\sigma(\sum_{i=1}^{n} p_i B_i \otimes B_i - S) < 1$$

$$S := \frac{1}{m^2}(\begin{bmatrix} \mathsf{1}_m \\ \mathsf{0}_m \end{bmatrix} \otimes \begin{bmatrix} \mathsf{1}_m \\ \mathsf{0}_m \end{bmatrix})(\begin{bmatrix} \mathsf{1}_m^\intercal & \mathsf{1}_m^\intercal \end{bmatrix} \otimes \begin{bmatrix} \mathsf{1}_m^\intercal & \mathsf{1}_m^\intercal) \end{bmatrix})$$

Rate of Convergence

Theorem 9

If the algorithm converges in second moment and the times between transmission times follow a distribution ρ then the convergence rate is given by:

$$\int_0^\infty e^{\alpha t} \rho(dt) = \frac{1}{r_1} \qquad r_1 := r_\sigma(\sum_{i=1}^{n_p} p_i B_i \otimes B_i - S)$$

Conclusion and Future Work

- We introduce a novel algorithm that converges in expectation to the average consensus.
- Convergence tests are provided for the first and second moment.
- The convergence rate is calculated for the second moment.
- In the future we would like to extend the results to stronger stability notions.
- Consider general graphs (i.e., directed graphs).
- Use distributed optimization to find the best parameter values.

Thank you!