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Measure and convergence: special subsets of the real line and their generalizations

PhD dissertation

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Author's declaration:
Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

September 15, 2017 .....
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The dissertation is ready to be reviewed.

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Abstract

In this thesis I present different notions of special subsets of the real line and their properties, in particular of those related to measure and convergence. We search for answers to open questions in this subject and we consider generalizations of known facts in the case of a larger cardinality, i.e. in the generalized Cantor space 2^κ, for an uncountable cardinal κ, equipped with the topology generated by sets of extensions of partial functions.

First (Chapter 2), we discuss special subsets of the Cantor space 2^ω. The theory of special subsets is already well developed (see [Miller, 1984] and [Bukovský, 2011]). I introduce two notions of such sets, which were not considered before: the class of perfectly null sets and the class of sets which are perfectly null in the transitive sense ([Korch and Weiss, 2016]). These classes may play the role of duals on the measure side to the corresponding classes on the category side. We investigate their properties, and although the main problem of whether the classes of perfectly null sets and universally null sets are consistently different remains open, we prove some results related to this question and study their version on the category side.

Next (Chapter 3), we study problems related to Egorov's Theorem, which describes a relation between convergence and measure. Egorov's Theorem can be generalized to some notions of ideal convergences (see e.g. [Mrożek, 2009]), and T. Weiss has proven ([Weiss, 2004]) that the generalized Egorov's statement (i.e. the theorem without the assumption on measurability) is independent from ZFC. Integrating both ideas, we prove that the generalized Egorov's statement as well as its negation are consistent with ZFC in different cases of ideal convergence ([Korch, 2017b]).

Many of the classical notions of special subsets of 2^ω can be considered in the case of the generalized Cantor space 2^κ. Although the theory of the generalized Cantor space 2^κ has recently been broadly developed (see e.g. [Laguzzi et al., 2016]), the theory of special subsets of 2^κ seems to be largely omitted from those considerations. We study those classes of sets in this setting ([Korch and Weiss, 2017]). It turns out that many of properties of subsets of 2^ω can be easily proved in 2^κ, although sometimes one has to use some additional set-theoretic assumptions (Chapter 4). Next we deal with less common classes of small sets in 2^κ (Chapter 5).

In Chapter 6, I present different types of convergence of κ-sequences of functions $2^\kappa \to 2^\kappa$, and study properties of special subsets of 2^κ related to the notion of convergence ([Korch, 2017a]). We relate those properties to the

sequence selection principles. We also consider convergence of sequences of points and functions with respect to an ideal on κ (Chapter 7).

Finally, to relate measure and convergence properties in 2^κ, we study the possibility of introducing Egorov's Theorem in 2^κ. Since no method of constructing measure in 2^κ which fulfils all reasonable requirements is known, we consider the properties such set-function should have to enable the proof of Egorov's Theorem. I leave the question of existence of such a function which satisfies some additional reasonable conditions open. Every κ-strongly null set is null with respect to such a set function which satisfies some additional properties. We study also the ideal version of Egorov's Theorem in 2^κ.

Key words: special subsets, measure, convergence, category, generalized Cantor space, Egorov's Theorem, perfectly null set, ideal convergence

AMS Mathematics Subject Classification:

→ 03. Mathematical logic and foundations

→ E. Set theory

→ 05. Other combinatorial set theory
→ 10. **Ordinal and cardinal numbers**
→ 15. Descriptive set theory
→ 17. Cardinal characteristics of the continuum
→ 20. Other classical set theory (including functions, relations, and set algebra)
→ 35. **Consistency and independence results**
→ 55. Large cardinals

→ 26. Real functions

→ A. Functions of one variable

→ 03. **Foundations: limits and generalizations, elementary topology of the line**

→ E. Miscellaneous topics

→ 35. Nonstandard analysis

→ 28. Measure and integration

→ A. Classical measure theory

→ 05. **Classes of sets (Borel fields, σ-rings, etc.), measurable sets, Suslin sets, analytic sets**

→ 12. Contents, measures, outer measures, capacities

→ 20. **Measurable and nonmeasurable functions, sequences of measurable functions, modes of convergence**

→ B. Set functions, measures and integrals with values in abstract spaces

→ 15. Set functions, measures and integrals with values in ordered spaces

→ C. Set functions and measures on spaces with additional structure

→ 10. Set functions and measures on topological groups or semigroups, Haar measures, invariant measures

→ E. Miscellaneous topics in measure theory

→ 05. Nonstandard measure theory

→ 15. **Other connections with logic and set theory**

→ 40. Sequences, series, summability

→ A. Convergence and divergence of infinite limiting processes

→ 30. **Convergence and divergence of series and sequences of functions**

→ 35. **Ideal and statistical convergence**

→ 54. General topology

→ A. Generalities

→ 20. Convergence in general topology (sequences, filters, limits, convergence spaces, etc.)

→ 25. Cardinality properties (cardinal functions and inequalities, discrete subsets)

→ 35. Consistency and independence results

→ F. Special properties

→ 99. None of the above, but in this section

Streszczenie

W niniejszej pracy rozważam różne pojęcia specjalnych podzbiorów prostej i ich właściwości, w szczególności te związane z miarą lub zbieżnością. Poszukuję odpowiedzi na otwarte pytania w tym zakresie oraz rozważam uogólnienia znanych faktów na wyższe liczby kardynalne, tj. w uogólnionej przestrzeni Cantora 2^κ, dla nieprzeliczalnej liczby kardynalnej κ, rozważanej z topologią generowaną przez zbiory przedłużeń funkcji częściowych.

Po pierwsze (Rozdział 2) rozważam specjalne podzbiory przestrzeni Cantora 2^ω. Teoria specjalnych podzbiorów jest oczywiście już znacząco rozwinięta (patrz [Miller, 1984] i [Bukovský, 2011]). W tej pracy wprowadzam dwie, do tej pory nierozważane, klasy takich zbiorów: klasę zbiorów doskonale miary zero oraz klasę zbiorów doskonale miary zero w sensie tranzytywnym. Te klasy mogą odgrywać rolę dualną po stronie miary do odpowiednich klas zbiorów po stronie kategorii ([Korch and Weiss, 2016]). Badam właściwości tych klas i, mimo że główny problem, czy klasy zbiorów doskonale miary zero i uniwersalnie miary zero są niesprzecznie różne, pozostaje nierozwiązany, to dowodzę twierdzeń powiązanych z tym pytaniem i rozważam ich wersje po stronie kategorii.

Następnie (Rozdział 3) badam problemy związane z twierdzeniem Jegorowa, które łączy ze sobą właściwości związane ze zbieżnością i miarą. Twierdzenie Jegorowa może być uogólnione na przypadek zbieżności ideałowej (patrz np. [Mrożek, 2009]), natomias T. Weiss udowodnił ([Weiss, 2004]), że uogólnione stwierdzenie Jegorowa (tj. twierdzenie bez założenia o mierzalności) jest niesprzeczne z ZFC. Łącząc oba pomysły, dowodzę, że uogólnione stwierdzenie Jegorowa i jego zaprzeczenie są niesprzeczne z ZFC dla różnych przypadków zbieżności ideałowych ([Korch, 2017b]).

Wiele klasycznych pojęć specjalnych podzbiorów w 2^ω może być uogólniona na przypadek uogólnionej przestrzeni Cantora 2^κ. Mimo że teoria uogólnionej przestreni Cantora 2^κ była w ostatnim czasie znacząco rozwijana (patrz np. [Laguzzi et al., 2016]), to teoria specjalnych podzbiorów 2^κ wydaje się być w znacznej części pomiajana w tych rozważaniach. W niniejszej pracy badam te klasy zbiorów w takim przypadku ([Korch and Weiss, 2017]). Okazuje się, że wiele własności zachodzących w 2^ω można łatwo wykazać dla 2^κ, choć czasem niezbędne są dodatkowe teorio-mnogościowe założenia (Rozdział 4). Następnie zajmuję się mniej znanymi klasami małych zbiorów w 2^κ (Rozdział 5).

W rozdziale 6 rozważam różne rodzaje zbieżności κ-ciągów funkcji $2^\kappa \to 2^\kappa$ i badam właściwości specjalnych podzbiorów 2^κ związanych ze zbieżnością

([Korch, 2017a]). Łączę te właściwości z właściwościami wyboru podciągów. Rozważam także zbieżność względem ideału na κ (Rozdział 7).

Na koniec, łącząc właściwości związane z miarą i ze zbieżnością w 2^κ, rozważam możliwość wprowadzenia twierdzenia Jegorowa w przestrzeni 2^κ (Rozdział 8). Ponieważ nie znana jest metoda konstrukcji miary w przestrzeni 2^κ, która spełniałaby wszystkie sensowne wymagania, rozważam właściwości, które są niezbędne do udowodnienia odpowiednika twierdzenia Jegorowa. Kwestię istnienia odpowiedniej funkcji miarowej spełniającej dodatkowe założenia zostawiam jako pytanie otwarte. Przy pewnych dodatkowych założeniach każdy zbiór κ-silnie miary zero jest miary zero względem takiej funkcji. Badam także ideałową wersję Tw. Jegorowa w 2^κ.

Słowa kluczowe: specjalne podzbiory, miara, zbieżność, kategoria, uogólniona przestrzeń Cantora, twierdzenie Jegorowa, zbiór doskonale miary zero, zbieżność ideałowa

Klasyfikacja tematyczna według AMS:

- → 03. Logika matematyczna i podstawy matematyki
  - → E. Teoria mnogości
    - → 05. Inne aspekty kombinatorycznej teorii mnogości
    - → 10. **Liczby porządkowe i kardynalne**
    - → 15. Deskryptywna teoria mnogości
    - → 17. Kardynalne charakterystyki kontinuum
    - → 20. Inne aspekty klasycznej teorii mnogości (w tym funkcje, relacje i algebra zbiorów)
    - → 35. **Niesprzeczność i niezależność**
    - → 55. Duże liczby kardynalne
- → 26. Funkcje rzeczywiste
  - → A. Funkcje jednej zmiennej
    - → 03. **Podstawy: granice i uogólnienia, elementarna topologia prostej**
  - → E. Inne tematy
    - → 35. Niestandardowa analiza

→ 28. Miara i całka

- → A. Klasyczna teoria miary
  - → 05. **Klasy zbiorów (ciała borelowskie, σ-pierścienie, etc.), zbiory mierzalne, zbiory Suslina, zbiory analityczne**
  - → 12. Miary skończenie addytywne, miary, miary zewnętrzne, pojemności
  - → 20. **Mierzalne i niemierzalne funkcje, ciągi mierzalnych funkcji, rodzaje zbieżności**
- → B. Funkcje na zbiorach, miary i całki z wartościami w innych przestrzeniach
  - → 15. Funkcje na zbiorach, miary i całki z wartościami w przestrzeniach uporządkowanych
- → C. Funkcje na zbiorach i miary w przestrzeniach z dodatkową strukturą
  - → 10. Funkcje na zbiorach i miary w grupach lub półgrupach topologicznych, miary Haara, miary niezmiennicze
- → E. Inne tematy w teorii miary
  - → 05. Niestandardowa teoria miary
  - → 15. **Inne powiązania z logiką i teorią mnogości**

→ 40. Ciągi, szeregi, sumowalność

- → A. Zbieżność i rozbieżność nieskończonych procesów granicznych
  - → 30. **Zbieżność i rozbieżność szeregów i ciągów funkcji**
  - → 35. **Zbieżność ideałowa i statystyczna**

→ 54. Topologia ogólna

- → A. Podstawy
  - → 20. Zbieżność w topologii ogólnej (ciągi, filtry, granice, przestrzenie zbieżności, etc.)
  - → 25. Własności związane z mocą (funkcje kardynale i nierówności, podzbiory dyskretne)
  - → 35. Niesprzeczność i niezależność
- → F. Specjalne właściwości
  - → 99. Żadne z powyższych, ale w tej sekcji

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Contents

Chapter 1

Introduction and preliminaries

The aim of this thesis is to consider different notions of special subsets of the real line and their properties, in particular of those related to measure and convergence. I consider generalizations of known facts in the case of a larger cardinality, i.e. in the generalized Cantor space 2^κ for an uncountable cardinal κ.

In the studies of the structure of the real line an important role is played by the notions of measure and category. Many questions arise from the duality between measure and category and the abrupt lack of it in some cases, which is shown in the seminal book of J. Oxtoby [Oxtoby, 1971].

In this chapter, I describe the motivation of this work and I provide some preliminary notions and facts required to understand it. The main notions and notation used in this thesis are included here along with a brief introduction to the historical background, which is included for the sake of completeness and can be omitted by a reader who is acknowledged with set theory of the real line.

1.1 Set theory, measure and category

For centuries, mathematicians in their studies encountered the concept of infinity. It was usually treated with slight suspicion and reservation because everything in the real world seemed to be finite. Finally, in the second half of the 19th century the study of the infinity found its way to the core of mathematics.

It all really started with the works of Georg Cantor. He considered a subset of a real line to be countable if its elements can be set in a sequence enumerated by the natural numbers and proved in [Cantor, 1874] that while the set of all algebraic numbers is countable, the set of all reals is not of this form. Indeed,

given a sequence of real numbers $\langle a_n \rangle_{n\in\omega}$ (where ω denotes the first infinite ordinal number, i.e. the set of all natural numbers) one can fix a closed interval $I_0 \subseteq \mathbb{R}$ such that $a_0 \notin I_0$, and then inductively choose a closed interval $I_{n+1} \subseteq I_n$ such that $a_{n+1} \notin I_{n+1}$. Since all the considered intervals are closed, $X = \bigcap_{n\in\omega} I_n$ is not empty. Moreover X does not contain any point of the sequence $\langle a_n \rangle_{n\in\omega}$, so there exists a real number which is not an element of the sequence.

This observation can be seen as the first of a series of properties which can be used to divide the class of all sets of reals into two subclasses – small sets and bigger sets. In this case the small sets are those which are countable. But it was not long till another such notion was defined. At the turn of the 19-th century, French mathematicians E. Borel and H. Lebesgue, who were studying properties of functions (see [Borel, 1898] and [Lebesgue, 1902]), defined measure (now called Lebesgue measure) and obviously, the class of null sets (i.e. sets of measure zero), and R. Baire among other notions considered category and meagre sets in his doctoral thesis [Baire, 1899]. Although I assume that the reader is familiar with those notions, I recall them briefly below.

A subset A of the real line $\mathbb{R}$ is open if it is a union of open intervals. A set is closed if it is a complement of an open set. Furthermore, a set is a G_δ-set (respectively, a F_σ-set) if it is an intersection (respectively, a union) of at most countably many open (respectively, closed) sets. Finally, the family of **Borel sets** is the least family of sets which contains all the open sets and is closed under taking countable unions and complements. A set A is **analytic** if there exists a Borel set $B \subseteq \mathbb{R}\times\mathbb{R}$, and $A = \pi_1[B]$.

If A is a subset of the real line, then the **outer measure of** A is

$$m^*(A) = \inf\left\{\sum_{i=0}^{\infty} |b_i - a_i| \colon A \subseteq \bigcup_{i\in\omega} (a_i; b_i)\right\},$$

where $(a; b)$ denotes the open interval with endpoints a and b, i.e. the outer measure of a set A is the infimum of possible sum of lengths of a family of intervals which cover A. It is easy to verify that if $A \subseteq B$, then $m^*(A) \subseteq m^*(B)$, and that if $A = \bigcup_{i=0}^{\infty} A_i$,

$$m^*(A) \leq \sum_{i=0}^{\infty} m^*(A_i).$$

If $\mu^*(A) = 0$, then the set A is called a **null set**. The family of all null sets will be denoted by $\boldsymbol{\mathcal{N}}$. A set $A \subseteq \mathbb{R}$ is said to be **measurable** if there exist Borel sets B_1, B_2 such that $B_1 \subseteq A \subseteq B_2$ and $B_2 \setminus B_1$ is null. The outer measure m^* restricted to the family of all measurable sets is denoted by m and called **Lebesgue measure**. If $\langle A_i \rangle_{i=0}^{\infty}$ is a sequence of pairwise-disjoint measurable sets, then

$$m\left(\bigcup_{i=0}^{\infty} A_i\right) = \sum_{i=0}^{\infty} m(A_i).$$

In particular, a union of countably many null sets is a null set. Obviously, $m(\mathbb{R}) = \infty$, so $\mathbb{R}$ is not null.

More generally a function $\mu\colon \mathcal{M} \to [0, \infty]$ with $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ closed under taking countable unions is called a **measure** if $\mu(\varnothing) = 0$, and if $\{A_i \colon i \in \omega\} \subseteq \mathcal{M}$ is a countable collection of pairwise disjoint sets, then

$$\mu\left(\bigcup_{i\in\omega} A_i\right) = \sum_{i\in\omega} \mu(A_i).$$

A set A is **null with respect to** μ if there exists a set $M \in \mathcal{M}$ such that $A \subseteq M$, and $\mu(M) = 0$.

A measure μ is **Borel** if $\mathcal{M}$ is the collection of all Borel sets. It is **finite** if $\mu(A) < \infty$ for all $A \in \mathcal{M}$, and is **strictly positive** if $\mu(U) > 0$ for any open $U \in \mathcal{M}$. Finally, μ is **diffused** if $\mu(\{x\}) = 0$ for any $x \in \mathbb{R}$.

A set A is **nowhere dense** if its closure has empty interior. A set is **meagre** if it is a union of countably many nowhere dense set. The family of all meagre sets will be denoted by $\boldsymbol{\mathcal{M}}$. Obviously, a union of countably many meagre sets is a meagre set. Baire proved that an intersection of countably many open dense sets of the real line is dense. This property is called **Baire Theorem**. In particular, every non-empty open set is not meagre, and therefore neither is the whole real line $\mathbb{R}$. A set $A \subseteq \mathbb{R}$ is said to have the **property of Baire** if there exist Borel sets B_1, B_2 such that $B_1 \subseteq A \subseteq B_2$ and $B_2 \setminus B_1$ is meagre.

Later on Cantor's theory of cardinalities and also the theory of the real line were quickly developed. We shall say that two sets A and B are of the same cardinality ($|A| = |B|$) if there exists a bijection $f\colon A \to B$, and $|A| \leq |B|$ if there exists a one-to-one function $g\colon A \to B$. Although one can consider two sets of the same cardinality without defining the notion of cardinal numbers, this notion can be formalized using ordinals. An **ordinal** α is a transitive set (meaning every its element is also its subset) well ordered by the relation $\subseteq$ (i.e. if $\beta, \gamma \in \alpha$, then either $\beta \subseteq \gamma$ or $\gamma \subseteq \beta$, and if $A \subseteq \alpha$ with $A \neq \varnothing$, then there exists the least element in A). If $\alpha \subseteq \beta$ are ordinals, we usually write $\leq$ instead of $\subseteq$. It is easy to check that if α, β are ordinals, then either $\alpha \leq \beta$ or $\beta \leq \alpha$, and also that $\alpha < \beta$ if and only if $\alpha \in \beta$. Moreover if $\langle P, \leq \rangle$ is a well ordered set, then there exists an ordinal α which is order isomorphic to it (i.e. there exists a bijection $f\colon P \to \alpha$ such that for any $p, q \in P$, $p \leq q$ if and only if $f(p) \leq f(q)$).

Notice that $\varnothing$ (also denoted by 0) is the least ordinal. Given an ordinal α, we can define its successor $\alpha + 1 = \alpha \cup \{\alpha\}$. An ordinal β such that there is no ordinal α such that $\beta = \alpha + 1$ will be called a **limit ordinal**. If β is a limit ordinal, then $\beta = \bigcup_{\alpha<\beta} \alpha$. All the finite ordinals can be therefore denoted by

natural numbers, and $n = \{0, 1, \ldots, n-1\}$. The set of all finite ordinals (i.e. the set of natural numbers) is hence also an ordinal and will be denoted by ω.

If α, β are ordinals, then by $\alpha + \beta$ we denote the ordinal which is order-isomorphic to the set $\{0\} \times \alpha \cup \{1\} \times \beta$ ordered by the lexicographic order. The ordinal order-isomorphic to the set $\alpha \times \beta$ ordered by the lexicographic order is denoted by $\alpha \cdot \beta$.

An ordinal κ is a **cardinal number** if there is no ordinal α such that $|\kappa| = |\alpha|$, but $\alpha < \kappa$. Therefore all finite ordinals (i.e. natural numbers) are cardinals, and ω (also denoted in this context by $\aleph_0$) is the first infinite cardinal. The axiom of choice is equivalent to Zermelo Theorem, which states that every set can be well-ordered. Therefore under the axiom of choice for every set X, there exists a cardinal number κ, which has the same cardinality as X (in this case we write $|X| = \kappa$). If κ is a cardinal, then κ^+ is a cardinal such that there is no cardinal number λ such that $\kappa < \lambda < \kappa^+$. A cardinal number λ such that there exists κ with $\lambda = \kappa^+$ is called a successor cardinal. All the other cardinal numbers are called **limit cardinals**. Obviously, every infinite cardinal is a limit ordinal.

If A, B are sets, then the set of all functions $f: A \to B$ is denoted by B^A, and abusing the notation, I denote by κ^λ the cardinal number of the same cardinality as the set of all functions $f: \lambda \to \kappa$. The set of all subsets $\mathcal{P}(A)$ of a set A is closely related to the set 2^A of **characteristic functions**, i.e. $B \subseteq A$ has its characteristic function $\chi_B: A \to \{0, 1\}$ defined in the following way:

$$\chi_B(a) = \begin{cases} 0 & \text{, if } a \notin B \\ 1 & \text{, if } a \in B \end{cases}.$$

If $s \in 2^A, t \in 2^B$, I shall write $s \subseteq t$ if $A \subseteq B$ and $t \restriction A = s$. For this reason I write $a^{-1}[\{1\}]$ to describe a subset of A given by characteristic function $a \in 2^A$.

G. Cantor in his well-known theorem (see [Cantor, 1892]) using diagonal argument proved that for any set A, we have $|A| < |\mathcal{P}(A)|$. This means that for every cardinal κ, $\kappa < 2^\kappa$. A cardinal κ is a **strong limit cardinal** if for all cardinals $\lambda < \kappa$, $2^\lambda < \kappa$. If $\langle P, \leq \rangle$ is a partially order set, then the **cofinality** $\mathrm{cof}(\langle P, \leq \rangle)$ is the least possible cardinality of a subset B of A such that for any $a \in A$, there exists $b \in B$ with $a \leq b$. If κ is a cardinal we write simply $\mathrm{cof}\kappa$ for the cofinality of $\langle \kappa, \leq \rangle$. If $\mathrm{cof}\kappa = \kappa$, κ is called a **regular cardinal**. Otherwise, it is called **singular**. A cardinal κ is **weakly inaccessible** if it is an uncountable regular limit cardinal. A weakly inaccessible cardinal which is a strong limit cardinal is said to be **strongly inaccessible**.

All the cardinals can be indexed by ordinal numbers in the sense that for an ordinal number α, $\aleph_\alpha$ is the only cardinal such that

$$\langle \{\lambda < \kappa : \lambda \text{ is an infinite cardninal}\}, \leq \rangle$$

is order-isomorphic to $\langle \alpha, \leq \rangle$. Therefore, $\aleph_0 = \omega$, and $\aleph_1 = \aleph_0^+$ is the first uncountable cardinal, and for any ordinal α, $\aleph_{\alpha+1} = \aleph_\alpha^+$. Moreover, for a limit ordinal α, $\aleph_\alpha = \bigcup_{\beta<\alpha} \aleph_\beta$. The cardinality of the set of real numbers $|\mathbb{R}| = 2^\omega$ is denoted by $\mathfrak{c}$. The cardinal number $\aleph_\alpha$ can also be denoted by ω_α when considered as an ordinal.

If κ, λ are cardinals, then the cardinality of the set $\kappa \times \{0\} \cup \lambda \times \{1\}$ will be denoted by $\kappa + \lambda$, and $\kappa \cdot \lambda$ stands for the cardinality of $\kappa \times \lambda$. Since every cardinal is an ordinal number this abuses the notation. It should be known from the context whether the above arithmetic operations are considered in the ordinal or cardinal sense. For more detailed study of arithmetic of ordinal and cardinal numbers, the reader is related to [Jech, 2006] or [Kunen, 2006].

A subset $A \subseteq \kappa$ is a **club (closed unbounded set)** if $\bigcup(A \cap \alpha) \in A$ for all limit ordinals $\alpha < \kappa$, and for all $\alpha < \kappa$, there exists $\beta \in A$ with $\alpha \leq \beta < \kappa$. Notice that an intersection of fewer than κ clubs is a club in κ. A set $S \subseteq \kappa$ is **stationary** in κ if for every club A, $S \cap A \neq \varnothing$. $\boldsymbol{\mathcal{NS}}_\kappa$ denotes the ideal of non-stationary sets in κ.

Set theory quickly became axiomatized. The standard set of axioms is the Zermelo-Fraenkel axioms (and the axiom of choice), denoted here by **ZFC**. The big breakthrough in set theory was proving that some statements are independent from this set of axioms. We say that a statement is consistent with ZFC if there exists a model of ZFC in which this statement holds (assuming that ZFC is consistent itself). A statement is independent from ZFC if it and its negation are consistent with ZFC. The first and the most famous proof of independence considered the **continuum hypothesis** (CH), which states that $\mathfrak{c} = \aleph_1$, and was first raised by Cantor, and included in the famous Hilbert's list of problems from 1900 ([Hilbert, 1900]). The proof of indepencence of CH was completed by Cohen in [Cohen, 1963] and [Cohen, 1964]. This proof introduced a method of proving independence of statements from ZFC called forcing, which allowed to prove an independence from ZFC of many more statements. The reader is referred to [Jech, 2006, Jech, 1986] or [Kunen, 2006] for an introduction to forcing methods.

In particular, many results presented in this thesis are consistency results. Usually I achieve them by proving an implication from other statements which are known to be consistent with ZFC. Apart from CH, I use **generalized continuum hypothesis** ($2^\kappa = \kappa^+$ for any cardinal κ, GCH), Jesen's diamond and statements concerning cardinal coefficients (which will be introduced later on).

Throughout this thesis I use standard set-theoretic notation. In particular, $A \triangle B$ denotes the symmetric difference $((A \setminus B) \cup (B \setminus A))$ between A and B, $\mathrm{dom} f$ denotes the domain of a function f, $f \restriction X$ denotes the restriction of

f to $X \subseteq \mathrm{dom} f$. The image of a set A under function f is denoted by $f[A]$, and the preimage of B under f by $f^{-1}[B]$. f^{-1} denotes the inverse function of f. $\langle a_\alpha \rangle_{\alpha<\xi}$ is a (transfinite) sequence of length ξ. The cardinality of a set A is denoted by $|A|$. The set of all functions $A \to B$ (i.e. the set of all sequences of elements of B indexed by elements of A) is denoted by B^A. $[A]^\kappa$ is the set of all subsets of A of cardinality κ, and $[A]^{<\kappa}$ is the set of all subsets of A of cardinality less than κ. Similarly, $A^{<\kappa}$ denotes the set of all sequences of length less than κ. If $s \in A^\alpha$, then $\mathrm{len}(s) = \alpha$. If s is a sequence of length α, then $s^\frown a$ is a sequence of length $\alpha + 1$ with $s(\alpha) = a$. $X \times Y$ is the set of all pairs $\{\langle x, y\rangle : x \in X, y \in Y\}$. The projection onto first (respectively, second) coordinate is denoted π_1 (respectively, π_2), $\pi_1(x, y) = x$.

A partially ordered set $\langle P, \leq\rangle$ satisfies **κ-chain condition** if every antichain in P has cardinality less than κ ($A \subseteq P$ is an **antichain** if for every $a, b \in A$ there is no $c \in P$ such that $c \leq a$ and $c \leq b$).

I also use standard topological notation. In particular, $\mathrm{int} A$ denotes the interior of a set A, and $\mathrm{cl} A$ denotes its closure. If A is a subset of a metric space, then $\mathrm{diam} A$ denotes the diameter of A, i.e. the supremum of the distance between two points of A. A function f is **continuous** if the preimage under f of every open set is open. It is a **homeomorphism** if f is a continuous bijection with continuous inverse f^{-1}. Finally, f is **measurable** (respectively, **Lebesgue measurable**) if every the preimage under f of every open set is Borel (respectively, m-measurable), and f is a **Borel isomorphism** if it is a measurable bijection with measurable inverse. A set is **perfect** if it is closed and does not have any isolated points.

A collection of open sets $\mathcal{U}$ is an **open cover** of a set X if $\bigcup \mathcal{U} \supseteq X$. Open cover $\mathcal{V} \subseteq \mathcal{U}$ is a **subcover** of $\mathcal{U}$. If $\mathcal{U}, \mathcal{V}$ are open covers of X, then $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for any $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$.

1.2 Set theory of the real line

Set theory of the real line deals with the set-theoretic properties of the reals. Many of those properties can be considered also in the unit interval (which is here denoted as $\boldsymbol{I}$), the Cantor space 2^ω or the Baire space ω^ω.

1.2.1 The Cantor space

The Cantor space 2^ω can be seen as a countable product of two-point discrete spaces. Therefore, the basic closed open set in 2^ω (spaces with clopen base are called zero-dimensional) is determined by a finite sequence $w \in 2^{<\omega}$. It is

denoted by $[w]$,

$$[w] = \{f \in 2^\omega : f{\restriction}\mathrm{len}(w) = w\}.$$

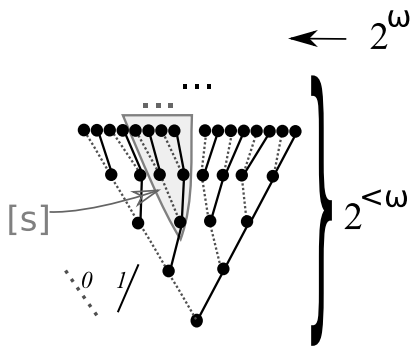

Figure 1.1: Basic set $[s]$ in the Cantor space for $s = 01$.

If F is a set of partial functions $\omega \to 2$, the expression $[F]$ denotes

$$\bigcup_{f\in F} \{x \in 2^\omega : x{\restriction}\mathrm{dom} f = f\}.$$

The Cantor space has a natural metric d defined in the following way:

$$d(x, y) = \frac{1}{2^n},$$

where $n = \min\{k \in \omega : x(n) \neq y(n)\}$, for $x, y \in 2^\omega$ and $x \neq y$. Obviously, $d(x, y) = 0$ for $x = y$.

The Cantor space is also be treated as a vector space over $\mathbb{Z}_2$. In particular, for $A, B \subseteq 2^\omega$, let $A + B = \{t + s : t \in A, s \in B\}$.

A set $T \subseteq 2^{<\omega}$ is called a **tree** if for all $t \in T$ and $s \subseteq t$, we have $s \in T$. A tree T is pruned if for all $t \in T$, there exists $s \in T$ with $t \subsetneq s$. If P is a closed set in 2^ω, there is a pruned tree $T_P \subseteq 2^{<\omega}$ such that the set of all infinite branches of T_P (usually denoted by $[T_P]$) equals P. If T is a pruned tree, then $[T]$ is perfect if and only if for any $w \in T$, there exist $w', w'' \in T$ such that $w \subseteq w', w \subseteq w''$, but $w' \not\subseteq w''$ and $w'' \not\subseteq w'$. Such a tree is called a **perfect tree**.

If $w \in 2^n$, and $a, b \in \omega$, with $a \leq b \leq n$, then by $w[a, b] \in 2^{b-a+1}$ I denote a finite sequence such that $w[a, b](i) = w(a + i)$ for $i \leq b - a$. If $\langle s_0, s_1, \ldots s_k\rangle$ is a finite sequence of natural numbers less than n, then $w\langle s_0, s_1, \ldots s_k\rangle \in 2^{k+1}$ denotes a sequence such that $w\langle s_0, s_1, \ldots s_k\rangle(i) = w(s_i)$ for any $i \leq k$. Let $\boldsymbol{Q}_0 = \{t \in 2^\omega : \exists_{m\in\omega} \forall_{n>m} f(n) = 0\}$, and $\boldsymbol{Q}_1 = \{t \in 2^\omega : \exists_{m\in\omega} \forall_{n>m} f(n) = 1\}$.

A finite sequence $w \in T_P$ is called a **branching point** of a perfect set P if $w^\frown 0, w^\frown 1 \in T_P$. A branching point is on level $i \in \omega$ if there exist i branching

points below it. The set of all branching points of P on level i will be denoted by $\mathrm{Split}_i(P)$ and

$$\mathrm{Split}(P) = \bigcup_{i\in\omega} \mathrm{Split}_i(P).$$

Let

$$s_i(P) = \min\{\mathrm{len}(w)\colon w \in \mathrm{Split}_i(P)\}$$

and

$$S_i(P) = \max\{\mathrm{len}(w)\colon w \in \mathrm{Split}_i(P)\}.$$

For $i > 0$, we say that $w \in T_P$ is on level i in P (denoted by $l_P(w) = i$) if there exist $v, t \in T_P$ such that $v \subsetneq w \subseteq t$, $v \in \mathrm{Split}_{i-1}(P), t \in \mathrm{Split}_i(P)$. We say that $w \in T_P$ is on level 0 if $w \subseteq t$ where $t \in \mathrm{Split}_0(P)$ (see Figure 1.2). For $w \in T_P$, let $[w]_P = [w] \cap P$.

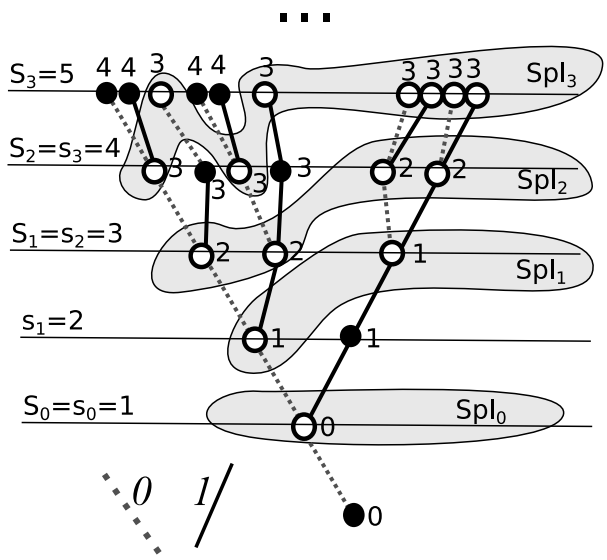

Figure 1.2: Branching points in a tree marked in white. The numbers in the labels of nodes are their levels.

Let P be a perfect set in 2^ω and $h_P\colon 2^\omega \to P$ be the homeomorphism given by the order isomorphism of $2^{<\omega}$ and $\mathrm{Split}(P)$. We call this homeomorphism the **canonical homeomorphism of** P (see Figure 1.3).

On 2^ω we consider the product **measure**, denoted here also by m, i.e. the measure such that $m([s]) = 1/2^k$ for any $s \in 2^k$, $k \in \omega$. Notice that this is the Haar measure in 2^ω, as it is invariant with respect to translations.

Finally, sometimes it seems convenient to use the lexicographical order $\leq_{\mathrm{lex}}$ on 2^ω, for $a, b \in 2^\omega$, $a \leq_{\mathrm{lex}} b$ if and only if $a = b$ or $a(n) = 0$, $b(n) = 1$ for $n \in \omega$ such that for all $m < n$, $a(m) = b(m)$. For $a, b \in 2^\omega$, let $[a, b) = \{x \in 2^\omega\colon a \leq_{\mathrm{lex}} x <_{\mathrm{lex}} b\}$. Moreover, for $a \in 2^\omega$, let $[a, \infty) = \{x \in 2^\omega\colon a \leq_{\mathrm{lex}} x\}$.

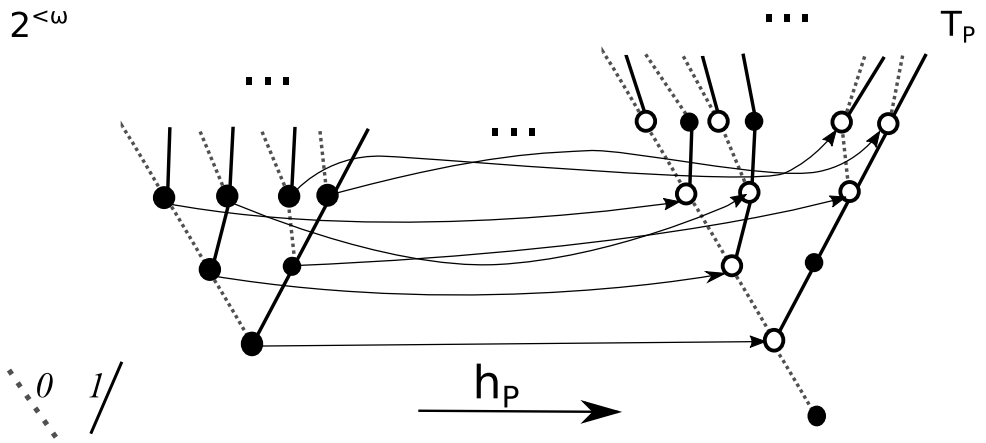

Figure 1.3: The order isomorphism of $2^{<\omega}$ and $\mathrm{Split}(P)$, which gives the canonical homeomorphism $h_P: 2^\omega \to P$ for an exemplary P.

1.2.2 The Baire space

The **Baire space** ω^ω is the countable product of discrete countable spaces. Similarly as in the Cantor space sets of form $[s] = \{f \in \omega^\omega : s \subseteq f\}$, $s \in \omega^{<\omega}$ constitute a basis of its topology. Notice also that $A \subseteq \omega^\omega$ is compact if and only if there exists $f \in \omega^\omega$ such that A is bounded by f , i.e. $a(n) \leq f(n)$ for all $n \in \omega$. In particular, every compact set is meagre. Therefore $K_\sigma \subseteq \boldsymbol{\mathcal{M}}$, where K_σ denotes the family of all countable unions of bounded sets. It is easy to see that K_σ is the family of all sets which are eventually bounded.

In the products of the form ω^S and $(\omega^S)^T$ we consider the partial orderings, denoted by the same symbol $\leq$, given by $x \leq y$ if $x(s) \leq y(s)$ for $x, y \in \omega^S$, $s \in S$, and $\phi \leq \psi$ if $\phi(t) \leq \phi(t)$ for $\phi, \psi \in (\omega^S)^T$, where $\phi(t), \psi(t) \in \omega^S$. We say that a function $o: X \to P$ from a set X into a partially ordered set P is **cofinal** if for every $p \in P$ there exists $x \in X$ such that $p \leq o(x)$.

Additionally, on ω^ω we define the **order of eventual domination**, as follows:

$$f \leq^* g \Leftrightarrow \exists_{m \in \omega} \forall_{n > m} f(n) \leq g(n).$$

One can also equip ω^ω with a measure. I will use a **measure** m such that

$$m([w]) = \prod_{i=0}^{\mathrm{len}(w)-1} \frac{1}{2^{w(i)+1}},$$

where $w \in \omega^{<\omega}$.

1.2.3 Equivalences in set theory of the real line

In many considerations in set theory of the real line it makes no difference which of the spaces: $\mathbb{R}$, the unit interval, 2^ω or ω^ω we take as the underlying space. This is because, if X, Y are any two of those spaces, there exists a homeomorphism $f: X \setminus Q_X \to Y \setminus Q_Y$, where Q_X, Q_Y are countable, and $f[N] \in \boldsymbol{\mathcal{N}}$ if and only if $N \in \boldsymbol{\mathcal{N}}$, $f[M] \in \boldsymbol{\mathcal{M}}$ if and only if $M \in \boldsymbol{\mathcal{M}}$. Indeed:

$\boldsymbol{I} \to \mathbb{R}$**:** Let $f: \boldsymbol{I} \setminus \{0,1\} \to \mathbb{R}$, with

$$f(x) = \frac{x - \frac{1}{2}}{x(x-1)}.$$

To see that $f[N] \in \boldsymbol{\mathcal{N}}$ for $N \in \boldsymbol{\mathcal{N}}$, notice that if $N \subseteq [x, x+1]$ with $x \in \mathbb{R}$, then $N \in \boldsymbol{\mathcal{N}}$ if and only if $f^{-1}[N] \in \boldsymbol{\mathcal{N}}$.

$2^\omega \to \boldsymbol{I}$**:** Let

$$Q = \left\{ \sum_{i=0}^{n} \frac{a_i}{2^{i+1}} : \langle a_i \rangle_{i \leq n} \in 2^{n+1}, n \in \omega \right\}.$$

Then $f: 2^\omega \setminus (\boldsymbol{Q}_0 \cup \boldsymbol{Q}_1) \to \boldsymbol{I} \setminus Q$ such that

$$f(a) = \sum_{i=0}^{\infty} \frac{a(i)}{2^{i+1}}$$

is the desired homeomorphism. Notice also that it preserves measure.

$\omega^\omega \to 2^\omega$**:** Let $f: \omega^\omega \to 2^\omega \setminus \boldsymbol{Q}_1$ be such that for $w \in \omega^\omega$,

$$f(w) = \underbrace{1 \ldots 1}_{w(0)} \,\hat{}\, 0 \,\hat{}\, \underbrace{1 \ldots 1}_{w(1)} \,\hat{}\, 0 \,\hat{}\, \ldots.$$

Then f is a homeomorphism. Moreover, it is measure preserving.

In particular, this justifies using the same notation $\boldsymbol{\mathcal{N}}$ for the ideals of null sets in all of those spaces, and $\boldsymbol{\mathcal{M}}$ for the ideals of meagre sets in all of those spaces.

1.2.4 The duality between measure and category

One can observe a stunning duality between measure and category (see [Bukovský, 2011], [Oxtoby, 1971]). Many results on the category side can be reformulated and proved on the measure side, and also the other way along. This was brilliantly noticed in [Sierpiński, 1934b], and [Sierpiński, 1934a]. Sierpiński discovered a principle which was later formulated in a stronger version

by Erdős, and which partially explains this duality. Sierpiński-Erdős Duality Theorem states that assuming CH, there exists $f\colon \mathbb{R} \to \mathbb{R}$, which is one-to-one, and $f = f^{-1}$ such that $f[A] \in \boldsymbol{\mathcal{N}}$ if and only if $A \in \boldsymbol{\mathcal{M}}$. Therefore (under CH), if P is a sentence constructed only out of the notion of null sets, meagre sets and pure set theory, then P holds if and only if the sentence with swapped notions of null and meagre sets holds. On the other hand, the above mapping f cannot be a measurable function (see [Oxtoby, 1971]). This implies that the Duality Theorem cannot be generalized to include the notion of measurability and the property of Baire.

But surprisingly, one can formulate a number of theorems with those notions which have their duals. Sometimes their proof are also very similar. But this is not always the case. The proofs of Fubini's Theorem, which states that if $A \subseteq \mathbb{R}^2$ is of measure zero, then

$$\{x \in \mathbb{R}\colon \{y \in \mathbb{R}\colon \langle x, y\rangle \in A\} \notin \boldsymbol{\mathcal{N}}\} \in \boldsymbol{\mathcal{N}},$$

and the proof of Kuratowski-Ulam Theorem, which states the same for meagre sets, are indeed far from identical (proofs of those theorems can be found, for example in [Oxtoby, 1971]). Furthermore, the duality may fail even more. For example, the category analogue of Egorov's Theorem (see below) is simply false.

1.2.5 Cardinal coefficients

Another such examples are the results concerning the **cardinal coefficients**.

A collection $\mathcal{I}$ of subsets of a set X is called a **family of thin sets** if $\{x\} \in \mathcal{I}$ for every $x \in X$, $X \notin \mathcal{I}$, and for every $A \in \mathcal{I}$ and $B \subseteq A$, $B \in \mathcal{I}$.

If $\mathcal{I}$ is a family of thin sets, let

$$\operatorname{add}(\mathcal{I}) = \min\left\{|\mathcal{A}|\colon \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\right\},$$

$$\operatorname{cof}(\mathcal{I}) = \min\left\{|\mathcal{A}|\colon \mathcal{A} \subseteq \mathcal{I} \wedge \forall_{A\in\mathcal{I}} \exists_{B\in\mathcal{A}} A \subseteq B\right\},$$

$$\operatorname{cov}(\mathcal{I}) = \min\left\{|\mathcal{A}|\colon \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\right\},$$

$$\operatorname{non}(\mathcal{I}) = \min\left\{|A|\colon A \subseteq X \wedge A \notin \mathcal{I}\right\}.$$

Obviously, we are interested in the above coefficients regarding the ideals $\boldsymbol{\mathcal{N}}$, and $\boldsymbol{\mathcal{M}}$. The two following cardinals also play an important role:

$$\mathfrak{b} = \min\{|\mathcal{A}|\colon \mathcal{A} \subseteq \omega^\omega \wedge \neg\exists_{f\in\omega^\omega} \forall_{g\in A} g \leq^* f\},$$

and

$$\mathfrak{d} = \min\{|\mathcal{A}|\colon \mathcal{A} \subseteq \omega^\omega \wedge \forall_{f\in\omega^\omega} \exists_{g\in A} f \leq^* g\},$$

which are called the **bounding and dominating number**, respectively.

The most important result regarding the cardinal coefficients of $\boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{M}}$ is Bartoszyński Theorem ([Bartoszyński, 1984]), which is yet another example of the fail of the duality. It states that $\mathrm{add}(\boldsymbol{\mathcal{N}}) \leq \mathrm{add}(\boldsymbol{\mathcal{M}})$ and $\mathrm{cof}(\boldsymbol{\mathcal{M}}) \leq \mathrm{cof}(\boldsymbol{\mathcal{N}})$. Taking into account other results due to Rothberger, Miller, Truss, Fremlin, and Bartoszyński, we receive the following (Table 1.1) well-known Cichoń diagram (see [Bartoszyński and Judah, 1995] and [Bukovský, 2011]).

$$\begin{array}{ccccccccc}
 & & \mathrm{cov}(\boldsymbol{\mathcal{N}}) & \leq & \mathrm{non}(\boldsymbol{\mathcal{M}}) & \leq & \mathrm{cof}(\boldsymbol{\mathcal{M}}) & \leq & \mathrm{cof}(\boldsymbol{\mathcal{N}}) & \leq & \mathfrak{c} \\
 & & & & \vee & & \vee & & & & \\
 & & \vee & & \mathfrak{b} & \leq & \mathfrak{d} & & \vee & & \\
 & & & & \vee & & \vee & & & & \\
\aleph_1 & \leq & \mathrm{add}(\boldsymbol{\mathcal{N}}) & \leq & \mathrm{add}(\boldsymbol{\mathcal{M}}) & \leq & \mathrm{cov}(\boldsymbol{\mathcal{M}}) & \leq & \mathrm{non}(\boldsymbol{\mathcal{N}}) & &
\end{array}$$

Table 1.1: Cichoń diagram.

In [Bartoszyński and Judah, 1995] one can find examples of models of ZFC in which the above cardinal coefficients have desired values $\leq \aleph_2$.

For further results in set theory of the real line the reader is referred to [Bukovský, 2011], [Bartoszyński and Judah, 1995] and [Cichoń et al., 1995].

1.3 Special subsets of the real line

The theory of special subsets of the real line is concerned with sets which are very small.

1.3.1 Special subsets related to measure and category

Among classes of special subsets of the real line, the classes of perfectly meager sets and universally null sets play an important role. A set is **perfectly meager** if it is meager relative to any perfect set, here denoted by $\mathrm{P}\boldsymbol{\mathcal{M}}$ (the concept first appeared in [Lusin, 1914]). A set is **universally null** if it is null with respect to any possible finite diffused Borel measure, denoted here by $\mathrm{U}\boldsymbol{\mathcal{N}}$ (this property was studied first in [Sierpiński and Marczewski, 1936]). Those classes were considered to be dual (see [Miller, 1984]), though some differences between them have been observed. For example, the class of universally null sets is closed under taking products (see [Miller, 1984]), but it is consistent with ZFC that this is not the case for perfectly meager sets (see [Pawlikowski, 1989] and [Recław, 1991a]).

In [Zakrzewski, 2000], P. Zakrzewski proved that two other earlier defined (see [Grzegorek, 1984] and [Grzegorek, 1980]) classes of sets, and smaller then

P$\boldsymbol{\mathcal{M}}$, coincide and are dual to U$\boldsymbol{\mathcal{N}}$. Therefore, he proposed to call this class the **universally meagre** sets (denoted by U$\boldsymbol{\mathcal{M}}$). A set $A \subseteq 2^\omega$ is universally meagre if every Borel isomorphic image of A in 2^ω is meagre.

In the paper [Nowik et al., 1998], the authors introduced a notion of perfectly meager sets in the transitive sense (denoted here by P$\boldsymbol{\mathcal{M}}'$), which turned out to be stronger than the classic notion of perfectly meager sets. A set $X \subseteq 2^\omega$ is **perfectly meagre in the transitive sense** if for any perfect set P, there exists an F_σ-set $F \supseteq X$ such that for any t, the set $(F + t) \cap P$ is a meager set relative to P. Further properties of P$\boldsymbol{\mathcal{M}}'$ sets were investigated in [Nowik, 1996], [Nowik and Weiss, 2000a], [Nowik and Weiss, 2001] and [Nowik and Weiss, 2000b], but still there are some open questions related to the properties of this class. This notion was motivated by its relation to the algebraic sums of sets belonging to different classes of small subsets of 2^ω, and by the obvious fact that a set $X \subseteq 2^\omega$ is perfectly meagre if and only if for any perfect set P, there exists an F_σ-set $F \supseteq X$ such that $F \cap P$ is meagre in P.

A set A is called **strongly null** (strongly of measure zero) if for any sequence of positive $\varepsilon_n > 0$, there exists a sequence of open sets $\langle A_n \rangle_{n\in\omega}$, with $\mathrm{diam} A_n < \varepsilon_n$ for $n \in \omega$, and such that $A \subseteq \bigcup_{n\in\omega} A_n$. I denote the class of such sets by S$\boldsymbol{\mathcal{N}}$. The idea was introduced for the first time in [Borel, 1919], and **Borel conjectured** that all S$\boldsymbol{\mathcal{N}}$ sets are countable. This hypothesis turned out to be independent from ZFC (see [Laver, 1976]). It is easy to see that a set A is strongly null if and only if for any sequence of positive $\varepsilon_n > 0$, there exists a sequence of open sets $\langle A_n \rangle_{n\in\omega}$, with $\mathrm{diam} A_n < \varepsilon_n$ for $n \in \omega$, and such that

$$A \subseteq \bigcap_{m\in\omega} \bigcup_{n>m} A_n.$$

Galvin, Mycielski and Solovay (in [Galvin et al., 1973]) proved that a set $A \in$ S$\boldsymbol{\mathcal{N}}$ (in 2^ω) if and only if for any meagre set B, there exists $t \in 2^\omega$ such that $A \cap (B + t) = \varnothing$. Therefore, one can consider a dual class of sets. A set A is called **strongly meagre** (strongly first category, denoted by S$\boldsymbol{\mathcal{M}}$) if for any null set B, there exists $t \in 2^\omega$ such that $A \cap (B + t) = \varnothing$.

We shall say that a set $L \subseteq 2^\omega$ is a **κ-Lusin set** if for any meagre set X, $|L \cap X| < \kappa$, but $|L| \geq \kappa$. An $\aleph_1$-Lusin set is simply called a **Lusin set**. This idea was introduced independently in [Lusin, 1914] and [Mahlo, 1913]. The existence of a Lusin set is independent from ZFC. It is easy to see that under CH such a set exists. Indeed, enumerate all closed nowhere dense sets and inductively take a point form a complement of each such set distinct from all the points chosen so far. The same can be easily done if $\mathrm{cov}(\boldsymbol{\mathcal{M}}) = \mathrm{cof}(\boldsymbol{\mathcal{M}}) = \aleph_1$ (see [Bukovský, 2011]).

Analogously, an uncountable set $S \subseteq 2^\omega$ is a **Sierpiński set** (introduced in [Sierpiński, 1924]) if for any null set X, $S \cap X$ is countable.

The above classes can be seen as two sequences of decreasing families of sets: for category and measure, as shown in the Table 1.2.

category	$\mathrm{P}\boldsymbol{\mathcal{M}}$	$\supseteq$	$\mathrm{U}\boldsymbol{\mathcal{M}}$	$\supseteq$	$\mathrm{P}\boldsymbol{\mathcal{M}}'$	$\supseteq$	$\mathrm{S}\boldsymbol{\mathcal{M}}$	$\supseteq$	Sierpiński sets
measure			$\mathrm{U}\boldsymbol{\mathcal{N}}$		$\supseteq$		$\mathrm{S}\boldsymbol{\mathcal{N}}$	$\supseteq$	Lusin sets

Table 1.2: Classes of special subsets of the real line.

Finally, a set A is called **null-additive** ($A \in \boldsymbol{\mathcal{N}}^*$) if for any null set X, $A+X$ is null. A set A is called **meagre-additive** ($A \in \boldsymbol{\mathcal{M}}^*$) if for any meagre set X, $A + X$ is meagre (see e.g. [Weiss, 2009] and [Bartoszyński and Judah, 1995]). Every null additive set is meagre additive which follows from the well-known **Shelah's characterization of null-additive sets** ([Shelah, 1995], see also [Bartoszyński and Judah, 1995][Theorem 2.7.18(3)]). If $Y \in \boldsymbol{\mathcal{N}}^*$ and $F\colon \omega \to \omega$ is any increasing function, then there exists a sequence $\langle I_n \rangle_{n\in\omega}$ such that $I_n \subseteq 2^{[F(n),F(n+1))}$, $|I_n| \leq n$ and

$$Y \subseteq \bigcup_{k\in\omega} \bigcap_{n\geq k} Y_n,$$

where $x \in Y_n$ if and only if $x{\restriction}[F(n), F(n+1)) \in I_n$ (see Figure 1.4), and the following **characterization of meagre-additive sets**. A set $X \in \boldsymbol{\mathcal{M}}^*$ ([Bartoszyński and Judah, 1995][Theorem 2.7.17]) if and only if for every increasing $f \in \omega^\omega$, there exists $g \in \omega^\omega$ and $y \in 2^\omega$ such that for all $x \in X$, there exists $m \in \omega$ such that for every $n > m$, there exists $k_n \in \omega$ with $g(n) \leq f(k_n) < f(k_n + 1) \leq g(n+1)$ and such that

$$x{\restriction}[f(k_n), f(k_n+1)) = y{\restriction}[f(k_n), f(k_n+1)).$$

1.3.2 Families of perfect subsets of 2^ω

A perfect set P will be called a **balanced perfect** set if $s_{i+1}(P) > S_i(P)$ for all $i \in \omega$. This definition generalizes the notion of uniformly perfect set, which can be found in [Brendle et al., 2008].

A perfect set P is **uniformly perfect** if for any $i \in \omega$, either $2^i \cap T_P \subseteq \mathrm{Split}(P)$ or $2^i \cap \mathrm{Split}(P) = \varnothing$. If additionally, in a uniformly perfect set P,

$$\forall_{w,v\in T_P} \left(\mathrm{len}(v) = \mathrm{len}(w) \Rightarrow \forall_{j\in\{0,1\}}(w^\frown j \in T_P \Rightarrow v^\frown j \in T_P)\right),$$

then P is called a **Silver perfect** set (see for example [Kysiak et al., 2007]).

A perfect set $P \subseteq 2^\omega$ is a **Laver perfect set** if there exists $s \in T_P$ such that for all $t \in T_P$, either $t \subseteq s$, or

$$\left|\left\{n \in \omega\colon t^\frown \underbrace{0\ldots0}_{n} {}^\frown 1 \in T_P\right\}\right| = \aleph_0.$$

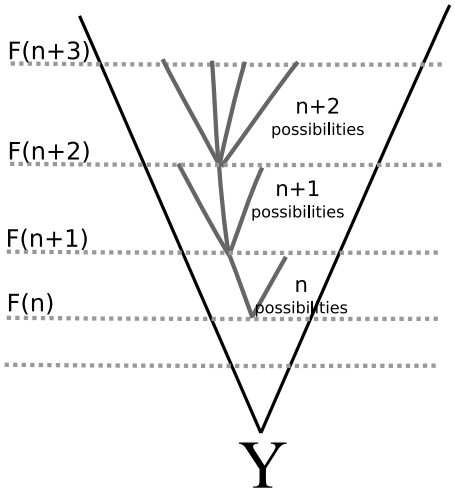

Figure 1.4: Shelah's characterization of null-additive sets.

Similarly, a perfect set $P \subseteq 2^\omega$ is a **Miller perfect set** if for every $s \in T_P$ there exists $t \in T_P$ such that $s \subseteq t$, and

$$\left|\left\{n \in \omega : t^\frown \underbrace{0 \ldots 0}_{n} {}^\frown 1 \in T_P\right\}\right| = \aleph_0.$$

A set $A \subseteq 2^\omega$ such that for any perfect set P there exists a perfect set $Q \subseteq P$ such that $A \cap Q = \varnothing$ is called an s_0**-set** ([Marczewski, 1935]).

We say that a set A is a v_0**-set** if for every Silver perfect set P, there exists a Silver perfect set $Q \subseteq P$ such that $Q \cap A = \varnothing$ (see [Kysiak et al., 2007]).

A set $A \subseteq 2^\kappa$ is l_0**-set** (respectively, m_0**-set**) if for every Laver (respectively, Miller) perfect set P, there exists a Laver (respectively, Miller) perfect set $Q \subseteq P$ such that $Q \cap A = \varnothing$ (see [Kysiak and Weiss, 2004]).

1.3.3 Other notions of special subsets

An open cover $\mathcal{U}$ of a set A is **proper** if $A \notin \mathcal{U}$. From now on we assume that all considered covers are proper.

An open cover $\mathcal{U}$ of a set A such that for any $C \in [A]^{<\omega}$ there exists $U \in \mathcal{U}$ such that $C \subseteq U$, is called ω**-cover**, and is called γ**-cover** if

$$A \subseteq \bigcup_{n\in\omega} \bigcap_{m\geq n} U_m.$$

The family of all ω-covers (respectively, γ-covers) of A is denoted by $\Omega(A)$ (respectively, $\Gamma(A)$). The family of all open covers of A is denoted by $\mathcal{O}(A)$.

The underlying set is often omitted in this notation if it is clear from the context.

Moreover, in Chapters 4 and 5 we consider in the generalized Cantor space the analogues of the following classes of special subsets of the real line (or the Cantor space) (see [Miller, 1984] and [Bukovský, 2011]):

set concentrated on a set C, i.e. a set A such that $A \setminus U$ is countable for every open U with $C \subseteq U$. Notice that every set concentrated on a countable set is $S\mathcal{N}$ ([Rothberger, 1939]),

λ-set, i.e. a set A such that every countable $B \subseteq A$ is a relative G_δ-set ([Kuratowski, 1933]). Every λ-set is perfectly meagre,

λ'-set, i.e. a set A such that for every countable B, $A \cup B$ is a λ-set. Obviously, every λ'-set is a λ-set,

σ-set, i.e. a set A such that any its relative F_σ-subset is also a relative G_δ-set ([Marczewski, 1930]),

Q-set, i.e. a set A such that every its subset is a relative F_σ-set ([Fleissner, 1978]). Every Q-set is a σ-set,

porous set, i.e. a set A such that $\operatorname{por}(x, A) > 0$ for all $x \in A$, where

$$\operatorname{por}(x, A) = \limsup_{r \to 0^+} \frac{\sup\{h \geq 0 \colon \exists_{y \in \mathbb{R}} [y - h, y + h] \subseteq [x - r, x + r] \setminus A\}}{r}$$

(see [Zajíček, 1987]),

γ-set i.e. a set A such that if for every open ω-cover $\mathcal{U}$, there exists $\mathcal{V} \subseteq \mathcal{U}$ which is a γ-cover ([Gerlits and Nagy, 1982]),

$SR^{\mathcal{N}}$, i.e. a set Y such that for every Borel set $H \subseteq 2^\omega \times 2^\omega$ such that $H_x = \{y \in 2^\omega \colon \langle x, y \rangle \in H\}$ is null for any $x \in 2^\omega$, $\bigcup_{x \in Y} H_x$ is null as well ([Bartoszyński and Judah, 1994]),

Ramsey null set, i.e a set A such that for any $n \in \omega$, $s \in 2^n$ and $S \in [\omega \setminus n]^\omega$, there exists $S' \in [S]^\omega$ such that $[s, S'] \cap A = \varnothing$, where if $s \in 2^n$, $n \in \omega$ and $S \in [\omega \setminus n]^\omega$, then

$$[s, S] = \{x \in 2^\omega \colon s^{-1}[\{1\}] \subseteq x^{-1}[\{1\}] \subseteq s^{-1}[\{1\}] \cup S \wedge |x^{-1}[\{1\}] \cap S| = \omega\}$$

(see [Plewik, 1986]),

T'-set, i.e. a set A such that there exists a sequence $\langle l_n\rangle_{n\in\omega}\in\omega^\omega$ such that for every increasing sequence $\langle d_n\rangle_{n\in\omega}\in\omega^\omega$ with $d_0=0$, there exists a sequence $\langle e_n\rangle_{n\in\omega}\in\omega^\omega$, and

$$H_n\in\left[2^{d_{e_n+1}\setminus d_{e_n}}\right]^{\leq l_{e_n}},$$

for all $n\in\omega$ such that

$$A\subseteq\{x\in 2^\omega\colon\forall_{m\in\omega}\exists_{n>m}x\restriction(d_{e_n+1}\setminus d_{e_n})\in H_n\}$$

(see [Nowik and Weiss, 2002] and also introduced in different context in [Repický, 1997]).

1.3.4 Selection principles

If $\mathcal{A}$ and $\mathcal{B}$ are families of covers of a topological space X, then X has $S_1(\mathcal{A},\mathcal{B})$ **principle** if for every sequence $\langle\mathcal{U}_n\rangle_{n\in\omega}\in\mathcal{A}^\omega$, there exists $\mathcal{U}=\{U_n\colon n\in\omega\}$ with $U_n\in\mathcal{U}_n$, for all $n\in\omega$ such that $\mathcal{U}\in\mathcal{B}$. X has $U_{<\omega}(\mathcal{A},\mathcal{B})$ **principle** if for every sequence $\langle\mathcal{U}_n\rangle_{n\in\omega}\in\mathcal{A}^\omega$ such that for every $n\in\omega$ if $\mathcal{W}\subseteq\mathcal{U}_n$ is finite, then $\mathcal{W}$ is not a cover, there exists $\langle\mathcal{U}_n\rangle_{n\in\omega}$ such that $\mathcal{U}_n\in[\mathcal{U}]^{<\omega}$, and $\{\bigcup\mathcal{U}_n\colon n\in\omega\}\in\mathcal{B}$. The covering principles were first systematically studied in [Scheepers, 1996].

It can be proven that a set X is a γ-set if and only if X satisfies $S_1(\Omega,\Gamma)$.

A set X is said to have the **Menger property** ([Menger, 1924]) if it satisfies $U_{<\omega}(\mathcal{O},\mathcal{O})$. It has the **Hurewicz property** ([Hurewicz, 1927]) if it satisfies $U_{<\omega}(\mathcal{O},\Gamma)$. Finally, it has the **Rothberger property**,([Rothberger, 1938]) if it satisfies $S_1(\mathcal{O},\mathcal{O})$.

1.3.5 The Lusin function

The **Lusin function** $\mathcal{L}\colon\omega^\omega\to 2^\omega$ is a continuous one-to-one function with measurable inverse such that if L is a Lusin set, then $\mathcal{L}[L]$ is perfectly meager. It was defined in [Lusin, 1933], and extensively described in [Sierpiński, 1934a]. To get the Lusin function we construct a system $\langle P_s\colon s\in\omega^{<\omega}\rangle$ of perfect sets such that for $s\in\omega^{<\omega}$ and $n,m\in\omega$:

(a)
$$\operatorname{diam}P_s\leq\frac{1}{2^{\operatorname{len}(s)}},$$

(b) $P_{s^\frown n}\subseteq P_s$ is nowhere dense in P_s,

(c) $\bigcup_{k\in\omega}P_{s^\frown k}$ is dense in P_s,

(d) if $n\neq m$, then $P_{s^\frown n}\cap P_{s^\frown m}=\varnothing$.

Next, for $x \in \omega^\omega$, we set $\mathcal{L}(x)$ to be the only point of $\bigcap_{n\in\omega} P_{x\restriction n}$. One can prove that $\mathcal{L}$ is a continuous and one-to-one function. Furthermore, if $Q \subseteq 2^\omega$ is a perfect set, then

$$\mathcal{L}^{-1}[\bigcup\{P_s\colon P_s \text{ is nowhere dense in } Q\}]$$

contains an open dense set (see also [Miller, 1984]). Moreover, it is easy to prove that $\mathcal{L}^{-1}$ is a function of the first Baire class.

1.4 Convergence and ideals

1.4.1 Convergence of a sequence of real functions and the Egorov's Theorem

Recall that a sequence $\langle f_n\rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ is **pointwise convergent** ($f_n \to f$) on a set $A \subseteq \boldsymbol{I}$ to a function $f\colon \boldsymbol{I} \to \boldsymbol{I}$ if for any $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$. In other words, if

$$\forall_{x\in A}\forall_{\varepsilon>0}\exists_{n\in\omega}\forall_{m\geq n}|f_m(x) - f(x)| \leq \varepsilon.$$

If

$$\forall_{\varepsilon>0}\exists_{n\in\omega}\forall_{m\geq n}\forall_{x\in A}|f_m(x) - f(x)| \leq \varepsilon,$$

we say that the sequence $\langle f_n\rangle_{n\in\omega}$ **converges uniformly** on a set $A \subseteq \boldsymbol{I}$ to f ($f_n \rightrightarrows f$).

The important part of considerations in this thesis is related to the well-known Egorov's Theorem. Let us recall that the classic **Egorov's Theorem** (originally proved in [Egorov, 1911], see also e.g. [Oxtoby, 1971]) states that given a sequence of Lebesgue measurable functions (we restrict our attention to the real functions $\boldsymbol{I} \to \boldsymbol{I}$) which is pointwise convergent on $\boldsymbol{I}$ and $\varepsilon > 0$, one can find a measurable set $A \subseteq \boldsymbol{I}$ with $m(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A.

This theorem plays a crucial role in this thesis, therefore I recall its proof (see e.g. [Oxtoby, 1971]). Let $\langle f_n\rangle_{n\in\omega}$ be a sequence of measurable functions $\boldsymbol{I} \to \boldsymbol{I}$ such that $f_n \to f$ on $\boldsymbol{I}$, and let $\varepsilon > 0$. Let

$$E_{n,k} = \left\{x \in \boldsymbol{I}\colon \exists_{i\geq n}|f_i(x) - f(x)| \geq \frac{1}{2^k}\right\}.$$

Notice that $E_{n,k}$ is a measurable set for every $n, k \in \omega$. Moreover, $E_{n+1,k} \subseteq E_{n,k}$, for any $n, k \in \omega$, and since $f_n \to f$, we get that $\bigcap_{n\in\omega} E_{n,k} = \varnothing$, for all $k \in \omega$. Therefore, for each $k \in \omega$, there exists $n_k \in \omega$ such that

$$m\left(E_{n_k,k}\right) \leq \frac{\varepsilon}{2^{k+1}}.$$

Let

$$B = \bigcup_{k\in\omega} E_{n_k,k},$$

and $A = \boldsymbol{I} \setminus B$. Then $|f_i(x) - f(x)| < 1/2^k$, for any $i \geq n_k$, and $x \in A$, because $A \subseteq \boldsymbol{I} \setminus E_{n_k,k}$. Thus, $f_n \rightrightarrows f$ on A, and $m(A) \geq 1 - \varepsilon$, because $m(B) \leq \varepsilon$.

There are also other notions of convergence of sequence of functions. A sequence $\langle f_n\rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ **converges quasi-normally** (introduced in [Császár and Laczkovich, 1975] and again in [Bukovská, 1991], see also [Bukovský, 2011]) on a set $A \subseteq \boldsymbol{I}$ ($f_n \xrightarrow{QN} f$) to a function $f\colon \boldsymbol{I} \to \boldsymbol{I}$ if there exists a sequence $\langle \varepsilon_i\rangle_{i\in\omega} \in (0,\infty)^\omega$ such that $\varepsilon_i \to 0$, and

$$\forall_{x\in A}\exists_{n\in\omega}\forall_{m\geq n}|f_m(x) - f(x)| \leq \varepsilon_m.$$

1.4.2 Ideals and convergence with respect to an ideal

We can also define a notion of convergence of a sequence of functions with respect to a given ideal I on ω. An **ideal** I on a set X is a collection of subsets of X such that

a) if $A \in I$, and $B \subseteq A$, then $B \in I$,

b) if $A, B \in I$, then $A \cup B \in I$,

c) $X \notin I$.

Given an ideal I on ω and a sequence $\langle x_n\rangle_{n\in\omega} \in \mathbb{R}^\omega$ we say that the sequence **converges** to a point $x \in \mathbb{R}$ **with respect to** I ($x_n \to_I x$) if for every $\varepsilon > 0$,

$$\{n \in \omega\colon |x_n - x| > \varepsilon\} \in I.$$

This idea was introduced in [Katětov, 1968], see also [Kostyrko et al., 2000], and [Nurray and Ruckle, 2000].

Notice that, the classical convergence is just the convergence with respect to the ideal $\mathrm{Fin} = [\omega]^{<\omega}$.

There is also an another way of introducing a notion of convergence with respect to an ideal I on ω. A sequence $\langle x_n\rangle_{n\in\omega} \in \mathbb{R}^\omega$ I^***-converges** to a point $x \in \mathbb{R}$ ($x_n \to_{I^*} x$) if there exists $C \in I$ such that the sequence $\langle x_n\rangle_{n\in(\omega\setminus C)}$ converges to x in the usual sense (see [Kostyrko et al., 2000]).

An ideal is **admissible** if it contains all the singletons. I will assume this about all the ideals discussed in this thesis.

An ideal I is **countably generated** (satisfies the chain condition) if there exists a sequence $\langle C_i\rangle_{i\in\omega}$ of elements of I such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$.

If $\rightsquigarrow$ and $\not\rightarrow$ are two notions of convergence, then we say that a sequence of functions converges with respect to $\rightsquigarrow \cup \not\rightarrow$ if it converges with respect to $\rightsquigarrow$ or with respect to $\not\rightarrow$.

If $A \subseteq \omega$ with $A \neq \omega$, then $\langle A \rangle = \mathcal{P}(A)$ is an ideal on ω.

If I, J are ideals on ω, then $I \vee J = \{A \cup B \colon A \in I \wedge B \in J\}$ is the least ideal containing I and J. If $\mathcal{I}$ is a family of ideals on ω, we denote by $\bigvee \mathcal{I}$ the least ideal containing $\bigcup \mathcal{I}$.

Given an ideal I on ω and a sequence $\langle I_n \rangle_{n \in \omega}$ of ideals of ω, we can consider an ideal $I\text{-}\prod_{n\in\omega} I_n$ on ω^2 called the **I-product of the sequence of ideals** $\langle I_n \rangle_{n \in \omega}$ and define it in the following way. For any $A \subseteq \omega^2$,

$$A \in I\text{-}\prod_{n\in\omega} I_n \Leftrightarrow \{n \in \omega \colon A_{(n)} \notin I_n\} \in I,$$

where $A_{(n)} = \{m \in \omega \colon \langle n, m \rangle \in A\}$ (see [Mrożek, 2010]). If $I_n = J$ for any $n \in \omega$, we usually denote $I\text{-}\prod_{n\in\omega} I_n$ as $I \times J$.

The ideal $\sum_{n\in\omega} I_n$ (called **the sum of a sequence of ideals** $\langle I_n \rangle_{n \in \omega}$), where $\langle I_n \rangle_{n \in \omega}$ is a sequence of ideals of ω, is an ideal on $\omega \times \omega$ defined in the following way. For any $A \subseteq \omega^2$,

$$A \in \sum_{n\in\omega} I_n \Leftrightarrow \forall_{n\in\omega} A_{(n)} \in I_n.$$

Finally, given an ideal I on ω and a sequence $\langle I_n \rangle_{n \in \omega}$ of ideals on ω, we consider an ideal

$$I\text{-}\underline{\lim}_{n\in\omega} I_n = \{A \subseteq \omega \colon \{n \in \omega \colon A \notin I_n\} \in I\}.$$

on ω called the **I-limit of the sequence of ideals** $\langle I_n \rangle_{n \in \omega}$.

Fix a bijection $b \colon \omega^2 \to \omega$ and a bijection $a_\beta \colon \omega \to \beta \setminus \{0\}$ for any limit $\beta < \omega_1$. The **ideals $\mathbf{Fin}^\alpha$**, $\alpha < \omega_1$, are defined inductively (see [Mrożek, 2010]) in the following way. Let $\mathrm{Fin}^1 = \mathrm{Fin}$ be the ideal of finite subsets of ω. We set

$$\mathrm{Fin}^{\alpha+1} = \{b[A] \colon A \in \mathrm{Fin} \times \mathrm{Fin}^\alpha\},$$

and for limit $\beta < \omega_1$, let

$$\mathrm{Fin}^\beta = \left\{ b[A] \colon A \in \mathrm{Fin}\text{-}\prod_{i\in\omega} \mathrm{Fin}^{a_\beta(i)} \right\}.$$

An ideal I is **analytic** if $\{\chi_C \colon C \in I\}$ is analytic as a subset of 2^ω.

Finally, an ideal I is a **P-ideal** if for any sequence $\langle A_i \rangle_{i\in\omega} \in I^\omega$ of mutually disjoint sets, there exists a sequence $\langle B_i \rangle_{i\in\omega}$ such that $A_i \vartriangle B_i$ is finite for all $i \in \omega$, and $\bigcup_{i\in\omega} B_i \in I$.

By the well-known **result of Solecki** ([Solecki, 1999]) if I is an analytic P-ideal, then $I = \mathrm{Exh}(\phi)$, where ϕ is a **lower semicontinuous submeasure**. A function $\phi\colon 2^\omega \to [0,\infty]$ is a lower semicontinuous submeasure (see also [Mrożek, 2009]) if it satisfies the following conditions:

(1) $\phi(\varnothing) = 0$,

(2) $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, for any $A, B \subseteq \omega$,

(3) $\phi(A) = \lim_{n\to\omega} \phi(A \cap n)$, for any $A \subseteq \omega$,

and,

$$\mathrm{Exh}(\phi) = \{A \subseteq \omega\colon \lim_{n\to\infty} \phi(A \setminus n) = 0\}.$$

We also consider the following partial ordering of ideals on ω:

Rudin-Keisler partial ordering, $I \leq_{RK} J$ if there exists $g\colon \omega \to \omega$ such that $I = \{A \subseteq \omega\colon g^{-1}[A] \in J\}$,

Rudin-Blass partial ordering, $I \leq_{RB} J$ if there exists $g\colon \omega \to \omega$ which is finite-to-one such that $I = \{A \subseteq \omega\colon g^{-1}[A] \in J\}$.

1.4.3 Convergence of a sequence of functions with respect to an ideal

Analogously to the classical convergence, we get different notions of convergence of a sequence $\langle f_n\rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ with respect to an ideal I on ω, which were introduced in [Balcerzak et al., 2007] and [Das and Chandra, 2013]:

pointwise ideal, $f_n \to_I f$ if and only if

$$\forall_{\varepsilon>0}\forall_{x\in A}\{n \in \omega\colon |f_n(x) - f(x)| \geq \varepsilon\} \in I,$$

quasi-normal ideal, $f_n \xrightarrow{QN}_I f$ if and only if there exists a sequence $\langle \varepsilon_i\rangle_{i\in\omega} \in (0,\infty)^\omega$ such that $\varepsilon_i \to_I 0$ and

$$\forall_{x\in A}\{n \in \omega\colon |f_n(x) - f(x)| \geq \varepsilon_n\} \in I,$$

uniform ideal, $f_n \rightrightarrows_I f$ if and only if

$$\forall_{\varepsilon>0}\exists_{B\in I}\forall_{x\in A}\{n \in \omega\colon |f_n(x) - f(x)| \geq \varepsilon\} \subseteq B.$$

The quasi-normal convergence with respect to an ideal I is also sometimes called I-equal convergence.

Yet another idea is to use the dual filter ($F = \{\omega \setminus C : C \in I\}$) to define convergence notions. In this approach we get the following notions of convergence of a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ on $A \subseteq \boldsymbol{I}$ (see [Das et al., 2014]):

I^*-pointwise, $f_n \to_{I^*} f$ if and only if for all $x \in A$, there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i}(x) \to f(x)$,

I^*-quasi-normal, $f_n \xrightarrow{QN}_{I^*} f$ if and only if there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \xrightarrow{QN} f$ on A,

I^*-uniform, $f_n \rightrightarrows_{I^*} f$ if and only if there exists $M = \{m_i : i \in \omega\} \subseteq \omega$, $m_{i+1} > m_i$ for $i \in \omega$ such that $\omega \setminus M \in I$ and $f_{m_i} \rightrightarrows f$ on A.

The above notions can be further generalized. Let $J \subseteq I$ be ideals. If $A \subseteq \boldsymbol{I}$ and $\langle f_n \rangle_{n\in\omega}$ is a sequence of functions $\boldsymbol{I} \to \boldsymbol{I}$, we have the following notions of convergence (see [Mačaj and Sleziak, 2000], [Repický, 2017]).

(J, I)-pointwise, $f_n \to_{J,I} f$ if and only if for all $x \in A$, there exists $N \in I$ such that for all $\varepsilon > 0$,

$$\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon\} \in J \vee \langle N \rangle,$$

(J, I)-quasi-normal, $f_n \xrightarrow{QN}_{J,I} f$ if and only if there exists $N \in I$ and a sequence $\langle \varepsilon_n \rangle_{n\in\omega}$ such that $\varepsilon_n \to_{J\vee\langle N\rangle} 0$, and for all $x \in A$,

$$\{n \in \omega : |f_n(x) - f(x)| \geq \varepsilon_n\} \in J \vee \langle N \rangle.$$

(J, I)-uniform, $f_n \rightrightarrows_{J,I} f$ if and only if there exists $N \in I$ and $f_n \rightrightarrows_{J\vee\langle N\rangle} f$ on A.

Notice that $\to_{I,I} = \to_I$, $\xrightarrow{QN}_{I,I} = \xrightarrow{QN}_I$, and $\rightrightarrows_{I,I} = \rightrightarrows_I$. Moreover, $\to_{\mathrm{Fin},I} = \to_{I^*}$, $\xrightarrow{QN}_{\mathrm{Fin},I} = \xrightarrow{QN}_{I^*}$, and $\rightrightarrows_{\mathrm{Fin},I} = \rightrightarrows_{I^*}$.

To avoid confusion notice also that the above notions are different from the notion of $(J - I)$-quasi-normal convergence which is considered in [Filipów and Staniszewski, 2014] and [Filipów and Staniszewski, 2015].

Therefore we have the following implications between notions of convergence for ideals $J \subseteq I$.

$$\begin{array}{ccccccc}
\to_{\mathrm{Fin}} & \Rightarrow & \to_{I^*} & \Rightarrow & \to_{J,I} & \Rightarrow & \to_I \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\xrightarrow{QN}_{\mathrm{Fin}} & \Rightarrow & \xrightarrow{QN}_{I^*} & \Rightarrow & \xrightarrow{QN}_{J,I} & \Rightarrow & \xrightarrow{QN}_I \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\rightrightarrows_{\mathrm{Fin}} & \Rightarrow & \rightrightarrows_{I^*} & \Rightarrow & \rightrightarrows_{J,I} & \Rightarrow & \rightrightarrows_I
\end{array}$$

Let I be an analytic P-ideal. Fix a lower continuous submeasure ϕ such that $I = \mathrm{Exh}(\phi)$. We have the following notions of convergence (see [Mrożek, 2009]) of a sequence $\langle f_n \rangle_{n\in\omega}$ of functions $\boldsymbol{I} \to \boldsymbol{I}$ on a set $A \subseteq \boldsymbol{I}$:

pointwise ideal, $f_n \to_I f$ if and only if

$$\forall_{\varepsilon>0}\forall_{x\in A}\exists_{k\in\omega}\phi(\{n \in \omega\colon |f_n(x) - f(x)| \geq \varepsilon\} \setminus k) < \varepsilon,$$

equi-ideal, $f_n \twoheadrightarrow_I f$ if and only if

$$\forall_{\varepsilon>0}\exists_{k\in\omega}\forall_{x\in A}\phi(\{n \in \omega\colon |f_n(x) - f(x)| \geq \varepsilon\} \setminus k) < \varepsilon,$$

uniform ideal, $f_n \rightrightarrows_I 0$ if and only if

$$\forall_{\varepsilon>0}\exists_{k\in\omega}\phi\left(\left\{n \in \omega\colon \sup_{x\in A} |f_n(x) - f(x)| \geq \varepsilon\right\} \setminus k\right) < \varepsilon.$$

It was proved in [Mrożek, 2009] that these notions of convergence are independent from the submeasure representation of I. Moreover, the pointwise ideal and uniform ideal convergences can be expressed without the notion of a submeasure and they coincide with the notions of ideal convergences defined above for any ideal I on ω. Obviously, $f_n \rightrightarrows_I 0 \Rightarrow f_n \twoheadrightarrow_I 0 \Rightarrow f_n \to_I 0$.

1.4.4 Generalizations of Egorov's Theorem

Given two notions of convergence with respect to an ideal, we can ask whether the classic Egorov's Theorem holds for those two notions of convergence in the sense of whether the weaker convergence implies the stronger convergence on a subset of arbitrarily large measure. The answer may often be negative as in the case of uniform and pointwise convergence for many analytic P-ideals (see [Mrożek, 2009, Theorem 3.4]). But one can also consider other types of convergence, e.g. equi-ideal convergence. And, for example, in the case of analytic P-ideal so called weak Egorov's Theorem for ideals (between equi-ideal and pointwise ideal convergence) was proved by N. Mrożek (see [Mrożek, 2009, Theorem 3.1]).

The measuability assumption in this theorem seem to play an important role. Actually, it is interesting whether one can drop the assumption on measurability of the functions in the classic Egorov's Theorem. A statement which says that given any sequence of functions $\boldsymbol{I} \to \boldsymbol{I}$ which is pointwise convergent and $\varepsilon > 0$, there exists a set $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$ such that the sequence converges uniformly on A, is called the **generalized Egorov's statement**. T. Weiss in his manuscript (see [Weiss, 2004]) proved that it is independent

from ZFC, and this fact was used in [Di Biase et al., 2007]. Then R. Pinciroli studied the method of T. Weiss more systematically (see [Pinciroli, 2006]). For example, he related it to cardinal coefficients: $\mathrm{non}(\mathcal{N})$, $\mathfrak{b}$ and $\mathfrak{d}$. In particular, he proved that $\mathrm{non}(\mathcal{N}) < \mathfrak{b}$ implies that the generalized Egorov's statement holds, but if, for example, $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$, then it fails.

1.4.5 Special subsets related to the notion of convergence

One can define notions of special subsets related to convergence of sequences functions (introduced in [Bukovský et al., 1991], see also [Bukovský, 2011][Chapter 8.3]). In this thesis we consider generalizations of following notions:

QN-set, i.e. a set $A \subseteq \boldsymbol{I}$ such that if $\langle f_n \rangle_{n\in\omega}$ is a sequence of continuous functions $A \to \boldsymbol{I}$ such that $f_n \to 0$ on A, then $f_n \xrightarrow{QN} 0$ on A,

weak QN-set (wQN-set), i.e. a set $A \subseteq \boldsymbol{I}$ such that if $\langle f_n \rangle_{n\in\omega}$ is a sequence of continuous functions $A \to \boldsymbol{I}$ such that $f_n \to 0$ on A, then there exists an increasing sequence $\langle k_n \rangle_{n\in\omega} \in \omega^\omega$ such that $f_{k_n} \xrightarrow{QN} 0$ on A,

mQN-set, i.e. a set $A \subseteq \boldsymbol{I}$ such that if $\langle f_n \rangle_{n\in\omega}$ is a sequence of continuous functions $A \to \boldsymbol{I}$ such that $f_n \to 0$ on A, and for all $x \in A$, $f_{n+1}(x) < f_n(x)$ for all $n \in \omega$, then $f_n \xrightarrow{QN} 0$ on A.

Such properties of a set A can be translated to covering properties of A and properties of sequences in $C_p(A)$ (the space of continuous real functions over A with the topology of pointwise convergence), e.g. A is a QN-set if and only if for every sequence of γ-covers $\langle \mathcal{U}_n : n \in \omega \rangle$, there exist finite sets $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup_{n\in\omega}(\mathcal{U}_n \setminus \mathcal{V}_n)$ is a γ-cover. On the other hand, the above property of A holds if and only if for any $x \in C_p(A)$ and any sequence $\left\langle \langle x_{n,m} \rangle_{m\in\omega} \right\rangle_{n\in\omega}$ such that for all $n \in \omega$, $\lim_{m\to\infty} x_{n,m} = x$, there exists a sequence $\langle y_m \rangle_{m\in\omega}$ such that $\lim_{m\to\infty} y_m = x$ and for all $n \in \omega$, $\{x_{n,m} : m \in \omega\} \subseteq^* \{y_m : m \in \mathbb{N}\}$, where $X \subseteq^* Y$ means that all but finite number of elements of X are in Y. For more details, see e.g. [Bukovský, 2011].

Analogous notions for ideal convergence of real functions can also be defined. They were studied in [Das and Chandra, 2013], [Šupina, 2016] and [Chandra, 2016].

1.5 Introducing the generalized Cantor space 2^κ

In this thesis I consider the generalized Cantor space 2^κ for an infinite cardinal $\kappa > \omega$ and study special subsets of this space. In the recent years the theory of the generalized Cantor and Baire spaces was extensively developed (see, e.g. [Lücke et al., 2016], [Friedman et al., 2014], [Friedman and Laguzzi, 2014], [Laguzzi, 2012], [Laguzzi, 2015], [Shelah, 2012], [Shelah and Cohen, 2016], [Friedman, 2010], [Friedman, 2013], [Friedman, 2014] and many other). An important part of the research in this subject is an attempt to transfer the results in set theory of the real line to those spaces (the list of open questions can be found in [Laguzzi et al., 2016]). Despite the rapid development in this theory, the author is not aware of any significant research in the subject of special subsets in 2^κ. Known results are related mainly to the ideal of strongly null sets (see [Halko, 1996] and [Halko and Shelah, 2001]).

Throughout this thesis, unless it is stated otherwise, I assume that κ is an uncountable regular cardinal number and $\kappa > \omega$.

1.5.1 Preliminaries

We consider the space 2^κ, called **κ-Cantor space (or the generalized Cantor space)**, endowed with so called bounded topology with basis $\{[x]\colon x \in 2^{<\kappa}\}$, where for $x \in 2^{<\kappa}$,

$$[x] = \{f \in 2^\kappa \colon f \restriction \mathrm{dom} x = x\}.$$

If we additionally assume that $\kappa^{<\kappa} = \kappa$, this basis has cardinality κ. This assumption proves to be very convenient when considering the generalized Cantor space, and is assumed throughout this thesis, unless stated otherwise (see e.g. [Friedman et al., 2014]).

The space 2^κ will also be treated as a vector space over $\mathbb{Z}_2$. In particular, for $A, B \subseteq 2^\kappa$, let $A + B = \{t + s \colon t \in A, s \in B\}$. Let $\mathbf{0} \in 2^\kappa$ be such that $\mathbf{0}(\alpha) = 0$ for all $\alpha < \kappa$, let $\mathbf{1} \in 2^\kappa$ be such that $\mathbf{1}(\alpha) = 1$ for all $\alpha < \kappa$, and let $\boldsymbol{Q} = \{x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} x(\beta) = 0\}$.

Notice that if $x \in 2^\alpha$, with $\alpha < \kappa$, then

$$2^\kappa \setminus [x] = \bigcup_{\beta<\alpha} \left[x \restriction \beta \frown (x(\beta) + 1)\right].$$

So, $2^\kappa \setminus [x]$ is also open. Therefore, the basis defined above consists of clopen sets. Notice also that an intersection of less than κ of basic sets is a basic set or the empty set. Therefore, an intersection of less than κ open sets is still open. Notice also that there are 2^κ closed sets in this space.

Additionally, under the assumption $\kappa^{<\kappa} = \kappa$, there exists a family $\mathcal{F}$ of subsets of κ such that $|\mathcal{F}| = 2^\kappa$, and for all $A, B \in \mathcal{F}$, $|A \cap B| < \kappa$ if $A \neq B$. Indeed, let $b: 2^{<\kappa} \to \kappa$ be a bijection. Then

$$\mathcal{F} = \{b[\{x\restriction\alpha : \alpha < \kappa\}] : x \in 2^\kappa\}$$

is such a family.

A T_1 topological space is said to be **κ-additive** if for any $\alpha < \kappa$, an intersection of an α-sequence of open subsets of this space is open. Various properties of κ-additive spaces were considered by R. Sikorski in [Sikorski, 1950]. The generalized Cantor space is an example of a κ-additive space. It is also easy to see that every κ-additive topological space X with clopen basis of cardinality κ, is homeomorphic to a subset of 2^κ.

A set $T \subseteq 2^{<\kappa}$ will be called a **tree** if for all $t \in T$ and $\alpha < \mathrm{len}(t)$, $t\restriction\alpha \in T$ as well. A branch in a tree is a maximal chain in it. For a tree T, let

$$[T]_\kappa = \{x \in 2^\kappa : \forall_{\alpha<\kappa} x\restriction\alpha \in T\}.$$

It is easy to see that A is closed if and only if $A = [T]_\kappa$ for some tree $T \subseteq 2^{<\kappa}$. Indeed, if $A = [T]_\kappa$ and T is a tree, then if $x \notin A$, there exists $\alpha < \kappa$ such that $x\restriction\alpha \notin T$. Therefore $[x\restriction\alpha] \subseteq 2^\kappa \setminus A$, so A is closed. On the other hand, if A is closed, let $T = \{x\restriction\alpha : x \in A, \alpha < \kappa\}$. Then, if $a \in 2^\kappa$, and $a\restriction\alpha \in T$ for all $\alpha < \kappa$, we have that $a \in A$, since A is closed. A tree $T \subseteq 2^{<\kappa}$ such that $A = [T]_\kappa$ is denoted by T_A.

A node $s \in T \subseteq 2^{<\kappa}$ will be called a **branching point** of T if $s^\frown 0, s^\frown 1 \in T$. The set of all branching points of a tree T is denoted by $\mathrm{Split}(T)$. For $\alpha < \kappa$, $t \in \mathrm{Split}_\alpha(T)$ if $\langle \{s \subsetneq t : s \in \mathrm{Split}(T)\}, \subseteq\rangle$ is order isomorphic with α.

The family of **κ-Borel sets** is the smallest family of subsets of 2^κ containing all open sets and closed under complementation, and under taking intersections of size κ. The family of such sets is denoted here by $\mathcal{B}_\kappa$. A function $f: 2^\kappa \to 2^\kappa$ is κ-measurable if for every $s \in 2^{<\kappa}$, $f^{-1}[[s]] \in \mathcal{B}_\kappa$.

We say that a set is **κ-meagre** if it is a union of at most κ nowhere dense (in the bounded topology) sets. Notice also that the generalization of theBaire category theorem holds, namely 2^κ is not κ-meagre (see [Sikorski, 1950, Theorem xv]). The family of all κ-meagre sets in 2^κ is denoted by $\boldsymbol{\mathcal{M}}_\kappa$.

Notice also that if $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ is a sequence of points in 2^κ such that for all $\xi < \kappa$, there exists $\delta_\xi < \kappa$ such that for all $\delta_\xi \leq \alpha, \beta < \kappa$, $x_\alpha\restriction\xi = x_\beta\restriction\xi$, then there exists $x \in 2^\kappa$ which is a (topological) limit of $\langle x_\alpha\rangle_{\alpha<\kappa}$ (i.e. for every open set U with $x \in U$, there exists $\xi < \kappa$ such that for all $\xi < \alpha < \kappa$, $x_\alpha \in U$). Indeed, take

$$x = \bigcup_{\xi<\kappa} x_{\delta_\xi}\restriction\xi.$$

Obviously, if $C \subseteq 2^\kappa$ is closed, and $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ is a sequence of points of C with limit $x \in 2^\kappa$, then $x \in C$ as well. Therefore, if $\langle C_\alpha \rangle_{\alpha\in\kappa}$ is a sequence of non-empty closed sets such that $C_\beta \subseteq C_\alpha$, when $\alpha < \beta < \kappa$ such that there exists an increasing sequence $\langle \xi_\alpha \rangle_{\alpha\in\kappa} \in \kappa^\kappa$ and $\langle s_\alpha \rangle_{\alpha\in\kappa} \in (2^{<\kappa})^\kappa$ such that $C_\alpha \subseteq [s_\alpha]$ and $s_\alpha \in 2^{\xi_\alpha}$, then there exists $x \in 2^\kappa$ such that

$$\bigcap_{\alpha<\kappa} C_\alpha = \{x\}.$$

Indeed, let $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ be any sequence of points such that $x_\alpha \in C_\alpha$, for any $\alpha \in \kappa$, then there exists a limit of this sequence x. But $x \in C_\alpha$ for any $\alpha < \kappa$, because $\langle x_\beta \rangle_{\alpha\leq\beta<\kappa}$ is a sequence of points in C_α.

Obviously, spaces $2^\kappa \times 2^\kappa$ and 2^κ are homeomorphic, and the canonical homeomorphism between them is given by the canonical well-ordering of $2\times\kappa$, $g\colon 2\times\kappa \to \kappa$.

1.5.2 Cardinal coefficients in 2^κ

A statement $2^\kappa = \kappa^+$ is the **Continuum Hypothesis for** κ and denoted by CH_κ.

Recall that $\diamondsuit_\kappa(E)$ for $E \subseteq \kappa$ is the following principle: there exists a sequence $\langle S_\alpha \rangle_{\alpha\in E}$ such that $S_\alpha \subseteq \alpha$ for all $\alpha \in E$, and the set

$$\{\alpha \in E\colon X \cap \alpha = S_\alpha\}$$

is stationary subset of κ for every $X \subseteq \kappa$ (see e.g. [Jech, 2006][Chapter 23]). The principle $\diamondsuit_\kappa(\kappa)$ is simply denoted by $\diamondsuit_\kappa$ (and called the **diamond principle for** κ).

If $f, g \in \kappa^\kappa$, then we write $f \leq^\kappa g$ if there exists $\alpha < \kappa$ such that for all $\beta < \kappa$ if $\beta > \alpha$, then $f(\beta) \leq g(\beta)$. In this case we say that f is **eventually dominated** by g.

Analogously, as in the case of ω^ω one can define cardinals related to the ordering $\leq^\kappa$. The two following cardinals also play an important role:

$$\mathfrak{b}_\kappa = \min\{|\mathcal{A}|\colon \mathcal{A} \subseteq \kappa^\kappa \wedge \neg\exists_{f\in\kappa^\kappa} \forall_{g\in A} g \leq^\kappa f\},$$

and

$$\mathfrak{d}_\kappa = \min\{|\mathcal{A}|\colon \mathcal{A} \subseteq \kappa^\kappa \wedge \forall_{f\in\kappa^\kappa} \exists_{g\in A} f \leq^\kappa g\},$$

which are called the **bounding and dominating number for** κ, respectively. Obviously, $\kappa < \mathfrak{b}_\kappa \leq \mathfrak{d}_\kappa \leq 2^\kappa$.

1.5.3 κ-Compactness

Not all the results of theory of the real line can be easily generalized to the case of 2^κ. One of the main obstacles is the notion of compactness. We shall say that a topological space X is **κ-compact** (or κ-Lindelöf) if every open cover of X has a subcover of cardinality less than κ (see [Monk and Scott, 1964], [Hung and Negrepontis, 1973]). Obviously, the Cantor space 2^ω is ω-compact (i.e. compact in the traditional sense). But it is not always the case that 2^κ is κ-compact. Recall that a cardinal number κ is weakly compact if it is uncountable and for every two-colour colouring of the set of all two-element subsets of κ, there exists a set $H \subseteq \kappa$ of cardinality κ, which is homogeneous (every two-element subset of H have the same colour in the considered colouring) (see [Jech, 2006]). Recall that every weakly compact cardinal is strongly inaccessible. Actually, the generalized Cantor space 2^κ is κ-compact if and only if κ is a weakly compact cardinal (see [Monk and Scott, 1964]).

And there is even more to that. If κ is not weakly compact, then all reasonable κ-additive spaces are homeomorphic. Precisely, if κ is not weakly compact, then every completely regular κ-additive topological space X without isolated points such that there exists a family of open sets $\mathcal{B}$ in X such that:

(1) the family of all intersections of less than κ sets from $\mathcal{B}$ is a basis of the topology of X,

(2) if $\mathcal{C} \subseteq \mathcal{B}$ is such that for any $n \in \omega$ and any $C_0, C_1, \ldots C_n \in \mathcal{C}$, $\bigcap_{i=0}^{n} C_n \neq \varnothing$, then $\bigcap \mathcal{C} \neq \varnothing$,

(3) $|\mathcal{B}| \leq 2^{<\kappa}$,

(4) $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$, where for any $\alpha < \kappa$, $\mathcal{B}_\alpha$ is a partition of X into open sets,

is homeomorphic to 2^κ (see [Hung and Negrepontis, 1974, Theorem 2.3] and [Hung, 1972]). On the other hand, if κ is weakly compact, then a completely regular κ-additive space X without isolated points is homeomorphic to 2^κ if and only if there exists a family of open sets $\mathcal{B}$ in X satisfying conditions (1)-(3) and also:

(4') $\mathcal{B} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$, where for any $\alpha < \kappa$, $\mathcal{B}_\alpha$ is a partition of X into open sets, and $|\mathcal{B}_\alpha| < \kappa$.

I will refer to the above theorem as the **Hung-Negrepontis characterization**. In particular, the generalized Cantor space 2^κ and the generalized Baire spaces κ^κ are homeomorphic if and only if κ is not a weakly compact cardinal.

Also notice that every κ-additive regular space is zero-dimensional (see [Sikorski, 1950]). Indeed, if $\langle G_n \rangle_{n\in\omega}$ is a sequence of open sets such that $\mathrm{cl} G_{n+1} \subseteq G_n$, for all $n \in \omega$, then $\bigcap_{n\in\omega} G_n$ is a clopen set.

1.5.4 Perfect sets in 2^κ

A set $P \subseteq 2^\kappa$ is a **perfect set** if it is closed and has no isolated points. A tree $T \subseteq 2^{<\kappa}$ is **perfect** if for any $t \in T$, there exists $s \in T$ such that $t \subseteq s$ and $s \in \mathrm{Split}(T)$. Notice that a set $P \subseteq 2^\kappa$ is perfect if and only if T_P is a perfect tree.

A perfect tree T will be called **κ-perfect** if for every limit $\beta < \kappa$, and $t \in 2^\beta$ such that $t{\restriction}\alpha \in T$, we have $t \in T$. Notice that every κ-perfect tree is order-isomorphic with $2^{<\kappa}$. A set $P \subseteq 2^\kappa$ is **κ-perfect** if $P = [T]_\kappa$ for a κ-perfect tree T. Obviously, every κ-perfect set is perfect. On the other hand, the converse does not hold.

Notice that if $x \in [T]_\kappa$, and T is a κ-perfect tree, then for all $\alpha < \kappa$, $\{x{\restriction}\beta : \beta < \kappa\} \cap \mathrm{Split}_\alpha(T) \neq \varnothing$.

For example if $s \in 2^\omega$ is such that $s(n) = 0$ for all $n \in \omega$, then $2^\kappa \setminus [s]$ is a perfect set but is not κ-perfect.

Another major difference between 2^κ and 2^ω is the perfect set property of analytic set. In 2^ω every uncountable analytic set contains a perfect set. On the other hand, the generalization of this theorem for 2^κ may not be true even for closed sets. There may even exist a perfect set which do not contain a κ-perfect set. Recall that a tree $T \subseteq 2^{<\kappa}$ is a **κ-Kurepa** tree if:

(1) $|[T]_\kappa| > \kappa$,

(2) if α is uncountable, then $|T \cap 2^\alpha| \leq |\alpha|$.

If T is a κ-Kurepa tree, then $[T]_\kappa$ is an example of a closed set of cardinality bigger than κ, with no κ-perfect subsets (see e.g. [Laguzzi et al., 2016, Friedman, 2010]).

Fortunately, one can see that every κ-comeagre set contains a κ-perfect set. Indeed, if $G = \bigcup_{\alpha<\kappa} G_\alpha$ with G_α nowhere dense, we choose by induction $\langle t_s \rangle_{s \in 2^{<\kappa}}$ such that $t_s \in 2^{<\kappa}$, and for $s, s' \in 2^{<\kappa}$, $s \subsetneq s'$ if and only if $t_s \subsetneq t_{s'}$. Indeed, let $t_\varnothing$ be such that $[t_\varnothing] \cap G_0 = \varnothing$. Then, given t_s, $s \in 2^\alpha$, let $t'_s \supsetneq t_s$ be such that $[t'_s] \cap G_{\alpha+1} = \varnothing$. Set $t_{s^\frown 0} = t'^\frown_s 0$ and $t_{s^\frown 1} = t'^\frown_s 1$. For limit $\beta < \kappa$, and $s \in 2^\beta$, let $t'_s = \bigcup_{\alpha<\beta} t_{s{\restriction}\alpha}$. Let $t_s \supsetneq t'_s$ be such that $[t'_s] \cap G_\beta = \varnothing$. Finally, let

$$T = \bigcup_{\alpha<\kappa} \{t \in 2^{<\kappa} : t \subseteq t_s, s \in 2^\alpha\}.$$

Obviously, T is a κ-perfect tree, so $P = [T]_\kappa$ is a κ-perfect subset of $2^\kappa \setminus G$.

Chapter 2

Special subsets in 2^ω: perfectly null sets

In this chapter we study classes of special subsets of the Cantor space 2^ω related to measure and category. The theory of special subsets of the real line was introduced in Section 1.3, and is described in [Miller, 1984] and [Bukovský, 2011]. We use the notation and notions related to the Cantor space 2^ω defined in section 1.2. Questions considered in this chapter arise mainly by applying the principle of duality between measure and category to some known notions and their properties. Most of the results presented here have been published in [Korch and Weiss, 2016].

Consider the notions of special subsets in 2^ω related to measure and category. Table 2.1 represents those notions and the inclusions between them.

category	$\mathrm{P}\boldsymbol{\mathcal{M}}$	$\supseteq$	$\mathrm{U}\boldsymbol{\mathcal{M}}$	$\supseteq$	$\mathrm{P}\boldsymbol{\mathcal{M}}'$	$\supseteq$	$\mathrm{S}\boldsymbol{\mathcal{M}}$
measure	$\boxed{?}$		$\mathrm{U}\boldsymbol{\mathcal{N}}$		$\boxed{?}$		$\mathrm{S}\boldsymbol{\mathcal{N}}$

Table 2.1: Classes of special subsets of the real line.

The classes $\mathrm{P}\boldsymbol{\mathcal{M}}$ and $\mathrm{P}\boldsymbol{\mathcal{M}}'$ were left without a counterpart, and in this chapter we introduce two new classes of special subsets of the real line: the class of perfectly null sets and the class of sets which are perfectly null in the transitive sense. These classes may play the role of duals to the corresponding classes on the category side. We state the main problem of whether there exists a perfectly null set which is not universally null, which remains open. Nevertheless, pursuing this problem we consider some simpler classes and their category counterparts.

2.1 Measure on a perfect subset of 2^ω

We start by defining a canonical measure on a perfect set $P \subseteq 2^\omega$. Let $A \subseteq P$ be such that $h_P^{-1}[A]$ is measurable in 2^ω, where $h_P: 2^\omega \to P$ is the canonical homeomorphism on P. We define

$$\mu_P(A) = m(h_P^{-1}[A]).$$

Measure μ_P will be called **the canonical measure on** P. A set $A \subseteq P$ such that $\mu_P(A) = 0$ will be called P**-null**, a set measurable with regard to μ_P will be called P-measurable. Sometimes measure μ_P will be considered as a measure on the whole 2^ω by setting $\mu_P(A) = \mu_P(A \cap P)$ for $A \subseteq 2^\omega$ such that $A \cap P$ is P-measurable. On P one can define the outer measure $\mu_P^*(A) = m^*(h_P^{-1}[A])$.

The same idea of the canonical measure on a perfect set was used in [Burke and Miller, 2005].

Now, we prove some simple properties of P-null sets.

Proposition 2.1 ([Korch and Weiss, 2016])**.** *If $Q, P \subseteq 2^\omega$ are perfect sets such that $Q \subseteq P$, and $A \subseteq Q$, then $\mu_P^*(A) \leq \mu_Q^*(A)$. In particular, every Q-null set $A \subseteq Q$ is also P-null.*

Proof: Notice that if $w \in T_P$ is on level i in P, then $\mu_P([w]_P) = 1/2^i$. If $Q \subseteq P$ is perfect, then $T_Q \subseteq T_P$, and therefore if $w \in T_Q$, then $l_Q(w) \leq l_P(w)$, so $\mu_Q([w]_Q) \geq \mu_P([w]_P)$. □

Proposition 2.2 ([Korch and Weiss, 2016])**.** *If $Q, P \subseteq 2^\omega$ are perfect sets such that $Q \subseteq P$, and A is a Q-measurable subset of Q, then it is P-measurable.*

Proof: If A is Q-measurable, there exists a Borel set $B \subseteq 2^\omega$ such that $B \cap Q \subseteq A$ and $\mu_Q(A \setminus B) = 0$, so $\mu_P(A \setminus B) = 0$. Let $B' = B \cap Q$. B' is Borel, $\mu_P(A \setminus B') = \mu_P(A \setminus B) = 0$ and $B' \subseteq A$. □

Corollary 2.3 ([Korch and Weiss, 2016])**.** *If $P \subseteq 2^\omega$ is perfect, and $Q_n \subseteq P$ for $n \in \omega$ are perfect sets such that*

$$\mu_P(\bigcup_n Q_n) = 1$$

and $A \subseteq P$ is such that for any $n \in \omega$, $A \cap Q_n$ is Q_n-measurable, then A is P-measurable and

$$\mu_P(A) \leq \sum_{n \in \omega} \mu_{Q_n}(A \cap Q_n).$$

In particular, if for all $n \in \omega$, $A \cap Q_n$ is Q_n-null, then A is P-null.

□

We will also need the following lemma.

Lemma 2.4 ([Korch and Weiss, 2016]). *Let $P \subseteq 2^\omega$ be a perfect set, $k \in \omega$ and $X \subseteq 2^\omega$ be such that for all $t \in P$, there exist infinitely many $n \in \omega$ such that there is $w \in 2^k$ with $[t{\restriction}n{}^\frown w]_P \subseteq P \setminus X$. Then $\mu_P(X) = 0$.*

Proof: Notice that if $k = 0$, then $X \cap P = \varnothing$, so we can assume that $k > 0$. We prove by induction that for any $m \in \omega$, there exists a finite set $S_m \subseteq T_P$ such that

$$X \cap P \subseteq \bigcup_{s \in S_m} [s]_P,$$

and

$$\sum_{s \in S_m} \frac{1}{2^{l_P(s)}} \leq \left(\frac{2^k - 1}{2^k}\right)^m.$$

Let $S_0 = \{\varnothing\}$. Given S_m, for each $s \in S_m$ and each $t \in P$ such that $s \subseteq t$, we can find $s_{s,t} \in T_P$ such that $s \subseteq s_{s,t} \subseteq t$ and $w_{s,t} \in 2^k$ with $[s_{s,t}{}^\frown w_{s,t}]_P \subseteq P \setminus X$. Therefore, since $[s]_P$ is compact, we can find a finite set $A_s \subseteq P$ such that $[s]_P = \bigcup_{t \in A_s} [s_{s,t}]_P$ and $[s_{s,t}]_P \cap [s_{s,t'}]_P = \varnothing$ if $t, t' \in A_s$ and $t \neq t'$. Let

$$S_{m+1} = \left\{s_{s,t}{}^\frown w \colon s \in S_m \wedge t \in A_s \wedge w \in 2^k \setminus \{w_{s,t}\}\right\} \cap T_P.$$

We have that

$$X \cap P \subseteq \bigcup_{s \in S_{m+1}} [s]_P.$$

Notice also that for $s \in S_m$,

$$\sum_{t \in A_s} \frac{1}{2^{l_P(s_{s,t})}} = \frac{1}{2^{l_P(s)}}.$$

Moreover, if $t \in A_s$, then

$$\sum_{w \in 2^k \setminus \{w_{s,t}\}} \frac{1}{2^{l_P(s_{s,t}{}^\frown w)}} \leq \frac{2^k - 1}{2^k} \cdot \frac{1}{2^{l_P(s_{s,t})}}.$$

Therefore,

$$\sum_{s \in S_{m+1}} \frac{1}{2^{l_P(s)}} \leq \frac{2^k - 1}{2^k} \cdot \sum_{s \in S_m} \frac{1}{2^{l_P(s)}} \leq \left(\frac{2^k - 1}{2^k}\right)^{m+1},$$

which concludes the induction argument.

Thus,

$$\mu_P(X) \leq \left(1 - \frac{1}{2^k}\right)^m$$

for any $m \in \omega$, and so $\mu_P(X) = 0$. □

2.2 Perfectly null sets

2.2.1 The definition and basic properties

We shall say that $A \subseteq 2^\omega$ is **perfectly null** if it is P-null for any perfect set $P \subseteq 2^\omega$. The class of perfectly null sets will be denoted by $\mathrm{P}\boldsymbol{\mathcal{N}}$.

We prove some basic properties of the above class of sets.

Proposition 2.5 ([Korch and Weiss, 2016])**.** *The following conditions are equivalent for a set $A \subseteq 2^\omega$:*

(1) *A is perfectly null,*

(2) *for every perfect $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but $P \setminus A \neq \varnothing$,*

(3) *there exists $n \in \omega$ such that for every $w \in 2^n$ and every perfect $P \subseteq [w]$, $A \cap P$ is P-null.*

Proof: Notice that if $A \cap P$ is P-measurable with $\mu_P(A \cap P) > 0$, then we can find a closed uncountable set F such that $F \subseteq A \cap P$. Therefore, there is a perfect set $Q \subseteq F$ and $Q \subseteq A$, so $Q \setminus A = \varnothing$. Moreover, given any perfect set P we have

$$P = \bigcup_{w \in 2^n \cap T_P} [w]_P,$$

and for any $w \in 2^n$ such that $w \in T_P$, the set $[w]_P$ is perfect. □

2.2.2 The main open problem

We have the following obvious fact.

Proposition 2.6 ([Korch and Weiss, 2016])**.** *$\mathrm{U}\boldsymbol{\mathcal{N}} \subseteq \mathrm{P}\boldsymbol{\mathcal{N}}$.*

Proof: Let $A \subseteq 2^\omega$ be universally null, and let P be perfect. Let λ be a measure on 2^ω such that $\lambda(B) = \mu_P(B \cap P)$ for any Borel set $B \subseteq 2^\omega$. Then $\lambda(A) = 0$, so A is P-null. □

Unfortunately, we still do not know the answer to the following question.

Question 2.7 ([Korch and Weiss, 2016])**.** *Is it consistent with ZFC that $\mathrm{U}\boldsymbol{\mathcal{N}} \neq \mathrm{P}\boldsymbol{\mathcal{N}}$?*

On the category side every proof of the consistency of the fact that $\mathrm{U}\boldsymbol{\mathcal{M}} \neq \mathrm{P}\boldsymbol{\mathcal{M}}$ known to me uses the idea of the Lusin function or similar arguments. The Lusin function is a continuous one-to-one function with measurable inverse and maps Lusin sets into perfectly meagre sets (see Section 1.3). Given such

a function it easy to see that if there exists a Lusin set L, then $\mathrm{U}\boldsymbol{\mathcal{M}} \neq \mathrm{P}\boldsymbol{\mathcal{M}}$. This should be clear since $\mathrm{U}\boldsymbol{\mathcal{M}}$ is a class closed under taking Borel isomorphic images, so $\mathcal{L}[L] \in \mathrm{P}\boldsymbol{\mathcal{M}} \setminus \mathrm{U}\boldsymbol{\mathcal{M}}$.

Therefore, to prove $\mathrm{P}\boldsymbol{\mathcal{N}} \neq \mathrm{U}\boldsymbol{\mathcal{N}}$, we possibly need some analogue of the Lusin function.

Question 2.8 ([Korch and Weiss, 2016])**.** *Is there an analogue of the Lusin function for perfectly null sets?*

But even if such an analogue exists, it cannot be constructed by a method similar way to the Lusin's argument.

Proposition 2.9 ([Korch and Weiss, 2016])**.** *Let $\mathcal{S}\colon \omega^\omega \to 2^\omega$ be a function such that there exists a sequence $\langle P_s \colon s \in \omega^{<\omega} \rangle$ such that for $s \in \omega^{<\omega}$, $P_s \subseteq 2^\omega$ is a perfect set, and for $n, m \in \omega$:*

(a) $n \neq m \Rightarrow P_{s^\frown n} \cap P_{s^\frown m} = \varnothing$,

(b) $P_{s^\frown n} \subseteq P_s$,

(c) $diam(P_s) \leq 1/2^{len(s)}$,

and $\mathcal{S}(x)$ is the only element of $\bigcap_{n\in\omega} P_{x\restriction n}$. Then there exists a perfect set $Q \subseteq 2^\omega$ such that

$$m\left(\mathcal{S}^{-1}[\bigcup\{P_s \colon s \in \omega^{<\omega} \wedge \mu_Q(P_s) = 0\}]\right) < 1.$$

Proof: We define $T \subseteq \omega^{<\omega}$ inductively as follows: in the n-th step we construct $T_n = T \cap \omega^n$ such that $|T_n| < \omega$ for all $n \in \omega$. Let $T_0 = \{\varnothing\}$. Assume that T_n is constructed and $w \in T_n$. Let $M_w \geq 2$ be such that $2^{M_w} \geq 2^{n+2} \cdot |T_n| \cdot m([w])$ and $T_{n+1} = \{w^\frown k \colon w \in T_n \wedge k \in \omega \wedge k < M_w\}$.

Therefore, if $w \in T_n$, then

$$\begin{aligned} m\left([w] \setminus \bigcup\{w^\frown k \colon k < M_w\}\right) &= m\left(\bigcup\{w^\frown k \colon k \geq M_w\}\right) = \\ &= m([w]) \cdot \sum_{i=M_w}^{\infty} \frac{1}{2^{i+1}} = \frac{m([w])}{2^{M_w}} \leq \frac{1}{2^{n+2}|T_n|}. \end{aligned}$$

Thus, for all $n \in \omega$,

$$m\left(\bigcup\{[s] \colon s \in T_n\} \setminus \bigcup\{[s] \colon s \in T_{n+1}\}\right) \leq \frac{1}{2^{n+2}},$$

so

$$m\left(\bigcup\{[s] \colon s \notin T\}\right) = m\left(\bigcup_{n\in\omega} \left(\bigcup\{[s] \colon s \in T_n\} \setminus \bigcup\{[s] \colon s \in T_{n+1}\}\right)\right) \leq \frac{1}{2}.$$

Let

$$Q = \bigcap_{n\in\omega} \bigcup_{s\in T_n} P_s.$$

Obviously, Q is a closed set. Moreover, if $s \in T$, there exists $w \in 2^{<\omega}$, $[w]_Q \subseteq P_s$. It should be clear since for all $n \in \omega$, $\{P_s : s \in T_n\}$ is a finite collection of disjoint perfect sets, and $Q \subseteq \bigcup_{s\in T_n} P_s$. Therefore, Q is perfect and $\mu_Q(P_s) > 0$. On the other hand, if $s \notin T$, then $P_s \cap Q = \varnothing$, so $\mu_Q(P_s) = 0$. Therefore, if $\mathcal{S}(x) \in P_s$ and $\mu_Q(P_s) = 0$, then $s \notin T$ and $x \in [s]$, so

$$m(\mathcal{S}^{-1}[\bigcup\{P_s : s \in \omega^{<\omega} \wedge \mu_Q(P_s) = 0\}]) = m(\bigcup\{[s] : s \notin T\}) \leq \frac{1}{2}.$$

□

2.2.3 Homeomorphisms of 2^ω

Notice the following easy observation.

Lemma 2.10. *Let $X \subseteq 2^\omega$ be a perfectly null set, and let $P \subseteq 2^\omega$ be a perfect set. Then $h_P^{-1}[X] \in P\boldsymbol{\mathcal{N}}$.*

Proof: If $Q \subseteq 2^\omega$ is a perfect set, then $h_P[Q] \subseteq P$ is also a perfect set. Therefore,

$$\mu_{h_P[Q]}(X) = \mu_Q(h_P^{-1}[X]) = 0.$$

□

Obviously, for every diffused Borel measure μ (we will always assume that $\mu(2^\omega) = 1$), there exists a Borel isomorphism of 2^ω mapping μ to the Lebesgue measure (see e.g. [Marczewski, 1937, Theorem 4.1(ii)]). Therefore, if the class $P\boldsymbol{\mathcal{N}}$ is closed under Borel automorphisms of 2^ω, then $U\boldsymbol{\mathcal{N}} = P\boldsymbol{\mathcal{N}}$, which motivates the following question.

Question 2.11 ([Korch and Weiss, 2016])**.** *Is the class $P\boldsymbol{\mathcal{N}}$ closed under homeomorphisms of 2^ω onto itself?*

It is a well-known fact (see e.g. [Oxtoby, 1971]) that on $\boldsymbol{I}$ for every strictly positive diffused Borel measure μ, there exists a homeomorphism of $\boldsymbol{I}$ mapping μ to the Lebesgue measure. It is easy to see that it is not the case for 2^ω. Indeed, the countable set of values of a measure on closed open sets is constant under homeomorphisms of 2^ω, but can be different for various measures on 2^ω. On the other hand, if we are interested only in the ideal of null sets, we get the following.

Theorem 2.12. *Let μ be a strictly positive diffused Borel measure on 2^ω. There exists a homeomorphism $h\colon 2^\omega \to 2^\omega$ such that $A \subseteq 2^\omega$ is null with respect to μ if and only if $h[A]$ is null (with respect to the Lebesgue measure).*

Proof: Let μ be a strictly positive diffused Borel measure on 2^ω. We construct by induction a Cantor scheme $\phi\colon 2^{<\omega} \to \{[a,b)\colon a \in \boldsymbol{Q}, b \in \boldsymbol{Q} \cup \{\infty\}\}$ such that

$$\left|\mu(\phi(s)) - \frac{1}{2^n}\right| < \frac{1}{2^{n+1}},$$

for all $s \in 2^n$, and $n \in \omega$. Additionally, we shall construct ϕ in such a way, to ensure that

$$\lim_{n\to\infty} \operatorname{diam}(\phi(x\restriction n)) = 0$$

for all $x \in 2^\omega$.

Namely, let $\phi(\varnothing) = 2^\omega = [0,\infty)$. Assume that $\phi(s) = [a_s, b_s)$ for $s \in 2^n$, $n \in \omega$, and $\operatorname{diam}([a_s,b_s)) = 1/2^m$. Let $c_s \in \boldsymbol{Q} \cap [a_s,b_s)$ be such that $\operatorname{diam}([a_s,c_s)) = 1/2^{m+1}$. Notice that also $\operatorname{diam}([c_s,b_s)) = 1/2^{m+1}$. Let $n_s > 0$ be the minimal natural number such that there exists $m_s \in \omega$, $1 \le m_s < 2^{n_s}$

$$\left|\mu([a_s,c_s)) - \frac{m_s}{2^{n+n_s}}\right| < \frac{m_s}{2^{n+n_s+1}}$$

and

$$\left|\mu([c_s,b_s)) - \frac{2^{n_s}-m_s}{2^{n+n_s}}\right| < \frac{2^{n_s}-m_s}{2^{n+n_s+1}}.$$

Such n_s exists, because if N is such that $1/2^N < \mu([a_s,c_s)) < \mu([a_s,b_s)) - 1/2^N$, and $|\mu([a_s,b_s)) - 1/2^n| < 1/2^N$, then there exists $m \in \omega$ such that

$$\left|\mu([a_s,c_s)) - \frac{m}{2^{n+N}}\right| < \frac{m}{2^{n+N+1}}$$

and

$$\left|\mu([c_s,b_s)) - \frac{2^{n_s}-m}{2^{n+N}}\right| < \frac{2^{n_s}-m}{2^{n+N+1}},$$

see also Figure 2.1.

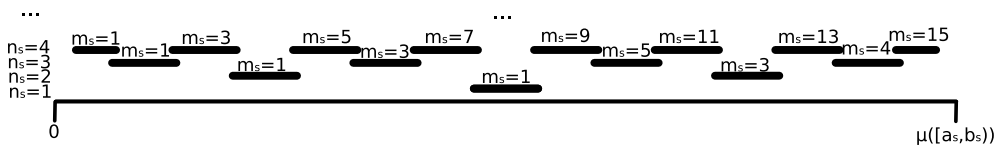

Figure 2.1: Theorem 2.12. Figure for $\mu([a_s,b_s)) = 0.7\frac{1}{2^{n-1}}$.

Let $d_{s,0} = a_s, d_{s,m_s} = c_s$ and $d_{s,2^{n_s}} = b_s$, and find $d_{s,i} \in \boldsymbol{Q} \cap [a_s, b_s)$ for $i \in 2^{n_s} \setminus \{0, m_s\}$ such that for all $i \in 2^{n_s}$, $d_{s,i} <_{\text{lex}} d_{s,i+1}$, and

$$\left|\mu([d_{s,i}, d_{s,i+1})) - \frac{1}{2^{n+n_s}}\right| < \frac{1}{2^{n+n_s+1}}.$$

Let $\{s_i : 0 \leq i < 2^{n_s}\} = 2^{n_s}$ be the enumeration of 2^{n_s} with respect to the lexicographical order. Set $\phi(s^\frown s_i) = [d_{s,i}, d_{s,i+1})$, for $0 \leq i < 2^{n_s}$. Also set

$$\phi(s^\frown t) = \bigcup_{k \in \{i \in 2^{n_s} : t \subseteq s_k\}} \phi(s^\frown s_i),$$

for all $t \in 2^{<n_s} \setminus \{\varnothing\}$.

Notice also that for all $t \in 2^{<n_s}$, $\operatorname{diam}(\phi(s^\frown t)) \leq \operatorname{diam}\phi(s)$, and $\operatorname{diam}(\phi(s^\frown s_i)) < \operatorname{diam}(\phi(s))$. Therefore, ϕ is indeed a Cantor scheme, and for every $x \in 2^\omega$, $\lim_{n\to\infty} \operatorname{diam}\phi(x\restriction n) = 0$.

Hence, for $x \in 2^\omega$, let $h(x)$ be the only element of $\bigcap_{n\in\omega} \phi(x\restriction n)$. Since $\{\phi(s) : s \in 2^{<\omega}\}$ is a basis of topology in 2^ω, and

$$\frac{m([s])}{2} = \frac{1}{2^{n+1}} < \mu(\phi(s)) < \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3m([s])}{2},$$

we get that for every measurable $A \subseteq 2^\omega$,

$$\frac{m(h[A])}{2} \leq \mu(A) \leq \frac{3m(h[A])}{2}.$$

Thus, $\mu(A) = 0$ if and only if $m(h[A]) = 0$. □

Therefore, we get the following corollary.

Corollary 2.13. *Let μ be a strictly positive diffused Borel measure on 2^ω such that $\mu(2^\omega) = 1$. There exists a perfect set P such that $\mu(P) = 1$, and a homeomorphism $g : P \to P$ such that $A \subseteq 2^\omega$ is null with respect to μ if and only if $g[A]$ is P-null.*

Proof: Let

$$Y = \bigcup\{[s] : s \in 2^{<\omega} \wedge \mu([s]) = 0\},$$

and let P be a perfect set such that $2^\omega \setminus Y = P \cup C$, there $C \subseteq 2^\omega$ is a countable set. Now consider measure $\mu \circ h_P$ on 2^ω, and apply to this measure Theorem 2.12 to get a homeomorphism h. Let $g = h_P \circ h \circ h_P^{-1}$. □

Corollary 2.14. *If the class $P\boldsymbol{\mathcal{N}}$ is closed under homeomorphisms of 2^ω, then $U\boldsymbol{\mathcal{N}} = P\boldsymbol{\mathcal{N}}$.*

Proof: Assume that $X \subseteq 2^\omega$ is perfectly null, and let μ be a diffused Borel measure on 2^ω. Without a loss of generality, assume that $\mu(2^\omega) = 1$. Apply Corollary 2.13 to get a perfect set P such that $\mu(P) = 1$, and a homeomorphism $g: P \to P$ such that $A \subseteq 2^\omega$ is null with respect to μ if and only if $g[A]$ is P-null. Notice that $g = h_P \circ h \circ h_P^{-1}$, where $h: 2^\omega \to 2^\omega$ is such that $A \subseteq 2^\omega$ is null with respect to $\mu \circ h_P$ if and only if $h[A]$ is null (with respect to the Lebesgue measure). By Lemma 2.10, $h_P^{-1}[X] \in \mathrm{P}\boldsymbol{\mathcal{N}}$, therefore $h \circ h_P^{-1}[X] \in \mathrm{P}\boldsymbol{\mathcal{N}}$, so $m(h \circ h_P^{-1}[X]) = 0$. Therefore, $\mu \circ h_P(h_P^{-1}[X]) = 0$, so $\mu(X) = 0$. □

2.2.4 Simple perfect sets

To understand what may happen in the solution of the main open problem which was mentioned above, we restrict our attention to some special subfamilies of all perfect sets. This leads to an important result in Theorem 2.27.

A set that is null in any balanced (respectively, uniformly, Silver) perfect set will be called **balanced perfectly null** (respectively, **uniformly perfectly null**, **Silver perfectly null**) (see Section 1.3.2 for all the necessary definitions). The class of such sets will be denoted by $\mathrm{bP}\boldsymbol{\mathcal{N}}$ (respectively, $\mathrm{uP}\boldsymbol{\mathcal{N}}$, $\mathrm{vP}\boldsymbol{\mathcal{N}}$). Obviously, $\mathrm{P}\boldsymbol{\mathcal{N}} \subseteq \mathrm{bP}\boldsymbol{\mathcal{N}} \subseteq \mathrm{uP}\boldsymbol{\mathcal{N}} \subseteq \mathrm{vP}\boldsymbol{\mathcal{N}}$.

Lemma 2.15 ([Korch and Weiss, 2016])**.** *There exists a perfect set E such that for every balanced perfect set B, we have either $\mu_B(E) = 0$ or $\mu_E(B) = 0$.*

Proof: Consider $K = \{000, 001, 011, 111\} \subseteq 2^3$ and a perfect set $E \in 2^\omega$ such that $x \in E$ if and only if $x[3k, 3k+2] \in K$ for every $k \in \omega$ (see Figure 2.2). Let B be a balanced perfect set. Imagine now how T_B looks like in a K-block of T_E (see Figure 2.2, where T_B is shown as doted lines). Let $k \in \omega$ and $w \in T_E \cap 2^{3k}$. The following two situations are possible. Either $\{w^\frown s : s \in K\} \subseteq T_B$ (possibility (a)), or alternatively $\{w^\frown s : s \in K\} \setminus T_B \neq \varnothing$ (possibility (b)).

Assume that for all $t \in E$, there exist infinitely many $k \in \omega$ such that $\{t{\restriction}3k^\frown s : s \in K\} \setminus T_B \neq \varnothing$ (case (b)). Then, by Lemma 2.4, $\mu_E(B) = 0$. On the other hand, assume that there exists $t \in E$ such for all but finite $k \in \omega$, we have $\{t{\restriction}3k^\frown s : s \in K\} \subseteq T_B$ (case (a)). It follows that there exists $i \in \omega$ such that B has a branching point of length j for all $j \geq i$, so $s_{j+1}(B) \leq S_j(B) + 1$, for any $j \geq i$. And since B is a balanced perfect set, this implies that $s_j(B) = S_j(B)$ and $s_{j+1}(B) = s_j(B) + 1$ for any $j > i$. In other words, for $w \in T_B \cap 2^i$, $B \cap [w] = [w]$, and hence, for any $v \in T_B \cap 2^{3k}$ with $3k > i$, there exists $w \in 2^3$ such that $v^\frown w \in T_B \setminus T_E$. It follows that $\mu_B(E) = 0$, by Lemma 2.4. □

Proposition 2.16 ([Korch and Weiss, 2016])**.** *Suppose that there exists a Sierpiński set. Then $\mathrm{P}\boldsymbol{\mathcal{N}} \subsetneq \mathrm{bP}\boldsymbol{\mathcal{N}}$.*

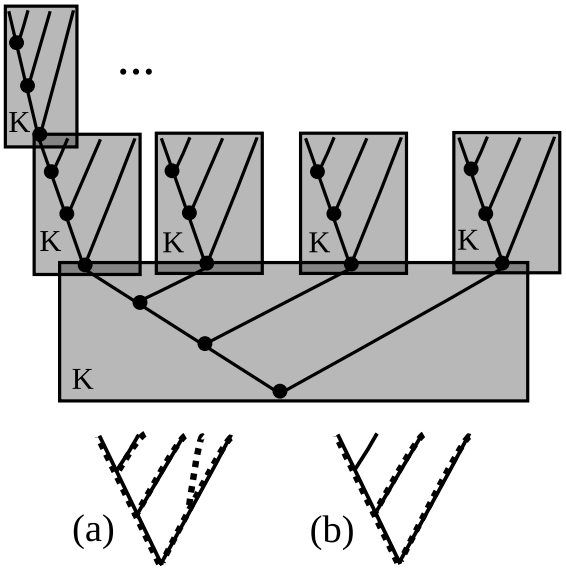

Figure 2.2: Proof of Lemma 2.15.

Proof: Let E be the perfect set defined in Lemma 2.15, and let $S \subseteq E$ be a Sierpiński set with respect to μ_E. Obviously, S is not perfectly null. But if B is a balanced perfect set, then either $\mu_B(E) = 0$, so $\mu_B(S) = 0$, or $\mu_E(B) = 0$, so $S \cap B$ is countable. Thus, $\mu_B(S) = 0$. So $S \in \mathrm{bP}\boldsymbol{\mathcal{N}} \setminus \mathrm{P}\boldsymbol{\mathcal{N}}$. □

It is easy to see that we get an analogue of the Lusin function for balanced perfectly null sets (cf. Question 2.8).

Corollary 2.17. *There exists a function $\mathcal{S}\colon 2^\omega \to 2^\omega$ which is a Borel isomorphism onto its range and such that if $S \subseteq 2^\omega$ is Sierpiński set, then $\mathcal{S}[S] \in bP\boldsymbol{\mathcal{N}}$.*

Proof: Let $\mathcal{S}\colon 2^\omega \to 2^\omega$ be defined as $\mathcal{S}(x) = h_E(x)$, where E is the perfect set defined in Lemma 2.15. Obviously, if $S \subseteq 2^\omega$ is a Sierpiński set, then $h_E[S] \subseteq E$ is a Sierpiński set with respect to μ_E. Thus, $h_E[S] \in \mathrm{bP}\boldsymbol{\mathcal{N}}$ (see the proof of Proposition 2.16). □

Proposition 2.18 ([Korch and Weiss, 2016])**.** *$bP\boldsymbol{\mathcal{N}} \subsetneq uP\boldsymbol{\mathcal{N}} \subsetneq vP\boldsymbol{\mathcal{N}}$.*

Proof: The first inclusion is proper, because if we take any balanced perfect set B such that for each $i \in \omega$, we have $|\mathrm{Split}(B) \cap 2^i| = 1$ and any uniformly perfect set U, then $\mu_U(B) \leq (n+1)/2^n$ for any $n \in \omega$, so B is U-null. Thus, $B \in uPN \setminus \mathrm{bP}\boldsymbol{\mathcal{N}}$.

To see that the second inclusion is proper, notice that the uniformly perfect set $U = \{\alpha \in 2^\omega \colon \forall_{i\in\omega} \alpha(2i+1) = \alpha(2i)\}$ is null in every Silver perfect set. Indeed, let S be a Silver perfect set. Let $i \in \omega$ be such that for every $w \in 2^{2i} \cap S$,

$w \in \mathrm{Split}(S)$, or for every $w \in 2^{2i+1} \cap S$, $w \in \mathrm{Split}(S)$. The following two cases are possible:

(a) for every $w \in 2^{2i} \cap S$, $w \in \mathrm{Split}(S)$, so $w^\frown 0, w^\frown 1 \in T_S$. Then $w^\frown 0^\frown 1 \in T_S$ or $w^\frown 0^\frown 0 \in T_S$. In the first case $w^\frown 0^\frown 1 \in T_S \setminus T_U$. In the second $w^\frown 1^\frown 0 \in T_S$, but $w^\frown 1^\frown 0 \notin T_U$.

(b) for every $w \in 2^{2i} \cap S$, $w \notin \mathrm{Split}(S)$. Without a loss of generality, assume that $w^\frown 0 \in T_S$. Then $w^\frown 0 \in \mathrm{Split}(S)$, and $w^\frown 0^\frown 1 \in T_S \setminus T_U$.

Since there exist infinitely many $i \in \omega$ such that $2^{2i} \cap S \subseteq \mathrm{Split}(S)$ or $2^{2i+1} \cap S \subseteq \mathrm{Split}(S)$, Lemma 2.4 can be applied to get that $\mu_S(U) = 0$. □

Proposition 2.19 ([Korch and Weiss, 2016])**.** *The following conditions are equivalent for a set $A \subseteq 2^\omega$:*

(1) A is perfectly null,

(2) for every perfect set $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but for every balanced perfect set $Q \subseteq 2^\omega$, $Q \setminus A \neq \varnothing$,

(3) for every perfect set $P \subseteq 2^\omega$, $A \cap P$ is P-measurable and $A \in bP\boldsymbol{\mathcal{N}}$.

Proof: Notice that there exists a balanced perfect set in every perfect set. Therefore, in the proof of Proposition 2.5 we can require that the perfect set Q is balanced. □

Notice that, even if a set is P-measurable for any perfect set and does not contain any uniformly perfect set, it needs not to be perfectly null. An example of such a set is the set B from the proof of Proposition 2.18.

Proposition 2.20 ([Korch and Weiss, 2016])**.**

(a) $A \in bP\boldsymbol{\mathcal{N}}$ if and only if for every balanced perfect $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but $P \setminus A \neq \varnothing$.

(b) $A \in uP\boldsymbol{\mathcal{N}}$ if and only if for every uniformly perfect $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but $P \setminus A \neq \varnothing$.

(c) $A \in vP\boldsymbol{\mathcal{N}}$ if and only if for every Silver perfect $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but $P \setminus A \neq \varnothing$.

Proof: We proceed as in the proof of Proposition 2.5. For uniformly and Silver perfect sets we use [Kysiak et al., 2007, Lemma 2.4], which states that there exists a Silver perfect set in every set of positive Lebesgue measure, and we notice that if P is a uniformly (respectively, Silver) perfect set, and $h_P \colon 2^\omega \to P$ is the canonical homeomorphism, then the image of any Silver perfect set is uniformly (respectively, Silver) perfect. □

2.2.5 Perfectly null sets and s_0 and v_0 ideals

Proposition 2.21 ([Korch and Weiss, 2016]). *$P\mathcal{N} \subseteq bP\mathcal{N} \subseteq s_0$.*

Proof: Indeed, if P is perfect and $X \in \mathrm{bP}\mathcal{N}$, let $B \subseteq P$ be a balanced perfect set. Then $\mu_B(B \setminus X) = 1$, so there exists a closed set $F \subseteq B \setminus X$ of positive measure. Therefore, it is uncountable, and there exists a perfect set $Q \subseteq F \subseteq P \setminus X$. □

Obviously, $\mathrm{uP}\mathcal{N} \not\subseteq s_0$ (see the proof of Proposition 2.18).

Proposition 2.22 ([Korch and Weiss, 2016]). *$P\mathcal{N} \subseteq vP\mathcal{N} \subseteq v_0$.*

Proof: Let $P \subseteq 2^\omega$ be a Silver perfect set, and let $X \in \mathrm{vP}\mathcal{N}$. Notice that the image of any Silver perfect set under the canonical homeomorphism $h_P\colon 2^\omega \to P$ is a Silver perfect set. Since $m(2^\omega \setminus h_P^{-1}[X]) = 1$, there exists a Silver perfect set $Q \subseteq 2^\omega \setminus h_P^{-1}[X]$ (see [Kysiak et al., 2007, Lemma 2.4]). So, $h_P[Q] \subseteq P \setminus X$ is a Silver perfect set. □

M. Scheepers (see [Scheepers, 1993]) proved that if X is a measure zero set with s_0 property, and S is a Sierpiński set, then $X + S$ is also an s_0-set. Therefore, we easily obtain the following proposition.

Proposition 2.23 ([Korch and Weiss, 2016]). *The algebraic sum of a Sierpiński set and a perfectly null set is an s_0-set.*

□

2.2.6 Products

We consider $\mathrm{P}\mathcal{N}$ sets in the product $2^\omega \times 2^\omega$ using the natural homeomorphism $h\colon 2^\omega \times 2^\omega \to 2^\omega$ defined as $h(x, y) = \langle x(0), y(0), x(1), y(1), \ldots\rangle$.

It is consistent with ZFC that the product of two perfectly meager sets is not perfectly meager (see [Recław, 1991a], [Pawlikowski, 1989]). If the answer to the Problem 2.7 is positive, then it makes sense to ask the following question.

Question 2.24 ([Korch and Weiss, 2016]). *Is the product of any two perfectly null sets perfectly null?*

Although this problem still remains open, in the easier case of Silver perfect sets, the answer is in the affirmative. First, notice the following simple lemma.

Lemma 2.25 ([Korch and Weiss, 2016]). *Let $P, Q \subseteq 2^\omega$ be perfect sets. Then $\mu_{P\times Q} = \mu_P \times \mu_Q$. In particular, if $X \subseteq 2^\omega \times 2^\omega$ is such that $\pi_1[X]$ is P – null, then $\mu_{P\times Q}(X) = 0$.*

Proof: First, we shall prove that for any $n \in \omega$ and any $v \in 2^{2n}$,

$$\mu_{P\times Q}\left([v]_{P\times Q}\right) = \frac{1}{2^{l_P(w_P)}} \cdot \frac{1}{2^{l_Q(w_Q)}},$$

where $w_P, w_Q \in 2^n$ are such that for any $i < n$, $w_P(i) = v(2i)$ and $w_Q(i) = v(2i+1)$. This assertion can be proved by induction on n. For $n = 0$, we get $v = w_P = w_Q = \varnothing$, and

$$\mu_{P\times Q}([v]_{P\times Q}) = 1 = \frac{1}{2^{l_P(w_P)}} \cdot \frac{1}{2^{l_Q(w_Q)}}.$$

Now consider $v \in 2^{2(n+1)}$. Then

(a) if both $w_P{\restriction}n$ and $w_Q{\restriction}n$ are branching points in P and Q respectively (so $l_P(w_P) = l_P(w_P{\restriction}n) + 1$ and $l_Q(w_Q) = l_Q(w_Q{\restriction}n) + 1$), then $v{\restriction}2n \in \mathrm{Split}(P \times Q)$ and $v{\restriction}2n+1 \in \mathrm{Split}(P \times Q)$, and so $\mu_{P\times Q}([v]_{P\times Q}) = 1/2 \cdot 1/2 \cdot \mu_{P\times Q}([v{\restriction}2n]_{P\times Q}) = 1/2 \cdot 1/2 \cdot 1/2^{l_P(w_P{\restriction}n)} \cdot 1/2^{l_Q(w_Q{\restriction}n)} = 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}$.

(b) if $w_P{\restriction}n$ or $w_Q{\restriction}n$, but not both, is a branching point in P or Q respectively, we may assume without a loss of generality that $w_P{\restriction}n \in \mathrm{Split}(P)$ and $w_Q{\restriction}n \notin \mathrm{Split}(Q)$ (so $l_P(w_P) = l_P(w_P{\restriction}n)+1$ and $l_Q(w_Q) = l_Q(w_Q{\restriction}n)$). Then $v{\restriction}2n \in \mathrm{Split}(P\times Q)$, but $v{\restriction}2n+1 \notin \mathrm{Split}(P\times Q)$, and so $\mu_{P\times Q}([v]_{P\times Q}) = 1/2 \cdot 1 \cdot \mu_{P\times Q}([v{\restriction}2n]_{P\times Q}) = 1/2 \cdot 1 \cdot 1/2^{l_P(w_P{\restriction}n)} \cdot 1/2^{l_Q(w_Q{\restriction}n)} = 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}$.

(c) if $w_P{\restriction}n \notin \mathrm{Split}(P)$ and $w_Q{\restriction}n \notin \mathrm{Split}(Q)$ (so $l_P(w_P) = l_P(w_P{\restriction}n)$ and $l_Q(w_Q) = l_Q(w_Q{\restriction}n)$), then $v{\restriction}2n, v{\restriction}2n+1 \notin \mathrm{Split}(P \times Q)$, and so $\mu_{P\times Q}([v]_{P\times Q}) = 1 \cdot 1 \cdot \mu_{P\times Q}([v{\restriction}2n]_{P\times Q}) = 1/2^{l_P(w_P{\restriction}n)} \cdot 1/2^{l_Q(w_Q{\restriction}n)} = 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}$,

which concludes the induction argument. Since every open set in $P \times Q$ is a countable union of sets of form $[v]_{P\times Q}$, with $v \in 2^{2n}$, $n \in \omega$, this concludes the proof of the Lemma. □

Proposition 2.26 ([Korch and Weiss, 2016])**.** *If $X, Y \in vP\mathcal{N}$, then $X \times Y \in vP\mathcal{N}$ in $2^\omega \times 2^\omega$.*

Proof: Fix a Silver perfect set P. Recall that such a set is uniquely defined by a sequence $\langle a_n \rangle_{n\in\omega}$, $a_n \in \{-1, 0, 1\}$ such that $\{n \in \omega : a_n = -1\}$ is infinite, T_P splits on all branches at length $n \in \omega$ if and only if $a_n = -1$, and $t(n) = a_n$ for all $t \in P$ for any other $n \in \omega$. Let T_1 be a tree which splits on all branches at length n if and only if $a_{2n} = -1$, and $t(n) = a_{2n}$ for any $t \in [T_1]$ for any other $n \in \omega$. Finally, let T_2 be a tree which splits on all branches at length n if and only if $a_{2n+1} = -1$, and $t(n) = a_{2n+1}$ for any $t \in [T_2]$ for any other $n \in \omega$. Let

$P_1 = [T_1]$ and $P_2 = [T_2]$. If $\{2n \in \omega : a_n = -1\}$ is infinite, then P_1 is a Silver perfect set. On the other hand, if $\{2n \in \omega : a_n = -1\}$ is finite, then P_1 is also finite. Analogously, if $\{2n+1 \in \omega : a_n = -1\}$ is infinite, then P_2 is a Silver perfect set. On the other hand, if $\{2n+1 \in \omega : a_n = -1\}$ is finite, then P_2 is also finite. Moreover, $P = P_1 \times P_2$.

If P_1 and P_2 are Silver perfect sets, then by Lemma 2.25, $\mu_P(X \times Y) = 0$.

The other case is when P_1 or P_2, but not both, is finite. Without a loss of generality, we may assume that P_2 is finite. Then $P = \bigcup_{t \in P_2} P_1 \times \{t\}$. Obviously, for any $t \in Y$, $\mu_{P_1 \times \{t\}}(X \times Y) = \mu_{P_1}(X) = 0$, so by Corollary 2.3, also $\mu_P(X \times Y) = 0$. □

On the other hand, it is consistent with ZFC that the classes uP$\mathcal{N}$ and bP$\mathcal{N}$ are not closed under taking products.

Theorem 2.27 ([Korch and Weiss, 2016])**.** *If there exists a Sierpiński set, then there are $X, Y \in bP\mathcal{N}$ such that $X \times Y \notin uP\mathcal{N}$.*

Proof: Let $J \subseteq 2^8$ be as shown in Figure 2.3 ($J = \{$00000000, 00010111, 00101011, 00111111, 01001010, 01011111, 01101011, 01111111, 10000101, 10010111, 10101111, 10111111, 11001111, 11011111, 11101111, 11111111$\}$). Let P be a perfect set such that $x \in P$ if and only if for all $n \in \omega$, $x[8n, 8n+7] \in J$. Obviously, P is a uniformly perfect set. Let $Q = \pi_1[P]$. Notice that $x \in Q$ if and only if for all $n \in \omega$, $x[4n, 4n+3] \in L$, where $L = \{$0000, 0001, 0011, 0111, 1000, 1001, 1011, 1111$\} \subseteq 2^4$ (see Figure 2.3 and Table 2.2).

Notice that L consists of two K-blocks (see the proof of Lemma 2.15) joined by an additional root.

Also, if B is a balanced perfect set, then $\mu_B(Q) = 0$ or $\mu_Q(B) = 0$. The argument is the same as in the proof of Lemma 2.15, namely there are two possibilities. If for all $t \in Q$, there exist infinitely many $k \in \omega$ such that $\{t{\restriction}_{4k}{}^\frown s : s \in L\} \setminus T_B \neq \varnothing$, then by Lemma 2.4, $\mu_Q(B) = 0$. If it is not the case, there exists $t \in Q$ such that for all but finite $k \in \omega$, we have $\{t{\restriction}_{4k}{}^\frown s : s \in L\} \subseteq T_B$. It follows that there exists $i \in \omega$ such that B has a branching point of length j for all $j \geq i$, so $s_{j+1}(B) \leq S_j(B) + 1$, for any $j \geq i$. And since B is a balanced perfect set, it implies that $s_j(B) = S_j(B)$ and $s_{j+1}(B) = s_j(B) + 1$ for any $j > i$. In other words, for $w \in T_B \cap 2^i$, $B \cap [w] = [w]$, and therefore for any $v \in T_B \cap 2^{4k}$ with $3k > i$, there exists $w \in 2^4$ such that $v{}^\frown w \in T_B \setminus T_Q$. It follows that $\mu_B(Q) = 0$, by Lemma 2.4.

Moreover, if A is Q-null, then $A \times 2^\omega$ is P-null. Indeed, if $n \in \omega$ and $w \in L$,

$$\mu_Q\left(\{x \in Q : x[4n, 4n+3] = w\}\right) = \frac{\left|\{s \in J : w = s\langle 0, 2, 4, 6\rangle\}\right|}{16} = \mu_P\left(\pi_1^{-1}\left[\{x \in Q : x[4n, 4n+3] = w\}\right]\right),$$

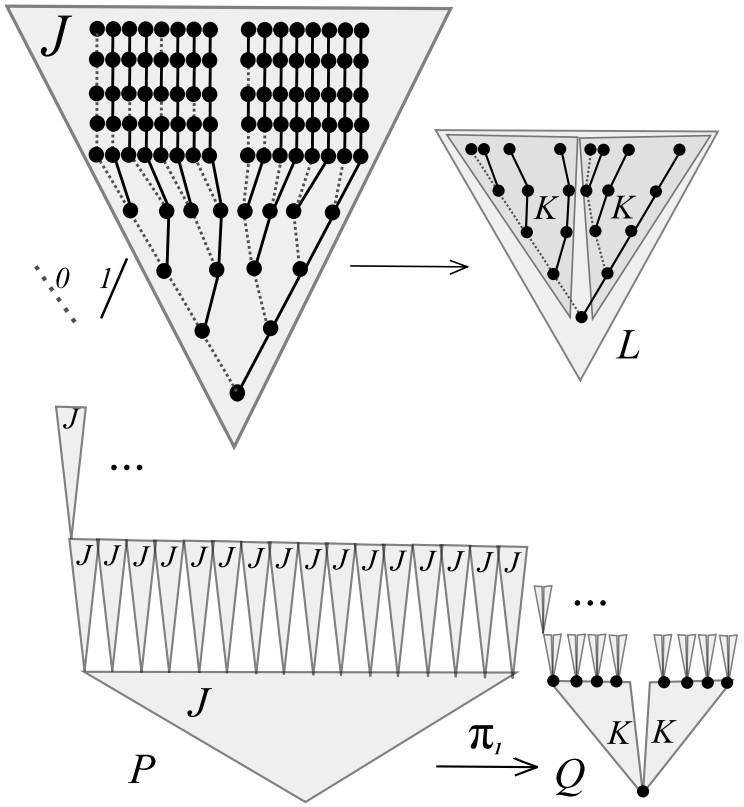

Figure 2.3: Proof of Theorem 2.27.

(see Table 2.2). Therefore, if $\varepsilon > 0$ and $\langle w_i \rangle_{i\in\omega}$ is a sequence such that $w_i \in T_Q$, $\bigcup_{i\in\omega}[w_i]_Q$ covers A and

$$\sum_{i\in\omega} \mu_Q\left([w_i]_Q\right) \leq \varepsilon,$$

then $\mu_P\left(\pi_1^{-1}\left[[w_i]_Q\right]\right) = \mu_Q\left([w_i]_Q\right)$, so

$$\bigcup_{i\in\omega} \pi_1^{-1}\left[[w_i]_Q\right]$$

is a covering of $A \times 2^\omega$ of measure μ_P not greater than ε.

Let $S \subseteq P$ be a Sierpiński set with respect to μ_P, and let $X = \pi_1[S] \subseteq Q$. Suppose that B is a balanced perfect set. Then either $\mu_B(Q) = 0$, so $\mu_B(X) = 0$, or $\mu_Q(B) = 0$, so $\mu_P(\pi_1^{-1}[Q \cap B]) = 0$. In the latter case, $S \cap \pi_1^{-1}[Q \cap B]$ is countable, so $X \cap B$ is countable and $\mu_B(X) = 0$. Hence $X \in \mathrm{bP}\boldsymbol{\mathcal{N}}$.

Notice also that $\pi_2[P] = Q$ as well (see Table 2.3). So analogously, one can check that $Y = \pi_2[S] \in \mathrm{bP}\boldsymbol{\mathcal{N}}$.

But $S \subseteq X \times Y$, so $X \times Y$ is not P-null, and therefore $X \times Y \notin \mathrm{uP}\boldsymbol{\mathcal{N}}$. □

$s \in J$	$w = s\langle 0,2,4,6\rangle$ $\in L$	μ_Q $\{x \in Q: x[4n,4n+3] = w\}$	μ_P $\pi_1^{-1}[\{x \in Q: x[4n,4n+3] = w\}]$
00000000	0000	1/16	1/16
00010111	0001	1/16	1/16
00101011	0111		
00111111	0111	1/4	4/16
01101011	0111		
01111111	0111		
01001010	0011	1/8	2/16
01011111	0011		
10000101	1000	1/16	1/16
10010111	1001	1/16	1/16
10101111	1111		
10111111	1111	1/4	4/16
11101111	1111		
11111111	1111		
11001111	1011	1/8	2/16
11011111	1011		

Table 2.2: Proof of Theorem 2.27.

The above result seems to be interesting as it resembles the argument of Recław (see [Recław, 1991a]) which proves that if there exists a Lusin set, then the class of perfectly meager sets is not closed under taking products. Recław in his proof actually constructs a perfect set $D \subseteq 2^\omega \times 2^\omega$ and shows that given a Lusin set $L \subseteq D$, its projections are perfectly meager. The same happens in the above proof where we consider a Sierpiński set and the class $\mathrm{bP}\boldsymbol{\mathcal{N}}$. Nevertheless, we still do not know whether it can be done in the case of the class $\mathrm{P}\boldsymbol{\mathcal{N}}$.

2.3 Sets meagre in simple perfects sets

Analogously, as in the case of measure, we say that a set which is meagre in any balanced (respectively, uniformly, Silver) perfect set is **balanced perfectly meagre** (respectively, **uniformly perfectly meagre**, **Silver perfectly meagre**). The class of such sets is here denoted by $\mathrm{bP}\boldsymbol{\mathcal{M}}$ (respectively, $\mathrm{uP}\boldsymbol{\mathcal{M}}$, $\mathrm{vP}\boldsymbol{\mathcal{M}}$). Obviously, $\mathrm{P}\boldsymbol{\mathcal{M}} \subseteq \mathrm{bP}\boldsymbol{\mathcal{M}} \subseteq \mathrm{uP}\boldsymbol{\mathcal{M}} \subseteq \mathrm{vP}\boldsymbol{\mathcal{M}}$.

Lemma 2.28. *There exists a perfect set E such that for every balanced perfect set B, we have either $B \cap E$ is nowhere dense in E, or it is nowhere dense in B.*

Proof: The set E defined in the proof of Lemma 2.15 has also the considered property. Indeed, if B is a balanced perfect set, and for all $t \in E$, there exist

$s \in J$	$w = s\langle 1,3,5,7\rangle \in L$
00000000	0000
00010111	0111
00111111	0111
10010111	0111
10111111	0111
00101011	0001
01101011	1001
01111111	1111
01011111	1111
11011111	1111
11111111	1111
01001010	1000
10000101	0011
10101111	0011
11101111	1011
11001111	1011

Table 2.3: $\pi_2[P] = Q$.

infinitely many $k \in \omega$ such that $\{t\restriction 3k^\frown s\colon s \in K\} \setminus T_B \neq \varnothing$, then obviously B is nowhere dense in E. On the other hand, if there exists $t \in E$ such for all but finite $k \in \omega$, we have $\{t\restriction 3k^\frown s\colon s \in K\} \subseteq T_B$, then as before for $w \in T_B \cap 2^i$, $B \cap [w] = [w]$, and therefore E is nowhere dense in B. □

Proposition 2.29. *Suppose that there exists a Lusin set. Then $P\mathcal{M} \subsetneq bP\mathcal{M}$.*

Proof: The reasoning is the same as in the proof of Proposition 2.16. □

Proposition 2.30. *$bP\mathcal{M} \subsetneq uP\mathcal{M} \subsetneq vP\mathcal{M}$.*

Proof: The set B defined in the proof of Proposition 2.18 is obviously also an example of set in $\mathrm{uP}\mathcal{M} \setminus \mathrm{bP}\mathcal{M}$.

Similarly, $U = \{\alpha \in 2^\omega\colon \forall_{i\in\omega}\alpha(2i+1) = \alpha(2i)\} \in \mathrm{vP}\mathcal{M} \setminus \mathrm{uP}\mathcal{M}$. □

Proposition 2.31.

(1) $bP\mathcal{M} \subseteq s_0$,

(2) $vP\mathcal{M} \subseteq v_0$.

Proof: The reasoning is the same as in the proofs of Propositions 2.21 and 2.22. □

Similarly as in the measure case, we have the following Proposition.

Proposition 2.32. *If $X, Y \in vP\mathcal{M}$, then $X \times Y \in vP\mathcal{M}$ in $2^\omega \times 2^\omega$.*

Proof: As in the proof of Proposition 2.26, notice that if $P \subseteq 2^\omega \times 2^\omega$ is a Silver perfect set, then $P = P_1 \times P_2$, where either P_1, P_2 are Silver perfect sets, or one of them is a Silver perfect set, and the other one is countable. Without a loss of generality assume that P_1 is a Silver perfect set. Then X is meagre in P_1, and hence $X \times Y$ is meagre in $P_1 \times P_2 = P$. □

On the other hand, it is not the case for uP$\mathcal{M}$ sets.

Proposition 2.33. *If there exists a Lusin set, then there are $X, Y \in bP\mathcal{M}$ such that $X \times Y \notin uP\mathcal{M}$.*

Proof: Consider set P defined in the proof of Theorem 2.27, and a Lusin set $N \subseteq P$. Analogously, $Q = \pi_1[P] = \pi_2[P]$ is such that if B is a balanced perfect set, then either $B \cap Q$ is nowhere dense in B, or is nowhere dense in Q. Moreover, it is easy to see that if A is nowhere dense in Q, $A \times 2^\omega$ is nowhere dense in P. Therefore if $X = \pi_1[N] \subseteq Q$, and B is a balanced perfect set, then either $Q \cap B$ is nowhere dense in B, and therefore X is nowhere dense in B, or $Q \cap B$ is nowhere dense in Q. Hence, $\pi_1^{-1}[Q \cap B]$ is nowhere dense in P, and thus has a countable intersection with N. Thus $X \cap B$ is countable. As before, the proof that $Y = \pi_2[N] \in$ bP$\mathcal{M}$ is analogous. □

2.4 Bartoszyński's small sets with respect to μ_P

In [Bartoszyński and Judah, 1995][Section 2.5.A] a collection of sets, which here will be called **Bartoszyński's small sets** is defined. A set $A \subseteq 2^\omega$ is Bartoszyński's small if there exists a sequence $\langle a_n \rangle_{n\in\omega} \in ([\omega]^{<\omega})^\omega$ of pairwise disjoint finite sets which is a partition of ω and $\langle J_n \rangle_{n\in\omega}$ such that $J_n \subseteq 2^{a_n}$ such that

$$A \subseteq \bigcap_{n\in\omega} \bigcup_{m>n} \{x \in 2^\omega : x \restriction a_m \in J_m\},$$

and

$$\sum_{n\in\omega} \frac{|J_n|}{2^n} < \infty.$$

We say that such a set is **interval small** if that $a_n = k_{n+1} \setminus k_n$, for all $n \in \omega$, and an increasing sequence $\langle k_n \rangle_{n\in\omega} \in \omega^\omega$ with $k_0 = 0$. This assumption also appears in [Bartoszyński and Judah, 1995].

In [Bartoszyński and Judah, 1995], the authors prove that every null set is a union of two Bartoszyński's small sets (see [Bartoszyński and Judah, 1995][Theorem 2.5.7]).

2.4.1 First approach

The crucial role in the construction of two small sets out of a null set (see [Bartoszyński and Judah, 1995][Theorem 2.5.7]) is played by Lemma 2.5.1 (or Corollary 2.5.2). This Lemma cannot be stated for the measure μ_P in a fully straightforward way.

The following lemma is an analogue of Lemma 2.5.1 and Corollary 2.5.2 from [Bartoszyński and Judah, 1995].

Proposition 2.34. *Assume that P is perfect and $X \subseteq P$. Consider the following properties:*

(1) For all $n \in \omega$, there exists $F_n \subseteq 2^{S_n(P)}$ such that

$$\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$$

and

$$X \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} [F_m].$$

(2) $\mu_P(X) = 0$.

(3) For all $n \in \omega$ there exists $F_n \subseteq 2^{s_n(P)}$ such that

$$\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$$

and

$$X \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} [F_m].$$

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof: The proof is quite straightforward. For (1) $\Rightarrow$ (2) notice that

$$\mu_P\left(\bigcup_{m\geq n} [F_m]\right) \leq \sum_{m\geq n} \mu_P[F_m] \leq \sum_{m\geq n} \frac{|F_m|}{2^m}$$

which converges to 0 when $n \to \infty$.

For (2) $\Rightarrow$ (3) let G_n be an open set such that $X \subseteq G_n$, and

$$\mu_P(G_n) < \frac{1}{2^n}.$$

We write G_n in a form of a union of disjoint clopen sets $G_n = \bigcup_m [s_n^m]$, where for each n, m there exists k such that $\mathrm{len}(s_n^m) = s_k(P)$. Let

$$F_n = \{s \in 2^{s_n(P)} \cap T_P : \exists_{k,l} s = s_k^l\}.$$

Then obviously,

$$X \subseteq \bigcap_{n\in\omega} \bigcup_{m\geq n} [F_m],$$

and

$$\sum_{n\in\omega} \frac{|F_n|}{2^n} \leq \sum_{n\in\omega} \sum_{s\in F_n} \mu_P([s]) \leq \sum_{m\in\omega} \mu_P(G_m) \leq 1.$$

□

Proposition 2.35. *The implications of Proposition 2.34 cannot be reversed.*

Proof: To see that (2) $\not\Rightarrow$ (1) consider a perfect set P such that $s \in T_P$ if and only if $s(0) = 0$ or if $s(0) = s(2) = 1$ and $s(2n+1) = 0$ for all $n \in \omega$. Notice that for $n > 0$, $S_n(P) = 2(n+1)$. Consider $Q \subseteq P$ such that $s \in T_Q$ if and only if $s(0) = 0$ and $s(2n+1) = 0$ for all $n \in \omega$. Obviously $\mu_P(Q) = 0$. Assume that there all $n \in \omega$ there exists $F_n \subseteq 2^{S_n(P)}$ such that

$$\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$$

and

$$Q \subseteq Y = \bigcap_{n} \bigcup_{m\geq n} [F_m].$$

Since Q is a perfect set we can consider $\mu_Q(Y)$. Notice that for $n > 1$,

$$\mu_Q(F_n) \leq \frac{|F_n|}{2^{\frac{S_n(P)}{2}}} \leq \frac{|F_n|}{2^{n+1}}.$$

So $\mu_Q(Y) = 0$, which is a contradiction since we assumed that $Q \subseteq Y$.

For (3) $\not\Rightarrow$ (2) let P be such that $s \in T_P$ if and only if $s(0) = 0$ or if $s(0) = 1$ and $s(2n+1) = 0$ for all $n \in \omega$. Notice that $s_n(P) = n$. Let $F_{2n} = \{s \in 2^{2n} : s(0) = 1 \wedge s \in T_P\}$ for $n > 0$ and $F_{2n+1} = F_0 = \varnothing$. Hence for $n > 0$, $|F_{2n}| = 2^n$ Therefore,

$$\sum_{n\in\omega} \frac{|F_n|}{2^{s_n(P)}} \leq \sum_{n>0} \frac{2^n}{2^{2n}} = 1,$$

but

$$\mu_P(\bigcap_{n\in\omega} \bigcup_{m\geq n} [F_m]) = \frac{1}{2}.$$

□

Proposition 2.36. *The implications of Proposition 2.34 can be reversed in the case of balanced perfect set.*

Proof: If P is a balanced perfect set, and $w \in 2^{s_{n+1}(P)} \cap T_P$ then $\mu_P([w]) \leq 1/2^n$, and if $w \in 2^{S_n(P)} \cap T_P$, then $\mu_P([w]) = 1/2^n$. □

Proposition 2.35 shows that we cannot hope to reach an analogue of Bartoszyński's small sets in this approach.

2.4.2 Second approach

But fortunately we can use the canonical homeomorphism h_P and consider Bartoszyński's small sets in the whole 2^ω.

Lemma 2.37. *Let P be a perfect set, and $S \subseteq 2^\omega$ be an interval small set. Then there exists an increasing function $g{:}\omega \to \omega$ and a sequence of sets $\langle K_n \rangle_{n\in\omega}$ such that $K_n \subseteq 2^{[g(n),g(n+1))}$ and:*

(a)
$$S \subseteq \bigcap_{k\in\omega} \bigcup_{n\geq k} h_P[[K_n]],$$

(b)
$$\frac{|K_n|}{2^{g(n+1)-g(n)}} \leq \frac{1}{2^n}$$
for $n \in \omega$,

(c)
$$S_P(g(n)) + 1 < s_P(g(n+1)),$$

where $x \in [K_n]$ if and only if $x\restriction[g(n), g(n+1)) \in K_n$.

Proof: Since S is interval small there exists an increasing $f{:}\omega \to \omega$ and a sequence $\langle J_n \rangle$, $n \in \omega$, $J_n \subseteq 2^{[f(n),f(n+1))}$ such that

$$S \subseteq \bigcap_{k\in\omega} \bigcup_{n\geq k} [J_n]$$

and

$$\sum_{n\in\omega} \frac{|J_n|}{2^{f(n+1)-f(n)}} < \infty.$$

Let $\alpha{:}\omega \to \omega$ be an increasing function such that for $n \in \omega$,

(i)
$$\frac{\prod_{i=\alpha(n)}^{\alpha(n+1)-1} |J_i|}{2^{f(\alpha(n+1))-f(\alpha(n))}} \leq \frac{1}{2^n},$$

(ii) $S_P(f(\alpha(n))) + 1 < s_P(f(\alpha(n+1)))$.

Such α exists because

$$\sum_{n\in\omega} \frac{|J_n|}{2^{f(n+1)-f(n)}} < \infty.$$

Let $g(n) = f(\alpha(n))$ and let $K_n \subseteq 2^{[g(n),g(n+1))}$ be such that for $k \in [\alpha(n), \alpha(n+1))$, $K_n \cap 2^{[f(k+1)-f(k))} = J_k$. □

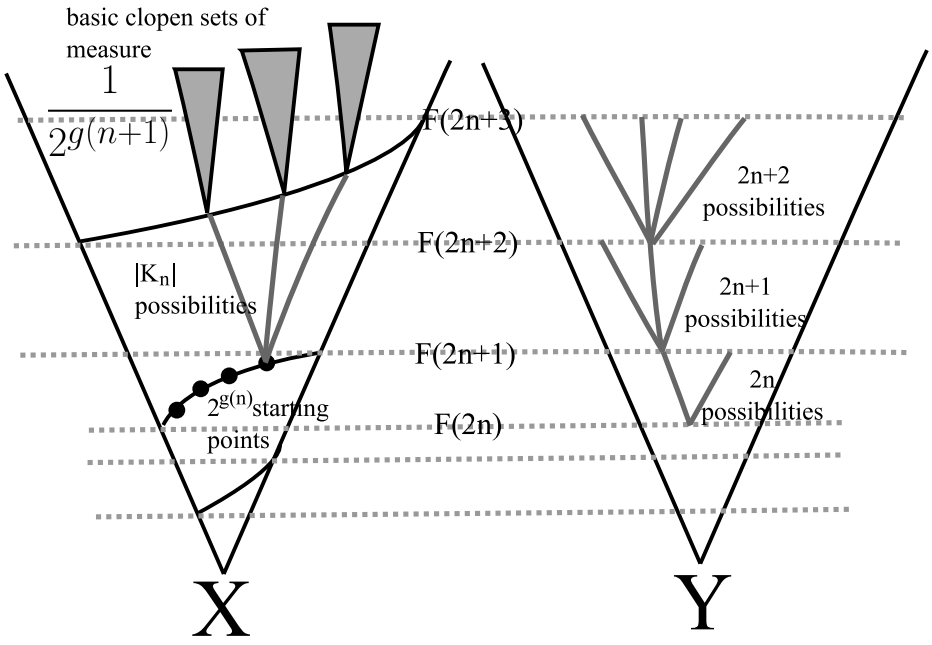

Figure 2.4: Proof of Lemma 2.38 and Proposition 2.41.

Lemma 2.38. *Let P be a perfect set, and $S \subseteq 2^\omega$ be an interval small set. Then there exists an increasing function $F\colon \omega \to \omega$ and $\langle L_n \rangle_{n\in\omega}$ such that $L_n \subseteq 2^{[F(2n),F(2n+3))}$, and*

(a)

$$\mu_P([L_n]) \leq \frac{1}{2^n},$$

(b)

$$h_P^{-1}[S] \subseteq \bigcap_{k\in\omega} \bigcup_{n\geq k} [L_n].$$

Proof: Use the previous Lemma and let $F(2n) = s_P(g(n))$ and $F(2n+1) = S_P(g(n)) + 1$ for $n \in \omega$. Let

$$L_n = \left\{ w\restriction [F(2n), F(2n+3))\colon w \in 2^\omega \wedge h_P^{-1}(w) \in [K_n] \right\}.$$

Notice that

$$\mu_P([L_n]) = 2^{g(n)} \cdot |K_n| \cdot \frac{1}{2^{g(n+1)}} = \frac{|K_n|}{2^{g(n+1)-g(n)}} \leq \frac{1}{2^n}$$

(see the left side of Figure 2.4). □

This motivates the following definition. $X \subseteq P$ will be called **small in** P if there exist an increasing function $F\colon \omega \to \omega$ and a sequence of sets $\langle L_n \rangle_{n\in\omega}$ such that $L_n \subseteq 2^{[F(2n),F(2n+3))}$, and

(1) $\mu_P([L_n]) \leq \frac{1}{2^n}$,

(2) $X \subseteq \bigcap_{k\in\omega}\bigcup_{n\geq k}[L_n]$.

We get easily the following property.

Proposition 2.39. *If $X \subseteq 2^\omega$ and $X \cap P$ is small in P, then $\mu_P(X) = 0$.*

□

Corollary 2.40. *If $X \subseteq P$ is P-null, then $X \subseteq A_1 \cup A_2$, where A_1, A_2 are small in P.*

Proof: Notice that $h_P^{-1}[X] \subseteq 2^\omega$ is null with respect to the Lebesgue measure, so it is a union of two interval small sets (see [Bartoszyński and Judah, 1995][Theorem 2.5.7]). Now use Lemma 2.38. □

Notice that the assumption that $X \subseteq P$ is crucial. The above approach cannot capture what is happening outside of P in any simple way.

Proposition 2.41. *Let $X \subseteq P$ be a small set in P and Y be a additively null set. Then $X + Y$ is P-null.*

Proof: Recall the Shelah characterization of a null-additive set (see Section 1.3). If $Y \in \boldsymbol{\mathcal{N}}^*$ and $F{:}\omega \to \omega$ is any increasing function, then there exists a sequence $I_n \subseteq 2^{[F(n),F(n+1))}$ such that $|I_n| \leq n$ and

$$Y \subseteq \bigcup_{k\in\omega}\bigcap_{n\geq k} Y_n,$$

where $x \in Y_n$ if and only if $x\restriction[F(n), F(n+1)) \in I_k$.

Let $F{:}\omega \to \omega$ be an increasing function such that there exists a sequence of sets $\langle L_n\rangle_{n\in\omega}$ such that $L_n \subseteq 2^{[F(2n),F(2n+3))}$, and:

(1) $\mu_P([L_n]) \leq \frac{1}{2^n}$,

(2) $X \subseteq \bigcap_{k\in\omega}\bigcup_{n\geq k}[L_n]$.

Notice that

$$X + Y \subseteq \bigcap_{k\in\omega}\bigcup_{n\geq k}\left([L_n] + (Y_{2n} \cap Y_{2n+1} \cap Y_{2n+2})\right).$$

We conclude (see the right side of the figure 2.4) that

$$\mu_P\left([L_n] + (Y_{2n} \cap Y_{2n+1} \cap Y_{2n+2})\right) \leq \frac{2n\cdot(2n+1)\cdot(2n+2)}{2^n}$$

But

$$\sum_{n\in\omega}\frac{2n\cdot(2n+1)\cdot(2n+2)}{2^n}$$

is convergent, so $X + Y$ is P-null. □

2.5 Perfectly null sets in the transitive sense

2.5.1 The definition

Obviously, a set is perfectly null if and only if for any perfect set P, there exists a G_δ set $G \supseteq X$ such that $\mu_P(G) = 0$. We define the following new class of special sets.

We call a set X **perfectly null in the transitive sense** if for any perfect set P, there exists a G_δ set $G \supseteq X$ such that for any t, the set $(G + t) \cap P$ is P-null. The class of sets which are perfectly null in the transitive sense will be denoted by $\mathrm{P}\mathcal{N}'$.

We do not know whether this class of sets forms a σ-ideal.

Similarly we define **ideals: $\mathbf{bP\mathcal{N}'}$, $\mathbf{uP\mathcal{N}'}$ and $\mathbf{vP\mathcal{N}'}$.**

Proposition 2.42 ([Korch and Weiss, 2016])**.** *The following sequence of inclusions holds:*

$$\begin{array}{ccccccc} P\mathcal{N}' & \subseteq & bP\mathcal{N}' & \subsetneq & uP\mathcal{N}' & \subsetneq & vP\mathcal{N}' \\ \cap| & & \cap| & & \cap| & & \cap| \\ P\mathcal{N} & \subseteq & bP\mathcal{N} & \subsetneq & uP\mathcal{N} & \subsetneq & vP\mathcal{N} \end{array}$$

Proof: The above inclusions follow immediately from the definitions. The sets B and U defined in the proof of Proposition 2.18 are obviously also in $\mathrm{uP}\mathcal{N}' \setminus \mathrm{bP}\mathcal{N}'$ and $\mathrm{vP}\mathcal{N}' \setminus \mathrm{uP}\mathcal{N}'$, respectively. □

2.5.2 $\mathrm{P}\mathcal{N}'$ sets and other classes of special subsets

In [Nowik, 1996], [Nowik and Weiss, 2000a], [Nowik and Weiss, 2001] and [Nowik and Weiss, 2000b] the authors prove that $\mathrm{S}\mathcal{M} \subseteq \mathrm{P}\mathcal{M}' \subseteq \mathrm{U}\mathcal{M}$, and that it is consistent with ZFC that those inclusions are proper. Therefore, we study the relation between the class $\mathrm{P}\mathcal{N}'$ and the classes of strongly null sets and universally null sets.

Theorem 2.43 ([Korch and Weiss, 2016])**.** *Every strongly null set is perfectly null in the transitive sense.*

Proof: Let X be a strongly null set, and let P be a perfect set. If $w \in T_P$ and $\mathrm{len}(w) = S_n(P) + 1$, then $\mu_P([w]_P) \leq 1/2^{n+1}$. It is a well-known fact that if a set A is strongly null, we can obtain a sequence of open sets of any given sequence of diamiters, the union of which covers X in such a way that every point of A is covered by infinitely many sets from this sequence (see, e.g.

[Bukovský, 2011]). Therefore, let $\langle A_n : n \in \omega \rangle$ be a sequence of open sets such that

$$X \subseteq \bigcap_{m\in\omega} \bigcup_{n\geq m} A_n$$

and

$$\operatorname{diam}(A_n) \leq \frac{1}{2^{S_n(P)+1}}.$$

Let $t \in 2^\omega$ be arbitrary. Let $B_n = (A_n + t) \cap P$. We have that $B_n \subseteq [w_n]_P$, where $w_n \in T_P$ and $\operatorname{len}(w_n) = S_n(P) + 1$. Therefore, $\mu_P(B_n) \leq 1/2^{n+1}$. But

$$(X + t) \cap P \subseteq (\bigcap_{m\in\omega} \bigcup_{n\geq m} A_n + t) \cap P \subseteq \bigcap_{m\in\omega} \bigcup_{n\geq m} B_n,$$

and

$$\mu_P(\bigcap_{m\in\omega} \bigcup_{n\geq m} B_n) = 0,$$

so X is perfectly null in the transitive sense. □

The following problem still remains open.

Question 2.44 ([Korch and Weiss, 2016])**.** *Does there exist a $P\boldsymbol{\mathcal{N}}'$ set which is not strongly null?*

In particular, I have not been able to answer the following question.

Question 2.45 ([Korch and Weiss, 2016])**.** *Does there exist an uncountable $P\boldsymbol{\mathcal{N}}'$ set in every model of ZFC?*

In [Nowik and Weiss, 2000b], the authors prove that $\mathrm{P}\boldsymbol{\mathcal{M}}' \subseteq \mathrm{U}\boldsymbol{\mathcal{M}}$. One can ask a natural question of whether the following is true. The answer is still not known.

Question 2.46 ([Korch and Weiss, 2016])**.** *$P\boldsymbol{\mathcal{N}}' \subseteq U\boldsymbol{\mathcal{N}}$?*

If this inclusion holds in ZFC, then it is consistent with ZFC that it is proper. Motivated by [Recław, 1991b, Theorem 1], we get the following theorem.

Theorem 2.47 ([Korch and Weiss, 2016])**.** *If there exists a universally null set of cardinality $\mathfrak{c}$, then there exists $Y \in U\boldsymbol{\mathcal{N}} \setminus bP\boldsymbol{\mathcal{N}}' \subseteq U\boldsymbol{\mathcal{N}} \setminus P\boldsymbol{\mathcal{N}}'$.*

Proof: As in [Nowik and Weiss, 2000b], we apply the ideas presented in [Recław, 1991b] in the case of subsets of 2^ω. Notice that there exists a perfect set $P \subseteq 2^\omega$ which is linearly independent over $\mathbb{Z}_2$. Indeed, define $\varphi : 2^{<\omega} \to 2^{<\omega}$ by induction. Let $\varphi(\varnothing) = \varnothing$. Given $\varphi(w) = v \in 2^{<\omega}$ for $w \in 2^{<\omega}$ with $n = |w|$, let $\varphi(w^\frown 0) = v^\frown \varepsilon_{2k}^{2^{n+1}}$ and $\varphi(w^\frown 1) = v^\frown \varepsilon_{2k+1}^{2^{n+1}}$, where $\varepsilon_l^m = 0 \ldots 010 \ldots 0$ is of length m

with 1 on the l-th position, and $k \in \omega$ is the natural number binary notation of which is given by w. For example, $\varphi(0) = 10$, $\varphi(1) = 01$, $\varphi(00) = 101000$, $\varphi(01) = 100100$, $\varphi(10) = 010010$, $\varphi(11) = 010001$, $\varphi(000) = 10100010000000$, and so on. Now, notice that $\langle [\varphi(w)] \rangle_{w \in 2^{<\omega}}$ is a Cantor scheme, so define

$$P = \bigcup_{\alpha \in 2^\omega} \bigcap_{n \in \omega} [\varphi(\alpha \restriction_n)].$$

Let $\alpha_1, \ldots, \alpha_n \in P$ be pairwise non-equal. There exists $l \in \omega$ such that for any $i, j \leq n, i \neq j$, $\alpha_i \restriction 2^l - 2 \neq \alpha_j \restriction 2^l - 2$. Then $\alpha_1, \ldots, \alpha_n$ restricted to $[2^l - 2, 2^{l+1} - 2)$ are basis vectors of 2^l. Thus, P is linearly independent over $\mathbb{Z}_2$. The existence of such a set follows also from Kuratowski-Mycielski Theorem (see [Kechris, 1995, Theorem 19.1]).

Next, we follow the argument from [Recław, 1991b]. Let C, D be perfect and disjoint subsets of P. We can require the set D to be a balanced perfect set. Assume that $X \subseteq C$ is a universally null set and $|X| = \mathfrak{c}$. Let $\langle B_x : x \in X \rangle$ enumerate all G_δ sets. For every $x \in X$, let $y_x \in x + D$ be such that $y_x \notin B_x$ if only $(D + x) \setminus B_x \neq \varnothing$. Otherwise, choose any $y_x \in x + D$. Put $Y = \{y_x : x \in X\}$.

Notice that $+ : C \times D \to C + D$ is a homeomorphism. Obviously, $+$ is continuous and open on $C \times D$. Since $(C + C) \cap (D + D) = \{0\}$ (because P is linearly independent), we have that $+$ is one-to-one. Since

$$\pi_1 \left[+^{-1}[Y] \right] = \pi_1 \left[\{ \langle x, d_x \rangle : x + d_x = y_x \wedge x \in X \} \right] = X$$

is universally null, Y is universally null as well.

Now, we prove that Y is not perfectly null in the transitive sense. Indeed, if $B_x \supseteq Y$ is a G_δ set, then $y_x \in B_x$, so $(D + x) \setminus B_x = \varnothing$ and $D \cap (B_x + x) = D$. Therefore, $\mu_D(D \cap (B_x + x)) = 1$. □

Corollary 2.48 ([Korch and Weiss, 2016]). *If non($\mathcal{N}$) = $\mathfrak{c}$, then $P\mathcal{N}' \neq U\mathcal{N}$.*

Proof: If non($\mathcal{N}$) = $\mathfrak{c}$, then there exists a universally null set of cardinality $\mathfrak{c}$ (see [Bukovský, 2011, Theorem 8.8]). □

Taking into account Proposition 2.6, we have the following.

Corollary 2.49 ([Korch and Weiss, 2016]). *If non($\mathcal{N}$) = $\mathfrak{c}$, $P\mathcal{N}' \neq P\mathcal{N}$.*

□

The class of perfectly meager sets in the transitive sense is closed under taking products (see [Nowik and Weiss, 2000b]). We still do not know whether this holds for P$\mathcal{N}'$ sets.

Question 2.50 ([Korch and Weiss, 2016]). *Let $X, Y \in P\mathcal{N}'$. Is it always true that $X \times Y \in P\mathcal{N}'$?*

The answer is in the positive for vP$\mathcal{N}'$ sets.

Proposition 2.51 ([Korch and Weiss, 2016]). *Let $X, Y \in vP\mathcal{N}'$. Then $X \times Y \in vP\mathcal{N}'$.*

Proof: Follows easily from the proof of Proposition 2.26. □

2.5.3 Additive properties of P$\mathcal{N}'$ sets

We investigate some additive properties of the class of sets perfectly null in the transitive sense.

Proposition 2.52 ([Korch and Weiss, 2016]). *Let $A \subseteq 2^\omega$ be open, μ be any Borel diffused measure on 2^ω and $0 \leq \varepsilon < 1$. Then the set $A_\varepsilon = \{t \in 2^\omega : \mu(A+t) > \varepsilon\}$ is also open.*

Proof: Assume that A is open, and let $A = \bigcup_{n\in\omega}[s_n]$. If $A_\varepsilon = \varnothing$, it is obviously open. Otherwise, let $t_0 \in A_\varepsilon$. There exists $N \in \omega$ such that

$$\mu(\bigcup_{n\leq N}[s_n] + t_0) > \varepsilon.$$

Let $M = \max\{\mathrm{len}(s_n) : n \leq N\}$. For any $t \in 2^\omega$ such that $t{\restriction}M = t_0{\restriction}M$,

$$\mu(A + t) \geq \mu(\bigcup_{n\leq N}[s_n] + t) = \mu(\bigcup_{n\leq N}[s_n] + t_0) > \varepsilon.$$

So A_ε is open. □

Lemma 2.53 ([Korch and Weiss, 2016]). *Let μ be a Borel diffused measure on 2^ω and $G \subseteq 2^\omega$ be a G_δ set. Let $Y \in \mathcal{N}^*$ be such that for every Borel map $\varphi : Y \to \omega^\omega$, there exists $\alpha \in \omega^\omega$ such that for every $y \in Y$, $\varphi(y) \leq^* \alpha$. Moreover, assume that for all $y \in Y$, $\mu(G + y) = 0$. Then $\mu(G + Y) = 0$.*

Proof: Let $G = \bigcap_{m\in\omega} G_m$, where for any $m \in \omega$, G_m is open and $G_{m+1} \subseteq G_m$. For $m \in \omega$, let $G_m = \bigcup_{i\in\omega}[w_{i,m}]$, with $w_{i,m} \in 2^{<\omega}$, $\mathrm{len}(w_{i,m}) > m$, and for $i \neq j$, $[w_{i,m}] \cap [w_{j,m}] = \varnothing$. Let

$$F_n = \{w_{i,m} : i, m \in \omega \wedge \mathrm{len}(w_{i,m}) = n\} \subseteq 2^n.$$

Notice that

$$G = \bigcap_{m\in\omega}\bigcup_{n\geq m}[F_n].$$

Let $\varphi : Y \to \omega^\omega$ be a function defined as follows:

$$\varphi(y)(k) = \min\left\{i \in \omega : \mu\left(\bigcup_{n\geq i}[F_n + y{\restriction}n]\right) \leq \frac{1}{2^{k+1}\cdot k!}\right\}$$

Notice that φ is well defined, as $\mu(G+y)=0$ for any $y\in Y$. By Proposition 2.52, the set

$$\varphi^{-1}\left[\{\gamma\in\omega^\omega\colon\gamma(k)>i\}\right]=\left\{y\in Y\colon\mu\left(\bigcup_{n\geq i}[F_n]+y\right)>\frac{1}{2^{k+1}\cdot k!}\right\}$$

is open for any $i,k\in\omega$, and therefore φ is Borel, so there exists strictly increasing $\alpha\in\omega^\omega$ such that for every $y\in Y$, $\varphi(y)\leq^*\alpha$. For $p\in\omega$, set $Y_p=\{y\in Y\colon\forall_{k\geq p}\varphi(y)(k)\leq\alpha(k)\}$.

Recall now the characterization of a null-additive set due to S. Shelah (see [Bartoszyński and Judah, 1995, Theorem 2.7.18(3)]). $A\in\boldsymbol{\mathcal{N}}^*$ if and only if for any increasing function $F\colon\omega\to\omega$, there exists a sequence $\langle I_q\rangle_{q\in\omega}$ such that for $q\in\omega$, $I_q\subseteq 2^{[F(q),F(q+1))}$, $|I_q|\leq q$ and

$$A\subseteq\bigcup_{r\in\omega}\bigcap_{q\geq r}[I_q].$$

Set $p\in\omega$, and apply the above characterization for Y_p and the function α. There exists a sequence $\langle I^p_q\rangle_{q\in\omega}$ such that for $q\in\omega$, $I^p_q\subseteq 2^{[\alpha(q),\alpha(q+1))}$, $|I^p_q|\leq q$ and

$$Y_p\subseteq\bigcup_{r\in\omega}\bigcap_{q\geq r}[I^p_q].$$

For $r\in\omega$, let

$$Y_{p,r}=Y\cap\bigcap_{q\geq r}[I^p_q].$$

Therefore, $Y_p=\bigcup_{r\in\omega}Y_{p,r}$. For any $q>r$, put

$$K_{p,q,r}=\{y\restriction\alpha(q+1)\colon y\in Y_{p,r}\}.$$

Notice that $K_{p,q,r}$ has at most

$$2^{\alpha(r)}\cdot\prod_{n=r}^{q}|I^p_n|=2^{\alpha(r)}\cdot\prod_{n=r}^{q}n\leq 2^{\alpha(r)}\cdot q!$$

elements.

Obviously, $Y=\bigcup_{p,r\in\omega}Y_{p,r}$, so it is sufficient to prove that $\mu(G+Y_{p,r})=0$ for any $p,r\in\omega$. Notice that for $p,r\in\omega$,

$$\begin{aligned}G+Y_{p,r}&=\bigcup_{y\in Y_{p,r}}G+y=\bigcup_{y\in Y_{p,r}}\bigcap_{m\in\omega}\bigcup_{n\geq m}[F_n+y\restriction_n]\subseteq\bigcap_{m\in\omega}\bigcup_{\substack{y\in Y_{p,r}\\ n\geq m}}[F_n+y\restriction_n]\\&=\bigcap_{m\in\omega}\bigcup_{\substack{y\in Y_{p,r}\\ q\geq m}}\bigcup_{\alpha(q)\leq n<\alpha(q+1)}[F_n+y\restriction_n]\subseteq\bigcap_{m\geq p}\bigcup_{q\geq m}\bigcup_{\substack{\alpha(q)\leq n<\alpha(q+1)\\ w\in K_{p,q,r}}}[F_n+w\restriction_n].\end{aligned}$$

Recall that if $w \in K_{p,q,r}$, then $w = y{\restriction}_{\alpha(q+1)}$ for some $y \in Y_{p,r} \subseteq Y_p$, thus for any $k \geq p$, $\alpha(k) \geq \varphi(y)(k)$, so

$$\mu\left(\bigcup_{n\geq\alpha(k)} [F_n + y{\restriction}_n]\right) \leq \frac{1}{2^{k+1}\cdot k!}.$$

In particular,

$$\mu\left(\bigcup_{n\geq\alpha(q)} [F_n + y{\restriction}_n]\right) \leq \frac{1}{2^{q+1}\cdot q!},$$

so

$$\mu\left(\bigcup_{q\geq m}\bigcup_{\substack{\alpha(q)\leq n<\alpha(q+1)\\ w\in K_{p,q,r}}} [F_n + w{\restriction}_n]\right) \leq 2^{\alpha(r)}\cdot\sum_{q\geq m}\frac{q!}{2^{q+1}q!} = \frac{2^{\alpha(r)}}{2^m}.$$

Therefore,

$$\mu(G + Y_{p,r}) \leq \frac{2^{\alpha(r)}}{2^m}$$

for any $m \in \omega$, so $\mu(G + Y_{p,r}) = 0$, for any $p, r \in \omega$. □

Theorem 2.54 ([Korch and Weiss, 2016])**.** *Let $X \in P\boldsymbol{\mathcal{N}}'$, and let Y be an $SR^{\boldsymbol{\mathcal{N}}}$ set. Then $X + Y \in P\boldsymbol{\mathcal{N}}$.*

Proof: This theorem is an easy consequence of Lemma 2.53. Indeed, by [Bartoszyński and Judah, 1994, Theorem 3.8] if Y is an $SR^{\boldsymbol{\mathcal{N}}}$ set, then $Y \in \boldsymbol{\mathcal{N}}^*$ and every Borel image of Y into ω^ω is bounded. Let P be perfect. Apply Lemma 2.53 to measure μ_P, the set Y and a G_δ set G such that $X \subseteq G$, and for all $t \in 2^\omega$, $\mu_P(G + t) = 0$. □

In [Nowik et al., 1998], the authors prove that $\mathrm{S}\boldsymbol{\mathcal{N}} + \mathrm{P}\boldsymbol{\mathcal{M}}' \subseteq s_0$. The question of whether the measure analogue is true still remains open.

Question 2.55 ([Korch and Weiss, 2016])**.** *$S\boldsymbol{\mathcal{M}} + P\boldsymbol{\mathcal{N}}' \subseteq s_0$?*

Notice that a weaker statement which says that the algebraic sum of a Sierpiński set and a $\mathrm{P}\boldsymbol{\mathcal{N}}'$ set is an s_0-set holds by Proposition 2.23.

2.6 Universally null sets in the transitive sense

Theorem 2.43 can be also formulated in a slightly stronger form.

We will call a set X **universally null in the transitive sense** ($\mathrm{U}\boldsymbol{\mathcal{N}}'$) if for any diffused Borel measure μ there exists a G_δ set $G \supseteq X$ such that for any t, the set $\mu(G + t) = 0$.

Obviously $\mathrm{U}\boldsymbol{\mathcal{N}}' \subseteq \mathrm{P}\boldsymbol{\mathcal{N}}'$, so the following theorem is stronger than a similar Theorem 2.43 proven before.

Theorem 2.56. *Every strongly null set is universally null in the transitive sense.*

Proof: Let X be a strongly null set and μ a diffused Borel measure. Let

$$S_n(\mu) = \min\left\{k \in \omega \colon \forall_{w \in 2^k} \mu([w]) \leq \frac{1}{2^n}\right\}.$$

$S_n(\mu)$ is well defined because μ is a diffused measure and 2^ω is compact. Now proceed as in the proof of Theorem 2.43. Let $\langle A_n \colon n \in \omega \rangle$ be a sequence of open sets such that

$$X \subseteq \bigcap_{m \in \omega} \bigcup_{n \geq m} A_n$$

and

$$\operatorname{diam}(A_n) \leq \frac{1}{2^{S_n(\mu)}}.$$

Let $t \in 2^\omega$ be arbitrary. Let $B_m = A_m + t$. We have that $B_n \subseteq [w]$, where $w \in 2^{S_n(\mu)}$. Therefore $\mu_P(B_n) \leq 1/2^n$. But

$$X + t \subseteq \bigcap_{m \in \omega} \bigcup_{n \geq m} (A_n + t) = \bigcap_{m \in \omega} \bigcup_{n \geq m} B_n$$

and

$$\mu\Big(\bigcap_{m \in \omega} \bigcup_{n \geq m} B_n\Big) = 0,$$

so X is universally null in the transitive sense. □

We also state the following observation.

Proposition 2.57. *Every $U\mathcal{N}'$ set is universally null. If there exists a universally null set of cardinality $\mathfrak{c}$, then there exists $Y \in U\mathcal{N} \setminus U\mathcal{N}'$.*

Proof: The first part of the Proposition follows immediately from the definition. The second is a corollary of Theorem 2.47. □

The following problems have not been solved.

Question 2.58. *Is $U\mathcal{N}'$ a proper subclass of $P\mathcal{N}'$?*

Question 2.59. *Is $S\mathcal{N}$ a proper subclass of $U\mathcal{N}'$?*

Question 2.60. *Is the class $U\mathcal{N}'$ closed under taking products?*

Chapter 3

Generalized ideal Egorov's statement

In this chapter we consider the generalized Egorov's statement (Egorov's Theorem without the assumption on measurability of the functions, see [Weiss, 2004]) in the case of an ideal convergence and a number of different types of ideal convergence notions. We prove that in those cases the generalized Egorov's statement is independent from ZFC. Most of the results presented here have been published in [Korch, 2017b].

It is assumed that the reader is familiar with preliminaries and notions presented in Section 1.4.

3.1 Generalization of Pinciroli's method

We start by a generalization of the method presented by R. Pinciroli (see [Pinciroli, 2006], and also [Repický, 2008]). The core of this method can be generalized to the following theorem.

Theorem 3.1 ([Korch, 2017b])**.** *Assume that* $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$*. Let* $\Phi \in (\omega^\omega)^{\boldsymbol{I}}$*. Then for any* $\varepsilon > 0$*, there exists* $A \subseteq \boldsymbol{I}$ *such that* $m^*(A) \geq 1 - \varepsilon$ *and* Φ *is bounded on* A*.*

Proof: We follow the arguments of Pinciroli (see [Pinciroli, 2006]).

Assume that $\mathrm{non}(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$. Notice that this statement holds for example in a model obtained by $\aleph_2$-iteration with countable support of Laver forcing (see e.g. [Bartoszyński and Judah, 1995]). Also it can be easily proven that under this assumption there exists a set $Y \subseteq \boldsymbol{I}$ of cardinality less that $\mathfrak{b}$ such that $m^*(Y) = 1$. Indeed, if $N \subseteq \boldsymbol{I}$ is a set of positive outer measure with $|N| < \mathfrak{b}$,

then let $Y = \{x + y : x \in N, y \in \mathbb{Q}\}$, where + denotes addition modulo 1. Then Y has outer measure 1 under the Zero-One Law.

Therefore, every function $\varphi : \boldsymbol{I} \to \omega^\omega$ maps Y onto a K_σ-set, where K_σ denotes the σ-ideal of subsets of ω^ω generated by the compact (equivalently bounded) sets. We get that $\Phi[Y] \in K_\sigma$. Assume that $\Phi[Y] \subseteq \bigcup_{n\in\omega} B_n$ with each B_n bounded. Let $A_n = \Phi^{-1}[\bigcup_{i=0}^{n} B_i]$. Therefore, $\Phi[A_n]$ is bounded, and for any $\varepsilon > 0$, there exists $n \in \omega$ such that $m^*(A_n) \geq 1 - \varepsilon$. □

For a sequence of functions $f_n : \boldsymbol{I} \to \boldsymbol{I}$ and subsets $A \subseteq \boldsymbol{I}$, we consider a notion of convergence $f_n \looparrowright f$ on A. We assume that if $B \subseteq A$ and $f_n \looparrowright f$ on A, then $f_n \looparrowright f$ on B. We write $f_n \looparrowright f$ provided that $f_n \looparrowright f$ on $\boldsymbol{I}$. Let $\mathcal{F} \subseteq \{\langle f_n\rangle_{n\in\omega} : \forall_{n\in\omega} f_n : \boldsymbol{I} \to \boldsymbol{I}\}$ be an arbitrary family of sequences of functions.

We consider two hypotheses between $\mathcal{F}$ and $\looparrowright$:

$(H^{\Rightarrow}(\mathcal{F}, \looparrowright))$ There exists $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$, then $F \looparrowright 0$ on A.

$(H^{\Leftarrow}(\mathcal{F}, \looparrowright))$ There exists cofinal $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$, if $F \looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$.

Theorem 3.2 ([Korch, 2017b])**.** *Assume that* $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$*, and* $H^{\Rightarrow}(\mathcal{F}, \looparrowright)$*. Then for any* $\langle f_n\rangle_{n\in\omega} \in \mathcal{F}$ *and any* $\varepsilon > 0$*, there exists* $A \subseteq \boldsymbol{I}$ *such that* $m^*(A) \geq 1 - \varepsilon$ *and* $f_n \looparrowright 0$ *on* A*.*

Proof: Apply Theorem 3.1 for $o(\langle f_n\rangle_{n\in\omega})$ given by $H^{\Rightarrow}(\mathcal{F}, \looparrowright)$. □

Now, notice that there exists a model of ZFC in which $\mathrm{non}(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$, and there exists $\mathfrak{c}$-Lusin set. To get this model it suffices to iterate $\aleph_2$-times Cohen forcing with finite supports over a model of GCH (see [Bartoszyński and Judah, 1995, Model 7.5.8 and Lemma 8.2.6]).

Theorem 3.3 ([Korch, 2017b])**.** *Assume that* $non(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$*, and that there exists a* $\mathfrak{c}$*-Lusin set. If* $H^{\Leftarrow}(\mathcal{F}, \looparrowright)$ *holds, then there exist* $\langle f_n\rangle_{n\in\omega} \in \mathcal{F}$ *and* $\varepsilon > 0$ *such that for all* $A \subseteq \boldsymbol{I}$ *with* $m^*(A) \geq 1 - \varepsilon$*,* $f_n \not\looparrowright 0$ *on* A*.*

Proof: Again, we follow the arguments of Pinciroli (see [Pinciroli, 2006]). Let $Z \subseteq \omega^\omega$ be a $\mathfrak{c}$-Lusin set. Since every compact set is meagre in ω^ω, every K_σ set is also meagre. Therefore, if $A \subseteq Z$ is a K_σ set, then $|A| < \mathfrak{c}$. Let $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ be a cofinal function given by $H^{\Leftarrow}(\mathcal{F}, \looparrowright)$. Let φ be a bijection between $\boldsymbol{I}$ and Z. Finally, let $\langle f_n\rangle_{n\in\omega} = F \in \mathcal{F}$ be such that $o(F) \geq \varphi$.

To get a contradiction, assume that for every $i \in \omega$, there exists $A_i \subseteq \boldsymbol{I}$ such that $m^*(A_i) \geq 1 - 1/2^i$ and $f_n \looparrowright 0$ on A_i. Let $A = \bigcup_{i\in\omega} A_i$. For any $i \in \omega$, $o(F)[A_i]$ is bounded because $f_n \looparrowright 0$ on A_i, and so $\varphi[A_i]$ is bounded since $o(F) \geq \varphi$. Therefore, $\varphi[A] \in K_\sigma$ and $|A| = |\varphi[A]| < \mathfrak{c}$ because $\varphi[A] \subseteq Z$. This is a contradiction because $m^*(A) = 1$ and $\mathrm{non}(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$. □

The following theorem was proved by R. Pinciroli in [Pinciroli, 2006].

Corollary 3.4 ([Pinciroli, 2006, Korch, 2017b])**.**

(1) Assume that $non(\mathcal{N}) < \mathfrak{b}$*. Then for any* $\langle f_n \rangle_{n\in\omega}$ *such that* $f_n : \boldsymbol{I} \to \boldsymbol{I}$ *for* $n \in \omega$*, and* $f_n \to 0$*, and any* $\varepsilon > 0$*, there exists* $A \subseteq \boldsymbol{I}$ *such that* $m^*(A) \geq 1-\varepsilon$ *and* $f_n \rightrightarrows 0$ *on* A*.*

(2) On the other hand, assume that $non(\mathcal{N}) = \mathfrak{c}$*, and that there exists a* $\mathfrak{c}$*-Lusin set. Then there exist* $\langle f_n \rangle_{n\in\omega}$ *such that* $f_n : \boldsymbol{I} \to \boldsymbol{I}$ *for* $n \in \omega$*, and* $f_n \to 0$*, and* $\varepsilon > 0$ *such that for all* $A \subseteq \boldsymbol{I}$ *with* $m^*(A) \geq 1-\varepsilon$*,* $f_n \not\rightrightarrows 0$ *on* A*.*

Proof: Let $\langle f_n \rangle_{n\in\omega}$ be such that $f_n \to 0$. Set $\varepsilon_n = 1/2^n$, $n \in \omega$. Consider $\mathcal{F} = \{\langle f_n \rangle_{n\in\omega} : \forall_{n\in\omega} f_n : \boldsymbol{I} \to \boldsymbol{I} \wedge f_n \to 0\}$ and $\looparrowright = \rightrightarrows$. Define $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ in the following way. Let

$$oF(x)(n) = \min\{m \in \omega : \forall_{l\geq m} f_l(x) \leq \varepsilon_n\}.$$

We get exactly the reasoning and the results of R. Pinciroli (see [Pinciroli, 2006]). He shows that the above function o proves that both $H^{\leftarrow}(\mathcal{F}_{\to}, \rightrightarrows)$ and $H^{\Rightarrow}(\mathcal{F}_{\to}, \rightrightarrows)$ hold, and then proves Theorems 3.2 and 3.3 in this particular case. □

In next sections we apply the method used in the proof of Corollary 3.4. Assume that we are given two notions of convergence of sequences of functions $f_n \rightsquigarrow f$ and $f_n \looparrowright f$ such that $f_n \looparrowright f$ implies $f_n \rightsquigarrow f$. We take

$$\mathcal{F}_{\rightsquigarrow} = \{\langle f_n \rangle_{n\in\omega} : \forall_{n\in\omega}\ f_n : \boldsymbol{I} \to \boldsymbol{I} \wedge f_n \rightsquigarrow 0\}$$

and we apply Theorem 3.2 and Theorem 3.3 with a suitable function $o : \mathcal{F}_{\rightsquigarrow} \to (\omega^\omega)^{\boldsymbol{I}}$ to get a conclusion on the stronger convergence $f_n \looparrowright 0$ of sequences from $\mathcal{F}_{\rightsquigarrow}$.

3.2 Pointwise and equi-ideal convergence (for analytic P-ideals)

Let I be an analytic P-ideal and $f_n : \boldsymbol{I} \to \boldsymbol{I}$, $n \in \omega$. By the well-known result of Solecki $I = \mathrm{Exh}(\phi)$ ([Solecki, 1999]), where ϕ is a lower semicontinuous submeasure (see Section 1.4).

It was proved in [Mrożek, 2009] that the ideal version of Egorov's Theorem holds (in the case of analytic P-ideals) between equi-ideal and pointwise ideal convergence, i.e. if $\langle f_n \rangle_{n\in\omega}$ is a sequence of measurable functions with $f_n \to_I 0$ on $\boldsymbol{I}$ and $\varepsilon > 0$, then there exists $A \subseteq \boldsymbol{I}$ such that $m(A) \geq 1-\varepsilon$ and $f_n \twoheadrightarrow_I 0$ on

A. Moreover, it was proved that the ideal version of Egorov's Theorem (in the case of analytic P-ideals) does not hold between uniform ideal and pointwise ideal convergence except for the trivial and pathological cases (see below, and also [Mrożek, 2010]).

Fix ϕ such that $I = \mathrm{Exh}(\phi)$. Notice that since I is a proper ideal, $\lim_{i\to\infty}\phi(\omega\setminus i) > 0$. If $\lim_{i\to\infty}\phi(\omega\setminus i) < \infty$, let

$$\varepsilon_n = \frac{\lim_{i\to\infty}\phi(\omega\setminus i)}{2^{n+1}}$$

for $n \in \omega$. Otherwise, set $\varepsilon_n = 1/2^{n+1}$. To use the method described in the previous section, we state the following definition. For a sequence of functions $F = \langle f_n\rangle_{n\in\omega}, f_n\colon \boldsymbol{I}\to\boldsymbol{I}$ such that $f_n \to_I 0$, let $o_\phi F \in (\omega^\omega)^{\boldsymbol{I}}$, and

$$(o_\phi F)(x)(n) = \min\{k\in\omega\colon \phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k) < \varepsilon_n\}.$$

The function $o_\phi\colon \mathcal{F}_{\to_I} \to (\omega^\omega)^{\boldsymbol{I}}$ is well defined, because for each $n\in\omega$, $\{k\in\omega\colon \phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k) < \varepsilon_n\}$ is not empty since $f_n\to_I 0$.

Lemma 3.5 ([Korch, 2017b])**.** *Let $F = \langle f_n\rangle_{n\in\omega}$ be a sequence of functions such that $f_n\colon \boldsymbol{I}\to\boldsymbol{I}$. Then $f_n \twoheadrightarrow_I 0$ on $A\subseteq\boldsymbol{I}$ if and only if $(o_\phi(\langle f_n\rangle_{n\in\omega}))[A]$ is bounded in ω^ω. In particular, $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.*

Proof: By definition, $f_n \twoheadrightarrow_I 0$ on A if and only if for any $n\in\omega$, there exists $k\in\omega$ such that for all $x\in A$, $\phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k) < \varepsilon_n$. This is true if and only if there exists a sequence $\langle k_n\rangle_{n\in\omega}$ of natural numbers such that for any $n\in\omega$ and $x\in A$, $\phi(\{m\in\omega\colon f_m(x)\geq\varepsilon_n\}\setminus k_n) < \varepsilon_n$, which holds if and only if for all $x\in A$, $(o_\phi F)(x)(n)\leq k_n$. □

Corollary 3.6 ([Korch, 2017b])**.** *Assume that $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$. Let I be any analytic P-ideal, $\varepsilon > 0$, and let $F = \langle f_n\rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I}\to\boldsymbol{I}$ for $n\in\omega$, be such that $f_n\to_I 0$. Then there exists $A\subseteq\boldsymbol{I}$ with $m^*(A)\geq 1-\varepsilon$ such that $f_n\twoheadrightarrow_I 0$ on A (the ideal version of the generalized Egorov's statement between equi-ideal and pointwise ideal convergence for analytic P-ideals is consistent with ZFC).*

Proof: Apply Theorem 3.2 and Lemma 3.5. □

Lemma 3.7 ([Korch, 2017b])**.** *For any $\varphi\colon \boldsymbol{I}\to\omega^\omega$, there exists $F = \langle f_n\rangle_{n\in\omega}$, $f_n\colon\boldsymbol{I}\to\boldsymbol{I}$ for $n\in\omega$ with $f_n\to_I 0$ such that $o_\phi F\geq\varphi$. In particular, $H^{\Leftarrow}(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds.*

Proof: Fix $x\in\boldsymbol{I}$. Notice that $\phi(\omega\setminus n)$ is a decreasing sequence with limit greater or equal to $2\varepsilon_0 > 0$, so $\phi(\omega\setminus n)\geq 2\varepsilon_0 > 0$ for any $n\in\omega$. Therefore, for each $m, n\in\omega$, there exists $k > n$ such that $\phi(k\setminus n) > \varepsilon_m$. Let $\langle k_i\rangle_{i\in\omega}$ be

an increasing sequence such that $k_0 = 0$ and $\phi(k_{i+1} \setminus \varphi(x)(i)) > \varepsilon_i$, $i \in \omega$. Set $f_j(x) = \varepsilon_i$ if $k_i \leq j < k_{i+1}$. Then $f_m(x) \geq \varepsilon_n$ if and only if $m < k_{n+1}$. Hence, if $\phi(\{m \in \omega : f_m(x) \geq \varepsilon_n\} \setminus k) < \varepsilon_n$, then $k \geq \varphi(x)(n)$, so $(o_\phi F)(x)(n) \geq \varphi(x)(n)$ for any $n \in \omega$.

This proves that o is a cofinal function. Therefore by Lemma 3.5, the property $H^{\leftarrow}(\mathcal{F}_{\to_I}, \twoheadrightarrow_I)$ holds. □

Corollary 3.8 ([Korch, 2017b]). *Assume that non($\mathcal{N}$) = $\mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any analytic P-ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n : \boldsymbol{I} \to \boldsymbol{I}$ for $n \in \omega$ with $f_n \to_I 0$ and $\varepsilon > 0$ such that for every $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\to_I 0$ on A (the negation of the ideal version of the generalized Egorov's statement between equi-ideal and pointwise ideal convergence for analytic P-ideals is consistent with ZFC).*

Proof: We use Theorem 3.3 and Lemma 3.7. □

An analytic P-ideal $I = \mathrm{Exh}(\phi)$ is **non-pathological** (see [Mrożek, 2009]) if for every $A \subseteq \omega$,

$$\phi(A) = \sup\{\mu(A) : \mu \text{ is a measure on } \omega \wedge \mu \leq \phi\}.$$

[Mrożek, 2009][Example 3.3] proves that the classic Egorov's statement between $\to_I$ and $\rightrightarrows_I$ does not hold if I is a non-pathological analytic P-ideal which is not isomorphic to Fin or $\sum_{i\in\omega} \mathrm{Fin}$. We analyse this proof to find property which distinguishes sequences of functions $\langle f_n \rangle_{n\in\omega}$ such that $f_n \to_I 0$, but there is $\varepsilon > 0$ such that $f_n \not\to_A 0$ for any $A \subseteq \boldsymbol{I}$ with $m(A) > 1 - \varepsilon$.

As in [Mrożek, 2009][Example 3.3], notice that if $I = \mathrm{Exh}(\phi)$ and is not isomorphic to Fin or $\sum_{i\in\omega} \mathrm{Fin}$, then there exists $A \subseteq \omega$ with $A \notin I$ such that $\lim_{n\in A} \phi(\{n\}) = 0$ ([Mrożek, 2009][Lemma 2.5]). Without a loss of generality assume that $\lim_{n\to\infty} \phi(A \setminus n) > 1$. Therefore we can construct by induction a sequence of finite pairwise disjoint subsets $\langle A_n \rangle_{n\in\omega} \in ([\omega]^{<\omega})^\omega$ such that $\phi(A_n) > 1$, for all $n \in \omega$, but $\phi(\{i\}) < 1/2^n$ if $i \in A_n, n \in \omega$. Let $|A_n| = k_n$, and let $A = \{a_{i,n} : i < k_n\}$. Also, since ϕ is non-pathological, there exists a sequence of measures $\langle \mu_n \rangle_{n\in\omega}$ on ω such that $\mu_n(A_n) = 1$ for all $n \in \omega$, and $\mu_n \leq \phi$.

Now, assume that $F = \langle f_n \rangle_{n\in\omega} \in \mathcal{F}_{\to_I}$. Let

$$I_{i,n,m}(F) = \left\{ x \in \boldsymbol{I} : f_{a_{i,m}}(x) > \frac{1}{2^n} \right\},$$

for $n \in \omega$ and $i < k_n$.

Fix $x \in \boldsymbol{I}$. Since $f_n \to_I 0$, we get that for all $n \in \omega$, there exists $k \in \omega$ such that

$$\phi\left(\left\{ m \in A : f_m(x) > \frac{1}{2^n} \right\} \setminus k \right) < \frac{1}{2^n}.$$

Thus for all $n \in \omega$, there exists $k \in \omega$,

$$\mu_n\left(\left\{m \in A\colon f_m(x) > \frac{1}{2^n}\right\} \setminus k\right) < \frac{1}{2^n}.$$

Therefore, for all $n \in \omega$, there exists $N \in \omega$ such that for all $m > N$,

$$\mu_n\left(\left\{i < k_m\colon f_{a_{i,m}}(x) > \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}.$$

Thus, for all $n \in \omega$, there exists $N \in \omega$ such that for all $m > N$,

$$\mu_n\left(\{i < k_m\colon x \in I_{i,n,m}(F)\}\right) < \frac{1}{2^n}.$$

Hence, let $o_F\colon \boldsymbol{I} \to \omega^\omega$ be defined as follows, let

$$o_F(x)(n) = \min\left\{N \in \omega\colon \forall_{m>N}\mu_n\left(\{i < k_m\colon x \in I_{i,n,m}(F)\}\right) < \frac{1}{2^n}\right\}.$$

We get the following.

Proposition 3.9. *Let $M \subseteq \boldsymbol{I}$. If o_F is unbounded on M, then then $f_n \not\rightrightarrows_I 0$ on M.*

Proof: Notice that if o_F is not bounded on $M \subseteq \boldsymbol{I}$, then there exists $n \in \omega$ such that for infinitely many $m \in \omega$,

$$\mu_n\left(\{i < k_m\colon M \cap I_{i,n,m}(F) \neq \varnothing\}\right) \geq \frac{1}{2^n},$$

But, assume that $f_n \rightrightarrows_I 0$ on a set $M \subseteq \boldsymbol{I}$. Hence, for any $n \in \omega$, there exists $k \in \omega$ such that

$$\phi\left(\left\{m \in A\colon \sup\{f_m(x)\colon x \in M\} > \frac{1}{2^n}\right\} \setminus k\right) < \frac{1}{2^n}.$$

Thus for all $n \in \omega$, there exists $k \in \omega$,

$$\mu_n\left(\left\{m \in A\colon \sup\{f_m(x)\colon x \in M\} > \frac{1}{2^n}\right\} \setminus k\right) < \frac{1}{2^n}.$$

Therefore, for all $n \in \omega$, there exists $N \in \omega$ such that for all $m > N$,

$$\mu_n\left(\left\{i < k_m\colon \sup\left\{f_{a_{i,m}}(x)\colon x \in M\right\} > \frac{1}{2^n}\right\}\right) < \frac{1}{2^n}.$$

Hence, for all $n \in \omega$, there exists $N \in \omega$ such that for all $m > N$,

$$\mu_n\left(\{i < k_m\colon M \cap I_{i,n,m}(F) \neq \varnothing\}\right) < \frac{1}{2^n}.$$

□

3.3 Countably generated ideals

Notice that in the case of countably generated ideals the generalized Egorov's statement holds between uniform ideal and quasi-normal ideal convergence (see [Das et al., 2014, Theorem 3.2]).

Let us therefore compare the pointwise and uniform ideal convergences. First, we show that the classic version (for measurable functions) of Egorov's Theorem holds in the case of convergence with respect to a countably generated ideal.

Theorem 3.10 ([Korch, 2017b])**.** *If $I \subseteq 2^\omega$ is a countably generated ideal, and $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$, $n \in \omega$ are Lebesgue-measurable functions such that $f_n \to_I 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq \boldsymbol{I}$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_I 0$ on $\boldsymbol{I} \setminus B$.*

Proof: Assume that I is countably generated and fix sets $\langle C_i \rangle_{i\in\omega}$ such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$. For $n, k \in \omega$, let

$$E_{n,k} = \left\{x \in \boldsymbol{I}\colon \left\{m \in \omega\colon f_m(x) > \frac{1}{2^k}\right\} \setminus C_n \neq \varnothing\right\}.$$

Notice that

$$E_{n,k} = \bigcup_{m\in\omega\setminus C_n} \left\{x \in \boldsymbol{I}\colon f_m(x) > \frac{1}{2^k}\right\}$$

is measurable for each $n, k \in \omega$. Moreover, $E_{n+1,k} \subseteq E_{n,k}$ and $\bigcap_{n\in\omega} E_{n,k} = \varnothing$ for all $k \in \omega$. Let $\varepsilon > 0$. For each $k \in \omega$, there exists $n_k \in \omega$ such that

$$m(E_{n_k,k}) \leq \frac{\varepsilon}{2^{k+1}}.$$

Let $B = \bigcup_{k\in\omega} E_{n_k,k}$. So $m(B) \leq \varepsilon$, and if $x \notin B$, then

$$\left\{m \in \omega\colon f_m(x) > \frac{1}{2^k}\right\} \subseteq C_{n_k},$$

for any $k \in \omega$, so $f_n \rightrightarrows_I 0$ on $\boldsymbol{I} \setminus B$. □

Let us consider the generalized Egorov's statement in this setting. The results presented below were proved by Joanna Jureczko using the method of T. Weiss (see [Weiss, 2004]) directly. We continue to apply the generalization of Pinciroli's method as presented above.

Assume that I is countably generated, and fix sets $\langle C_i \rangle_{i\in\omega}$ such that $C_i \subseteq C_{i+1}$ for all $i \in \omega$ and for every $A \in I$, there exists $k \in \omega$ such that $A \subseteq C_k$. We can assume that $C_{i+1} \setminus C_i \neq \varnothing$ for all $i \in \omega$.

If $F = \langle f_n \rangle_{n\in\omega}$, $f_n \to_I 0$, we define

$$(o_{\langle C_i\rangle}F)(x)(n) = \min\left\{k \in \omega\colon \left\{m \in \omega\colon f_m(x) > \frac{1}{2^n}\right\} \subseteq C_k\right\}.$$

Notice that if $A \subseteq \boldsymbol{I}$, then $f_n \rightrightarrows_I 0$ on A if and only if $(o_{\langle C_i\rangle}F)[A]$ is bounded, and so $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds. Therefore, we get the following theorem.

Corollary 3.11 ([Korch, 2017b])**.** *Assume that* $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$*. Let I be any countably generated ideal, and let $\varepsilon > 0$. Let $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$, for $n \in \omega$ be such that $f_n \to_I 0$. Then there exists $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_I 0$ on A (the ideal version of the generalized Egorov's statement between uniform ideal and pointwise ideal convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 3.2. □

Lemma 3.12 ([Korch, 2017b])**.** *For any $\varphi\colon \boldsymbol{I} \to \omega^\omega$ there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$, $f_n \to_I 0$ for $n \in \omega$ such that $o_{\langle C_i\rangle}F = \varphi$. In particular, $H^{\Leftarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$ holds.*

Proof: Without a loss of generality we can assume that $\varphi(x)$ is increasing for all $x \in \boldsymbol{I}$. Let $x \in \boldsymbol{I}$. Let $f_j(x) = 1/2^n$ if and only if

$$j \in C_{\varphi(x)(n+1)} \setminus C_{\varphi(x)(n)}.$$

□

Corollary 3.13 ([Korch, 2017b])**.** *Assume that* $non(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$*, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any countably generated ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$ for $n \in \omega$ with $f_n \to_I 0$, and $\varepsilon > 0$ such that for all $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\rightrightarrows_I 0$ on A (the negation of the ideal version of the generalized Egorov's statement between uniform ideal and pointwise ideals convergence for countably generated ideal is consistent with ZFC).*

Proof: Apply Theorem 3.3 and Lemma 3.12. □

3.4 I^*-convergence for countably generated ideals

As before, let $\langle f_n \rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$, and let I be an ideal on ω.

Notice that for any ideal I, the generalized Egorov's statement holds between I^*-uniform and I^*-quasinormal convergence (see [Das et al., 2014, Theorem 3.3]).

Let us therefore compare the pointwise and uniform ideal convergences. First, we show that the classic version (for measurable functions) of Egorov's Theorem holds in the case of I^*-convergence with respect to a countably generated ideal I.

Theorem 3.14 ([Korch, 2017b]). *If $I \subseteq 2^\omega$ is a countably generated ideal and $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$, $n \in \omega$ are Lebesgue-measurable functions such that $f_n \to_{I^*} 0$ and $\varepsilon > 0$, then there exists a measurable set $B \subseteq \boldsymbol{I}$ such that $m(B) \leq \varepsilon$ and $f_n \rightrightarrows_{I^*} 0$ on $\boldsymbol{I} \setminus B$.*

Proof: Assume that I is countably generated and fix $\langle C_n \rangle_{n\in\omega}$ such that for all $A \in I$, there exists $n \in \omega$ with $A \subseteq C_n$. Let $\omega \setminus C_n = \{m_{i,n} \colon i \in \omega\}$, $m_{i+1,n} > m_{i,n}$, $i, n \in \omega$, and

$$F_n = \left\{ x \in \boldsymbol{I} \colon \lim_{i\in\omega} f_{m_{i,n}}(x) = 0 \right\}$$

Obviously, $F_n \subseteq F_{n+1}$ for $n \in \omega$ and $\bigcup_{n\in\omega} F_n = \boldsymbol{I}$. Moreover,

$$F_n = \bigcap_{i\in\omega} \bigcup_{j\in\omega} \bigcap_{k\geq j} \left\{ x \in \boldsymbol{I} \colon f_{m_{k,n}}(x) < \frac{1}{2^i} \right\}$$

is measurable. Therefore, there exists $N \in \omega$ such that $m(F_N) \geq 1 - \varepsilon/2$. Now apply the classic Egorov's Theorem for the set F_N, $\left\langle f_{m_{i,N}} \right\rangle_{i\in\omega}$ and $\varepsilon/2$ to get a set $A \subseteq F_N$ such that $f_{m_{i,N}}$ converges uniformly on $F_N \setminus A$ and $m(A) < \varepsilon/2$. Let $B = A \cup (\boldsymbol{I} \setminus F_N)$. We get that $f_n \rightrightarrows_{I^*} 0$ on $\boldsymbol{I} \setminus B$ and $m(B) \leq \varepsilon$. □

Let us consider the generalized Egorov's statement in this setting. Assume that I is countably generated and fix $\langle C_n \rangle_{n\in\omega}$ such that for all $A \in I$, there exists $n \in \omega$ such that $A \subseteq C_n$. Let $F = \langle f_n \rangle_{n\in\omega}$ be such that $f_n \to_{I^*} 0$. Let $F = \langle f_n \rangle_{n\in\omega}$ be such that $f_n \to_{I^*} 0$. For $x \in \boldsymbol{I}$ define $o_{\langle C_i \rangle}(F)(x) = \psi \in \omega^\omega$ by

$$\psi(0) = \min \left\{ n \in \omega : \langle f_m \rangle_{m\in\omega\setminus C_n} \to 0 \right\},$$
$$\psi(n) = \min \left\{ m \in \omega : \forall_{\substack{l\in\omega\setminus C_{\psi(0)} \\ l>m}} \; f_l(x) < \frac{1}{2^n} \right\}, \quad n > 0.$$

Obviously, $o_{\langle C_i \rangle}F$ is bounded if and only if $f_n \rightrightarrows_{I^*} 0$, and so the property $H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ holds.

Therefore, we get the following theorem.

Corollary 3.15 ([Korch, 2017b]). *Assume that $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$. Let I be any countably generated ideal, and let $\varepsilon > 0$ and $F = \langle f_n \rangle_{n\in\omega}$, $f_n\colon \boldsymbol{I} \to \boldsymbol{I}$ for $n \in \omega$, with $f_n \to_{I^*} 0$. Then there exists $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_{I^*} 0$ on A (the ideal version of the generalized Egorov's statement between uniform I^* and pointwise I^* convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 3.2. □

Lemma 3.16 ([Korch, 2017b]). *For any $\varphi: \boldsymbol{I} \to \omega^\omega$, there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n: \boldsymbol{I} \to \boldsymbol{I}$, $f_n \to_{I^*} 0$ for $n \in \omega$ such that $o_{\langle C_i\rangle} F \geq \varphi$. In particular, the condition $H^{\leftarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ holds.*

Proof: It is enough to prove the lemma for φ such that $\varphi(x)$ is increasing for all $x \in \boldsymbol{I}$. Let $x \in \boldsymbol{I}$. Let $\omega \setminus C_{\varphi(x)(0)} = \{m_i : i \in \omega\}$, $m_{i+1} > m_i$ for $i \in \omega$. Let $f_j(x) = 1$ for $j \in C_{\varphi(x)(0)}$ and let $f_j(x) = 1/2^n$ if

$$j \in (\omega \setminus C_{\varphi(x)(0)}) \cap \{i \in \omega : \varphi(x)(n) \leq i < \varphi(x)(n+1)\}.$$

□

Corollary 3.17 ([Korch, 2017b]). *Assume that $non(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$, and that there exists a $\mathfrak{c}$-Lusin set. Let I be any countably generated ideal. Then there exists $F = \langle f_n \rangle_{n\in\omega}$, $f_n: \boldsymbol{I} \to \boldsymbol{I}$ for $n \in \omega$, with $f_n \to_{I^*} 0$, and $\varepsilon > 0$ such that for all $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$, $f_n \not\rightrightarrows_{I^*} 0$ on A (the negation of the ideal version of the generalized Egorov's statement between uniform I^* and pointwise I^* convergence for countably generated ideals is consistent with ZFC).*

Proof: Apply Theorem 3.3 and Lemma 3.16. □

3.5 Ideals Fin^α

In [Mrożek, 2010, Theorem 3.25], N. Mrożek proves that ideal Fin^α for any $\alpha < \omega_1$ satisfies Egorov's Theorem for ideals (between uniform ideal and pointwise ideal convergences).

Let $\mathcal{F}_\alpha = \mathcal{F}_{\to_{\mathrm{Fin}^\alpha}}$. We get the following theorem.

Theorem 3.18 ([Korch, 2017b]). *Assume that $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$. Let $0 < \alpha < \omega_1$, and let $\varepsilon > 0$ and $F = \langle f_n \rangle_{n\in\omega}$, $f_n: \boldsymbol{I} \to \boldsymbol{I}$ for $n \in \omega$, with $f_n \to_{Fin^\alpha} 0$. Then there exists $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1 - \varepsilon$ such that $f_n \rightrightarrows_{Fin^\alpha} 0$ on A (the ideal version of the generalized Egorov's statement between uniform Fin^α and pointwise Fin^α convergence is consistent with ZFC).*

Proof: Fix a bijection $b: \omega^2 \to \omega$ and a bijection $a_\beta: \omega \to \beta \setminus \{0\}$ for any limit $\beta < \omega_1$. We define $o_\alpha: \mathcal{F}_\alpha \to (\omega^\omega)^{\boldsymbol{I}}$ in the following way. Let $\varepsilon_n = 1/2^n$ for $n \in \omega$, and let

$$\mathcal{F}_\alpha^n = \{\langle f_n \rangle_{n\in\omega} : \forall_{n\in\omega} f_n : \boldsymbol{I} \to \boldsymbol{I} \wedge \forall_{x\in\boldsymbol{I}} \{q \in \omega : f_q(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\}.$$

First, define $o_\alpha^n: \mathcal{F}_\alpha^n \to (\omega^\omega)^{\boldsymbol{I}}$, $n \in \omega, 0 < \alpha < \omega_1$, by induction on α. Let

$$M_{1,n,x} = \min\{p \in \omega : \forall_{q\geq p} f_q(x) < \varepsilon_n\},$$

and let

$$(o_1^n F)(x)(k) = M_{1,n,x}$$

be a constant sequence. Given o_α^n, let

$$M_{\alpha+1,n,x} = \min\left\{p \in \omega \colon \forall_{q\geq p}\{m \in \omega \colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha\right\},$$

and

$$(o_{\alpha+1}^n F)(x)(k) = \begin{cases} M_{\alpha+1,n,x} & \text{for } k = b(p,q), \\ & p < M_{\alpha+1,n,x}+1, q \in \omega, \\ \left(o_\alpha^n \left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega}\right)(x)(q), & \text{for } k = b(p,q), \\ & p \geq M_{\alpha+1,n,x}+1, q \in \omega. \end{cases}$$

This definition is correct, since $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}_\alpha^n$ for $p \geq M_{\alpha+1,n,x}+1$.

Moreover, for limit $\beta < \omega_1$, let

$$M_{\beta,n,x} = \min\left\{p \in \omega \colon \forall_{q\geq p}\{m \in \omega \colon f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}_{a_\beta(q)}\right\}$$

and

$$(o_\beta^n F)(x)(k) = \begin{cases} M_{\beta,n,x} & \text{for } k = b(p,q), \\ & p < M_{\beta,n,x}+1, q \in \omega, \\ \left(o^n_{a_\beta(p-1)} \left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega}\right)(x)(q), & \text{for } k = b(p,q), \\ & p \geq M_{\beta,n,x}+1, q \in \omega. \end{cases}$$

This definition is correct, since, for each $p \geq M_{\beta,n,x}+1$, $\left\langle f_{b(p-1,r)}\right\rangle_{r\in\omega} \in \mathcal{F}^n_{a_\beta(p-1)}$.

Notice that $\mathcal{F}_\alpha \subseteq \mathcal{F}_\alpha^n$, for any $n \in \omega$. Therefore, finally let

$$(o_\alpha F)(x)(k) = (o_\alpha^n F)(x)(m),$$

for $k = b(n,m)$, $n, m \in \omega$.

Now, notice that if $F = \langle f_r\rangle_{r\in\omega} \in \mathcal{F}_\alpha$, and $o_\alpha F$ is bounded on a set $A \subseteq \boldsymbol{I}$, then $f_r \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. Indeed, if $o_\alpha F$ is bounded, then for each $n \in \omega$, $o_\alpha^n F$ is bounded. If so, $\{m \in \omega \colon \sup_{x\in A} f_m(x) \geq \varepsilon_n\} \in \mathrm{Fin}^\alpha$, for all $n \in \omega$. We fix $n \in \omega$ and prove this statement by induction on $\alpha < \omega_1$. Let $(o_\alpha^n F)(x)(k) < a_{k,n}$ for all $x \in A, k \in \omega$ and some $\langle a_{k,n}\rangle_{k\in\omega} \in \omega^\omega$. If $\alpha = 1$, we get $f_q(x) < \varepsilon_n$ for all $x \in A$ and all $q \geq a_0$, so

$$\left\{m \in \omega \colon \sup_{x\in A} f_m(x) \geq \varepsilon_n\right\} \in \mathrm{Fin}.$$

Now, assume that the statement holds for some $\alpha < \omega_1$. Then for all $x \in A$, $M_{\alpha+1,n,x} < a_{b(0,0)}$, so for all $p \geq a_{b(0,0)}$, $o^n_\alpha \left\langle f_{b(p-1,r)} \right\rangle_{r\in\omega}$ is bounded by $\left\langle a_{b(p,q)} \right\rangle_{q\in\omega}$, and thus by the induction hypothesis,

$$\left\{ r \in \omega : \sup_{x\in A} f_{b(p-1,r)} \geq \varepsilon_n \right\} \in \mathrm{Fin}^\alpha$$

for all $p \geq a_{b(0,0)}$. Therefore,

$$\left\{ m \in \omega : \sup_{x\in A} f_m(x) \geq \varepsilon_n \right\} \in \mathrm{Fin}^{\alpha+1}.$$

Analogous reasoning can be easily applied for limit $\beta < \omega_1$. This proves that $H^{\Rightarrow}(\mathcal{F}_\alpha, \rightrightarrows_{\mathrm{Fin}^\alpha})$ holds.

Therefore, by Theorem 3.2, there exists $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1-\varepsilon$ such that $f_n \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. □

Theorem 3.19 ([Korch, 2017b])**.** *Assume that non*$(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$*, and that there exists a* $\mathfrak{c}$*-Lusin set. Let* $0 < \alpha < \omega_1$*. Then there exist* $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}_\alpha$ *and* $\varepsilon > 0$ *such that for all* $A \subseteq \boldsymbol{I}$ *with* $m^*(A) \geq 1-\varepsilon$*,* $f_n \not\rightrightarrows_{Fin^\alpha} 0$ *on* A *(the negation of the ideal version of the generalized Egorov's statement between uniform* Fin^α *and pointwise* Fin^α *convergence for countably generated ideals is consistent with ZFC).*

Proof: As before, let $\varepsilon_n = 1/2^n$, $n \in \omega$. This time, we define o_α in a different way then in the previous proof. Namely, let

$$(o_\alpha F)(x)(n) = M_{\alpha,n,x},$$

where $M_{\alpha,n,x}$ is defined as in the previous proof. Notice that if $F = \langle f_n \rangle_{n\in\omega}$ is such that $f_n \rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on a set $A \subseteq \boldsymbol{I}$, then

$$\left\{ m \in \omega : \sup_{x\in A} f_m(x) \geq \varepsilon_n \right\} \in \mathrm{Fin}^\alpha$$

for all $n \in \omega$. If $\alpha = 1$, this means that

$$\min\{p \in \omega : \forall_{q\geq p} f_q(x) < \varepsilon_n\} = M_{1,n,x} = o_1 F(x)(n)$$

is bounded on A. If α is a limit ordinal, then for all $n \in \omega$, there exists M_n such that for all $q \geq M_n$, $\{m \in \omega : f_{b(q,m)}(x) \geq \varepsilon_n\} \in \mathrm{Fin}_{a_\alpha(q)}$. In other words, $M_{\alpha,n,x} = o_\alpha F(x)$ is bounded on A. Similar argument can be used in the case of a successor ordinal $\alpha > 1$.

Moreover, fix any $\varphi : \boldsymbol{I} \to \omega^\omega$. Without a loss of generality, assume that for $x \in \boldsymbol{I}$, $\varphi(x)$ is increasing. There exists $F = \langle f_n \rangle_{n\in\omega} \in \mathcal{F}$ such that $o_\alpha(F) \geq \varphi$. It is obvious for $\alpha = 1$. For $\alpha > 1$, let $f_n(x) = \varepsilon_k$ for $k = b(i,j)$, $\varphi(x)(k) \leq n < \varphi(x)(k+1)$. Therefore $H^{\Leftarrow}(\mathcal{F}_\alpha, \rightrightarrows_{\mathrm{Fin}^\alpha})$ holds.

In conclusion, by Theorem 3.3, there exist $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}$ and $\varepsilon > 0$ such that for all $A \subseteq \boldsymbol{I}$ with $m^*(A) \geq 1-\varepsilon$, $f_n \not\rightrightarrows_{\mathrm{Fin}^\alpha} 0$ on A. □

3.6 Even more generalized approach and open problems

In [Repický, 2017], Miroslav Repický generalized further my results presented above. In this section I present some of his results along with further open problems.

3.6.1 Preliminaries

For a sequence of functions $f_n : \boldsymbol{I} \to \boldsymbol{I}$ and subsets $A \subseteq \boldsymbol{I}$, we consider notion of convergence $f_n \looparrowright f$ on A. As before, we assume that if $B \subseteq A$ and $f_n \looparrowright f$ on A, then $f_n \looparrowright f$ on B, and write $f_n \looparrowright f$ provided that $f_n \looparrowright f$ on $\boldsymbol{I}$. Let $\mathcal{F} \subseteq \{\langle f_n\rangle_{n\in\omega} : \forall_{n\in\omega} f_n : \boldsymbol{I} \to \boldsymbol{I}\}$ be an arbitrary family of sequences of functions. A mapping $o{:}\mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ is said to be measurability preserving, if for any sequence of measurable functions $\langle f_n\rangle_{n\in\omega} \in \mathcal{F}$, $o(f)$ is measurable as well.

Apart from hypothesis $(H^{\Rightarrow}(\mathcal{F}, \looparrowright))$ and $(H^{\Leftarrow}(\mathcal{F}, \looparrowright))$, we consider other hypotheses between $\mathcal{F}$ and $\looparrowright$:

$(\bar{H}^{\Rightarrow}(\mathcal{F}, \looparrowright))$ There exists $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$, then $F \looparrowright 0$ on A.

$(\bar{H}^{\Leftarrow}(\mathcal{F}, \looparrowright))$ There exists $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ which is cofinal (with respect to $\leq$) such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$, if $F \looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$.

$(M^{\Rightarrow}(\mathcal{F}, \looparrowright))$ There exists measurability preserving $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$, then $F \looparrowright 0$ on A.

$(M^{\Leftarrow}(\mathcal{F}, \looparrowright))$ There exists measurability preserving cofinal $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$, if $F \looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$.

$(\bar{M}^{\Rightarrow}(\mathcal{F}, \looparrowright))$ There exists measurability preserving $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$, then $F \looparrowright 0$ on A.

$(\bar{M}^{\Leftarrow}(\mathcal{F}, \looparrowright))$ There exists measurability preserving $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ which is cofinal (with respect to $\leq$) such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$, if $F \looparrowright 0$ on A, then $o(F)[A]$ is bounded in $(\omega^\omega, \leq^*)$.

Obviously, we get the following implications.

Proposition 3.20.

$$\begin{array}{ccc} \bar{H}^{\Rightarrow}(\mathcal{F},\not\to) & \Rightarrow & H^{\Rightarrow}(\mathcal{F},\not\to) \\ \Uparrow & & \Uparrow \\ \bar{M}^{\Rightarrow}(\mathcal{F},\not\to) & \Rightarrow & M^{\Rightarrow}(\mathcal{F},\not\to) \end{array} \qquad \begin{array}{ccc} H^{\Leftarrow}(\mathcal{F},\not\to) & \Rightarrow & \bar{H}^{\Leftarrow}(\mathcal{F},\not\to) \\ \Uparrow & & \Uparrow \\ M^{\Leftarrow}(\mathcal{F},\not\to) & \Rightarrow & \bar{M}^{\Leftarrow}(\mathcal{F},\not\to) \end{array}$$

□

As before, assume that we are given two notions of convergence of sequences of functions $f_n \rightsquigarrow f$ and $f_n \not\to f$ such that $f_n \not\to f$ implies $f_n \rightsquigarrow f$, and take

$$\mathcal{F}_{\rightsquigarrow} = \{\langle f_n\rangle_{n\in\omega} \colon \forall_{n\in\omega}\ f_n : \boldsymbol{I} \to \boldsymbol{I} \wedge f_n \rightsquigarrow 0\}.$$

Notice the following observation.

Corollary 3.21 ([Repický, 2017])**.** *Assume that $M^{\Rightarrow}(\mathcal{F}_{\rightsquigarrow},\not\to)$ holds. Then for every sequence of measurable functions $F = \langle f_n\rangle_n \in \omega \in \mathcal{F}_{\rightsquigarrow}$, and $\varepsilon > 0$, there exists a measurable set $A \subseteq \boldsymbol{I}$ such that $m(A) \geq 1-\varepsilon$, and $f \not\to 0$ on A (i.e. the classical Egorov's statement holds between $\rightsquigarrow$ and $\not\to$).*

Proof: We reformulate the proof of Egorov's Theorem. Take measurability preserving $o : \mathcal{F} \to (\omega^\omega)^{\boldsymbol{I}}$ such that for every $F \in \mathcal{F}$ and every $A \subseteq \boldsymbol{I}$ if $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$, then $F \not\to 0$ on A. Let

$$E_{n,k} = \{x \in \boldsymbol{I} \colon o(F)(x)(k) > n\}.$$

Notice that $E_{n,k}$ is a Borel set for every $n, k \in \omega$. Moreover, $E_{n+1,k} \subseteq E_{n,k}$, for any $n, k \in \omega$, and $\bigcap_{n\in\omega} E_{n,k} = \varnothing$, for all $k \in \omega$. Therefore, for each $k \in \omega$, there exists $n_k \in \omega$ such that

$$m\left(E_{n_k,k}\right) \leq \frac{\varepsilon}{2^{k+1}}.$$

Let

$$B = \bigcup_{k\in\omega} E_{n_k,k},$$

and $A = \boldsymbol{I} \setminus B$. Then for all $x \in A$, $o(F)(x)(k) \leq n_k$ for all $k \in \omega$. Therefore, $o(F)[A]$ is bounded in $(\omega^\omega, \leq)$, and so $f_n \not\to 0$ on A, and $m(A) \geq 1-\varepsilon$, because $m(B) \leq \varepsilon$. □

Recall also Theorems 3.2 and 3.3.

Corollary 3.22 ([Korch, 2017b])**.** *Assume that $H^{\Rightarrow}(\mathcal{F}_{\rightsquigarrow},\not\to)$ holds. If $non(\boldsymbol{\mathcal{N}}) < \mathfrak{b}$, then for every sequence of functions $F = \langle f_n\rangle_n \in \omega \in \mathcal{F}_{\rightsquigarrow}$, and $\varepsilon > 0$, there exists a set $A \subseteq \boldsymbol{I}$ such that $m^*(A) \geq 1-\varepsilon$, and $f \not\to 0$ on A (i.e. the generalized Egorov's statement holds between $\rightsquigarrow$ and $\not\to$).*

Proof: See Theorem 3.2. □

Corollary 3.23 ([Korch, 2017b]). *Assume that* $non(\boldsymbol{\mathcal{N}}) = \mathfrak{c}$*, and that there exists a* $\mathfrak{c}$*-Lusin set. If* $\bar{H}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$ *holds, then there exist* $\langle f_n \rangle_{n\in\omega} \in \mathcal{F}_{\rightsquigarrow}$ *and* $\varepsilon > 0$ *such that for all* $A \subseteq \boldsymbol{I}$ *with* $m^*(A) \geq 1 - \varepsilon$*,* $f_n \not\rightarrow 0$ *on* A *(i.e. the generalized Egorov's statement fails between* $\rightsquigarrow$ *and* $\not\rightarrow$*).*

Proof: The proof of Theorem 3.3 is also valid if we have that $o(F)[A_i]$ is bounded in $(\omega^\omega, \leq^*)$. □

3.6.2 Repický's results

Let I be an ideal on ω. In [Repický, 2017] the property $\bar{H}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$, where $\rightsquigarrow$ and $\not\rightarrow$ are various notions of convergence with respect to I is considered. In particular, it is proven that if $\rightsquigarrow$ is any notion of convergence weaker than $\rightarrow$, and $\not\rightarrow$ is stronger than $\rightrightarrows_I \cup \xrightarrow{QN}_{I^*}$, then $\bar{H}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$ holds. This result along with Corollary 3.23 implies immediately Corollaries 3.13, 3.17, and Theorem 3.19.

Actually, the function obtained in the proof of this observation witnesses $\bar{M}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$, and we have the following.

Corollary 3.24. *Assume that* I *is an ideal on* ω*, and* $\rightsquigarrow$ *is any notion of convergence weaker than* $\rightarrow$*, and* $\not\rightarrow$ *is stronger than* $\rightrightarrows_I \cup \xrightarrow{QN}_{I^*}$*, then* $\bar{M}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$ *holds.*

Proof: In the proof of [Repický, 2017][Lemma 2.1] the function $o\colon \mathcal{F}_{\rightsquigarrow} \to (\omega^\omega)^{\boldsymbol{I}}$ such that

$$o(F)(x)(n) = \begin{cases} \min(C_{f,x,n}), & \text{if } C_{f,x,n} \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases}$$

where $C_{f,x,n} = \{m \in \omega\colon f_m(x) < 1/2^n\}$, is considered. It is proven that under the assumptions of this Corollary this function witnesses $\bar{H}^{\leftarrow}(\mathcal{F}_{\rightsquigarrow}, \not\rightarrow)$. But it is easy to see that it is also measure preserving. □

M. Repický considers also the closure properties of classes $\mathcal{I}$ of all ideals I which satisfy respectively one of the properties: $H^{\Rightarrow}(\mathcal{F}_{\rightarrow_I}, \rightrightarrows_I)$, $M^{\Rightarrow}(\mathcal{F}_{\rightarrow_I}, \rightrightarrows_I)$, $H^{\Rightarrow}(\mathcal{F}_{\rightarrow_{I^*}}, \rightrightarrows_{I^*})$ or $M^{\Rightarrow}(\mathcal{F}_{\rightarrow_{I^*}}, \rightrightarrows_{I^*})$. Those results are summarized in Table 3.1, where $b\colon \omega\times\omega \to \omega$ is a fixed bijection. The results along with Corollaries 3.21 and 3.22 encompass Theorems 3.10, 3.14, 3.18, and Corollaries 3.11 and 3.15.

$\mathcal{I}:$	$M^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$	$H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$	$M^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$	$H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$
Fin $\in$	✓	✓	✓	✓
$B \subseteq \omega$ is coinfinite, then $\langle B \rangle \in$	✓	✓	✓	✓
downward $\leq_{RK}$ closed	✓	✓		
downward $\leq_{RB}$ closed	✓	✓	✓	✓
$\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, then $b[\sum_{n\in\omega} I_n] \in$	✓	✓	✓	✓
$\mathcal{J} \in [\mathcal{I}]^\omega$, then $\bigcap \mathcal{I} \in$	✓	✓	✓	✓
$I, J \in \mathcal{I}$, J is a P-ideal, then $I \vee J \in$		✓		✓
$\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, then $\bigvee\{I_n : n \in \omega\} \in$				✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$		✓		✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of analytic ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$		✓	✓	✓
$\langle I_n \rangle_{n\in\omega}$ is an increasing sequence of Borel ideals from $\mathcal{I}$, then $\bigvee\{I_n : n \in \omega\} \in$	✓	✓	✓	✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, $b[I\text{-}\prod_{n\in\omega} I_n] \in$		✓		✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega}$ is a sequence of analytic ideals from $\mathcal{I}$, $b[I\text{-}\prod_{n\in\omega} I_n] \in$	✓	✓	✓	✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega} \in \mathcal{I}^\omega$, $I\text{-}\underline{\lim}_{n\in\omega} I_n \in$		✓		✓
$I \in \mathcal{I}$, $\langle I_n \rangle_{n\in\omega}$ is a sequence of analytic ideals from $\mathcal{I}$, $I\text{-}\underline{\lim}_{n\in\omega} I_n \in$	✓	✓	✓	✓

Table 3.1: [Repický, 2017][Theorems 3.2-3.5].

3.6.3 Open problems

Question 3.25. *Is there any possible condition, which implies that classic Egorov's statement (measurable version) does not hold for a given ideal in ZFC (cf. Proposition 3.9)?*

Question 3.26. *Are there any examples of ideals which prove that the classes of all ideals satisfying* $M^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $M^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$, *and* $H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ *are pairwise distinct?*

Question 3.27. *Is there an ideal* I *such that* $\bar{H}^{\Leftarrow}\left(\mathcal{F}_{\to_I}, \xrightarrow{QN}_I\right)$ *does not hold?*

Chapter 4

Special subsets of 2^κ: simple generalizations

The aim of this chapter is to generalize to the case of 2^κ different notions of special subsets defined for 2^ω (see Section 1.3), and check their properties and relations between them. Most of the results presented here have their counterparts in the standard case of 2^ω, and if so I give a reference in the form (ω: [n]).

The results presented in this chapter consist of relatively simple generalizations of some results summarized in [Miller, 1984] and [Bukovský, 2011] to the case of the generalized Cantor space.

The generalized Cantor space, preliminaries and related notions were introduced in Section 1.5.

The results of this chapter are to be included in [Korch and Weiss, 2017].

4.1 Lusin sets for κ

Let $\kappa < \lambda \leq 2^\kappa$. A set $L \subseteq 2^\kappa$ such that $|L| \geq \lambda$, and if $X \subseteq 2^\kappa$ is any κ-meager set, then $|X \cap L| < \lambda$ will be called a **λ-κ-Lusin set**. A κ^+-κ-Lusin set is simply called a **Lusin set for κ**.

Theorem 4.1 (ω: [Bukovský, 2011])**.** *If $\lambda = cov(\boldsymbol{\mathcal{M}}_\kappa) = cof(\boldsymbol{\mathcal{M}}_\kappa)$, then there exists a λ-κ-Lusin set.*

Proof: The proof is straightforward as in the case $\kappa = \omega$. Let $\langle A_\alpha : \alpha < \lambda \rangle$ be a sequence of κ-meagre sets such that for every κ-meagre set A, there exists $\alpha < \kappa$ such that $A \subseteq A_\alpha$. Inductively, for $\alpha < \lambda$, choose

$$x_\alpha \in 2^\kappa \setminus \left(\{x_\beta : \beta < \alpha\} \cup \bigcup_{\beta<\alpha} A_\beta \right).$$

The above is always possible because a complement of a union of $< \lambda$ κ-meagre sets is always not empty and even of cardinality $\geq \lambda$, because for every $x \in 2^\kappa$, $\{x\}$ is κ-meagre. Now, set $L = \{x_\alpha : \alpha < \lambda\}$ to get a λ-κ-Lusin set. □

Immediately, we get the following corollary.

Corollary 4.2 (ω: [Miller, 1984, Bukovský, 2011]). *Assume CH_κ. Then there exists a Lusin set for κ in 2^κ.*

□

On the other hand, the existence of a λ-κ-Lusin set constrains the value of $\mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)$.

Proposition 4.3 (ω: [Bukovský, 2011]). *Assume that λ is a regular cardinal and $\kappa < \lambda \leq 2^\kappa$. If L is a λ-κ-Lusin set, then $|L| \leq cov(\boldsymbol{\mathcal{M}}_\kappa)$.*

Proof: Let L be a λ-κ-Lusin set, and let $\langle A_\alpha \rangle_{\alpha < \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)}$ be a sequence of κ-meagre sets such that $\bigcup_{\alpha < \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)} A_\alpha = 2^\kappa$. Notice that

$$L = \bigcup_{\alpha < \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)} (A_\alpha \cap L)\,.$$

But for any $\alpha < \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)$, $|L \cap A_\alpha| < \lambda$. Since $\lambda \leq |L|$, we get that $|L| \leq \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)$. □

Corollary 4.4 (ω: [Bukovský, 2011]). *Assume that λ is a regular cardinal, $\kappa < \lambda \leq 2^\kappa$, and that there exists a λ-κ-Lusin set L. Then $non(\boldsymbol{\mathcal{M}}_\kappa) \leq \lambda \leq cov(\boldsymbol{\mathcal{M}}_\kappa)$.*

□

4.2 Sets of κ-strong measure zero

A set $A \subseteq 2^\kappa$ will be called **κ-strongly measure zero** ($\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$) if for every $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists $\langle x_\alpha \rangle_{\alpha<\kappa}$ such that $x_\alpha \in 2^{\xi_\alpha}$, $\alpha < \kappa$ and $A \subseteq \bigcup_{\alpha<\kappa}[x_\alpha]$ (see also [Halko, 1996] and [Halko and Shelah, 2001]). Obviously if $A \in [2^\kappa]^\kappa$, then $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$.

The well-known characterization of strongly null sets can be generalized to κ.

Proposition 4.5 (ω: [Bukovský, 2011]). *If $A \in S\boldsymbol{\mathcal{N}}_\kappa$, and $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists $\langle x_\alpha \rangle_{\alpha<\kappa} \in (@^\kappa)^\kappa$ such that $x_\alpha \in 2^{\xi_\alpha}$ for all $\alpha < \kappa$, and*

$$A \subseteq \bigcap_{\alpha<\kappa} \bigcup_{\alpha<\beta<\kappa} [x_\beta].$$

Proof: Let $\langle X_\alpha \rangle_{\alpha<\kappa} \in [\kappa]^\kappa$ be a sequence of pairwise disjoint sets such that $\bigcup_{\alpha<\kappa} X_\kappa = \kappa$. Since $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$, for all $\alpha < \kappa$, there exist $\langle x_\beta \rangle_{\beta \in X_\alpha} \in (2^\kappa)^{X_\alpha}$ such that $A \subseteq \bigcup_{\beta \in X_\alpha} [x_\beta \restriction \xi_\beta]$. Then

$$A \subseteq \bigcap_{\alpha<\kappa} \bigcup_{\alpha<\beta<\kappa} [x_\beta \restriction \xi_\beta].$$

□

In particular, the family of $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ sets forms a κ^+-complete ideal.

Notice also that $2^\kappa \notin \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. Indeed, assume otherwise, and take $\langle a_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that $2^\kappa = \bigcup_{\alpha<\kappa} [a_\alpha \restriction \alpha + 1]$. Let $x \in 2^\kappa$ be such that $x(\alpha) = a_\alpha(\alpha) + 1$. Then

$$x \in 2^\kappa \setminus \bigcup_{\alpha<\kappa} [a_\alpha \restriction \alpha + 1],$$

which is a contradiction.

The **Generalized Borel Conjecture for κ (GBC(κ))** states that $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa = [2^\kappa]^{\leq\kappa}$.

Some properties of this class of sets were considered in [Halko and Shelah, 2001]. In particular, it is proven that if κ is a successor cardinal, then $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ is a $\mathfrak{b}_\kappa$-additive ideal. Under Generalized Martin Axiom for κ (GMA(κ), see [Shelah, 1978]), $\mathfrak{b}_\kappa = 2^\kappa$, so then $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ is 2^κ-additive. Finally, it is proven that GBC(κ) fails for successor κ.

We study some other properties of κ-strong measure zero sets.

Proposition 4.6 (ω_1: [Halko, 1996])**.** *Assume that κ is a strongly inaccessible cardinal. Then the family of all closed subsets of 2^κ which are not $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ does not satisfy 2^κ-chain condition.*

Proof: Let $X \in [\kappa]^\kappa$, and let

$$A_X = \{x \in 2^\kappa : \forall_{\alpha \in \kappa \setminus X} x(\alpha) = 0\}.$$

Notice that A_X is a closed set in 2^κ, and moreover $A_X \notin \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. Indeed, consider $X' = \{\alpha + 1 : \alpha \in X\}$. Let $\langle x_\alpha \rangle_{\alpha \in X} \in (2^\kappa)^X$ be any sequence, and let $x \in 2^\kappa$ be such that

$$x(\alpha) = \begin{cases} x_\alpha(\alpha) + 1, & \text{if } \alpha \in X, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$x \in A_X \setminus \bigcup_{\alpha \in X} [x_\alpha \restriction \alpha + 1],$$

so $A_X \notin \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$.

Since $\kappa^{<\kappa} = \kappa$, we can take a family $\mathcal{F}$ of subsets of κ such that $|\mathcal{F}| = 2^\kappa$, and for all $X, Y \in \mathcal{F}$, $|X \cap Y| < \kappa$ if $X \neq Y$ (see Section 1.5). Consider the family $\mathcal{A} = \{A_X : X \in \mathcal{F}\}$. If $X, Y \in \mathcal{F}$ are such that $X \neq Y$, then

$$A_X \cap A_Y = \left\{x \in 2^\kappa : \forall_{\alpha \in \kappa \setminus (X \cap Y)} x(\alpha) = 0\right\}.$$

so $|A_X \cap A_Y| = 2^\lambda < \kappa$ (κ is strongly inaccessible), for some $\lambda < \kappa$, because $|X \cap Y| < \kappa$. Thus $A_X \cap A_Y \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$, and $\mathcal{A}$ is an antichain of size 2^κ in the family of all closed subsets of 2^κ which are not $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. □

Proposition 4.7 (ω: [Miller, 1984])**.** *Assume CH_κ. Then there exists a Lusin set for κ L such that $L \times L \notin S\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: Let $\langle X_\alpha : \alpha < \kappa^+ \rangle$ be an enumeration of all closed nowhere dense sets, and let $\{y_\alpha : \alpha < \kappa^+\} = 2^\kappa$. Inductively, for $\alpha < \kappa^+$, choose

$$x_\alpha, x'_\alpha \in 2^\kappa \setminus \left(\{x_\beta : \beta < \alpha\} \cup \{x'_\beta : \beta < \alpha\} \cup \bigcup_{\beta < \alpha} X_\beta \right)$$

such that $x_\alpha + x'_\alpha = y_\alpha$. This is possible, since

$$F_\alpha = \{x_\beta : \beta < \alpha\} \cup \{x'_\beta : \beta < \alpha\} \cup \bigcup_{\beta < \alpha} X_\beta$$

is κ-meagre, so $(y_\alpha + F_\alpha) \cup F_\alpha$ is also κ-meagre. Thus, there exists $x_\alpha \notin (y_\alpha + F_\alpha) \cup F_\alpha$. Let $x'_\alpha = x_\alpha + y_\alpha$. Then also $x'_\alpha \notin F_\alpha$.

Obviously L is a Lusin set for κ. Nevertheless, $L \times L$ is not a $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ set. Indeed, let $f : 2^\kappa \times 2^\kappa \to 2^\kappa$ be given by $f(x, x') = x + x'$. Notice that if $\alpha < \kappa$ is a limit ordinal, then $g(0, \alpha) = \alpha$, where g is the canonical well-ordering of $2 \times \kappa$. Therefore, if $x \in 2^\beta$, for $\omega \leq \beta < \kappa$, then $[x]$ when considered as a subset of $2^\kappa \times 2^\kappa$ is contained in $[y] \times [z]$, where $y, z \in 2^\alpha$ with α a limit ordinal such that $\alpha \leq \beta < \alpha + \omega$. This implies that $f[[x]] \subseteq [y + z]$, and thus if $X \subseteq 2^\kappa \times 2^\kappa$ is κ-strongly null, then $f[X]$ is $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ as well. But $f[L] = 2^\kappa$, so $L \notin \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. □

Next, we study the possibility of generalization of Galvin, Mycielski and Solovay ([Galvin et al., 1973]) characterization of strongly null sets. One of the implications can be generalized under no additional assumptions.

Proposition 4.8 (ω: [Miller, 1984, Bukovský, 2011])**.** *Let A be such that for any nowhere dense set F, there exists $x \in 2^\kappa$ such that $(x + A) \cap F = \varnothing$. Then, A is $S\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: Let $\{\xi_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$. Fix an enumeration

$$\{x_\alpha : \alpha < \kappa\} = \boldsymbol{Q},$$

and let

$$F = 2^\kappa \setminus \bigcup_{\alpha<\kappa} [x_\alpha{\restriction}\xi_\alpha].$$

Since F is nowhere dense, there exists $x \in 2^\kappa$ such that $(x + A) \cap F = \varnothing$. Therefore,

$$A \subseteq \bigcup_{\alpha<\kappa} [(x_\alpha + x){\restriction}\xi_\alpha].$$

□

The reversed implication can be generalized if κ is a weakly compact cardinal.

Lemma 4.9 (ω: [Miller, 1984, Bukovský, 2011])**.** *Assume that κ is weakly compact. For any closed nowhere dense set $C \subseteq 2^\kappa$ and $s \in 2^{<\kappa}$, there exists $\xi < \kappa$ and $F \subseteq \{s' \in 2^{<\kappa} : s \subsetneq s'\}$ with $|F| < \kappa$ such that for any $t \in 2^\xi$, there exists $s' \in F$ such that $([s'] + [t]) \cap C = \varnothing$.*

Proof: Let $x \in 2^\kappa$. Since $x + C$ is nowhere dense, we can find $s_x \supsetneq s$ such that $[s_x] \cap (x + C) = \varnothing$. Let $\alpha_x = \mathrm{len}(s_x)$. Then

$$([x{\restriction}\alpha_x] + [s_x]) \cap C = \varnothing.$$

The family $\{[x{\restriction}\alpha_x] : x \in 2^\kappa\}$ is an open covering of 2^κ, and since κ is weakly compact, there exists $\lambda < \kappa$ and a sequence $\langle x_\alpha\rangle_{\alpha<\lambda}$ such that $\{[x_\alpha{\restriction}\alpha_{x_\alpha}] : \alpha < \lambda\}$ covers 2^κ. Let $F = \{s_{x_\alpha} : \alpha < \lambda\}$, and $\xi = \bigcup_{\alpha<\lambda} \alpha_{x_\alpha} < \kappa$. If $t \in 2^\xi$, then there exists $\alpha < \lambda$ such that $x_\alpha{\restriction}\alpha_{x_\alpha} \subseteq t$, so $[t] \subseteq [x_\alpha{\restriction}\alpha_{x_\alpha}]$. Therefore,

$$([s_{x_\alpha}] + [t]) \cap C = \varnothing.$$

□

Theorem 4.10 (ω: [Miller, 1984, Bukovský, 2011])**.** *Assume that κ is a weakly compact cardinal, and $A \subseteq 2^\kappa$ is $S\mathcal{N}_\kappa$. Then for any κ-meagre set F, there exists $x \in 2^\kappa$ such that $(x + A) \cap F = \varnothing$.*

Proof: Let $F = \bigcup_{\alpha<\kappa} C_\alpha$ with C_α closed nowhere dense sets. We can assume that $C_\alpha \subseteq C_\beta$ if $\alpha < \beta$.

We construct inductively a tree $T \subseteq \kappa^{<\kappa}$, along with sequences $\langle\delta_u\rangle_{u\in T}, \langle\xi_u\rangle_{u\in T} \in \kappa^T$, and $\langle s_u\rangle_{u\in T} \in (2^{<\kappa})^T$ such that:

(a) if $u \in T \cap \kappa^\beta$, $\beta < \kappa$, then $\{u' \in T \cap \kappa^{\beta+1} : u \subseteq u'\} = \{u^\frown\alpha : \alpha < \delta_u\}$,

(b) for any $u, u' \in T$ if $u \subsetneq u'$, then $s_u \subsetneq s_{u'}$,

(c) for any $u \in T \cap \kappa^\beta$, $\beta < \kappa$, and $t \in 2^{\xi_u}$, there exists $\alpha < \delta_u$ such that

$$([s_{u^\frown\alpha}] + [t]) \cap C_\beta = \varnothing.$$

Precisely, let $s_\varnothing = \varnothing$. If $u \in T \cap \kappa^\beta$, $\beta < \kappa$, apply Lemma 4.9 to C_β and s_u to get $\xi_u < \kappa$ and $F_u \subseteq \{s' \in 2^{<\kappa} : s \subseteq s'\}$ with $|F| = \delta_u < \kappa$ such that for any $t \in 2^{\xi_u}$, there exists $s' \in F_u$, so that $([s'] + [t]) \cap C_\beta = \varnothing$. Fix an enumeration $F_u = \{s'_{u,\alpha} : \alpha < \delta_u\}$, and put $\{u' \in T \cap \kappa^{\beta+1} : u \subseteq u'\} = \{u^\frown\alpha : \alpha < \delta_u\}$. For all $\alpha < \delta_u$, let $s_{u^\frown\alpha} = s'_{u,\alpha}$. If $\beta < \kappa$ is a limit ordinal, let

$$T \cap \kappa^\beta = \{u \in \kappa^\beta : \forall_{\alpha<\beta} u{\restriction}\alpha \in T\}.$$

Also, for $u \in T \cap \kappa^\beta$, let $s_u = \bigcup_{\alpha<\beta} s_{u\restriction\alpha}$.

Next, define $\langle\delta_\alpha\rangle_{\alpha<\kappa}, \langle\xi_\alpha\rangle_{\alpha<\kappa}$ in the following way. For $\alpha < \kappa$, let

$$\delta_\alpha = \bigcup_{u\in T\cap\kappa^\alpha} \delta_u,$$

and

$$\xi_\alpha = \bigcup_{u\in T\cap\kappa^\alpha} \xi_u.$$

Notice that for all $\alpha < \kappa$, $\delta_\alpha, \xi_\alpha < \kappa$. Indeed, if it is the case for $\alpha < \kappa$, then $|T \cap \kappa^{\alpha+1}| = \delta_\alpha < \kappa$, so $\delta_{\alpha+1}, \xi_{\alpha+1} < \kappa$ since κ is regular. If α is a limit ordinal, then $T \cap \kappa^\alpha \subseteq \delta^\alpha$ with $\delta = \bigcup_{\beta<\alpha} \delta_\beta < \kappa$. And $\delta^\alpha < \kappa$, because κ is strongly inaccessible (every weakly compact cardinal is strongly inaccessible).

A is $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. Therefore, there exists $\langle x_\alpha\rangle_{\alpha\in\kappa}$ such that $x_\alpha \in 2^{\xi_\alpha}$, $\alpha \in \kappa$ and

$$A \subseteq \bigcap_{\beta<\kappa} \bigcup_{\beta<\alpha<\kappa} [x_\alpha].$$

By induction construct $y \in \kappa^\kappa$ such that:

(a) for all $\alpha < \kappa$, $y{\restriction}\alpha \in T$,

(b) for all $\alpha < \kappa$, $\big([s_{y\restriction(\alpha+1)}] + [x_\alpha]\big) \cap C_\alpha = \varnothing$.

Precisely, let $y(\alpha) < \delta_{y\restriction\alpha}$ be such that

$$\big([s_{y\restriction\alpha^\frown y(\alpha)}] + [x_\alpha]\big) \cap C_\alpha = \varnothing.$$

Notice that if α is a limit ordinal, then

$$y{\restriction}\alpha = \bigcup_{\beta<\alpha} y{\restriction}\beta \in T.$$

Finally, let

$$x = \bigcup_{\alpha<\kappa} s_{y\restriction\alpha} \in 2^\kappa.$$

Notice that for all $\beta \leq \alpha < \kappa$, we get $(x + [x_\alpha]) \cap C_\beta = \varnothing$. Therefore,

$$(x + A) \cap F = \varnothing.$$

□

The above propositions imply two following corollaries (see [Bukovský, 2011, Corollary 8.14]).

Proposition 4.11 (ω: [Bukovský, 2011]). *Assume that κ is weakly compact, and $A, B \subseteq 2^\kappa$ are such that $|A| < add(\boldsymbol{\mathcal{M}}_\kappa)$ and $B \in S\boldsymbol{\mathcal{N}}_\kappa$. Then $A \cup B \in S\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: As in the proof of [Bukovský, 2011, Corollary 8.14], assume that $\mathbf{0} \in A \cap B$. Let F be κ-meagre. Then $(A \cup B) + F \subseteq B + A + F \neq 2^\kappa$, by Theorem 4.10, because $A + F$ is κ-meagre. Thus, $A \cup B$ is $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ by Proposition 4.8. □

Proposition 4.12 (ω: [Bukovský, 2011]). *If $A \subseteq 2^\kappa$, and $|A| < cov(\boldsymbol{\mathcal{M}}_\kappa)$, then $A \in S\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: Indeed, if F is κ-meagre, then $A + F = \bigcup_{a \in A} a + F \neq 2^\kappa$. Therefore by Proposition 4.8, $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. □

4.3 κ^+-Concentrated sets

Furthermore, a set $A \subseteq 2^\kappa$ will be called **λ-concentrated on a set** $B \subseteq 2^\kappa$ (for $\kappa < \lambda \leq 2^\kappa$) if for any open set G such that $B \subseteq G$, we have $|A \setminus G| < \lambda$.

The relation between concentrated sets, Lusin sets, and strongly null sets can be easily generalized to κ.

Proposition 4.13 (ω: [Miller, 1984, Bukovský, 2011]). *A set $A \subseteq 2^\kappa$ is a Lusin set for κ if and only if $|A| > \kappa$ and is κ^+-concentrated on every dense set $D \subseteq 2^\kappa$ with $|D| = \kappa$.*

Proof: Indeed, if A is a Lusin set for κ, then $|A| > \kappa$ and moreover, if $D \subseteq 2^\kappa$ is dense with $|D| = \kappa$, and $G \supseteq D$ is open, then G is a dense open set, so $|A \setminus G| = |(2^\kappa \setminus G) \cap A| \leq \kappa$.

On the other hand, let $A \subseteq 2^\kappa$ with $|A| > \kappa$ be a set κ^+-concentrated on every dense set $D \subseteq 2^\kappa$ with $|D| = \kappa$ and let $X \subseteq 2^\kappa$ be a nowhere dense set. Since X is contained in a closed nowhere dense set, $2^\kappa \setminus X \supseteq G$, where G is a dense open set. But there exists a basis of size κ, so there is a dense set $D \subseteq G$ with $|D| = \kappa$, and hence A is κ^+-concentrated on D. Thus, $|A \setminus G| \leq \kappa$, so A is a Lusin set for κ. □

Proposition 4.14 (ω: [Miller, 1984, Bukovský, 2011]). *If a set $A \subseteq 2^\kappa$ is κ^+-concentrated on a set B such that $|B| \leq \kappa$, then $A \in S\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: Fix an enumeration of B, $B = \{b_\alpha : \alpha < \kappa\}$. Let $I = \{\xi_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$, and let $f : \kappa \times \{0,1\} \to \kappa$ be a bijection. Moreover, let

$$G = \bigcup_{\alpha < \kappa} \left[b_\alpha \restriction \xi_{f(\alpha,0)} \right].$$

Then $|A \setminus G| \leq \kappa$, so let $A \setminus G = \{c_\alpha : \alpha < \kappa\}$. Therefore,

$$A \subseteq \bigcup_{\alpha<\kappa} \left[b_\alpha \restriction \xi_{f(\alpha,0)}\right] \cup \bigcup_{\alpha<\kappa} \left[c_\alpha \restriction \xi_{f(\alpha,1)}\right],$$

which proves that $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. □

Corollary 4.15 (ω: [Miller, 1984, Bukovský, 2011])**.** *Every Lusin set for κ is $S\boldsymbol{\mathcal{N}}_\kappa$*

□

On the other hand, we get the following.

Proposition 4.16 (ω: [Miller, 1984])**.** *Assume CH_κ. Then there exists a set $A \subseteq 2^\kappa$ such that $A \in S\boldsymbol{\mathcal{N}}_\kappa$, but A is not κ^+-concentrated on any $B \subseteq 2^\kappa$ with $|B| \leq \kappa$.*

Proof: Let $\langle X_\alpha : \alpha < \kappa^+ \rangle$ be an enumeration of all closed nowhere dense sets. Inductively, for $\alpha < \kappa^+$, choose a perfect nowhere dense set P_α such that

$$P_\alpha \cap \left(\bigcup_{\beta<\alpha} P_\beta \cup \bigcup_{\beta<\alpha} X_\alpha \right) = \varnothing.$$

Choosing such a set is possible since every co-meagre set contains a κ-perfect set (see Section 1.5).

Therefore, for any $\alpha < \beta < \kappa^+$, P_α is a perfect nowhere dense set, and $P_\alpha \cap P_\beta = \varnothing$. Moreover, if X is κ-meagre, then there exists $\xi < \kappa^+$ such that

$$M \cap \bigcup_{\xi<\beta<\kappa^+} P_\beta = \varnothing.$$

Let $I = \{\xi_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$, and let

$$f : [\kappa]^{<\kappa} \times \{0\} \cup \kappa \times \kappa \times \{1\} \to \kappa$$

be a bijection. For $s \in [\kappa]^{<\kappa}$, let $\chi_s \in 2^\kappa$ be the characteristic function of s, and let

$$G = \bigcup_{s\in[\kappa]^{<\kappa}} \left[\chi_s \restriction \xi_{f(s,0)}\right].$$

Notice that G is open and dense, and therefore there exists $\xi < \kappa^+$ such that

$$\bigcup_{\xi<\beta<\kappa^+} P_\beta \subseteq G.$$

Let $L_\alpha \subseteq P_\alpha$ be a Lusin set relativized to P_α, $\alpha < \kappa^+$, and let $A = \bigcup_{\alpha<\kappa^+} L_\alpha$. Let $g : \xi + 1 \to \kappa$ be an injection. Since for all $\beta < \kappa^+$, we have $L_\beta \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$, let $\langle x_{\alpha,\beta} \in 2^{<\kappa} : \alpha < \kappa, \beta \leq \xi \rangle$, be such that for all $\beta \leq \xi$,

$$L_\beta \subseteq \bigcup_{\alpha<\kappa} \left[x_{\alpha,\beta} \restriction \xi_{f(\alpha,g(\beta),1)}\right].$$

Then

$$A \subseteq \bigcup_{\xi<\beta<\kappa^+} P_\beta \cup \bigcup_{\beta\leq\xi} L_\beta \subseteq \bigcup_{s\in[\kappa]^{<\kappa}} \left[\chi_s{\restriction}\xi_{f(s,0)}\right] \cup \bigcup_{\beta\leq\xi}\bigcup_{\alpha<\kappa} \left[x_{\alpha,\beta}{\restriction}\xi_{f(\alpha,g(\beta),1)}\right],$$

so $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$.

On the other hand, if $B \subseteq 2^\kappa$ with $|B| \leq \kappa$, then there exists $\alpha < \kappa^+$ such that $P_\alpha \cap B = \emptyset$. Therefore, $G = 2^\kappa \setminus P_\alpha$ is an open set such that $B \subseteq G$, but

$$A \setminus G = A \cap P_\alpha = L_\alpha,$$

and $|L_\alpha| > \kappa$. □

Proposition 4.17 (ω: [Bukovský, 2011])**.** *If $A \subseteq 2^\kappa$ is $cov(\boldsymbol{\mathcal{M}}_\kappa)$-concentrated on a $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ set, then A is also $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$.*

Proof: Let $f{:}\,2\times\kappa \to \kappa$ be a bijection and $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$. Let $A \subseteq 2^\kappa$ be $\mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)$-concentrated on a $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ set B. There exists a sequence $\langle a_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that

$$B \subseteq G = \bigcup_{\alpha<\kappa} [a_\alpha{\restriction}\xi_{f(0,\alpha)}].$$

G is open, therefore $|A \setminus G| < \mathrm{cov}(\boldsymbol{\mathcal{M}}_\kappa)$. By Proposition 4.12, $A \setminus G \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$, so there exists a sequence $\langle b_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that

$$A \setminus G \subseteq \bigcup_{\alpha<\kappa} [b_\alpha{\restriction}\xi_{f(1,\alpha)}].$$

Therefore,

$$A \subseteq \bigcup_{\alpha<\kappa} [a_\alpha{\restriction}\xi_{f(0,\alpha)}] \cup \bigcup_{\alpha<\kappa} [b_\alpha{\restriction}\xi_{f(1,\alpha)}].$$

□

4.4 Perfectly κ-meagre sets and κ-λ-sets

A set $A \subseteq 2^\kappa$ is a **κ-λ-set** if for any $B \subseteq A$ with $|B| \leq \kappa$ there exists a sequence $\langle B_\alpha\rangle_{\alpha<\kappa}$, where $B_\alpha \subseteq 2^\kappa$ are open, and $\bigcap_{\alpha<\kappa} B_\alpha \cap A = B$.

Furthermore, a set $A \subseteq 2^\kappa$ will be called **perfectly κ-meagre** ($\mathrm{P}\boldsymbol{\mathcal{M}}_\kappa$) if for every perfect $P \subseteq 2^\kappa$, $A \cap P$ is κ-meagre relatively to P. Additionally, a set $A \subseteq 2^\kappa$ will be called **κ-perfectly κ-meagre** ($\mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$) if for every κ-perfect $P \subseteq 2^\kappa$, $A \cap P$ is κ-meagre relatively to P. Obviously, if $A \in \mathrm{P}\boldsymbol{\mathcal{M}}_\kappa$, then $A \in \mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$.

Proposition 4.18 (ω: [Miller, 1984, Bukovský, 2011])**.** *Every κ-λ-set $A \subseteq 2^\kappa$ is perfectly κ-meagre.*

Proof: Let $P \subseteq 2^\kappa$ be a perfect set and $A \cap P \neq \varnothing$. Since there exists a base of size κ, we can find a set $B \subseteq P \cap A$ with $|B| \leq \kappa$ which is dense in $P \cap A$. Let $\langle B_\alpha \rangle_{\alpha<\kappa}$ be a sequence of open sets such that $\bigcap_{\alpha<\kappa} B_\alpha \cap A = B$. Therefore,

$$P \cap A \subseteq B \cup \bigcup_{\alpha<\kappa} (P \cap A \setminus B_\alpha)$$

is κ-meagre in P. □

On the other hand, since not every κ-analytic subset of 2^κ has to have κ-Baire property (see e.g. [Friedman, 2013]), it is not clear whether there always exists a $\mathrm{P}\boldsymbol{\mathcal{M}}_\kappa$ set of cardinality greater then κ.

Question 4.19. *Is there a set $A \subseteq 2^\kappa$ such that $|A| = \kappa^+$ and $A \in P\boldsymbol{\mathcal{M}}_\kappa$ in every model of ZFC.*

A set A will be called a **κ-λ'-set** if for any F such that $|F| \leq \kappa$, $A \cup F$ is a κ-λ-set.

Proposition 4.20 (ω: [Miller, 1984])**.** *A union of κ many κ-λ'-sets is a κ-λ'-set.*

Proof: Indeed, let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a sequence of κ-λ'-sets, and let F be such that $|F| \leq \kappa$. Then, let $\langle G_{\alpha,\beta} \rangle_{\alpha,\beta<\kappa}$ be a collection of open sets such that

$$F = (A_\alpha \cup F) \cap \bigcap_{\beta<\kappa} G_{\alpha,\beta},$$

for any $\alpha < \kappa$. We have that

$$F = \left(F \cup \bigcup_{\alpha<\kappa} A_\alpha \right) \cap \bigcap_{\alpha,\beta<\kappa} G_{\alpha,\beta}.$$

□

Proposition 4.21 (ω: [Miller, 1984])**.** *If X, Y are κ-λ sets, then $X \times Y$ is also a κ-λ set.*

Proof: Let $F \subseteq X \times Y$ be such that $|F| \leq \kappa$. Then $F_1 = \pi_1[F]$ and $F_2 = \pi_2[F]$ are also at most of cardinality κ. Let $\langle G_{\alpha,1} \rangle_{\alpha<\kappa}$ and $\langle G_{\alpha,2} \rangle_{\alpha<\kappa}$ be such that

$$F_1 = X \cap \bigcap_{\alpha<\kappa} G_{\alpha,1}$$

and

$$F_2 = Y \cap \bigcap_{\alpha<\kappa} G_{\alpha,2}.$$

We obtain

$$F = X \times Y \cap \bigcap_{\alpha<\kappa} G_{\alpha,1} \times 2^\kappa \cap \bigcap_{\alpha<\kappa} 2^\kappa \times G_{\alpha,2} \cap \bigcap_{x\in F_1\times F_2\setminus F} (2^\kappa \times 2^\kappa \setminus \{x\}).$$

□

The above proposition can be proven analogously for κ-λ' sets.

A set $A \subseteq 2^\kappa$ is a **κ-s_0-set** if for any κ-perfect $P \subseteq 2^\kappa$, there exists a κ-perfect set $Q \subseteq P$ such that $Q \cap A = \varnothing$.

Proposition 4.22 (ω: [Miller, 1984, Bukovský, 2011])**.** *Every $\mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$ set is a κ-s_0-set.*

Proof: Let P be κ-perfect, and $A \in \mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$. There exists a homeomorphism $h\colon P \to 2^\kappa$. Then $h[A \cap P]$ is κ-meagre, so there exists a κ-perfect set $Q' \subseteq 2^\kappa \setminus h[A]$. Then $Q = h^{-1}[Q']$ is a κ-perfect set included in $P \setminus A$. □

Similar proposition can be proven for $\mathrm{P}\boldsymbol{\mathcal{M}}_\kappa$ sets.

Proposition 4.23 (ω: [Miller, 1984, Bukovský, 2011])**.** *Every $\mathrm{P}\boldsymbol{\mathcal{M}}_\kappa$ set is an s_0-set.*

Proof: If $G = \bigcup_{\alpha<\kappa} G_\alpha \subseteq P$ with G_α nowhere dense in P, we construct by induction a partial function $F\colon 2^{<\kappa} \to T_P$ such that for $s, s' \in \mathrm{dom}F$, $s \subsetneq s'$ if and only if $F(s) \subsetneq F(s')$. Indeed, let $F(\varnothing)$ be such that $[F(\varnothing)] \cap G_0 = \varnothing$. Then, given $F(s)$, $s \in 2^\alpha \cap \mathrm{dom}F$, let $t_s \supsetneq F(s)$ be such that $[t_s] \cap G_{\alpha+1} = \varnothing$. Set $F(s^\frown 0) = t_s{}^\frown 0$ and $F(s^\frown 1) = t_s{}^\frown 1$. For limit $\beta < \kappa$, and $s \in 2^\beta$ such that $s{\restriction}\alpha \in \mathrm{dom}F$ for all $\alpha < \beta$, let $t_s = \bigcup_{\alpha<\beta} F(s{\restriction}\alpha)$. If $t_s \in T_P$, then let $F(s) \supsetneq t_s$ be such that $F(s) \cap G_\beta = \varnothing$. Otherwise, $s \notin \mathrm{dom}F$. Notice that since G_α is κ-meagre, for any $s \in 2^{<\beta} \cap \mathrm{dom}F$ there exists $s' \in 2^\beta \cap \mathrm{dom}F$ such that $s \subseteq s'$. Finally, let

$$T_Q = \{t \in 2^{<\kappa}\colon t \subseteq F(s), s \in \mathrm{dom}F\}.$$

Obviously, $T_Q \subseteq T_P$ is a perfect tree, so $Q = [T_Q]_\kappa$ is a perfect subset of $P \setminus G$. □

Notice that a set having only κ-meagre homeomorphic images may not be perfectly κ-meagre.

Proposition 4.24 (ω: [Miller, 1984])**.** *There exists a set $A \subseteq 2^\kappa$ which is not $P_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$, but its every homeomorphic image is κ-meagre.*

Proof: Let $P \subseteq 2^\kappa$ be a κ-meagre κ-perfect set, e.g. $P = \{x \in 2^\kappa\colon \forall_{\alpha<\kappa} x(\alpha + 1) = 0\}$. Let $\langle P_\xi\rangle_{\xi<2^\kappa}$ be an enumeration of all κ-perfect subsets of P. Find inductively $\langle x_\xi\rangle_{\xi<2^\kappa}$ and $\langle y_\xi\rangle_{\xi<2^\kappa}$ such that $x_\xi \neq y_\xi$, and

$$x_\xi, y_\xi \in P_\xi \setminus \bigcup_{\eta<\xi} \{x_\eta, y_\eta\},$$

for all $\xi < 2^\kappa$. Finally, let

$$A = \boldsymbol{Q} \cup \bigcup_{\xi<2^\kappa} \{x_\xi\}.$$

Notice that A is not $\mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$, as it is not a κ-s_0-set. Indeed, there is no κ-perfect $Q \subseteq P$ such that $Q \cap A = \varnothing$. But if $s \in 2^{<\kappa}$, then $[s^\frown 1] \cap P = \varnothing$, so every open set contains an open subset U such that $|U \cap A| \leq \kappa$. Therefore if h is a homeomorphism, then $h[A]$ has also this property. In particular, for $s \in 2^{<\kappa}$ let $t_s \in 2^{<\kappa}$ be such that $s \subseteq t_s$, and $|h[A] \cap [t_s]| \leq \kappa$. Then

$$A' = \bigcup_{s\in 2^{<\kappa}} (h[A] \cap [t_s])$$

is of cardinality at most κ, and $h[A] \setminus A'$ is nowhere dense. □

On the other hand, for κ-λ-sets we get the following.

Proposition 4.25 (ω: [Miller, 1984])**.** *Let $A, B \subseteq 2^\kappa$, and assume that $f\colon A \to B$ is a one-to-one continuous map. If B is a κ-λ set, then A is also a κ-λ-set.*

Proof: Indeed, let $C \subseteq A$ and $|C| \leq \kappa$. then $f[C] \subseteq B$ is also of cardinality at most κ, and there exists a sequence of open sets $\langle G_\alpha\rangle_{\alpha<\kappa}$ such that

$$B \cap \bigcap_{\alpha<\kappa} G_\alpha = f[C].$$

But since f is one-to-one, we get

$$A \cap \bigcap_{\alpha<\kappa} f^{-1}[G_\alpha] = C.$$

□

A similar statement can be proven for κ-λ'-sets.

Proposition 4.26 (ω: [Miller, 1984])**.** *Let $X, Y \subseteq 2^\kappa$, and assume that $f\colon X \to Y$ is a continuous map. Let $A \subseteq X$ and $B \subseteq Y$ such that B is a κ-λ'-set, and $f\restriction A$ is one-to-one onto B. Then A is also a κ-λ' set.*

Proof: The proof is similar to the proof in the case $\kappa = \omega$. Namely, let $C \subseteq X$ with $|C| < \kappa$. Then there exists a sequence of open sets $\langle G_\alpha\rangle_{\alpha<\kappa}$ such that $(B \cup f[C]) \cap G = f[C]$, where $G = \bigcap_{\alpha<\kappa} G_\alpha$. Therefore,

$$f^{-1}[G] = f^{-1}[B \cap f[C]] \cup f^{-1}[G \setminus B] = (A \cap C) \cup f^{-1}[G \setminus B],$$

because f is one-to-one on A. This implies that

$$f^{-1}[G] \cap (A \cup C) = (A \cap C) \cup (f^{-1}[G] \cap C) = C.$$

□

4.5 κ-σ-Sets

A set $A \subseteq 2^\kappa$ will be called **κ-σ-set** if for any sequence of closed sets $\langle F_\alpha \rangle_{\alpha<\kappa}$, there exists a sequence of open sets $\langle G_\alpha \rangle_{\alpha<\kappa}$ such that

$$A \cap \bigcup_{\alpha<\kappa} F_\alpha = A \cap \bigcap_{\alpha<\kappa} G_\alpha.$$

Proposition 4.27 (ω: [Bukovský, 2011])**.** *Every κ-σ-set is $P\mathcal{M}_\kappa$.*

Proof: Let A be a κ-σ set, and let $P \subseteq 2^\kappa$ be a perfect set, and assume that $P \cap A \neq \varnothing$. Let $C \in [A \cap P]^{\leq\kappa}$ be such that for all $s \in 2^{<\kappa}$ if $[s] \cap P \cap A \neq \varnothing$, then $[s] \cap C \neq \varnothing$. There exists a sequence of open sets $\langle G_\alpha \rangle_{\alpha<\kappa}$ such that

$$C = A \cap \bigcap_{\alpha<\kappa} G_\alpha.$$

Therefore $C \subseteq G_\alpha$, for any $\alpha < \kappa$. Thus, for all $\alpha < \kappa$, $A \setminus G_\alpha$ is nowhere dense in P. Since C is nowhere dense in itself, we have that

$$A = C \cup (A \setminus C) = C \cup A \setminus \bigcap_{\alpha<\kappa} G_\alpha = C \cup \bigcup_{\alpha<\kappa} (A \setminus G_\alpha)$$

is κ-meagre in P. □

4.6 κ-Q-sets

A set $A \subseteq 2^\kappa$ will be called **κ-Q-set** if for any set $B \subseteq A$, there exists a sequence of closed sets $\langle F_\alpha \rangle_{\alpha<\kappa}$ such that

$$A \cap \bigcup_{\alpha<\kappa} F_\alpha = B.$$

Obviously, every κ-Q-set is a κ-σ-set.

Corollary 4.28 (ω: [Bukovský, 2011])**.** *Every κ-Q-set is $P\mathcal{M}_\kappa$.*

□

4.7 κ-Porous sets

If $A \subseteq 2^\kappa$, $\beta < \kappa$, and $x \in 2^\kappa$, let

$$\gamma_\kappa(x, \beta, A) = \min\left(\{\kappa\} \cup \{\alpha < \kappa \colon \exists_{y \in 2^\kappa} [y{\restriction}\alpha] \subseteq [x{\restriction}\beta] \setminus A\}\right).$$

Furthermore, let

$$\mathrm{por}_\kappa(x, A) = \bigcap_{\gamma<\kappa} \bigcup_{\gamma<\beta<\kappa} \min\{\alpha \leq \kappa \colon \beta \cdot \alpha \geq \gamma_\kappa(x, \beta, A)\}.$$

A set $A \subseteq 2^\kappa$ is called **κ-porous** if for every $x \in A$, $\mathrm{por}_\kappa(x, A) < \kappa$.

Proposition 4.29 (ω: [Bukovský, 2011]). *If $A \subseteq 2^\kappa$ is κ-porous, then it is nowhere dense.*

Proof: Let $A \subseteq 2^\kappa$ be a κ-porous set, and let $s \in 2^{<\kappa}$ be such that $[s] \cap A \neq \emptyset$. Let $x \in A \cap [s]$. There exists $\mathrm{len}(s) \leq \beta < \kappa$ such that $\gamma_\kappa(x, \beta, A) < \kappa$. Therefore, there exists $\beta < \alpha < \kappa$ and $y \in 2^\kappa$ such that

$$[y{\restriction}\alpha] \subseteq [x{\restriction}\beta] \setminus A \subseteq [s] \setminus A.$$

□

4.8 Cover selection principles in 2^κ

In this section we study analogues of cover selection properties for subsets of 2^κ.

4.8.1 κ-γ-Sets

A family of open subsets $\mathcal{U}$ of a set X will be called a **κ-cover** of X if for any $A \in [X]^{<\kappa}$ there exists $U \in \mathcal{U}$ such that $A \subseteq U$. It is a **γ-κ-cover** if $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$, and

$$X \subseteq \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} U_\beta.$$

Notice that every subsequence of length κ of a κ-γ-cover is still a κ-γ-cover.

The family of all κ-covers of X will be denoted by $\Omega_\kappa(X)$, and the family of all κ-γ-covers will be denoted by $\Gamma_\kappa(X)$. The family of all open covers of size κ of X, is denoted by $\mathcal{O}_\kappa(X)$. The underlying set can be omitted in this notation if it is apparent from the context. We always assume that the covers which are considered are proper, i.e. the set itself is never an element of its cover.

$X \subseteq 2^\kappa$ will be called a **κ-γ-set** if for every open κ-cover $\mathcal{U}$ of X there exists a sequence $\langle U_\alpha \rangle_{\alpha<\kappa} \in \mathcal{U}^\kappa$ such that $\{U_\alpha : \alpha < \kappa\}$ is a κ-γ-cover.

If $\mathcal{A}, \mathcal{B}$ are families of open covers of a set X, we shall say that it has $S_1^\kappa(\mathcal{A}, \mathcal{B})$ **property** if for every sequence $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$, there exists a sequence $\langle U_\alpha \rangle_{\alpha<\kappa}$ such that $U_\alpha \in \mathcal{U}_\alpha$, for all $\alpha < \kappa$, and $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$.

We aim to prove that similarly to the case $\kappa = \omega$, κ-γ-sets can be characterized in terms of selection principles. First we need the following easy observation.

Lemma 4.30 (ω: [Bukovský, 2011]). *Let X be a subset of a κ-additive topological space, and $\mathcal{A}, \mathcal{B}$ be any families of open covers of cardinality κ of X such that:*

(a) if $\mathcal{V} \in \mathcal{B}$ is a refinement of an open cover $\mathcal{U}$, then there exists $\mathcal{U}' \subseteq \mathcal{U}$ with $\mathcal{U}' \in \mathcal{B}$,

(b) if $\beta < \kappa$, and $\langle \mathcal{U}_\alpha \rangle_{\alpha<\beta} \in \mathcal{A}^\beta$, then there exists $\mathcal{U} \in \mathcal{A}$ such that $\mathcal{U}$ is a refinement of $\mathcal{U}_\alpha$ for every $\alpha < \beta$,

(c) if $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$, and $\mathcal{V}_\beta = \{V_{\alpha,\beta} : \alpha < \gamma_\beta\}$ for $\beta < \kappa$ and $\langle \gamma_\beta \rangle_{\beta\in\kappa} \in \kappa^\kappa$ are such that $U_\beta \subseteq V_{\alpha,\beta}$ for all $\beta < \kappa, \alpha < \gamma_\beta$, then $\bigcup_{\beta<\kappa} \mathcal{V}_\beta \in \mathcal{B}$.

Then X satisfies $S_1^\kappa(\mathcal{A},\mathcal{B})$ if and only if for every $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ such that $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$, for all $\alpha < \beta < \kappa$, there exists $\langle U_\alpha \rangle_{\alpha<\kappa}$ such that $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$, and $U_\alpha \in \mathcal{U}_\alpha$ for all $\alpha < \kappa$.

Proof: Let X be a set satisfying the premise of the Lemma 4.30 along with families $\mathcal{A}$ and $\mathcal{B}$ and such that for every $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ such that $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$, for all $\alpha < \beta < \kappa$, there exists $\langle U_\alpha \rangle_{\alpha<\kappa}$ such that $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$, and $U_\alpha \in \mathcal{U}_\alpha$ for all $\alpha < \kappa$.

Let $\langle \mathcal{W}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ be arbitrary. By induction we construct $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ such that $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$ and $\mathcal{W}_\alpha$, for all $\alpha < \beta < \kappa$. Hence, there exists $\langle U_\alpha \rangle_{\alpha<\kappa}$ such that $\{U_\alpha : \alpha < \kappa\} \in \mathcal{B}$, and $U_\alpha \in \mathcal{U}_\alpha$ for all $\alpha < \kappa$. For all $\alpha < \kappa$, let $O_\alpha \in \mathcal{W}_\alpha$ be such that $U_\alpha \subseteq O_\alpha$. Then $\{U_\alpha : \alpha < \kappa\}$ is a refinement of $\{O_\alpha : \alpha < \kappa\}$ thus there exists $A \subseteq \kappa$ such that $\{O_\alpha : \alpha \in A\} \in \mathcal{B}$.

Now, choose $\langle V_\alpha \rangle_{\alpha<\kappa}$ such that $V_\alpha = O_\alpha$ if $\alpha \in A$, and $V_\alpha \in \mathcal{W}_\alpha$ be such that $O_\beta \subseteq V_\alpha$ for $\beta = \min A \setminus \alpha$. Then $\{V_\alpha : \alpha < \kappa\} \in \mathcal{B}$, and for any $\alpha < \kappa$, $V_\alpha \in \mathcal{W}_\alpha$. □

Lemma 4.31 (ω: [Bukovský, 2011])**.** *If $X \subseteq 2^\kappa$, then $\mathcal{A} = \Omega_\kappa(X)$ and $\mathcal{B} = \Gamma_\kappa(X)$ satisfy the premise of Lemma 4.30.*

Proof: Recall that an intersection of less than κ open sets in 2^κ is still open. The rest of the proof is obvious. □

Theorem 4.32 (ω: [Bukovský, 2011])**.** *A set $X \subseteq 2^\kappa$, with $|X| \geq \kappa$ is a κ-γ-set if and only if it has $S_1^\kappa(\Omega_\kappa, \Gamma_\kappa)$.*

Proof: As in the case $\kappa = \omega$, choose a sequence of distinct points $\langle x_\alpha \rangle_{\alpha<\kappa} \in X^\kappa$. Assume that $\langle \mathcal{W}_\alpha \rangle_{\alpha<\kappa} \in (\Omega_\kappa(X))^\kappa$ is a sequence of covers such that for $\alpha < \beta$, $\mathcal{W}_\beta$ is a refinement of $\mathcal{W}_\alpha$. Let

$$\mathcal{U} = \{U \setminus \{x_\alpha\} : U \in \mathcal{W}_\alpha, \alpha \in \kappa\}.$$

Notice that $\mathcal{U}$ is a κ-cover of X. Since X is a κ-γ-set, there exists a κ-γ-cover $\mathcal{V} \subseteq \mathcal{U}$. Let $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$, and let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that $V_\alpha = U_\alpha \setminus \{x_{\xi_\alpha}\}$

with $U_\alpha \in \mathcal{W}_{\xi_\alpha}$. Notice that $|\{\xi_\alpha : \alpha < \kappa\}| = \kappa$. Indeed, if this is not the case, an ordinal $\gamma < \kappa$ occurs cofinitely many times in the sequence $\langle \xi_\alpha \rangle_{\alpha \in \kappa}$, thus

$$x_\gamma \notin \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} V_\beta.$$

Therefore, let $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that $\langle \xi_{\delta_\alpha} \rangle_{\alpha<\kappa}$ is a strictly increasing sequence. Notice that $\langle V_{\delta_\alpha} \rangle_{\alpha<\kappa}$ is also a κ-γ-cover of X.

Let $A = \{\xi_{\delta_\alpha} : \alpha < \kappa\}$, and choose $\langle W_\alpha \rangle_{\alpha<\kappa}$ such that $W_{\xi_\alpha} = V_\alpha$ if $\xi_\alpha \in A$, and otherwise choose $W_\alpha \in \mathcal{W}_\alpha$ such that $V_\beta \subseteq W_\alpha$ for $\beta = \min A \setminus \alpha$. Then $\{W_\alpha : \alpha < \kappa\} \in \Gamma_\kappa$, and for any $\alpha < \kappa$, $W_\alpha \in \mathcal{W}_\alpha$. Therefore, by Lemmas 4.30 and 4.31, X satisfies $S_1^\kappa(\Omega_\kappa, \Gamma_\kappa)$. □

Corollary 4.33 (ω: [Bukovský, 2011]). *Every κ-γ-set satisfies $S_1^\kappa(\Gamma_\kappa, \Gamma_\kappa)$.*

Proof: Obviously, every κ-γ-cover is a κ-cover. □

Finally, we prove that every union of κ many closed subsets of κ-γ-set is κ-γ-set as well.

Proposition 4.34 (ω: [Bukovský, 2011]). *A κ-union of closed subsets of a κ-γ-set is a κ-γ-set.*

Proof: Let $F = \bigcup_{\alpha<\kappa} F_\alpha$ with $F_\alpha \subseteq X$, where X is a κ-γ-set and F_α are closed in X. Assume that for $\alpha < \beta < \kappa$, $F_\alpha \subseteq F_\beta$, and let $\mathcal{U}$ be a κ-cover of F. For any $\alpha < \kappa$,

$$\mathcal{U}_\alpha = \{U \cup (X \setminus F_\alpha) : U \in \mathcal{U}\}$$

is a κ-cover of X. Thus, by Theorem 4.32, there exists a sequence $\langle U_\alpha \rangle_{\beta<\kappa}$ such that $U_\alpha \in \mathcal{U}_\alpha$, and

$$X \subseteq \bigcup_{\gamma<\kappa} \bigcap_{\gamma<\beta<\kappa} U_\beta.$$

Let $\langle V_\alpha \rangle_{\alpha,\beta<\kappa} \in \mathcal{U}^\kappa$ be such that $U_\alpha = V_\alpha \cup (X \setminus F_\alpha)$.

Then

$$F \subseteq \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} V_\beta,$$

because if $x \in F$, there exists $\alpha < \kappa$ such that $x \notin X \setminus F_\beta$ for all $\beta < \kappa$ with $\alpha < \beta$. Thus,

$$x \in \bigcap_{\alpha<\beta<\kappa} V_\beta.$$

□

4.8.2 κ-Hurewicz property

A cover $\mathcal{U}$ of a set X is **essentially of size** κ if for every $\mathcal{V} \in [\mathcal{U}]^{<\kappa}$, $X \setminus \bigcup \mathcal{V} \neq \varnothing$.

We will say that a set X satisfies $U^{\kappa}_{<\kappa}(\mathcal{A},\mathcal{B})$ **principle** if for every sequence $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in \mathcal{A}^\kappa$ of covers essentially of size κ, there exists $\langle \mathcal{V}_\alpha \rangle_{\alpha<\kappa}$ such that $\mathcal{V}_\alpha \in [\mathcal{U}_\alpha]^{<\kappa}$ for all $\alpha<\kappa$, and $\{\bigcup \mathcal{V}_\alpha : \alpha<\kappa\} \in \mathcal{B}$.

A set X has κ-**Hurewicz property** if it satisfies $U^{\kappa}_{<\kappa}(\mathcal{O}_\kappa,\Gamma_\kappa)$ principle.

Proposition 4.35 (ω: [Bukovský, 2011])**.** *If X satisfies $S^{\kappa}_1(\Gamma_\kappa,\Gamma_\kappa)$, then it has κ-Hurewicz property.*

Proof: Assume that $\langle \mathcal{U}_\alpha \rangle_{\alpha\in\kappa}$ is a sequence of open covers of X which are essentially of size κ. Let $\mathcal{U}_\beta = \{U_{\beta,\alpha} : \alpha<\kappa\}$, for all $\beta<\kappa$, and let $V_{\beta,\alpha} = \bigcup_{\gamma<\alpha} U_{\beta,\gamma}$ for all $\alpha,\beta<\kappa$.

Notice that, for any $\beta<\kappa$, $\langle V_{\beta,\alpha} \rangle_{\alpha<\kappa}$ is a κ-γ-cover of X. Indeed, if there exists

$$x \in X \setminus \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\gamma<\kappa} V_{\beta,\gamma} = X \setminus \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\gamma<\kappa} \bigcup_{\delta<\gamma} U_{\beta,\delta},$$

then $x \notin U_{\beta,\delta}$ for all $\delta<\kappa$.

Thus, there exists a sequence $\langle \xi_\alpha \rangle_{\alpha\in\kappa} \in \kappa^\kappa$ such that $\{V_{\alpha,\xi_\alpha} : \alpha<\kappa\}$ is a κ-γ-cover. For $\alpha<\kappa$, let $\mathcal{V}_\alpha = \{U_{\alpha,\beta} : \beta<\xi_\alpha\}$. Then

$$\{\bigcup \mathcal{V}_\alpha : \alpha<\kappa\} = \{V_{\alpha,\xi_\alpha} : \alpha<\kappa\}$$

is the desired κ-γ-cover. □

Corollary 4.36 (ω: [Bukovský, 2011])**.** *If X is a κ-γ-set, it has κ-Hurewicz property.*

□

On the other had, no Lusin set in κ can have κ-Hurewicz property. Indeed, we have the following.

Lemma 4.37 (ω: [Bukovský, 2011])**.** *If $A \subseteq 2^\kappa$ with empty interior has κ-Hurewicz property, then A is κ-meagre.*

Proof: Let $\{s_\alpha : \alpha<\kappa\} = 2^{<\kappa}$, and let $\{x_\alpha : \alpha<\kappa\}$ be such that $x_\alpha \in [s_\alpha] \setminus A$ for all $\alpha<\kappa$, and let $U_{\alpha,\beta} = 2^\kappa \setminus [x_\alpha{\restriction}\beta]$. Finally, let $\mathcal{U}_\alpha = \{U_{\alpha,\beta} : \beta<\kappa\}$ for $\alpha<\kappa$. For $\alpha<\kappa$, $\mathcal{U}_\alpha$ is an increasing open cover of A, which is essentially of size κ.

Since A has κ-Hurewicz property, there exists $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\{U_{\alpha,\xi_\alpha} : \alpha<\kappa\}$$

is a κ-γ-cover of A. In other words,

$$A = \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} U_{\beta,\xi_\beta}.$$

Obviously,

$$\bigcap_{\alpha<\beta<\kappa} U_{\beta,\xi_\beta} = \bigcap_{\alpha<\beta<\kappa} 2^\kappa \setminus [x_\beta{\restriction}\xi_\beta] = 2^\kappa \setminus \bigcup_{\alpha<\beta<\kappa} [x_\beta{\restriction}\xi_\beta]$$

is a nowhere dense set for any $\alpha < \kappa$. Hence, A is κ-meagre. □

Corollary 4.38 (ω: [Bukovský, 2011]). *If $\kappa < \lambda \leq 2^\kappa$, and $L \subseteq 2^\kappa$ is a λ-κ-Lusin set, then L does not have κ-Hurewicz property.*

□

4.8.3 κ-Menger property

A set has **κ-Menger property** if it satisfies $U^\kappa_{<\kappa}(\mathcal{O}_\kappa, \mathcal{O}_\kappa)$ principle.

Despite that every Lusin set for κ lacks κ-Hurewicz property (see Corollary 4.38, it has κ-Menger property.

Proposition 4.39 (ω: [Bukovský, 2011]). *Let $L \subseteq 2^\kappa$ be a Lusin set in κ. Then L has κ-Menger property.*

Proof: Let $\{s_\alpha : \alpha < \kappa\} = \{s \in 2^{<\kappa} : [s] \cap L \neq 0\}$, and let $\{x_\alpha : \alpha < \kappa\}$ be such that $x_\alpha \in [s_\alpha] \cap L$ for all $\alpha < \kappa$.

Let $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa}$ be a sequence of open covers essentially of size κ. For $\alpha < \kappa$, let $U_\alpha \in \mathcal{U}_\kappa$ be such that $x_\alpha \in U_\alpha$. Then, $L \setminus \bigcup_{\alpha<\kappa} U_\alpha$ is nowhere dense, hence $|L \setminus \bigcup_{\alpha<\kappa} U_\alpha| \leq \kappa$. Thus let

$$L \setminus \bigcup_{\alpha<\kappa} U_\alpha = \{y_\alpha : \alpha < \kappa\}.$$

For all $\alpha < \kappa$, let $V_\alpha \in \mathcal{U}_\alpha$ be such that $y_\alpha \in V_\alpha$. Let $\mathcal{V}_\alpha = \{U_\alpha, V_\alpha\}$, for $\alpha < \kappa$. Then $\{\bigcup_{\alpha<\kappa} \mathcal{V}_\alpha : \alpha < \kappa\}$ is an open cover of L. □

4.8.4 κ-Rothberger property

A set has **κ-Rothberger property** if it satisfies $S^\kappa_1(\mathcal{O}_\kappa, \mathcal{O}_\kappa)$ principle. Obviously, this property implies κ-Menger property.

Proposition 4.40 (ω: [Bukovský, 2011]). *If $A \subseteq 2^\kappa$ has κ-Rothberger property, then $A \in S\mathcal{N}_\kappa$.*

Proof: Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a sequence of ordinals. For $\alpha < \kappa$, let $\mathcal{U}_\alpha = \{[s] : s \in 2^{\xi_\alpha}\}$. Since A has κ-Rothberger property, we get that there exist $\langle s_\alpha \rangle_{\alpha<\kappa}$ such that $s_\alpha \in 2^{\xi_\alpha}$ for all $\alpha < \kappa$, and $\{[s_{\xi_\alpha}] : \alpha < \kappa\}$ is a cover of A. □

Corollary 4.41 (ω: [Bukovský, 2011]). *The generalized Cantor space 2^κ does not have κ-Rothberger property.*

□

Proposition 4.14 can be formulated in a stronger form.

Proposition 4.42 (ω: [Bukovský, 2011]). *If $A \subseteq 2^\kappa$ is κ^+-concentrated on a set $B \subseteq 2^\kappa$ with $|B| \leq \kappa$, then A has κ-Rothberger property.*

Proof: We modify the proof of Proposition 4.14. Fix an enumeration of B, $B = \{b_\alpha : \alpha < \kappa\}$. Let $\langle \mathcal{U}_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{O}_\kappa)^\kappa$ be a sequence of open covers of size κ, and let $f : \kappa \times \{0,1\} \to \kappa$ be a bijection. For all $\alpha < \kappa$, let $\mathcal{U}_\alpha = \{U_{\alpha,\beta} : \beta < \kappa\}$. Let $\langle \xi_\alpha \rangle_{\alpha\in\kappa} \in \kappa^\kappa$ be such that $b_\alpha \in U_{f(\alpha,0),\xi_\alpha}$ for all $\alpha < \kappa$. Moreover, let $G = \bigcup_{\alpha<\kappa} U_{f(\alpha,0),\xi_\alpha}$. Then $|A \setminus G| \leq \kappa$, so let $A \setminus G = \{c_\alpha : \alpha < \kappa\}$.

Find $\langle \delta_\alpha \rangle_{\alpha\in\kappa} \in \kappa^\kappa$ such that $c_\alpha \in U_{f(\alpha,1),\delta_\alpha}$ for all $\alpha < \kappa$. Then,

$$A \subseteq \bigcup_{\alpha<\kappa} U_{f(\alpha,0),\xi_\alpha} \cup \bigcup_{\alpha<\kappa} U_{f(\alpha,1),\delta_\alpha}.$$

□

This allows us to formulate a stronger version of Proposition 4.39.

Corollary 4.43 (ω: [Bukovský, 2011]). *Every Lusin set for κ has κ-Rothberger property.*

Proof: By Proposition 4.13, every Lusin set for κ satisfies the premise of Proposition 4.42. □

Lemma 4.44 (ω: [Bukovský, 2011]). *If $X \subseteq 2^\kappa$, then $\mathcal{A} = \mathcal{O}_\kappa(X)$ and $\mathcal{B} = \mathcal{O}_\kappa(X)$ satisfy the premise of Lemma 4.30.*

□

Theorem 4.45 (ω: [Bukovský, 2011]). *Every κ-γ-set of cardinality $\geq \kappa$ has κ-Rothberger property.*

Proof: Assume that $X \subseteq 2^\kappa$ is a κ-γ-set, and let $\langle \mathcal{U}_\alpha \rangle_{\alpha\in\kappa}$ be a sequence of open covers of X of size κ such that $\mathcal{U}_\beta$ is a refinement of $\mathcal{U}_\alpha$ for all $\alpha < \beta$. Let $\langle a_\alpha \rangle_{\alpha<\kappa} \in X^\kappa$ be a sequence of distinct points. Let $b : \langle \bigcup_{\alpha<\kappa} \{\alpha\} \times \alpha, \leq_{\text{lex}} \rangle \to \kappa$ be the order isomorphism.

For $\alpha < \kappa$, let

$$\mathcal{V}_\alpha = \left\{ \bigcup_{\beta<\alpha} U_\beta \setminus \{a_\alpha\} : \langle U_\beta \rangle_{\beta<\alpha} \text{ such that } \forall_{\beta<\alpha} U_\beta \in \mathcal{U}_{b(\alpha,\beta)} \right\},$$

and let $\mathcal{V} = \bigcup_{\alpha<\kappa} \mathcal{V}_\alpha$.

Notice that if $B \subseteq X$ is such that $|B| = \lambda < \kappa$, then there exists $\alpha < \kappa$ such that $\lambda < \alpha$, and $a_\alpha \notin B$. Let $B = \{b_\alpha : \alpha < \lambda\}$. For $\beta < \lambda$, let $U_\beta \in \mathcal{U}_{b(\alpha,\beta)}$ be such that $b_\beta \in U_\beta$, and for $\lambda \leq \beta < \alpha$, let $U_\beta \in \mathcal{U}_{b(\alpha,\beta)}$ be arbitrary. Then

$$B \subseteq \bigcup_{\beta\in\alpha} U_\beta \in \mathcal{V}_\alpha \subseteq \mathcal{V}.$$

Thus, $\mathcal{V}$ is a κ-cover of X.

Since X is a κ-γ-set, there exist a κ-γ-cover $\langle V_\alpha \rangle_{\alpha<\kappa} \in \mathcal{V}^\kappa$. Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that $V_\alpha \in \mathcal{V}_{\xi_\alpha}$. Notice that $|\{\xi_\alpha : \alpha < \kappa\}| = \kappa$, because for all $\alpha < \kappa$, $a_\alpha \notin \bigcup \mathcal{V}_\alpha$. Therefore, there exists an increasing sequence $\langle \delta_\alpha \rangle_{\alpha\in\kappa}$ such that $\langle \xi_{\delta_\alpha} \rangle_{\alpha\in\kappa}$ is strictly increasing. Then $\langle V_{\delta_\alpha} \rangle_{\alpha<\kappa}$ is a κ-γ-cover as well.

For $\alpha < \kappa$, let $\langle U_{\alpha,\beta} \rangle_{\beta<\xi_{\delta_\alpha}}$ be such that $U_{\alpha,\beta} \in \mathcal{U}_{b(\xi_{\delta_\alpha},\beta)}$, for $\alpha < \kappa$, $\beta < \xi_{\delta_\alpha}$, and

$$V_{\delta_\alpha} = \bigcup_{\beta<\xi_{\delta_\alpha}} U_\beta \setminus \{a_{\xi_{\delta_\alpha}}\}.$$

Let $A = \{b(\xi_{\delta_\alpha}, \beta) : \alpha < \kappa, \beta < \xi_{\delta_\alpha}\}$, and choose $\langle W_\alpha \rangle_{\alpha<\kappa}$ such that $W_\alpha = U_{\beta,\gamma} \in \mathcal{U}_\alpha$ if $\alpha \in A$ and $\alpha = b(\beta, \gamma)$. If $\alpha \notin A$, choose $W_\alpha \in \mathcal{U}_\alpha$ be such that $W_\alpha \supseteq W_\beta$ for $\beta = \min A \setminus \alpha$. Then $\{W_\alpha : \alpha < \kappa\} \in \mathcal{O}_\kappa$, and for any $\alpha < \kappa$, $W_\alpha \in \mathcal{U}_\alpha$. Therefore, by Lemmas 4.30 and 4.44, X satisfies $S_1^\kappa(\mathcal{O}_\kappa, \mathcal{O}_\kappa)$. □

Corollary 4.46 (ω: [Bukovský, 2011])**.** *Every κ-γ-set is κ-strongly null.*

Proof: Follows by Corollary 4.40. □

Corollary 4.47. *The generalized Cantor space 2^κ is not a κ-γ-set.*

□

Thus, no κ-perfect subset of 2^κ is a κ-γ-set. Nevertheless, the following question remains unanswered.

Question 4.48. *Is there a closed subset of 2^κ which is a κ-γ-set?*

We finish by proving a Lemma which becomes useful in the next chapter.

Lemma 4.49 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a weakly inaccessible cardinal. Let $A \subseteq 2^\kappa$ be a κ-γ set which is not closed. Then there exists $B \in [\kappa]^\kappa$ such that for all $C \in [B]^\kappa$, $\chi_C \notin A$.*

Proof: Let $A \subseteq 2^\kappa$ be a κ-γ set, and let $b\colon \bigcup_{\alpha<\kappa}\{\alpha\}\times\alpha \to \kappa$ be a bijection. Notice that $2^\kappa \setminus A$ is not an open set. Therefore, there exists $y \in 2^\kappa \setminus A$ such that $A \cap [y\restriction\alpha] \neq \emptyset$, for any $\alpha < \kappa$. Choose inductively a sequence $\langle x_\alpha\rangle_{\alpha<\kappa} \in A^\kappa$ such that if for $\alpha, \beta < \kappa$, $x_\alpha = x_\beta$ only if $\alpha = \beta$, and for every $\gamma < \kappa$ there exists $\alpha < \kappa$ such that $y\restriction\gamma = x_\alpha\restriction\gamma$. To achieve this, take any $x_0 \in A$, and for $\alpha < \kappa$, let

$$\xi = \bigcup_{\beta<\alpha}\bigcup\{\gamma<\kappa\colon y\restriction\gamma = x_\beta\restriction\gamma\}.$$

Let $x_\alpha \in A \cap [y\restriction\xi+1]$.

If $I \subseteq \kappa$ and $s \in 2^I$, let $[s]$ denote $\{x \in 2^\kappa\colon x\restriction I = s\}$. For $\alpha < \kappa$, let

$$\mathcal{U}_\alpha = \left\{\bigcup_{s\in S}[s] \cap A \setminus \bigcup_{\alpha\leq\beta<\kappa}\{x_\beta\}\colon S \in \left[2^{b[\{\alpha\}\times\alpha]}\right]^{<|\alpha|}\right\},$$

and let $\mathcal{U} = \bigcup_{\alpha<\kappa}\mathcal{U}_\alpha$. Notice that $\mathcal{U}$ is a κ-cover of Y, because κ is weakly inaccessible. Therefore, we have $\langle U_\alpha\rangle_{\alpha<\kappa} \in \mathcal{U}^\kappa$ such that

$$A \subseteq \bigcup_{\alpha<\kappa}\bigcap_{\alpha<\beta<\kappa} U_\beta.$$

But since

$$x_\alpha \notin \bigcup_{\beta<\alpha}\bigcup\mathcal{U}_\beta,$$

for all $\alpha \in \kappa$, we get that for any $\alpha < \kappa$, there exists $\xi < \kappa$ such that for all $\xi < \beta < \kappa$, there exists $\alpha < \gamma < \kappa$ such that $U_\beta \in \mathcal{U}_\gamma$. Therefore, we can choose inductively increasing sequences $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $\langle\delta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $U_{\xi_\alpha} \in \mathcal{U}_{\delta_\alpha}$, for any $\alpha < \kappa$.

Fix $\alpha < \kappa$, and let $S_\alpha \in \left[2^{b[\{\delta_\alpha\}\times\delta_\alpha]}\right]^{<|\delta_\alpha|}$ be such that

$$U_{\xi_\alpha} = \bigcup_{s\in S_\alpha}[s] \cap A \setminus \bigcup_{\delta_\alpha\leq\beta<\kappa}\{x_\beta\}.$$

There exists $\eta_\alpha < \delta_\alpha$ such that $\{b(\delta_\alpha, \eta_\alpha)\} \notin S_\alpha$. Let

$$B = \{b(\delta_\alpha, \eta_\alpha)\colon \alpha < \kappa\}.$$

Then, for all $C \in [B]^\kappa$, $\chi_C \notin A$. Indeed, if $C \in [B]^\kappa$, then for every $\alpha < \kappa$, there is $\alpha < \beta < \kappa$ such that

$$C \cap b[\{\delta_\beta\}\times\delta_\beta\}] = \{b(\delta_\beta, \eta_\beta)\} \notin S_\beta.$$

For such β, $\chi_C \notin U_{\xi_\beta}$, therefore for all $\alpha < \kappa$,

$$\chi_C \notin \bigcap_{\alpha<\beta<\kappa} U_{\xi_\beta},$$

and hence

$$\chi_C \notin \bigcup_{\alpha<\kappa}\bigcap_{\alpha<\beta<\kappa} U_\beta \supseteq A.$$

□

Chapter 5

Generalization of other notions of small sets in 2^κ

In this chapter we present generalizations of some less common notions of small sets.

Some of the results presented here have their counterparts in the standard case of 2^ω (or $\omega_1^{\omega_1}$), and if so, we give a reference in the form (ω: [n]) (or (ω_1: [n])).

In this chapter we use notation and notions described in Sections 1.3, 1.5, and Chapter 4.

The results of this chapter are to be included in [Korch and Weiss, 2017].

5.1 X-small sets

In this section we present some generalizations of the results from [Halko, 1996, Chapter 4].

If $X \subseteq \kappa$, then a set $A \subseteq 2^\kappa$ will be called X**-small** if there exists $\langle a_\alpha \rangle_{\alpha \in X} \in (2^\kappa)^X$ such that

$$A \subseteq \bigcup_{\alpha \in X} [a_\alpha \restriction \alpha].$$

Notice that A is $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ if it is X-small for any $X \in [\kappa]^\kappa$.

Consider the following ordering on $[\kappa]^\kappa$. For $X, Y \subseteq [\kappa]^\kappa$, let $X < Y$ (respectively, $X \leq Y$) if and only if there exists a bijection $F\colon X \to Y$ such that for all $\alpha \in X$, $\alpha < F(\alpha)$ (respectively, $\alpha \leq F(\alpha)$). Let $X + 1 = \{\alpha + 1 \colon \alpha \in X\}$. Notice that if $X < Y$, then $X + 1 \leq Y$.

Let $X, Y \in [\kappa]^\kappa$ be such that $X < Y$. Then, the family of Y-small sets is a proper subfamily of X-small sets (see [Halko, 1996]). Indeed, it is sufficient to prove that there exists a X-small set which is not a $(X + 1)$-small. Assume

that $A \subseteq \bigcup_{\alpha\in X}[a_\alpha{\restriction}\alpha]$ with $\langle a_\alpha\rangle_{\alpha\in X} \in (2^\kappa)^X$. We can assume that if $\beta,\alpha \in X$ with $\beta > \alpha$, then $a_\beta \notin [a_\alpha{\restriction}\alpha]$. To obtain a contradiction assume that

$$A \subseteq B = \bigcup_{\alpha\in X}[b_\alpha{\restriction}\alpha+1]$$

with $\langle b_\alpha\rangle_{\alpha\in X} \in (2^\kappa)^X$. Then consider $x \in 2^\kappa$ such that

$$x(\alpha) = \begin{cases} a_{\min X}(\alpha), & \text{if } \alpha < \min X,\\ b_\alpha(\alpha)+1, & \text{if } \alpha \in X,\\ 0, & \text{otherwise.}\end{cases}.$$

Notice that $x \in [a_{\min A}{\restriction}\min A] \subseteq A$, but $x \notin B$, which is a contradiction.

Let $\lambda < \kappa$. We say that a set $A \subseteq 2^\kappa$ is **λ-X-small** for $X \subseteq \kappa$ if there exists $\langle a_{\alpha,\beta}\rangle_{\alpha\in X,\beta<\lambda} \in ((2^\kappa)^X)^\lambda$ such that

$$A \subseteq \bigcup_{\alpha\in X}\bigcup_{\beta<\lambda}[a_{\alpha,\beta}{\restriction}\alpha].$$

$A \subseteq 2^\kappa$ is **$\mathcal{X}$-null** for $\mathcal{X} \subseteq 2^\kappa$ if for all $X \in \mathcal{X}$, A is X-small, and **λ-$\mathcal{X}$-null** if for all $X \in \mathcal{X}$, A is λ-X-small. Obviously, A is $\mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$ if and only if A is $[\kappa]^\kappa$-null.

The notion of λ-$\mathcal{X}$-null sets for $\mathcal{X} \subseteq [\kappa]^\lambda$ does not depend precisely on $\mathcal{X}$. Indeed, we get the following proposition.

Proposition 5.1 (ω_1: [Halko, 1996])**.** *Let $\lambda < \kappa$. A set $A \subseteq 2^\kappa$ is λ-$\{\{\alpha\}\colon\alpha < \kappa\}$-null in 2^κ if and only if it is $[\kappa]^\lambda$-null.*

Proof: Let $\lambda < \kappa$, and assume that A is λ-$\{\{\alpha\}\colon\alpha < \kappa\}$-null. Let $X = \{\xi_\beta\colon\beta < \lambda\} \in [\kappa]^\lambda$ and $\alpha = \bigcup X$. Obviously, $\alpha < \kappa$. Therefore, there exists a sequence $\langle a_\beta\rangle_{\beta<\lambda}$ such that

$$A \subseteq \bigcup_{\beta<\lambda}[a_\beta{\restriction}\alpha] \subseteq \bigcup_{\beta<\lambda}[a_\beta{\restriction}\xi_\beta],$$

so A is X-small.

On the other hand, assume that A is $[\kappa]^\lambda$-null and $\alpha < \kappa$. Then let $X = \{\alpha+\beta\colon\beta < \lambda\} \in [\kappa]^\lambda$. There exists a sequence $\langle a_\beta\rangle_{\beta<\lambda}$ such that

$$A \subseteq \bigcup_{\beta<\lambda}[a_\beta{\restriction}\alpha+\beta] \subseteq \bigcup_{\beta<\lambda}[a_\beta{\restriction}\alpha],$$

so A is λ-$\{\alpha\}$-small. □

A set $A \subseteq 2^\kappa$ will be called **small in** 2^κ if there exists $\lambda < \kappa$ such that A is λ-$\{\{\alpha\}\colon\alpha < \kappa\}$-null. Obviously, every $A \subseteq 2^\kappa$ with $|A| < \kappa$ is small in 2^κ.

Notice that every small set in 2^κ is κ-strongly null.

Proposition 5.2 (ω_1: [Halko, 1996])**.** *Let $A \subseteq 2^\kappa$ be small in 2^κ. Then $A \in S\mathcal{N}_\kappa$.*

Proof: Let $\lambda < \kappa$ be such that A is λ-$\{\{\alpha\}: \alpha < \kappa\}$-null. Therefore, by Proposition 5.1, A is $[\kappa]^\lambda$-null. Let $X = \{\xi_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$. There exists a sequence $\langle a_\alpha \rangle_{\alpha<\lambda} \in (2^\kappa)^\lambda$ such that $A \subseteq \bigcup_{\alpha<\lambda}[a_\alpha \restriction \xi_\alpha]$. For $\lambda \leq \alpha < \kappa$ set $a_\alpha = \mathbf{0}$. We get that $A \subseteq \bigcup_{\alpha<\kappa}[a_\alpha \restriction \xi_\alpha]$. □

Proposition 5.3 (ω_1: [Halko, 1996])**.** *A set $A \subseteq 2^\kappa$ is $S\mathcal{N}_\kappa$ if and only if there exists $\lambda < \kappa$ such that A is λ-$[\kappa]^\kappa$-null.*

Proof: If $A \subseteq 2^\kappa$ is $\mathrm{S}\mathcal{N}_\kappa$, it is obviously λ-$[\kappa]^\kappa$-null for all $\lambda < \kappa$. Assume that $\lambda < \kappa$, and $A \subseteq 2^\kappa$ is λ-$[\kappa]^\kappa$-null. Let $X = \{\xi_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$. Let $b: \lambda \times \kappa \to \kappa$ be a bijection, and for all $\alpha < \kappa$, let $X_\alpha = \{\xi_{b(\beta,\alpha)} : \beta < \lambda\} \in [\kappa]^\lambda$. Let $\delta_\alpha = \bigcup X_\alpha$, for $\alpha < \kappa$. Finally, let $Y = \{\delta_\alpha : \alpha < \kappa\} \in [\kappa]^\kappa$. We can find $\langle x_{\alpha,\beta} \rangle_{\alpha<\kappa,\beta<\lambda} \in (2^\kappa)^{\kappa\times\lambda}$ such that

$$A \subseteq \bigcup_{\alpha<\kappa} \bigcup_{\beta<\lambda} [x_{\alpha,\beta} \restriction \delta_\alpha].$$

For $\alpha < \kappa$, let $z_\alpha = x_{b^{-1}(\alpha)}$. Then

$$A \subseteq \bigcup_{\alpha<\kappa} [z_\alpha \restriction \delta_{\pi_2(b^{-1}(\alpha))}] \subseteq \bigcup_{\alpha<\kappa} [z_\alpha \restriction \xi_\alpha].$$

□

Proposition 5.4. *Let $X \subseteq \kappa$ be such that $0 \notin X$ and $X \cap Lim = \varnothing$. If $A \subseteq 2^\kappa$ is X-small, then $|2^\kappa \setminus A| = 2^\kappa$.*

Proof: Let $\langle x_\alpha \rangle_{\alpha\in X} \in (2^\kappa)^X$ be such that $A \subseteq \bigcup_{\alpha\in X}[x_\alpha \restriction \alpha]$. Consider the set

$$B = \{x \in 2^\kappa : \forall_{\alpha<\kappa} (\alpha + 1 \in X \Rightarrow x(\alpha) = x_{\alpha+1}(\alpha) + 1)\}.$$

Then for all $\alpha \in X$, $B \cap [x_\alpha \restriction \alpha] = \varnothing$. Thus, $B \cap A = \varnothing$. Furthermore, B contains a set homeomorphic to 2^κ, so $|2^\kappa \setminus A| = 2^\kappa$. □

Next we study a connection between the diamond principle for κ (see section 1.5) and the notion of C-smallness for closed unbounded or stationary sets $C \subseteq 2^\kappa$.

For $E \subseteq \kappa$, $A \subseteq 2^\kappa$, $\mathcal{I} \subseteq [\kappa]^{\leq\kappa}$, let $\diamondsuit_\kappa(E, A, \mathcal{I})$ denote the following principle: there exists a sequence $\langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\kappa$ such that for all $x \in A$,

$$\{\alpha \in E : x \restriction \alpha = s_\alpha\} \notin \mathcal{I}.$$

Notice the following easy observation.

Proposition 5.5. *If $A \subseteq 2^\kappa$, and $E \subseteq \kappa$, then $\diamondsuit_\kappa(E, A, \{\varnothing\})$ if and only if A is E-null.*

Proof: Indeed, $\diamondsuit_\kappa(E, A, \{\varnothing\})$ if and only if for all $x \in A$,

$$\{\alpha \in E : x{\restriction}\alpha = s_\alpha\} \neq \varnothing.$$

□

Proposition 5.6 (ω_1: [Halko, 1996]). *Let $E \subseteq \kappa$. The principle $\diamondsuit_\kappa(E, 2^\kappa, \boldsymbol{\mathcal{NS}}_\kappa)$ holds if and only if $\diamondsuit_\kappa(E)$ holds.*

Proof: Let $\langle s_\alpha \rangle_{\alpha\in E} \in (2^{<\kappa})^\kappa$ be such that for all $x \in 2^\kappa$, $\{\alpha \in E : x{\restriction}\alpha = s_\alpha\}$ is stationary in κ. Let $S_\alpha = s_\alpha^{-1}[\{1\}] \cap \alpha$, for $\alpha \in E$, and let $X \subseteq \kappa$. Then

$$\{\alpha \in E : X \cap \alpha = S_\alpha\} = \{\alpha \in E : \chi_X {\restriction}\alpha = s_\alpha\}$$

is a stationary subset of κ, so $\diamondsuit_\kappa(E)$ holds.

Similarly, if $\langle S_\alpha \rangle_{\alpha<\kappa} \in ([\kappa]^{<\kappa})^\kappa$ is such that for any $X \subseteq \kappa$, $\{\alpha \in E : X \cap \alpha = S_\alpha\}$ is stationary, let $s_\alpha = \chi_{S_\alpha \cap \alpha}$, for $\alpha < \kappa$. This sequence witnesses $\diamondsuit_\kappa(E, 2^\kappa, \boldsymbol{\mathcal{NS}}_\kappa)$. □

Proposition 5.7 (ω_1: [Halko, 1996]). *Assume $\diamondsuit_\kappa$. If C is a closed unbounded set in κ, then 2^κ is C-small.*

Proof: By Proposition 5.6, there exists a sequence $\langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\kappa$ such that for all $x \in 2^\kappa$, $\{\alpha \in \kappa : x{\restriction}\alpha = s_\alpha\}$ is stationary in κ. Therefore, if C is a closed unbounded set in κ, then $\{\alpha \in C : x{\restriction}\alpha = s_\alpha\}$ is stationary, thus non-empty for all $x \in 2^\kappa$. Therefore, $2^\kappa = \bigcup_{\alpha\in C}[s_\alpha]$. □

Proposition 5.8 (ω_1: [Halko, 1996]). *Let $E \subseteq \kappa$, and assume $\diamondsuit_\kappa(E)$. Then 2^κ is E-small.*

Proof: By Proposition 5.6, there exists a sequence $\langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\kappa$ such that for all $x \in 2^\kappa$, $\{\alpha \in E : x{\restriction}\alpha = s_\alpha\}$ is stationary in κ. So it is not empty, and $2^\kappa = \bigcup_{\alpha\in E}[s_\alpha]$. □

Corollary 5.9 (ω_1: [Halko, 1996]). *Assume $V = L$. Then 2^κ is X-small for every stationary set $X \subseteq \kappa$.*

Proof: Recall that $V = L$ implies $\diamondsuit_\kappa(X)$ for every stationary set $X \subseteq \kappa$ (see [Kunen, 2006, Exercise VI.14]). Therefore, by Proposition 5.8, 2^κ is small for every stationary $X \subseteq \kappa$. □

The whole space 2^κ can be presented as a union of a κ-meagre set, and a $\mathcal{X}$-null set for $\mathcal{X} \in [[\kappa]^\kappa]^\kappa$.

Proposition 5.10 (ω_1: [Halko, 1996]). *Let $\mathcal{X} \in [[\kappa]^\kappa]^\kappa$. There exist $A, B \subseteq 2^\kappa$ such that A is $\mathcal{X}$-null and B is κ-meagre, and $A \cup B = 2^\kappa$.*

Proof: Let $\mathcal{X} \in [[\kappa]^\kappa]^\kappa$. Let $\boldsymbol{Q} = \{q_\alpha : \alpha < \kappa\}$ and let $\mathcal{X} = \{X_\alpha : \alpha < \kappa\}$, and $X_\alpha = \{x_{\alpha,\beta} : \beta < \kappa\}$ be enumerations. For $\alpha < \kappa$, take

$$A_\alpha = \bigcup_{\beta<\kappa} [q_\alpha \restriction x_{\alpha,\beta}].$$

Notice that $2^\kappa \setminus A_\alpha$ is nowhere dense, therefore, if $A = \bigcap_{\alpha<\kappa} A_\alpha$, then $2^\kappa \setminus A$ is κ-meagre. Obviously, A is $\mathcal{X}$-null. □

On the other hand, we have the following.

Proposition 5.11 (ω_1: [Halko, 1996])**.** *Every set which is small in 2^κ is nowhere dense.*

Proof: Let $\lambda < \kappa$ be such that $A \subseteq 2^\kappa$ is $\lambda - \{\{\alpha\} : \alpha < \kappa\}$-null. Let $s \in 2^\beta$ with $\beta < \kappa$, and let $\xi = \beta + \lambda$. There exists $\langle x_\alpha \rangle_{\alpha<\lambda} \in (2^\kappa)^\lambda$ such that

$$A \subseteq \bigcup_{\alpha<\lambda} [x_\alpha \restriction \xi].$$

But $|\{x \restriction \xi : x \in [s]\}| = 2^\lambda$, thus there exists $t \in 2^\xi$ such that $s \subseteq t$, and $[t] \cap A = \varnothing$. □

But not every nowhere dense set in 2^κ is small in 2^κ.

Proposition 5.12 (ω_1: [Halko, 1996])**.** *There exists a nowhere dense set $A \subseteq 2^\kappa$ which is not κ-strongly null.*

Proof: Let $\langle \xi_\alpha \rangle \in \kappa^\kappa$ be an increasing sequence of limit ordinals. Let

$$A = \{x \in 2^\kappa : \forall_{\alpha<\kappa} x(\xi_\alpha) = 0\}.$$

Obviously, A is nowhere dense. Assume that $A \in \mathrm{S}\boldsymbol{\mathcal{N}}_\kappa$. Then there exists $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that $A \subseteq \bigcup_{\alpha<\kappa} [x_\alpha \restriction \xi_\alpha + 1]$. Let $x \in 2^\kappa$ be such that $x(\xi_\alpha + 1) = x_\alpha(\xi_\alpha + 1) + 1$ for all $\alpha < \kappa$, and $x(\beta) = 0$ for $\beta \notin \{\xi_\alpha : \alpha \in \kappa\}$. Then $x \in A$, but $x \notin \bigcup_{\alpha<\kappa} [x_\alpha \restriction \xi_\alpha + 1]$, which is a contradiction. □

5.2 κ-Meagre additive sets

In this section we present some generalizations of results concerning meagre additive sets. We start by generalizing the combinatorial characterization of meagre sets (see [Bartoszyński and Judah, 1995, Theorem 2.2.4]).

Proposition 5.13 (ω: [Bartoszyński and Judah, 1995])**.** *Assume that κ is strongly inaccessible, and $A \subseteq 2^\kappa$ is a κ-meagre set. Then there exists $y \in 2^\kappa$ and an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that*

$$A \subseteq \left\{x \in 2^\kappa : \exists_{\beta<\kappa} \forall_{\beta<\gamma<\kappa} \exists_{\xi_\gamma \le \xi < \xi_{\gamma+1}} x(\xi) \neq y(\xi)\right\}.$$

Proof: Let $A \subseteq \bigcup_{\alpha<\kappa} F_\alpha$ with F_α closed nowhere dense for all $\alpha < \kappa$. Additionally, we assume that if $\alpha < \beta < \kappa$, then $F_\alpha \subseteq F_\beta$. We define $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $y \in 2^\kappa$ by induction. Let $\xi_0 = 0$. Assume that $\eta < \kappa$, and ξ_η and $y{\restriction}\eta$ are defined. Let $\langle t_{\alpha,\eta} \rangle_{\alpha<\delta_\eta}$ be an enumeration of 2^η. Notice that $\delta_\eta < \kappa$, since κ is assumed to be strongly inaccessible. Define inductively $\langle s_{\alpha,\eta} \rangle_{\alpha<\delta_\eta}$ such that

(a) if $\alpha \leq \beta < \delta_\eta$, then $s_{\alpha,\eta} \subseteq s_{\beta,\eta}$,

(b) $[t_{\alpha,\eta}{}^\frown s_{\alpha,\eta}] \cap F_\eta = \varnothing$.

Let $s_\eta = \bigcup_{\alpha<\delta_\eta} s_{\alpha,\eta}$, and let $\mathrm{len}(s_\eta) = \gamma_\eta$. Obviously, $\gamma_\eta < \kappa$. Set $\xi_{\eta+1} = \xi_\eta + \gamma_\eta$ and $y(\xi_\eta + \alpha) = s_\eta(\alpha)$ for $\alpha < \gamma_\eta$. If $\eta < \kappa$ is a limit ordinal set $\xi_\eta = \bigcup_{\alpha<\eta} \xi_\alpha$.

It follows that if $x \in 2^\kappa$, and the set of all $\gamma < \kappa$ such that for all ξ such that $\xi_\gamma \leq \xi < \xi_{\gamma+1}$, we have $x(\xi) = y(\xi)$, is cofinal in κ, then for all $\alpha < \kappa$, there exists $\gamma < \kappa$ with $\gamma \geq \alpha$, and $x \notin F_\gamma$. Therefore, $x \notin \bigcup_{\alpha<\kappa} F_\alpha \supseteq A$. □

A set $A \subseteq 2^\kappa$ will be called **κ-meagre additive** if for any κ-meagre set F, $A + F$ is κ-meagre. The family of all κ-meagre additive sets we denote by $\boldsymbol{\mathcal{M}}^*_\kappa$.

By Proposition 4.8, we immediately get the following Corollary.

Corollary 5.14. *Every κ-meagre additive set is κ-strongly null.*

□

The following theorem is a generalization of the characterization of meagre-additive sets ([Bartoszyński and Judah, 1995, Theorem 2.7.17], see also Section 1.3).

Proposition 5.15 (ω: [Bartoszyński and Judah, 1995])**.** *Assume that κ is strongly inaccessible, and $X \subseteq 2^\kappa$. Then $X \in \boldsymbol{\mathcal{M}}^*_\kappa$ if and only if for every increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ such that*

$$X \subseteq \left\{ x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1} \wedge \forall_{\xi_\gamma \leq \delta < \xi_{\gamma+1}} x(\delta) = z(\delta) \right) \right\}.$$

Proof: Assume that $X \in \boldsymbol{\mathcal{M}}^*_\kappa$, and $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$. Let

$$B = \left\{ y \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\xi_\beta \leq \delta < \xi_{\beta+1}} y(\delta) \neq 0 \right\}.$$

Obviously, B is κ-meagre, so $X + B$ is also κ-meagre, and $X + B = \bigcup_{x \in X} B_x$, where

$$B_x = \left\{ y \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\xi_\beta \leq \delta < \xi_{\beta+1}} y(\delta) \neq x(\delta) \right\}.$$

By Proposition 5.13, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ such that

$$X + B \subseteq C = \left\{ a \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\eta_\beta \leq \delta < \eta_{\beta+1}} a(\delta) \neq z(\delta) \right\}.$$

Therefore, for any $x \in X$, $B_x \subseteq C$. Similarly to [Bartoszyński and Judah, 1995, Lemma 2.7.5], we prove that there exists $\alpha < \kappa$ such that for all $\alpha < \beta < \kappa$, there exists $\gamma < \kappa$ such that $\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1}$ and for all $\xi_\gamma \leq \delta < \xi_{\gamma+1}$, we get $x(\delta) = z(\delta)$.

Indeed, let

$$S = \left\{\beta < \kappa \colon \neg\exists_{\gamma<\kappa} \left(\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1} \wedge \forall_{\xi_\gamma \leq \delta < \xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

To obtain a contradiction, assume that for all $\alpha < \kappa$, $S \setminus \alpha \neq \varnothing$. Let $S = \{\sigma_\alpha \colon \alpha < \kappa\}$, and let $S' = \{\sigma_\alpha \colon \alpha < \kappa \wedge \alpha \text{ is a limit ordinal}\}$. Finally, let

$$D = \left\{\alpha < \kappa \colon \exists_{\beta \in S'} \eta_\beta \leq \alpha < \eta_{\beta+1}\right\}.$$

Notice that if for $\beta < \kappa$, $\{\delta < \kappa \colon \xi_\beta \leq \delta < \xi_{\beta+1}\} \subseteq D$, then there exists $\xi_\beta \leq \delta < \xi_{\beta+1}$ such that $x(\delta) \neq z(\delta)$. Let $y \in 2^\kappa$ be such that

$$y(\delta) = \begin{cases} z(\delta), & \text{if } \delta \in D, \\ x(\delta) + 1, & \text{otherwise.} \end{cases}$$

Then $y \in B_x$, but $y \notin C$, which is a contradiction.

Therefore,

$$X \subseteq \left\{x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1} \wedge \forall_{\xi_\gamma \leq \delta < \xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

Conversely, assume that $X \subseteq 2^\kappa$ is such that for every sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exist a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ such that

$$X \subseteq \left\{x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1} \wedge \forall_{\xi_\gamma \leq \delta < \xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

Let F be κ-meagre. Then, by Proposition 5.13 we get a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $y \in 2^\kappa$ such that

$$F \subseteq F' = \left\{a \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\xi_\beta \leq \delta < \xi_{\beta+1}} a(\delta) \neq y(\delta)\right\}.$$

Let $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ be such that

$$X \subseteq \left\{x \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \eta_{\beta+1} \wedge \forall_{\xi_\gamma \leq \delta < \xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

Then

$$X + F \subseteq X + F' \subseteq \left\{a \in 2^\kappa \colon \exists_{\alpha<\kappa} \forall_{\alpha<\beta<\kappa} \exists_{\eta_\beta \leq \delta < \eta_{\beta+1}} a(\delta) \neq y(\delta) + z(\delta)\right\},$$

which is a κ-meagre set. Therefore, $X \in \boldsymbol{\mathcal{M}}^*_\kappa$. □

Notice that this implies that under the same assumption every κ-meagre additive set is $\mathrm{P}_{\boldsymbol{\kappa}}\boldsymbol{\mathcal{M}}_\kappa$.

Proposition 5.16 (ω: [Kysiak et al., 2007])**.** *Assume that κ is a strongly inaccessible cardinal. Then every κ-meagre additive set is $P_\kappa\boldsymbol{\mathcal{M}}_\kappa$.*

Proof: Let $A \in \boldsymbol{\mathcal{M}}^*_\kappa$, and let $P \subseteq 2^\kappa$ be a κ-perfect set. By induction we construct a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $\xi_0 = 0$, and for $\alpha < \kappa$,

$$\xi_{\alpha+1} = \bigcup_{t \in T_P \cap 2^{\xi_\alpha}} \min\{\mathrm{len}(s) : t \subseteq s \in \mathrm{Split}(T_P)\} + 1.$$

Finally, for limit $\alpha < \kappa$, let $\xi_\alpha = \bigcup_{\beta<\alpha} \xi_\beta$.

By Proposition 5.15, we can find a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ and $z \in 2^\kappa$ such that

$$A \subseteq \bigcup_{\alpha<\kappa} \left\{ x \in 2^\kappa : \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \le \xi_\gamma < \xi_{\gamma+1} \le \eta_{\beta+1} \wedge \forall_{\xi_\gamma \le \delta < \xi_{\gamma+1}} x(\delta) = z(\delta) \right) \right\}.$$

Let $\alpha < \kappa$, and let $s \in T_P$. Fix $s' \in T_P$ such that $s \subseteq s'$, and for some $\beta < \alpha$, $\mathrm{len}(s') = \eta_\beta$. Let

$$\gamma_0 = \min\{\gamma < \kappa : \eta_\beta \le \xi_\gamma < \xi_{\gamma+1} \le \eta_{\beta+1}\}$$

and

$$\gamma_1 = \bigcup \{\gamma < \kappa : \eta_\beta \le \xi_\gamma < \xi_{\gamma+1} \le \eta_{\beta+1}\} + 1.$$

Inductively, we construct a sequence $\langle t_\delta \rangle_{\gamma_0 \le \delta \le \gamma_1}$ such that for all $\gamma_0 \le \delta \le \delta' \le \gamma_1$, $t_\delta \in T_P \cap 2^{\xi_\delta}$, $t_\delta \subseteq t_{\delta'}$, and $\exists_{\xi_\delta \le \xi < \xi_{\delta+1}} t_{\delta+1}(\xi) \neq z(\xi)$. Indeed, let $t_{\gamma_0} \in T_P$ be such that $s \subseteq t_{\gamma_0}$, and $\mathrm{len}(t_{\gamma_0}) = \xi_{\gamma_0}$. Given t_δ, by definition of $\langle \xi_\alpha \rangle$, one can find $t_{\delta+1} \supseteq t_\delta$ such that $\exists_{\xi_\delta \le \xi < \xi_{\delta+1}} t_{\delta+1}(\xi) \neq z(\xi)$, because $|\{t \in T_P \cap 2^{\xi_{\delta+1}} : t \supseteq t_\delta\}| \ge 2$. For limit $\delta < \kappa$, set any $t_\delta \supseteq \bigcup_{\gamma_0 \le \xi < \delta} t_\xi$ such that $\mathrm{len}(t_\delta) = \xi_\delta$.

Then,

$$[t_{\gamma_1}] \cap P \cap \left\{ x \in 2^\kappa : \forall_{\alpha<\beta<\kappa} \exists_{\gamma<\kappa} \left(\eta_\beta \le \xi_\gamma < \xi_{\gamma+1} \le \eta_{\beta+1} \wedge \forall_{\xi_\gamma \le \delta < \xi_{\gamma+1}} x(\delta) = z(\delta) \right) \right\}$$

is empty, and hence A is κ-meagre in P. □

5.3 κ-Ramsey null sets

In this section we generalize some results presented in [Nowik and Weiss, 2002].

For $\alpha < \kappa$, $s \in 2^\alpha$ and $S \in [\kappa \setminus \alpha]^\kappa$, let

$$[s, S] = \{x \in 2^\kappa : s^{-1}[\{1\}] \subseteq x^{-1}[\{1\}] \subseteq s^{-1}[\{1\}] \cup S \wedge |x^{-1}[\{1\}] \cap S| = \kappa\}.$$

A set $A \subseteq 2^\kappa$ will be called **κ-Ramsey null** ($\kappa - CR_0$) if for any $\alpha < \kappa$, $s \in 2^\alpha$ and $S \in [\kappa \setminus \alpha]^\kappa$, there exists $S' \in [S]^\kappa$ such that $[s, S'] \cap A = \varnothing$.

It is a well-known fact that the ideal of Ramsey null subsets of 2^ω is a σ-ideal (see e.g. [Halbeisen, 2011]). We do not know whether the analogue holds for κ-Ramsey null sets.

Question 5.17. *Is the ideal of κ-Ramsey null subsets of 2^κ κ^+-complete?*

Theorem 5.18 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a weakly inaccessible cardinal. Then every κ-γ-set which is not closed in 2^κ is κ-Ramsey null.*

Proof: The proof is similar to the proof of [Nowik and Weiss, 2002, Theorem 2.1]. Namely, let $A \subseteq 2^\kappa$ be a κ-γ-set, and $\delta < \kappa$, $s \in 2^\delta$ and

$$S = \{\xi_\alpha : \alpha < \kappa\} \in [\kappa \setminus \delta]^\kappa.$$

Let

$$E = \{x \in 2^\kappa : s^{-1}[\{1\}] \subseteq x^{-1}[\{1\}] \subseteq s^{-1}[\{1\}] \cup S\} = s_0 + S_0,$$

where $s_0 = s \cup \{\langle \beta, 0\rangle : \beta \in \kappa \setminus \delta\}$ and $S_0 = \{f \cup \{\langle \beta, 0\rangle : \beta \notin S\} : f \in 2^S\}$. Notice that S_0 is a closed set in 2^κ, and so is E. Moreover, $\varphi : 2^\kappa \to E$ given by the following expression

$$\varphi(x) = s_0 + \chi_{\{\xi_\alpha : x(\alpha)=1 \wedge \alpha<\kappa\}}$$

is a homeomorphism.

By Proposition 4.34, $E \cap A$ is a κ-γ set, and therefore so is $\varphi^{-1}[E \cap A]$. By Lemma 4.49, there exists $B \in [\kappa]^\kappa$ such that for all $C \in [B]^\kappa$, $\chi_C \notin \varphi^{-1}[E \cap A]$, which means that $\varphi(\chi_C) \notin A$. Let $S' = \{\xi_\alpha : \alpha \in B\}$. Then $S' \in [S]^\kappa$, and $[s, S'] = \{\varphi(\chi_C) : C \in [B]^\kappa\}$. Thus, $[s, S'] \cap A = \varnothing$. □

Lemma 5.19 (ω: [Nowik and Weiss, 2002])**.** *If $A, B \subseteq 2^\kappa$, then*

$$2^\kappa \setminus (A + 2^\kappa \setminus B) = \{x \in 2^\kappa : x + A \subseteq B\}.$$

Proof: The proof of [Nowik and Weiss, 2002][Lemma 4.1] is valid for any vector space over $\mathbb{Z}_2$. □

Proposition 5.20 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is strongly inaccessible, and $A \subseteq 2^\kappa$ is a κ-meagre set. Then there exists a κ-meagre set $B \subseteq 2^\kappa$ such that $A + (2^\omega \setminus B)$ is κ-Ramsey null.*

Proof: By Proposition 5.13, we get $z \in 2^\kappa$ and a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $A \subseteq A'$, where

$$A' = \left\{x \in 2^\kappa : \exists_{\beta<\kappa} \forall_{\beta<\gamma<\kappa} \exists_{\xi_\gamma \le \xi < \xi_{\gamma+1}} z(\xi) \neq x(\xi)\right\}.$$

Fix a bijection $b : \kappa \times 2 \to \kappa$. Let

$$B = \left\{x \in 2^\kappa : \exists_{\beta<\kappa} \forall_{\beta<\alpha<\kappa} \exists_{\gamma \in ((\xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}) \cup (\xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}))} x(\gamma) \neq z(\gamma)\right\}$$

Let $\eta < \kappa$, $s \in 2^\eta$ and $S \in [\kappa \setminus \eta]^\kappa$. We shall find $S' \subseteq S$ such that

$$[s, S'] \cap (A' + (2^\omega \setminus B)) = \varnothing.$$

Let $S' \in [S]^\kappa$ be such that for all $\alpha < \kappa$ such that $\xi_{b(\alpha,0)}, \xi_{b(\alpha,1)} > \alpha$,

$$\left|\left(\left(\xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}\right) \cup \left(\xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}\right)\right) \cap S'\right| \leq 1.$$

Let $v \in [s, S']$, and assume that $v = a + b$ for some $a \in A'$, $b \in 2^\omega \setminus B$. Thus,

(a) there exists $\xi < \kappa$ such that for all $\xi < \alpha < \kappa$, there exists $\gamma_0 \in \xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}$ and $\gamma_1 \in \xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}$ such that $a(\gamma_0) \neq z(\gamma_0)$ and $a(\gamma_1) \neq z(\gamma_1)$,

(b) for every $\delta < \kappa$, there exists $\delta < \alpha < \kappa$ such that for all $\beta \in \left(\xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}\right) \cup \left(\xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}\right)$, $b(\beta) = z(\beta)$.

Hence, there exists $\alpha < \kappa$ such that

(i) there exists at most one $\eta \in \left(\xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}\right) \cup \left(\xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}\right)$ such that $v(\eta) = 1$,

(ii) there exists $\gamma_0 \in \xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}$ and $\gamma_1 \in \xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}$ such that $a(\gamma_0) \neq z(\gamma_0)$ and $a(\gamma_1) \neq z(\gamma_1)$,

(iii) for all $\beta \in \left(\xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}\right) \cup \left(\xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}\right)$, $b(\beta) = z(\beta)$.

Then, either for all $\beta \in \xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}$, $v(\eta) = 0$, or for all $\beta \in \xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}$, $v(\eta) = 0$. Hence, either for all $\beta \in \xi_{b(\alpha,0)+1} \setminus \xi_{b(\alpha,0)}$, $a(\eta) = b(\eta)$, or for all $\beta \in \xi_{b(\alpha,1)+1} \setminus \xi_{b(\alpha,1)}$, $a(\eta) = b(\eta)$. This is a contradiction, thus,

$$[s, S'] \subseteq 2^\kappa \setminus (A' + (2^\kappa \setminus B)).$$

Hence, $A + (2^\kappa \setminus B) \subseteq A' + (2^\kappa \setminus B)$ is κ-Ramsey null. □

We get the following theorem.

Theorem 5.21 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is strongly inaccessible, $cov(\kappa - CR_0) \geq 2^\kappa$, and $add(\boldsymbol{\mathcal{M}}_\kappa) = 2^\kappa$. Then there exists a κ-meagre additive set which is not κ-Ramsey null.*

Proof: Let $\{F_\alpha : \alpha < 2^\kappa\}$ be an enumeration of all closed nowhere dense sets in 2^κ, and $[\kappa]^\kappa = \{X_\alpha : \alpha < 2^\kappa\}$. We construct a sequence $\langle x_\alpha \rangle_{\alpha < 2^\kappa} \in (2^\kappa)^\kappa$ by induction. For $\alpha < 2^\kappa$, using Proposition 5.20, choose a κ-meagre set $B_\alpha \subseteq 2^\kappa$ such that $F_\alpha + (2^\omega \setminus B_\alpha)$ is κ-Ramsey null. Choose any

$$x_\alpha \in \{\chi_S : S \in [X_\alpha]^\kappa\} \setminus \bigcup_{\beta < \alpha} (F_\beta + (2^\omega \setminus B_\beta)).$$

Such x_α exists, because $\mathrm{cov}(\kappa - CR_0) \geq 2^\kappa$.

Let $A = \{x_\alpha : \alpha < 2^\kappa\}$. Obviously, A is not κ-Ramsey null, because for all $S \in [\kappa]^\kappa$, there exists $S' \in [S]^\kappa$ such that $\chi_{S'} \in A$.

Moreover, if F is nowhere dense, then let $\alpha < 2^\kappa$ be such that $F \subseteq F_\alpha$. For every $\beta < \alpha$,

$$x_\beta \notin F_\alpha + (2^\omega \setminus B_\alpha),$$

thus by Lemma 5.19,

$$x_\beta + F_\alpha \subseteq B_\alpha.$$

Hence,

$$A + F \subseteq A + F_\alpha = \bigcup_{\beta \leq \alpha} (x_\beta + M_\alpha) \cup \bigcup_{\alpha < \beta < 2^\kappa} (x_\beta + M_\alpha) = \bigcup_{\beta \leq \alpha} (x_\beta + M_\alpha) \cup B_\alpha,$$

which is κ-meagre, since $\mathrm{add}(\boldsymbol{\mathcal{M}}_\kappa) = 2^\kappa$. □

5.4 κ-T'-sets

A definition of a T'-set was given in [Nowik and Weiss, 2002] (see also section 1.3). We provide a generalization of this notion in case of 2^κ. A set $A \subseteq 2^\kappa$ is here called **κ-T'-set** if there exists a sequence of cardinal numbers $\langle \lambda_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for every increasing sequence $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $\delta_0 = 0$, and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ for limit α, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$H_\alpha \in \left[2^{\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}}\right]^{\leq \lambda_{\eta_\alpha}},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \left\{x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in H_\alpha\right\}.$$

Similarly to [Nowik and Weiss, 2002] we prove some equivalent characterizations of this class of sets.

Proposition 5.22 (ω: [Nowik and Weiss, 2002])**.** *A set $A \subseteq 2^\kappa$ is a κ-T'-set if and only if there exists a sequence of cardinal numbers $\langle \lambda_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for every increasing sequences $\langle \delta_{0,\alpha} \rangle_{\alpha<\kappa}, \langle \delta_{1,\alpha} \rangle_{\alpha<\kappa} \in \kappa^\kappa$, with $\delta_{0,\alpha} < \delta_{1,\alpha} \leq \delta_{0,\alpha+1}$ for all $\alpha < \kappa$, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and*

$$I_\alpha \in \left[2^{\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}}\right]^{\leq \lambda_{\eta_\alpha}},$$

for all $\alpha < \kappa$, so that

$$A \subseteq \left\{x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \in I_\alpha\right\}.$$

Proof: Obviously, a set which fulfils the above condition is a κ-T'-set. On the other hand, if $A \subseteq 2^\kappa$ is a κ-T'-set, then let $\langle \lambda_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a sequence of cardinals given by the definition of a κ-T'-set. Let

$$\delta_\alpha = \bigcup_{\beta<\alpha} \delta_{1,\beta},$$

for $\alpha < \kappa$. There exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$H_\alpha \in \left[2^{\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}}\right]^{\leq \lambda_{\eta_\alpha}},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \{x \in 2^\kappa \colon \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x{\restriction}(\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in H_\alpha\}.$$

Notice that $(\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \subseteq (\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha})$. Let

$$I_\alpha = \{f{\restriction}(\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \colon f \in H_\alpha\},$$

for $\alpha < \kappa$. Obviously, $|I_\alpha| \leq |H_\alpha| \leq \lambda_{\eta_\alpha}$, and

$$A \subseteq \{x \in 2^\kappa \colon \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x{\restriction}(\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \in I_\alpha\}.$$

□

Proposition 5.23 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a weakly inaccessible cardinal. A set $A \subseteq 2^\kappa$ is a κ-T'-set if and only if for every increasing sequence $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $\delta_0 = 0$, and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ for limit $\alpha < \kappa$, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for limit $\beta < \kappa$, $\eta_\beta = \bigcup_{\alpha<\beta} \eta_\alpha$, and*

$$J_\alpha \in \left[2^{\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}}\right]^{\leq |\eta_\alpha|},$$

for all $\alpha < \kappa$, so that

$$A \subseteq \{x \in 2^\kappa \colon \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x{\restriction}(\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in J_\alpha\}.$$

Proof: Obviously, a set which fulfils the above condition is κ-T'-set. On the other hand, if $A \subseteq 2^\kappa$ is κ-T'-set, then let $\langle \lambda_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a sequence of cardinals given by the definition of a κ-T'-set. Since κ is weakly inaccessible, we can assume that $\langle \lambda_\alpha \rangle_{\alpha<\kappa}$ is strictly increasing and $\bigcup_{\alpha<\beta} \lambda_\alpha = \lambda_\beta$ for limit $\beta < \kappa$. Let $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an increasing sequence. Let $\langle \delta'_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be the following sequence: $\delta'_0 = 0$,

$$\delta'_{\alpha+1} = \delta_{\lambda_\alpha+1},$$

and $\delta'_\alpha = \bigcup_{\beta<\alpha} \delta'_\beta$, when α is a limit ordinal.

There exists a sequence $\langle \eta'_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$H_\alpha \in \left[2^{\delta'_{\eta'_\alpha+1} \setminus \delta'_{\eta'_\alpha}} \right]^{\leq \lambda_{\eta'_\alpha}},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \left\{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta'_{\eta'_\alpha+1} \setminus \delta'_{\eta'_\alpha}) \in H_\alpha \right\}.$$

One can also assume that $\eta'_\beta = \bigcup_{\alpha<\beta} \eta'_\alpha$ for all limit $\beta < \kappa$. Let $\eta_\alpha = \lambda_{\eta'_\alpha}$. Notice that $\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha} \subseteq \delta'_{\eta'_\alpha+1} \setminus \delta'_{\eta'_\alpha}$. Thus, let

$$J_\alpha = \{ f \restriction (\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) : f \in H_\alpha \}.$$

We get that

$$|J_\alpha| \leq |H_\alpha| \leq \lambda_{\eta'_\alpha} = \eta_\alpha,$$

and

$$A \subseteq \{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in J_\alpha \}.$$

□

Corollary 5.24 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a weakly inaccessible cardinal. A set $A \subseteq 2^\kappa$ is a κ-T'-set if and only if for every increasing sequences $\langle \delta_{0,\alpha} \rangle_{\alpha<\kappa}, \langle \delta_{1,\alpha} \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $\delta_{0,\alpha} < \delta_{1,\alpha} \leq \delta_{0,\alpha+1}$ for all $\alpha < \kappa$, there exists a sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for limit $\beta < \kappa$, $\bigcup_{\alpha<\beta} \eta_\alpha = \eta_\beta$, and*

$$I_\alpha \in \left[2^{\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}} \right]^{\leq |\eta_\alpha|},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \in I_\alpha \}.$$

□

Proposition 5.25 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a weakly inaccessible cardinal. The class of κ-T'-sets forms a κ^+-complete ideal of subsets of 2^κ.*

Proof: Let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a sequence of κ-T'-sets, and let sequences $\langle \delta_{0,\alpha} \rangle_{\alpha<\kappa}, \langle \delta_{1,\alpha} \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be increasing sequences such that $\delta_{0,\alpha} < \delta_{1,\alpha} \leq \delta_{0,\alpha+1}$ for all $\alpha < \kappa$. Inductively construct sequences $\langle \eta_{\alpha,\beta} \rangle_{\alpha,\beta<\kappa} \in \kappa^{\kappa\times\kappa}$ and $\langle J_{\alpha,\beta} \rangle_{\alpha,\beta<\kappa}$ such that:

(a) if $\beta_1 < \beta_2 < \kappa$, then $\{ \eta_{\alpha,\beta_2} : \alpha < \kappa \} \subseteq \{ \eta_{\alpha,\beta_1} : \alpha < \kappa \}$,

(b) $\{\eta_{\alpha,\beta}\colon \alpha < \kappa\}$ is a closed unbounded set in κ for every $\beta < \kappa$,

(c) $J_{\alpha,\beta} \in \left[2^{\delta_{1,\eta_{\alpha,\beta}} \setminus \delta_{0,\eta_{\alpha,\beta}}}\right]^{\leq |\eta_{\alpha,\beta}|}$, for all $\alpha, \beta < \kappa$,

(d) $A_\beta \subseteq \left\{x \in 2^\kappa \colon \forall_{\gamma<\kappa} \exists_{\gamma<\alpha<\kappa} x{\restriction}(\delta_{1,\eta_{\alpha,\beta}} \setminus \delta_{0,\eta_{\alpha,\beta}}) \in J_{\alpha,\beta}\right\}$, for all $\beta < \kappa$.

To obtain the above, inductively construct a sequence $\langle I_\alpha \rangle_{\alpha<\kappa} \in ([\kappa]^\kappa)^\kappa$ such that $I_0 = \kappa$, and let $I_{\beta+1} = \{\eta_{\alpha,\beta}\colon \alpha < \kappa\}$. Moreover, for limit $\alpha < \kappa$, let $I_\alpha = \bigcap_{\beta<\alpha} I_\beta$. Obviously, I_α is then closed unbounded.

Now, for $\beta < \kappa$, by Corollary 5.24, we can get $\langle \eta'_{\alpha,\beta} \rangle_{\alpha<\kappa}$ and $\langle J_{\alpha,\beta} \rangle_{\alpha<\kappa}$ for sequences $\langle \delta_{0,\zeta_{\alpha,\beta}} \rangle_{\alpha\in\kappa}$ and $\langle \delta_{0,\zeta_{\alpha,\beta}} \rangle_{\alpha\in\kappa}$, where $\{\zeta_{\alpha,\beta}\colon \alpha < \kappa\} = I_\beta$ is the increasing enumeration, i.e. such that

$$J_{\alpha,\beta} \in \left[2^{\delta_{1,\zeta_{\eta'_{\alpha,\beta},\beta}} \setminus \delta_{0,\zeta_{\eta'_{\alpha,\beta},\beta}}}\right]^{\leq \left|\eta'_{\alpha,\beta}\right|},$$

for all $\alpha < \kappa$, and

$$A_\beta \subseteq \left\{x \in 2^\kappa \colon \forall_{\gamma<\kappa} \exists_{\gamma<\alpha<\kappa} x{\restriction}\left(\delta_{1,\zeta_{\eta'_{\alpha,\beta},\beta}} \setminus \delta_{0,\zeta_{\eta'_{\alpha,\beta},\beta}}\right) \in J_{\alpha,\beta}\right\}.$$

and $\eta'_\beta = \bigcup_{\alpha<\beta} \eta'_\alpha$ for all limit $\beta < \kappa$. Now, let $\eta_{\alpha,\beta} = \zeta_{\eta'_{\alpha,\beta},\beta}$, for $\alpha < \kappa$. We get

$$|J_{\alpha,\beta}| \leq \left|\eta'_{\alpha,\beta}\right| \leq |\eta_{\alpha,\beta}|,$$

for all $\alpha < \kappa$.

Let

$$I = \bigcup_{\beta<\kappa} \{\zeta_{\alpha,\beta}\colon \alpha < \beta\}.$$

Notice that $|I| = \kappa$, and for all $\beta < \kappa$ there exists $\gamma < \kappa$ such that $I \setminus \gamma \subseteq I_\beta$. Let $\{\zeta_\alpha\colon \alpha < \kappa\} = I$ be the increasing enumeration of I. Let

$$J_\alpha = \bigcup_{\beta<\alpha} \left\{g \in 2^{\delta_{1,\zeta_\alpha} \setminus \delta_{0,\zeta_\alpha}} \colon \exists_{f\in J_{\gamma,\beta}} g{\restriction}\mathrm{dom} f = f \wedge \zeta_\alpha = \eta_{\gamma,\beta} \wedge \gamma < \kappa\right\},$$

for $\alpha < \kappa$. Notice that $J_\alpha \subseteq 2^{\delta_{1,\zeta_\alpha} \setminus \delta_{0,\zeta_\alpha}}$, and

$$|J_\alpha| \leq |\alpha| \cdot |\zeta_\alpha| \leq |\zeta_\alpha|,$$

for $\omega \leq \alpha < \kappa$. Finally, notice that

$$\bigcup_{\alpha<\kappa} A_\alpha \subseteq \{x \in 2^\kappa \colon \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x{\restriction}(\delta_{1,\zeta_\alpha} \setminus \delta_{0,\zeta_\alpha}) \in J_\alpha\}.$$

Thus, by Corollary 5.24, $\bigcup_{\alpha<\kappa} A_\alpha$ is a κ-T'-set. □

Proposition 5.26 (ω: [Nowik and Weiss, 2002]). *If $A, B \subseteq 2^\kappa$ are κ-T'-sets, then $A+B$ is also a κ-T'-set.*

Proof: Let $\langle \lambda^A_\alpha \rangle_{\alpha<\kappa}, \langle \lambda^B_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be sequences of cardinals given by the definition of κ-T'-sets for A and B, respectively. Let

$$\lambda_\alpha = \max\{\lambda^A_\alpha, \lambda^B_\alpha, \aleph_0\},$$

for $\alpha < \kappa$. Let $\langle \delta_{0,\alpha} \rangle_{\alpha<\kappa}, \langle \delta_{1,\alpha} \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be sequences such that $\delta_{0,\alpha} < \delta_{1,\alpha} \leq \delta_{0,\alpha+1}$ for all $\alpha < \kappa$. By Proposition 5.22, we get a sequence $\langle \eta^A_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$I^A_\alpha \in \left[2^{\delta_{1,\eta^A_\alpha} \setminus \delta_{0,\eta^A_\alpha}} \right]^{\leq \lambda^A_{\eta^A_\alpha}},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \left\{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{1,\eta^A_\alpha} \setminus \delta_{0,\eta^B_\alpha}) \in I^A_\alpha \right\}.$$

Let $\delta^B_{0,\alpha} = \delta_{0,\eta^A_\alpha}$, and $\delta^B_{1,\alpha} = \delta_{1,\eta^A_\alpha}$, for $\alpha < \kappa$. Again, by Proposition 5.22, we get a sequence $\langle \eta^B_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$I^B_\alpha \in \left[2^{\delta^B_{1,\eta^B_\alpha} \setminus \delta^B_{0,\eta^B_\alpha}} \right]^{\leq \lambda^B_{\eta^B_\alpha}},$$

for all $\alpha < \kappa$ such that

$$B \subseteq \left\{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction \left(\delta^B_{1,\eta^B_\alpha} \setminus \delta^B_{0,\eta^B_\alpha} \right) \in I^B_\alpha \right\}.$$

Let $\eta_\alpha = \eta^A_{\eta^B_\alpha}$, for $\alpha < \kappa$, and let

$$I_\alpha = I^A_{\eta^B_\alpha} + I^B_\alpha \subseteq 2^{\delta^B_{1,\eta^B_\alpha} \setminus \delta^B_{0,\eta^B_\alpha}} = 2^{\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}},$$

for $\alpha < \kappa$. Notice that $|I_\alpha| \leq \lambda_\alpha$, for all $\alpha < \kappa$, and

$$A + B \subseteq \left\{ x \in 2^\kappa : \forall_{\beta<\kappa} \exists_{\beta<\alpha<\kappa} x \restriction (\delta_{1,\eta_\alpha} \setminus \delta_{0,\eta_\alpha}) \in I_\alpha \right\},$$

so by Proposition 5.22, $A+B$ is a κ-T'-set. □

Proposition 5.27 (ω: [Nowik and Weiss, 2002]). *Assume that κ is a strongly inaccessible cardinal. Then every κ-γ-set is a κ-T'-set.*

Proof: Assume that $A \subseteq 2^\kappa$ is a κ-γ-set, and let $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a sequence such that $\delta_0 = 0$, and $\delta_\alpha = \bigcup_{\beta<\alpha} \delta_\beta$ for limit α. Let

$$I_\alpha = \left[2^{\delta_{\alpha+1} \setminus \delta_\alpha} \right]^{\leq |\alpha|},$$

for $\alpha < \kappa$, and let

$$U_{\alpha,S} = \{x \in 2^\kappa : x\restriction(\delta_{\alpha+1} \setminus \delta_\alpha) \in S\},$$

for $\alpha < \kappa$, and $S \in I_\alpha$. Obviously, $\mathcal{U} = \{U_{\alpha,S} : \alpha < \kappa \wedge S \in I_\alpha\}$ is an open κ-cover of 2^κ. Therefore, there exists a sequence $\langle V_\alpha\rangle_{\alpha<\kappa} \in \mathcal{U}^\kappa$ such that

$$A \subseteq \bigcup_{\alpha<\kappa} \bigcap_{\alpha<\beta<\kappa} V_\beta.$$

Since κ is strongly inaccessible, for every $\beta, \gamma < \kappa$, there exist $\gamma \leq \alpha < \kappa$ and $\beta \leq \delta < \kappa$ such that $V_\alpha = U_{\delta,S}$, with $S \in I_\delta$. Therefore, there exist increasing sequences $\langle\xi_\alpha\rangle_{\alpha<\kappa}, \langle\eta_\alpha\rangle_{\alpha<\kappa}$ such that $V_{\xi_\alpha} = U_{\eta_\alpha,S_\alpha}$, where $S_\alpha \in I_{\eta_\alpha}$. Thus,

$$A \subseteq \{x \in 2^\kappa : \forall_{\beta<\kappa}\exists_{\beta<\alpha<\kappa} x\restriction(\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in S_\alpha\}.$$

Hence, A is a κ-T'-set. □

Proposition 5.28 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a strongly inaccessible cardinal. Then every κ-T'-set is κ-meagre additive.*

Proof: Let $A \subseteq 2^\kappa$ be a κ-T'-set, and let $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an increasing sequence. Let $\langle\zeta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a sequence such that $\zeta_0 = 0$, $\zeta_{\alpha+1} = \zeta_\alpha + \alpha$, and $\zeta_\alpha = \bigcup_{\beta<\alpha} \zeta_\beta$, for limit $\alpha < \kappa$.

Let $\delta_\alpha = \xi_{\zeta_\alpha}$. By Proposition 5.23, there exists a sequence $\langle\eta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$, and

$$J_\alpha \in \left[2^{\delta_{\eta_\alpha+1}\setminus\delta_{\eta_\alpha}}\right]^{\leq|\eta_\alpha|},$$

for all $\alpha < \kappa$ such that

$$A \subseteq \{x \in 2^\kappa : \forall_{\beta<\kappa}\exists_{\beta<\alpha<\kappa} x\restriction(\delta_{\eta_\alpha+1} \setminus \delta_{\eta_\alpha}) \in J_\alpha\}.$$

For $\beta < \kappa$ let $\{j_{\alpha,\beta}, \alpha < \eta_\alpha\} = J_\beta$ be an enumeration. Let $z \in 2^\kappa$ be the following:

$$z(\gamma) = \begin{cases} j_{\alpha,\beta}(\gamma), & \text{if } \xi_{\zeta_{\eta_\beta}+\alpha} \leq \gamma < \xi_{\zeta_{\eta_\beta}+\alpha+1}, \alpha, \beta < \kappa, \\ 0, & \text{otherwise.} \end{cases}$$

We have that

$$A \subseteq \left\{x \in 2^\kappa : \exists_{\alpha<\kappa}\forall_{\alpha<\beta<\kappa}\exists_{\gamma<\kappa}\left(\delta_\beta \leq \xi_\gamma < \xi_{\gamma+1} \leq \delta_{\beta+1} \wedge \forall_{\xi_\gamma\leq\delta<\xi_{\gamma+1}} x(\delta) = z(\delta)\right)\right\}.$$

Thus, by Proposition 5.15, A is κ-meagre additive. □

Therefore, we get the following.

Corollary 5.29 (ω: [Nowik and Weiss, 2002])**.** *Assume that κ is a strongly inaccessible cardinal. Then every κ-γ-set is κ-meagre additive.*

□

On the other hand, recall that if κ is strongly inaccessible, $\mathrm{cov}(\kappa - CR_0) \geq 2^\kappa$, and $\mathrm{add}(\boldsymbol{\mathcal{M}}_\kappa) = 2^\kappa$, then there exists a κ-meagre additive set which is not κ-Ramsey null (Theorem 5.21), but ever κ-γ-set is κ-Ramsey-null (Theorem 5.18). Thus, under those conditions the above implication cannot be reversed.

5.5 κ-v_0-Sets

A κ-perfect set P is a **κ-Silver perfect** if for all $\alpha < \kappa$ and any $i \in \{0,1\}$,

$$\exists_{s \in 2^\alpha \cap T_P} s^\frown i \in T_P \Rightarrow \forall_{s \in 2^\alpha \cap T_P} s^\frown i \in T_P.$$

A set $A \subseteq 2^\kappa$ is a **κ-v_0-set** if for all κ-Silver perfect set $P \subseteq 2^\kappa$, there exists a κ-Silver perfect set $Q \subseteq P$ such that $A \cap Q = \varnothing$. The notion of κ-v_0 sets was considered in [Laguzzi, 2015]. We study the relation between this notion and other notions of special subsets of 2^κ.

Proposition 5.30 (ω: [Halbeisen, 2003])**.** *Assume that κ is a strongly inaccessible cardinal. Then every κ-comeagre subset of 2^κ contains a κ-Silver perfect set.*

Proof: Let $A \subseteq 2^\kappa$ be κ-meagre, and by Proposition 5.13, we get $z \in 2^\kappa$ and a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$A \subseteq \left\{ x \in 2^\kappa : \exists_{\beta<\kappa} \forall_{\beta<\gamma<\kappa} \exists_{\xi_\gamma \leq \xi < \xi_{\gamma+1}} z(\xi) \neq x(\xi) \right\}.$$

Let

$$Q = \{ x \in 2^\kappa : \forall_{\alpha \in \mathrm{Lim}} \forall_{\xi_\alpha \leq \xi < \xi_{\alpha+1}} x(\xi) = z(\xi) \}.$$

Then $Q \subseteq 2^\kappa \setminus A$, and Q is a κ-Silver perfect set. □

Corollary 5.31 (ω: [Kysiak et al., 2007])**.** *Assume that κ is a strongly inaccessible cardinal. Then every κ-perfectly κ-meagre set in 2^κ is a κ-v_0-set.*

Proof: Notice that for every κ-Silver perfect set $P \subseteq 2^\kappa$, there exists a natural homeomorphism $h : P \to 2^\kappa$ such that $Q \subseteq 2^\kappa$ is a κ-Silver perfect set if an only if $h^{-1}[Q]$ is κ-Silver perfect. The corollary follows from Proposition 5.30. □

Proposition 5.32 (ω: [Kysiak et al., 2007])**.** *Every κ-strongly null set in 2^κ is a κ-v_0-set.*

Proof: Let $P \subseteq 2^\kappa$ be a κ-Silver perfect set, and $A \in \mathrm{S}\mathcal{N}_\kappa$. Let

$$S = \{\mathrm{len}(s) \colon s \in \mathrm{Split}(T_P)\}.$$

Let $b \colon \kappa \times \{0,1\} \to S$ be a bijection, and let $X = f[\kappa \times \{0\}]$. Let $\langle x_\alpha \rangle_{\alpha \in X} \in (2^\kappa)^X$ be such that

$$A \subseteq \bigcup_{\alpha \in X} [x_\alpha \restriction \alpha + 1].$$

Then

$$Q = \{x \in P \colon \forall_{\alpha \in X} x(\alpha) = x_\alpha(\alpha) + 1\}$$

is a κ-Silver perfect set such that $Q \subseteq P$, and $Q \cap A = \varnothing$. □

A κ-perfect set $P \subseteq 2^\kappa$ is a **κ-Laver perfect set** if there exists $s \in T_P$ such that for all $t \in T_P$, either $t \subseteq s$, or

$$|\{\alpha < \kappa \colon t^\frown[\mathbf{0} \restriction \alpha]^\frown 1 \in T_P\}| = \kappa.$$

Similarly, a κ-perfect set $P \subseteq 2^\kappa$ is a **κ-Miller perfect set** if for every $s \in T_P$ there exists $t \in T_P$ such that $s \subseteq t$, and

$$|\{\alpha < \kappa \colon t^\frown[\mathbf{0} \restriction \alpha]^\frown 1 \in T_P\}| = \kappa.$$

A set $A \subseteq 2^\kappa$ is **κ-l_0-set** (respectively, **κ-m_0-set**) if for every κ-Laver (respectively, κ-Miller) κ-perfect set P, there exists a κ-Laver (respectively, κ-Miller) κ-perfect set $Q \subseteq P$ such that $Q \cap A = \varnothing$.

We leave the following question to be a subject of further research.

Question 5.33. *What is the relation between κ-l_0-sets (respectively, κ-m_0-sets) with other notions of special subsets of 2^κ?*

Chapter 6

Convergence in 2^κ

In this section we deal with the convergence of sequences of functions on 2^κ and special subsets related to this notion.

We use notions and notations related to the generalized Cantor space 2^κ introduced in section 1.5. For introduction to theory of convergence of real functions and related special subsets see section 1.4.

The results of this chapter are to be included in [Korch, 2017a].

6.1 Preliminaries

Recall that a sequence $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ converges to $x \in 2^\kappa$ ($x_\alpha \to_\kappa x$) if for all $\beta < \kappa$, there exists $\gamma < \kappa$ such that for all $\gamma \leq \alpha < \kappa$, $x_\alpha \in [x\restriction\beta]$.

We say that a sequence $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ has **κ-Cauchy property** if for any $\xi < \kappa$, there exists $\delta < \kappa$ such that for all $\alpha, \beta \in \kappa \setminus \delta$, $x_\alpha \in [x_\beta\restriction\xi]$.

Obviously, we get the following fact.

Proposition 6.1. *A sequence $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ has κ-I-Cauchy property if and only if there exists $x \in 2^\kappa$ such that $x_\alpha \to_\kappa x$.*

Proof: Assume that $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, and for $\xi < \kappa$, let $\delta_\xi < \kappa$ be such that for all $\alpha, \beta \in \kappa \setminus \delta$, $x_\alpha \in [x_\beta\restriction\xi]$. Then let $x = \bigcup_{\alpha<\kappa} x_{\alpha+1}\restriction\delta_\alpha$. Obviously, $x_\alpha \to_\kappa x$. The other implication is trivial. □

A sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ is **κ-pointwise convergent** to a function $f\colon 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \to_\kappa f$) on $A \subseteq 2^\kappa$ if

$$\forall_{x\in A}\forall_{\beta<\kappa}\exists_{\gamma<\kappa}\forall_{\gamma\leq\alpha<\kappa} f_\alpha(x) \in [f(x)\restriction\beta].$$

Similarly, we say that such a sequence of functions **converges κ-uniformly** to $f\colon 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \rightrightarrows_\kappa f$) on $A \subseteq 2^\kappa$ if

$$\forall_{\beta<\kappa}\exists_{\gamma<\kappa}\forall_{x\in A}\forall_{\gamma\leq\alpha<\kappa} f_\alpha(x) \in [f(x)\restriction\beta].$$

Finally, we say that a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ **converges κ-quasi-normally** to a function $f: 2^\kappa \to 2^\kappa$ (denoted by $f_\alpha \xrightarrow{QN}_\kappa f$) on $A \subseteq 2^\kappa$ if there exists an unbounded non-decreasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\forall_{x \in A} \exists_{\beta<\kappa} \forall_{\beta \leq \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha].$$

6.2 Properties of κ-quasi-normal and κ-uniform convergence

First, notice that κ-quasi-normal convergence implies κ-pointwise convergence.

Proposition 6.2. *If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-quasi-normally to a function $f: 2^\kappa \to 2^\kappa$, then $f_\alpha \to_\kappa f$.*

Proof: Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an unbounded non-decreasing sequence such that

$$\forall_{x \in 2^\kappa} \exists_{\beta<\kappa} \forall_{\beta \leq \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha],$$

and let $\xi \in \kappa$. Then let $\beta < \kappa$ be such that $\xi < \xi_\beta$. We get that

$$\forall_{x \in 2^\kappa} \exists_{\beta<\kappa} \forall_{\beta \leq \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha] \subseteq [f(x) \restriction \xi].$$

□

Proposition 6.3 (ω: [Bukovský, 2011])**.** *If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-quasi-normally to a function $f: 2^\kappa \to 2^\kappa$, then for any increasing sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists an increasing sequence $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that*

$$\forall_{x \in 2^\kappa} \exists_{\beta<\kappa} \forall_{\beta \leq \alpha < \kappa} f_{\delta_\alpha}(x) \in [f(x) \restriction \eta_\alpha].$$

Proof: Assume that there exists an unbounded non-decreasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\forall_{x \in 2^\kappa} \exists_{\beta<\kappa} \forall_{\beta \leq \alpha < \kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha],$$

and let $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an increasing sequence. Let $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an increasing sequence such that $\xi_{\delta_\alpha} > \eta_\alpha$ for any $\alpha < \kappa$. Then, for $\alpha < \kappa$ and any $x \in 2^\kappa$,

$$f_{\delta_\alpha}(x) \in [f(x) \restriction \xi_{\delta_\alpha}] \subseteq [f(x) \restriction \eta_\alpha].$$

□

Notice the following property of κ-uniform convergence.

Proposition 6.4. *Assume that $\lambda < \kappa$, and $\langle A_\alpha \rangle_{\alpha<\lambda} \in (\mathcal{P}(2^\kappa))^\lambda$. If a sequence of functions $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-uniformly to a function $f: 2^\kappa \to 2^\kappa$ on A_α for all $\alpha < \lambda$, then $f_\alpha \rightrightarrows_\kappa f$ on $\bigcup_{\alpha<\lambda} A_\alpha$.*

Proof: Let $\beta < \kappa$, and let $\langle \gamma_\alpha \rangle_{\alpha<\lambda} \in \kappa^\kappa$ be such that for all $x \in A_\alpha$, and for all $\xi < \kappa$ such that $\xi > \gamma_\alpha$, $f_\xi(x) \in [f(x) \restriction \beta]$. Let $\gamma = \bigcup_{\alpha<\lambda} \gamma_\alpha$. Obviously, $\gamma < \kappa$, since κ is regular, and for all $x \in \bigcup_{\alpha<\lambda} A_\alpha$, and $\xi > \gamma$, $f_\xi(x) \in [f(x) \restriction \beta]$. □

Proposition 6.5 (ω: [Bukovský, 2011])**.** *Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, and $f: 2^\kappa \to 2^\kappa$. The following conditions are equivalent:*

(1) $f_\alpha \xrightarrow{QN}_\kappa f$ *on* $A \subseteq 2^\kappa$,

(2) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, *and for all* $\alpha < \kappa$, $f_\alpha \rightrightarrows_\kappa f$ *on* A_α,

(3) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ *for all* $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ *for limit* $\beta < \kappa$, *and for all* $\alpha < \kappa$, $f_\alpha \rightrightarrows_\kappa f$ *on* A_α.

Proof: (2) and (3) are equivalent due to Propostion 6.4.

Assume that there exists an unbounded non-decreasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\forall_{x\in A} \exists_{\beta<\kappa} \forall_{\beta\leq\alpha<\kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha].$$

For $\beta < \kappa$, let

$$A_\beta = \{x \in A: \forall_{\beta\leq\alpha<\kappa} f_\alpha(x) \in [f(x) \restriction \xi_\alpha]\}.$$

Obviously, $\bigcup_{\alpha<\kappa} A_\alpha = A$. Also, for any $\beta < \kappa$, and $\xi < \kappa$, find $\gamma < \kappa$ such that $\gamma > \beta$, and $\xi < \xi_\gamma$. Then, for all $\gamma < \alpha < \kappa$, we get that $f_\alpha(x) \in [f(x) \restriction \gamma]$, for all $x \in A_\beta$, thus $f_\alpha \rightrightarrows_\kappa f$ on A_β.

Assume now that there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\alpha < \kappa$, $f_\alpha \rightrightarrows_\kappa f$ on A_α.

For $\gamma, \delta < \kappa$, let

$$\xi_{\delta,\gamma} = \bigcap_{\alpha\geq\delta} \bigcap \left\{\beta < \kappa: \forall_{x\in A_\gamma} f_\alpha(x) \in [f(x) \restriction \beta]\right\}.$$

Notice that for all $\gamma < \kappa$, $\langle \xi_{\delta,\gamma} \rangle_{\delta<\kappa}$ is a non-decreasing unbounded sequence. Also, if $\gamma < \gamma' < \kappa$, then for any $\delta < \kappa$, $\xi_{\delta,\gamma} \geq \xi_{\delta,\gamma'}$.

Hence, we can find an increasing sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for all $\gamma < \kappa$ if $\eta_\gamma \leq \delta < \kappa$, then $\xi_{\delta,\gamma} > \gamma$. Since $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, we can require also that $\bigcup_{\alpha<\beta} \eta_\alpha = \eta_\beta$ for limit $\beta < \kappa$.

Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that $\xi_\alpha = 0$ for $\alpha < \eta_0$, and $\xi_\alpha = \beta$ for all $\eta_\beta \leq \alpha < \eta_{\beta+1}$. Obviously, $\langle \xi_\alpha \rangle_{\alpha<\kappa}$ is unbounded and non-decreasing.

Let $x \in A$. There exists $\beta < \kappa$ such that for all $\beta \leq \gamma < \kappa$, $x \in A_\gamma$. Therefore, for all $\eta_\beta \leq \alpha < \kappa$,

$$f_\alpha(x) \in [f(x)\restriction \xi_{\alpha,\gamma}] \subseteq [f(x)\restriction \xi_\alpha],$$

where $\gamma < \kappa$ is such that $\eta_\gamma \leq \alpha < \eta_{\gamma+1}$. Hence, $f_\alpha \xrightarrow{QN}_\kappa f$. □

In particular, we get the following.

Corollary 6.6. *If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-uniformly to a function $f: 2^\kappa \to 2^\kappa$, then $f_\alpha \xrightarrow{QN}_\kappa f$.*

□

On the other hand, if a sequence converges κ-quasi normally on every element of a family of less than $\mathfrak{b}_\kappa$ subsets of 2^κ, it converges on its union.

Proposition 6.7 (ω: [Bukovský, 2011])**.** *Let $A \subseteq 2^\kappa$, and $A = \bigcup_{\alpha<\lambda} A_\alpha$ for $\lambda < \mathfrak{b}_\kappa$. If a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $A \to 2^\kappa$ converges κ-quasi normally on A_α to $f: A \to 2^\kappa$, for all $\alpha < \lambda$, then $f_\alpha \xrightarrow{QN}_\kappa f$ on A.*

Proof: Let $\langle \xi_{\alpha,\delta} \rangle_{\alpha<\kappa,\delta<\lambda}$ be such that for all $\delta < \lambda$

$$\forall_{x\in A_\delta} \exists_{\beta<\kappa} \forall_{\beta\leq\alpha<\kappa} f_\alpha(x) \in [f(x)\restriction \xi_{\alpha,\delta}].$$

Assume that for all $\delta < \lambda$, $\langle \xi_{\alpha,\delta} \rangle_{\alpha<\kappa}$ is non-decreasing.

For $\delta < \lambda$, construct inductively $x_\delta \in \kappa^\kappa$ such that for $\alpha < \kappa$,

$$x_\delta(\alpha+1) = \bigcap \{\beta < \kappa : \xi_{\beta,\delta} > \alpha+1 \wedge \beta > x_\delta(\alpha)\},$$

and $x_\delta(\beta) = \bigcup_{\alpha<\beta} x_\delta(\alpha)$ for limit $\beta < \kappa$.

But since $\lambda < \mathfrak{b}_\kappa$, there exists $x \in \kappa^\kappa$ such that for all $\delta < \lambda$, $x_\delta \leq^\kappa x$. Thus, for all $\delta < \lambda$ if $x(\gamma) < \alpha$, then $\xi_{\alpha,\delta} > \gamma + 1$.

Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that

$$\xi_\alpha = \begin{cases} 0, & \text{if } \alpha < x_0, \\ \beta, & \text{if } \forall_{\gamma<\beta} x(\gamma) < \alpha \wedge \alpha < x(\beta). \end{cases}$$

Hence, for all $\alpha, \delta < \kappa$, $\xi_\alpha \leq \xi_{\alpha,\delta}$.

Fix $x \in A$, and $\delta < \lambda$ such that $x \in A_\delta$. Then there exists $\beta < \kappa$ such that for all $\alpha > \beta$,

$$f_\alpha(x) \in [f(x)\restriction \xi_{\alpha,\delta}] \subseteq [f(x)\restriction \xi_\alpha].$$

□

Obviously, one can find sequences of functions which distinguish between different notions of convergence.

Proposition 6.8. *There exists a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \to_\kappa f$ with $f\colon 2^\kappa \to 2^\kappa$, but $f_\alpha \overset{QN}{\not\to}_\kappa f$.*

Proof: Indeed, let $\langle\langle \xi_{\alpha,x}\rangle_{\alpha<\kappa}\rangle_{x\in 2^\kappa}$ be an enumeration of all increasing sequences of ordinals $< \kappa$ such that for all limit $\gamma < \kappa$, $\bigcup_{\alpha<\gamma}\xi_{\alpha,x} = \xi_{\gamma,x}$ for all $x \in 2^\kappa$.

Let $\langle \eta_{\alpha,x}\rangle_{x\in 2^\kappa,\alpha<\kappa}$ be such that for $x \in 2^\kappa, \alpha < \kappa$, $\eta_{\alpha,x} = \beta$ if $\alpha \leq \xi_{\beta,x}$ and for all $\delta < \beta$, $\alpha > \xi_{\delta,x}$. Notice that $\langle \eta_{\alpha,x}\rangle_{\alpha<\kappa}$ is an unbounded non-decreasing sequence for all $x \in 2^\kappa$.

Then let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be defined in the following way. Let

$$f_\alpha(x)(\beta) = \begin{cases} 0, & \text{if } \beta < \eta_{\alpha,x} \\ 1, & \text{otherwise.} \end{cases}$$

By definition, $f_\alpha \to_\kappa \mathbf{0}$, but there is no sequence $\langle \xi_\alpha\rangle_{\alpha<\kappa}$ such that for all $x \in 2^\kappa$, there exists $\delta < \kappa$ such that $f_\alpha(x) \in [\mathbf{0}\restriction\xi_\alpha]$ for all $\alpha < \kappa$ with $\alpha > \delta$. □

6.3 Extending convergent sequence of functions

In this section, we prove that if $P \subseteq 2^\kappa$ is a κ-perfect set, then every sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of continuous functions $P \to 2^\kappa$ can be extended to a sequence of functions defined on the whole space 2^κ.

We start by proving the following proposition.

Proposition 6.9. *If $P \subseteq 2^\kappa$ is a κ-perfect set, and $f\colon P \to 2^\kappa$ is continuous, then there exists a continuous function $F\colon 2^\kappa \to 2^\kappa$ such that $F\restriction P = f$.*

Proof: Let $\langle s_\alpha\rangle_{\alpha<\kappa} \in (2^{<\kappa})^\lambda$ with $\lambda \leq \kappa$ be such that $2^\kappa \setminus P = \bigcup_{\alpha<\lambda}[s_\alpha]$, and $[s_\alpha] \cap [s_\beta] = \emptyset$, for any $\alpha < \beta < \lambda$. Let $\langle \xi_\alpha\rangle_{\alpha<\lambda} \in \kappa^\lambda$ be such that for $\alpha < \lambda$,

$$\xi_\alpha = \bigcup\{\xi \in \kappa\colon [s_\alpha\restriction\xi] \cap P \neq \emptyset\}.$$

Notice that since P is κ-perfect, for every $\alpha < \lambda$, $[s_\alpha\restriction\xi_\alpha] \cap P \neq \emptyset$, and therefore choose any $\langle x_\alpha\rangle_{\alpha<\lambda} \in (2^\kappa)^\lambda$ such that $x_\alpha \in [s_\alpha\restriction\xi_\alpha] \cap P$ for any $\alpha < \lambda$.

Let

$$F(x) = \begin{cases} f(x), & \text{if } x \in P, \\ f(x_\alpha), & \text{if } x \in [s_\alpha], \alpha < \lambda. \end{cases}$$

Then obviously, $F\restriction P = f$. Moreover F is continuous. Indeed, if $x \in [s_\alpha]$ for some $\alpha < \lambda$, then obviously, F is continuous at x, because it is constant on $[s_\alpha]$. On the other hand, if $x \in P$, and $t \in 2^{<\kappa}$ are such that $f(x) \in [t]$, then since f is continuous, we can get $s \in T_P$ such that $x \in [s]$ and $f[[s] \cap P] \subseteq [t]$.

But if $y \in [s] \setminus P$, then there exists $y' \in [s] \cap P$ such that $F(y) = f(y)$. Thus, $F[[s]] \subseteq [t]$ as well. □

Therefore, we get the following Corollary.

Corollary 6.10. *If $P \subseteq 2^\kappa$ is a κ-perfect set, and $\langle f_\alpha \rangle_{\alpha<\kappa}$ is a sequence of continuous functions $P \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on P, then there exists a sequence $\langle F_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $2^\kappa \to 2^\kappa$ such that $F_\alpha \to_\kappa 0$ on 2^κ, and for all $\alpha < \kappa$, $F_\alpha \restriction P = f_\alpha$.*

Proof: Let $\langle s_\alpha \rangle_{\alpha<\kappa} \in (2^{<\kappa})^\lambda$ with $\lambda \leq \kappa$ be such that $2^\kappa \setminus P = \bigcup_{\alpha<\lambda} [s_\alpha]$, and $[s_\alpha] \cap [s_\beta] = \varnothing$, for any $\alpha < \beta < \lambda$. Then for $\alpha < \lambda$, let $f'_\alpha : P \cup \bigcup_{\beta<\alpha} [s_\beta]$ be such that

$$f'_\alpha(x) = \begin{cases} f_\alpha(x), & \text{if } x \in P, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

If $\lambda < \kappa$, let $f'_\alpha(x) = \mathbf{0}$ for all $x \in 2^\kappa$, and $\lambda \leq \alpha < \kappa$.

Notice that f'_α is continuous, because $\bigcup_{\beta<\alpha} [s_\beta]$ is a closed open set. By Proposition 6.9, for $\alpha < \kappa$, let $F_\alpha : 2^\kappa \to 2^\kappa$ be a continuous function such that $F_\alpha \restriction P \cup \bigcup_{\beta<\alpha} [s_\beta] = f'_\alpha$. Obviously, $F_\alpha(x) \to_\kappa \mathbf{0}$ for all $x \in 2^\kappa$. □

As in the standard case, κ-uniform limit of a sequence of continuous functions is continuous as well.

Proposition 6.11. *Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_\kappa f$ on A, where $f : A \to 2^\kappa$. Then f is continuous on A.*

Proof: Assume otherwise that f is not continuous in $x \in A$. Therefore, there exists $\xi < \kappa$ such that for every $\alpha < \kappa$, there exists $x_\alpha \in [x \restriction \alpha] \cap A$ with $f(x_\alpha) \notin [f(x) \restriction \xi]$. But also there exists $\delta < \kappa$ such that for all $\alpha < \kappa$ with $\alpha > \delta$, and for every $y \in A$, $f_\alpha(y) \in [f(y) \restriction \xi]$. For such α and any $\beta < \kappa$, $f_\alpha(x_\beta) \in [f(x_\beta) \restriction \xi]$. But $[f(x_\beta) \restriction \xi] \cap [f(x) \restriction \xi] = \varnothing$, so $f_\alpha(x_\beta) \notin [f(x) \restriction \xi]$. On the other hand, $f_\alpha(x) \in [f(x) \restriction \xi]$, which implies that f_α is not continuous, and brings to a contradiction. □

The above fact can be used to prove that there exists a sequence of functions which converges κ-quasi-normally, but not κ-uniformly.

Proposition 6.12. *There exists a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \xrightarrow{QN}_\kappa f$ with $f : 2^\kappa \to 2^\kappa$, but $f_\alpha \not\rightrightarrows_\kappa f$.*

Proof: Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence defined as follows,

$$f_\alpha(x) = \begin{cases} \mathbf{1}, & \text{if } x \in [\mathbf{1} \restriction \alpha], \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

and let $f{:}2^\kappa \to 2^\kappa$, be such that $f(\mathbf{1}) = \mathbf{1}$ and $f(x) = \mathbf{0}$ for any $x \in 2^\kappa \setminus \{\mathbf{1}\}$.

Notice that $f_\alpha \xrightarrow{QN}_\kappa f$. Indeed, for every $x \in 2^\kappa$ if $x \in [\mathbf{1}{\restriction}\alpha] \setminus [\mathbf{1}{\restriction}\alpha+1]$, $\alpha < \kappa$, then for all $\beta > \alpha$, $f_\beta(x) \in [f(x){\restriction}\alpha]$.

On the other hand, by Proposition 6.11, $f_\alpha \not\rightrightarrows_\kappa f$, because f is not continuous. □

Notice also that if a sequence of continuous functions converges κ-uniformly on a set $A \subseteq 2^\kappa$, then it converges κ-uniformly on $\mathrm{cl}A$.

Proposition 6.13. *Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_\kappa f$ on A, where $f \in 2^\kappa \to 2^\kappa$ is continuous. Then $f_\alpha \rightrightarrows_\kappa f$ on clA.*

Proof: Indeed, let $\xi < \kappa$, and assume that for $\delta < \kappa$, for all $\alpha < \kappa$ such that $\alpha > \delta$, and all $x \in A$, $f_\alpha(x) \in [f(x){\restriction}\xi]$. Let $y \in \mathrm{cl}A$. Then for any $\beta < \kappa$ there exists $x_\beta \in A$ such that $x_\beta \in [y{\restriction}\beta]$. But, for all $\beta < \kappa$ and $\alpha > \delta$, $f_\alpha(x_\beta) \in [f(x_\beta){\restriction}\xi]$. Since f is continuous there exists $\delta' < \kappa$ such that for all $\beta < \kappa$ with $\beta > \delta$, $f(x_\beta) \in [f(y){\restriction}\xi]$, hence for $\alpha > \delta, \beta > \delta'$, $f_\alpha(x_\beta) \in [f(y){\restriction}\xi]$. But for every $\alpha < \kappa$, f_α is continuous, thus $f_\alpha(y) \in [f(y){\restriction}\xi]$, for all $\alpha > \delta$. □

Those properties can be used to prove that there is a κ-convergent sequence of functions which converges κ-uniformly only on nowhere dense sets.

Proposition 6.14 (ω: [Bukovský, 2011])**.** *There exists a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of continuous functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on 2^κ, but if $A \subseteq 2^\kappa$ is such that there exists $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $f_{\xi_\alpha} \rightrightarrows_\kappa \mathbf{0}$, then A is nowhere dense.*

Proof: Let $2^{<\kappa} = \{s_\alpha{:}\alpha < \kappa\}$ be an enumeration. We construct $\langle f_\alpha\rangle_{\alpha<\kappa}$ as follows. If $x \in 2^\kappa$, and $\alpha, \beta < \kappa$, then let $f_\alpha(x)(\beta) = 1$ if

(a) $x{\restriction}(\mathrm{len}(s_\beta)+1) = s_\beta{}^\frown 1$, for all $0 < \gamma < \alpha$,

(b) $x(\mathrm{len}(s_\beta)+1+\gamma) = 0$,

(c) and $x(\mathrm{len}(s_\beta)+1+\alpha) = 1$.

Otherwise, let $f_\alpha(x)(\beta) = 0$.

Notice that f_α is continuous for every $\alpha < \kappa$, and $f_\alpha \to_\kappa \mathbf{0}$.

But if $A \subseteq 2^\kappa$ is such that there exists $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \rightrightarrows_\kappa \mathbf{0}$ on A, then also $\mathrm{cl}A$ has this property (see Proposition 6.13). Assume that $B = \mathrm{int}(\mathrm{cl}A) \neq \emptyset$. Then there exists $x \in B$ such that $x{\restriction}(\mathrm{len}(s_\beta)+1) = s_\beta{}^\frown 1$, and $x(\alpha) = 0$, for all $\mathrm{len}(s_\beta) < \alpha < \kappa$. But as B is open, there also exists $\delta < \kappa$ such that for all $\delta < \xi < \kappa$, there is $x_\xi \in B$ such that $x_\xi{\restriction}(\mathrm{len}(s_\beta)+1) = s_\beta{}^\frown 1$, for all $0 < \gamma \leq \xi$, $x_\xi(\mathrm{len}(s_\beta)+\gamma) = 0$, and $x_\xi(\mathrm{len}(s_\beta)+1+\xi) = 1$. But then for all $\delta < \xi < \kappa$, $f_\xi(x_\xi) \notin [\mathbf{0}{\restriction}\beta+1]$, which is a contradiction. □

6.4 Special subsets of 2^κ related to convergence

Similarly to the case $\kappa = \omega$ (see section 1.4), we define some classes of special subsets of 2^κ related to convergence.

A set $A \subseteq 2^\kappa$ is a **κ-QN-set,** if any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, converges also κ-quasi-normally ($f_\alpha \xrightarrow{QN}_\kappa \mathbf{0}$ on A).

A set $A \subseteq 2^\kappa$ is a **κ-weak QN-set (κ-wQN-set),** if for any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, there exists an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \xrightarrow{QN}_\kappa \mathbf{0}$ on A.

A set $A \subseteq 2^\kappa$ is a **κ-mQN-set,** if any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, and for all $x \in A$, and $\alpha < \beta < \kappa$

$$\bigcup \{\gamma < \kappa \colon \forall_{\delta<\gamma} f_\alpha(x)(\delta) = 0\} \leq \bigcup \{\gamma < \kappa \colon \forall_{\delta<\gamma} f_\beta(x)(\delta) = 0\},$$

converges also κ-quasi-normally ($f_\alpha \xrightarrow{QN}_\kappa \mathbf{0}$ on A).

6.4.1 Basic properties

Proposition 6.15 (ω: [Bukovský, 2011])**.** *If $A \subseteq 2^\kappa$ is a κ-wQN-set, and $\langle f_\alpha \rangle_{\alpha<\kappa}$ is a sequence of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on A, then for any increasing sequence $\langle \eta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists an increasing sequence $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that*

$$\forall_{x\in A} \exists_{\beta<\kappa} \forall_{\beta\leq\alpha<\kappa} f_{\delta_\alpha}(x) \in [f(x) \restriction \eta_\alpha].$$

Proof: Since, A a κ-wQN-set, there exists $\langle x_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \xrightarrow{QN}_\kappa \mathbf{0}$ on A. Hence, by Proposition 6.3, there exists an increasing sequence $\langle \beta_\alpha \rangle_{\alpha<\kappa}$ such that

$$\forall_{x\in A} \exists_{\beta<\kappa} \forall_{\beta\leq\alpha<\kappa} f_{\xi_{\beta_\alpha}}(x) \in [f(x) \restriction \eta_\alpha].$$

Set $\delta_\alpha = \xi_{\beta_\alpha}$. □

The following corollary is immediately implied by Corollary 6.10.

Corollary 6.16. *If $A \subseteq 2^\kappa$ is a κ-QN-set (respectively, κ-mQN-set), and $P \subseteq 2^\kappa$ is κ-perfect, then $P \cap A$ is κ-QN-set (respectively, κ-mQN-set) as well.*

□

By Proposition 6.7, we immediately get the following.

Corollary 6.17 (ω: [Bukovský, 2011])**.** *If $\lambda < \mathfrak{b}_\kappa$, and $\langle A_\alpha \rangle_{\alpha<\lambda}$ is a sequence of κ-QN-sets (respectively, κ-mQN-sets), then $\bigcup_{\alpha<\lambda} A_\alpha$ is κ-QN-set (respectively, κ-mQN-set) as well.*

□

Therefore, we get the following corollaries.

Corollary 6.18. *If $\lambda < \mathfrak{b}_\kappa$, and $\langle P_\alpha\rangle_{\alpha<\lambda}$ is a sequence of κ-perfect sets, and A is a κ-QN-set (respectively, κ-mQN-set), then $A \cap \bigcup_{\alpha<\lambda} P_\alpha$ is κ-QN-set (respectively, κ-mQN-set) as well.*

□

Corollary 6.19. *If $X \subseteq 2^\kappa$, and $|X| < \mathfrak{b}_\kappa$, then X is a κ-QN-set.*

□

Finally, let us annotate that the whole 2^κ is not κ-wQN-set.

Proposition 6.20 (ω: [Bukovský, 2011])**.** *The generalized Cantor space 2^κ is not a κ-wQN-set.*

Proof: For $x \in 2^\kappa$, $\alpha < \kappa$, let $\delta_{x,\alpha}$ be an ordinal order isomorphic to $\langle\{\gamma \leq \alpha : x(\gamma) = 1\}, \leq\rangle$. Let $\langle f_\alpha\rangle_{\alpha\in\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, defined in the following way. For $\alpha, \beta < \kappa, x \in 2^\kappa$, let

$$f_\alpha(x)(\beta) = \begin{cases} 0, & \text{if } x(\alpha) = 0, \\ 1, & \text{if } x(\alpha) = 1, \beta \geq \delta_{x,\alpha}, \\ 0, & \text{if } x(\alpha) = 1, \beta < \delta_{x,\alpha}. \end{cases}$$

Obviously, f_α is continuous for any $\alpha < \kappa$. It is also easy to check that $f_\alpha \to_\kappa \mathbf{0}$.

To obtain a contradiction, assume that there exists an increasing sequence $\langle\eta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\eta_\alpha} \xrightarrow{QN}_\kappa \mathbf{0}$. Thus, there exists an increasing sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $\{\xi_\alpha : \alpha < \kappa\} \subseteq \{\eta_\alpha : \alpha < \kappa\}$, and for all $x \in X$, there exists $\delta < \kappa$ such that for all $\alpha < \kappa$ with $\alpha > \delta$, $f_{\xi_\alpha}(x) \in [\mathbf{0}\restriction\alpha + 1]$.

Let $x \in 2^\kappa$ be such that $x(\beta) = 1$ if $\beta \in \{\xi_\alpha : \alpha < \kappa\}$, and $x(\beta) = 0$, otherwise. Then, for all $\alpha < \kappa$, $\delta_{x,\xi_\alpha} = \alpha$, and hence $f_{\xi_\alpha}(\alpha) = 1$, so $f_{\xi_\alpha}(x) \notin [\mathbf{0}\restriction\alpha+1]$, which is a contradiction. □

Actually, every κ-wQN-set in 2^κ has to be κ-perfectly κ-meagre.

Proposition 6.21 (ω : [Bukovský, 2011])**.** *If $X \subseteq 2^\kappa$ is a κ-wQN-set, then $X \in P_\kappa\mathcal{M}_\kappa$.*

Proof: Let $P \subseteq 2^\kappa$ be a κ-perfect set. Thus, it is homeomorphic to 2^κ, so by Proposition 6.14, we get a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of continuous functions $P \to 2^\kappa$ such that $f_\alpha \to_\kappa \mathbf{0}$ on P, and if $A \subseteq P$ is such that there exists $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $f_{\xi_\alpha} \not\to_\kappa \mathbf{0}$, then A is nowhere dense in P.

By Corollary 6.16, we get that $X \cap P$ is a κ-wQN-set. Thus there exists a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \xrightarrow{QN}_\kappa \mathbf{0}$ on $X \cap P$.

Therefore, by Proposition 6.5, there exist a sequence $\langle A_\alpha \rangle_{\alpha<\kappa}$ of sets such that $\bigcup_{\alpha<\kappa} A_\alpha = X \cap P$, and $f_{\xi_\alpha} \not\rightrightarrows_\kappa \mathbf{0}$ on A_α for all $\alpha < \kappa$. Hence, for all $\alpha < \kappa$, A_α is nowhere dense in P, and so $X \cap P$ is κ-meagre in P. □

6.4.2 κ-Sequence selection properties

In this section we consider the space of all continuous functions $X \to 2^\kappa$, where $X \subseteq 2^\kappa$. This space is denoted here by $C_p^\kappa(X)$ when considered along with notion of κ-pointwise convergence.

We say that $C_p^\kappa(X)$ has **κ-sequence selection property** if for every $\langle f_{\alpha,\beta} \rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ such that $\langle f_{\alpha,\beta} \rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$ for all $\alpha < \kappa$, there exist $\langle \xi_\alpha \rangle_{\alpha<\kappa}, \langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha,\delta_\alpha} \to_\kappa \mathbf{0}$.

Moreover, if for every $\langle f_{\alpha,\beta} \rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ such that for all $\alpha < \kappa$, $\langle f_{\alpha,\beta} \rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$, there exists $\langle g_\alpha \rangle_{\alpha<\kappa} \in (C_p^\kappa(X))^\kappa$ such that $g_\alpha \to_I \mathbf{0}$, and

- for all $\alpha < \kappa$,
$$|\{f_{\alpha,\beta} : \beta \in \kappa\} \setminus \{g_\beta : \beta < \kappa\}| < \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_1)$,
- for all $\alpha < \kappa$,
$$|\{f_{\alpha,\beta} : \beta \in \kappa\} \cap \{g_\beta : \beta < \kappa\}| = \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_2)$,
- $$|\{\alpha < \kappa : |\{f_{\alpha,\beta} : \beta \in \kappa\} \cap \{g_\beta : \beta < \kappa\}| = \kappa\}| = \kappa,$$
then $C_p^\kappa(X)$ has **property** $\kappa - (\alpha_3)$,
- $$|\{\alpha < \kappa : \{f_{\alpha,\beta} : \beta \in \kappa\} \cap \{g_\beta : \beta < \kappa\} \neq \varnothing\}| = \kappa,$$
then $C_p^\kappa(X)$ **has property** $\kappa - (\alpha_4)$.

Obviously, we have the following implications between the above properties:

$$\kappa - (\alpha_1) \Rightarrow \kappa - (\alpha_2) \Rightarrow \kappa - (\alpha_3) \Rightarrow \kappa - (\alpha_4)$$

The above properties are closely related to notions of special subsets considered before. The following Proposition is an analogue of the standard case.

Theorem 6.22 (ω: [Bukovský, 2011])**.** *The following conditions are equivalent.*

(1) X *is a* κ*-wQN-set,*

(2) $C_p^\kappa(X)$ *has* κ*-sequence selection property,*

(3) $C_p^\kappa(X)$ *has* $\kappa - (\alpha_2)$ *property,*

(4) $C_p^\kappa(X)$ *has* $\kappa - (\alpha_3)$ *property,*

(5) $C_p^\kappa(X)$ *has* $\kappa - (\alpha_4)$ *property.*

Proof:

(1)⇒(2): Let X be κ-wQN set, and assume that $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ is such that for every $\alpha \in \kappa$, $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$. Let $\langle g_\beta\rangle_{\beta<\kappa}$ be a sequence of functions $X \to 2^\kappa$ defined in the following way:

$$g_\beta(x)(\alpha) = \begin{cases} 0, & \text{if } \forall_{\gamma<\alpha} f_{\alpha,\beta}(x)(\gamma) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Notice that for all $\beta < \kappa$, g_β is a continuous function. Indeed, if $g_\beta(x) \in [\mathbf{0}\restriction\alpha]$ for some $\alpha < \kappa$, then for all $\gamma < \alpha$, $x \in f_{\gamma,\beta}^{-1}[[\mathbf{0}\restriction\gamma]]$, which is an open set.

Moreover, if $\xi < \kappa$, and $x \in X$, then for every $\gamma < \xi$, let ξ_γ be such that $f_{\gamma,\beta}(x)(\delta) = 0$ for all $\delta < \xi$ and $\beta > \xi_\gamma$. Let $\eta = \bigcup_{\gamma<\xi} \xi_\gamma$. Then for all $\beta > \eta$, $g_\beta(x) \in [\mathbf{0}\restriction\xi]$, thus $g_\beta \to_\kappa \mathbf{0}$.

Recall that X is a κ-wQN-space, so there exists $\langle\delta_\beta\rangle_{\beta<\kappa} \in \kappa^\kappa$ such that $g_{\delta_\beta} \xrightarrow{QN}_\kappa \mathbf{0}$ in X. By Proposition 6.3, there exists an increasing sequence $\langle\delta'_\beta\rangle_{\beta\in\kappa} \in \kappa^\kappa$ such that $\{\delta'_\beta : \beta \in \kappa\} \subseteq \{\delta_\beta : \beta < \kappa\}$, and

$$\forall_{x\in 2^\kappa} \exists_{\gamma<\kappa} \forall_{\gamma\le\beta<\kappa} g_{\delta'_\beta}(x) \in [\mathbf{0}\restriction(\beta+1)].$$

But $g_{\delta'_\beta}(x) \in [\mathbf{0}\restriction(\beta+1)]$ implies that $f_{\beta,\delta'_\beta}(x) \in [\mathbf{0}\restriction\beta]$. Hence, $f_{\beta,\delta'_\beta} \xrightarrow{QN}_\kappa \mathbf{0}$.

(2)⇒ (3): Let $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ be such that for all $\alpha < \kappa$, $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$. Let $b : \kappa \times \kappa \to \kappa$ be a bijection, and let $\langle f'_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ be such that for $\alpha, \beta, \gamma < \kappa$, $f'_{\alpha,\beta} = f_{\gamma,\beta}$ if there exists $\delta < \kappa$ such that $b(\gamma, \delta) = \alpha$.

Now, let $\langle g_{\alpha,\beta}\rangle_{\alpha,\beta} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ be defined as follows. For $\alpha,\beta,\gamma<\kappa$, and $x\in X$, let

$$g_{\alpha,\beta}(x)(\gamma)=\begin{cases}0, & \text{if } \forall_{\delta\leq\alpha} f'_{\delta,\beta}(x)(\gamma)=0,\\ 1, & \text{otherwise.}\end{cases}$$

As before, it is easy to see that for all $\alpha,\beta<\kappa$, $g_{\alpha,\beta}$ is a continuous function.

Moreover, notice that for all $\alpha<\kappa$, $\langle g_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$, because for all $\alpha<\kappa$, $\langle f'_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$.

Thus, by κ-sequence selection property, there exist $\langle\xi_\alpha\rangle_{\alpha<\kappa},\langle\delta_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ such that $g_{\xi_\alpha,\delta_\alpha}\to_\kappa\mathbf{0}$. It is easy to see that we can require $\langle\xi_\alpha\rangle_{\alpha<\kappa},\langle\delta_\alpha\rangle_{\alpha<\kappa}$ to be increasing.

Let $\langle g_\alpha\rangle_{\alpha<\kappa}$ be such that for all $\alpha<\kappa$, $g_\alpha=f'_{\alpha,\delta_\alpha}$. Notice that then for $x\in X$, $\beta\in\kappa$, $g_\alpha(x)(\beta)=f'_{\alpha,\delta_\alpha}(x)(\beta)=0$ whenever $g_{\xi_\alpha,\delta_\alpha}(x)(\beta)=0$. Hence, since $g_{\xi_\alpha,\delta_\alpha}\to_\kappa\mathbf{0}$, $g_\alpha\to_\kappa\mathbf{0}$ as well. But also for all $\alpha<\kappa$,

$$|\{f_{\alpha,\beta}:\beta\in\kappa\}\cap\{g_\beta:\beta<\kappa\}|=\kappa.$$

(3)⇒(4)⇒(5): is obvious.

(5)⇒(1): Let $\langle f_\alpha\rangle_{\alpha<\kappa}\in(C_p^\kappa(X))^\kappa$ be such that $f_\alpha\to_\kappa\mathbf{0}$. Let $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa}$ be defined as follows. For $x\in X$, $\alpha,\beta,\gamma<\kappa$ such that $\gamma>0$, let

$$f_{\alpha,\beta}(x)(\gamma)=f_{\alpha+\beta}(x)(\alpha+\gamma),$$

and $f_{\alpha,\beta}(x)(0)=0$ if $f_{\alpha+\beta}\in[\mathbf{0}\restriction\alpha]$, and $f_{\alpha,\beta}(x)(0)=1$, otherwise.

Notice that $f_{\alpha,\beta}$ is continuous for all $\alpha,\beta<\kappa$. Moreover, for every $\alpha<\kappa$, $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$.

Since $\kappa-(\alpha_4)$ holds, we get $\langle\xi_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ and an increasing sequence $\langle\eta_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ such that $f_{\xi_\alpha,\eta_\alpha}\to_\kappa\mathbf{0}$. Notice that by induction, one can choose sequences $\langle\xi'_\alpha\rangle_{\alpha<\kappa},\langle\eta'_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ such that $\{\xi'_\alpha:\alpha<\kappa\}\subseteq\{\xi_\alpha:\alpha<\kappa\}$, $\{\eta'_\alpha:\alpha<\kappa\}\subseteq\{\eta_\alpha:\alpha<\kappa\}$, and $\langle\xi'_\alpha+\eta'_\alpha\rangle_{\alpha<\kappa}$ is an increasing sequence.

Notice that since $f_{\xi'_\alpha,\eta'_\alpha}\to_\kappa\mathbf{0}$ for any $x\in X$, there exists $\delta<\kappa$ such that for all $\alpha>\delta$, $f_{\xi'_\alpha,\eta'_\alpha}(x)(0)=0$. Thus, for such α, $f_{\xi'_\alpha+\eta'_\alpha}(x)\in[\mathbf{0}\restriction\xi'_\alpha]$. Hence, $f_{\xi'_\alpha+\eta'_\alpha}\xrightarrow{QN}_\kappa\mathbf{0}$. □

On the other hand, property $\kappa-(\alpha_1)$ of $C_p^\kappa(X)$ is equivalent to X being a κ-QN-space.

Theorem 6.23 (ω: [Bukovský, 2011])**.** *The following conditions are equivalent.*

(1) X is a κ-QN-set,

(2) $C_p^\kappa(X)$ has $\kappa - (\alpha_1)$ property.

Proof:

(1)⇒(2): Let $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta\in\kappa} \in (C_p^\kappa(X))^{\kappa\times\kappa}$ be such that for any $\alpha < \kappa$, $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-pointwise to $\mathbf{0}$.

Let $\langle g_\beta\rangle_{\beta<\kappa}$ be a sequence of functions $X \to 2^\kappa$ defined in the following way:

$$g_\beta(x)(\alpha) = \begin{cases} 0, & \text{if } \forall_{\gamma<\alpha} f_{\alpha,\beta}(x)(\gamma) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Notice that for all $\beta < \kappa$, g_β is a continuous function, and $g_\beta \to_\kappa \mathbf{0}$.

Since X is a κ-QN-set, we get an unbounded non-decreasing sequence $\langle \xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that

$$\forall_{x\in X}\exists_{\beta<\kappa}\forall_{\beta\le\alpha<\kappa} g_\alpha(x) \in [\mathbf{0}\restriction\xi_\alpha].$$

Let $\langle \delta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ be an increasing sequence such that for all $\alpha < \kappa$, $\xi_{\delta_\alpha} > \alpha$.

Fix a bijection $b\colon \kappa \to \bigcup_{\alpha<\kappa}\{\alpha\}\times\{\kappa\setminus\delta_\alpha\}$, and let $\langle f_\alpha\rangle_{\alpha<\kappa} \in (C_p^\kappa(X))^\kappa$ be such that $f_\alpha = f_{b(\alpha)}$.

Obviously, for all $\alpha < \kappa$, $|\{f_{\alpha,\beta}\colon \beta \in \kappa\} \setminus \{f_\beta\colon \beta < \kappa\}| < \kappa$, so it suffices to prove that $f_\alpha \to_\kappa \mathbf{0}$. Let $x \in X$, and $\xi \in \kappa$. We can find $\eta < \kappa$ such that $\eta > \xi$, and for all $\alpha < \kappa$ with $\alpha \ge \eta$, $g_\alpha(x) \in [\mathbf{0}\restriction\xi_\alpha]$. Moreover, we can get $\zeta < \kappa$ such that for all $\beta < \kappa$ with $\beta \ge \zeta$, and $\alpha < \eta$, $f_{\alpha,\beta}(x) \in [\mathbf{0}\restriction\xi]$.

Then, if $\alpha > \eta$, and $\beta \ge \delta_\alpha$, we have that $\beta > \delta_\eta$, and

$$g_\beta(x) \in [\mathbf{0}\restriction\xi_\beta] \subseteq [\mathbf{0}\restriction\xi_{\delta_\alpha}] \subseteq [\mathbf{0}\restriction\alpha].$$

Thus, for such $\alpha, \beta < \kappa$, we get that $f_{\alpha,\beta}(x) \in [\mathbf{0}\restriction\alpha] \subseteq [\mathbf{0}\restriction\xi]$. Hence, if

$$\langle\alpha,\beta\rangle \in \bigcup_{\eta<\alpha<\kappa} \{\alpha\}\times(\kappa\setminus\delta_\alpha) \cup \eta\times(\kappa\setminus\zeta),$$

then $f_{\alpha,\beta}(x) \in [\mathbf{0}\restriction\xi]$. Therefore, there exists $\gamma < \kappa$ such that for all $\alpha < \kappa$ such that $\alpha > \gamma$, $f_\alpha(x) \in [\mathbf{0}\restriction\xi]$.

(2)⇒(1): Let $\langle f_\alpha\rangle_{\alpha<\kappa} \in (C_p^\kappa(X))^\kappa$ be such that $f_\alpha \to_\kappa \mathbf{0}$. Consider a sequence $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa}$ defined as follows. For $x \in X$, $\alpha,\beta,\gamma<\kappa$ such that $\gamma>0$, let

$$f_{\alpha,\beta}(x)(\gamma) = f_\beta(x)(\alpha+\gamma),$$

and $f_{\alpha,\beta}(x)(0)=0$ if $f_\beta \in [\mathbf{0}\restriction\alpha]$, and $f_{\alpha,\beta}(x)(0)=1$, otherwise.

Since X possesses $\kappa-(\alpha_1)$-property, we get $\langle g_\alpha\rangle_{\alpha<\kappa} \in (C_p^\kappa(X))^\kappa$ such that $g_\alpha \to_\kappa \mathbf{0}$, and for every $\alpha<\kappa$, $|\{f_{\alpha,\beta}\colon \beta\in\kappa\}\setminus\{g_\beta\colon\beta<\kappa\}|<\kappa$. Hence, let $\langle\eta_\alpha\rangle \in \kappa^\kappa$ be an increasing sequence such that for all $\alpha<\kappa$, $\eta_\alpha>0$, and $\beta\geq\eta_\alpha$, $f_{\alpha,\beta}\in\{g_\gamma\colon\gamma<\kappa\}$. Let $\langle h_\alpha\rangle_{\alpha<\kappa}\in(C_p^\kappa)^\kappa$ be defined in the following way.

$$h_\gamma = \begin{cases} g_\gamma, & \text{if } g_\gamma = f_{\alpha,\beta}, \gamma\geq\alpha, \beta\geq\bigcup_{\zeta<\alpha}\eta_\zeta,\\ g_\gamma, & \text{if } g_\gamma = f_{\alpha,\beta}, \beta<\bigcup_{\zeta<\alpha}\eta_\zeta,\\ f_{\alpha,0}, & \text{if } g_\gamma = f_{\alpha,\beta}, \gamma\geq\alpha, \beta<\bigcup_{\zeta<\alpha}\eta_\zeta. \end{cases}$$

Moreover, let $\langle\xi_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ be such that $\xi_\alpha=\beta<\kappa$ if and only if $\alpha\leq\eta_\beta$, and for all $\gamma<\beta$, $\eta_\gamma<\alpha$.

Fix $x\in X$, and let $\delta<\kappa$ be such that for all $\beta>\delta$, $h_\beta(x)(0)=0$. If $\alpha\geq\eta_\delta$, let $\beta<\kappa$ be such that $\beta\geq\delta$, and $\alpha\leq\eta_\beta$, and for all $\gamma<\beta$, $\eta_\gamma<\alpha$. Then there exist $\gamma,\eta<\kappa$ such that $\gamma\geq\beta$ and $f_{\beta,\alpha}=h_\gamma$. Hence, for such β,α, $f_{\beta,\alpha}(x)(0)=0$, and so for $\alpha\geq\eta_\delta$, $f_\alpha(x)\in[\mathbf{0}\restriction\beta]=[\mathbf{0}\restriction\xi_\alpha]$. Thus, $f_\alpha \xrightarrow{QN}_\kappa \mathbf{0}$. □

A set $A\subseteq 2^\kappa$ has **κ-quasi-normal sequence selection property** if for any sequence $\langle f_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa}$ of functions $2^\kappa\to 2^\kappa$ such that $\langle f_{\alpha,\beta}\rangle_{\beta<\kappa}$ converges κ-quasi-normally to $\mathbf{0}$, and for every $\alpha,\beta<\kappa$, $f_{\alpha,\beta}$ is continuous, there exists $\langle\xi_\alpha\rangle_{\alpha<\kappa}\in\kappa^\kappa$ such that $f_{\alpha,\xi_\alpha}\xrightarrow{QN}_\kappa\mathbf{0}$.

6.4.3 Relation to cover selection properties in 2^κ

In the classical theory (see [Bukovský, 2011]), wQN-sets and QN-set are closely related to cover selection properties. I do not know whether such relation exists also in the generalized Cantor space. In particular, I leave those two problems as open questions.

Question 6.24. *Is every set satisfying $S_1^\kappa(\Gamma_\kappa,\Gamma_\kappa)$ principle, a κ-wQN-set?*

Question 6.25. *Does every κ-QN-set satisfy $S_1^\kappa(\Gamma_\kappa,\Gamma_\kappa)$ principle?*

Chapter 7

Ideal convergence in 2^κ

In this chapter we study notions of convergence of sequences of functions $2^\kappa \to 2^\kappa$ with respect to an ideal on κ.

The reader is expected to read the previous chapter first, we also use notions and notation defined in sections 1.4 and 1.5.

The results of this chapter are to be included in [Korch, 2017a].

7.1 Preliminaries

7.1.1 κ-I-convergence of sequences of points

If I is an ideal on κ, then we say that a sequence $\langle x_\alpha \rangle \in (2^\kappa)^\kappa$ **κ-converges to a point $x \in 2^\kappa$ with respect to the ideal I** ($x_\alpha \to_{\kappa\text{-}I} x$), if for any $\beta < \kappa$

$$\{\alpha < \kappa : x_\alpha \notin [x \restriction \beta]\} \in I.$$

Similarly, a sequence $\langle x_\alpha \rangle \in (2^\kappa)^\kappa$ **κ-I^*-converges to a point** $x \in 2^\kappa$ ($x_\alpha \to_{\kappa\text{-}I^*} x$), if there exists $B \in I$ such that $x_{\eta_\alpha} \to_I x$, where $\{\eta_\alpha : \alpha < \kappa\} = \kappa \setminus B$ is the increasing enumeration.

Notice that if $I = [\kappa]^{<\kappa}$, then both the convergence notions from above coincide with κ-convergence discussed in the previous chapter.

I will assume that every considered ideal I on κ is admissible.

An ideal I on κ is **κ-generated** if there exists a sequence $\langle C_\alpha \rangle_{\alpha<\kappa}$ of elements of I such that for every $A \in I$, there exists $\alpha < \kappa$ such that $A \subseteq C_\alpha$.

If $\lambda \leq \kappa$, we say that an ideal I on κ is **λ-complete** if for any $\mu < \lambda$, and $\mathcal{A} \in [I]^\mu$, $\bigcup \mathcal{A} \in I$.

We say that an ideal I on κ is **κ-admissible** if $[\kappa]^{<\kappa} \subseteq I$.

The following propositions are easy observations.

Proposition 7.1 (ω : [Kostyrko et al., 2000]). *If I is a κ-admissible ideal on κ, $\langle x_\alpha\rangle \in (2^\kappa)^\kappa$, $x \in 2^\kappa$, and $x_\alpha \to_\kappa x$, then $x_\alpha \to_{\kappa-I} x$.*

□

Proposition 7.2 (ω : [Balcerzak et al., 2007]). *If I is a κ-admissible ideal on κ, $\langle x_\alpha\rangle \in (2^\kappa)^\kappa$, $x \in 2^\kappa$, and $x_\alpha \to_{\kappa-I^*} x$, then $x_\alpha \to_{\kappa-I} x$.*

□

Proposition 7.3 (ω : [Kostyrko et al., 2000]). *If I is an ideal on κ, and $\langle x_\alpha\rangle \in (2^\kappa)^\kappa$, $x, y \in 2^\kappa$, with $x_\alpha \to_\kappa x$ and $x_\alpha \to_\kappa y$, then $x = y$.*

Proof: Assume that $x \neq y$. If $\xi < \kappa$ is such that $x \notin [y\restriction\xi]$, then

$$\{\alpha < \kappa\colon x_\alpha \notin [x\restriction\xi]\} = A \in I,$$

$\{\alpha < \kappa\colon x_\alpha \notin [y\restriction\xi]\} = B \in I$. But $2^\kappa \setminus (A \cup B) \neq \varnothing$, hence we get a contradiction.
□

Proposition 7.4 (ω : [Kostyrko et al., 2000]). *If I is a κ-admissible ideal on κ, $\langle x_\alpha\rangle \in (2^\kappa)^\kappa$, and $x \in 2^\kappa$ are such that for any increasing sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$, there exists an increasing sequence $\langle\eta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $\{\eta_\alpha\colon \alpha < \kappa\} \subseteq \{\xi_\alpha\colon \alpha < \kappa\}$ with $x_{\eta_\alpha} \to_{\kappa-I} x$, then $x_\alpha \to_{\kappa-I} x$.*

Proof: Assume that $x_\alpha \not\to_{\kappa-I} x$, and let $\xi < \kappa$ be such that

$$A = \{\alpha < \kappa\colon x_\alpha \notin [x\restriction\xi]\} \notin I.$$

Since I is κ-admissible, $|A| = \kappa$. Let $A = \{\xi_\alpha\colon \alpha < \kappa\}$ be the increasing enumeration of A. Obviously, there is no $\langle\eta_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ with $\{\eta_\alpha\colon \alpha < \kappa\} \subseteq \{\xi_\alpha\colon \alpha < \kappa\}$ such that $x_{\eta_\alpha} \to_{\kappa-I} x$. □

Proposition 7.5 (ω : [Kostyrko et al., 2000]). *If I is an ideal on κ such that there exists $A \in I$ with $|A| = \kappa$ and $|\kappa \setminus A| = \kappa$, then there exist $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, $x, y \in 2^\kappa$ and an increasing sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $x_\alpha \to_{\kappa-I} x$, $x_{\xi_\alpha} \to_{\kappa-I} y$, but $x \neq y$.*

Proof: For $\alpha < \kappa$, let

$$x_\alpha = \begin{cases} \mathbf{0}, & \text{if } \alpha \in A, \\ \mathbf{1}, & \text{otherwise.} \end{cases}$$

Then, if $A = \{\xi_\alpha\colon \alpha < \kappa\}$ is the increasing enumeration, then obviously, $x_\alpha \to_{\kappa-I} \mathbf{1}$, but $x_{\xi_\alpha} \to_{\kappa-I} \mathbf{0}$. □

Proposition 7.6 (ω : [Kostyrko et al., 2000]). *There exist a κ-admissible κ-complete ideal I, $x \in 2^\kappa$ and $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that $x_\alpha \to_{\kappa-I} x$, but $x_\alpha \not\to_{\kappa-I^*} x$.*

Proof: Let $b\colon \kappa \times \kappa \to \kappa$ be a bijection, and let

$$I = \{A \subseteq \kappa\colon |\{\alpha < \kappa\colon A \cap b[\{\alpha\} \times \kappa] \neq \varnothing\}| < \kappa\}.$$

Obviously, I is a κ-admissible ideal on κ. Moreover, I is κ-complete. For $\alpha, \beta < \kappa$, let also

$$z_\alpha(\beta) = \begin{cases} 0, & \text{if } \beta \leq \alpha, \\ 1, & \text{otherwise.} \end{cases}$$

Obviously, $z_\alpha \to_{\kappa-I} \mathbf{0}$.

For $\alpha < \kappa$, let $x_\alpha = z_\beta$ if $\alpha \in b[\{\beta\} \times \kappa]$. Notice that if $\xi < \kappa$, then

$$\{\alpha < \kappa\colon x_\alpha \notin [\mathbf{0}\restriction\xi]\} \subseteq b[(\xi+1) \times \kappa] \in I.$$

Hence, $x_\alpha \to_{\kappa-I} \mathbf{0}$.

On the other hand, to achieve a contradiction, assume that $x_\alpha \to_{\kappa-I^*} \mathbf{0}$. Then there exists $A \in I$ such that $\langle x_{\xi_\alpha} \rangle_{\alpha<\kappa}$ κ-converges to $\mathbf{0}$, where $\{\xi_\alpha\colon \alpha < \kappa\} = \kappa \setminus A$ is the increasing enumeration. But there exists $\xi < \kappa$ such that $A \subseteq b[\xi \times \kappa]$, and then $b[\{\xi+1\} \times \kappa] \subseteq \kappa \setminus A$. Since $|b[\{\xi+1\} \times \kappa]| = \kappa$, for all $\beta < \kappa$, there exists $\alpha < \kappa$ with $\alpha > \beta$ such that $x_\alpha \notin [\mathbf{0}\restriction(\xi+1)]$, which is a contradiction. □

Finally, every κ-I-convergent sequence has a subsequence which κ-converges.

Proposition 7.7 (ω: [Balcerzak et al., 2007]). *If I is a κ-admissible ideal on κ, $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, and $x \in 2^\kappa$ are such that $x_\alpha \to_{\kappa-I} x$, then there exists an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $x_{\xi_\alpha} \to_\kappa x$.*

Proof: Indeed, construct $\langle \xi_\alpha \rangle_{\alpha<\kappa}$ by induction. Given ξ_β for all $\beta < \gamma$, let

$$A = \{\alpha < \kappa\colon x_\alpha \notin [x\restriction\gamma]\} \cup \bigcup_{\beta<\gamma} \xi_\beta \in I.$$

Thus let $\xi_\gamma \in \kappa \setminus A$ be arbitrary. It is easy to see that $x_{\xi_\alpha} \to_\kappa x$. □

Similarly as in the case of ideals on ω we consider a property analogous to P-ideal property.

Proposition 7.8 (ω : [Balcerzak et al., 2007]). *Let I be an ideal on κ. The following statements are equivalent.*

(1) For any sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$, there exists $B \in I$ such that for every $\alpha < \kappa$, $|A_\alpha \setminus B| < \kappa$.

(2) For any sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in I^\kappa$, there exists a sequence $\langle B_\alpha\rangle_{\alpha<\kappa}$ such that $|A_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B_\alpha \in I$.

Proof: The proof is similar to the standard case (see [Balcerzak et al., 2007]).

(1)⇒(2): If $\langle A_\alpha\rangle_{\alpha<\kappa} \in I^\kappa$, there exists $B \in I$ such that for every $\alpha < \kappa$, $|A_\alpha \setminus B| < \kappa$. For $\alpha < \kappa$, let $B_\alpha = A_\alpha \cap B$. Then $A_\alpha \triangle B_\alpha = A_\alpha \setminus B$ is of cardinality less than κ for any $\alpha < \kappa$. Moreover, for all $\alpha < \kappa$, $B_\alpha \subseteq B$, so $\bigcup_{\alpha<\kappa} B_\alpha \subseteq B \in I$.

(2)⇒(1): Let $\langle A_\alpha\rangle_{\alpha<\kappa} \in I^\kappa$ be a sequence of pairwise disjoint sets. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. There exists a sequence $\langle B_\alpha\rangle_{\alpha<\kappa}$ such that $|A'_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B_\alpha \in I$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha \in I$. Then for any $\alpha < \kappa$, $A_\alpha \setminus B \subseteq A_\alpha \setminus B_\alpha$ is of cardinality less than κ.

□

An ideal I on κ is called a **κ-P-ideal** if it satisfies the above properties.

Proposition 7.9. *If I is a κ-complete ideal on κ, then it is a κ-P-ideal if and only if*

(3) For any sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in I^\kappa$ of mutually disjoint sets, there exists a sequence $\langle B_\alpha\rangle_{\alpha<\kappa}$ such that $|A_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B_\alpha \in I$.

Proof:

(2)⇒(3): Obvious.

(3)⇒(2): Let $\langle A_\alpha\rangle_{\alpha<\kappa} \in I^\kappa$, and for $\alpha < \kappa$, let $A'_\alpha = A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta$. There exists a sequence $\langle B'_\alpha\rangle_{\alpha<\kappa}$ such that $|A'_\alpha \triangle B'_\alpha| < \kappa$ for all $\alpha < \kappa$, and $\bigcup_{\alpha<\kappa} B'_\alpha \in I$. For $\alpha < \kappa$, take $B_\alpha = \bigcup_{\beta<\alpha} B'_\beta$. Since I is κ-complete, $B_\alpha \in I$, for all $\alpha < \kappa$. Moreover, for any $\alpha < \kappa$,

$$A_\alpha \triangle B_\alpha \subseteq \bigcup_{\beta<\alpha} (A'_\beta \triangle B'_\beta)$$

is a union of less than κ sets of cardinality less than κ, hence is of cardinality less then κ. Moreover,

$$\bigcup_{\alpha<\kappa} B_\alpha = \bigcup_{\alpha<\kappa} B'_\alpha \in I.$$

□

Proposition 7.10 (ω: [Kostyrko et al., 2000])**.** *Let I be a κ-complete ideal on κ. The following two properties are equivalent.*

(1) For every sequence $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$*, and* $x \in 2^\kappa$*,* $x_\alpha \to_{\kappa-I} x$ *if and only if* $x_\alpha \to_{\kappa-I^*} x$*.*

(2) I *is a* κ*-P-ideal.*

Proof:

(1)⇒(2): For $\alpha, \beta < \kappa$, let

$$z_\alpha(\beta) = \begin{cases} 0, & \text{if } \beta \leq \alpha, \\ 1, & \text{otherwise.} \end{cases}$$

Obviously, $z_\alpha \to_{\kappa-I} \mathbf{0}$. Let $\langle A_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$ be a sequence of mutually disjoint sets such that $\bigcup_{\alpha<\kappa} A_\alpha = \kappa$, and for $\alpha < \kappa$, let $x_\alpha = z_\beta$ if $\alpha \in A_\beta$.

Let $\xi < \kappa$. Then

$$\{\alpha < \kappa : x_\alpha \notin [\mathbf{0} \restriction \xi]\} = \bigcup_{\alpha \leq \xi} A_\alpha \in I.$$

Hence, $x_\alpha \to_{\kappa-I} \mathbf{0}$, thus by assumption $x_\alpha \to_{\kappa-I^*} \mathbf{0}$.

Therefore, let $B \in I$ be such that $x_{\xi_\alpha} \to_\kappa \mathbf{0}$, where $\{\xi_\alpha : \alpha < \kappa\} = \kappa \setminus B$ is the increasing enumeration. For $\alpha < \kappa$, let $B_\alpha = B \cap A_\alpha$. We get that

$$\bigcup_{\alpha<\kappa} B_\alpha \subseteq B \in I.$$

Notice also that $|(\kappa \setminus B) \cap A_\alpha| < \kappa$ for any $\alpha < \kappa$. Hence, $A_\alpha \triangle B_\alpha = A_\alpha \setminus B_\alpha$ is of cardinality less than κ. Therefore, by Proposition 7.9 I is a κ-P-ideal.

(2)⇒(1): Assume that I is a κ-P-ideal, and $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, $x \in 2^\kappa$ are such that $x_\alpha \to_{\kappa-I} x$. For $\alpha < \kappa$, let

$$A_\alpha = \{\beta < \kappa : x_\beta \notin [x \restriction \alpha]\}.$$

Then, by Proposition 7.8, there exists a sequence $\langle B_\alpha \rangle_{\alpha<\kappa}$ such that $|A_\alpha \triangle B_\alpha| < \kappa$ for all $\alpha < \kappa$, and $B = \bigcup_{\alpha<\kappa} B_\alpha \in I$.

Let $\xi < \kappa$. Since $|A_\xi \triangle B_\xi| < \kappa$, there exists $\delta < \kappa$ such that $B_\xi \cap (\kappa \setminus \delta) = A_\xi \cap (\kappa \setminus \delta)$. Thus, for all $\alpha \in \kappa \setminus B$ such that $\alpha > \delta$, $x_\alpha \in [x \restriction \xi]$. Hence, $x_\alpha \to_{\kappa-I^*} x$.

On the other hand, if $x_\alpha \to_{\kappa-I^*} x$, then $x_\alpha \to_{\kappa-I} x$ by Proposition 7.2.

□

A sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ is said to be **κ-I-unbounded** if for any $\delta < \kappa$,

$$\{\alpha < \kappa : \xi_\alpha < \delta\} \in I.$$

Notice the following easy observation.

Proposition 7.11. *Let I be an ideal on κ. A sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ is κ-I-unbounded if and only if a sequence $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ defined in the following way:*

$$x_\alpha(\beta) = \begin{cases} 0, & \text{if } \beta < \xi_\alpha \\ 1, & \text{otherwise.} \end{cases}$$

κ-I-converges to $\mathbf{0}$.

□

We say that a sequence $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ has the **κ-I-Cauchy property** if for any $\xi < \kappa$, there exists $B \in I$ such that for all $\alpha, \beta \in \kappa \setminus B$, $x_\alpha \in [x_\beta \restriction \xi]$.

Obviously, we get the following fact.

Proposition 7.12 (ω : [Dems, 2004])**.** *Let I be a κ-complete ideal on κ. A sequence $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ has κ-I-Cauchy property if and only if there exists $x \in 2^\kappa$ such that $x_\alpha \to_{\kappa-I} x$.*

Proof: Assume that $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, $x \in 2^\kappa$, and $x_\alpha \to_{\kappa-I} x$. Let $\xi < \kappa$, and let $A = \{\alpha < \kappa : x_\alpha \notin [x \restriction \xi]\} \in I$. Therefore, if $\alpha, \beta \in \kappa \setminus A$, then $x_\alpha, x_\beta \in [x \restriction \xi]$, thus $x_\alpha \in [x_\beta \restriction \xi]$.

On the other hand, if $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ has κ-I-Cauchy property. For $\xi < \kappa$, let $A_\xi \in I$ be such that for all $\alpha, \beta \in \kappa \setminus A_\xi$, $x_\alpha \in [x_\beta \restriction \xi]$. For $\xi < \kappa$, let $B_\xi = \bigcup_{\alpha \leq \xi} A_\alpha$. Let $\langle \xi_\alpha \rangle_{\alpha<\kappa}$ be such that for all $\alpha < \kappa$, $\xi_\alpha \in \kappa \setminus B_\alpha$.

Now, notice that for all $\alpha < \beta < \kappa$, $x_{\xi_\alpha}, x_{\xi_\beta} \in \kappa \setminus B_\alpha$, thus $x_{\xi_\alpha} \in [x_{\xi_\beta} \restriction \alpha]$. Hence, $x = \bigcup_{\alpha<\kappa}(x_{\xi_\alpha} \restriction \alpha)$ is an element of 2^κ, and if $\xi < \kappa$, then

$$\{\alpha < \kappa : x_\alpha \notin [x \restriction \xi]\} \subseteq B_\xi \in I.$$

Thus, $x_\alpha \to_{\kappa-I} x$. □

Obviously, by Proposition 6.1 we get the following fact.

Proposition 7.13. *If I is an ideal on κ, and $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, then the following properties are equivalent.*

(1) There exists $x \in 2^\kappa$ such that $x_\alpha \to_{\kappa-I^*} x$.

(2) There exists $A \in I$ such that $\langle x_\alpha \rangle_{\alpha \in \kappa \setminus A}$ has κ-Cauchy property.

□

It is also obvious that the latter property implies κ-I-Cauchy property.

Proposition 7.14 (ω : [Balcerzak et al., 2007])**.** *If I is a κ-admissible ideal on κ, and $\langle x_\alpha \rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$ is such that there exists $A \in I$ such that $\langle x_\alpha \rangle_{\alpha \in \kappa \setminus A}$ has κ-Cauchy property, then $\langle x_\alpha \rangle_{\alpha<\kappa}$ has κ-I-Cauchy property.*

□

In the case of κ-P-ideals this implication can be reversed.

Proposition 7.15 (ω : [Balcerzak et al., 2007]). *If I is a κ-admissible κ-P-ideal on κ, and $\langle x_\alpha\rangle_{\alpha<\kappa} \in (2^\kappa)^\kappa$, then there exists $A \in I$ such that $\langle x_\alpha\rangle_{\alpha\in\kappa\setminus A}$ has κ-Cauchy property if and only if $\langle x_\alpha\rangle_{\alpha<\kappa}$ has κ-I-Cauchy property.*

Proof: For $\xi < \kappa$, let $A_\xi \in I$ be such that for all $\alpha, \beta \in \kappa \setminus A_\xi$, $x_\alpha \in [x_\beta \restriction \xi]$. Since I is a κ-P-ideal, there exists $B \in I$ such that for all $\xi < \kappa$, $|A_\xi \setminus B| < \kappa$. Let $\{\xi_\alpha : \alpha < \kappa\} = \kappa \setminus B$ be the increasing enumeration.

Then $\langle x_{\xi_\alpha}\rangle_{\alpha<\kappa}$ satisfies κ-Cauchy condition. Indeed, if $\xi < \kappa$, then let $\delta < \kappa$ be such that $A_\xi \setminus B \subseteq \xi_\delta$. Thus for all $\alpha, \beta \in \kappa \setminus \delta$, $x_{\xi_\alpha}, x_{\xi_\beta} \notin B \cup A_\xi$, so $x_{\xi_\alpha} \in [x_{\xi_\beta} \restriction \xi]$. □

7.1.2 κ-I-convergence of sequences of functions

Using the notions defined above we can define different types of ideal convergence, similarly to the case $\kappa = \omega$.

A sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ **converges with respect to an ideal I on κ on a set $A \subseteq 2^\kappa$**:

κ-pointwise ideal, $f_\alpha \to_{\kappa\text{-}I} f$ if and only if

$$\forall_{\xi<\kappa}\forall_{x\in A}\{\alpha < \kappa : f_\alpha(x) \notin [f(x)\restriction\xi]\} \in I,$$

κ-quasi-normal ideal, $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ if and only if there exists a sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ which is κ-I-unbounded and

$$\forall_{x\in A}\{\alpha < \kappa : f_\alpha(x) \notin [f(x)\restriction\xi_\alpha]\} \in I,$$

κ-uniform ideal, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ if and only if

$$\forall_{\xi\in\kappa}\exists_{B\in I}\forall_{x\in A}\{\alpha < \kappa : f_\alpha(x) \notin [f(x)\restriction\xi]\} \subseteq B.$$

κ-I^*-pointwise, $f_\alpha \to_{\kappa\text{-}I^*} f$ if and only if for all $x \in A$, there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha}(x) \to_\kappa f(x)$ on A,

κ-I^*-quasi-normal, $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I^*} f$ if and only if there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha} \xrightarrow{QN}_\kappa f$ on A,

κ-I^*-uniform, $f_\alpha \rightrightarrows_{\kappa\text{-}I^*} f$ if and only if there exists $M = \{m_\alpha : \alpha < \kappa\} \subseteq \kappa$, $m_\beta \geq m_\alpha$ for all $\alpha < \beta < \kappa$ such that $\kappa \setminus M \in I$, and $f_{m_\alpha} \rightrightarrows_\kappa f$ on A.

If $J \subseteq I$ are ideals on κ, and $N \in I$, let $J \vee \langle N \rangle = \{A \cup N : A \in J\}$. If $A \subseteq 2^\kappa$ and $\langle f_\alpha \rangle_{\alpha<\kappa}$ is a sequence of functions $2^\kappa \to 2^\kappa$, we have the following notions of convergence.

κ-(J, I)-pointwise, $f_\alpha \to_{\kappa-J,I} f$ if and only if for all $x \in A$, there exists $N \in I$ such that for all $\xi < \kappa$,

$$\{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi]\} \in J \vee \langle N \rangle,$$

κ-(J, I)-quasi-normal, $f_\alpha \xrightarrow{QN}_{\kappa-J,I} f$ if and only if there exists $N \in I$ and a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa}$, which is κ-$J \vee \langle N \rangle$-unbounded such that for all $x \in A$,

$$\{\alpha < \kappa : f_\alpha(x) \in [f(x) \restriction \xi_\alpha]\} \in J \vee \langle N \rangle.$$

κ-(J, I)-uniform, $f_\alpha \rightrightarrows_{\kappa-J,I} f$ if and only if there exists $N \in I$ and $f_\alpha \rightrightarrows_{\kappa-J\vee\langle N\rangle} f$ on A.

7.2 Relation between different notions of κ-ideal convergence

7.2.1 Properties of κ-I-quasi-normal convergence

In this section, we generalize some results of [Das et al., 2014] to the case of ideal convergence in 2^κ. Let I be an ideal on κ.

First notice that κ-I-quasi-normal convergence implies κ-I-pointwise convergence.

Proposition 7.16. *If $\langle f_\alpha \rangle_{\alpha<\kappa}$ is a sequence of functions $2^\kappa \to 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$ such that $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on $A \subseteq 2^\kappa$. Then $f_\alpha \to_{\kappa-I} f$ on A.*

Proof: Let $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ be a κ-I-unbounded sequence such that

$$\forall_{x\in A} \{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi_\alpha]\} \in I,$$

and let $\xi \in \kappa$. We get that for all $x \in A$,

$$\{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi]\} \subseteq \{\alpha < \kappa : \xi_\alpha < \xi\} \cup \{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi_\alpha]\} \in I.$$

□

On the other hand, κ-I-uniform convergence implies κ-I-quasi-normal convergence.

Proposition 7.17 (ω: [Das et al., 2014]). *If $\langle f_\alpha\rangle_{\alpha<\kappa}$ is a sequence of functions $2^\kappa \to 2^\kappa$, and $f\colon 2^\kappa \to 2^\kappa$ such that $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on $A \subseteq 2^\kappa$. Then $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A.*

Proof: Let $\langle \xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that

$$\xi_\alpha = \bigcap_{x\in A}\bigcup\{\beta<\kappa\colon f_\alpha(x) \in [f(x)\restriction\beta]\}.$$

Notice that for any $\delta<\kappa$, there exists $B \in I$ such that for all $x \in A$, $\{\alpha<\kappa\colon f_\alpha(x) \notin [f(x)\restriction\delta]\} \subseteq B$. Hence,

$$\{\alpha<\kappa\colon \xi_\alpha<\delta\} = \{\alpha<\kappa\colon \exists_{x\in A} f_\alpha(x) \notin [f(x)\restriction\delta]\} = \bigcup_{x\in A}\{\alpha<\kappa\colon f_\alpha(x)\notin[f(x)\restriction\delta]\} \subseteq B \in I,$$

thus, $\langle\xi_\alpha\rangle_{\alpha<\kappa}$ is κ-I-unbounded. By definition, for all $x \in A$, $f_\alpha(x) \in [f(x)\restriction\xi_\alpha]$, so $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A. □

Thus, we get the following corollary.

Proposition 7.18. *If $\langle f_\alpha\rangle_{\alpha<\kappa}$ is a sequence of functions $2^\kappa \to 2^\kappa$, and $f\colon 2^\kappa \to 2^\kappa$ such that $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on $A \subseteq 2^\kappa$, then there exists a κ-I-unbounded sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that for all $x \in A$, $\{\alpha<\kappa\colon f_\alpha(x)\notin[f(x)\restriction\xi_\alpha]\} = \varnothing$.*

Proof: See the proof of Proposition 7.17. □

The following proposition is an easy observation.

Proposition 7.19. *Let I be a λ^+-complete ideal on κ for $\lambda<\kappa$, and let $\langle A_\alpha\rangle_{\alpha<\lambda} \in (\mathcal{P}(2^\kappa))^\lambda$. If a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges κ-I-uniformly to a function $f\colon 2^\kappa \to 2^\kappa$ on A_α for all $\alpha<\lambda$, then $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on $\bigcup_{\alpha<\lambda} A_\alpha$.*

□

Therefore, we get the following.

Corollary 7.20. *Let I be a κ-complete ideal and $A \subseteq 2^\kappa$. Then the following conditions are equivalent.*

(1) there exists a sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\beta<\kappa$, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on A_β,

(2) there exists a sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha<\beta<\kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta<\kappa$, and for all $\beta<\kappa$, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on A_β.

□

The following proposition establishes relation between the above properties and κ-I-quasi normal convergence.

Proposition 7.21 (ω: [Das et al., 2014]). *Let I be an κ-complete ideal on κ. If $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ is such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β, then $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on the whole A.*

Proof: By Proposition 7.18, we get $\langle \xi_{\alpha,\delta}\rangle_{\alpha,\delta\in\kappa} \in \kappa^\kappa$ such that for all $\delta < \kappa$, $\langle \xi_{\alpha,\delta}\rangle_{\alpha<\kappa}$ is a κ-I-unbounded sequence, and for all $x \in A_\delta$,

$$\{\alpha < \kappa\colon f_\alpha(x) \notin [f(x)\restriction\xi_{\alpha,\delta}]\} = \varnothing.$$

Fix $\langle B_\delta\rangle_{\delta<\kappa} \in I^\kappa$ such that $\{\alpha < \kappa\colon \xi_{\alpha,\delta} < \delta\} \subseteq B_\delta$. Such B_δ exists because for all $\delta < \kappa$, $\langle \xi_{\alpha,\delta}\rangle_{\alpha<\kappa}$ is a κ-I-unbounded sequence.

Now, for $\delta < \kappa$, let $C_\delta = \bigcup_{\gamma\le\delta} B_\gamma$. Since I is κ-complete, $C_\delta \in I$, for all $\delta < \kappa$.

Finally, let $\langle \xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ be such that

$$\xi_\alpha = \begin{cases} \beta, & \text{if } \forall_{\gamma<\beta}\alpha \notin C_\gamma \wedge \alpha \in C_\beta, \\ \bigcup_{\gamma\le\alpha} \xi_{\alpha,\gamma} + 1, & \text{if } \alpha \notin \bigcup_{\gamma<\kappa} C_\gamma. \end{cases}$$

Notice that $\langle \xi_\alpha\rangle_{\alpha<\kappa}$ is κ-I-unbounded. Indeed, if $\delta < \kappa$, then

$$\{\alpha < \kappa\colon \xi_\alpha < \delta\} \subseteq C_\delta \in I.$$

Moreover, for every $x \in A$, let $\delta < \kappa$ be such that $x \in A_\delta$. Then,

$$\{\alpha < \kappa\colon f_\alpha(x) \notin [f(x)\restriction\xi_{\alpha,\delta}]\} = \varnothing,$$

but if $\xi_\alpha < \xi_{\alpha,\delta}$, then $\alpha \in C_\delta \cup \delta$, thus,

$$\{\alpha < \kappa\colon f_\alpha(x) \notin [f(x)\restriction\xi_\alpha]\} \subseteq \{\alpha < \kappa\colon f_\alpha(x) \notin [f(x)\restriction\xi_{\alpha,\delta}] \wedge \xi_\alpha \ge \xi_{\alpha,\delta}\} \cup \{\alpha < \kappa\colon \xi_\alpha < \xi_{\alpha,\delta}\} = \varnothing \cup C_\delta \cup \delta \in I.$$

Therefore, $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on the whole A □

We need an additional assumption to prove the converse.

Proposition 7.22 (ω: [Das et al., 2014]). *Let I be a κ-generated, κ-complete ideal on κ. If $f_\alpha \xrightarrow{QN}_{\kappa-I} f$ on $A \subseteq 2^\kappa$, then there exists $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$ such that for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β.*

Proof: Let $\langle C_\alpha \rangle_{\alpha<\kappa} \in I^\kappa$ be such that $C_\alpha \subseteq C_\beta$, for all $\alpha < \beta$, and for every $B \in I$, there exists $\alpha < \kappa$ such that $B \subseteq C_\alpha$.

Since $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A, there exists a sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$, which is κ-I-unbounded, and $\langle \delta_x \rangle_{x\in A} \in \kappa^A$ such that for all $x \in A$,

$$\{\alpha < \kappa : f_\alpha(x) \notin [f(x) \restriction \xi_\alpha]\} \subseteq C_{\delta_x}.$$

Let $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(A))^\kappa$ be defined in the following way. For $\alpha < \kappa$, let

$$A_\alpha = \{x \in A : \delta_x = \alpha\}.$$

Obviously, $A = \bigcup_{\alpha<\kappa} A_\alpha$.

Also, for every $\delta < \kappa$, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on A_δ. Indeed, let $\xi < \kappa$. Then $C = \{\alpha < \kappa : \xi_\alpha < \xi\} \in I$. Thus for all $x \in A_\delta$,

$$\{\alpha < \kappa : f_\alpha \notin [f(x) \restriction \xi]\} \subseteq$$
$$\{\alpha < \kappa : \xi_\alpha < \xi\} \cup \{\alpha < \kappa : f_\alpha \notin [f(x) \restriction \xi_\alpha] \wedge \xi_\alpha \geq \xi\} \subseteq C \cup C_\delta \in I.$$

□

We therefore get the following corollary.

Corollary 7.23 (ω: [Das et al., 2014])**.** *Let I be a κ-generated, κ-complete ideal on κ, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, $A \subseteq 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$. The following conditions are equivalent:*

(1) $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A,

(2) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on A_β,

(3) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta < \kappa$, and for all $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa\text{-}I} f$ on A_β.

□

In particular, we get the following corollary.

Corollary 7.24. *Let I be a κ-generated, κ-complete ideal on κ, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, $A \subseteq 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$. If there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\beta < \kappa$, $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A_β, then $f_\alpha \xrightarrow{QN}_{\kappa\text{-}I} f$ on A.*

□

Moreover, in this case we can require the sets A_α to be closed, if the sequence consists of continuous functions.

Proposition 7.25 (ω: [Das et al., 2014]). *Let I be a κ-generated, κ-complete ideal on κ, and let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, $A \subseteq 2^\kappa$, and $f\colon 2^\kappa \to 2^\kappa$. The conditions of Corollary 7.23 are equivalent to:*

(4) there exists a sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ of closed in A sets such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\beta<\kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β,

(5) there exists a sequence $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ of closed in A sets such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ for all $\alpha<\beta<\kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ for limit $\beta<\kappa$, and for all $\beta<\kappa$, $f_\alpha \rightrightarrows_{\kappa-I} f$ on A_β.

Proof: Notice that obviously (5) $\Rightarrow$ (4) $\Rightarrow$ (2), and the union of less than κ closed sets is closed, thus (4) $\Rightarrow$ (5). To see that (1) $\Rightarrow$ (4), notice that the sets A_α defined in the proof of Proposition 7.22 can actually be described in the following way:

$$A_\alpha = \{x \in A\colon \forall_{\alpha,\beta\notin C_\alpha}\, \alpha<\beta \Rightarrow f_\alpha(x) \in [f_\beta(x)\restriction\alpha]\}$$

and are closed, if f_α is continuous for all $\alpha<\kappa$. □

Notice also the following fact.

Proposition 7.26 (ω: [Balcerzak et al., 2007]). *If I is a κ-admissible ideal on κ, $\langle f_\alpha\rangle_{\alpha<\kappa}$ is a sequence of functions $2^\kappa \to 2^\kappa$, and $f\colon 2^\kappa \to 2^\kappa$ is such that $f_\alpha \rightrightarrows_{\kappa-I} f$ on $X \subseteq 2^\kappa$, then there exists an increasing sequence $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \rightrightarrows_\kappa f$ on X.*

Proof: In the same way as in the proof of Proposition 7.7, construct $\langle\xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ by induction. Given ξ_β for all $\beta<\gamma$, let

$$A = \{\alpha<\kappa\colon \forall_{x\in X} f_\alpha(x) \notin [f(x)\restriction\gamma]\} \cup \bigcup_{\beta<\gamma} \xi_\beta \in I.$$

Thus let $\xi_\gamma \in \kappa\setminus A$ be arbitrary. It is easy to see that $f_{\xi_\alpha} \rightrightarrows_\kappa f$ on X. □

Thus, similarly to the classical case, a κ-I uniform limit of continuous function has to be continuous.

Corollary 7.27 (ω: [Balcerzak et al., 2007]). *Let I be an ideal on κ, and let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_{\kappa-I} f$ on A, where $f\colon A \to 2^\kappa$. Then f is continuous on A.*

Proof: We use Propositions 7.26 and 6.11. □

One can find sequences of functions which distinguish different notions of κ-I-convergence.

Proposition 7.28 (ω: [Das et al., 2014])**.** *Let I be a κ-admissible ideal on κ. There exists a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \to_{\kappa\text{-}I} f$ with $f\colon 2^\kappa \to 2^\kappa$, but $f_\alpha \overset{QN}{\not\longrightarrow}_{\kappa\text{-}I} f$.*

Proof: The example constructed in the proof of Proposition 6.8 is also valid here. Indeed, for every κ-I unbounded sequence $\langle \xi_\alpha\rangle_{\alpha<\kappa} \in \kappa^\kappa$ there exists an increasing unbounded sequence $\langle \eta_\alpha\rangle_{\alpha<\kappa}$ with $\eta_\beta = \bigcup_{\alpha<\beta}\eta_\alpha$ for all limit $\alpha<\kappa$ such that

$$\{\alpha<\kappa\colon \xi_\alpha < \eta_\alpha\} \notin I.$$

Therefore, $f_\alpha \overset{QN}{\not\longrightarrow}_{\kappa\text{-}I} \mathbf{0}$, but since $f_\alpha \to_\kappa \mathbf{0}$, we get that $f_\alpha \to_{\kappa\text{-}I} \mathbf{0}$. □

Proposition 7.29 (ω: [Das et al., 2014])**.** *Let I be an ideal on κ. There exists a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \overset{QN}{\longrightarrow}_{\kappa\text{-}I} f$ with $f\colon 2^\kappa \to 2^\kappa$, but $f_\alpha \not\rightrightarrows_{\kappa\text{-}I} f$.*

Proof: Because of Corollary 7.27, the example constructed in Proposition 6.12 is valid also in this case. □

7.2.2 Properties of κ-I^*-quasi-normal convergence

Let I be an ideal on κ. By Proposition 6.2 and Corollary 6.6, we immediately get the following implications.

Corollary 7.30. *Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, $A \subseteq 2^\kappa$ and $f\colon 2^\kappa \to 2^\kappa$.*

(a) if $f_\alpha \rightrightarrows_{\kappa\text{-}I^*} f$ on A, then $f_\alpha \overset{QN}{\longrightarrow}_{\kappa\text{-}I^*} f$ on A,

(b) if $f_\alpha \overset{QN}{\longrightarrow}_{\kappa\text{-}I^*} f$ on A, then $f_\alpha \to_{\kappa\text{-}I^*} f$ on A.

□

By Proposition 6.11, we also immediately get the following fact.

Corollary 7.31 (ω: [Balcerzak et al., 2007])**.** *Let I be an ideal on κ, and let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $A \subseteq 2^\kappa$. Assume that $f_\alpha \rightrightarrows_{\kappa\text{-}I^*} f$ on A, where $f\colon A \to 2^\kappa$. Then f is continuous on A.*

□

The following Proposition is an easy observation.

Proposition 7.32. *Let I be a λ^+-complete ideal on κ for $\lambda<\kappa$, and let $\langle A_\alpha\rangle_{\alpha<\lambda} \in (\mathcal{P}(2^\kappa))^\lambda$. If a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ converges $\kappa - I^*$-uniformly to a function $f\colon 2^\kappa \to 2^\kappa$ on A_α for all $\alpha<\lambda$, then $f_\alpha \rightrightarrows_{\kappa\text{-}I^*} f$ on $\bigcup_{\alpha<\lambda} A_\alpha$.*

Proof: Let $\langle M_\delta\rangle_{\delta<\lambda}$ be such that for all $\delta<\lambda$, $\kappa\setminus M_\delta\in I$, $M_\delta=\{m_{\alpha,\delta}:\alpha<\kappa\}\subseteq\kappa$, $m_{\beta,\delta}>m_{\alpha,\delta}$ for all $\alpha<\beta<\kappa$, and $f_{m_{\alpha,\delta}}\rightrightarrows_\kappa f$ on A_δ. Then let $M=\bigcap_{\delta<\lambda}M_\delta$. We have that $\kappa\setminus M\in I$. Let $M=\{m_\alpha:\alpha<\kappa\}\subseteq\kappa$ be an enumeration such that $m_\beta>m_\alpha$ for all $\alpha<\beta<\kappa$. Then for all $\delta<\lambda$, $f_{m_\alpha}\rightrightarrows_\kappa f$ on A_δ. Hence, by Proposition 6.4, $f_{m_\alpha}\rightrightarrows_\kappa f$ on $\bigcup_{\delta<\lambda}A_\delta$. Thus, $f_\alpha\rightrightarrows_{\kappa\text{-}I^*} f$ on $\bigcup_{\delta<\lambda}A_\delta$. □

A similar Proposition holds for κ-I^*-quasi-normal convergence.

Proposition 7.33. *Let I be a λ^+-complete ideal on κ for $\lambda<\kappa$, and let $\langle A_\alpha\rangle_{\alpha<\lambda}\in(\mathcal{P}(2^\kappa))^\lambda$. If a sequence $\langle f_\alpha\rangle_{\alpha<\kappa}$ of functions $2^\kappa\to 2^\kappa$ converges κ-I^*-quasi normally to a function $f:2^\kappa\to 2^\kappa$ on A_α for all $\alpha<\lambda$, then $f_\alpha\xrightarrow{QN}_{\kappa\text{-}I^*} f$ on $\bigcup_{\alpha<\lambda}A_\alpha$.*

Proof: Again, let $\langle M_\delta\rangle_{\delta<\lambda}$ be such that for all $\delta<\lambda$, $\kappa\setminus M_\delta\in I$, $M_\delta=\{m_{\alpha,\delta}:\alpha<\kappa\}\subseteq\kappa$, $m_{\beta,\delta}>m_{\alpha,\delta}$ for all $\alpha<\beta<\kappa$, and $f_{m_{\alpha,\delta}}\xrightarrow{QN}_\kappa f$ on A_δ. Then let $M=\bigcap_{\delta<\lambda}M_\delta$. We have that $\kappa\setminus M\in I$. Let $M=\{m_\alpha:\alpha<\kappa\}\subseteq\kappa$ be an enumeration such that $m_\beta\geq m_\alpha$ for all $\alpha<\beta<\kappa$. Then for all $\delta<\lambda$, $f_{m_\alpha}\xrightarrow{QN}_\kappa f$ on A_δ. Hence, by Proposition 6.7, $f_{m_\alpha}\xrightarrow{QN}_\kappa f$ on $\bigcup_{\delta<\lambda}A_\delta$. Thus, $f_\alpha\xrightarrow{QN}_{\kappa\text{-}I^*} f$ on $\bigcup_{\delta<\lambda}A_\delta$. □

Proposition 7.34 (ω: [Das et al., 2014])**.** *Let I be a κ-complete ideal on κ. If $f_\alpha\xrightarrow{QN}_{\kappa\text{-}I^*} f$ on $A\subseteq 2^\kappa$, then there exists $\langle A_\alpha\rangle_{\alpha<\kappa}\in(\mathcal{P}(2^\kappa))^\kappa$ such that $A=\bigcup_{\alpha<\kappa}A_\alpha$, and for all $\beta<\kappa$, $f_\alpha\rightrightarrows_{\kappa\text{-}I^*} f$ on A_β.*

Proof: Since $f_\alpha\xrightarrow{QN}_{\kappa\text{-}I^*} f$ on $A\subseteq 2^\kappa$, let $B\in I$ be such that $\langle f_{\xi_\alpha}\rangle_{\alpha\in\kappa}$ converges κ-quasi-normally to f on A, where $\{\xi_\alpha:\alpha<\kappa\}=\kappa\setminus B$ is the increasing enumeration. By Proposition 6.5, there exists $\langle A_\alpha\rangle_{\alpha<\kappa}$ such that $A=\bigcup_{\alpha<\kappa}A_\alpha$, and $\langle f_{\xi_\alpha}\rangle_{\alpha\in\kappa}$ converges κ-uniformly on A_α for all $\alpha<\kappa$. Thus, $f_\alpha\rightrightarrows_{\kappa\text{-}I^*} f$ on A_α, for all $\alpha<\kappa$. □

The reverse implication holds for κ-P-ideals.

Proposition 7.35 (ω: [Das et al., 2014])**.** *If I is a κ-admissible κ-P-ideal on κ, and $\langle A_\alpha\rangle_{\alpha<\kappa}\in(\mathcal{P}(2^\kappa))^\kappa$ is such that $A=\bigcup_{\alpha<\kappa}A_\alpha$, $A_\alpha\subseteq A_\beta$ for all $\alpha<\beta<\kappa$, $\bigcup_{\alpha<\beta}A_\alpha=A_\beta$ for limit $\beta<\kappa$, and for all $\alpha<\kappa$, $f_\alpha\rightrightarrows_{\kappa\text{-}I^*} f$ on A_α, then $f_\alpha\xrightarrow{QN}_{\kappa\text{-}I^*} f$ on the whole A.*

Proof: Notice that there exists a sequence $\langle B_\alpha\rangle_{\alpha<\kappa}\in I^\kappa$ such that for all $\alpha<\kappa$, $\langle f_{\xi_{\alpha,\beta}}\rangle_{\beta<\kappa}$ converges κ-uniformly to f on A_α, where $\{\xi_{\alpha,\beta}:\beta<\kappa\}=\kappa\setminus B_\alpha$ is the increasing enumeration.

Since I is a κ-P-ideal, take $B\in I$ such that for every $\alpha<\kappa$, $|B_\alpha\setminus B|<\kappa$. Notice that as I is κ-admissible, for every $\alpha<\kappa$, $f_{\xi_\beta}\rightrightarrows_\kappa f$ on A_α, where

$\{\xi_\alpha : \alpha < \kappa\} = \kappa \setminus B$ is the increasing enumeration. Hence, by Proposition 6.5, $f_{\xi_\alpha} \xrightarrow{QN}_\kappa f$ on the whole A. Thus, $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ on A. □

We therefore get the following corollary.

Corollary 7.36 (ω: [Das et al., 2014])**.** *Let I be a κ-complete κ-P-ideal on κ, $A \subseteq 2^\kappa$, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$. The following conditions are equivalent:*

(1) $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ *on* A,

(2) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, *and for all* $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ *on* A_β,

(3) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ *for all* $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ *for limit* $\beta < \kappa$, *and for all* $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ *on* A_β.

□

In particular, we get the following corollary.

Corollary 7.37. *Let I be a κ-complete κ-P-ideal on κ, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$. If there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$, and for all $\alpha < \kappa$, $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ on A_α, then $f_\alpha \xrightarrow{QN}_{\kappa-I^*} f$ on A.*

□

As before, in this case we can require the sets A_α to be closed, the sequence consists of continuous functions.

Proposition 7.38 (ω: [Das et al., 2014])**.** *Let I be a κ-complete κ-P-ideal on κ, $A \subseteq 2^\kappa$, and let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of continuous functions $2^\kappa \to 2^\kappa$, and $f : 2^\kappa \to 2^\kappa$. The conditions of Corollary 7.23 are equivalent to:*

(4) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *of closed in A sets such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, *and for all* $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ *on* A_β,

(5) there exists a sequence $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{P}(2^\kappa))^\kappa$ *of closed in A sets such that* $A = \bigcup_{\alpha<\kappa} A_\alpha$, $A_\alpha \subseteq A_\beta$ *for all* $\alpha < \beta < \kappa$, $\bigcup_{\alpha<\beta} A_\alpha = A_\beta$ *for limit* $\beta < \kappa$, *and for all* $\beta < \kappa$, $f_\alpha \rightrightarrows_{\kappa-I^*} f$ *on* A_β.

Proof: Obviously (5) ⇒ (4) ⇒ (3), and (4) ⇒ (5) since a union of less than κ closed sets is closed, and I is κ-complete.

To see (1) ⇒ (4), notice that the sets A_α defined in the proof of Proposition 6.5 are closed, if f_α is continuous for every $\alpha < \kappa$. □

Proposition 7.39 (ω: [Das et al., 2014])**.** *Let I be a κ-admissible ideal on κ. There exists a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \to_{\kappa-I^*} f$ with $f\colon 2^\kappa \to 2^\kappa$, but $f_\alpha \overset{QN}{\not\longrightarrow}_{\kappa-I^*} f$.*

Proof: The example constructed in the proof of Proposition 6.8 is also valid here. It is easy to see that $f_\alpha \overset{QN}{\not\longrightarrow}_{\kappa-I^*} \mathbf{0}$, but since $f_\alpha \to_\kappa \mathbf{0}$, we get that $f_\alpha \to_{\kappa-I^*} \mathbf{0}$. □

Proposition 7.40 (ω: [Das et al., 2014])**.** *Let I be an ideal on κ. There exists a sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of functions $2^\kappa \to 2^\kappa$ such that $f_\alpha \overset{QN}{\longrightarrow}_{\kappa-I^*} f$ with $f\colon 2^\kappa \to 2^\kappa$, but $f_\alpha \not\rightrightarrows_{\kappa-I^*} f$.*

Proof: Because of Corollary 7.31, the example constructed in Proposition 6.12 is valid also in this case. □

7.2.3 $\kappa - (J, I)$-convergence

Let $J \subseteq I$ be ideals on κ. By Propositions 7.16 and 7.17, we immediately get the following implications.

Corollary 7.41. *Let $\langle f_\alpha \rangle_{\alpha<\kappa}$ be a sequence of functions $2^\kappa \to 2^\kappa$, $A \subseteq 2^\kappa$ and $f\colon 2^\kappa \to 2^\kappa$.*

(a) if $f_\alpha \rightrightarrows_{\kappa-J,I} f$ on A, then $f_\alpha \overset{QN}{\longrightarrow}_{\kappa-J,I} f$ on A,

(b) if $f_\alpha \overset{QN}{\longrightarrow}_{\kappa-J,I} f$ on A, then $f_\alpha \to_{\kappa-J,I} f$ on A.

□

Therefore we have the following implications between notions of convergence for ideals $J \subseteq I$ on κ.

Corollary 7.42. *If $J \subseteq I$ are κ-admissible ideals on κ, then*

$$\begin{array}{ccccccc}
\to_\kappa & \Rightarrow & \to_{\kappa-I^*} & \Rightarrow & \to_{\kappa-J,I} & \Rightarrow & \to_{\kappa-I} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\overset{QN}{\longrightarrow}_\kappa & \Rightarrow & \overset{QN}{\longrightarrow}_{\kappa-I^*} & \Rightarrow & \overset{QN}{\longrightarrow}_{\kappa-J,I} & \Rightarrow & \overset{QN}{\longrightarrow}_{\kappa-I} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\rightrightarrows_\kappa & \Rightarrow & \rightrightarrows_{\kappa-I^*} & \Rightarrow & \rightrightarrows_{\kappa-J,I} & \Rightarrow & \rightrightarrows_{\kappa-I}
\end{array}$$

□

7.3 Special subsets related to κ-ideal convergence

Let I, J be ideals on κ.

A set $A \subseteq 2^\kappa$ is a **κ-(I,J)-QN-set,** if any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_{\kappa-I} \mathbf{0}$ on A, it converges also κ-J-quasi-normally ($f_\alpha \xrightarrow{QN}_{\kappa-J} \mathbf{0}$ on A).

A set $A \subseteq 2^\kappa$ is a **κ-weak QN-set ($\kappa-(I,J)$-wQN-set),** if for any sequence $\langle f_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $A \to 2^\kappa$ such that $f_\alpha \to_{\kappa-I} \mathbf{0}$ on A, there exists an increasing sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $f_{\xi_\alpha} \xrightarrow{QN}_{\kappa-J} \mathbf{0}$ on A.

If $I = [\kappa]^{<\kappa}$ in the above definition we write simply κ-J-QN-set (respectively, κ-J-wQN-set).

Analogous notions for ideal convergence of real functions were studied in [Das and Chandra, 2013, Šupina, 2016] and [Chandra, 2016]. We generalize some of results of [Das and Chandra, 2013].

First notice the following fact.

Lemma 7.43. *If I is a κ-admissible ideal on κ, $P \subseteq 2^\kappa$ is a κ-perfect set, I is a κ-admissible ideal on κ, and $\langle f_\alpha \rangle_{\alpha<\kappa}$ is a sequence of continuous functions $P \to 2^\kappa$ such that $f_\alpha \to_{\kappa-I} \mathbf{0}$ on P, then there exists a sequence $\langle F_\alpha \rangle_{\alpha<\kappa}$ of continuous functions $2^\kappa \to 2^\kappa$ such that $F_\alpha \to_{\kappa-I} \mathbf{0}$ on 2^κ, and for all $\alpha < \kappa$, $F_\alpha \restriction P = f_\alpha$.*

Proof: The proof of Corollary 6.10 is valid here, since I is a κ-admissible ideal. □

Thus we immediately get the following corollary.

Corollary 7.44 (ω: [Das and Chandra, 2013])**.** *If I, J are ideals on κ, and J is κ-admissible, $A \subseteq 2^\kappa$ is a κ-(I,J)-QN-set, and $P \subseteq 2^\kappa$ is κ-perfect, then $P \cap A$ is $\kappa-(I,J)$-QN-set as well.*

□

So, by Corollary 7.24, we get the following fact.

Corollary 7.45 (ω: [Das and Chandra, 2013])**.** *If I, J are ideals on κ, and J is κ-admissible and κ-generated, $\langle P_\alpha \rangle_{\alpha<\kappa}$ is a sequence of κ-perfect sets, and A is a κ-(I,J)-QN-set, then $A \cap \bigcup_{\alpha<\kappa} P_\alpha$ is κ-(I,J)-QN-set as well.*

□

The study of $\kappa-(J,I)$-QN-sets and $\kappa-(J,I)$-wQN-sets will be also a matter of further research.

Question 7.46. *Describe $\kappa-(J,I)$-QN-sets and $\kappa-(J,I)$-wQN-sets in terms of κ-sequence selection or κ-cover selection principles.*

Chapter 8

κ-Proto-measure and Egorov's Theorem in 2^κ and its generalizations

In this chapter we relate measure and convergence properties in 2^κ, and we study the possibility of introducing an analogue of Egorov's Theorem. Since no satisfactory method of introducing measure on 2^κ is known, we devise a notion of κ-proto measure.

We use notion and notation defined in sections 1.4 and 1.5, as well as in the previous two chapters.

The results of this chapter are to be included in [Korch, 2017a].

8.1 Known approaches to introduce measure in 2^κ

It is clear that all notions related to measure need to be devised anew, because the product measure is not a solution (it gives only a σ-algebra, while we need κ-additivity). Some properties of σ-ideals on 2^κ were studied in [Kraszewski, 2001]. Various approaches have been considered to define the notions related to the measure in 2^κ. One can define measure as a function into a linearly ordered set L endowed with the operation $\sum_{\alpha<\kappa}$ (see [Laguzzi, 2012]). Unfortunately, this definition does not meet many expectations, for example there exist sequences $\langle a_\alpha\rangle_{\alpha<\kappa}, \langle b_\alpha\rangle_{\alpha<\kappa} \in L^\kappa$, $\alpha < \kappa$ such that $a_\alpha < b_\alpha$ for any $\alpha < \kappa$, but

$$\sum_{\alpha<\kappa} a_\alpha > \sum_{\alpha<\kappa} b_\alpha.$$

The analogue of the random-real forcing obtained by this method also does not have some of the expected properties, e.g. it is not κ^κ-bounding. Additionally, this approach assumes that the set of limit ordinals $\alpha < \kappa$ such that $2^\alpha = \kappa$ is of cardinality κ, so κ is not an inaccessible cardinal.

The natural idea is to use the Sikorski-Klaua structure of generalized reals $\mathbb{R}_\kappa$, which which was independently constructed by Sikorski ([Sikorski, 1948, Sikorski, 1949]) and Klaua ([Klaua, 1959, Klaua, 1960]). One can find even more details on this structure in [Klaua, 1994, Cowles and LaGrange, 1983] and [Cantini, 1979]. This structure can be successfully used to introduce a metric analogue in 2^κ, but the author of this thesis is unable to construct a measure analogue with values in $\mathbb{R}_\kappa$.

Therefore, the other way is to try to define a forcing with the properties analogous to the properties of the random-real forcing (for inaccessible cardinals, see [Friedman and Laguzzi, 2014], [Shelah, 2012], and [Shelah and Cohen, 2016]), which is a forcing related to the algebra of measurable sets, i.e. which is κ^+-c.c., $<\kappa$-closed, κ^κ-bounding, and does not have the Sacks property. Obviously, since our aim is to introduce Egorov's Theorem in 2^κ, this approach is not sufficient in our case.

In this chapter we give a definition of a κ-proto-measure which has only properties which are sufficient to prove Egorov's Theorem. Unfortunately, we leave the question of existence of κ-proto-measure satisfying some additional reasonable conditions open.

8.2 κ-Proto-measure

8.2.1 The definition

A triple $\langle \mathbb{L}, \mu, L\rangle$ will be called a **κ-proto-measure** if

(1) $\langle \mathbb{L}, \leq\rangle$ is a linear order with the least element.

(2) $\mu\colon \mathcal{B}_\kappa \to \mathbb{L}$ is a function defined on the family of κ-Borel subsets of 2^κ with values in $\mathbb{L}$.

(3) If $\langle A_\alpha\rangle_{\alpha<\kappa} \in (\mathcal{B}_\kappa)^\kappa$ is such that $\bigcap_{\alpha<\kappa} A_\alpha = \varnothing$, and for all $\alpha < \alpha' < \kappa$, $A_{\alpha'} \subseteq A_\alpha$, then for every $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$, there exists $\delta < \kappa$ such that $\mu(A_\delta) < \xi$,

(4) $L\colon (\mathbb{L} \setminus \{\min \mathbb{L}\}) \times \kappa \to \mathbb{L} \setminus \{\min \mathbb{L}\}$,

(5) For all $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$ if $\langle A_\alpha\rangle_{\alpha<\kappa} \in (B_\kappa)^\kappa$ is such that $\mu(A_\alpha) \leq L(\xi, \alpha)$ for all $\alpha < \kappa$, then

$$\mu\left(\bigcup_{\alpha<\kappa} A_\alpha\right) \leq \xi.$$

A κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ is **diffused** if for every $x \in 2^\kappa$, $\mu(\{x\}) = \min \mathbb{L}$. It is **increasing** if for every $A, B \in \mathcal{B}_\kappa$ such that $A \subseteq B$, $\mu(A) \leq \mu(B)$. Finally, it is **strictly positive** if for every $s \in 2^{<\kappa}$, $\mu([s]) > \min \mathbb{L}$.

A set $A \subseteq 2^\kappa$ is **μ-null** if there exists $B \in \mathcal{B}_\kappa$ such that $A \subseteq B$ and $\mu(B) = \min \mathbb{L}$. The collection of all μ-null subsets of 2^κ is denoted by $\mathcal{N}_\mu$. A set $A \subseteq 2^\kappa$ is **μ-measurable** if there exists $B \in \mathcal{B}_\kappa$ such that $A \vartriangle B$ is μ-null.

If $\lambda \leq 2^\kappa$ is a cardinal, then a κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ is **λ-null-complete** if for every $\beta < \lambda$, and sequence $\langle A_\alpha \rangle_{\alpha<\beta}$ of μ-null sets, $\bigcup_{\alpha<\beta} A_\alpha$ is μ-null as well. It is **null-good** if for every $A, B \in \mathcal{B}_\kappa$ if A is μ-null, $\mu(A \cup B) = \mu(B)$.

A κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ is **basically transition-invariant** if for any $\alpha < \kappa$, and $t, s \in 2^\alpha$, $\mu([s]) = \mu([t])$.

8.2.2 Basic properties

We prove some basic properties of κ-proto-measures.

Proposition 8.1. *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is a diffused, null-good κ-proto-measure. Then for every $x \in 2^\kappa$, and $\xi \in \mathbb{L}$, there exists $\alpha < \kappa$ such that $\mu([x{\restriction}\alpha]) < \xi$.*

Proof: Indeed, for $x \in 2^\kappa$, consider $\langle [x{\restriction}\alpha] \setminus \{x\} \rangle_{\alpha<\kappa}$. □

Corollary 8.2. *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is an increasing, diffused, null-good κ-proto-measure. Then for every $x \in 2^\kappa$, $\langle \mu([x{\restriction}\alpha]) \rangle_{\alpha<\kappa}$ is either coinitial in $\mathbb{L} \setminus \{\min \mathbb{L}\}$, or eventually constant and equal* $\min \mathbb{L}$.

Proof: Indeed, the sequence $\langle \mu([x{\restriction}\alpha]) \rangle_{\alpha<\kappa}$ is non-increasing. If it is not eventually constant and equal to $\min \mathbb{L}$, then for every $\xi \in \mathbb{L}$, we get by Proposition 8.1 that there exists $\alpha < \kappa$ such that $\min \mathbb{L} < \mu([x{\restriction}\alpha]) < \xi$, thus it is a coinitial sequence. □

Corollary 8.3. *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is an increasing, strictly positive, diffused, null-good κ-proto-measure. Then $\mathbb{L} \setminus \{\min \mathbb{L}\}$ has coinitiality κ (i.e. there exists a coinitial sequence of length κ in $\mathbb{L} \setminus \{\min \mathbb{L}\}$).*

□

Proposition 8.4. *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is a null-good κ-proto-measure, and $A \in \mathcal{N}_\mu$. Then for all $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{B}_\kappa)^\kappa$ such that $\bigcap_{\alpha<\kappa} A_\alpha = A$, and for all $\alpha < \alpha' < \kappa$, $A_{\alpha'} \subseteq A_\alpha$, and for all $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$, there exists $\delta \in \kappa$ such that $\mu(A_\delta) < \xi$,*

Proof: Consider the sequence $\langle A_\alpha \setminus A \rangle_{\alpha<\kappa}$. □

8.2.3 Examples

Let us start with the most trivial example. If $\mathbb{L} = \{0\}$, and μ is a constant function, then obviously $\langle \mathbb{L}, \mu, \varnothing \rangle$ is a κ-proto-measure. A κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ such that $\mu(2^\kappa) = \min \mathbb{L}$ is **trivial**. In the consideration below we always assume that a κ-proto-measure which is considered is not trivial.

Slightly better example can be constructed for $\mathbb{L} = \langle \kappa + 1, \geq \rangle$ (the set of ordinals $\leq \kappa$ with the reversed order, $\min \mathbb{L} = \kappa$), and a fixed $p \in 2^\kappa$. In this case, for $A \in \mathcal{B}_\kappa$, let

$$\mu_p(A) = \begin{cases} \bigcap \{\alpha < \kappa \colon [p{\restriction}\alpha] \subseteq A\}, & \text{if } \exists_{\alpha<\kappa} [p{\restriction}\alpha] \subseteq A, \\ \kappa, & \text{otherwise.} \end{cases}$$

Obviously, if $\langle A_\alpha \rangle_{\alpha<\kappa} \in (\mathcal{B}_\kappa)^\kappa$ is such that $\bigcap_{\alpha<\kappa} A_\alpha = \varnothing$, and for all $\alpha < \alpha' < \kappa$, $A_{\alpha'} \subseteq A_\alpha$, then for all $\alpha < \kappa$, there exists $\beta < \kappa$ such that for all, $[p{\restriction}\alpha] \setminus A_\beta \neq \varnothing$. Let $L \colon \kappa \times \kappa \to \kappa$, be such that for all $\alpha, \beta < \kappa$, $L(\alpha, \beta) = \alpha$. Then $\langle \mathbb{L}, \mu_\kappa, L \rangle$ is an increasing and diffused κ-proto-measure. On the other hand, it is not null-good ($\{p\}, 2^\kappa \setminus \{p\} \in \mathcal{N}_{\mu_p}$, but $\mu_p(2^\kappa) = 0 \neq \min \mathbb{L}$).

To see yet another example fix $S \subseteq 2^\kappa$ along with enumeration $\{s_\alpha \colon \alpha < \lambda\} = S$. Let $\mathbb{L} = \lambda$, and

$$\mu(A) = \min\{\alpha < \lambda \colon s_\alpha \notin A\}.$$

Then set $L(\alpha, \beta) = \alpha$ for $\alpha < \lambda \setminus \{0\}$ and $\beta < \kappa$. Indeed, if $\langle A_\beta \rangle_{\beta<\kappa}$ is a sequence of subsets of 2^κ such that $\bigcap_{\beta<\kappa} A_\beta = \varnothing$ and $A_\beta \subseteq A_{\beta'}$ for $\beta' \leq \beta$, then for every $\alpha < \lambda$, there exists $\xi < \kappa$ such that $s_\alpha \notin A_\beta$ for all $\beta < \kappa$ with $\beta \geq \xi$. Thus, $\mu(A_\beta) < \alpha$ for all $\beta \geq \xi$. Notice that this is an example of an increasing and null-good κ-proto-measure. On the other hand, it is not diffused, nor basically transition-invariant.

We do not know whether there exists a κ-proto-measure which fulfils more of the reasonable requirements.

Question 8.5. *Does there exist a non-trivial κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$ which is*

(a) increasing, diffused and null-good?

(b) increasing, diffused and κ-null complete?

(c) diffused and such that $\mathbb{L} = \mathbb{R}_\kappa$ (where $\mathbb{R}_\kappa$ is the Sikorski-Klaua structure of generalized reals ([Sikorski, 1948], [Sikorski, 1949], [Klaua, 1959], [Klaua, 1960], [Klaua, 1994], [Cowles and LaGrange, 1983] and [Cantini, 1979]), and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha \rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ with $A_\alpha \subseteq A_{\alpha'}$ for $\alpha < \alpha' < \beta$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) \colon \alpha < \beta\}?$$

(d) diffused and such that for every bounded $A \subseteq \mathbb{L}$, there exists $\sup A \in \mathbb{L}$, and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha \rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ such that for $\alpha < \alpha' < \beta$, $A_\alpha \subseteq A_{\alpha'}$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) : \alpha < \beta\}?$$

(e) increasing, diffused, and basically-transition invariant?

8.3 Special subsets related to κ-proto-measure

Notice that under certain assumptions, every κ-strongly null set is μ-null with respect to a κ-proto-measure $\langle \mathbb{L}, \mu, L \rangle$.

Proposition 8.6. *Assume that $\langle \mathbb{L}, \mu, L \rangle$ is an increasing, diffused, null-good, κ^+-null complete, basically translation-invariant κ-proto-measure. Then every κ-strongly null set is μ-null.*

Proof: Let $A \subseteq 2^\kappa$ be κ-strongly null. Let $x \in 2^\kappa$. By Proposition 8.2 one can find $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \mathbb{L}^\kappa$ which is a coinitial in $\mathbb{L} \setminus \min L$ sequence, or eventually equal to $\min \mathbb{L}$ along with $\langle \delta_\alpha \rangle_{\alpha<\kappa} \in \kappa^\kappa$ such that $\mu([x{\restriction}\delta_\alpha]) = \xi_\alpha$. since $\langle \mathbb{L}, \mu, L \rangle$ is basically transition-invariant we get that for every $\alpha < \kappa$ and $s \in 2^{\delta_\alpha}$, $\mu([s]) = \xi_\alpha$.

If the sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \mathbb{L}^\kappa$ is eventually equal to $\min \mathbb{L}$, let $\eta < \kappa$ be such that for all $\eta < \alpha < \kappa$, $\xi_\alpha = \min \mathbb{L}$. Then for $\langle \delta_\alpha \rangle_{\eta<\delta<\kappa}$ choose $\langle x_\alpha \rangle_{\eta<\alpha<\kappa} \in (2^\kappa)^{\kappa\setminus\eta}$ such that

$$X \subseteq \bigcup_{\eta<\alpha<\kappa} [x_\alpha{\restriction}\delta_\alpha].$$

Then $\mu([x_\alpha{\restriction}\delta_\alpha]) = \min \mathbb{L}$ for every $\eta < \alpha < \kappa$, thus A is μ-null.

On the other hand, assume that $\langle \xi_\alpha \rangle_{\alpha<\kappa} \in \mathbb{L}^\kappa$ is coinitial in $\mathbb{L} \setminus \min \mathbb{L}$. Fix $\alpha < \kappa$. One can find $\langle \eta_{\alpha,\beta} \rangle_{\beta<\kappa} \in \kappa^\kappa$ such that $\mu([x{\restriction}\eta_{\alpha,\beta}]) \leq L(\xi_\alpha, \beta)$ for all $\beta < \kappa$. Then for all $\beta < \kappa$, and $s \in 2^{\eta_{\alpha,\beta}}$, $\mu([s]) \leq L(\xi_\alpha, \beta)$. And for $\langle \eta_{\alpha,\beta} \rangle_{\beta<\kappa}$, choose $\langle x_{\alpha,\beta} \rangle_{\beta<\kappa}$ such that

$$X \subseteq \bigcup_{\beta<\kappa} [x_{\alpha,\beta}{\restriction}\eta_{\alpha,\beta}].$$

Then

$$\mu\left(\bigcup_{\beta<\kappa} [x_{\alpha,\beta}{\restriction}\eta_{\alpha,\beta}]\right) \leq \xi_\alpha,$$

but since ξ_α is a coinitial sequence, we get

$$\mu\left(\bigcap_{\alpha<\kappa}\bigcup_{\beta<\kappa}[x_{\alpha,\beta}\restriction\eta_{\alpha,\beta}]\right)=\min\mathbb{L}.$$

Obviously,

$$A\subseteq\bigcap_{\alpha<\kappa}\bigcup_{\beta<\kappa}[x_{\alpha,\beta}\restriction\eta_{\alpha,\beta}].$$

Hence, A is μ-null. □

Proposition 8.7. *Let κ be a weakly compact cardinal. Assume that $\langle\mathbb{L},\mu,L\rangle$ is an increasing, diffused, null-good κ-proto-measure. Then every κ-strongly null set is μ-null.*

Proof: Let $A\subseteq 2^\kappa$ be a κ-strongly null set. By Proposition 8.2, for every $x\in 2^\kappa$, $\langle\mu([x\restriction\alpha])\rangle_{\alpha<\kappa}$ is either coinitial in $\mathbb{L}\setminus\{\min\mathbb{L}\}$, or eventually constant and equal $\min\mathbb{L}$.

If for every $x\in 2^\kappa$, $\langle\mu([x\restriction\alpha])\rangle_{\alpha<\kappa}$ is eventually constant and equal $\min\mathbb{L}$, then (since under the above assumptions 2^κ is κ-compact), one can find $\eta<\kappa$ such that for all $\eta<\alpha<\kappa$, and $s\in 2^\alpha$, $\mu([s])=\min\mathbb{L}$. Then let $\langle x_\alpha\rangle_{\eta<\alpha<\kappa}\in(2^\kappa)^{\kappa\setminus\eta}$ be such that

$$A\subseteq\bigcup_{\eta<\alpha<\kappa}[x_\alpha\restriction\alpha].$$

Obviously,

$$\mu\left(\bigcup_{\eta<\alpha<\kappa}[x_\alpha\restriction\alpha]\right)=\min\mathbb{L},$$

thus A is μ-null.

On the other hand, assume that there exists $x\in 2^\kappa$ such that $\langle\mu([x\restriction\alpha])\rangle_{\alpha<\kappa}$ is coinitial in $\mathbb{L}\setminus\{\min\mathbb{L}\}$. Since 2^κ is κ-compact space under the above assumptions, for every $\xi\in\mathbb{L}\setminus\{\min\mathbb{L}\}$ one can find $\delta_\xi<\kappa$ such that for all $\delta_\xi<\alpha<\kappa$, and $s\in 2^\alpha$, $\mu([s])<\xi$. Fix $\alpha<\kappa$, and let for all $\beta<\kappa$, $\delta_{\alpha,\beta}$ be such that for all $\delta_{\alpha,\beta}\leq\gamma<\kappa$, and $s\in 2^\gamma$, $\mu([s])\leq L(\mu([x\restriction\alpha],\beta)$. Find $\langle x_{\alpha,\beta}\rangle\in(2^\kappa)^\kappa$ such that

$$A\subseteq\bigcup_{\beta<\kappa}[x_{\alpha,\beta}\restriction\delta_{\alpha,\beta}].$$

Notice that

$$\mu\left(\bigcup_{\beta<\kappa}[x_{\alpha,\beta}\restriction\delta_{\alpha,\beta}]\right)\leq\mu([x\restriction\alpha]),$$

and since $\langle\mu([x\restriction\alpha])\rangle_{\alpha<\kappa}$ is coinitial in $\mathbb{L}\setminus\{\min\mathbb{L}\}$, and $\langle\mathbb{L},\mu,L\rangle$ is increasing,

$$\mu\left(\bigcap_{\alpha<\kappa}\bigcup_{\beta<\kappa}[x_{\alpha,\beta}\restriction\delta_{\alpha,\beta}]\right)=\min\mathbb{L}.$$

But

$$A \subseteq \bigcap_{\alpha<\kappa} \bigcup_{\beta<\kappa} [x_{\alpha,\beta} \restriction \delta_{\alpha,\beta}],$$

thus it is μ-null. □

8.4 Egorov's Theorem in 2^κ

8.4.1 κ-Convergence

Given a κ-proto-measure it is easy to prove an analogue of Egorov's Theorem.

Theorem 8.8. *Let $\langle \mathbb{L}, \mu L\rangle$ be a κ-proto-measure, and let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ such that is κ-pointwise convergent on $X \subseteq 2^\kappa$ to $\mathbf{0}$ with $X \in \mathcal{B}_\kappa$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \subseteq X$, $A \in \mathcal{B}_\kappa$ with $\mu(X \setminus A) \leq \xi$ such that the sequence converges κ-uniformly on A.*

Proof: For $\alpha, \beta < \kappa$. Let

$$E_{\alpha,\beta} = \{x \in 2^\kappa \colon \exists_{\gamma\geq\alpha} f(\gamma) \notin [\mathbf{0}\restriction\beta]\}.$$

Notice that $E_{\alpha,\beta}$ is a κ-Borel set for every $\alpha, \beta < \kappa$. Moreover, if $\alpha < \alpha' < \kappa$, and $\beta < \kappa$, then $E_{\alpha',\beta} \subseteq E_{\alpha,\beta}$. Since $f_\alpha \to_\kappa \mathbf{0}$, we get that $\bigcap_{\alpha<\kappa} E_{\alpha,\beta} = \varnothing$, for all $\beta < \kappa$. Therefore, for each $\beta < \kappa$, there exists $\xi_\beta < \kappa$ such that

$$\mu\left(E_{\xi_\beta,\beta}\right) \leq L(\xi, \beta).$$

Let

$$B = \bigcup_{\beta<\kappa} E_{\xi_\beta,\beta},$$

and $A = X \setminus B$. Then for any $\beta < \kappa$, $f_\alpha(x) \in [\mathbf{0}\restriction\beta]$, for any $\alpha < \kappa$ with $\alpha \geq \xi_\beta$, and $x \in A$. This is because $A \subseteq X \setminus E_{\xi_\beta,\beta}$. Thus, $f_\alpha \rightrightarrows_\kappa \mathbf{0}$ on A, and $\mu(X \setminus A) \leq \xi$. □

8.4.2 Ideal version of Egorov's Theorem in 2^κ for κ-generated ideals

8.4.2.1 κ-I-convergence

Let I be a ideal on κ. Then we get the following.

Theorem 8.9. *Assume that I is a κ-generated κ-complete ideal on κ, and $\langle \mathbb{L}, \mu, L\rangle$ is a κ-proto-measure. Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ such that is κ-I-pointwise convergent on 2^κ to $\mathbf{0}$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \in \mathcal{B}_\kappa$ with $\mu(2^\kappa \setminus A) \leq \xi$ such that the sequence converges κ-I-uniformly on A.*

Proof: Assume that I is κ-generated and fix sets $\langle C_\alpha \rangle_{\alpha<\kappa}$ such that $C_\alpha \subseteq C_{\alpha'}$ for all $\alpha < \alpha' < \kappa$ and for every $C \in I$, there exists $\alpha < \kappa$ such that $C \subseteq C_\alpha$. For $\alpha, \beta < \kappa$, let

$$E_{\alpha,\beta} = \{x \in 2^\kappa : \{\gamma < \kappa : f_\gamma(x) \notin [\mathbf{0}\restriction\beta]\} \setminus C_\alpha \neq \varnothing\}.$$

Notice that

$$E_{\alpha,\beta} = \bigcup_{\alpha\in\kappa\setminus C_\alpha} \{x \in 2^\kappa : f_\alpha(x) \notin [\mathbf{0}\restriction\beta]\}$$

is κ-Borel for each $\alpha, \beta \in \omega$. Moreover, $E_{\alpha',\beta} \subseteq E_{\alpha,\beta}$ for all $\alpha < \alpha' < \kappa$, and $\bigcap_{\alpha<\kappa} E_{\alpha,\beta} = \varnothing$ for all $\beta < \kappa$. Hence, for each $\beta < \kappa$, there exists $\xi_\beta < \kappa$ such that

$$\mu(E_{\xi_\beta,\beta}) \leq L(\xi, \beta).$$

Let $B = \bigcup_{\beta<\kappa} E_{\xi_\beta,\beta}$. So $\mu(B) \leq \xi$, and if $x \notin B$, then

$$\{\gamma < \kappa : f_\gamma(x) \notin [\mathbf{0}\restriction\beta]\} \subseteq C_{\xi_\beta},$$

for any $\beta < \kappa$, so $f_n \rightrightarrows_{\kappa-I} \mathbf{0}$ on $A = 2^\kappa \setminus B$. □

8.4.2.2 κ-I^*-convergence

We get also a similar theorem for I^*-convergence.

Theorem 8.10. *Assume that I is a κ-generated κ-admissible ideal on κ, and $\langle \mathbb{L}, \mu, L\rangle$ is a κ-proto-measure. Let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a sequence of κ-measurable functions $2^\kappa \to 2^\kappa$ such that is κ-I^*-pointwise convergent on 2^κ to $\mathbf{0}$, and let $\xi \in \mathbb{L} \setminus \{\min \mathbb{L}\}$. Then there exists a set $A \in \mathcal{B}_\kappa$ with $\mu(2^\kappa \setminus A) \leq \xi$ such that the sequence converges κ-I^*-uniformly on A.*

Proof: Fix $\langle C_\alpha\rangle_{\alpha<\kappa}$ such that for all $C \in I$, there exists $\alpha < \kappa$ with $C \subseteq C_\alpha$. Let $\omega \setminus C_\beta = \{\delta_{\alpha,\beta} : \alpha < \kappa\}$ be the increasing enumeration, and let

$$F_\beta = \left\{x \in 2^\kappa : \lim_{\alpha<\kappa} f_{\delta_{\alpha,\beta}}(x) = \mathbf{0}\right\}$$

Obviously, for $\beta < \beta' < \omega$, $F_\beta \subseteq F_{\beta'}$ for, and $\bigcup_{\beta<\kappa} F_\beta = 2^\kappa$. Moreover,

$$F_\beta = \bigcap_{\alpha<\kappa} \bigcup_{\gamma<\kappa} \bigcap_{\gamma\leq\eta<\kappa} \left\{x \in 2^\kappa : f_{\delta_{\eta,\beta}}(x) \in [\mathbf{0}\restriction\alpha]\right\}$$

is κ-Borel. Let $F'_\alpha = 2^\kappa \setminus F_\alpha$, for all $\alpha < \kappa$. There exists $\eta < \kappa$ such that $\mu(F'_\eta) \leq L(\xi, 0)$.

Now apply the proof of Theorem 8.8 for the set F_η, and sequence $\left\langle f_{\delta_{\alpha,\eta}}\right\rangle_{\alpha<\kappa}$ to get sets $\langle E_{\alpha,\beta}\rangle_{\alpha,\beta<\kappa}$ such that for each $\beta < \kappa$, there exists $\xi_\beta < \kappa$ such that

$$\mu\left(E_{\xi_\beta,\beta}\right) \leq L(\xi, \beta + 1).$$

Let

$$B = F'_\eta \cup \bigcup_{\beta<\kappa} E_{\xi_\beta,\beta},$$

and $A = X \setminus B$. We get that $f_\alpha \Rightarrow_{\kappa-I^*} 0$ on A and $\mu(B) \leq \xi$. □

Chapter 9

Conclusions, further development and open problems

In this chapter I summarise the main results of this thesis. I point out directions of further research and collect the main open problems in one place.

9.1 The real line

The theory of special subsets of the real line and the theory of convergence of sequences of real functions are relatively well developed. In this thesis I have presented some further developments is two subjects: perfectly null sets (Chapter 2) and generalized Egorov's statement for ideals (Chapter 3).

The idea of constructing perfectly null sets comes from the observed duality between measure and category and the lack of notion dual to the notion of perfectly meagre set. We have defined such a notion and studied its properties (see e.g. Proposition 2.5). Nevertheless, the answer to the main problem in this chapter remains unknown.

Question 2.7. *Is it consistent with ZFC that there exists a perfectly null set which is not universally null? In particular, is the class of perfectly null sets closed under taking products?*

Pursuing the answer to the above problem we have shown that if there exists a measure analogue of the Lusin function it cannot be constructed in an analogous way (see Proposition 2.9). Also, if the class $\mathrm{P}\boldsymbol{\mathcal{N}}$ is closed under homeomorphisms, then $\mathrm{U}\boldsymbol{\mathcal{N}} = \mathrm{P}\boldsymbol{\mathcal{N}}$ (Corollary 2.14). Finally, we have considered some simpler classes of perfect subsets in which analogous problems can be at least partially solved (see Theorem 2.27).

Next we studied analogues of the small sets considered by Bartoszyński ([Bartoszyński and Judah, 1995]) with respect to the canonical measure on perfect sets. Two approaches were presented, out of which the second one seems to be more promising. In particular, we have shown that every set which is null in a perfect set P can be presented as a union of two small sets in P (see Corollary 2.40). We also studied additive properties of small sets in P (see Proposition 2.41).

Finally, we constructed an analogue ($\mathrm{P}\boldsymbol{\mathcal{N}}'$) of the class of perfectly meagre in the transitive sense sets ($\mathrm{P}\boldsymbol{\mathcal{M}}'$). It is known that every strongly meagre set is $\mathrm{P}\boldsymbol{\mathcal{M}}'$, and every $\mathrm{P}\boldsymbol{\mathcal{M}}'$ set is universally meagre, and that it is consistent with ZFC that those inclusions are proper. We have proved some of the analogous results on the measure side. Every strongly null set is perfectly null in the transitive sense (see Theorem 2.43), and under certain set-theoretical assumptions, there exists a universally null set which is not $\mathrm{P}\boldsymbol{\mathcal{N}}'$ (see Theorem 2.47). The other two remain open.

Question 2.46. *$P\boldsymbol{\mathcal{N}}' \subseteq U\boldsymbol{\mathcal{N}}$?*

Question 2.44. *Does there exist a $P\boldsymbol{\mathcal{N}}'$ set, which is not strongly null? In particular, does there exist an uncountable $P\boldsymbol{\mathcal{N}}'$ set in every model of ZFC?*

Since the consideration of $\mathrm{P}\boldsymbol{\mathcal{M}}'$ class started with its additive properties, we have also studied additive properties of $\mathrm{P}\boldsymbol{\mathcal{N}}'$ sets (see Theorem 2.54). Nevertheless, the main problem remains unsolved.

Question 2.55. *If $A \in S\boldsymbol{\mathcal{M}}$, and $B \in P\boldsymbol{\mathcal{N}}'$, is $A + B$ an s_0-set?*

In Chapter 3, we studied the second subject in set theory of real line, which concerns generalizations of Egorov's Theorem. Previously, it has been known that Egorov's Theorem without assumption on measurability (so called generalized Egorov's statement) is consistent with ZFC, as is its negation. Also ideal version of Egorov's Theorem (with the measurability assumption) was studied for different notions of ideal convergence. Therefore, we studied the generalized Egorov's statement in the case of different notions of ideal convergence. By generalizing the method of Pinciroli (Theorems 3.2 and 3.3) we proved that both the ideal version of the generalized Egorov's statement and its negation are consistent with ZFC:

(a) between pointwise and equi-convergence with respect to analytic P-ideals (Corollaries 3.6 and 3.8),

(b) between pointwise and uniform convergence with respect to countably generated ideals (Corollaries 3.11 and 3.13),

(c) between pointwise-I^* and uniform-I^* convergence with respect to countably generated ideals (Corollaries 3.15 and 3.17),

(d) between pointwise and uniform convergence with respect to ideals of the form Fin^α, $\alpha < \omega_1$ (Corollaries 3.18 and 3.19).

Additionally, I proved Egorov's Theorem (with measurability assumption) (Theorems 3.10 and 3.14) in the cases in which it was not proven before ((b) and (c)).

This generalization of Pinciroli method gives a combinatorial properties denoted by $(H^{\Rightarrow}(\mathcal{F}, \not\to))$ and $(H^{\Leftarrow}(\mathcal{F}, \not\to))$ which imply that the generalized Egorov's statement (respectively, its negation) is consistent with ZFC. Later on, M. Repický ([Repický, 2017]) further generalized my results by studying the closure properties of classes of ideals satisfying those properties. He also introduced an analogous property $(M^{\Rightarrow}(\mathcal{F}, \not\to))$ which implies that Egorov's Theorem (with measurability assumption) holds for convergence with respect to such ideal.

Nevertheless, the research in this topic needs to be continued to answer some open problems. I have stated three such questions.

Question 3.25. *Is there any possible condition, which implies that classic Egorov's statement (measurable version) does not hold for a given ideal in ZFC (cf. Proposition 3.9)?*

Question 3.26. *Are there any examples of ideals which prove that the classes of all ideals satisfying $M^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $H^{\Rightarrow}(\mathcal{F}_{\to_I}, \rightrightarrows_I)$, $M^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$, and $H^{\Rightarrow}(\mathcal{F}_{\to_{I^*}}, \rightrightarrows_{I^*})$ are pairwise distinct?*

Question 3.27. *Is there an ideal I such that $\bar{H}^{\Leftarrow}\left(\mathcal{F}_{\to_I}, \xrightarrow{QN}_I\right)$ does not hold?*

9.2 In the generalized Cantor space

In the subsequent chapters we studied the generalized Cantor space 2^κ, where κ is an uncountable regular cardinal. This space is equipped with a basis of closed open sets of form

$$[s] = \{x \in 2^\kappa : x{\restriction}\mathrm{len}(s) = s\}.$$

Throughout this thesis we have assumed that $\kappa^{<\kappa} = \kappa$, thus this basis is of cardinality κ.

In Chapter 4 we introduced simple notions of special subsets in 2^κ:

(a) λ-κ-Lusin sets, and we proved that such a set exists if $\lambda = \text{cov}\boldsymbol{\mathcal{M}}_\kappa = \text{cof}\boldsymbol{\mathcal{M}}_\kappa$ (Theorem 4.1), thus it exists under CH_κ.

(b) κ-strongly measure zero sets, and we proved that one of the implications is the analogue of Galvin-Mycielski-Soloway holds (Proposition 4.8). The other implication was proven when κ is weakly compact (Theorem 4.10).

(c) κ^+-concentrated sets, and we proved that Lusin sets in κ are exactly the sets κ^+-concentrated on every dense subset (Proposition 4.13), and that every set κ^+ concentrated on a set of cardinality $\leq \kappa$ is κ-strongly null (Proposition 4.14), thus every Lusin set for κ is κ-strongly null.

(d) κ-perfectly κ-meagre, perfectly κ-meagre, and κ-λ sets, and among other properties we proved that every κ-λ-set is perfectly κ-meagre (Proposition 4.18), on the other hand, we do not know if there exists such a non-trivial set in every model of ZFC.

Question 4.19. *Is there a set $A \subseteq 2^\kappa$ such that $|A| = \kappa^+$ and $A \in P\boldsymbol{\mathcal{M}}_\kappa$ in every model of ZFC.*

(e) κ-σ-sets, and we proved that every such set is perfectly κ-meagre (Proposition 4.27),

(f) κ-Q-sets,

(g) κ-porous sets.

We also studied the generalization of selection properties in κ. In particular, we proved that every κ-γ-set satisfies $S_1^\kappa(\Omega_\kappa, \Gamma_\kappa)$ (Theorem 4.32), and that every set which has $S_1^\kappa(\Gamma_\kappa, \Gamma_\kappa)$ principle has κ-Hurewicz property (Proposition 4.35). Hence, every κ-γ-set has κ-Hurewicz property (Corollary 4.36). On the other hand, we proved that every λ-κ-Lusin set does not have this property (Corollary 4.38, although it has κ-Menger property (Proposition 4.39). Obviously, if a set has κ-Rothberger property, then it is κ-strongly null set. Moreover, if a set is κ^+-concentrated on a set of cardinality less than κ, then it has κ-Rothberger property (Propositions 4.40 and 4.42). Hence, the whole 2^κ does not have this property. Also every κ-γ-set satisfies κ-Rothberger property (Theorem 4.45). In particular, the whole space 2^κ cannot be a κ-γ-set.

In Chapter 5, we studied in 2^κ versions of less known notions of special subsets. We have introduced:

(a) X-small sets, which follow the idea of small sets in $\omega_1^{\omega_1}$ presented in [Halko, 1996]. We proved that every set which is small in 2^κ is κ-strongly

null as well (Proposition 5.2). Under $\diamondsuit_\kappa$, 2^κ is C-small for every closed unbounded set C, and under $V = L$, 2^κ is X-small for every stationary set X (Propositions 5.7 and 5.9). We also showed that every set small in 2^κ is nowhere dense, but the reversed implication does not hold (Propositions 5.11 and 5.12).

(b) κ-meagre additive sets, and we proved a combinatorial characterization of κ-meagre additive sets (Proposition 5.15) for strongly inaccessible κ. This characterization implies that every κ-meagre additive set is κ-perfectly κ-meagre (Proposition 5.16).

(c) κ-Ramsey null sets, and in particular we proved that every κ-γ-set is κ-Ramsey null if κ is weakly inaccessible (Proposition 5.18). On the other hand, we were not able to determine the additivity of this ideal.

Question 5.17. *Is the ideal of κ-Ramsey null subsets of 2^κ κ^+-complete?*

(d) κ-T'-sets, and we proved various characterizations of this notion (Propositions 5.22, 5.23 and 5.24). The class of κ-T'-sets forms a κ^+-complete ideal (Proposition 5.25, and an algebraic sum of two κ-T'-sets is still a κ-T'-set. For strongly inaccessible κ, we proved that every κ-γ-set is a κ-T'-set (Proposition 5.27), and that every κ-T'-set is a κ-meagre additive set (Proposition 5.28). Thus, if κ is strongly inaccessible, every κ-γ-set is κ-meagre additive. Under some additional assumptions this inclusion cannot be reversed (see Theorem 5.21).

(e) κ-v_0-sets, and we proved that if κ is a strongly inaccessible cardinal, then every κ-perfectly κ-meagre set is a κ-v_0-set (Corollary 5.31). Also every κ-strongly null set is a κ-v_0-set (Proposition 5.32).

We have left as aśubject for further research the following issue.

Question 5.33. *What is the relation between κ-l_0-sets (respectively, κ-m_0-sets) with other notions of special subsets of 2^κ?*

In Chapter 6, we introduced and studied the convergence of κ sequences of functions $2^\kappa \to 2^\kappa$. We considered κ-uniform convergence, which implies κ-quasi-normal convergence (Proposition 6.6), which itself implies κ-pointwise convergence (Proposition 6.2). We have given examples of sequences of functions which separate those notions (Propositions 6.8 and 6.12). We proved that similarly to the standard case, κ-quasi normal convergence is equivalent to existence of a partition of the underlying set into κ many subsets on which we have κ-uniform convergence (Proposition 6.5). On the other hand, if a sequence of functions converges κ-quasi-normally on every set from a collection of

less than $\mathfrak{b}_\kappa$ sets, it converges κ-quasi-normally on its union (Proposition 6.7). Finally, we proved that a κ-uniform convergent sequence of continuous functions converges to a continuous function (Proposition 6.11).

We also studied special subsets of 2^κ related to convergence of sequences of functions, i.e. κ-QN-sets, κ-wQN-sets and κ-mQN-sets. We gave some basic properties of such sets (Corollaries 6.17 and 6.16), and also we proved that every κ-wQN-set is κ-perfectly κ-meagre (Proposition 6.21). We characterized κ-wQN-sets and κ-QN-sets in terms of κ-sequence selection properties (Theorems 6.22 and 6.23). The following issues will be a subject to further research.

Question 6.24. *Is every set satisfying $S_1^\kappa(\Gamma_\kappa, \Gamma_\kappa)$ principle a κ-wQN-set?*

Question 6.25. *Does every κ-QN-set satisfy $S_1^\kappa(\Gamma_\kappa, \Gamma_\kappa)$ principle?*

Further on, in Chapter 7, we studied the notions of κ-I-convergence and κ-I^*-convergence of sequences of points of 2^κ for an ideal I on κ. We started by proving some simple properties (Propositions 7.3 and 7.7). Obviously I^*-convergence implies I-convergence, but this implication can be reversed (Proposition 7.10) if and only if I is a κ-P-ideal (Propostions 7.8 and 7.9). Finally, we studied properties related to κ-I-Cauchy property (Propositions 7.12-7.15).

The notions of κ-I-convergence and κ-I^*-convergence of points of 2^κ allowed us to study different notions of ideal convergence of functions $2^\kappa \to 2^\kappa$. In particular, κ-I-uniform convergence implies κ-I-quasi-normal convergence, which itself implies κ-I-pointwise convergence (Propositions 7.16 and 7.17). Similarly, κ-I^*-uniform convergence implies κ-I^*-quasi-normal convergence, which itself implies κ-I^*-pointwise convergence (Corollary 7.30). All those implications cannot be reversed (Propositions 7.29 and 7.28). We have also proven that if a sequence of function converges κ-I-uniformly on every set from a collection of κ subsets of 2^κ, it converges κ-I-quasi-normally on its union (Proposition 7.21). This implication can be reversed for κ-generated ideals (Proposition 7.22). Similarly, if I is κ-P-ideal, then if a sequence of function converges κ-I^*-uniformly on every set from a collection of κ subsets of 2^κ, it converges κ-I^*-quasi-normally on its union (Proposition 7.35). This implication can be reversed not only for κ-P-ideals (Proposition 7.34). Finally, we have proven that if a sequence of continuous functions converges κ-I-uniformly or κ-I^*-uniformly, then the limit is continuous as well (Propositions 7.27 and 7.31).

We also considered κ-(I, J)-QN-sets and κ-(I, J)-wQN-sets and proved some of their basic properties (e.g. Proposition 7.44). Nevertheless, the following important subject will be a subject of future research.

Question 7.46. *Describe $\kappa-(J,I)$-QN-sets and $\kappa-(J,I)$-wQN-sets in terms of κ-sequence selection or κ-cover selection principles.*

In the final chapter (Chapter 8), we studied the possibility of introducing Egorov's Theorem in 2^κ. To achieve this we need a measure analogue in 2^κ. Since no satisfactory concept is known, we define a notion of κ-proto-measure with properties which suffice to prove an analogue of Egorov's Theorem (Theorem 8.8) and also analogue theorems for I-convergence and I^*-convergence in the case of κ-generated ideals on κ (Theorems 8.9 and 8.10).

We have discussed some properties of κ-proto-measures (e.g. Proposition 8.1 and Corollary 8.2), and proved that every κ-strongly null set is μ-null if μ is a proto-measure which satisfies some additions conditions and either κ is weakly compact or μ is transitive-invariant (Propositions 8.6 and 8.7). Although, some simple κ-proto-measure exist, we were not able to find a κ-proto-measures which is more complex. The existence of such κ-proto-measure is important in the light of proven theorems.

Question 8.5. *Does there exist a non-trivial κ-proto-measure $\langle \mathbb{L}, \mu, L\rangle$ which is*

(a) increasing, diffused and null-good?

(b) increasing, diffused and κ-null complete?

(c) diffused and such that $\mathbb{L} = \mathbb{R}_\kappa$ (where $\mathbb{R}_\kappa$ is the Sikorski-Klaua structure of generalized reals ([Sikorski, 1948], [Sikorski, 1949], [Klaua, 1959], [Klaua, 1960], [Klaua, 1994], [Cowles and LaGrange, 1983] and [Cantini, 1979]), and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha\rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ such that for $\alpha < \alpha' < \beta$, $A_\alpha \subseteq A_{\alpha'}$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) : \alpha < \beta\}?$$

(d) diffused and such that for every bounded $A \subseteq \mathbb{L}$, there exists $\sup A \in \mathbb{L}$, and such that for every limit ordinal $\beta < \kappa$, and any sequence $\langle A_\alpha\rangle_{\alpha<\beta} \in (\mathcal{B}_\kappa)^\beta$ such that for $\alpha < \alpha' < \beta$, $A_\alpha \subseteq A_{\alpha'}$, we have

$$\mu\left(\bigcup_{\alpha<\beta} A_\alpha\right) = \sup\{\mu(A_\alpha) : \alpha < \beta\}?$$

(e) increasing, diffused, and basically transition-invariant?

To sum up, it is possible to study theory of special subsets and convergence in 2^κ, although one has to make additional assumptions very often or define notions which are more abstract or intricate than their classical counterparts. Therefore, there is still a wide range of possibilities for further research in this topic, and this thesis, I hope, lays the groundwork in those cases.

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