Existence and regularity theory in weighted Sobolev spaces and applications

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‘‘I firmly believe that love [of a subject or hobby] is a better teacher than a sense of duty- at least for me.’’ - Albert Einstein

This thesis is dedicated to my beloved parents: my mother Bijali Dhara, my father Rabindranath Dhara as well as my dear wife Paramita Chatterjee, who always keep inspiring me. I also dedicate this dissertation to my elder brother Lakshmi Narayan Dhara. I want to give special thanks to all of my friends for their constant support and encouragement.

Abstract

In the thesis we discuss several questions related to the study of degenerate, possibly nonlinear PDEs of elliptic type. At first we discuss the equivalent conditions between the validity of weighted Poincaré inequalities, structure of the functionals on weighted Sobolev spaces, isoperimetric inequalities and the existence and uniqueness of solutions to the degenerate nonlinear elliptic PDEs with nonhomogeneous boundary condition, having the form:

$$\begin{cases} \operatorname{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) = x^*, \\ u - w \in W^{1,p}_{\rho,0}(\Omega), \end{cases} \tag{0.0.1}$$

involving any given $x^* \in (W^{1,p}_{\rho,0}(\Omega))^*$ and $w \in W^{1,p}_{\rho}(\Omega)$, where $u \in W^{1,p}_{\rho}(\Omega)$ and $W^{1,p}_{\rho}(\Omega)$ denotes certain weighted Sobolev space, $W^{1,p}_{\rho,0}(\Omega)$ is the completion of $\mathcal{C}_0^\infty(\Omega)$. As a next step, we undertake a natural question how to interpret the nonhomogenous boundary conditions in weighted Sobolev spaces, when the natural analytical tools, like trace embedding theorems, are missing. Our further goal is to contribute to solvability and uniqueness for degenerate elliptic PDEs with nonhomogenous boundary condition being the extension of (0.0.1). In addition to the monotonicity method used in the first step of our discussion for the problem (0.0.1), we also exploit Lax-Miligram theorem to treat the linear problem like:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) + B(x)\cdot\nabla u(x) + C(x)u(x) = x^* \;\text{ for a.e. } x \in \Omega, \\ u(x) = g(x) \;\text{ for a.e. } \; x \in \partial\Omega, \end{cases}$$

as well as Ekeland's Variational Principle and Boccardo-Murat techniques to consider problem like:

$$\begin{cases} -\operatorname{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = x^*, \\ u - z \in X, \end{cases}$$

where $p > 1$, $\lambda > 0$, and the operator $\mathcal{L}_\lambda u := -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u$ is non-monotone.

For the study of the nonhomogeneous BVPs, we apply recent results due to Kałamajska and myself, where we constructed trace extension operator from weighted Orlicz-Slobodetskii spaces defined on the boundary of the domain to weighted Orlicz-Sobolev spaces in the domain. Information on the spectrum of the corresponding differential operator is also derived. Moreover, some nonexistence and nonuniqueness results are also analyzed.

Key words and phrases: weighted Poincaré inequality, weighted Orlicz-Sobolev spaces, weighted Orlicz-Slobodetskii spaces, isoperimetric inequalities, weighted Sobolev spaces, p-Laplace equation, Baire Category method, extension operator, nonhomogeneous boundary value problem, trace theorem, degenerate elliptic PDEs, upper and lower bounds of eigenvalues, two weighted Poincaré inequality, eigenvalue problems, nonexistence

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Declaration

Author's declaration:
Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

March 3, 2017 . . . .
Date *Raj Narayan Dhara*

Supervisor's declaration:
The dissertation is ready to be reviewed.

March 3, 2017 . . . .
Date *dr hab. Agnieszka Kałamajska, prof. UW*

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Contents

Chapter 1

Introduction

Weighted Sobolev and Orlicz-Sobolev spaces are important tools in many disciplines of the theoretical and applied mathematics. They are used in the study of degenerate partial differential equations, i.e., equations with various type of singularities in the coefficients, it is natural to look solutions in weighted Sobolev space, see e.g. [17, 30, 35, 36, 41, 52, 56, 65, 93].. The nonlinear degenerate partial differential equations arise naturally in many fundamental problems such as complex analysis, differential geometry, dynamical systems, elasticity, fluid mechanics, optimization, relativity and string theory.

Some of recent developments, including results, ideas and techniques, in the study of degenerate partial differential equations are already surveyed and analyzed. On the other hand, most of the important problems of nonlinear degenerate partial differential equations are truly challenging and still open. This requires further new ideas, techniques, and deserve our special attentions.

In this thesis my emphasis will be on exploring and/or developing unified mathematical approaches adapting the new ideas and techniques.

Below I present few examples of problems which arise naturally in mathematical models and deserve special interests also from the theoretical point of view, because of the missing analytical tools.

a. Oceanographic models: In [4], [13] one finds the following PDE

$$-\mathrm{div}\left(\frac{1}{\mathrm{dist}(x,\partial\Omega)}\nabla u\right) = g$$

where the domain Ω can be either bounded or unbounded. We remark that the weight function: $\rho(x) = \frac{1}{\mathrm{dist}(x,\partial\Omega)}$ is not integrable in Ω.

b. Diffusion in a potential field: We start from the equation

$$-\mathrm{div}(\nabla v - bv) + v = g \quad \text{in } \mathbb{R}^N,$$

in which b is a potential vector field and $b = \nabla\Phi$. Setting

$$\rho = e^{\Phi}, \quad v = u\rho \;\text{ and }\; g = \rho f,$$

we arrive at equation

$$-\mathrm{div}(\rho A\nabla u) + \rho u = \rho f, \quad f \in \mathcal{C}_0^\infty(\mathbb{R}^N)$$

for u. Physical examples of potentials Φ show that the condition $\rho^{-1} \in L^1_{\mathrm{loc}}$ does not always hold, see [93].

c. Heston model in mathematical finance: Consider the PDE (1.1.1) with A in the following type (see [36, 42])

$$Av := -\frac{y}{2}(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy}) - (c_0 - q - \frac{y}{2})v_x + \kappa(\theta - y)v_y + c_0 v.$$

The generator of this process with killing, called the elliptic Heston operator, is a second-order, degenerate-elliptic partial differential operator, where the degeneracy in the operator symbol is proportional to the distance to the boundary of the half-plane. In mathematical finance, solutions to obstacle problem for the elliptic Heston operator correspond to value functions for perpetual American-style options on the underlying asset.

d. Generators of diffusion processes employed in stochastic volatility models in mathematical finance, of diffusion processes in mathematical biology, and the study of porous media (see [36] and references therein)

$$Av := -x_d \mathrm{tr}(aD^2 v) - b \cdot Dv + cv \quad \text{ on } \Omega, \; v \in C^\infty(\Omega).$$

e. Matukuma equation which appears in astrophysics:

$$\Delta u + \frac{1}{1+|x|^2} u^q = 0, \qquad q > 1,$$

describing the dynamics of globular clusters of stars [63]. The existence results and qualitative properties for Matukuma-Dirichlet problems which are generalized version of Makutuma equation and read as follows:

$$\begin{cases} -\mathrm{div}(|\mathrm{x}|^{\alpha}\mathrm{m}(|\nabla \mathrm{u}|)\nabla \mathrm{u}) + \frac{|x|^{s-b}}{(1+|x|^b)^{s/b}} g(u) = 0 \ \text{ in } B(0,R), \\ u = 0 \quad \text{on } \partial B(0,R), \end{cases}$$

were studied, for example in [29, 78].

Not much is known about the degenerate PDEs involving the non integrable weights.

One group of experts is applying the existing tools, whereas the other one is constructing general tools, but not using. It may because of the lack of contacts between specialists in the theory and specialists in the applied branches of PDEs. Therefore finding the bridge between the applied and theoretical issues seem to be very important.

1.1 Degenerate elliptic PDEs in the weighted setting

Elliptic PDEs

Let us consider the following problem:

$$\begin{cases} Av = f & \text{in} \quad \Omega \\ v = g & \text{in} \quad \partial\Omega, \end{cases} \tag{1.1.1}$$

where $f : \Omega \to \mathbb{R}$ is a given source function, the function $g : \partial\Omega \to \mathbb{R}$ prescribes some Dirichlet boundary condition and A is the given elliptic operator, possibly nonlinear.

We observe that the role of a weight function consists in describing the behaviour at zero and infinity of the given functions which belong to the weighted L^p spaces when it approaches the point.

Some selected related contributions

In case when A is linear, elliptic in the classical sense, having smooth coefficients with functions f, g belong to the appropriate Sobolev and Slobodetskii type spaces in the domain and on its boundary respectively, the elliptic regularity theory (see [37]) is rather closed. The solutions of such type PDEs are investigated in the framework of classical Sobolev spaces $W^{m,p}(\Omega)$.

In case when A is the degenerate elliptic operator the investigations have to be provided in the weighted Sobolev spaces $W^{m,p}(\Omega, \rho dx)$ where ρ is the given weight. In such cases, for smooth domains, the existence and regularity of the solutions had already been developed in the second half of 20th century by S. Agmon, Besov, A. Douglis, Fabes et al., Illin, V. A. Kondrat'ev, J. L. Lions, E. Magenes, Maz'ya, Venkatesha Murthy, J. Nečas, L. Nirenberg, Nikolskii, I. G. Petrovskii, M. Schechter, Guido Stampacchia, Stredulinsky, M.I.Vishik.

There are not much known in the degenerate settings. This is because the typical tools:
- Hardy/Poincaré inequalities;
- trace type theorems (the well posedness with respect to g);
- Gagliardo-Nirenberg inequalities

are missing in the literature in more general form.

To confirm how little theory is known, let us consider an operator in the following form:

$$Av := -x_d \mathrm{tr}(aD^2 v) - b \cdot Dv + cv \quad \text{on } \Omega,\ v \in C^\infty(\Omega), \tag{1.1.2}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the coefficients of A are given by a matrix-valued function, $a = (a^{ij}) : \overline{\Omega} \to \mathcal{F}^+(d)$, a vector field, $\mathcal{F}^+(d) \subset \mathbb{R}^{d\times d}$, the subset of non-negative definite matrices, $b = (b^i) : \overline{\Omega} \to \mathbb{R}^d$ and a function, $c : \overline{\Omega} \to \mathbb{R}$. The operator " $-A$" is the generator of a degenerate-diffusion process with killing.

The authors in recent paper [36] have studied the problem (1.1.2) and obtained the a priori interior Schauder estimates and higher-order Hölder regularity up to the boundary - as measured by certain weighted Hölder spaces, $C_s^{k,2+\alpha}(\overline{\Omega})$ (see Definition 2.3 in [36]) using so-called the Daskalopoulos-Hamilton-Hölder cycloidal distance function s, see section 2 in [4].
Another related approach was provided in the paper by Fabes, Kenig Serapioni in [35], where the following weak ellipticity condition is satisfied:

$$(a\xi, \xi) \geq \rho(x)|\xi|^2.$$

and weight function ρ belongs to the Muckenhoupt class A_p (definition is given in Section 1.2).

In case of A_p class regularity theory is rather well developed up to now. On the other hand, trace embedding theorems in weighted Sobolev spaces are not well known even in the class A_p weights. Some recent works by Tyulenev e.g. [87] can be found in this direction. In particular it is impossible to consider the problem (1.1.1) in the degenerate setting as it is not clear how to choose the proper function space for g so that the problem is well posed.

Example application of weighted Sobolev spaces are in the shape optimization problems in [51] where the author used so called Kondratiev type spaces. The author studied conical diffraction problems with non-smooth grating structures. They prove the existence, uniqueness and regularity results for solutions in Kondratiev type spaces. An a priori estimate that follows from these results is then used to prove shape differentiability of solutions. One of the tools there to examine the regularity of the degenerate elliptic type problems in Kondratiev space. More applications can be found in the models describing diffusion process in a potential field (see [93] for further study), diffusion process with killing [36], where in most situations only homogeneous boundary data were considered.

According to studies in [3], we have a regularity result for the Poisson problem

$$-\Delta u = f, \quad u_{|\partial\mathbb{P}} = g,$$

on a polyhedral domain $\mathbb{P} \subset \mathbb{R}^3$ using the Babuška-Kondratiev spaces $\mathcal{K}_\alpha^m(\mathbb{P})$. These are weighted Sobolev spaces in which the weight is given by the distance to the set of edges [53, 7]. In particular, it has been shown that there is no loss of $\mathcal{K}^m_\alpha(\mathbb{P})$-regularity for solutions of strongly elliptic systems with smooth coefficients. Degenerate PDEs were also applied in shape optimization theory in the paper [80].

The degenerate PDEs have been studied rather extensively in recent years.

Great mathematicians investigated degenerate PDEs

I would also like to add some informations about the winners of prizes, popular authors who has been worked in this area:

1. *Why degenerate PDEs in science and the real life:* On ICM'2006 in Madrid, J. L. Vazquez in his plenary talk discussed the perspectives of the mathematical theory of nonlinear diffusion, focusing on the fast diffusion equation and porous medium equation, and underlying the connections between functional analysis, semigroup theory, physics of continuous media, probability and differential geometry. Number of citations: 6012.

2. *Why elliptic regularity theory is so inspiring:* L. Nirenberg (an applications of Gagliardo-Nirenberg inequlaities to the regularity theory also an aim of my further research) had great impact on modern analysis - he obtained prestigious Chern Medal in 2010 and Abel Prize 2015. His number of citations according to Mathematical Review: 11277.

3. *Why weighetd Sobolev spaces setting is of interest:* Vladimir Mazy'a. Author of 550 publications, considered as modern Euler. He has 95 contributors. Author of the papers dealing with weighted Sobolev spaces is regarded as one of fathers of this theory. In his works, he always takes into account the applications of the theories.

1.2 Brief description of my contributions

My dissertation is created on the basis of four articles [24, 26, 27, 28]. My research in this direction originated in papers [23], [25], obtained jointly with my supervisor Agnieszka Kałamajska, where we studied trace extension theorems from an weighted Orlicz-Slobedetskii space defined on the boundary of the domain to an weighted Orlicz-Sobolev space defined on the domain. In this thesis, we study following problems:

A) jointly with my supervisor, we analysed the structure of functionals acting on weighted Sobolev spaces and it's equivalence consequences to Poincaré inequality, isoperimetric inequality and the solvability of the degenerate PDEs, [24].

B) jointly with my supervisor, we discussed how to interpret the boundary data properly when the corresponding trace embedding theorem is missing, [26].

C) existence and uniqueness of solutions to the linear degenerate elliptic PDEs with nonhomogeneous boundary data, [27].

D) existence and uniqueness of the solutions to non-linear elliptic PDEs deduced from the two-weighted Poincaré inequality along with the nonexistence of positive solutions of such problems, [28].

I would be mostly interested in the setting involving weights outside the Muckenhoupt class of A_p (see [35, 69, 70]) as this class is so far better understood by Muckenhoupt itself, Fabes et al., Trudinger, Maz'ya and many others. Instead, we use the B_p-class of weights, introduced by Kufner and Opic in 1984 (see [55]), which is wider than A_p-class.

The thesis is organised as follows. We denote by $W^{1,p}_\rho(\Omega) = \{f \in L^1_{\rm loc}(\Omega) : \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\Big\}$ weighted Sobolev space subordinated to weight ρ, $W^{1,p}_{\rho,0}(\Omega)$ is the completion of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}_\rho(\Omega)$, $(W^{1,p}_{\rho,0}(\Omega))^*$ is the space of functionals on $W^{1,p}_{\rho,0}(\Omega)$. We also deal with two weighted Sobolev spaces $W^{1,p}_{b,\rho}(\Omega) = \Big\{f \in L^1_{\rm loc}(\Omega) : f \in L^p_b(\Omega), \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\Big\}$, $W^{1,p}_{b,\rho,0}(\Omega)$-the completion of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}_{b,\rho}(\Omega)$ and it's duals $\left(W^{1,p}_{b,\rho,0}(\Omega)\right)^*$.
Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p > 1$. We say that a positive weight ρ satisfies $B_p(\Omega)$-condition ($\rho \in B_p(\Omega)$) if $\rho^{-1/(p-1)} \in L^1_{\rm loc}(\Omega)$. We say $\rho \in A_p$, introduced by B. Muckenhoupt [69], if $\sup\limits_Q \left(\frac{1}{|Q|}\int_Q \rho\, dx\right)\left(\frac{1}{|Q|}\int_Q \rho^{-1/(p-1)}\, dx\right)^{p-1}$

$< \infty$, where $Q \subset \mathbb{R}^n$ are arbitrary cubes edges parallel to the coordinate axes and $|Q|$ is the volume of Q. In particular, we have $A_p \subset L^1_{\rm loc}(\mathbb{R}^n) \cap B_p(\mathbb{R}^n)$, see Remark 4.9 in [55].

In Chapter 2, we contribute to the problem **A)**. We consider the following problems:

(P) the validity of weighted Poincaré inequality:

$$\int_\Omega |f(x)|^p \rho(x)\, dx \leq C \int_\Omega |\nabla f(x)|^p \rho(x)\, dx,$$

where the function $f \in W^{1,p}_{\rho,0}(\Omega)$.

(R) representation of functionals: every functional acting on $W^{1,p}_{\rho,0}(\Omega)$ is weak divergence of certain function $g = (g_1, \dots, g_n) \in L^{p'}_\tau(\Omega, \mathbb{R}^n)$, where $\tau = \rho^{-1/(p-1)}$.

(I) validity of isoperimetric inequalities:

$$\mu(F) := \int_F \rho(x)\, dx \leq \beta \cdot W^{1,p}_\rho\text{-cap}(F, \Omega),$$

involving weighted Sobolev capacities $W^{1,p}_\rho$-cap(F, Ω) defined by (2.4.3).

(S) solvability of degenerated PDEs:

$$\begin{cases} \operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = x^*, \\ \qquad\qquad u - w \in W^{1,p}_{\rho,0}(\Omega), \end{cases}$$

involving any given $x^* \in (W^{1,p}_{\rho,0}(\Omega))^*$ and $w \in W^{1,p}_\rho(\Omega)$, where $u \in W^{1,p}_\rho(\Omega)$ is unknown.

We show that under some assumptions the above conditions are equivalent. It is worth to mention that the proof of the difficult implication **(P)**$\Rightarrow$**(R)** is based on the Baire method and was inspired by the techniques of Bogachev [10]. Sections 2.1 (introduction), 2.2 (preliminaries), 2.3 (**(P)**$\Longleftrightarrow$**(R)**) and 2.6 (conclusions) were obtained together with my supervisor. My alone contributions are to Section 2.5 (**(P)**$\Longleftrightarrow$**(S)**), further equivalent relation with isoperimetric inequalities in Section 2.4 ((**P**) $\iff$ (**I**)), constructions of Poincaré inequalities (Section 2.7) as well as finding application of the recent paper due to Skrzypczak [79] to construct certain Poincaré inequality.

Chapter 3 is based on the paper [26] and obtained together with my supervisor. It is dedicated to formulate properly the boundary value problems when we do not have the trace embedding theorem in the corresponding weighted Sobolev spaces. This helps to get the solvability and uniqueness of the degenerate nonhomogenous Dirichlet type BVPs. For instance, let us consider the following boundary value problem.

$$\begin{cases} -\mathrm{div}(\rho(x)\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2.1}$$

Suppose that $g \in Y$, Y is some function space on $\partial\Omega$. Suppose that there exists bounded operator:

$$\mathrm{Ext} : Y \to W^{1,2}_{\rho}(\Omega).$$

Then there exists $\Psi_g \in W^{1,2}_{\rho}(\Omega)$ such that $\Psi_g|_{\partial\Omega} = g$. Substitute: $v := u - \Psi_g$, then the problem (1.2.1) is equivalent to:

$$\begin{cases} Pv = f - P\Psi_g & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $Pw = -\mathrm{div}\,(\rho(x)\nabla w)$. This homogeneous problem can be solved by the standard methods, for example with the help of Lax-Miligram theorem. In particular, it also gives the existence of solution $u = v + \Psi_g$ of the nonhomogenous problem (1.2.1). It is not clear if that solution is independent of the chosen extension operator Ext. We have proven in [26] (as a special case of more general statement) that as far as the weight $\rho \in B_2(\Omega)$ is integrable, the constructed solution u of the nonhomogeneous problem does not depend on the choice of the extension operator. However, we must know what is the space Y and have given the extension operator Ext. Such extension operators were constructed in [23, 25] by Kałamajska and the author.

Chapter 4 is based on the paper [27]. We study the linear elliptic problem:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u(x)) + B(x)\cdot\nabla u(x) + C(x)u(x) = f(x) \ \text{ for a.e. } x \in \Omega, \\ u(x) = g(x) \ \text{ for a.e. } \ x \in \partial\Omega, \end{cases} \tag{1.2.2}$$

where the diffusion coefficient $A(x) = [a_{ij}(x)]_{i,j=1,\dots,n}$ is the symmetric matrix with measurable entries and satisfies the degenerate ellipticity condition:

$$c_1|\xi|^2\rho(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le c_2|\xi|^2\rho(x), \text{ for a. e. } x \in \Omega \text{ and every } \ \xi \in \mathbb{R}^n,$$

$c_1,\ c_2 > 0$ are given constants. Using Lax-Miligram theorem as well as our previous extension results in [23, 25, 26], we prove the existence and uniqueness of solution to (1.2.2). We also provide lower bound for spectrum of operator $P_1 u := -\mathrm{div}(A(x)\nabla u(x)) + B(x)\nabla u(x) + C(x)u(x)$, see Theorem 4.4.5. It looks that our conditions provided in Theorems 4.3.2 and 4.4.5 are almost optimal. Namely, when we reduce them to the classical case $\rho \equiv 1, A \equiv Id$, some of them are sharp, see Remarks 4.4.2, 4.4.4 and 4.4.7.

In Chapter 5, we show that the two-weighted Poincaré inequality:

$$\int_\Omega |u(x)|^p b(x)\,dx \le C_P \int_\Omega |\nabla u(x)|^p \rho(x)\,dx, \qquad \text{for any } u \in W^{1,p}_{b,\rho,0}(\Omega)$$

implies the solvability to the nonlinear eigenvalue problems:

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = x^*, \\ \qquad\qquad u - z \in W^{1,p}_{b,\rho,0}(\Omega), \end{cases}$$

where $x^* \in \left(W^{1,p}_{b,\rho,0}(\Omega)\right)^*$, $z \in W^{1,p}_{b,\rho}(\Omega)$ are taken arbitrarily and $\lambda \ge 0$ is sufficiently small, (see Theorem 5.3.4). In the case $\lambda > 0$, we can not apply techniques from Chapter 2, because the operator $-\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u$ is not monotone (see Remarks 5.4.2). Moreover, when $p \ne 2$, we can not apply Lax-Miligram method as in Chapter 4, because in that case we require Hilbert structure of the related Sobolev space $W^{1,p}_{b,\rho,0}(\Omega)$. Instead, we adapt the Ekeland's variational principle along with Boccardo-Murat almost everywhere convergence technique to our degenerate case. Moreover, we give an example of nonexistence of positive classical solutions for degenerate eigenvalue problem like:

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = \gamma f(u), \qquad \lambda > 0, \quad \gamma \in \mathbb{R}\setminus\{0\}, \\ \qquad\qquad u(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is bounded, starshaped with respect to the origin, $f : [0,\infty] \mapsto [0,\infty]$ is a continuous function satisfying certain assumptions, $\rho(x) = |x|^\alpha$,

$b(x) = |x|^{\alpha - p},\ \alpha < p - n,\ p > 1$, see Theorem 5.5.1. Results obtained there are adaptation of the methods from the paper by Azorero-Alonso [6] which dealt with weight functions $\rho \equiv 1,\ b(x) = 1/|x|^p$.

Chapter 2

Equivalency between Poincaré inequality and solvability

Poincaré inequality is an important tool to get a solvability of the partial differential equations. This is also not very difficult to see that Poincaré inequality gives the functional in the form of divergence in the respective dual spaces. In this chapter, which is new to discuss if these above relations are equivalent in the weighted settings. In this chain of relations the most interesting implication is how solvability or the divergence form of the functional may give the Poincaré inequality. The proof is inspired by the work of Bogachev and Shaposhnikov (see [10]) with the usage of the beautiful Baire theorem (see [8]). This work has been published as a research paper in a journal (see [24]). Here my contribution is to have further equivalent relation with isoperimetric inequalities and the related updated literature reviews, constructions of Poincaré inequalities (see Section 2.7) with application of the recent paper due to Skrzypczak [79]. Furthermore, I was involved with the assistance in edifying the research paper.

2.1 Introduction

Let $\Omega \subseteq \mathbf{R}^n$ be a given subset, $p > 1$ and $W^{1,p}_\rho(\Omega) = \Big\{ f \in L^1_{loc}(\Omega) : f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega) \Big\}$ where $\frac{\partial f}{\partial x_i}$ are distributional derivatives, be the weighted Sobolev space subordinated to the weight function ρ. For technical reasons (see Proposition 2.2.5) we assume that ρ together with $\rho^{-1/(p-1)}$ is locally integrable on Ω.
We are interested in the study of equivalent conditions for the validity of weighted Poincaré inequality:

(P)

$$\int_\Omega |f(x)|^p \rho(x)\, dx \le C \int_\Omega |\nabla f(x)|^p \rho(x)\, dx,$$

where the function f belongs to the completion of $C_0^\infty(\Omega)$ in $W^{1,p}_\rho(\Omega)$ denoted by $W^{1,p}_{\rho,0}(\Omega)$.

The conditions we derive include

(R) representation of functionals: every functional acting on $W^{1,p}_{\rho,0}(\Omega)$ is weak divergence of certain function $g = (g_1, \ldots, g_n) \in L^{p'}_\tau(\Omega, \mathbb{R}^n)$, where $\tau = \rho^{-1/(p-1)}$.

(I) validity of isoperimetric inequalities:

$$\mu(F) := \int_F \rho(x)\, dx \le \beta \cdot W^{1,p}_\rho\text{-cap}(F, \Omega),$$

involving weighted Sobolev capacities $W^{1,p}_\rho$-cap(F, Ω) defined by (2.4.3).

(S) solvability of degenerated PDEs:

$$\begin{cases} \operatorname{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) = x^*, \\ \qquad\qquad u - w \in W^{1,p}_{\rho,0}(\Omega), \end{cases}$$

involving any given $x^* \in (W^{1,p}_{\rho,0}(\Omega))^*$ and $w \in W^{1,p}_\rho(\Omega)$, where $u \in W^{1,p}_\rho(\Omega)$ is unknown.

Some of the implications may be known to the specialists. Namely, the equivalence $(P) \iff (I)$ is the special variant of the so-called isoperimetric estimates due to Maz'ya (see [65], Theorem 1 in Chapter 2.3.4 and [64] for the historical source). The implication $(P) \implies (S)$ is based on application of the Minty Browder Theorem ([15], [67], Theorem 26.A in [92]) applied to the monotone operator $Au := -\mathrm{div}\,(\rho|\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w))$, while the implication $(S) \implies (R)$ is obvious. The detailed discussion is presented in Sections 2.4 and 2.5. We could not find anywhere the implication $(R) \implies (P)$, which closes the chain of equivalences.

Here we present argument for the validity of $(R) \implies (P)$ which is based on Baire Cathegory method (Theorem 2.6.1). Similar techniques but in different context can be found in the paper [10]. Closing this implication is important for example because of the following reason. The implication $(R) \implies (P)$ implies $\sim (P) \implies \sim (R) \iff \sim (S)$. In particular if we know that the inequality (P) does not hold, we cannot expect solvability of (S) with any functional $x^* \in X^*$. As one can prove sometimes the non-existence of the weighted Poincaré inequalities (P), this way we hope to contribute in the analysis of degenerated PDEs. Motivated by the problem (S), in Section 2.7 we show some new constructions of Poincaré inequality (P) and its applications to problem (S). As one of the proposed tools we present an application of recent method of Skrzypczak [79], from where Poincaré inequalities follow as consequence of solvability of the nonlinear eigenvalue problem

(PDI)

$$-\Delta_{p_0} u_0 \geq C u_0^{p_0 - 1},$$

where $u_0 \in W^{1,p_0}_{loc}(\Omega)$ is nonnegative and $\Delta_q v := \mathrm{div}(|\nabla v|^{q-2}\nabla v)$ is the q-Laplacian, $p_0 > 1$.

It appears that solvability of this simple Partial Differential Inequality (PDI) and knowledge about function u_0 allows to construct certain family of weights $\rho_\beta(x) := (u_0(x))^{p_0-1-\beta}\chi_{u_0(x)>0}$ for which the degenerated Partial Differential Equation:

$$\begin{cases} \mathrm{div}\,(\rho_\beta(x)|\nabla u|^{p-2}\nabla u) = x^*, \\ \qquad\qquad u - w \in W^{1,p}_{0,\rho_\beta}(\Omega), \end{cases}$$

where $x^* \in (W^{1,p}_{0,\rho_\beta}(\Omega))^*$ and $w \in W^{1,p}_{\rho_\beta}(\Omega)$ are given, while $u \in W^{1,p}_{\rho_\beta}(\Omega)$ is unknown, has a solution for any given $p \geq p_0$.

Let us mention that every weight ρ being the A_p Muckenhoupt class satisfies Poincaré inequality **(P)** (see Section 2.7 definition of Muckenhoupt class and discussion). We refer e. g. to [18, 19, 20, 31, 32, 55, 56, 70] and to the references enclosed therein for the information about the validity of Poincaré inequality (P). For analysis of degenerated PDEs we refer for example to [30, 35, 41, 56, 84] and to their references.

We hope that the presented approach may contribute to the analysis of solvability of degenerated PDEs. We also wanted to focus on the importance to construct weights ρ which are admitted to the Poincaré inequality (P).

2.2 Notation and preliminaries

General assumptions. If not said otherwise we assume that $\Omega \subseteq \mathbb{R}^n$ is an open set. We assume that $p > 1$ and p' is its Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. If the function f is defined on some subset $A \subseteq \Omega$, by $f\chi_A$ we denote its extension by zero outside A.

Weights of class $B_p(\Omega)$. We will need the following definitions.

Definition 2.2.1 (positive weights)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{M}(\Omega)$ be the set of all Borel measurable functions. Elements of the set

$$W(\Omega) := \{\rho \in \mathcal{M}(\Omega) : \ 0 < \rho(x) < \infty, \text{ for a.e. } x \in \Omega\},$$

will be called positive weights.

Definition 2.2.2 (class $B_p(\Omega)$, Definition 1.4 in [55])**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p > 1$. We will say that a weight $\rho \in W(\Omega)$ satisfies $B_p(\Omega)$-condition $(\rho \in B_p(\Omega))$ if $\rho^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$.

The following simple observation results from Hölder inequality.

Proposition 2.2.3 (Theorem 1.5 in [55]). *Let $\Omega \subset \mathbb{R}^n$ be an open set, $p > 1$ and $\rho \in B_p(\Omega)$. Then $L^p_\rho(\Omega)$ is the subset of $L^1_{\mathrm{loc}}(\Omega)$.*

Weighted Sobolev spaces.
We will be dealing with the following definition of weighted Sobolev spaces.

Definition 2.2.4 (weighted *Sobolev space*). Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p > 1$ and $\rho \in W(\Omega)$ be the given weight. The linear set

$$W^{1,p}_\rho(\Omega) := \{f \in L^1_{loc}(\Omega) : f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\}, \tag{2.2.1}$$

where $\frac{\partial f}{\partial x_i}$ are distributional derivatives, equipped with the norm

$$\|f\|_{W^{1,p}_\rho(\Omega)} := \|f\|_{L^p_\rho(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega,\mathbb{R}^n)}, \tag{2.2.2}$$

will be called weighted Sobolev space.
The symbol $W^{1,p}_{\rho,0}(\Omega)$ will denote the completion of $C_0^\infty(\Omega)$ (of smooth compactly supported functions in Ω) in the norm of the space $W^{1,p}_\rho(\Omega)$.

For our convenience in the above definition we chose the following norm in the space $L^p_\rho(\Omega; \mathbb{R}^n)$:

$$\|(v_1, \dots, v_n)\|_{L^p_\rho(\Omega;\mathbb{R}^n)} := \left(\sum_{i=1}^n \|v_i\|^p_{L^p_\rho(\Omega)} \right)^{\frac{1}{p}}. \tag{2.2.3}$$

As noticed in [55], the linear set defined by (2.2.1) is not always complete in the norm $\|\cdot\|_{W^{1,p}_\rho(\Omega)}$. It becomes a Banach space under some special assumptions on the involved weight function ρ.

Proposition 2.2.5 (Theorem 2.1 in [55]). *Let $p > 1$, $\Omega \subseteq \mathbb{R}^n$ be an open set and $\rho \in B_p(\Omega)$. Then linear set $W^{1,p}_\rho(\Omega)$ defined by* (2.2.1) *equipped with the norm* (2.2.2) *is a Banach space. In particular $W^{1,p}_{\rho,0}(\Omega) \subseteq \{f \in L^1_{loc}(\Omega) : f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\}$.*

Remark 2.2.6. To our best knowledge the necessary and sufficient conditions for the completeness of linear set defined by (2.2.1) under the norm (2.2.2) are not known.

2.3 (P) $\iff$ (R)

The following statement is the generalization of Theorem 2 from [65], Section 1.1.15, where the statement was given in the unweighted setting. For the weighted setting we refer to Lemma 13.8 in [56], where the statement is given in the case $p = 2$. For reader's convenience we enclose the proof which follows by adaptation of their techniques.

Proposition 2.3.1 (representation of functionals on $W^{1,p}_{\rho,0}(\Omega)$)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\rho \in B_p(\Omega) \cap L^1_{loc}(\Omega)$, $X = W^{1,p}_{\rho,0}(\Omega), X^* = (W^{1,p}_{\rho,0}(\Omega))^*$. Then for any functional $x^* \in X^*$ there exist functions $g = (g_0, g_1, \ldots, g_n) \in L^{p'}_\tau(\Omega; \mathbb{R}^{n+1})$, where $\tau := \rho^{-\frac{1}{p-1}}$, such that for any $f \in X$ we have*

$$(x^*, f) = \int_\Omega f(x) g_0(x)\, dx - \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial x_i}(x) g_i(x)\, dx, \tag{2.3.1}$$

in other words

$$\langle x^*, f\rangle = \int_\Omega f(x) g_0(x)\, dx + (\operatorname{div} \bar{g}, f),$$

where $\bar{g} = (g_1, \ldots, g_n) \in L^{p'}_\tau(\Omega, \mathbb{R}^n)$.

Remark 2.3.2. 1) Let us note that under our assumptions we have $\tau^{-\frac{1}{p'-1}} = \rho \in L^1_{loc}(\Omega)$. Hence $\tau \in B_{p'}(\Omega)$ and so according to Proposition 2.2.3 we have $L^{p'}_\tau(\Omega) \subseteq L^1_{loc}(\Omega)$. In particular distributional derivatives of $g_1, \ldots, g_n$ are well defined and so the expression

$$(\operatorname{div} \bar{g}, f) := -\sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial x_i}(x) g_i(x)\, dx \tag{2.3.2}$$

has the distributional interpretation when $f \in C^\infty_0(\Omega)$. Moreover, by Hölder inequality

$$\left|\int_\Omega \frac{\partial f}{\partial x_i} g_i\, dx\right| = \left|\int_\Omega \frac{\partial f}{\partial x_i} \rho^{\frac{1}{p}} g_i \rho^{-\frac{1}{p}}\, dx\right| \le \left(\int_\Omega \left|\frac{\partial f}{\partial x_i}\right|^p \rho\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |g_i|^{p'} \tau\, dx\right)^{\frac{1}{p'}} \tag{2.3.3}$$

and the application of the norm (2.2.3) gives

$$|(\operatorname{div}\bar{g}, f)| \leq \|\nabla f\|_{L^p_\rho(\Omega;\mathbb{R}^n)} \|\bar{g}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)},$$

whenever $f \in X$. In particular $\operatorname{div}\bar{g}$ is an element of X^*, which restricted to $C_0^\infty(\Omega)$ agrees with the distributional divergence of $\bar{g}$.
2) The function $\bar{g}$ in Proposition 2.3.1 is not defined uniquely as for example we may add to $\bar{g}$ any smooth divergence free vector.
3) Finiteness of right hand side in (2.3.1) for any $f \in X$ and $g \in L^{p'}_\tau(\Omega, \mathbb{R}^n)$ follows from the computations enclosed in (2.3.3).
4) Property $\rho \in B_p(\Omega)$ was required for the completeness of the space X.
5) Simple computations (regularization by convolution and Lebesgue's Dominated Convergence Theorem) show that assumption $\rho \in L^1_{loc}(\Omega)$ implies that Lipschitz compactly supported functions belong to $W^{1,p}_{\rho,0}(\Omega)$.
6) Let $\rho \in B_p(\Omega)$ and $p > 1$. Since the embedding $W^{1,p}_\rho(\Omega) \ni f \mapsto (f, \nabla f) \in L^p_\rho(\Omega, \mathbf{R}^{n+1})$ identifies $W^{1,p}_\rho(\Omega)$ with the closed subspace in $L^p_\rho(\Omega, \mathbf{R}^{n+1})$- the reflexive Banach space, we conclude that then the space $W^{1,p}_\rho(\Omega)$, as well as its closed subspace $W^{1,p}_{\rho,0}(\Omega)$, are reflexive. This follows from general fact that any closed subspace of the reflexive Banach space is reflexive, see e.g. pp. 104–105 in [66].

Before we prove Proposition 2.3.1 we recall the following simple fact.

Proposition 2.3.3. *Let $d \in \mathbb{N}$, $\rho \in L^1_{loc}(\Omega) \cap W(\Omega)$ and $p > 1$. Then we have $(L^p_\rho(\Omega;\mathbb{R}^d))^* \sim L^{p'}_\tau(\Omega;\mathbb{R}^d)$ with $\tau = \rho^{-\frac{1}{p-1}}$ where the duality is expressed by*

$$\langle g, h\rangle = \sum_{i=1}^d \int_\Omega g_i h_i \, dx, \quad \text{where} \quad g \in L^p_\rho(\Omega;\mathbb{R}^d), \; h \in L^{p'}_\tau(\Omega;\mathbb{R}^d).$$

Proof. For $\rho \equiv 1$ the identification is obvious. In the general case we have $f \in L^p_\rho(\Omega) \iff f\rho^{\frac{1}{p}} \in L^p(\Omega)$. Hence, any functional $x^* \in (L^p_\rho(\Omega;\mathbb{R}^d))^*$ can be identified with $y^* \in (L^p(\Omega;\mathbb{R}^d))^*$ where

$$\langle x^*, f\rangle =: \langle y^*, \rho^{\frac{1}{p}} f\rangle = \sum_{i=1}^d \int_\Omega \tilde{g}_i(\rho^{\frac{1}{p}} f_i)\, dx = \sum_{i=1}^d \int_\Omega (\tilde{g}_i \rho^{\frac{1}{p}}) f_i \, dx,$$

for some $\tilde{g} := (\tilde{g}_1, \dots, \tilde{g}_d) \in L^{p'}(\Omega; \mathbb{R}^d)$. Note that $g = \rho^{\frac{1}{p}} \tilde{g} = (\rho^{\frac{1}{p}} \tilde{g}_1, \dots, \rho^{\frac{1}{p}} \tilde{g}_d) \in L^{p'}_\tau(\Omega; \mathbb{R}^d)$ and so $(L^p_\rho(\Omega; \mathbb{R}^d))^* \subseteq L^{p'}_\tau(\Omega; \mathbb{R}^d)$. The remaining inclusion $(L^p_\rho(\Omega; \mathbb{R}^d))^* \supseteq L^{p'}_\tau(\Omega; \mathbb{R}^d)$ follows from arguments as in (2.3.3). □

We are ready to prove Proposition 2.3.1.

Proof of Proposition 2.3.1. Let us define the linear mapping $\Psi : W^{1,p}_{\rho,0}(\Omega) \to L^p_\rho(\Omega; \mathbb{R}^{n+1})$ by

$$W^{1,p}_{\rho,0}(\Omega) \ni f \mapsto \Psi(f) := \left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

We observe that Ψ is an isometric embedding of $W^{1,p}_{\rho,0}(\Omega)$ onto some closed subspace $Y \subseteq L^p_\rho(\Omega; \mathbb{R}^{n+1})$. In particular, every functional $F \in (W^{1,p}_{\rho,0}(\Omega))^*$, is uniquely identified with an element $F^{'} \in Y^*$ by expression $\langle F^{'}, \Psi(f) \rangle = \langle F, f \rangle$, where $f \in W^{1,p}_{\rho,0}(\Omega)$. Since $Y \subseteq L^p_\rho(\Omega; \mathbb{R}^{n+1})$ is a closed, by the Hahn-Banach Extension Theorem (see e.g. Chapter IV, Section 1, page 102 in [91]) $F^{'}$ can be extended to $(L^p_\rho(\Omega; \mathbb{R}^{n+1}))^*$. We denote this extension by $\tilde{F}$. According to Proposition 2.3.3, there exist $\tilde{g} = (\tilde{g}_0, \tilde{g}_1, \dots, \tilde{g}_n) \in L^{p'}_\tau(\Omega; \mathbb{R}^{n+1})$, where $\tau = \rho^{-\frac{1}{p-1}}$, such that for and $h = (h_0, \dots, h_n) \in L^p_\rho(\Omega; \mathbb{R}^{n+1})$

$$\langle \tilde{F}, h \rangle = \sum_{j=0}^{n} \int_\Omega \tilde{g}_j(x) h_j(x)\, dx,$$

and it applies to $h = \Psi(f)$. This and the choice of $g := -\tilde{g}$ ends the proof of the proposition. □

One of our main goals is to prove the following statement.

Theorem 2.3.4 (Poincaré inequality and representation of functionals)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain, $\rho \in B_p(\Omega) \cap L^1_{loc}(\Omega)$, $\tau = \rho^{-\frac{1}{p-1}}$ is nonnegative a.e., $p > 1$, $X = W^{1,p}_{\rho,0}(\Omega), X^* = (W^{1,p}_{\rho,0}(\Omega))^*$. The following conditions are equivalent:*

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^p \rho(x)\, dx \le C \int_\Omega |\nabla f(x)|^p \rho(x)\, dx \qquad (2.3.4)$$

holds for every $f \in X$, with constant C independent on f.

ii) *For every $x^* \in X^*$ there exists function $g = (g_1, \dots, g_n) \in L^{p'}_{\tau}(\Omega; \mathbb{R}^n)$ such that for any $f \in X$*

$$\langle x^*, f \rangle = -\sum_{i=1}^{n} \int_{\Omega} \partial_i f(x) g_i(x)\, dx = \langle \operatorname{div} g, f \rangle.$$

iii) *norms: $\|f\|_1 := \|f\|_{L^p_\rho(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega;\mathbb{R}^n)}$ and $\|f\|_2 := \|\nabla f\|_{L^p_\rho(\Omega;\mathbb{R}^n)}$ are equivalent on X.*

Remark 2.3.5. The condition $\rho \in B_p(\Omega) \cap L^1_{loc}(\Omega)$ does not imply the validity of Poincaré inequality. One may consider for example the case of $\Omega = \mathbf{R}^n$ and the Gaussian measure where $\rho(x) = \frac{1}{(\sqrt{2\pi})^n} \exp(-|x|^2/2)$. Easy approximation argument shows that Poincaré inequality with ρ does not hold. For this we proceed by contradiction: consider $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi \equiv 1$ on $B(1)$ and $\phi \equiv 0$ outside $B(2)$, substitute the function $u_R := u\phi_R := u\phi(x/R)$ to the inequality (2.3.4) and let $R \to \infty$.

Remark 2.3.6. Zhikov in [94] considered the case $p = 2, \rho, \rho^{-1} \in L^1(\Omega)$. It is easy to observe that

$$H := W^{1,2}_{\rho,0}(\Omega) \subseteq W := \{u \in W^{1,1}_0(\Omega) : \int_\Omega |\nabla u(x)|^p \rho(x)\, dx < \infty\} \subseteq W^{1,1}_0(\Omega).$$

It is shown in Example 1 on page 669 that in general equality $H = W$ fails. The example is provided on the two dimensional unit ball. As noticed there the situation $H \neq W$ implies existence of the nonzero element $0 \not\equiv u \in W$, which is perpendicular to H in W, in particular for every $\phi \in C_0^\infty(\Omega)$ and more generally for every $\phi \in H$

$$\int_\Omega \nabla u \cdot \nabla \phi \rho\, dx = 0 \quad \text{and} \quad u|_{\partial\Omega} = 0. \tag{2.3.5}$$

Last property is interpreted in the sense of trace operator in $W^{1,1}(\Omega)$ (see e.g. [57], Theorem 6.4.2). In particular (2.3.5) holds for any $\phi \in H$ but not for any $\phi \in W$ (consider $\phi = u$).
In our setting, under the assumption $p = 2, \rho, \rho^{-1} \in L^1(\Omega)$, we deal only with elements of H and our zero boundary conditions (which could be perhaps denoted also by $u|_{\partial\Omega} = 0$) are precisely defined as $u \in W^{1,2}_{\rho,0}(\Omega)$. This might

not be the same as saying that $u|_{\partial\Omega} = 0$ in the sense of trace operator in $W^{1,1}(\Omega)$. If it would be so, it would imply $H = W$, which is not always the case.

Remark 2.3.7. The authors of [20] considered the two weighted Poincaré inequality

$$\left(\int_\Omega |u|^q w(x)\,dx\right)^{1/q} \le C\left(\int_\Omega |\nabla u|^p v(x)\,dx\right)^{1/p}, \qquad (2.3.6)$$

dealing with real valued function u defined on open set $\Omega \subseteq \mathbf{R}^n$ of finite measure, locally integrable nonnegative functions v, w and $p, q \in (1, +\infty)$. It is assumed that u extended by zero outside Ω belongs to $W^{1,1}(\mathbf{R}^n)$, which is interpreted as $u \equiv 0$ on $\partial\Omega$. Main result there, formulated as Theorem 2.1, gives sufficient conditions for the validity of (2.3.6) described in terms of rearrangements of v and w. In case $q > n/(n-1)$ result is sharp as results from Theorem 2.2 there.

We are interested in the case $p = q$, $v = w =: \rho$. Let for example Ω be the Lipschitz boundary domain (see [57], Definition 6.2.2) with finite measure. In that case the space of functions admissible to the inequality (2.3.6) with right hand side finite is

$$W^p := \{u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u(x)|^p \rho(x)\,dx < \infty\}. \qquad (2.3.7)$$

Indeed, let

$$K^p := \Big\{u \in L^1_{\mathrm{loc}}(\Omega) : \text{extended by zero outside } \Omega \text{ belongs to } W^{1,1}(\mathbf{R}^n) \text{ and } \int_\Omega |\nabla u(x)|^p \rho(x)\,dx < \infty\Big\}.$$

The fact that $W^p \subseteq K^p$ is obvious. To justify the inclusion $W^p \supseteq K^p$ we take any $u \in K^p$, so that u extended by zero outside Ω belongs to $W^{1,1}(\mathbf{R}^n)$. In particular this extension has prescribed trace $Tr : W^{1,1}(\Omega) \to L^1(\partial\Omega)$ defined on $\partial\Omega$ (again by Theorem 6.4.2 in [57]). Therefore trace of u is zero as it must the the same as the one defined from outside Ω. This implies that u belongs to $W_0^{1,1}(\Omega)$ as this is the kernel of trace operator defined on the boundary. As we pointed out already in Remark 2.3.6 the space W^p can be essentially larger than the space $W^{1,p}_{\rho,0}(\Omega)$.

Therefore in the paper [20], when considering the case $p = q$, $v = w =: \rho$, the authors deal with Poincaré type inequalities defined on the possibly larger space of admissible functions than we do.

Remark 2.3.8. When $n = 1$ the following statement allows to characterize weights admissible to (2.3.6) on the space W^p given by (2.3.7).

Proposition 2.3.9 ([20], Proposition 2.7 p.19)**.** *Inequality* (2.3.6) *holds for any open set $\Omega \subseteq \mathbf{R}$ having fixed finite measure, any weights w and v with prescribed rearrangements and any absolutely continuous function u defined on Ω such that $u \equiv 0$ on $\partial\Omega$ if and only if $w \in L^1$ and $v^{-1/(p-1)} \in L^1$.*

Before we prove Theorem 2.3.4 we recall Baire Theorem. Our formulation comes from [58], Chapter XIV, § 3.

Theorem 2.3.10 (Baire Theorem, [8])**.** *In a non-empty complete metric space the union*

$$E = F_1 \cup F_2 \cup \cdots \cup F_k \cup \cdots$$

of closed boundary sets can not fill the entire space; furthermore, this union is a boundary set.

We will be using the following well known corollary coming from Baire Theorem.

Corollary 2.3.11. *If X is a complete space and $X = \bigcup_n K_n$ where K_n are closed then at least one of the sets contains a ball.*

We are in the position to prove Theorem 2.3.4.

Proof of Theorem 2.3.4. Equivalence condition "$i) \iff iii)$" is obvious.
"$iii) \implies ii)$" : This part of the proof is based on the Banach-Hahn Theorem on extension of functionals and is the simple modification of the proof of Proposition 2.3.1. We omit the details and only give the hint.
Consider the mapping: $\Phi : X \to L^p_\rho(\Omega; \mathbb{R}^n)$, defined by expression

$$X \ni h \mapsto \nabla h \in L^p_\rho(\Omega; \mathbb{R}^n)$$

and let us choose the norm $\|\cdot\|_2$ on X. By part iii), Φ is an isometry between X and its range $Y := \mathrm{Im}\Phi$, which is closed subspace of $L^p_\rho(\Omega;\mathbb{R}^n)$. Now the proof follows the same lines as the proof of Proposition 2.3.1, with the only difference that first coordinate (contributed to the part involving g_0 and f) is omitted in the analysis because of iii).
"$ii) \implies i)$" : This part of the proof exploits the Baire Cathegory method and seems to us more difficult than the proof of part "$i) \implies ii)''$. The proof follows by steps.
Step 1.
We prove that there exists constant N such that for every element $x^* \in X^*$ there exists $g \in L^{p'}_\tau(\Omega;\mathbb{R}^n)$ satisfying $\operatorname{div} g = x^*$ and

$$\|g\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq N\|x^*\|_{X^*}. \tag{2.3.8}$$

The proof will follow by sequence of substeps.
Substep a).
Take $x^* \in X^*$ and consider sets

$$T_{x^*} := \{g \in L^{p'}_\tau(\Omega;\mathbb{R}^n) : x^* = \operatorname{div} g\}.$$

By the assumption ii) we easily see that the set T_{x^*} is nonempty, weakly closed and convex. By Mazur's Theorem (see e.g. Theorem 2, Section V in [91]) it is also strongly closed. There exists uniquely defined function $g_{x^*} \in T_{x^*}$ such that

$$\|g_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} = \min\{\|g\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} : g \in T_{x^*}\}.$$

This follows from general fact that for a normed strictly convex Banach space X (i.e. such that there are not segments laying on the boundary of any given ball) and any convex set $C \subseteq X$ every point of X has at most one nearest point in C, see e.g. p.112-113 in [16]. We use the fact that $L^{p'}_\tau(\Omega;\mathbb{R}^n)$ is the strictly convex space, what follows from Minkowski inequality.

Substep b). We show that:
a) triangle inequality is satisfied:

$$\|g_{x^*+y^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \|g_{x^*} + g_{y^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \|g_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} + \|g_{y^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)}; \tag{2.3.9}$$

b) for any $t \in \mathbb{R}$ we have

$$g_{tx^*} = tg_{x^*}. \tag{2.3.10}$$

Part a) is obvious, while for the proof of part b) we take $t \in \mathbb{R}$ and $x^* \in X^*$ and verify that $tg_{x^*} \in T_{tx^*}$. Suppose by contradiction that $tg_{x^*} \neq g_{tx^*}$. Consequently $\|g_{tx^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} < \|tg_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)}$. This is equivalent to inequality

$$\left\|\frac{1}{t} g_{tx^*}\right\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} < \|g_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)}. \tag{2.3.11}$$

On the other hand, $\frac{1}{t} g_{tx^*} \in T_{x^*}$, therefore $\|g_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \|\frac{1}{t} g_{tx^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)}$, what contradicts (2.3.11).

Substep c). For every $N \in \mathbb{N}$ we consider the following subsets in X^*

$$K_N := \{x^* \in X^* : \|g_{x^*}\|_{L^{p'}_\tau(\Omega,\mathbb{R}^n)} \leq N\|x^*\|_{X^*}\}.$$

We show that sets K_N possesses the following properties:
a) they are closed with respect to the strong topology of X^*;
b) $K_1 \subseteq K_2 \subseteq \ldots K_N$ and

$$X^* = \bigcup_{N \in \mathbb{N}} K_N \ ;$$

c) when $x^* \in K_N$ and $t \in \mathbb{R}$ then $tx^* \in K_N$.

Property b) is obvious. To prove part a) assume that $x^*_k \in K_N$ and $x^*_k \to x^*$ strongly in X^*, in particular $\|x^*_k\|_{X^*} \to \|x^*\|_{X^*}$ as $k \to \infty$. We deduce that sequence $\{g_{x^*_k}\}_{k \in \mathbb{N}}$ is bounded, therefore we can extract the subsequence $\{g_{x^*_{k_l}}\}_{l \in \mathbb{N}}$ converging weakly in $L^{p'}_\tau(\Omega;\mathbb{R}^n)$ to some g. By the very definition of $\operatorname{div} g$, given by (2.3.2), we verify that sequence $\{\operatorname{div} g_{x^*_{k_l}}\}_{l \in \mathbb{N}}$ converges to $\operatorname{div} g$ weakly in X^*, and at the same time it strongly converges to x^*. Therefore $g \in T_{x^*}$ and

$$\begin{aligned}\|g_{x^*}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \|g\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \liminf_{l \to \infty} \|g_{x^*_{k_l}}\|_{L^{p'}_\tau(\Omega;\mathbb{R}^n)} \leq \liminf_{l \to \infty} N\|x^*_{k_l}\|_{X^*} \\ = N\|x^*\|_{X^*}.\end{aligned}$$

This implies that $x^* \in K_N$ and ends the proof of part a).
Property c) is direct consequence of (2.3.10).

Substep d). We show that there exists $N \in \mathbb{N}$ and $r \in (0, \infty)$ such that $B(0, r) \subseteq K_N$. Indeed, according to Corollary 2.3.11 there exists $r \in (0, \infty)$,

$x_0^* \in X^*$ and $N_0 \in \mathbb{N}$ such that $\overline{B(x_0^*, r)} \subseteq K_{N_0}$. We prove that there exists $N \in \mathbb{N}$ such that $B(0,r) \subseteq K_N$, i.e. by changing possibly N_0 one can suppose x_0^* to be zero. Indeed, when $y^* \in X^*$ is such that $\|y^*\|_{X^*} = r$, we have

$$\begin{aligned}\|g_{y^*}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} &= \|g_{(y^*+x_0^*)-x_0^*}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} \overset{(2.3.9)}{\leq} \|g_{(y^*+x_0^*)}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} + \\ \|g_{-x_0^*}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} &\overset{(2.3.10)}{=} \|g_{(y^*+x_0^*)}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} + \|g_{x_0^*}\|_{L_\tau^{p'}(\Omega;\mathbb{R}^n)} \\ &\leq N_0\|y^*+x_0^*\|_{X^*} + N_0\|x_0^*\|_{X^*} \leq N_0\left(2\|x_0^*\|_{X^*} + r\right) \\ &= \frac{N_0\left(2\|x_0^*\|_{X^*}+r\right)}{r}\|y^*\|_{X^*} \leq N\|y^*\|_{X^*},\end{aligned}$$

where $N = \left[\frac{N_0\left(2\|x_0^*\|_{X^*}+r\right)}{r}\right] + 1$ and $[z]$ denotes the integer part of real number z. Therefore $\partial B(0,r) \subseteq K_N$. This implies that $B(0,r) \subseteq K_N$. Indeed, when $y^* \in B(0,r)$, we have $y^* = (\frac{\|y^*\|_{X^*}}{r}) \cdot (r\|\frac{y^*}{\|y^*\|_{X^*}}) =: t \cdot y_1$ where $y_1 \in \partial B(0,r)$. Therefore this property follows from Substep C), part c).

Substep e). We finish the proof of Step 1). By previous substep and property c) of Substep C) we obtain

$$K_N \supseteq \bigcup_{t>0} tB(0,r) = X^*,$$

which finishes the proof of Step 1.

Step 2. We finish the proof of the implication $ii) \implies i)$. Take any $f \in X$. By Banach-Hahn Theorem (Corollary 2 of Section IV.6 in [91]) there exists $x^* = x_f^* \in X^*$ satisfying $\|x^*\|_{X^*} = 1$ and $\|f\|_X = |(f, x^*)|$. According to already proven Step 1 there exists $g = (g_1, \ldots, g_n) \in L_\tau^{p'}(\Omega, \mathbb{R}^n)$ such that $\mathrm{divg} = x^*$ and inequality (2.3.8) holds. This implies

$$\begin{aligned}\|f\|_X &= |(f, x^*)| = |(f, \operatorname{div} g)| = \left|\int_\Omega (\nabla f(x), g(x))\, dx\right| \\ &\leq \|\nabla f\|_{L_\rho^p(\Omega,\mathbb{R}^n)}\|g\|_{L_\tau^{p'}(\Omega,\mathbb{R}^n)} \leq \|\nabla f\|_{L_\rho^p(\Omega,\mathbb{R}^n)} N\|x^*\|_{X^*} \\ &\qquad = N\|\nabla f\|_{L_\rho^p(\Omega,\mathbb{R}^n)}.\end{aligned}$$

This ends the proof of Step 2 and of the theorem. $\square$

Remark 2.3.12 (conditions (i)-(iii) and constant functions). One would think at first glance that conditions (i),(ii),(iii) are equivalent to the condition iv): $\chi_\Omega \notin X$ as the implication "$ii) \Longrightarrow iv)$" is obvious. The case of $\Omega = \mathbb{R}^n$ and $\rho \equiv 1$ shows that converse implication does not hold.

Remark 2.3.13 (the solutions of $\operatorname{div} g = x^*$). For solutions of equation $\operatorname{div} g = x^*$ in the unweighted setting we refer e.g. to the papers [11, 43, 22, 89] and their references.

2.4 (P) $\Longleftrightarrow$ (I)

In this section we provide the information about links of Poincaré inequality with isoperimetric inequalities due to Maz'ya [64]. Reach discussion of the isoperimetric inequalities is presented in [65], Chapter 2.3.4. We start with the following definition of the capacity.

Definition 2.4.1 ([65] Definition 2.2.1). *Let $p \geq 1$, F be a compact subset of a domain $\Omega \subseteq \mathbb{R}^n$ and let $\Phi(x,\xi) \geq 0$ be a continuous function defined on $\Omega \times \mathbb{R}^n$ which is homogeneous of degree one with respect to ξ, i. e. $\Phi(x,t\xi) = |t|\Phi(x,\xi)$ for any $t \in \mathbf{R}$. The number*

$$(p,\Phi)\text{-cap}(F,\Omega) := \inf \left\{ \int_\Omega [\Phi(x,\nabla f)]^p \, dx : \; f \in C_0^\infty(\Omega), \; f \geq 1 \text{ on } F \right\}$$

is called the (p,Φ)-capacity of F relative to Ω.

We focus on the following property of the capacities.

Theorem 2.4.2 (Corollary 1 in [65], Chapter 2.3.4). *Let $\Omega \subseteq \mathbf{R}^n$ be a given domain, $\Phi(x,\xi) \geq 0$ be a continuous function in $\Omega \times \mathbb{R}^n$ which is homogeneous of degree one with respect to ξ and μ be the given measure on Ω. Then we have.*

1. *If there exists a constant β such that for any compact set $F \subset \Omega$*

$$\mu(F)^{\alpha p} \leq \beta \cdot (p,\Phi)\text{-cap}(F,\Omega), \tag{2.4.1}$$

where $p \geq 1,\ \alpha > 0,\ \alpha p \leq 1$, *then for all* $f \in C_0^\infty(\Omega)$

$$\|f\|^p_{L^q(\Omega,\mu)} \leq C \int_\Omega [\Phi(x,\nabla f)]^p\, dx, \tag{2.4.2}$$

where $q = \alpha^{-1}$ *and* $C \leq p^p(p-1)^{1-p}\beta$.

2. *If* (2.4.2) *holds for any* $f \in C_0^\infty(\Omega)$ *with constant* C *not depending on* f, *then* (2.4.1) *is valid for every compact set* $F \subset \Omega$ *with* $\alpha = q^{-1}$ *and* $\beta \leq C$.

As an immediate corollary one obtains the following statement. It shows that Poincaré inequality is equivalent to the validity of isoperimetric inequality (2.4.5) stated below. Consequently isoperimetric inequality (2.4.5) is also equivalent to the validity of conditions (i)-(iii) in the formulation of Theorem 2.3.4.

Corollary 2.4.3 (Poincaré inequality and isoperimetric inequality)**.** *Assume that* $\Omega \subseteq \mathbf{R}^n$ *is a given domain,* $p \geq 1$, $\rho \in L^1_{loc}(\Omega)$ *be a given weight. Let us consider* $\Phi(x,\lambda) = \rho(x)^{1/p}|\lambda|$, *in particular,* $W^{1,p}_\rho\text{-cap}(F,\Omega) :=$

$$(p,\Phi)\text{-cap}(F,\Omega) = \inf\left\{\int_\Omega |\nabla f(x)|^p \rho(x)\, dx \ :\ f \in C_0^\infty(\Omega),\ f \geq 1 \text{ on } F\right\}. \tag{2.4.3}$$

The following conditions are equivalent.

i) *Poincaré inequality:*

$$\|f\|_{L^p_\rho(\Omega)} \leq C\|\nabla f\|_{L^p_\rho(\Omega)} \tag{2.4.4}$$

holds for every $f \in W^{1,p}_{\rho,0}(\Omega)$ *with constant* C *independent on* f;

iv) *There exists a constant* $\beta > 0$ *such that for any compact set* $F \subset \Omega$

$$\mu(F) := \int_F \rho(x)\, dx \leq \beta \cdot W^{1,p}_\rho\text{-cap}(F,\Omega). \tag{2.4.5}$$

Moreover, if (2.4.4) *holds with constant* C *then* (2.4.5) *holds with a constant* $\beta \leq C$, *while if* (2.4.5) *holds with constant* β *then* (2.4.4) *holds with a constant* $C \leq p(p-1)^{(1-p)/p}\beta$.

Proof. We apply Theorem 2.4.2 with given Φ and μ in the particular case: $\alpha = 1/p$, $p = q$. $\square$

Remark 2.4.4. In general the verification of isoperimetric inequality (2.4.5) is not possible on every compact set. Therefore the constructions of Poincaré inequalities (2.4.4) can serve as a tool to derive (2.4.5).

2.5 (P) $\Longleftrightarrow$ (S)

We are now to discuss the solvability of equation $\operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) = x^*$ in the weighted Sobolev space $X = W^{1,p}_\rho(\Omega)$, where $x^* \in X^*$. The arguments presented here are based on application of Minty Browder Theorem [15, 67] and so we are applying the theory of monotone operators. We start with the following definition.

Definition 2.5.1 (Definition 25.2. in [92])**.** Let $A : D(A) \subset X \to X^*$ be the given operator defined on a normed linear space X. Then A is called

monotone if
$$\langle Au - Av, u - v\rangle \geq 0 \quad \text{forall} \quad u, v \in D(A);$$

coercive if
$$\lim_{\|u\|\to\infty} \frac{< Au, u >}{\|u\|} = +\infty;$$

hemicontinuous if the map
$$t \to < A(u + tv), w >$$
is continuous on $[0, 1]$ for all $u, v, w \in X$.

We will be applying the following variant of the Minty Browder Theorem.

Theorem 2.5.2 (Theorem 26.A in [92])**.** *Let X be a real reflexive Banach space and $A : X \to X^*$ monotone, hemicontinuous and coercive. Then A is surjective.*

As the corollary we arrive at the following statement, which is further extension of Theorem 2.3.4. To our best knowledge the implication $v) \Longrightarrow i)$ was not known so far.

Theorem 2.5.3 (Poincaré inequality and solvability of elliptic PDE)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain, $\rho \in B_p(\Omega) \cap L^1_{loc}(\Omega)$, $\tau = \rho^{-\frac{1}{p-1}}$ is nonnegative a.e., $p > 1$, $X = W^{1,p}_{\rho,0}(\Omega), X^* = (W^{1,p}_{\rho,0}(\Omega))^*$. The following conditions are equivalent:*

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^p \rho(x)\, dx \leq C \int_\Omega |\nabla f(x)|^p \rho(x)\, dx$$

holds for every $f \in X$ with constant C independent on f.

v) *for any $x^* \in X^*$ and $w \in W^{1,p}_\rho(\Omega)$ there exists a function $u \in X$ which solves the equation*

$$\begin{cases} \operatorname{div}\left(\rho(x)|\nabla u|^{p-2}\nabla u\right) = x^*, \\ \qquad\qquad u - w \in X. \end{cases} \tag{2.5.1}$$

Proof. Let us consider equivalent conditions i)-iii) in Theorem 2.3.4.
The implication "$v) \Longrightarrow i)$" follows from $v) \Longrightarrow ii)$. For this we use the fact that $g := \rho|\nabla u|^{p-2}\nabla u \in L^{p'}_\tau(\Omega, \mathbf{R}^n)$. Indeed, as $||\nabla u|^{p-2}\nabla u| = |\nabla u|^{p-1} \in L^{p'}_\rho(\Omega)$, we have $|g|^{p'}\tau = |\nabla u|^p \rho^{-\frac{1}{p-1}}\rho^p = |\nabla u|^p\rho \in L^1(\Omega)$.
To complete the proof of the statement we will prove the implication $iii) \Longrightarrow v)$. For this we take $w \in X$ and apply Theorem 2.5.2 to the operator

$$Au := -\operatorname{div}\left(\rho|\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w)\right),$$

acting as functional on X by expression

$$\langle Au, v\rangle = \int_\Omega \rho|\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w)\cdot\nabla v\, dx.$$

Note that according to Remark 2.3.2, part 6), the space X is reflexive. Let us choose the norm $\|u\|_2$ on X and denote it by $\|\cdot\|$. Hölder inequality:

$$|\langle Au, v\rangle| \leq \left(\int_\Omega |\nabla u + \nabla w|^p \rho\, dx\right)^{1-\frac{1}{p}} \left(\int_\Omega |\nabla v|^p\rho\, dx\right)^{\frac{1}{p}} = \|u + w\|^{p-1}\|v\| \tag{2.5.2}$$

shows that Au is well defined functional on X, so that $A : X \to X^*$. Using the inequality (see e.g. [21], Lemma 2.1)

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \geq \gamma |a-b|^2 \left(|a|^{p-2} + |b|^{p-2}\right) \tag{2.5.3}$$

applied to $a = \nabla u_1 + \nabla w$ and $b = \nabla u_2 + \nabla w$ (where $\gamma > 0$ is independent on a, b), one easily shows that A is monotone.
To prove its coerciveness we provide the following computations:

$$\begin{aligned}
\frac{\langle Au, u\rangle}{\|u\|} &= \frac{\int_\Omega |\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w)\cdot \nabla u \rho \, dx}{\|u\|} \\
&= \frac{\|u+w\|^p}{\|u\|} - \frac{\int_\Omega |\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w)\cdot \nabla w \rho \, dx}{\|u\|} \\
&= \frac{\|u+w\|^p}{\|u+w\|}\frac{\|u+w\|}{\|u\|} + a(w,u) = \|w+u\|^{p-1} b(w,u) + a(w,u),
\end{aligned}$$

where

$$\begin{aligned}
a(w,u) &:= -\frac{\int_\Omega |\nabla u + \nabla w|^{p-2}(\nabla u + \nabla w)\cdot \nabla w \rho \, dx}{\|u\|}, \\
b(w,u) &:= \frac{\|u+w\|}{\|u\|} \overset{\|u\|\to\infty}{\longrightarrow} 1.
\end{aligned}$$

Applying (2.5.2) with $v = w$ we obtain

$$|a(w,u)| \leq \frac{\|u+w\|^{p-1}}{\|u\|}\|w\| = \|u+w\|^{p-2} b(w,u)\|w\|,$$

consequently

$$\begin{aligned}
\frac{\langle Au, u\rangle}{\|u\|} &\geq b(w,u)\left(\|u+w\|^{p-1} - \|u+w\|^{p-2}\|w\|\right) \\
&= b(w,u)\|u+w\|^{p-1}\left(1 - \frac{\|w\|}{\|u+w\|}\right) \overset{\|u\|\to\infty}{\to} \infty.
\end{aligned}$$

Hemicontinuity property is direct consequence of Lebesgue's Dominated Convergence Theorem.
According to Theorem 2.5.2, for any $x^* \in X^*$ equation

$$\begin{cases} \operatorname{div}\left(\rho(x)|\nabla f + \nabla w|^{p-2}\nabla(f+w)\right) = x^*, \\ \hfill f \in X, \end{cases} \tag{2.5.4}$$

admits the solution f. The substitution of $u := f + w$ gives the solution of the original problem (2.5.1). □

Remark 2.5.4. Similar techniques appear in the paper by Cavalheiro [17].

Remark 2.5.5. Solvability of problem (2.5.1) implies its uniqueness as an easy consequence of (2.5.3).

2.6 Conclusion. Main result

As direct conclusion resulting from Theorem 2.3.4, Corollary 2.4.3 and Theorem 2.5.2 we obtain the following result.

Theorem 2.6.1 (main result)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain, $\rho \in B_p(\Omega) \cap L^1_{loc}(\Omega)$, $\tau = \rho^{-\frac{1}{p-1}}$, $p > 1$, $X = W^{1,p}_{\rho,0}(\Omega), X^* = (W^{1,p}_{\rho,0}(\Omega))^*$. The following conditions are equivalent:*

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^p \rho(x)\, dx \leq C \int_\Omega |\nabla f(x)|^p \rho(x)\, dx \tag{2.6.1}$$

holds for every $f \in X$ with constant C independent on f.

ii) *For every $x^* \in X^*$ there exists function $g = (g_1, \ldots, g_n) \in L^{p'}_\tau(\Omega, \mathbb{R}^n)$ such that for any $f \in X$*

$$\langle x^*, f \rangle = -\sum_{i=1}^n \int_\Omega \partial_i f(x) g_i(x)\, dx = \langle \operatorname{div} g, f \rangle.$$

iii) *Norms: $\|f\|_1 := \|f\|_{L^p_\rho(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega, \mathbb{R}^n)}$ and $\|f\|_2 := \|\nabla f\|_{L^p_\rho(\Omega, \mathbb{R}^n)}$ are equivalent on X.*

iv) *There exists a constant β such that for any compact set $F \subset \Omega$*

$$\mu(F) := \int_F \rho(x)\, dx \leq \beta \cdot W^{1,p}_\rho\text{-cap}(F, \Omega). \tag{2.6.2}$$

v) *For any $x^* \in X^*$ and $w \in W^{1,p}_\rho(\Omega)$ there exists function $u \in W^{1,p}_\rho(\Omega)$ which solves the equation*

$$\begin{cases} \operatorname{div}(\rho(x)|\nabla u|^{p-2}\nabla u) & = & x^*, \\ u - w & \in & X. \end{cases}$$

Moreover, if (2.6.1) *holds with constant* C *then* (2.6.2) *holds with constant* $\beta \leq C$*, while if* (2.6.2) *holds with constant* β *then* (2.6.1) *holds with* $C \leq p(p-1)^{(1-p)/p}\beta$.

It appears that Poincaré inequalities self-improve and we have the following statement being direct conclusion resulting from the above result.

Theorem 2.6.2 (self-improving conditions)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be an arbitrary domain,* $\rho \in B_{p_0}(\Omega) \cap L^1_{loc}(\Omega)$, $\tau_p := \rho^{-\frac{1}{p-1}}$ *be nonnegative a.e.,* $1 < p, p_0 < \infty$, $X_p := W^{1,p}_{\rho,0}(\Omega)$, $X_p^* := (W^{1,p}_{\rho,0}(\Omega))^*$ *and assume that Poincaré inequality*

$$\int_\Omega |f(x)|^{p_0} \rho(x)\, dx \leq C \int_\Omega |\nabla f(x)|^{p_0} \rho(x)\, dx \tag{2.6.3}$$

holds for every $f \in X_{p_0}$. *Then for every* $1 < p_0 \leq p < \infty$ *the following conditions hold:*

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^{p} \rho(x)\, dx \leq C^{\frac{p}{p_0}} \left(\frac{p}{p_0}\right)^p \int_\Omega |\nabla f(x)|^{p} \rho(x)\, dx \tag{2.6.4}$$

holds for every $f \in X_p$.

ii) *For every* $x^* \in (X_p)^*$ *there exists function* $g = (g_1, \ldots, g_n) \in L^{p'}_{\tau_p}(\Omega, \mathbb{R}^n)$ *such that for any* $f \in X_p$

$$\langle x^*, f\rangle = -\sum_{i=1}^n \int_\Omega \partial_i f(x) g_i(x)\, dx = \langle \operatorname{div} g, f\rangle.$$

iii) *Norms:* $\|f\|_{1,p} := \|f\|_{L^p_\rho(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega,\mathbb{R}^n)}$ *and* $\|f\|_{2,p} := \|\nabla f\|_{L^p_\rho(\Omega,\mathbb{R}^n)}$ *are equivalent on* X_p.

iv) *There exists a constant* $\beta_p \leq C^{\frac{p}{p_0}} \left(\frac{p}{p_0}\right)^p$ *such that for any compact set* $F \subset \Omega$

$$\mu(F) := \int_F \rho(x)\, dx \leq \beta_p \cdot W^{1,p}_\rho\text{-cap}(F, \Omega).$$

v) *For any $x^* \in (X_p)^*$ and $w \in W^{1,p}_\rho(\Omega)$ there exists uniquely defined function $v \in W^{1,p}_\rho(\Omega)$ which solves the equation*

$$\begin{cases} \operatorname{div}\left(\rho(x)|\nabla v|^{p-2}\nabla v\right) = x^*, \\ \qquad\qquad\qquad v - w \in X_p. \end{cases}$$

Proof. We apply Theorem 2.6.1. It suffices to check that Poincaré inequality (2.6.3) implies (2.6.4) with every $p \geq p_0$. Observe that when $u \in C_0^\infty(\Omega)$ we have $|u|^{p/p_0} \in Lip(\Omega)$, and so $|u|^{p/p_0} \in X_p$ (see Remark 2.3.2, part 5)). The implication follows now from chain of inequalities:

$$\int_\Omega |u|^p \rho\, dx = \int_\Omega \left(|u|^{\frac{p}{p_0}}\right)^{p_0} \rho\, dx \leq C \int_\Omega \left|\nabla \left(|u|^{\frac{p}{p_0}}\right)\right|^{p_0} \rho\, dx$$

$$= C \int_\Omega \left|\frac{p}{p_0}|u|^{\frac{p}{p_0}-1} \mathrm{sgn} u \nabla u\right|^{p_0} \rho\, dx = C \left(\frac{p}{p_0}\right)^{p_0} \int_\Omega \left(|u|^{\frac{p}{p_0}-1}|\nabla u|\right)^{p_0} \rho\, dx$$

$$\leq C \left(\frac{p}{p_0}\right)^{p_0} \left(\int_\Omega |u|^p \rho\, dx\right)^{1-\frac{p_0}{p}} \left(\int_\Omega |\nabla u|^p \rho\, dx\right)^{\frac{p_0}{p}}.$$

In the last line above we have applied Hölder inequality with $q = \frac{p}{p_0}$ and $q' = \frac{p}{p-p_0}$. Now it suffices to rearrange the above inequality. □

Remark 2.6.3. Note that the condition $\rho \in B_{p_0}(\Omega)$ implies $\rho \in B_p(\Omega)$ for every $p \geq p_0$. Indeed, we have

$$\int_V \rho^{-\frac{1}{p-1}}\, dx = \int_V \left(\rho^{-\frac{1}{p_0-1}}\right)^{\frac{p_0-1}{p-1}} \cdot 1\, dx$$

and it suffices to apply Hölder inequality with $q = \frac{p-1}{p_0-1} > 1$ and q'.

Remark 2.6.4. Poincaré inequalities imply some other interesting inequalities as well, like for example interpolation inequalities ([18, 19, 45]) and Sobolev type inequalities ([38, 59]).

Open Problem 2.6.5. It would be interesting to know if there is some class of weights for which Poincaré inequality (2.6.3) implies Poincaré inequality $\int_\Omega |f(x)|^p \rho(x)\, dx \leq C_p \int_\Omega |\nabla f(x)|^p \rho(x)\, dx$, where p can be taken also smaller than p_0. Such question is inspired by the interesting result known in the metric setting [47], where the author deals with another type of Poincaré inequalities, the so called $(1,p)$-Poincaré inequalities which have the local form.

2.7 Constructions of Poincaré inequalities and applications

2.7.1 Constructions of Poincaré inequalities

New constructions.
In this subsection at first we focus mostly on the convenient new method to construct Hardy-type inequalities due to Skrzypczak [79]. The method shows how to construct the two-parameter family of Hardy-type inequality from solutions of the Partial Differential Inequality $-\Delta_p u \geq \Phi$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. We will need the following definitions.

Definition 2.7.1. *Let $p > 1$, Ω be any open subset of $\mathbb{R}^n$ and Ψ be the locally integrable function defined on Ω such that for every nonnegative compactly supported function $w \in W^{1,p}(\Omega)$,*

$$\int_\Omega \Psi w \, dx > -\infty.$$

Let $p > 1$, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $u \not\equiv 0$ a.e.. We say that

$$-\Delta_p u \geq \Psi \tag{2.7.1}$$

if for every non-negative compactly supported function $w \in W^{1,p}(\Omega)$, we have

$$\langle -\Delta_p u, w\rangle := \int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla w\rangle \, dx \geq \int_\Omega \Psi w \, dx.$$

Definition 2.7.2 ((Ψ, p)-condition)**.** *Suppose u and Ψ are as in Definition 2.7.1 and moreover there exists*

$$\sigma_0 := \inf\{\sigma \in \mathbb{R} : \Psi \cdot u + \sigma |\nabla u|^p \geq 0 \text{ a.e. in } \Omega \cap \{u > 0\}\} \in \mathbb{R}, \tag{2.7.2}$$

where we set $\inf \emptyset = +\infty$.

The following statement is of our particular interest.

Theorem 2.7.3 (Hardy inequality, [79] Theorem 4.1)**.** *Suppose that* $1 < p < \infty$ *and there exists a non-negative solution* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *to PDI* (2.7.1) *in the sense of Definition 2.7.1, where* Ψ *is locally integrable and satisfies* (Ψ, p) *with* $\sigma_0 \in \mathbb{R}$*, given by* (2.7.2)*. Assume further that* β *and* σ *are arbitrary numbers such that* $\beta > 0$ *and* $\beta > \sigma \geq \sigma_0$*. Then for every Lipschitz function* f *with compact support in* Ω*, we have*

$$\int_\Omega |f(x)|^p \ \mu_{1,\sigma,\beta}(dx) \leq \int_\Omega |\nabla f(x)|^p \ \mu_{2,\sigma,\beta}\, dx,$$

where $\mu_{1,\sigma,\beta}, \mu_{2,\sigma,\beta}$ *are two parameter families of measures, given by*

$$\begin{aligned}
\mu_{1,\sigma,\beta}(dx) &:= \left(\frac{\beta - \sigma}{p-1}\right)^{p-1} [\Psi \cdot u + \sigma |\nabla u|^p] \cdot u^{-\beta-1} \chi_{\{u>0\}}\, dx,\\
\mu_{2,\sigma,\beta}(dx) &:= u^{p-\beta-1} \chi_{\{|\nabla u| \neq 0\}}\, dx.
\end{aligned}$$

Our goal is to adopt this general method to the nonnegative solutions to the PDI: $-\Delta_p u \geq C u^{p-1}$. This leads to the following construction of Poincaré inequality.

Corollary 2.7.4 (construction of Poincaré inequality)**.** *Suppose that* $1 < p < \infty$ *and* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *is the non-negative solution to PDI* $-\Delta_p u \geq C u^{p-1}$ *in the sense of Definition 2.7.1. Assume further that* $\beta > 0$ *is an arbitrary number. Then for every Lipschitz function* f *with compact support in* Ω*, we have*

$$\int_\Omega |f(x)|^p \ \rho_\beta(x)\, dx \leq \frac{1}{C} \left(\frac{p-1}{\beta}\right)^{p-1} \int_\Omega |\nabla f(x)|^p \ \rho_\beta(x)\, dx, \qquad (2.7.3)$$

where ρ_β *is the one parameter weight, given by*

$$\rho_\beta(x) \ := \ (u(x))^{p-\beta-1}\, \chi_{u(x)>0}.$$

Proof. At first we note that $\Psi = C u^{p-1} \in L^{p/(p-1)}_{loc}(\Omega) \subseteq L^1_{loc}(\Omega)$ and $\sigma_0 \leq 0$ in (2.7.2). The choice of $\beta > 0, \sigma = 0$ and direct computations lead to the desired result. $\square$

Remark 2.7.5. Density argument (Remark 2.3.2, part 5)) shows that when $\rho_\beta \in L^1_{loc}(\Omega)$, then inequality (2.7.3) holds for every $f \in W^{1,p}_{0,\rho_\beta}(\Omega)$.

Remark 2.7.6 (derivation of classical Poincaré inequality with best constant)**.** Plugging $\beta := p-1$ we retrieve the classical Poincaré inequality: $\int_\Omega |f(x)|^p\,dx \le \frac{1}{C}\int_\Omega |\nabla f(x)|^p\,dx$. Moreover, it is achieved with best constant when $-\Delta_p u = Cu^{p-1}$. It is because when the nonnegative function u solves the PDE $-\Delta_p u = C|u|^{p-2}u$ with constant C, then C^{-1} is best constant in Poincaré inequality. Otherwise u should have to change its sign inside Ω (see [5] or Remark 1 on page 163 in [61]). Therefore Corollary 2.7.4 allows to retrieve classical Poincaré inequality with best constant as the special case.

Remark 2.7.7 (characterization of principal eigenvalue by the pointwise condition)**.** When $u \in W^{1,p}_\rho(\Omega)$ is nonnegative and solves $-\Delta_p u \ge C_1 u^{p-1}$ and $u_0 \in W^{1,p}_{\rho,0}(\Omega)$ solves $-\Delta_p u_0 = Cu_0^{p-1}$, we have $C_1 \le C$. This is because first inequality leads to Poincaré inequality with constant C_1^{-1}, while second one to Poincaré inequality with constant C^{-1} which is best constant, see our previous remark. Therefore $C_1^{-1} \ge C^{-1}$. This leads to the following interpretation of principal eigenvalue:

$$\begin{aligned} C := \sup\{C(u) = \inf_{x\in\Omega} \frac{-\Delta_p u}{u^{p-1}} \cdot \chi_{\{u>0\}} :& \\ u \in W^{1,p}_\rho(\Omega), u \ge 0, u \not\equiv 0 \text{ a.e.}, \Delta_p u \in L^1_{\mathrm{loc}}(\Omega), \Delta_p u \le 0\}.& \end{aligned}$$

A_p **weights.**

Our further discussion here is related to A_p weights. Suppose that $\rho \in A_p$ $(1 < p < 1)$, that is ρ satisfies the A_p Muckenhoupt condition ([69], page 214)

$$\sup_Q \frac{1}{|Q|}\int_Q \rho\,dx \left\{\frac{1}{|Q|}\int_Q \rho^{-1/p-1}\,dx\right\}^{p-1} < \infty,$$

where Q are cubes in $\mathbf{R}^n$. Clearly, $A_p \subseteq B_p(\Omega)$. Let Mf denote the Hardy-Littlewood maximal function of the locally integrable function f, i.e.

$$Mf(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B |f|\,dx,$$

where the supremum is taken over all balls containing x. Muckenhoupt Theorem (Theorem 9 on page 222 in [69], [82]) states that for $1 < p < 1$ the operator Mf is bounded in L^p if and only if $\rho \in A_p$.

The following statement is known (see Section 1.5, [41]). For reader convenience we present its direct proof which is based on Hedberg inequality from [40] and integral representation (see e.g. [65], Theorem 2, Section 1.1.10).

Proposition 2.7.8. *If ρ is an A_p weight and Ω is bounded domain then Poincaré inequality* (2.6.1) *holds with ρ and p.*

Proof. We recall the following facts.

Fact 2.7.9 ([40], Lemma.(a), page 506)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. If $1 < n$, then for all $x \in \mathbb{R}^n$ and $\delta > 0$*

$$\int_{|y-x|\leq\delta} |f(y)||y-x|^{1-n} dy \leq C\delta Mf(x), \text{ a.e.},$$

with a constant independent of f.

Fact 2.7.10 ([65], Theorem 2, Section 1.1.10., pp-19)**.** *For any $f \in C_0^\infty(\mathbf{R}^n)$ we have*

$$f(x) = \sum_{i=1}^n K_i * \frac{\partial f}{\partial x_i}(x) \text{ a.e., } \quad \text{where} \quad K_i(x) = -\frac{1}{n\omega_n}\frac{x_i}{|x|^n},$$

ω_n denotes the volume of the unit ball in $\mathbf{R}^n$.

Using the above facts, for any $f \in C_0^\infty(\Omega)$ where $\Omega \subseteq B(R)$ and every $x \in \Omega$, we have

$$\begin{aligned} |f(x)| &\leq \int_{\mathbf{R}^n} \frac{a}{|x-y|^{n-1}}|\nabla f(y)|dy = \int_{B(R)} \frac{a}{|x-y|^{n-1}}\chi_{\{|x-y|\leq 2R\}}|\nabla f(y)|dy \\ &= \left(\frac{a}{\{|z|^{n-1}\}}\chi_{|z|\leq 2R}\right) * |\nabla f|(x) \leq C_1 M(|\nabla f|\chi_\Omega)(x). \end{aligned}$$

This implies

$$\begin{aligned} \int_\Omega |f(x)|^p \rho(x)\, dx \leq & C_1^p \int_\Omega (M|\nabla f|(x))^p \rho(x)\, dx \leq C_1^p \int_{\mathbf{R}^n} (M|\nabla f|\chi_\Omega(x))^p \rho(x)\, dx \\ &\leq C_2 \int_{\mathbf{R}^n} (|\nabla f|\chi_\Omega(x))^p \rho(x)\, dx = C_2 \int_\Omega |\nabla f(x)|^p \rho(x)\, dx. \end{aligned}$$

Constants a, C_1, C_2 do not depend on x. This and the density argument complete the proof of the proposition. □

Remark 2.7.11. For discussion of the capacities in the weighted A_p setting we refer to [48].

2.7.2 Applications

Linking Corollary 2.7.4 with Theorem 2.6.2 we arrive at the following result, leading to the constructions of Poincaré inequalities, representation of functionals, as well as to the isoperimetric inequalities.

Theorem 2.7.12. *Suppose that $1 < p_0 < \infty$ and $u_0 \in W^{1,p_0}_{\mathrm{loc}}(\Omega)$ is the nonnegative solution to PDI: $-\Delta_{p_0} u_0 \geq C u_0^{p-1}$ in the sense of Definition 2.7.1. Let $\beta > 0$, $p > 1$ be arbitrary numbers,*

$$\rho_\beta(x) := (u_0(x))^{p_0-\beta-1}\, \chi_{u_0(x)>0}, \;\; \tau_{\beta,p}(x) = (u_0(x))^{\frac{\beta}{p-1} - \frac{p_0-1}{p-1}}\, \chi_{u_0(x)>0},$$

$$X_{\beta,p} := W^{1,p}_{0,\rho_\beta}(\Omega)$$

and assume that $\rho_\beta \in L^1_{loc}(\Omega) \cap B_{p_0}(\Omega)$.

Then for every $p \geq p_0$ we have:

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^p \rho_\beta(x)\, dx \leq C(p_0, \beta, p) \int_\Omega |\nabla f(x)|^p \rho_\beta(x)\, dx$$

holds for every $f \in X_{\beta,p}$, where $C(p_0,\beta,p) = [\frac{1}{C}\left(\frac{p-1}{\beta}\right)^{p-1}]^{\frac{p}{p_0}} (\frac{p}{p_0})^p$.

ii) *For every $x^* \in (X_{\beta,p})^*$ there exists function $g = (g_1, \ldots, g_n) \in L^{p'}_{\tau_{\beta,p}}(\Omega, \mathbb{R}^n)$, such that for any $f \in X_{\beta,p}$*

$$\langle x^*, f\rangle = -\sum_{i=1}^n \int_\Omega \partial_i f(x) g_i(x)\, dx = \langle \operatorname{div} g, f\rangle.$$

iii) *Norms: $\|f\|_{1,\beta,p} := \|f\|_{L^p_{\rho_\beta}(\Omega)} + \|\nabla f\|_{L^p_{\rho_\beta}(\Omega,\mathbb{R}^n)}$ and $\|f\|_{2,\beta,p} := \|\nabla f\|_{L^p_{\rho_\beta}(\Omega,\mathbb{R}^n)}$ are equivalent on $X_{\beta,p}$.*

iv) *There exists a constant $\bar{\beta}_p \leq C(p_0,\beta,p)$ such that for any compact set $F \subset \Omega$*

$$\mu_{\beta,p}(F) := \int_F \rho_\beta(x)\, dx \leq \bar{\beta}_p \cdot W^{1,p}_{\rho_\beta}\text{-cap}(F, \Omega).$$

v) *For any* $x^* \in X^*_{\beta,p}$ *and* $w \in W^{1,p}_{\rho_\beta}(\Omega)$ *there exists function* $u \in W^{1,p}_{\rho_\beta}(\Omega)$ *which solves the equation*

$$\begin{cases} \operatorname{div}\left(\rho_\beta(x)|\nabla u|^{p-2}\nabla u\right) = x^*, \\ \hfill u - w \in X_{\beta,p}. \end{cases}$$

Remark 2.7.13. The condition $\rho_\beta \in L^1_{loc}(\Omega) \cap B_{p_0}(\Omega)$ is satisfied if for example u is continuous and strictly positive inside Ω.

As direct consequence of Theorem 2.6.2 and Proposition 2.7.8 we obtain the following result.

Theorem 2.7.14. *Suppose that* $1 < p_0, p < \infty$, $\rho \in A_{p_0}$ *and* $\Omega \subseteq \mathbf{R}^n$ *is a bounded domain. Then statements i)-v) in Theorem 2.6.2 are satisfied with every* $p \geq p_0$.

Remark 2.7.15. Boundary value problem

$$\begin{cases} \operatorname{div}\left(\rho(x)|\nabla f + \nabla w|^{p-2}\nabla(f + w)\right) = x^*, \\ \hfill f \in X \end{cases}$$

is often presented as

$$\begin{cases} \operatorname{div}\left(\rho(x)|\nabla u|^{p-2}\nabla u\right) = x^*, \\ \hfill u = \bar{w} \in Y \text{ on } \partial\Omega, \end{cases}$$

where $u \in W^{1,p}_\rho(\Omega), \bar{w} \in Y$ and Y is some function space where functions are defined on the boundary of Ω (we interpret it as $\bar{w} = w|_{\partial\Omega}$). If we know that every element $\bar{w}$ of Y can be extended inside Ω to the function w which belongs to $W^{1,p}_\rho(\Omega)$, the problem can be transformed to (2.5.4) by substitution $f := u - w$. There are no many results about extension property in the general setting outside power weights. For results dealing with power weights we refer e.g. to [62, 88, 54, 90, 72], while for results in the general setting we refer to our recent results [23, 25].

Chapter 3

Proper formulation of the boundary condition when the trace embedding theorems are missing

This chapter is based on a joint work with my supervisor Agnieszka Kałamajska. My contribution is to formulate the necessary applications of the problem to the BVPs in an weighted sapce and get the uniqueness of BVPs regardless the choice of extension operators and the necessary edifications throughout the research paper. Here we propose certain interpretation of boundary condition $f = g$ on $\partial\Omega$, the boundary of Ω, when f belongs to weighted Sobolev space $W^{1,p}_{\rho}(\Omega)$ subordinated to the integrable weight $\rho(x)$ and g is defined on $\partial\Omega$, where $\Omega \subseteq \mathbf{R}^n$ is a given domain. It may happen that the trace embedding theorems are missing. Our proposed interpretation requires to have defined an extension operator $\mathrm{Ext} : X \to W^{1,p}_{\rho}(\Omega)$, where X is the relevant function space of functions defined on $\partial\Omega$. We show that the proposed interpretation does not depend on the choice of extension operator Ext. Moreover, we apply our result to obtain existence and uniqueness of solutions to an example of the boundary value problem involving degenerated p-Laplacian with non-homogeneous boundary condition. Existence of the extension operator within the class of weights $\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega))$ when

$X = W_\omega^{1-1/p,p}(\partial\Omega)$ is certain weighted Slobodetskii space and Ω is Lipschitz boundary domain, is confirmed by our previous results [23, 25].

3.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be the given domain, $\partial\Omega$ be its boundary, $\rho \in L^1(\Omega)$, $\rho \geq 0$, be the given weight function defined on Ω and $W_\rho^{1,p}(\Omega) := \Big\{ v \in L^1_{\rm loc}(\Omega) : v, \frac{\partial v}{\partial x_j} \in L^p_\rho(\Omega) \Big\}$ be the weighted Sobolev space, where $L^p_\rho(\Omega)$ is the weighted L^p-space defined on Ω. In this paper we are interested in two problems. At first, we consider the problem of proper formulation of boundary data $f = g$ on $\partial\Omega$ when function f belongs to weighted Sobolev space $W_\rho^{1,p}(\Omega)$ and g belongs to the well chosen function space X of functions defined on $\partial\Omega$. As main result of this approach we propose the interpretation of such boundary condition by exploiting the given extension operator $\mathrm{Ext} : X \to W_\rho^{1,p}(\Omega)$, then we prove independence of such interpretation on the choice of operator Ext. As next step we give an application of our first result to the solvability and uniqueness of an example of the nonhomogeneous boundary value problem which reads as follows:

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u(x)|^{p-2}\nabla u) = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1.1}$$

We start with the discussion of first approach.

When $\rho \equiv 1$, one has the classical trace operator $\mathrm{Tr} : W^{1,p}(\Omega) \mapsto W^{1-1/p,p}(\partial\Omega)$, being the surjection from Sobolev space $W^{1,p}(\Omega)$ to Slobodetskii space $W^{1-1/p,p}(\partial\Omega)$ [75, 81] provided that the domain Ω is sufficiently regular. When restricted to Lipschitz functions defined on $\overline{\Omega}$ it is the usual restriction $\mathrm{Tr}\, u = u|_{\partial\Omega}$. As Lipschitz functions are dense in $W^{1,p}(\Omega)$ for the sufficiently regular domain Ω, in that case we interpret the boundary data $u = g$ on $\partial\Omega$ as $\mathrm{Tr}\, u = g$, whenever $g \in W^{1-1/p,p}(\partial\Omega)$.

In general, situation at first glance we would like to follow the same schema. Namely, we would like to have defined linear continuous operator $\mathrm{Tr} : W_\rho^{1,p}(\Omega) \to X$, where X is the given function space of functions defined on the boundary of Ω, serving instead of $W^{1-1/p,p}(\partial\Omega)$ such that when u

is Lipschitz on $\overline{\Omega}$ then $\operatorname{Tr} u$ is the usual restriction of u to the boundary of Ω. Moreover, we would like to have Lipschitz functions dense in $W^{1,p}_{\rho}(\Omega)$. Unfortunately, in most situations this idea will not work as up to nowadays, except the situation of simple weights like e.g. powers of distances to the boundary of Ω, to some manifolds or points, the trace operator has not been found.

To overcome this difficulty we propose to involve the extension operator instead, which looks much easier to handle. Let X be the given function space of functions defined on $\partial\Omega$ and that there exists linear continuous operator

$$\operatorname{Ext} : X \to W^{1,p}_{\rho}(\Omega), \tag{3.1.2}$$

which obeys the following two properties: 1) when g is Lipschitz on $\partial\Omega$ then $\operatorname{Ext} g$ is the usual extension of g up to $\overline{\Omega}$, i.e. $\operatorname{Ext} g|_{\partial\Omega} = g$ and 2) Lipschitz functions are dense in X.
In general, we cannot directly restrict $\operatorname{Ext} g$ to the boundary of Ω when g is not Lipschitz. This is because we then only know that the function $\operatorname{Ext} g$ belongs to $W^{1,p}_{\rho}(\Omega)$, so it is defined only inside Ω.

We propose the following idea to interpret the boundary condition for a given $u \in W^{1,p}_{\rho}(\Omega)$. Having the extension operator $\operatorname{Ext} : X \to W^{1,p}_{\rho}(\Omega)$ as in (3.1.2) and $u \in W^{1,p}_{\rho}(\Omega),\ g \in X$, we will say that u equals g in the sense of operator Ext ($u \overset{\operatorname{Ext}}{=} g$) on the boundary of Ω when

$$u - \operatorname{Ext} g \in W^{1,p}_{\rho,0}(\Omega), \tag{3.1.3}$$

where $W^{1,p}_{\rho,0}(\Omega)$ denotes the completion of the $\mathcal{C}^{\infty}_0(\Omega)$ in $W^{1,p}_{\rho}(\Omega)$. The natural questions arise: *Question 1)* When does the operator $\operatorname{Ext} : X \to W^{1,p}_{\rho}(\Omega)$ exist? *Question 2)* When is the condition (3.1.3) independent of the choice of operator $\operatorname{Ext} : X \to W^{1,p}_{\rho}(\Omega)$?

To approach Question 1 we adopt our previous results from [23, 25], where the extension operator $\operatorname{Ext} : W^{1-1/p,p}_{\omega}(\partial\Omega) \to W^{1,p}_{\rho}(\Omega)$ between relevant weighted Slobodetskii space $W^{1-1/p,p}_{\omega}(\partial\Omega)$ of functions defined on $\partial\Omega$ (see Definition 3.2.5) and weighted Sobolev space of functions defined on Ω was constructed in case when Ω is a Lipschitz boundary domain, $\rho(x) = \tau(\operatorname{dist}(x, \partial\Omega))$ and $\omega(x,y) = \tau(|x-y|)$, under certain assumptions on the involved function τ (see Theorem 3.3.5).

To approach Question 2 we prove that when only $\rho \in L^1(\Omega)$, the condition $u \stackrel{\text{Ext}}{=} g$ on the boundary of Ω, defined in (3.1.3), is independent on the choice of extension operator Ext as in (3.1.2). The proof of this statement is based on the surprisingly simple fact that the function of distance to the boundary of Ω is Lipschitz, no matter what is the regularity of the domain Ω.

In last part of the paper, Section 3.5, we apply our previous results to problem (3.1.1). Here the problem can be reduced to the homogeneous BVP with zero boundary data where the unique solvability is proven, but still there is no guarantee that the solution is unique. This is because due to the reformulation argument it might depend on the choice of the extension operator. This however is impossible because of our result of Theorem 3.4.1 on independence of the interpretation of boundary data w/r to the choice of operator of extension. In particular, uniqueness of solutions to the nonhomogeneous BVP (3.1.1) requires the concept of Theorem 3.4.1, which is new.

For trace and extension results done in the weighted setting we refer to the historical paper by Nikolskii [75] and by Slobodetskii [81], to books [56, 71, 83], to works mostly done by the Russian school of Lizorkin [62], Vasarin [88], Portnov [77], Kudryavcev [54] (Section 9), Uspenski [90], Nečas [72], dealing with weight functions like $\mathrm{dist}(x, \partial\Omega)^\alpha$. See also [1, 39, 49, 50, 68, 73, 74, 85, 87, 86] for more recent related works. In particular, in recent studies [87, 86] exact trace theorems were obtained for the weight which depends on all of spatial coordinates and locally satisfies the Muckenhoupt condition. For the other results related to the A_p setting we refer also to: [1, 39, 49, 50, 68, 85]. We are interested in the possibly general approach and we do not assume that the admitted weight function is an A_p weight.

Let us mention that the interpretation of boundary condition $f = g$ on $\partial\Omega$ when function f belongs to the weighted Sobolev space is the longstanding open problem when one investigates the degenerated partial differential equations posed in $W^{1,p}_\rho(\Omega)$ with the given nonzero boundary data. For example, the function ρ can explode or converge to zero at the boundary of Ω. Moreover, in many cases the theory of existence of solutions to homogeneous boundary value problems like (3.1.1) with $g \equiv 0$ has been systematically undertaken but the authors are aware of the fact that this theory is avoiding the nonhomogeneous boundary problems because of lack of general trace/extension results. This difficulty has been emphasized in [30] on page 16 in the Introduction. For analysis of degenerated PDEs, we refer for example to [30, 35, 41, 56, 84] and the references enclosed therein.

3.2 Notation and preliminaries

Basic notation. If not said otherwise, we assume that $\Omega \subseteq \mathbb{R}^n$ is an open set and $\overline{\Omega}$ is its closure. If the function f is defined on some subset $A \subseteq \Omega$, by $f\chi_A$ we denote its extension by zero outside A. We denote by $\mathcal{C}_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω. We will be using the definition of Lipschitz boundary domains of class $\mathcal{C}^{0,1}$ as in [65], Section 1.1.9. In particular, when $\Omega \in \mathcal{C}^{0,1}$, by σ we denote Hausdorff measure defined on its boundary. By $\mathrm{Lip}(Z)$ we denote Lipschitz real functions defined on a set Z, contained in the Euclidean space. When function f is defined on domain A and $B \subseteq A$, by $f|_B$ we denote the usual restriction of f to B. When $1 < p < \infty$, by p' we denote it's Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Weights of class $B_p(\Omega)$. We will use the following definitions.

Definition 3.2.1 (positive weights, class $B_p(\Omega)$, Definition 1.4 in [55])**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{M}(\Omega)$ the set of all Borel - measurable functions.
1) Elements of set $W(\Omega) := \{\rho \in \mathcal{M}(\Omega) : \ 0 < \rho(x) < \infty, \text{ for a.e. } x \in \Omega\}$, will be called positive weights.
2) Let $p > 1$. We will say that a weight $\rho \in W(\Omega)$ satisfies $B_p(\Omega)$-condition ($\rho \in B_p(\Omega)$) if $\rho^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega)$.

The following simple observation results from Hölder inequality.

Proposition 3.2.2 (Theorem 1.5 in [55])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 < p < \infty$ and $\rho \in B_p(\Omega)$. Then $L^p_\rho(\Omega)$ is the subset of $L^1_{\mathrm{loc}}(\Omega)$.*

Weighted Sobolev spaces. We will be dealing with definition of weighted Sobolev space due to Kufner and Opic [55].

Definition 3.2.3 (weighted *Sobolev space*)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\rho \in W(\Omega)$, $p \in [1, \infty)$. The linear set

$$W_\rho^{1,p}(\Omega) := \{f \in L^1_{\text{loc}}(\Omega) : f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\}, \tag{3.2.1}$$

where $\frac{\partial f}{\partial x_i}$ are distributional derivatives, equipped with the norm

$$\|f\|_{W_\rho^{1,p}(\Omega)} := \|f\|_{L^p_\rho(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega,\mathbb{R}^n)}, \tag{3.2.2}$$

where $\|(v_1, \ldots, v_n)\|_{L^p_\rho(\Omega;\mathbb{R}^n)} := \left(\sum_{i=1}^n \|v_i\|^p_{L^p_\rho(\Omega)}\right)^{\frac{1}{p}}$, will be called weighted Sobolev space.
Symbol $W^{1,p}_{\rho,0}(\Omega)$ (respectively $W^{1,p}_{\rho,L}(\Omega)$) will denote the completion of set $C_0^\infty(\Omega)$ (respectively set of Lipschitz functions) in the norm of the space $W^{1,p}_\rho(\Omega)$.

As noticed in [55], the linear set defined by (3.2.1) is not always complete in the norm $\|\cdot\|_{W^{1,p}_\rho(\Omega)}$. It becomes a Banach space under the assumption that $\rho \in B_p(\Omega)$. This important property follows from Proposition 3.2.2 and is summarized in the following statement.

Proposition 3.2.4 (Theorem 1.11 in [55])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\rho \in B_p(\Omega)$. Then linear set $W^{1,p}_\rho(\Omega)$ defined by* (3.2.1) *equipped with the norm* (3.2.2) *is a Banach space. In particular, $W^{1,p}_{\rho,0}(\Omega) \subseteq \{f \in L^1_{loc}(\Omega) : f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega)\}$.*

Weighted Slobodetskii spaces defined on the boundary of the domain. We will adopt the definition from [25, 24] for the space denoted by $Y^{\Psi,\Phi}_\omega(\partial\Omega)$ in the special case when $\Psi(\lambda) = \Phi(\lambda) = |\lambda|^p$ and propose the more classical name for that case.

Definition 3.2.5 (weighted Slobodetskii space)**.** Let $1 < p < \infty$, $\Omega \subseteq \mathbf{R}^n$ be the given domain of class $\mathcal{C}^{0,1}$ and $\omega(x, y)$ be the given weight on $\partial\Omega \times \partial\Omega$. The space $W^{1-1/p,p}_\omega(\partial\Omega)$ consists of all measurable functions defined on $\partial\Omega$

for which the norm

$$\|u\|_{W_w^{1-\frac{1}{p},p}(\partial\Omega)} := \left(\int_{\partial\Omega} \int_{\partial\Omega} \left(\frac{|u(x)-u(y)|}{|x-y|} \right)^p \frac{\omega(x,y)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y) \right)^{\frac{1}{p}} + \left(\int_{\partial\Omega} |u(x)|^p d\sigma(x) \right)^{\frac{1}{p}}$$

is finite. The completion of Lipschitz functions in that space will be denoted by $W^{1-1/p,p}_{\omega,L}(\partial\Omega)$.

The special class of admitted weights. We will be dealing with the case of $\omega(x,y) = \tau(|x-y|)$ where τ is the given continuous function defined on $\mathbf{R}_+$. Of our special interest will be the following class of functions introduced in [25].

Definition 3.2.6 (Condition (τ))**.** We assume that $\tau : (0,\infty) \to \mathbb{R}_+$ is continuous, monotonic, $\int_0^1 \tau(t)dt < \infty$, τ satisfies one of the following conditions i) or ii) for small arguments s:

i) τ is nondecreasing, absolutely continuous, satisfies the Δ_2 condition and $s\tau'(s) \le F \cdot \tau(s)$, where $\frac{F}{n-1} < 1$;

ii) τ is nonincreasing and τ satisfies $\Delta_{\frac{1}{2}}$-condition i.e. $\tau(\frac{1}{2}s) < c\tau(s)$, where c is independent on s.

3.3 Extension operator and interpretation of boundary data

In this section we propose the new approach to interpret the boundary data of the given Sobolev function in the weighted setting. This is the case when the usual trace operator (see e.g. [57]) in most situations has not been found. To overcome this difficulty we exploit an extension operator, which seems much easier to handle.

Extension operator. We first introduce the following definition of an extension operator.

Definition 3.3.1 (extension operator)**.** Let $\Omega \subseteq \mathbf{R}^n$ be the given domain, $X = X(\partial\Omega)$ be the given function space of functions defined on $\partial\Omega$ and let $Y = Y(\Omega)$ be the given function space of functions defined on Ω. Suppose that Lipschitz functions defined on $\partial\Omega$ are dense in X. The linear continuous operator: $\mathrm{Ext} : X \to Y$ will be called *extension operator* if Ext restricted to $\mathrm{Lip}(\partial\Omega)$ is the usual extension to $\mathrm{Lip}(\overline{\Omega})$.

Remark 3.3.2. Formally the space Y consists of functions which are defined inside Ω only. Therefore the condition $\mathrm{Ext}|_{\mathrm{Lip}(\partial\Omega)} : \mathrm{Lip}(\partial\Omega) \to \mathrm{Lip}(\overline{\Omega})$ (as the mapping not the operator) precisely means that it maps Lipschitz functions defined on $\partial\Omega$ to Lipschitz functions defined on Ω in such a way that when $u \in \mathrm{Lip}(\partial\Omega)$ then $\mathrm{Ext}(u) \in \mathrm{Lip}(\Omega) \cap Y$ and can be extended to Lipschitz function $\widetilde{\mathrm{Ext}(u)}$ defined on $\overline{\Omega}$ in such a way that

$$\widetilde{\mathrm{Ext}(u)}|_{\partial\Omega} = u. \tag{3.3.1}$$

Remark 3.3.3. Let $X = W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega)$ (see Def. 3.2.5) and $Y = W^{1,p}_{\rho}(\Omega)$. If there exist continuous operator $\mathrm{Tr} : Y \overset{\text{onto}}{\to} X$, then one has $\mathrm{Tr}(u) = u|_{\partial\Omega}$ when $u \in Y \cap \mathrm{Lip}(\overline{\Omega})$. If further there exists operator $\mathrm{Ext} : X \to Y$ in the sense of Definition 3.3.1, then (3.3.1) is equivalent to the condition $\mathrm{Tr} \circ \mathrm{Ext}(u) = u$ for every $u \in X$ by the continuity of both involved operators and the density argument. In that case operator Ext is right inverse to the operator of trace Tr as in the classical setting.

Interpretation of boundary data. We introduce the following definition to interpret the boundary condition for weighted Sobolev space.

Definition 3.3.4 (interpretation of boundary data)**.** Let $\Omega \subseteq \mathbf{R}^n$ be the given domain, X be the given function space of functions defined on $\partial\Omega$ and let Y be the given function space of functions defined on Ω. Moreover, suppose that Lipschitz functions defined on $\partial\Omega$ are dense in X and that there exists the extension operator $\mathrm{Ext} : X \to Y$. Let $g \in X$ and $u \in Y$. We will say that u agrees with g on $\partial\Omega$ in the sense of operator Ext ($u \overset{\mathrm{Ext}}{=} g$ on $\partial\Omega$) if

$$u - \mathrm{Ext}(g) \in Y_0,$$

where Y_0 is the completion of smooth compactly supported functions in Y.

In next part of this section we will focus on the special choice of spaces X and Y: weighted Slobodetskii and Sobolev spaces, respectively, where the extension operator exists. This way we approach an answer on Question 1 posed in the Introduction.

Existence of extension operator between weighted Slobodetskii and Sobolev spaces. The following statement is direct consequence of the special case of Theorem 3.3 in [25], adopted to the situation $\Psi(\lambda) = \lambda^p$.

Theorem 3.3.5. *Suppose that $n \geq 2$, $\Omega \subseteq \mathbf{R}^n$ is a domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau\left(\operatorname{dist}(x, \partial\Omega)\right) \in L^1(\Omega) \cap B_p(\Omega)$, $\omega(x,y) = \tau(|x-y|)$ and $\tau : (0,\infty) \to \mathbb{R}_+$ satisfies Condition (τ) in Definition 3.2.6.*
Then there exists the liner extension operator $\operatorname{Ext} : W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega) \to W^{1,p}_{\rho}(\Omega)$ as in Definition 3.3.1 such that for every $u \in W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega)$ the function $\tilde{u} = \operatorname{Ext}(u)$ satisfies

$$\int_\Omega |\widetilde{u}(x)|^p \,\tau(\operatorname{dist}(x,\partial\Omega))\,dx \leq C_1 \int_{\partial\Omega} |u(x)|^p\,d\sigma(x)$$

$$\begin{aligned}\int_\Omega |\nabla\widetilde{u}(x)|^p \tau(\operatorname{dist}(x,\partial\Omega))\,dx \\ \leq C_2 \left(\int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{|u(x)-u(y)|}{|x-y|}\right)^p \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x)d\sigma(y) \right. \\ \left. + \int_{\partial\Omega} |u(x)|^p d\sigma(x)\right), \quad (3.3.2)\end{aligned}$$

and consequently,

$$\|\widetilde{u}\|_{L^p_\rho(\Omega)} \leq C_3\|u\|_{L^p(\partial\Omega)} \quad \text{and} \quad \|\nabla\widetilde{u}\|_{L^p_\rho(\Omega)} \leq C_4\|u\|_{W^{1-\frac{1}{p},p}_\omega(\partial\Omega)}. \qquad (3.3.3)$$

Constants C_1, C_2, C_3, C_4 are independent on u.

Proof. In Theorem 3.3 in [25] it was proven that (3.3.2) and (3.3.3) hold for every $u \in \operatorname{Lip}(\partial\Omega)$ and that $\widetilde{u} \in \operatorname{Lip}(\overline{\Omega})$. This together with the density argument closes the proof. $\square$

Remark 3.3.6. More general statement, where X is Orlicz-Slobodetskii space and Y is Orlicz-Sobolev space, follows from Theorem 3.3 in [25]. We omit its formulation which requires more involved definitions.

3.4 Independence of boundary condition with respect to the choice of the extension operator

In this section we come back to the problem of interpretation of boundary condition proposed in Definition 3.3.4. We will prove the following statement which is main result of this section and answers on Question 2 posed in the Introduction.

Theorem 3.4.1 (independence of boundary condition w/r to the choice of extension operator)**.** *Let Ω be a given bounded domain and $\rho \in B_p(\Omega) \cap L^1(\Omega)$. Moreover, let X be the given function space of functions defined on $\partial\Omega$ such that there exist two extension operators $\mathrm{Ext}_1, \mathrm{Ext}_2 : X \to W^{1,p}_{\rho}(\Omega)$. Then we have*

i) $\mathrm{Ext}_1(g) - \mathrm{Ext}_2(g) \in W^{1,p}_{\rho,0}(\Omega)$;

ii) $u \overset{\mathrm{Ext}_1}{=} g$ on $\partial\Omega \Leftrightarrow u \overset{\mathrm{Ext}_2}{=} g$ on $\partial\Omega$.

In particular, the interpretation of the boundary condition $u \overset{Ext}{=} g$ on the boundary of Ω proposed in Definition 3.3.4 is independent on the choice of extension operator.

Before we prove the statement we first solve the related problem by indicating the typical representant of the space $W^{1,p}_{\rho,0}(\Omega)$ in the case when $\rho \in L^1(\Omega)$.

Lemma 3.4.2. *Let $\Omega \subseteq \mathbf{R}^n$ be any domain, $\rho \in L^1(\Omega) \cap B_p(\Omega)$ and $p \geq 1$. Then for any Lipschitz function f defined on $\overline{\Omega}$ which is zero on the boundary of Ω, we have $f \in W^{1,p}_{\rho,0}(\Omega)$.*

Proof. Let us consider distance function

$$d(x) := \mathrm{dist}(x, \partial\Omega), \quad x \in \Omega.$$

We note that $d(x)$ is a Lipschitz function with Lipschitz constant 1. To see that, we take x and y in Ω and assume that $d(x) \geq d(y)$, where $d(x) = |x - z_1|, d(y) = |y - z_2|$ for some $z_1, z_2 \in \partial\Omega$. Then

$$|d(x) - d(y)| = d(x) - d(y) = |x - z_1| - |y - z_2| \leq |x - z_2| - |y - z_2| \leq |x - y|,$$

which shows Lipschitzity of $d(x)$. As distance function reaches its value along the lines, the Lipschitz constant cannot be smaller than 1. We will construct compactly supported Lipschitz functions which approximate f in $W^{1,p}_\rho(\Omega)$. For this, for a fixed $\delta > 0$ we consider the function

$$\eta_\delta(x) := \begin{cases} 0 & \text{if } \ d(x) \leq \delta/2, \\ (d(x) - \delta/2)2/\delta, & \text{if } \ \delta/2 \leq d(x) \leq \delta \\ 1 & \text{if } \ d(x) \geq \delta. \end{cases}$$

It is clear that

1) η_δ is Lipschitz with the Lipschitz constant $2/\delta$;

2) $\eta_\delta = 0$ for $d(x) \leq \frac{\delta}{2}$ and $\eta_\delta = 1$ for $d(x) \geq \delta$;

3) $0 \leq \eta_\delta \leq 1$ and $\eta_\delta(x) \to 1$ pointwise in Ω as $\delta \to 0$.

Moreover, the function

$$f_\delta(x) := f(x)\eta_\delta(x)$$

is Lipschitz and compactly supported in Ω. Now we show that $f_\delta(x) \to f(x)$ in $W^{1,p}_\rho(\Omega)$.

Convergence of f_δ in $L^p_\rho(\Omega)$ follows from Lebesgue's Dominated Convergence Theorem because $\eta_\delta(x) \to 1$ as $\delta \to 0$ and $f \in L^p_\rho(\Omega)$.

To prove that ∇f_δ converge to ∇f in $L^p_\rho(\Omega)$, we note that

$$|\nabla f_\delta(x) - \nabla f(x)| = |(\eta_\delta(x) - 1)\nabla f(x) + f(x)\nabla\eta_\delta(x)| \leq |1 - \eta_\delta(x)||\nabla f(x)| + |f(x)||\nabla\eta_\delta(x)|.$$

Hence

$$\int_\Omega |\nabla f_\delta(x) - \nabla f(x)|^p \rho\, dx \leq 2^{p-1} \left(\int_\Omega |1 - \eta_\delta(x)|^p |\nabla f(x)|^p \rho\, dx \right.$$
$$\left. + \int_\Omega |f|^p |\nabla \eta_\delta|^p \rho\, dx \right).$$

The first term above converges to 0 by Lebesgue's Dominated Convergence Theorem. To deal with the second term, we observe that $|f(x)| \leq \delta \|\nabla f\|_\infty$. Indeed, let $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ and let $x \in \Omega \setminus \Omega_\delta$, so that $d(x) < \delta$. There exists $y \in \partial\Omega$ such that $d(x) = |x - y|$ and, as $f(y) = 0$, we have

$$|f(x)| = |f(x) - f(y)| \leq \|\nabla f\|_\infty |x - y| \leq \delta \|\nabla f\|_\infty.$$

Since $\|\nabla \eta_\delta\|_\infty \leq 2/\delta$ in $\Omega \setminus \Omega_\delta$, we have

$$\begin{aligned}\int_\Omega |f|^p |\nabla \eta_\delta|^p \rho\, dx \leq& \frac{2^p}{\delta^p} \delta^p \|\nabla f\|_\infty^p \int_{\Omega_{\delta/2} \setminus \Omega_\delta} \rho\, dx \\ &= 2^p \|\nabla f\|_\infty^p \int_\Omega \chi_{\Omega_{\delta/2} \setminus \Omega_\delta}(x) \rho\, dx \overset{\delta \to 0}{\longrightarrow} 0.\end{aligned}$$

Let us show that when $\rho \in L^1(\Omega)$ then Lipschitz compactly supported functions belong to $W^{1,p}_{\rho,0}(\Omega)$. Indeed, let $\{\phi\}_\varepsilon$ be the standard molifier, so that

$$f_\varepsilon := (f * \phi_\varepsilon)(x) = \int_\Omega f(x - y) \phi_\varepsilon(y) dy \overset{\varepsilon \to 0}{\to} f \ a.e. \text{ and } |f_\varepsilon(x)| \leq \|f\|_\infty.$$

In particular, $|f_\varepsilon - f|$ converges to 0 almost everywhere and is dominated by $2\|f\|_\infty$, which belongs to $L^1_\rho(\Omega)$. Lebesgue's Dominated Convergence Theorem gives that $f_\varepsilon \to f$ in $L^p_\rho(\Omega)$. The same arguments with f_ε substituted by $\frac{\partial f_\varepsilon}{\partial x_i} = (\frac{\partial f}{\partial x_i} * \phi_\varepsilon)(x)$, where $i = 1, \ldots, n$, so that also $\frac{\partial f_\varepsilon}{\partial x_i} \to \frac{\partial f}{\partial x_i}$ in $L^p_\rho(\Omega)$. As f_ε's are smooth and compactly supported, we deduce that $f \in W^{1,p}_{\rho,0}(\Omega)$. This implies the assertion of the lemma.

□

We are now to prove Theorem 3.4.1.

Proof of Theorem 3.4.1. We observe that conditions i) and ii) are equivalent. To verify conclusion $i)$, we consider the sequence $g_m \in \mathrm{Lip}(\partial\Omega)$ such that $g_m \to g$ in X. Then $(\mathrm{Ext}_1 g_m - \mathrm{Ext}_2 g_m)\,|_{\partial\Omega} \equiv 0$. Therefore $f_m := \mathrm{Ext}_1 g_m - \mathrm{Ext}_2 g_m$ is Lipschitz function which vanishes on the boundary of Ω. By Lemma 3.4.2 we have $f_m \in W^{1,p}_{\rho,0}(\Omega)$. Moreover, as $\{f_m\}$ converges to $\mathrm{Ext}_1 g - \mathrm{Ext}_2 g$ in $W^{1,p}_{\rho}(\Omega)$ by the continuity of the two extension operators involved, we get $\mathrm{Ext}_1 g - \mathrm{Ext}_2 g \in W^{1,p}_{\rho,0}(\Omega)$, which finishes the proof. $\square$

Remark 3.4.3. For some more information about the space $W^{1,p}_{\rho,0}(\Omega)$, involving the information about functionals on $W^{1,p}_{\rho,0}(\Omega)$, the validity of Poincaré inequality on $W^{1,p}_{\rho,0}(\Omega)$, isoperimetric inequalities and solvability of problems involving degenerated p-Laplacian, we refer to our recent work [24].

Remark 3.4.4. The assumption $\rho \in L^1(\Omega)$ is still rather general as the weight function ρ can explode or converge to zero at the boundary of Ω. One could ask what happens when one relaxes the assumption $\rho \in L^1(\Omega)$. Note that when ρ is not integrable on Ω then there are no affine nonconstant functions which belong to $W^{1,p}_{\rho}(\Omega)$.

The interpretation of the boundary data under the special choice of spaces. Theorems 3.3.5 and 3.4.1 imply the following statement.

Theorem 3.4.5. *Suppose that $n \geq 2$, $\Omega \subseteq \mathbf{R}^n$ is a domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\mathrm{dist}\,(x, \partial\Omega)) \in L^1(\Omega) \cap B_p(\Omega)$, $\omega(x,y) = \tau(|x-y|)$ and $\tau : (0,\infty) \to \mathbb{R}_+$ satisfies Condition (τ) in Definition 3.2.6.*
Then there exists extension operator $\mathrm{Ext} : W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega) \to W^{1,p}_{\rho}(\Omega)$. *Moreover, if*
$\mathrm{Ext}_1 : W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega) \to W^{1,p}_{\rho}(\Omega)$ *is any other extension operator, then we have*

$$u - \mathrm{Ext}(g) \in W^{1,p}_{\rho,0}(\Omega) \Leftrightarrow u - \mathrm{Ext}_1(g) \in W^{1,p}_{\rho,0}(\Omega)$$

and the above condition is equivalent to the fact that $\mathrm{Ext}(g) - \mathrm{Ext}_1(g) \in W^{1,p}_{\rho,0}(\Omega)$. *In particular, the interpretation of boundary condition* $u \overset{Ext}{=} g$ on $\partial\Omega$ *proposed in Definition 3.3.4 is independent on the choice of extension operator between $W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega)$ and $W^{1,p}_{\rho}(\Omega)$.*

3.5 Application to the unique solvability of degenerate BVPs with non-homogeneous boundary condition

We are now to propose an example of the application of our results to the solvability of degenerate elliptic PDEs with nonhomogeneous boundary condition. Obviously, there are many ways to generalize this statement.

Theorem 3.5.1. *Suppose that $n \geq 2$, $\Omega \subseteq \mathbf{R}^n$ is a domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\operatorname{dist}(x, \partial\Omega)) \in L^1(\Omega) \cap B_p(\Omega)$, $\omega(x,y) = \tau(|x-y|)$ and $\tau : (0,\infty) \to \mathbb{R}_+$ satisfies Condition (τ) in Definition 3.2.6. Moreover, assume that Poincaré inequality*

$$\int_\Omega |v(x)|^p \rho(x)\, dx \leq C \int_\Omega |\nabla v(x)|^p \rho(x)\, dx \tag{3.5.1}$$

holds for every $v \in W^{1,p}_{\rho,0}(\Omega)$ with constant C independent on v.
Consider the following nonhomogeneous BVP

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla u(x)|^{p-2}\nabla u(x)) = f(x) & \text{in } \Omega, \\ u(x) = g & \text{on } \partial\Omega, \end{cases} \tag{3.5.2}$$

where $f \in \left(W^{1,p}_{\rho,0}(\Omega)\right)^*$, $g \in W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega)$. Then we have

a) *there exists an extension operator* $\operatorname{Ext} : W^{1-\frac{1}{p},p}_{\omega,L}(\partial\Omega) \longrightarrow W^{1,p}_{\rho,0}(\Omega)$ *and the boundary condition "$u = g$ on $\partial\Omega$" in* (3.5.2) *interpreted in the sense of Definition 3.3.4 is independent on the given operator* Ext*;*

b) *the problem* (3.5.2) *has a unique solution $u \in W^{1,p}_\rho(\Omega)$.*

Proof. Part **a**) of the statement is exactly Theorem 3.4.5. To prove part **b**), we substitute $v = v_{\mathrm{Ext}} := u - \operatorname{Ext}(g)$, and note that v solves the homogeneous problem:

$$\begin{cases} -\operatorname{div}(\rho(x)|\nabla v + \nabla \operatorname{Ext}(g)|^{p-2}(\nabla v + \nabla \operatorname{Ext}(g)) & = f \in \left(W^{1,p}_{\rho,0}(\Omega)\right)^*, \\ v \in W^{1,p}_{\rho,0}(\Omega). \end{cases}$$

For the solvability of the above homogeneous problem we refer for example to Theorem 5.3 from [24]. We note that $u_{\mathrm{Ext}} := v_{\mathrm{Ext}} + \mathrm{Ext}(g)$ solves the original problem

$$\begin{cases} Pu_{\mathrm{Ext}} = f \in \left(W^{1,p}_{\rho,0}(\Omega)\right)^*, \\ u_{\mathrm{Ext}} \overset{\mathrm{Ext}}{=} g \ \text{ on } \ \partial\Omega, \end{cases}$$

where $Pw = -\mathrm{div}(\rho(x)|\nabla w|^{p-2}\nabla w)$. It remains to prove that $u = u_{\mathrm{Ext}}$ is independent of the choice of the extension operator Ext given. Suppose that we have two solutions $u_1,\, u_2 \in W^{1,p}_{\rho}(\Omega)$ which solve the following problems, respectively

$$\begin{cases} Pu_1 = f \in \left(W^{1,p}_{\rho,0}(\Omega)\right)^*, \\ u_1 - \mathrm{Ext}_1(g) \in W^{1,p}_{\rho,0}(\Omega), \end{cases} \qquad \text{and} \qquad \begin{cases} Pu_2 = f \in \left(W^{1,p}_{\rho,0}(\Omega)\right)^*, \\ u_2 - \mathrm{Ext}_2(g) \in W^{1,p}_{\rho,0}(\Omega). \end{cases}$$

Thus we have

$$\begin{cases} Pu_1 - Pu_2 = 0 \text{ in } (W^{1,p}_{\rho,0}(\Omega))^*, \\ (u_1 - u_2) - (\mathrm{Ext}_1(g) - \mathrm{Ext}_2(g)) \in W^{1,p}_{\rho,0}(\Omega). \end{cases} \tag{3.5.3}$$

It is crucial that $\mathrm{Ext}_1(g) - \mathrm{Ext}_2(g)$ belongs to $W^{1,p}_{\rho,0}(\Omega)$ by Theorem 3.4.5. Therefore $u_1 - u_2$ also belongs to $W^{1,p}_{\rho,0}(\Omega)$ and we can evaluate first line in (3.5.3) on $u_1 - u_2$, getting

$$\begin{aligned} 0 &= \langle Pu_1 - Pu_2, u_1 - u_2\rangle = \langle Pu_1, u_1 - u_2\rangle - \langle Pu_2, u_1 - u_2\rangle \\ &= \int_\Omega \langle |\nabla u_1|^{p-2}\nabla u_1, \nabla u_1 - \nabla u_2\rangle \rho dx - \int_\Omega \langle |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle \rho dx \\ &= \int_\Omega \langle \Phi_p(\nabla u_1) - \Phi_p(\nabla u_2), \nabla u_1 - \nabla u_2\rangle \rho dx, \end{aligned}$$

where $\Phi_p(\lambda) = |\lambda|^{p-2}\lambda$ for $\lambda \in \mathbf{R}^n$. It is well known that for the fixed $a, b \in \mathbf{R}^n$ we have

$$\langle \Phi_p(a) - \Phi_p(b), a - b\rangle \geq c|a-b|^2 \left(|a| + |b|\right)^{p-2},$$

with constant c independent on a, b, see e.g. [21]. Therefore we have $\nabla(u_1 - u_2) = 0$ a.e.. As $u_1 - u_2 \in W^{1,p}_{\rho,0}(\Omega)$, by Poincaré inequality (3.5.1) we get $u_1 = u_2$ a. e., which finishes the proof. □

Remark 3.5.2. The solvability of (3.5.2) with $g \in X$, where X is a given function space, requires to have defined an extension operator $\mathrm{Ext} : X \to W^{1,p}_{\rho}(\Omega)$. Therefore it is important to know what is the possibly large function space X for which such extension operator exists.

Chapter 4

Existence of solutions of the linear degenerate elliptic equations

We prove existence and uniqueness of solution to the nonhomogeneous degenerate elliptic PDE of second order with boundary data in weighted Orlicz-Slobodetskii space. Our goal is to consider the possibly general assumptions on the involved constraints: the class of weights, boundary data, as well as the admitted coefficients. We also provide some estimates on the spectrum of our degenerate elliptic operator.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We consider a linear elliptic problem:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u(x)) + B(x) \cdot \nabla u(x) + C(x)u(x) = f(x) & \text{for a.e. } x \in \Omega, \\ u(x) = g(x) \ \text{ for a.e. } \ x \in \partial\Omega, \end{cases} \tag{4.1.1}$$

where the diffusion coefficient $A(x) = [a_{ij}(x)]_{i,j=1,\dots,n}$ is the symmetric matrix with measurable entries and satisfies the degenerate ellipticity condition:

$$c_1|\xi|^2\rho(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le c_2|\xi|^2\rho(x), \text{ for almost every } x \in \Omega$$
$$\text{and every } \ \xi \in \mathbb{R}^n,$$

$c_1,\ c_2 > 0$ are given constants. It is assumed that $\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega))$ is the given weight in the class of weights $B_2(\Omega)$ introduced by Kufner and Opic in [55] and $\tau : (0, \infty) \to (0, \infty)$ is the given function which obeys certain assumptions (the details are discussed in the Section 4.2). As ρ may have some local singularities, the considered PDE is the degenerated one. Because of the involved boundary data, the function g defined on $\partial\Omega$, which is not necessarily zero, our problem is the nonhomogeneous one.

Our goal is to find the possibly general set of assumptions on other involved constraints: the drift coefficient $B(x)$, the potential coefficient $C(x)$, as well as on the involved function g, which guarantee the solvability and uniqueness of our problem (4.1.1). For this we apply the Lax-Milgram Theorem.

Perhaps one of the main difficulties here is to find the proper Hilbert space $\widehat{H}$ and Y such that the problem will be well posed for $u \in \widehat{H}$ and $g \in Y$. This is because of the lack of tools to deal with the degenerated pde's. Here

$$\widehat{H} = W^{1,2}_{\rho}(\Omega) = \{u \in L^1_{\mathrm{loc}}(\Omega) : u, |\nabla u| \in L^2_{\rho}(\Omega)\}$$

is the weighted Sobolev space subordinated to the weight ρ and Y is the weighted Slobodetskii space, the completion of Lipschitz functions in the norm

$$\|g\|_{W^{\frac{1}{2},2}_{\omega}(\partial\Omega)} := \left(\int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{|g(x) - g(y)|}{|x-y|} \right)^2 \frac{\omega(x,y)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y) \right.$$
$$\left. + \int_{\partial\Omega} |g(x)|^2 d\sigma(x) \right)^{\frac{1}{2}}, \quad (4.1.2)$$

where $\omega(x, y)$ is given weight on $\partial\Omega \times \partial\Omega$, σ is the $n-1$ dimensional Hausdorff measure defined on $\partial\Omega$. Weights ρ and ω are related via:

$$\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega)), \quad \omega(x,y) = \tau(\mathrm{dist}(x,y)) = \tau(|x-y|).$$

To apply the Lax-Milgram theorem efficiently we need at first to transform the nonhomogeneous problem (4.1.1) to the homogeneous one:

$$\begin{cases} Pv = F & \text{in} \quad \Omega, \\ v \equiv 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1.3}$$

where

$$Pv := -\mathrm{div}(A(x)\nabla v(x)) + B(x) \cdot \nabla v(x) + C(x)v(x).$$

Let us skip for the moment the explanation how to derive (4.1.3) from (4.1.1) and focus on the problem $v \equiv 0$ on $\partial\Omega$. Note that the application of Lax-Milgram Theorem requires an assumption $v \in H$, where H is the given Hilbert space, which does not explicitly contain the information about the boundary data $v \equiv 0$ on $\partial\Omega$. Therefore this condition must be enclosed in the structure of H. The equation (4.1.3) is investigated in the space

$$H := W^{1,2}_{\rho,0}(\Omega) \text{ - the completion of } C_0^\infty(\Omega) \text{ in } W^{1,2}_{\rho}(\Omega).$$

This is how we interpret the zero boundary condition for v: v is the limit in H of such their elements which vanish in certain neighborhood of $\partial\Omega$ (form $C_0^\infty(\Omega)$).

To deal with (4.1.3) we look for $v \in H$ which solves the equation $Pv = F$ where $F \in H^*$ is the functional on H.

Now let us explain how to derive (4.1.3) from (4.1.1). For this we apply the new tool from recent papers of Kałamajska and the author, [23, 25]. Namely, we assume that $g \in Y$. For the space Y (see (4.1.2)) there exists a linear bounded operator: $\mathrm{Ext} : Y \to W^{1,2}_{\rho}(\Omega)$ as proven in [25], Theorem 3.1 (see also [23] where we deal with the case when Ω is the cube) such that when g is Lipschitz then $\mathrm{Ext}(g)|_{\partial\Omega} = g$. Hence there exists $\Psi_g := \mathrm{Ext}(g) \in W^{1,2}_{\rho}(\Omega)$ and the boundary condition $u = g$ on $\partial\Omega$ reads as $u - \Psi_g \in H$. The problem of correctness of such interpretation has been discussed in details in [26] and sketched in Section 4.2.
We substitute: $v := u - \Psi_g$. Then the problem (4.1.1) is equivalent to:

$$\begin{cases} Pv = f - P\Psi_g := F & \text{in} \quad \Omega, \\ \qquad\quad v & \in \quad H. \end{cases}$$

Simple verification gives $P\Psi_g \in H^*$, and when $f \in H^*$ we have $F \in H^*$. The verification is provided in details in the proof of Theorem 4.3.2, which asserts existence and uniqueness of solutions for (4.1.1) and is main result of this paper. We note that the presentation of our solution u of (4.1.1) requires the definition of the extension operator $\mathrm{Ext} : Y \to W_\rho^{1,2}(\Omega)$. The fact that this solution is independent of the choice of the extension operator Ext, does not follow directly classical arguments but involves main result of previous paper by Kałamajska and the author in [26] (see Theorem 4.2.19).

We also provide lower bound for spectrum of operator
$P_1 u := -\mathrm{div}(A(x)\nabla u(x)) + B(x)\nabla u(x) + C(x)u(x)$, see Theorem 4.4.5.

Let us mention that set of our conditions in Theorems 4.3.2 and 4.4.5 seems to be almost optimal. Namely, when we reduce them to the classical case $\rho \equiv 1, A \equiv Id$, some of the conditions are sharp, see Remarks 4.4.2, 4.4.4 and 4.4.7.

Some other related results can be found for example in [17, 29, 30, 35, 50, 56, 68]

4.2 Notation and preliminaries

4.2.1 Basic notation

If it is not said otherwise we assume that $\Omega \subseteq \mathbb{R}^n$ is an open set. In this paper we assume that $p = 2$, however most of the statements are true for $p > 1$ as well. When $p > 1$ by p' we denote it's Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We denote by $\mathcal{C}_0^\infty(\Omega)$, the space of infinitely differentiable function with compact support in Ω. The symbol $\langle \cdot, \cdot \rangle$ denotes the the duality action of $x^* \in X^*$ on $x \in X$, were by X^* we denote the dual space of X. We will be using definition of Lipschitz boundary domains of class $\mathcal{C}^{0,1}$ as in [65], Section 1.1.9. In particular, when $\Omega \in \mathcal{C}^{0,1}$, by σ we denote Hausdorff measure defined on its boundary.

4.2.2 Weights of class $B_2(\Omega)$.

We will need the following definitions.

Definition 4.2.1 (positive weights)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{M}(\Omega)$ the set of all Borel measurable functions. Elements of set $W(\Omega) := \{\rho \in \mathcal{M}(\Omega) : 0 < \rho(x) < \infty, \text{ for a.e. } x \in \Omega\}$, will be called positive weights on Ω.

Definition 4.2.2 (class $B_2(\Omega)$, Definition 1.4 in [55])**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We will say that a weight $\rho \in W(\Omega)$ satisfies $B_2(\Omega)$-condition ($\rho \in B_2(\Omega)$) if $\rho^{-1} \in L^1_{\mathrm{loc}}(\Omega)$.

The following simple observation results from Hölder inequality.

Proposition 4.2.3 (Theorem 1.5 in [55])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $\rho \in B_2(\Omega)$. Then $L^2_\rho(\Omega)$ is the subset of $L^1_{\mathrm{loc}}(\Omega)$.*

4.2.3 Weighted Sobolev spaces and representation of functionals

We will be dealing with the following definition of weighted Sobolev spaces.

Definition 4.2.4 (weighted *Sobolev space*)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\rho \in W(\Omega)$ be the given weight. The linear set

$$\left\{ f \in L^1_{\mathrm{loc}}(\Omega) : f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^2_\rho(\Omega) \right\}, \tag{4.2.1}$$

where $\frac{\partial f}{\partial x_i}$ are distributional derivatives, equipped with the norm

$$\|f\|_{W^{1,2}_\rho(\Omega)} := \|f\|_{L^2_\rho(\Omega)} + \|\nabla f\|_{L^2_\rho(\Omega,\mathbb{R}^n)}, \tag{4.2.2}$$

will be called weighted Sobolev space. We will denote it by $W^{1,2}_\rho(\Omega)$.
Symbol $W^{1,2}_{\rho,0}(\Omega)$ will denote the completion of set $C_0^\infty(\Omega)$ in the norm of the space $W^{1,2}_\rho(\Omega)$.

For our convenience in the above definition we chose the following norm in the space $L^2_\rho(\Omega;\mathbb{R}^n)$:

$$\|(v_1,\dots,v_n)\|_{L^2_\rho(\Omega;\mathbb{R}^n)} := \left(\sum_{i=1}^n \|v_i\|^2_{L^2_\rho(\Omega)}\right)^{\frac{1}{2}}. \tag{4.2.3}$$

As noticed in Theorem 1.11 from [55], the linear set defined by (4.2.1) is not always complete in the norm $\|\cdot\|_{W^{1,2}_\rho(\Omega)}$. It becomes a Banach space under some special assumptions on the involved weight function ρ.

The following fact is rather known.

Proposition 4.2.5 (Theorem 1.11 in [55])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\rho \in B_2(\Omega)$. Then linear set $W^{1,2}_\rho(\Omega)$ defined by (4.2.1) equipped with the norm (4.2.2) is a Banach space. In particular $W^{1,2}_{\rho,0}(\Omega) \subseteq \{f \in L^1_{\mathrm{loc}}(\Omega) : f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^2_\rho(\Omega)\} \subseteq W^{1,1}_{\mathrm{loc}}(\Omega)$.*

Remark 4.2.6. To our best knowledge the necessary and sufficient conditions for the completeness of linear set defined by (4.2.1) under the norm (4.2.2) are not known.

In case when ρ is not in $B_2(\Omega)$, we define $W^{1,2}_\rho(\Omega)$ as the completion of set defined by (4.2.1) in the norm (4.2.2). For simplicity we denote

$$\widehat{H} = W^{1,2}_\rho(\Omega),\ \widehat{H}^* = (W^{1,2}_\rho(\Omega))^*,\ H = W^{1,2}_{\rho,0}(\Omega),\ H^* = (W^{1,2}_{\rho,0}(\Omega))^*, \tag{4.2.4}$$
$$H_L = \text{the completion of Lipschitz function in the norm } \widehat{H}.$$

We equip H with the semi-norm $\|v\|_H = \|\nabla v\|_{L^2_\rho(\Omega;\mathbb{R}^n)}$. The following statement deals with functionals on $W^{1,2}_\rho(\Omega)$.

Proposition 4.2.7 (representation of functionals on $W^{1,2}_\rho(\Omega)$, Proposition 3.1 in [24])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain, $\rho \in B_2(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$. Then for any functional $x^* \in \widehat{H}^*$ there exist functions $g = (g_0, g_1, \dots, g_n) \in L^2_\tau(\Omega;\mathbb{R}^{n+1})$, where $\tau := \rho^{-1}$, such that for any $f \in \widehat{H}$ we have*

$$(x^*, f) = \int_\Omega f(x) g_0(x)dx - \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial x_i}(x) g_i(x)dx, \tag{4.2.5}$$

in other words

$$\langle x^*, f\rangle = \int_\Omega f(x)g_0(x)dx + (\mathrm{div}\bar{g}, f),$$

where $\bar{g} = (g_1, \ldots, g_n) \in L^2_\tau(\Omega, \mathbb{R}^n)$.

It appears that when H (see (4.2.4)) admits Poincaré inequality, then function g_0 can be omitted in the definition of function g representing functionals on that subspace. This follows from the following statement proven in [24]. Here we state it in the abbreviated version, dealing only with Sobolev spaces related to $p = 2$.

Theorem 4.2.8 (Poincaré inequality and representation of functionals, Theorem 3.4 in [24])**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be an arbitrary domain,* $\rho \in B_2(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$*,* $\tau = \rho^{-1}$ *is nonnegative almost everywhere. The following conditions are equivalent:*

i) *Poincaré inequality*

$$\int_\Omega |f(x)|^2\rho(x)dx \le C\int_\Omega |\nabla f(x)|^2\rho(x)dx$$

holds for every $f \in H$ *with constant* C *independent on* f*.*

ii) *For every* $x^* \in H^*$ *there exist function* $\bar{g} = (g_1, \ldots, g_n) \in L^2_\tau(\Omega; \mathbb{R}^n)$ *such that for any* $f \in H$

$$\langle x^*, f\rangle = -\sum_{i=1}^n \int_\Omega \partial_i f(x) g_i(x)dx = \langle \mathrm{div}\bar{g}, f\rangle. \tag{4.2.6}$$

iii) *the norm:* $\|f\|_1 := \|f\|_{L^2_\rho(\Omega)} + \|\nabla f\|_{L^2_\rho(\Omega;\mathbb{R}^n)}$ *and the semi-norm* $\|f\|_2 := \|\nabla f\|_{L^2_\rho(\Omega;\mathbb{R}^n)}$ *are equivalent on* H*.*

The following remark is in order.

Remark 4.2.9 ([24])**. 1)** Under the assumptions in Proposition 4.2.7 and Theorem 4.2.8 we have $\tau^{-1} = \rho \in L^1_{\mathrm{loc}}(\Omega)$. Hence $\tau \in B_2(\Omega)$ and so according to Proposition 4.2.3 we have $L^2_\tau(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$. In particular distributional derivatives of $g_1, \dots, g_n$ are well defined and so expression

$$(\text{div}\bar{g}, f) := -\sum_{i=1}^{n} \int_\Omega \frac{\partial f}{\partial x_i}(x) g_i(x) dx$$

has the distributional interpretation when $f \in C_0^\infty(\Omega)$. Moreover, by Hölder inequality

$$\left| \int_\Omega \frac{\partial f}{\partial x_i} g_i dx \right| = \left| \int_\Omega \frac{\partial f}{\partial x_i} \rho^{\frac{1}{2}} g_i \rho^{-\frac{1}{2}} dx \right| \leq \left(\int_\Omega \left| \frac{\partial f}{\partial x_i} \right|^2 \rho dx \right)^{\frac{1}{2}} \left(\int_\Omega |g_i|^2 \tau dx \right)^{\frac{1}{2}} \tag{4.2.7}$$

and the application of the norm (4.2.3) gives

$$|(\text{div}\bar{g}, f)| \leq \|\nabla f\|_{L^2_\rho(\Omega;\mathbb{R}^n)} \|\bar{g}\|_{L^2_\tau(\Omega;\mathbb{R}^n)}.$$

whenever $f \in H$. In particular div$\bar{g}$ is an element of H^*, which restricted to $C_0^\infty(\Omega)$ agrees with the distributional divergence of $\bar{g}$.
2) The function $\bar{g}$ in Proposition 4.2.7 and Theorem 4.2.8 is not defined uniquely as for example we may add to $\bar{g}$ any smooth divergence free vector.
3) Finiteness of right hand side in (4.2.5) (respectively in (4.2.6)) for any $f \in \widehat{H}$ (respectively any $f \in H$) and $g \in L^2_\tau(\Omega, \mathbb{R}^{n+1})$ (respectively any $\bar{g} \in L^2_\tau(\Omega, \mathbb{R}^n)$) follows from similar computations as the one enclosed in (4.2.7).
4) Property $\rho \in B_2(\Omega)$ was required for the completeness of the space $\widehat{H}$.
5) Simple computations (regularization by convolution and Lebesgue's Dominated Convergence Theorem) show that the assumption $\rho \in L^1_{\text{loc}}(\Omega)$ implies that Lipschitz compactly supported functions belong to H.

We also recall the following simple fact.

Proposition 4.2.10. *Let $d \in \mathbb{N}$, $\rho \in L^1_{\text{loc}}(\Omega) \cap W(\Omega)$. Then we have $(L^2_\rho(\Omega;\mathbb{R}^d))^* \sim L^2_\tau(\Omega;\mathbb{R}^d)$ with $\tau = \rho^{-1}$ where the duality is expressed by*

$$\langle g, h \rangle = \sum_{i=1}^{d} \int_\Omega g_i h_i dx, \quad \text{where} \;\; g \in L^2_\rho(\Omega;\mathbb{R}^d), \; h \in L^2_\tau(\Omega;\mathbb{R}^d).$$

4.2.4 Weighted Slobodetskii spaces on the boundary of the domain and one extension theorem

The following definition was introduced in [24]. We adopt it here for the space denoted in [24] by $Y_\omega^{\Psi,\Phi}(\partial\Omega)$ in the special case when $\Psi(\lambda) = \Phi(\lambda) = |\lambda|^2$. Because of the special structure chosen, we propose the more classical name for this particular case.

Definition 4.2.11. Let $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$ and ω be the given weight function defined on $\partial\Omega \times \partial\Omega$ subordinated to $2n-2$ dimensional Hausdorff measure. The space $W_\omega^{1/2,2}(\partial\Omega)$ consists of all measurable functions defined on $\partial\Omega$ for which the norm

$$\|u\|_{W_w^{\frac{1}{2},2}(\partial\Omega)} := \int_{\partial\Omega}\int_{\partial\Omega}\left(\frac{|u(x)-u(y)|}{|x-y|}\right)^2 \frac{\omega(x,y)}{|x-y|^{n-2}} d\sigma(x)d\sigma(y) + \int_{\partial\Omega} |u(x)|^2 d\sigma(x)$$

is finite, where $\omega(x,y)$ is given weight on $\partial\Omega \times \partial\Omega$. The completion of Lipschitz functions in that space will be denoted by $W_{\omega,L}^{1/2,2}(\partial\Omega)$

We will be dealing with the case of $\omega(x,y) = \tau(|x-y|)$ where τ is the given continuous function.
We need to introduce the following condition.

Definition 4.2.12 (Condition **C**)**.** The function $\tau : (0,\infty) \to \mathbb{R}_+$ satisfies the Condition **C** if it is continuous, monotonic, $\int_0^1 \tau(t)dt < \infty$ and τ satisfies one of the following conditions i) or ii) for small arguments:

i) τ is nondecreasing, absolutely continuous, satisfies the Δ_2 condition and $s\tau'(s) \le F \cdot \tau(s)$, where $F < n-1$;

ii) τ is nonincreasing and satisfies the $\Delta_{\frac{1}{2}}$-condition: $\tau(\frac{1}{2}s) < c\tau(s)$, where c is independent on s.

Example 4.2.13. Let $\rho(x) = \tau(\text{dist}(x,\partial\Omega))$. The following functions τ satisfy Condition **C**. Easy verification is left to the reader.

(a) $\tau \equiv 1$;

(b) $\tau(t) = t^\alpha$, $-1 < \alpha < n-1$;

(c) $\tau(t) = t^\alpha \left(\ln\left(2 + \frac{1}{t}\right)\right)^\beta$, $-1 < \alpha < n-1, \beta > 0$;

(d) $\tau(t) = (\log(2 + \frac{1}{t}))^{-\alpha}, \alpha > 0$;

(e) $\tau(t) = 1 - e^{\alpha t}$, $\alpha < 0$, $n > 2$.

The following statement is the special case of Theorem 3.3 in [25] adopted to the situation $\Psi(\lambda) = \lambda^2$, while for Ω is a cube analysed in Theorem 3.1 in [23].

Theorem 4.2.14 ([23, 25])**.** *Suppose that $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\operatorname{dist}(x, \partial\Omega)) \in L^1(\Omega)$ and $\tau : (0, \infty) \to \mathbb{R}_+$ satisfies Condition* **C** *(see Definition 4.2.12). Let $u : \partial\Omega \to \mathbf{R}$ be Lipschitz. Then there exists Lipschitz function $\widetilde{u} : \bar{\Omega} \to \mathbf{R}$ such that $\tilde{u} = u$ on $\partial\Omega$ and we have*

$$\int_\Omega |\widetilde{u}(x)|^2 \, \tau(\operatorname{dist}(x, \partial\Omega)) \, dx \leq C_1 \int_{\partial\Omega} |u(x)|^2 \, d\sigma(x);$$

$$\begin{aligned}\int_\Omega & |\nabla \widetilde{u}(x)|^2 \tau(\operatorname{dist}(x, \partial\Omega)) \, dx \\ & \leq C_2 \left(\int_{\partial\Omega} \int_{\partial\Omega} \left(\frac{|u(x) - u(y)|}{|x-y|} \right)^2 \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y) \right. \\ & \qquad \left. + \int_{\partial\Omega} |u(x)|^2 d\sigma(x) \right),\end{aligned}$$

consequently,

$$\|\widetilde{u}\|_{L^2_\rho(\Omega)} \leq C_3 \|u\|_{L^2(\partial\Omega)}, \qquad \|\widetilde{u}\|_{W^{1,2}_\rho(\Omega)} \leq C_5 \|u\|_{W^{\frac{1}{2},2}_\omega(\partial\Omega)},$$

where $\omega(x, y) = \tau(|x-y|)$ and positive constants C_1, C_2, C_3, C_5 are independent on u. Moreover, the mapping $u \mapsto \widetilde{u}$ is linear.

4.2.5 Interpretation of boundary values

It is a difficult problem to interpret the boundary condition $u = g$ on the boundary of a domain, knowing only that u belongs to weighted Sobolev space $W^{1,p}_\rho(\Omega)$ and g is defined on $\partial\Omega$. Note that, weight function $\rho(x)$ may converge to zero or to infinity, when x approaches the boundary of Ω. To overcome this problem, in [26], we have proposed the following definition which is based on the existence of extension operator defined below. Here we focus on its variant dealing with the choice of function spaces $X = W^{1/2,2}_\omega(\partial\Omega)$ and $Y = W^{1,2}_\rho(\Omega)$, while more general approach can be found in [26].

Definition 4.2.15 (extension operator)**.** Let $\Omega \in \mathbb{R}^n$ be an open set with Lipschitz boundary and $\rho \in L^1_{\text{loc}}(\Omega)\cap B_2(\Omega)$. The linear continuous operator: $\text{Ext} : W^{1/2,2}_{\omega,L}(\partial\Omega) \to W^{1,2}_\rho(\Omega)$ will be called extension operator if Ext maps Lipschitz functions defined on $\partial\Omega$ to Lipschitz functions defined on Ω in such a way that when $u \in \text{Lip}(\partial\Omega)$ then $\text{Ext}(u)$ can be extended to Lipschitz function $\widetilde{\text{Ext}(u)}$ defined on $\overline{\Omega}$ and we have

$$\widetilde{\text{Ext}(u)}|_{\partial\Omega} = u. \tag{4.2.8}$$

Here by $v|_{\partial\Omega}$ we denote the usual restriction of Lipschitz function defined on $\overline{\Omega}$ to $\partial\Omega$.

Remark 4.2.16. Let $X = W^{1-\frac{1}{2},2}_{\omega,L}(\partial\Omega)$ (see Def. 4.2.11) and $Y = W^{1,2}_\rho(\Omega)$. If there exist continuous operator $\text{Tr} : Y \overset{\text{onto}}{\to} X$, then one has $\text{Tr}(u) = u|_{\partial\Omega}$ when $u \in Y\cap\text{Lip}(\overline{\Omega})$. If further there exists operator $\text{Ext} : X\to Y$ in the sense of Definition 4.2.15, then (4.2.8) is equivalent to the condition $\text{Tr}\circ\text{Ext}(u) = u$ for every $u \in X$ by the continuity of both involved operators and the density argument. In that case operator Ext is right inverse to the operator of trace Tr as in the classical setting.

The classical example showing existence of such operator would be the case of $\rho \equiv 1$ and $\omega \equiv 1$. Less trivial example is presented below. It follows from the consequence of Theorem 4.2.14.

Corollary 4.2.17. Suppose that $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\text{dist}(x, \partial\Omega)) \in L^1(\Omega) \cap B_2(\Omega)$ and $\tau : (0, \infty) \to \mathbb{R}_+$ satisfies

Condition **C** (see Definition 4.2.12). Then there exists an extension operator $\operatorname{Ext} : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,2}_{\rho}(\Omega)$ and we have

$$\| \operatorname{Ext}(u)\|_{L^2_\rho(\Omega)} \le C_3 \|u\|_{L^2(\partial\Omega)}, \qquad \| \operatorname{Ext}(u)\|_{W^{1,2}_\rho(\Omega)} \le C_5 \|u\|_{W^{\frac{1}{2},2}_{\omega}(\partial\Omega)}, \tag{4.2.9}$$

Proof. Let $u \in W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega)$ and $u_n \in \operatorname{Lip}(\partial\Omega)$ be a Cauchy sequence which converges to u in the norm $W^{\frac{1}{2},2}_{\omega}(\partial\Omega)$. By Theorem 4.2.14, $\widetilde{u}_n$ is a Cauchy sequence in $W^{1,2}_\rho(\Omega)$ and so it converges to some $\widetilde{u} := \operatorname{Ext}(u) \in W^{1,2}_\rho(\Omega)$. Clearly the function $\operatorname{Ext}(u)$ does not depend on the choice of the sequence u_n converging to u. The condition (4.2.9) is satisfied. □

We propose the following interpretation of boundary data $u = g$ on $\partial\Omega$ when $u \in W^{1,2}_\rho(\Omega),\ g \in W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega)$.

Definition 4.2.18 (interpretation of boundary data in weighted Sobolev space, [26])**.** Let the assumptions of Corollary 4.2.17 be satisfied and let $u \in W^{1,2}_\rho(\Omega), g \in W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega)$. We will say that u agrees with g on $\partial\Omega$ in the sense of operator Ext($u \overset{\mathrm{Ext}}{=} g$ on $\partial\Omega$) if

$$u - \operatorname{Ext}(g) \in W^{1,2}_{\rho,0}(\Omega),$$

where $\operatorname{Ext} : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,2}_\rho(\Omega)$ is given by Corollary 4.2.17.

Suppose that $\operatorname{Ext}_1, \operatorname{Ext}_2 : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,2}_\rho(\Omega)$ are given two extension operators. Using Definition 4.2.18 one may define two classes of functions having trace g on the boundary.

$$Y_1 := \left\{ u \in W^{1,2}_\rho(\Omega) : u - \operatorname{Ext}_1(g) \in W^{1,2}_{\rho,0}(\Omega) \right\},$$
$$Y_2 := \left\{ u \in W^{1,2}_\rho(\Omega) : u - \operatorname{Ext}_2(g) \in W^{1,2}_{\rho,0}(\Omega) \right\}.$$

It is not clear why should we have $Y_1 = Y_2$ as it is not obvious that $\operatorname{Ext}_1(g) - \operatorname{Ext}_2(g) \in W^{1,2}_{\rho,0}(\Omega)$. The following theorem shows that Definition 4.2.18 is independent on the choice of extension operator $\operatorname{Ext} : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,2}_\rho(\Omega)$ under some special assumptions.

Theorem 4.2.19 (Theorem 4.5 in [26], independence of boundary condition w/r to the choice of extension operator)**.** *Suppose that* $\Omega \subseteq \mathbf{R}^n$ *is a bounded domain of class* $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega)) \in B_2(\Omega) \cap L^1(\Omega)$ *and* $\tau : (0, \infty) \to \mathbb{R}_+$ *satisfies Condition* $\mathbf{C}$ *(see Definition 4.2.12).*

Then there exists extension operator $\mathrm{Ext} : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,2}_{\rho}(\Omega)$. *Moreover, if* $\mathrm{Ext}_1 : W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega) \to W^{1,p}_{\rho}(\Omega)$ *is any other extension operator, then we have*

$$u - \mathrm{Ext}(g) \in W^{1,2}_{\rho,0}(\Omega) \Leftrightarrow u - \mathrm{Ext}_1(g) \in W^{1,2}_{\rho,0}(\Omega).$$

In particular, the interpretation of boundary condition $u \overset{Ext}{=} g$ *on* $\partial\Omega$ *from Definition 4.2.18 is independent on the choice of extension operator from* $W^{\frac{1}{2},2}_{\omega,L}(\partial\Omega)$ *to* $W^{1,2}_{\rho}(\Omega)$.

4.2.6 Other tools

As a key tool we will be applying Lax-Milgram theorem.

Theorem 4.2.20 (**Lax-Milgram Theorem** [60, 34])**.** *Assume that* H *is a Hilbert space and*

$$a : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exist positive constants α, β *such that the following conditions:*

i) continuity

$$|a[v, \phi]| \leq \alpha \|v\|_H \|\phi\|_H$$

ii) coercivity

$$a[v, v] \geq \beta \|v\|_H$$

hold. Moreover, let $F : H \to \mathbb{R}$ *be a bounded linear functional on* H.

Then there exists a unique element $v \in H$ *such that*

$$a[v, \phi] = \langle F, \phi \rangle$$

for all $\phi \in H$.

We will also use the Nikodym's ACL Characterization Theorem, see e.g. Theorem 1.1.3 from [65].

Theorem 4.2.21 (Nikodym ACL Characterization Theorem)**. i)** *Let* $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$*. Then for every* $i \in \{1, \ldots, n\}$ *and for almost every* $a \in \mathbb{R}^{n-i-1} \times \{0\} \times \mathbb{R}^i$ *the function*

$$\mathbb{R} \ni t \mapsto u(a + te_i) \tag{4.2.10}$$

is locally absolutely continuous on $\mathbb{R}$*, in particular, for almost every point* $x \in \mathbb{R}^n$ *the distributional derivative* $\frac{\partial u(x)}{\partial x_i}$ *is the same as the usual derivative at* x*.*

ii) *If for every* $i \in \{1, \ldots, n\}$ *and for almost every* $a \in \mathbb{R}^{n-i-1} \times \{0\} \times \mathbb{R}^i$ *the function (4.2.10) is locally absolutely continuous on* $\mathbb{R}$*, and all the derivatives* $\frac{\partial u(x)}{\partial x_i}$ *computed almost everywhere, together with the function* u*, are locally integrable on* $\mathbb{R}^n$*, then* u *belongs to* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$*.*

4.3 Main result

Let us consider the PDE:

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + B(x) \cdot \nabla u + C(x)u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.3.1}$$

We are now to investigate the existence and uniqueness of solution to (4.3.1) under the possibly weak assumptions on the coefficients which will be specified later.

In particular, when $u \in W^{1,2}_{\rho}(\Omega)$, left hand side in the first equation of (4.3.1) is well defined in the distributional sense. and it reads as:

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i}\, dx + \int_{\Omega} B(x) \cdot \nabla u \ \phi \ dx + \int_{\Omega} C(x) u \ \phi \ dx = \langle f, \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the action of functional from $(W^{1,2}_{\rho,0}(\Omega))^*$ on the element $\phi \in C_0^{\infty}(\Omega) \subseteq W^{1,2}_{\rho,0}(\Omega)$.

Second equation in (4.3.1) requires the interpretation of boundary condition for $u \in W^{1,2}_{\rho}(\Omega)$. We involve some extra assumptions on g and ρ, Definition 4.2.18 and Theorem 4.2.19 and the following assumptions deal with the weight function ρ and the matrix $A(x)$.

(H1) $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$, $\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega)) \in B_2(\Omega) \cap L^1(\Omega)$, and $\tau : (0, \infty) \to \mathbb{R}_+$ is continuous, monotonic, $\int_0^1 \tau(t)dt < \infty$, τ satisfies one of the following conditions i) or ii) for small arguments:

- **i)** τ is nondecreasing, absolutely continuous, satisfies the Δ_2 condition and $s\tau^{'}(s) \leq F \cdot \tau(s)$, where $F < n - 1$;
- **ii)** τ is nonincreasing and τ satisfies $\Delta_{\frac{1}{2}}$-condition ($\tau(\frac{1}{2}s) < c\tau(s)$), where c is independent on s.

(H2) Every function $w \in W^{1,2}_{\rho,0}(\Omega)$ satisfies Poincaré inequality:

$$\int_{\Omega} |w(x)|^2 \rho(x) dx \leq C_P \int_{\Omega} |\nabla w(x)|^2 \rho(x) dx,$$

with constant $C_P > 0$ independent of u.

(H3) The matrix $A(x) = [a_{ij}(x)]_{i,j=1,\cdots,n}$ satisfy the degenerate ellipticity condition

$$c_1 |\xi|^2 \rho(x) \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq c_2 |\xi|^2 \rho(x), \tag{4.3.2}$$

for all $\xi \in \mathbb{R}^n$ and almost everywhere $x \in \Omega$ and where $c_1, c_2 > 0$.

Remark 4.3.1. It follows from (4.3.2) that the assumptions **(H2)** and **(H3)** imply the following Poincaré inequality

(H2$'$) For every $v \in W^{1,2}_{\rho,0}(\Omega)$:

$$\int_{\Omega} |v(x)|^2 \rho(x) dx \leq C_A \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx,$$

where $C_A \leq \frac{C_P}{c_1}$ is given constant independent on v.

Constructions of Poincaré inequality can be found for examples in [44, 46, 79].

Our main result reads as follows:

Theorem 4.3.2. *Let $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$, $\omega(x,y) = \tau(|x-y|)$ and the above assumptions* **(H1)-(H3)** *hold. Moreover, let $B : \Omega \to \mathbb{R}^n, B \in L^1_{\text{loc}}(\Omega;\mathbb{R}^n)$ be measurable, $C : \Omega \to \mathbb{R}$ are such that for almost every $x \in \Omega$,*

$$-\frac{1}{2}\text{div}B(x) + C(x) \geq c_4\rho(x) \quad \text{in } \mathcal{D}', \tag{4.3.3}$$

i.e., for any $\phi \in \mathcal{C}_0^\infty(\Omega)$, $\phi \geq 0$

$$\int_\Omega \frac{1}{2}B(x)\cdot\nabla\phi + C(x)\phi(x)\ dx \geq c_4\int_\Omega \rho(x)\phi(x)\ dx,$$

and also

$$|B(x)|/\rho(x) \in L^\infty(\Omega), \tag{4.3.4a}$$
$$|C(x)|/\rho(x) \in L^\infty(\Omega), \tag{4.3.4b}$$
$$(c_1 + c_4C_P) > 0. \tag{4.3.4c}$$

Then for every $f \in (W^{1,2}_{\rho,0}(\Omega))^*$ and $g \in W^{1/2,2}_{\omega,L}(\partial\Omega)$ there is a unique solution $u \in W^{1,2}_\rho(\Omega)$ of Dirichlet problem

$$\begin{cases} -\text{div}\,(A(x)\nabla u) + B(x)\cdot\nabla u + C(x)u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.3.5}$$

Moreover, there exist positive constants D_1, D_2 independent on u, such that

$$\|u\|_{W^{1,2}_\rho(\Omega)} \leq D_1\|f\|_{\left(W^{1,2}_{\rho,0}(\Omega)\right)^*} + D_2\|g\|_{W^{\frac{1}{2},2}_\omega(\partial\Omega)}. \tag{4.3.6}$$

Proof. We consider the operators

$$\begin{aligned} P_0w &:= -\text{div}(A(x)\nabla w), \\ P_1w &:= P_0w + B(x)\nabla w + C(x)w, \end{aligned} \tag{4.3.7}$$

and define the bilinear form acting on $H \times H$ (see (4.2.4)) by the expression:

$$\begin{aligned} a[w,\phi] := (P_1w,\phi) &= \int_\Omega A(x)\nabla w\nabla\phi\ dx + \int_\Omega B(x)\nabla w\ \phi\ dx + \int_\Omega C(x)w\phi\ dx \\ &=: a_0[w,\phi] + a_1[w,\phi] + a_2[w,\phi], \end{aligned} \tag{4.3.8}$$

where

$$\begin{cases} a_0[w,\phi] := (-\mathrm{div}(A(x)\nabla w), \phi) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial \phi}{\partial x_i}\ dx, \\ a_1[w,\phi] := \int_\Omega B(x)\nabla w\ \phi\ dx, \\ a_2[w,\phi] := \int_\Omega C(x) w\phi\ dx. \end{cases} \tag{4.3.9}$$

Moreover, let the constants c_3, c_8 be such that $|B(x)| \le c_8\rho(x)$, $|C(x)| \le c_3\rho(x)$ and let us recall the notation from (4.2.4). Then we proceed by steps.

STEP 1. Reduction to the problem with homogeneous boundary condition. We have assumed that $g \in W^{1/2,2}_{\omega,L}(\partial\Omega)$. By Corollary 4.2.17 there exists a bounded operator: $\mathrm{Ext} : W^{1/2,2}_{\omega,L}(\partial\Omega) \to W^{1,2}_\rho(\Omega)$. Hence there exists $\Psi_g \in W^{1,2}_\rho(\Omega) = \widehat{H}$ such that $\Psi_g|_{\partial\Omega} = g$ in the sense of Definition 4.2.15. We substitute: $v := u - \Psi_g \in H$. Then the problem (4.3.5) is equivalent to:

$$\begin{cases} P_1 v = F \quad \text{in} \quad \Omega \\ \ v \in H, \end{cases} \tag{4.3.10}$$

where

$$F := f - P_1\Psi_g, \quad \Psi_g \in \widehat{H} \text{ and } \quad \Psi_g|_{\partial\Omega} = g. \tag{4.3.11}$$

In the preceding steps we will show the existence and uniqueness of solutions for the linear problem (4.3.10) by using Lax-Milgram Theorem.

STEP 2. We show that $P_1 w \in H^*$ for any $w \in \widehat{H}$, so that $F \in H^*$
Indeed, it is enough to prove that for any $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have

$$|(P_1 w, \phi)| \le \widetilde{c}_2 \|w\|_{\widehat{H}} \|\phi\|_H,$$

with some constant $\widetilde{c}_2 > 0$ independent of ϕ. For this we use the same arguments as in Remark 4.2.9, part 1), to get

$$\begin{aligned} |(P_0 w, \phi)| = |(-\mathrm{div}\,(A(x)\nabla w)\,, \phi)| &= |\langle A(x)\nabla w, \nabla\phi\rangle_{L^2(\Omega;\mathbb{R}^n)}| \\ &= \left| \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial w(x)}{\partial x_j} \frac{\partial \phi(x)}{\partial x_i}\ dx \right|. \end{aligned}$$

Define $\langle\langle b, d\rangle\rangle_x := \sum_{i,j=1}^n a_{ij}(x) b_j d_i$, which is a symmetric scalar product well defined for almost every x and for any $b = (b_i)$, $d = (d_i) \in \mathbb{R}^n$. By

Schwarz's inequality $|\langle\langle b,d\rangle\rangle_x|\leq\langle\langle b,b\rangle\rangle_x^{\frac{1}{2}}\langle\langle d,d\rangle\rangle_x^{\frac{1}{2}}$ and we choose $b_j=b_j(x)=\partial w(x)/\partial x_j,\ d_j=d_j(x)=\partial\phi(x)/\partial x_j$. Then $(P_0w,\phi)=\int_\Omega\langle\langle b(x),d(x)\rangle\rangle_x\,dx$ and

$$\langle\langle b,b\rangle\rangle_x=\sum_{i,j=1}^n a_{ij}(x)\frac{\partial w(x)}{\partial x_j}\frac{\partial w(x)}{\partial x_i}\overset{(4.3.2)}{\leq} c_2\rho(x)|\nabla w(x)|^2;\ \langle\langle d,d\rangle\rangle_x \leq c_2\rho(x)|\nabla\phi(x)|^2.$$

Consequently, by Schwarz's inequalities

$$\begin{cases} |(P_0w,\phi)|\leq\displaystyle\int_\Omega c_2\rho(x)|\nabla w(x)||\nabla\phi(x)|\;dx\leq c_2\|\nabla w\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\|\nabla\phi\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad= c_2\|\nabla w\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\|\phi\|_H,\\ \left|\displaystyle\int_\Omega B(x)\nabla w(x)\phi(x)\;dx\right|\leq c_8\left(\displaystyle\int_\Omega|\nabla w|^2\rho(x)dx\right)^{\frac{1}{2}}\left(\displaystyle\int_\Omega|\phi|^2\rho(x)dx\right)^{\frac{1}{2}}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\leq c_8C_P^{\frac{1}{2}}\|\nabla w\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\|\phi\|_H,\\ \left|\displaystyle\int_\Omega C(x)w(x)\phi(x)\;dx\right|\leq c_3\left(\displaystyle\int_\Omega w^2(x)\rho(x)dx\right)^{\frac{1}{2}}\left(\displaystyle\int_\Omega\phi^2(x)\rho(x)dx\right)^{\frac{1}{2}}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\leq c_3C_P^{\frac{1}{2}}\|w\|_{L^2_\rho(\Omega)}\|\phi\|_H. \end{cases} \tag{4.3.12}$$

We get for any $\phi\in H$,

$$|(P_1w,\phi)|\overset{(4.3.12)}{\leq}\left(c_3C_P^{\frac{1}{2}}\|w\|_{L^2_\rho(\Omega)}+(c_2+c_8C_P^{\frac{1}{2}})\|\nabla w\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\right)\|\phi\|_H\leq\widetilde{c}_2\|w\|_{\widehat{H}}\|\phi\|_H, \tag{4.3.13}$$

where $\widetilde{c}_2=\max\{c_2+c_8C_P^{\frac{1}{2}},c_3C_P^{\frac{1}{2}}\}$.

Step 3. We verify the continuity condition in Lax-Milgram theorem. This follows directly from (4.3.8), (4.3.13) reduced to $w\in H$ and Poincaré inequality **(H2)**.

Step 4. We verify coercivity condition for the form $a[v,\phi]:=\sum_{i=0}^2 a_i[v,\phi]$ (see (4.3.9)). We have for $v\in H$

$$a_0[v,v]\overset{(4.3.2)}{\geq}c_1\sum_{i,j=1}^n\int_\Omega\left|\frac{\partial v}{\partial x_i}\right|^2\rho(x)\;dx=c_1\|v\|_H^2. \tag{4.3.14}$$

Because of the continuity of the bi-linear form $a_1[\cdot,\cdot], a_2[\cdot,\cdot]$ and the density of $\mathcal{C}_0^\infty(\Omega)$ in H, it suffices to provide the estimations dealing with $v \in \mathcal{C}_0^\infty(\Omega)$. In such case we have $v^2 \in \mathcal{C}_0^\infty(\Omega)$, so by definition of weak divergence, we get

$$\begin{aligned}(a_1+a_2)[v,v] &:= \int_\Omega B(x)\nabla v(x)\cdot v(x)\ dx + C(x)v^2(x)\ dx \\ &= \int_\Omega \left(-\frac{1}{2}\mathrm{div}B(x) + C(x)\right) v^2(x)\ dx \geq c_4 C_P \|v\|_H^2. \end{aligned} \tag{4.3.15}$$

Finally, for $v \in H$,

$$a[v,v] := a_0[v,v] + a_1[v,v] + a_2[v,v] \overset{(4.3.14),(4.3.15)}{\geq} (c_1 + c_4 C_P)\,\|v\|_H^2, \tag{4.3.16}$$

which proves the coercivity provided that $(c_1 + c_4 C_P) > 0$.

Step 5. We conclude the existence and uniqueness for (4.3.5).
We recall that (4.3.5) is equivalent to (4.3.10) (Step 1), while Steps 2-4 together with Lax-Milgram Theorem imply existence and uniqueness for (4.3.10).

Step 6. We prove the a priori estimate (4.3.6) for solution to (4.3.5). For this we start with the a priori estimates for v being the solution of (4.3.10). Let $K := c_1 + c_4 C_P$. We have

$$\begin{aligned} K\|v\|_H^2 &\overset{(4.3.16)}{\leq} |a[v,v]| \overset{(4.3.8),(4.3.10)}{=} |\langle F, v\rangle| \overset{(4.3.11)}{=} |\langle f - P_1\Psi_g, v\rangle| \\ &\leq |\langle f, v\rangle| + |\langle P_1\Psi_g, v\rangle| \overset{(4.3.13)}{\leq} \|f\|_{H^*}\|v\|_H + \widetilde{c}_2\|\Psi_g\|_{\widehat{H}}\|v\|_H \\ &\overset{\textbf{Corollary } 4.2.17}{\leq} \left(\|f\|_{H^*} + \widetilde{c}_2 C_5 \|g\|_{W_\omega^{\frac{1}{2},2}(\partial\Omega)}\right)\|v\|_H. \end{aligned}$$

Therefore,

$$\|v\|_H \leq \frac{1}{K}\|f\|_{H^*} + \frac{\widetilde{c}_2 C_5}{K}\|g\|_{W_\omega^{\frac{1}{2},2}(\partial\Omega)}. \tag{4.3.17}$$

As $v = u - \Psi_g$, we obtain

$$\begin{aligned} \|u\|_{\widehat{H}} \leq \|v\|_{\widehat{H}} + \|\Psi_g\|_{\widehat{H}} &\overset{\textbf{Corollary } 4.2.17}{\leq} (1 + C_P)\|v\|_H + C_5\|g\|_{W_\omega^{\frac{1}{2},2}(\partial\Omega)} \\ &\overset{(4.3.17)}{\leq} \frac{1+C_P}{K}\|f\|_{H^*} + \left(\frac{\widetilde{c}_2(1+C_P)}{K} + 1\right) C_5\|g\|_{W_\omega^{\frac{1}{2},2}(\partial\Omega)}, \end{aligned}$$

which finishes the proof of Step 6

STEP 7. We prove that the solution is independent of the extension operator chosen. Paradoxically, it is not so obvious because the definition $u = g$ on $\partial\Omega$ involves the extension operator. Let v_1, $v_2 \in H$ be the unique solution guaranteed by Lax-Miligram theorem of the following two problems using two possibly different extensions Ψ_g^1 and Ψ_g^1 respectively,

$$\begin{cases} P_1 v_1 = F_1, \\ v_1 \in H, \end{cases} \qquad \text{and} \qquad \begin{cases} P_1 v_2 = F_2, \\ v_2 \in H, \end{cases} \tag{4.3.18}$$

with $F_1 = f - P_1\Psi_g^1$, $F_2 = f - P_1\Psi_g^2$. Problems (4.3.18) are equivalent to the fact that for

$$u_1 := v_1 + \Psi_g^1,\ u_2 := v_2 + \Psi_g^2,$$

we have

$$\begin{cases} P_1 u_1 = f, \\ u_1 - \Psi_g^1 \in H, \end{cases} \qquad \text{and} \qquad \begin{cases} P_1 u_2 = f, \\ u_2 - \Psi_g^2 \in H. \end{cases}$$

Note that $\Psi_g^1 - \Psi_g^2$ belongs to H by Theorem 4.2.19. Therefore, $u_1 - u_2$ also belongs to H and implies that under our assumptions $u = u_1 - u_2$ solves

$$\begin{cases} P_1 u = 0, \\ u \in H. \end{cases}$$

This implies that $u \equiv 0$ by Steps 2-6 dealing with solutions in H. Hence, $u_1 = u_2$. This ends the proof of the theorem. □

4.4 Final remarks

The following remarks recall the classical names of the operators we deal with.

Remark 4.4.1. When $B(x) \not\equiv 0,\ C(x) \equiv 0$, the operator P_1 defined in (4.3.7) is called *Orstein-Uhlenbeck* type operator. When $B(x) = 0,\ C(x) \not\equiv 0$, the operator P_1 is called *Schrödinger* type operator.

Our next remarks discuss the sharpness of the conditions posed on the coefficients and constants.

Remark 4.4.2 (relaxation of the assumption $(B/\rho) \in L^\infty(\Omega)$ in (4.3.4a))**.** In general, assumption $(B/\rho) \in L^\infty(\Omega)$ can be substituted by a weaker one to get a unique solution. Note that, assuming $\rho \equiv 1$, the second inequality of (4.3.12) can be modified as follows:

$$\left|\int_\Omega B(x)\cdot\nabla u\,\phi\right| dx \le \left(\int_\Omega |B(x)|^n\,dx\right)^{\frac{1}{n}} \left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |\phi|^{2^*}\,dx\right)^{\frac{1}{2^*}},$$
$$\le c_7\|B(\cdot)\|_n\|\nabla u\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\|\nabla\phi\|_{L^2_\rho(\Omega;\mathbb{R}^n)},$$

where c_7 is independet of ϕ coming from the Sobolev's inequality. In particular, if $B \in L^n(\Omega)$ without being bounded uniformly, we may have the existence and uniqueness of the problem. However, Zhikov in [95] showed a counter example confirming the non-uniqueness for the following incompressible diffusion equation

$$-\mathrm{div}(\nabla u + au) = f \in H^{-1}(\Omega), \quad u \in H^1_0(\Omega)$$

involving $a \in L^1(\Omega;\mathbb{R}^n)$ such that $\mathrm{div} a = 0$. The above equation can be compared with (4.3.5) with $\rho \equiv 1,\ A \equiv Id$ and $-a =: B,\ C \equiv 0$. Indeed, for $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$(-\mathrm{div}(\nabla u + au), \phi) = (-\mathrm{div}(Id\nabla u) - a\cdot\nabla u, \phi).$$

Remark 4.4.3 (relaxation on the assumption $(C(x)/\rho) \in L^\infty(\Omega)$ is possible)**.** Suppose $A(x) \equiv Id$, $\rho \equiv 1$, $B \equiv 0$ and $C(x) \in L^{\frac{n}{2}}(\Omega)$ is not bounded. For example, take $\Omega = B(0,1) \subseteq \mathbb{R}^n$, $C(x) = \frac{1}{|x|^\alpha}, \alpha < 2$. As we have $W_0^{1,2}(\Omega) \subseteq L^{2^*}(\Omega),\ 2^* = \dfrac{2n}{n-2}$, the last estimate in (4.3.12) can be improved by the following interpolation estimations.

$$
\begin{aligned}
\left|\int_\Omega C(x)w(x)\phi(x)\ dx\right| &\leq \left(\int_\Omega w^2(x)|C(x)|dx\right)^{\frac{1}{2}} \left(\int_\Omega \phi^2(x)|C(x)|dx\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega w^{2^*}(x)\,dx\right)^{\frac{1}{2^*}} \left((\int_\Omega |C(x)|^{(2^*/2)'}dx\right)^{\frac{1}{2(2^*/2)'}} \\
&\quad \left(\int_\Omega \phi^{2^*}(x)\,dx\right)^{\frac{1}{2^*}} \left(\int_\Omega |C(x)|^{(2^*/2)'}dx\right)^{\frac{1}{2(2^*/2)'}} \\
&\leq c_6\|C(\cdot)\|_{\frac{n}{2}}\|\nabla w\|_{L^2_\rho(\Omega;\mathbb{R}^n)}\|\nabla\phi\|_{L^2_\rho(\Omega;\mathbb{R}^n)},
\end{aligned}
$$

where c_6 is independent of w and ϕ coming form the Sobolev's inequality. In that case, the proof of Step 1 of Theorem 4.3.2 with the above modification instead of the last line of (4.3.12) lead to the existence and uniqueness as well while (4.3.4b) is not satisfied but the other assumptions (4.3.4a) with $c_4 = 1$ and (4.3.4c) hold.

Remark 4.4.4 (sharpness of the condition (4.3.4c))**.** Consider $\rho \equiv 1, A \equiv Id, B \equiv 0$, then equation (4.3.5) as

$$
\begin{cases} -\Delta u + C(x)u = 0 \ \text{ in } \Omega, \\ \qquad\qquad\quad u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{4.4.1}
$$

We remark that in that case the assumption $(c_1 + c_4 C_P) > 0$ is sharp. Indeed, when $(1 + c_4 C_P) = 0$ there is no uniqueness in general. For instance when $C(x) = -\frac{1}{C_P}$, equation (4.4.1) reads as

$$
\begin{cases} -\Delta u = \frac{1}{C_P}u \ \text{ in } \Omega, \\ \quad u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{4.4.2}
$$

After multiplying it by u and integrating it over Ω, we get

$$
\int_\Omega u^2\ dx = C_P \int_\Omega |\nabla u|^2\ dx,
$$

and so u achieves C_P in **(H3)** for $u \in W_0^{1,2}(\Omega)$. By Lindquivist Theory (see [61]) there exists such nontrivial u solving (4.4.2). This contradicts the uniqueness of the solution (4.4.1), as $u \equiv 0$ is another solution.

Our next result shows an application of Theorem 4.3.2 to the spectral theory.

Theorem 4.4.5 (application to the spectral theory)**.** *Let the assumptions* **(H1)-(H3)** *hold,* $B : \Omega \to \mathbb{R}^n$, $C : \Omega \to \mathbb{R}$ *and such that for almost every* $x \in \Omega$,

$$-\frac{1}{2}\mathrm{div}B(x) + C(x) \geq c_4\rho(x) \quad \textit{in } \mathcal{D}', \tag{4.4.3}$$

and

$$\inf_{x\in\Omega}\tfrac{F(x)}{\rho(x)} > -\infty, \quad \textit{where } F(x) := -\tfrac{1}{2}\mathrm{div}B(x) + C(x);$$
$$|B(x)|/\rho,\ C/\rho \in L^\infty(\Omega),\ (c_1 + c_4C_P) > 0.$$

Moreover, let us consider the operator

$$P_1u := -\mathrm{div}\,(A(x)\nabla u) + B(x)\cdot\nabla u + C(x)u$$

and the spectral problem

$$\begin{cases} P_1u = \lambda\rho u \\ u \in H, \end{cases} \tag{4.4.4}$$

where $\lambda \in \mathbb{R}^n$ *and* H *as in* (4.2.4)*.*
Then the spectrum of operator P_1 *lies in a half-line* $\left[\frac{c_1}{C_P} + \inf\limits_{x\in\Omega}\frac{F(x)}{\rho(x)}, +\infty\right)$.

Remark 4.4.6. As $|B(x)|/\rho,\ C/\rho \in L^\infty(\Omega)$, $B \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and $C \in L^1_{\mathrm{loc}}(\Omega)$. In particular, (4.4.3) is well defined in $\mathcal{D}'$.

Proof. Denote $a := \frac{c_1}{C_P} + \inf\limits_{x\in\Omega}\frac{F(x)}{\rho(x)}$. We will show when $\lambda < a$ there are no non-trivial solutions to the problem (4.4.4), *i.e.*, in such case the only solution to (4.4.4), $u \equiv 0$. For this we willl apply the Theorem 4.3.2. Let us fix $\lambda \in \mathbb{R}$, $\lambda < a$ and consider the problem (4.4.4). Obviously, this problem is equivalent to

$$\begin{cases} \widetilde{P}_1u := (P_1 - \lambda\rho)u = 0 \ \text{ in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{4.4.5}$$

By Theorem 4.3.2 applied to the operator $\widetilde{P}_1$, (4.4.5) admits a unique solution $u \equiv 0$ provided that

$$F_1(x) := F(x) - \lambda\rho \geq c_4\rho(x), \text{ where } F(x) := -\frac{1}{2}\mathrm{div}B(x) + C(x)$$

and $\quad |B(x)| \leq c_8\rho(x), \qquad |C(x)| \leq c_3\rho(x), \qquad (c_1 + c_4 C_P) > 0.$

(4.4.6a) (4.4.6b) (4.4.6c)

We note that $\inf\limits_{x\in\Omega} \frac{F(x)}{\rho(x)} > -\infty$ we can define $c_4 := \inf\limits_{x\in\Omega} \frac{F(x)}{\rho(x)} - \lambda$ which is a fintie number. The condition (4.4.6a) is obviously verified, as well as the condition (4.4.6b), beacuse

$$|C(x)/\rho(x) - \lambda| \leq |C/\rho| + \lambda < \infty.$$

The condition (4.4.6c) with the constant c_4 as defined above reads as:

$$c_1 + C_P \left(\inf_{x\in\Omega} \frac{F(x)}{\rho(x)} - \lambda \right) > 0, \text{ which is equivalent to}$$

$$a := \frac{c_1}{C_P} + \inf_{x\in\Omega} \frac{F(x)}{\rho(x)} > \lambda.$$

Hence (4.4.6c) is satisfied. Therefore the only solution to the problem (4.4.4) is $u \equiv 0$. This ends the proof of the theorem. This implies that the spectrum of operator P_1 in (4.4.4) lies in a half-line $\left[\frac{c_1}{C_P} + \inf\limits_{x\in\Omega} \frac{F(x)}{\rho(x)}, +\infty \right)$. □

Remark 4.4.7. In the standard case: $\rho \equiv 1,\ A \equiv Id,\ F \equiv 0$, the spectrum is contained in $\left[\frac{1}{C_P}, +\infty \right)$ and $\frac{1}{C_P}$ is exactly the first eigenvalue in (4.4.4) (see Definition 2.1 in [61]).

Theorem 4.4.8. *Let $\Omega \subseteq \mathbf{R}^n$ is any open domain and the assumption* **(H3)** *holds. If for every $f \in (W^{1,2}_{\rho,0}(\Omega))^*$ there is a unique solution $u \in W^{1,2}_{\rho}(\Omega)$ of Dirichlet problem*

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then any function $u \in W^{1,2}_{\rho,0}(\Omega)$ satisfies Poincaré inequality in **(H2)**.

Proof. It follows directly from the implication $ii) \Longrightarrow i)$ in Theorem 4.2.8. □

As a consequence we obtain the following result. It shows that under certain circumstances the solvability of the rather simpler problem (4.4.7) is equivalent to the solvability of the other class of the non-homogeneous boundary value problems in the from of (4.4.8).

Theorem 4.4.9. *Let $\Omega \subseteq \mathbf{R}^n$ is a bounded domain of class $\mathcal{C}^{0,1}$ and conditions* **(H1), (H3)** *hold. Then the following statements are equivalent.*

I) *For any $f \in (W^{1,2}_{\rho,0}(\Omega))^*$ there is a unique solution $u \in W^{1,2}_{\rho}(\Omega)$ of Dirichlet problem*

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4.7}$$

II) *Let $\omega(x,y) = \tau(|x-y|)$, $B : \Omega \to \mathbb{R}^n$, $C : \Omega \to \mathbb{R}$ such that* (4.3.3), (4.3.4) *are satisfied. Then for every $f \in (W^{1,2}_{\rho,0}(\Omega))^*$ and $g \in W^{1/2,2}_{\omega,L}(\partial\Omega)$ there is a unique solution $u \in W^{1,2}_{\rho}(\Omega)$ of Dirichlet problem*

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) + B(x)\cdot\nabla u + C(x)u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.4.8}$$

Proof. The implication $I) \implies II)$ follows from Theorem 4.4.8 and then Theorem 4.3.2 whereas the implication $II) \implies I)$ is obvious. $\square$

Chapter 5

On solvability of nonlinear eigenvalue problems for degenerate PDEs of elliptic type

Let $\Omega \subseteq \mathbb{R}^n$ be a given subset, $p > 1$. We show for the validity of Poincaré inequality $(P): \int_\Omega |u(x)|^p b(x)\, dx \leq C \int_\Omega |\nabla u(x)|^p \rho(x)\, dx$ implies the solvability of degenerated elliptic eigenvalue problems $(S): -\text{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda b(x)|u|^{p-2}u = x^*$. Our method exploit Ekeland's Variational Principle and deals with Palais-Smale sequences, which are adapted in the quite general assumptions in the involved weights $b, \rho \in B_p(\Omega)$. Nonexistence of positive solutions is also included.

5.1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, $p > 1$, $W^{1,p}_{b,\rho}(\Omega) = \Big\{ f \in L^1_{\text{loc}}(\Omega) : f \in L^p_b(\Omega), \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega) \Big\}$ denote the two weighted Sobolev space, $W^{1,p}_{b,\rho,0}(\Omega)$ be the completion of $\mathcal{C}^\infty_0(\Omega)$ in $W^{1,p}_{b,\rho}(\Omega)$ and let $\big(W^{1,p}_{b,\rho,0}(\Omega)\big)^*$ be its dual. In this paper we deal with existence of solutions to the following nonlinear elliptic PDEs

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = x^*, \\ \qquad\qquad u - z \in W^{1,p}_{b,\rho,0}(\Omega), \end{cases} \tag{5.1.1}$$

where $x^* \in \left(W^{1,p}_{b,\rho,0}(\Omega)\right)^*$, $z \in W^{1,p}_{b,\rho}(\Omega)$ are taken arbitrarily and $\lambda \geq 0$ is sufficiently small. We assume that the involved weights b, ρ are locally integrable, they belong to the B_p-class introduced by Kufner and Opic in [55], moreover, they admit the following Poincaré inequality

$$\int_\Omega |u(x)|^p b(x)\,dx \leq C_P \int_\Omega |\nabla u(x)|^p \rho(x)\,dx, \qquad \text{for any } u \in W^{1,p}_{b,\rho,0}(\Omega). \tag{P}$$

Our main result formulated in Theorem 5.3.4 is the proof of the existence of the solution to (5.1.1) where $0 \leq \lambda < 1/C_P$.

Our goal is to look for solutions to (5.1.1) under the possible minimal assumptions on the involved weight functions. While the usual approaches deal with the so-called *Muckenhoupt*'s A_p-class (see e.g. [35]), we use the essentially weaker assumtions that they belong to the B_p-class due to Kufner and Opic ([55], see also Section 5.2). Such weights might even not satisfy the doubling property, contrary to the A_p-weights. With our general assumptions many functional analytic tools are missing. For example, we cannot assume in general, that the embedding of $W^{1,p}_{b,\rho}(\Omega) \subseteq L^p_b(\Omega)$ given by (P) is compact. Our main concern is to prove the existence of solutions to eigenvalue problem (5.1.1), without using compactness arguments.

Our considerations exploit Ekeland's Variational Principle and deal with Palais-Smale sequences, which are adapted to our weighted setting. This approach was inspired by the work of Azorero and Alonzo [6] and by Boccardo and Murat [12]. We refer also to [27, 24], where the existence results in the weighted settings were obtained via Lax-Miligram theorem and Minty-Browder theorem, respectively. Let us additionally mention that usage of Lax-Miligram theorem requires Hilbert structure of the admitted space, consequently we have to assume that $p = 2$, while the usage of Minty-Browder theorem requires monotonicity of the involved operator $Au := -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u$. As we explain in Proposition 5.4.1, the operator A cannot not be monotone when $\lambda > 0$. In particular, the methods from papers [27, 24] are not applicable in our general setting. For some other existence results in degenerate setting we refer for example to [2, 9, 17, 21, 29, 56, 68, 95]. However, because of the huge number of contributions, it is not possible here to mention all the valuable related works.

The crucial tool in our considerations seems to be the validity of Poincaré inequality (P), as well as the information about the constant C_P in that inequality. For recent contributions related to constructions of Poincaré inequalities, we refer e.g. to [44, 79].

Our main result, existence of solutions to (5.1.1), is proven in Section 5.3. The preceding sections are devoted to discussions. More precisely, in Section 5.4 we discuss non-uniqueness of solutions to (5.1.1) and non-monotonicity of the corresponding operators. In Section 5.5, we give an example of equation similar to (5.1.1), which cannot admit any positive classical solution. For this, we propose certain generalisation of Derrick-Pohozaev type arguments which was inspired by the argument from the paper by Azorero and Alonso ([6]).

Degenerated pdes arise, for example, in the models describing diffusion process in a potential field (see e.g. [93] for motivation), in shape optimization theory [51], or in diffusion process with killing [36]. In most situations only homogeneous boundary data were considered. This might be because of lack of analytical tools needed to study the non-homogeneous boundary value problems in the degenerated settings. For contributions dealing with degenerated pdes and their motivations, we refer to books [30, 65].

5.2 Notations and preliminaries

Basic notation. If not said otherwise, we assume that $\Omega \subseteq \mathbb{R}^n$ is an open set and $\overline{\Omega}$ is its closure. We denote by $\mathcal{C}_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω, $\mathcal{D}'(\Omega)$ is the space of distribution. When $1 < p < \infty$, by p' we denote it's Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we denote $\Phi_p(\xi) := |\xi|^{p-2}\xi,\ \xi \in \mathbb{R}^n$.

Class of weights. We will need the following definitions.

Definition 5.2.1 (positive weights)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{M}(\Omega)$ be the set of all Borel measurable functions. Elements of the set

$$W(\Omega) := \{\rho \in \mathcal{M}(\Omega) : \ 0 < \rho(x) < \infty, \text{ for a.e. } x \in \Omega\},$$

will be called positive weights.

Definition 5.2.2 (class $B_p(\Omega)$, Definition 1.4 in [55])**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p > 1$. We will say that a weight $\rho \in W(\Omega)$ satisfies $B_p(\Omega)$-condition ($\rho \in B_p(\Omega)$) if $\rho^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega)$.

We have the following proposition.

Proposition 5.2.3 (Theorem 1.5 in [55])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, $p > 1$ and $\rho \in B_p(\Omega)$. Then $L^p_\rho(\Omega)$ is the subset of $L^1_{\text{loc}}(\Omega)$. Moreover, if $f_k \to f$ in $L^p_\rho(\Omega)$, then $f_k \to f$ in $L^1_{\text{loc}}(\Omega)$.*

Proof. First part of the proof we can find from Theorem 1.5 in [55]. For the second part, we have

$$\begin{aligned}\int_\Omega |f_k - f|\, dx &= \int_\Omega |f_k - f| \rho^{1/p} \rho^{-1/p}\, dx,\\ &\overset{\text{Hölder inequality}}{\leq} \left(\int_\Omega |f_k - f|^p \rho\, dx\right)^{1/p} \left(\int_\Omega \rho^{-1/(p-1)}\, dx\right)^{1/p'}.\end{aligned}$$

□

Weighted Sobolev spaces.
We will be dealing with the following definition of weighted Sobolev spaces due to Kufner and Opic (see [55]).

Definition 5.2.4 (two weighted *Sobolev space*)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open set, $p > 1$ and $b, \rho \in W(\Omega)$ be the given weights. The linear set

$$W^{1,p}_{b,\rho}(\Omega) = \left\{ f \in L^1_{\text{loc}}(\Omega) : f \in L^p_b(\Omega), \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \in L^p_\rho(\Omega) \right\}, \qquad (5.2.1)$$

where $\frac{\partial f}{\partial x_i}$ are distributional derivatives, equipped with the norm

$$\|f\|_{W^{1,p}_{b,\rho}(\Omega)} := \|f\|_{L^p_b(\Omega)} + \|\nabla f\|_{L^p_\rho(\Omega,\mathbb{R}^n)},$$

will be called weighted Sobolev space. The following example shows that in general, the set given by (5.2.1) is not a Banach space.

Example 5.2.5 (see Example 1.12 in [55])**.** Let us consider $\Omega = (-1, 1)$, $p = 2$ and $b(x) = |x|^2, \rho(x) = |x|^4$. It is easy to see $b, \rho \notin B_p(\Omega)$. We show that $W^{1,p}_{b,\rho}(\Omega)$ is incomplete by showing a certain Cauchy sequence $\{f_n\} \subseteq W^{1,p}_{b,\rho}(\Omega)$ which does not converge to a limit in the same space. For this let

$$f(x) := \begin{cases} 0 & \text{for } \ x \leq 0, \\ x^\gamma & \text{for } x > 0, \end{cases} \qquad \text{with} \quad \gamma \in (-3/2, -1].$$

It is clear that

$$f \notin L^1_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad f \in W^{1,2}_{b,\rho}(\Omega). \tag{5.2.2}$$

Let us define for $\delta \in (0, 1)$

$$\eta_\delta = \begin{cases} 0 & \text{for } \ x \in (-1, \delta/2], \\ \frac{2}{\delta}x - 1 & \text{for } \ x \in (\delta/2, \delta), \\ 1 & \text{for } \ x \in [\delta, 1), \end{cases} \qquad \text{and} \quad g_\delta(x) := f(x)\eta_\delta(x).$$

Then

$$\lim_{\delta \to 0} \|f - g_\delta\|_{W^{1,2}_{b,\rho}(\Omega)} = 0, \tag{5.2.3}$$

and $f_n := g_{1/n} \in W^{1,2}_{b,\rho}(\Omega)$ which forms a Cauchy sequence because of (5.2.3). Let us assume that $W^{1,2}_{b,\rho}(\Omega)$ is complete. Then there exists an element $f^* \in W^{1,2}_{b,\rho}(\Omega)$ such that

$$\|f^* - f_n\|_{W^{1,2}_{b,\rho}(\Omega)} \to 0 \quad \text{ as } \ n \to \infty.$$

Then by (5.2.3) we have $f = f^*$ a.e. in Ω and $f^* \in W^{1,2}_{b,\rho}(\Omega)$. Consequently, $f = f^* \in L^1_{\mathrm{loc}}(\Omega)$ which contradicts (5.2.2).

However, as proven in [55] (Theorem 2.1) if only $b \in W(\Omega)$ and $\rho \in B_p(\Omega)$, then $W^{1,p}_{b,\rho}(\Omega)$ is a Banach space.
The symbol $W^{1,p}_{b,\rho,0}(\Omega)$ will denote the completion of $C^\infty_0(\Omega) \cap W^{1,p}_{b,\rho}(\Omega)$ in the space $W^{1,p}_{b,\rho}(\Omega)$.

5.3 Solvability of the non-linear eigenvalue problem

We are now to discuss the solvability of equation

$$\operatorname{div}\left(\rho(x)|\nabla u|^{p-2}\nabla u\right) - \lambda\, b(x)|u|^{p-2}u = x^*$$

in the weighted Sobolev space $X = W^{1,p}_{b,\rho,0}(\Omega)$, where $x^* \in X^*$. The arguments presented here are based on application of *Ekeland Variational Principle* [33] and the Convergence Theorem due to Boccardo and Murat [12]. In particular, we are applying the theory of monotone operators (see e.g. [92]). We start by recalling the following definition.

Definition 5.3.1. Let X be a Banach space. A function is $J : X \to \mathbb{R} \cup \{\infty\}$ called *Gâteaux-differentiable* if at every point u_0 with $J(u_0) < +\infty$, there is a continuous linear functional $J'(u_0) \in X^*$ such that for every $v \in X$:

$$(d/dt)J(u_0 + tv)|_{t=0} = \langle J'(u_0), v\rangle.$$

Let us recall the *Variational Principle of Ekeland.*

Proposition 5.3.2 (Corollary 2.3 in [33])**.** *Let X be a Banach space, and $J : X \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. functional (i.e., $J(u) \leq \liminf J(u_k)$, whenever $u \in X$, $\{u_k\} \subseteq X$ with $u_k \to u$, see Definition 3.1, p-40 in [76]), Gâteaux-differentiable and such that $-\infty < \inf J < \infty$. Then there exists sequence $\{u_k\} \subseteq X$ such that*

$$J(u_k) \to \inf J, \qquad J'(u_k) \to 0 \quad in \quad X^*.$$

We need the following inequalities. For reader's convenience we enclose their proofs.

Lemma 5.3.3. *For $a, b \in \mathbb{R}^n$, $p \geq 1$,*

$$|a + b|^p \geq |b|^p + p|b|^{p-2}\langle b, a\rangle; \tag{5.3.1}$$
$$|a + b|^p \leq |b|^p + p|a + b|^{p-2}\langle a, a + b\rangle, \tag{5.3.2}$$

where the $\langle \cdot, \cdot\rangle$ denotes the inner product on the underlying space.

Proof. Let $\psi(\xi) = |\xi|^p,\ \xi \in \mathbb{R}^n$. Convexity of Ψ gives $|a+b|^p = \psi(a+b) \geq \psi(b) + \langle D\psi(b), a\rangle = |b|^p + p|b|^{p-2}\langle b, a\rangle$, which is (5.3.1). For (5.3.2), substitute $a := -a$ and $b := a + b$ to (5.3.1). □

We arrive at the following statement, which is our main result.

Theorem 5.3.4 (Poincaré inequality and solvability of non-linear eigenvalue problem)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $p > 1$; $b, \rho \in B_p(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$, $\widehat{X} = W^{1,p}_{b,\rho}(\Omega)$, $X = W^{1,p}_{b,\rho,0}(\Omega), X^* = (W^{1,p}_{b,\rho,0}(\Omega))^*$. Assume further that the Poincaré inequality holds:*

$$\int_\Omega |u(x)|^p b(x)\, dx \leq C_P \int_\Omega |\nabla u(x)|^p \rho(x)\, dx,$$

for every $u \in X$, with constant C_P independent on u. Then for any $x^* \in X^*$, $z \in \widehat{X}$ and any $0 \leq \lambda < 1/C_P$ there exist a function $u \in \widehat{X}$ which solves the equation

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = x^*, \\ \qquad\qquad u - z \in X. \end{cases} \tag{5.3.3}$$

Remark 5.3.5. The condition $u - z \in X$ is interpreted as $u = 0$ on $\partial\Omega$. One could also formulate this condition in the form $u = g$ on $\partial\Omega$, where g is a function defined on $\partial\Omega$, in the suitable function space Y. Moreover, the choice of the function space Y for which we will find $z \in X$ such that $z_{|\partial\Omega} = g$ on $\partial\Omega$ is a long standing open problem in the degenerated setting. We refer to recent approaches in this direction [23, 25, 26] and references therein.

Proof of Theorem 5.3.4. Let us rewrite the equation (5.3.3) in the following form:

$$\langle -\mathrm{div}\,(a(x)\Phi_p(\nabla u))\,, v\rangle + \langle \lambda b(x)\Phi_p(u), v\rangle - \langle x^*, v\rangle = 0, \tag{5.3.4}$$

where Φ_p is as in Notations, $v \in X$ is taken arbitrarily. Consider the energy functional

$$J(w) := \int_\Omega F(x, w, \nabla w)\, dx, \qquad \text{where } \ w \in X, \ \text{ and} \tag{5.3.5}$$

$$F(x, w, \nabla w) := (1/p)\rho(x)|\nabla(z + w)|^p - (\lambda/p)b(x)|z + w|^p - x^* w.$$

Now the proof follows the steps.

STEP 1: Easy verification shows that J is Gâteaux differentiable in X and for any $v \in X$

$$\langle J'(w), v\rangle = \langle -\mathrm{div}\,(\rho(x)\Phi_p(\nabla(z+w)))\,, v\rangle + \langle \lambda b(x)\Phi_p(z+w), v\rangle - \langle x^*, v\rangle. \tag{5.3.6}$$

In particular, left hand side in (5.3.4) interprets as $J'(u-z)=0$ as functional on X. We look for the stationary point of J. We notice that the functional J obeys the assumptions of Proposition 5.3.2. Moreover, J is coercive, because

$$\begin{aligned}
\frac{1}{p}\int_\Omega |\nabla(z+w)|^p\rho(x)\,dx &\overset{(5.3.1)}{\geq} \frac{1}{p}\int_\Omega |\nabla w|^p\rho(x)\,dx \\
&\quad + \int_\Omega |\nabla w|^{p-2}\langle \nabla w, \nabla z\rangle \rho(x)\,dx, \\
-\frac{\lambda}{p}\int_\Omega |z+w|^p b(x)\,dx &\overset{(5.3.2),\ \lambda>0}{\geq} -\frac{\lambda}{p}\int_\Omega |w|^p b(x)\,dx \\
&\quad - \lambda\int_\Omega |z+w|^{p-2}z(z+w)b(x)\,dx,
\end{aligned}$$

which give

$$J(w) \geq \frac{1-\lambda C_P}{p}\int_\Omega |\nabla w|^p\rho(x)\,dx + \gamma(w,z), \tag{5.3.7}$$

where $(1-\lambda C_P)/p > 0$ and

$$\begin{aligned}
\gamma(w,z) := \int_\Omega |\nabla w|^{p-2}\langle \nabla w, \nabla z\rangle\rho(x)\,dx - \lambda\int_\Omega |z+w|^{p-2}z(z+w)b(x)\,dx \\
-\langle x^*, w\rangle =: I + II + III.
\end{aligned}$$

For any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$,

$$\begin{aligned}
|I| &\leq \int_\Omega |\varepsilon_1 \nabla w|^{p-1}(|\nabla z|/\varepsilon_1^{(p-1)})\rho(x)\,dx, \\
&\overset{\text{Young's inequality}}{\leq} \frac{\varepsilon_1^p(p-1)}{p}\int_\Omega |\nabla w|^p\rho(x)\,dx + \frac{1}{p\varepsilon_1^{p(p-1)}}\int_\Omega |\nabla z|^p\rho(x)\,dx,
\end{aligned}$$

$$
\begin{aligned}
|II| \le 2^{p-1}\lambda \int_\Omega (|z|^{p-1} + |w|^{p-1})|z| b(x)\, dx &= 2^{p-1}\lambda \Bigg(\int_\Omega |z|^p b(x)\, dx \\
&\qquad + \int_\Omega |w|^{p-1}|z| b(x) \Bigg)\, dx, \\
&\le 2^{p-1}\lambda \Bigg(\int_\Omega |z|^p b(x)\, dx + \frac{1}{p\varepsilon_2^{p(p-1)}} \int_\Omega |z|^p b(x)\, dx \\
&\qquad + \frac{\varepsilon_2^p (p-1)}{p} \int_\Omega |w|^p b(x)\, dx \Bigg), \\
|III| \le \|x^*\|_{X^*} \|w\|_X \le \frac{\|x^*\|_{X^*}^{p'}}{p' \varepsilon_3^{p'}} + \frac{\varepsilon_3^p \|w\|_X^p}{p}.
\end{aligned}
$$

By choosing sufficiently small $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, we deduce from (5.3.7) and the above estimates that there exist constants $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that J satisfies the coercivity condition:

$$
J(w) \ge c_1 \int_\Omega |\nabla w|^p \rho(x)\, dx - c_2.
$$

By the *Variational Principle of Ekeland* (Proposition 5.3.2), we find a sequence $\{u_k\}_{k \in \mathbb{N}}$ which satisfies the *Palais-Smale* condition:

$$
J(w_k) \to \inf J \quad \text{and} \quad J'(w_k) \to 0 \ \text{ as } \ k \to \infty. \tag{5.3.8}
$$

Now proceed by substeps.
Substep 2.1: The coercivity of J implies the boundedness of $\{w_k\}_{k \in \mathbb{N}}$ in X. Passing eventually to a subsequence, we can find $w_0 \in X$ and assume that

(1) $\nabla w_k \rightharpoonup \nabla w_0$ in $L^p_\rho(\Omega; \mathbb{R}^n)$, and $w_k \rightharpoonup w$ in $L^p_b(\Omega)$.

Substep 2.2. We will show that

(2) there exists a subseqence $\{w_k\}$ such that $w_k \to w_0$ a.e. in Ω and in $L^1_{\text{loc}}(\Omega)$.

Indeed, let $\{\Omega_l\}$ be a countable sequence of Lipschitz boundary subdomains of Ω such that $\Omega = \cup_l \Omega_l$ and $\overline{\Omega}_l \subseteq \Omega$. By the diagonal procedure it sufices to show that $iii)$ holds with $\Omega = \Omega_l$. Let us fix l and denote $\Omega' = \Omega_l$. For simplicity $C_{\Omega'} > 0$ denotes a constant depending only on Ω', whose value may vary even in the same estimations. By Poincaré inequlaity there exist constants c_k's such that

$$\|w_k - c_k\|_{L^1(\Omega')} \leq C_{\Omega'} \|\nabla w_k\|_{L^1(\Omega')} \leq C \|\nabla w_k\|_{L^p_\rho(\Omega)},$$

where $C > 0$ is independent of k. Therefore the sequence $\{w_k - c_k\}$ is bounded in $W^{1,1}(\Omega')$. Rellich-Kondrachev theorem allows us to extract a subsequence denoted also by $\{w_k\}$ and find a function v such that $w_k - c_k \to v$ in $L^1(\Omega')$ and a.e.. Moreover, if $b \in B_p(\Omega)$

$$\|c_k\|_{L^1(\Omega')} \leq \|w_k - c_k\|_{L^1(\Omega')} + \|w_k\|_{L^1(\Omega')} \overset{b \in B_p}{\leq} \|w_k - c_k\|_{L^1(\Omega')} + C_{\Omega'} \|w_k\|_{L^p_b(\Omega)}.$$

This implies that the sequence $\{c_k\}$ is bounded and by futher extraction of the subsequence, we can assume that $c_k \to c$ for some constant $c \in \mathbb{R}$. Thus $w_k = w_k - c_k + c_k \to v + c$ a.e. and in $L^1_{\text{loc}}(\Omega)$. Hence, by the distributional argument $w_0 = v + c$ a.e..

Step 3: Let $k \in \mathbb{N}$ and $u_k = w_k + z,\ u = w_0 + z$. We show that $\nabla u_k \to \nabla u$ a.e. by adapting the methods due to Boccardo and Murat [12] to our weighted setting. We already know from (5.3.6) and (5.3.8) that:

$$J'(w_k) = -\operatorname{div}(\rho(x)\Phi_p(\nabla u_k)) - \lambda b(x)\Phi_p(u_k) - x^* \overset{k\to\infty}{\longrightarrow} 0 \text{ strongly in } X^*. \tag{5.3.9}$$

Fix a compact subset $K \subseteq \Omega$ and a let $\phi_K \in \mathcal{C}_0^\infty(\Omega)$ be such that $0 \leq \phi_K \leq 1$ in Ω and $\phi_K \equiv 1$ in K. Define for fixed $\eta > 0$,

$$T_\eta(s) := \begin{cases} s & \text{if} \quad |s| \leq \eta, \\ \eta s/|s| & \text{if} \quad |s| \geq \eta. \end{cases}$$

and let $v_k := T_\eta(w_k - w_0)\phi_K = T_\eta(u_k - u)\phi_K \in X$. Now the proof follows by substeps.

Substep 3.1. Note that $\sup_k \|v_k\|_X < \infty$ because $b, \rho \in L^1_{\text{loc}}(\Omega)$. Moreover, Lebesgue Dominated Convergence Theorem together with Substep 2.2 implies that $v_k \to 0$ strongly in $L^p_b(\Omega)$. Further computations shows that $v_k \rightharpoonup 0$ weakly in X.

Substep 3.2. We prove that

$$\int_\Omega \Phi_p(\nabla u_k)\nabla v_k \rho(x)\, dx \to 0 \quad \text{as} \quad k \to \infty.$$

Using previous substep and expression (5.3.9), we deduce that

$$\langle J'(w_k), v_k\rangle = \int_\Omega \Phi_p(\nabla u_k)\nabla v_k \rho(x)\,dx - \lambda \int_\Omega \Phi_p(u_k) v_k b(x)\,dx \\ -\langle x^*, v_k\rangle \overset{k\to\infty}{\longrightarrow} 0, \qquad (5.3.10)$$

$\langle x^*, v_k\rangle \to 0$ and $\int_\Omega \Phi_p(u_k)v_k b(x)\,dx \to 0$ as $k \to \infty$.
SUBSTEP 3.3. We prove that up to the choice of the subsequence

$$\nabla u_k - \nabla u \to 0 \text{ a.e..}$$

As $\nabla v_k \to 0$ in $L^p_\rho(\Omega)$, we have $\int_\Omega \Phi_p(\nabla u)\nabla v_k \rho\,dx \to 0$, and so

$$\int_\Omega [\Phi_p(\nabla w_k) - \Phi_p(\nabla u)]\nabla v_k\,\rho(x)\,dx \overset{k\to\infty}{\to} 0.$$

Moreover, $\nabla v_k = T'_\eta(u_k - u)(\nabla u_k - \nabla u)\phi_K + T_\eta(u_k - u)\nabla\phi_K$, $T'_\eta(u_k - u) = \chi_{\{|u_k - u| \le \eta\}}$ and

$$\int_\Omega [\Phi_p(\nabla w_k) - \Phi_p(\nabla u)]T'_\eta(u_k - u)(\nabla u_k - \nabla u)\phi_K\,\rho(x)\,dx \overset{k\to\infty}{\to} 0.$$

Further from the fact that $T_\eta(w_k - u) \to 0$ almost everywhere in Ω and $|[\Phi_p(w_k) - \Phi_p(u)]T_\eta(w_k - u)| \le 2^{p-1}(|w_k|^{p-1} + |u|^{p-1})\cdot\eta$ and with Lebesgue Dominated Convergence Theorem, we have

$$\int_\Omega [\Phi_p(\nabla w_k) - \Phi_p(\nabla u)]T_\eta(u_k - u)\nabla\phi_K\,\rho(x)\,dx \overset{k\to\infty}{\to} 0.$$

Finally, we have

$$\int_K [\Phi_p(\nabla u_k) - \Phi_p(\nabla u)]\nabla(u_k - u)\chi_{\{|u_k - u| < \eta\}}(x)\,\rho(x)\,dx \to 0,$$

and $\chi_{\{|u_k - u| < \eta\}}(x) \to 1$ a.e., it is easy to see that $\nabla u_k \to \nabla u$ a.e. in Ω, as $k \to \infty$.
STEP 4 We show that

$$-\mathrm{div}(\rho\Phi_p(\nabla u_k)) \rightharpoonup -\mathrm{div}(\rho\Phi_p(\nabla u)) \quad \text{weakly in } X^*, \qquad (5.3.11)$$

$$-\lambda b\Phi_p(u_k) \rightharpoonup -\lambda b\Phi_p(u) \quad \text{weakly in } X^*. \qquad (5.3.12)$$

We present the proof of (5.3.11) as the proof of (5.3.12) follows the same line. For which we take $\phi \in \mathcal{C}_0^\infty(\Omega)$ and we have

$$\langle -\mathrm{div}(\rho\Phi_p(\nabla u_k)), \phi\rangle = \int_\Omega \rho\Phi_p(\nabla u_k)\nabla\phi\, dx, \tag{5.3.13}$$

where $\Phi_p(\nabla u_k)$ is bounded in $L^{p'}_\rho(\Omega)$, therefore $\rho\Phi_p(\nabla u_k)\cdot\nabla\phi$ is equiintegrable. Indeed,

$$\begin{aligned}\int_{A\cap\Omega\cap\operatorname{supp}\phi} & \rho^{1/p}\rho^{1/p'}|\Phi_p(\nabla u_k)||\nabla\phi|\, dx \\ & \le \sup|\nabla\phi| \left(\int_{A\cap\Omega\cap\operatorname{supp}\phi}\rho\, dx\right)^{1/p}\left(\int_{A\cap\Omega\cap\operatorname{supp}\phi}\rho|\nabla u_k|^p\, dx\right)^{1/p'},\end{aligned}$$

and the term $\int_{A\cap\Omega\cap\operatorname{supp}\phi}\rho\, dx$ can be taken as small as we want by chosing the set A of small measure, while the expression $\int_{A\cap\Omega\cap\operatorname{supp}\phi}\rho|\nabla u_k|^p\, dx$ is uniformly bounded in k. The fact that $\nabla\Phi(\nabla u_k)\to\nabla\Phi(\nabla u)$ a.e. together with the Lebesgue Dominated Convergence Theorem we show that r.h.s. of (5.3.13) converges to

$$\int_\Omega \rho\Phi_p(\nabla u)\nabla\phi\, dx = \langle -\mathrm{div}(\rho\Phi_p(\nabla u)), \phi\rangle.$$

The density argument completes the proof of (5.3.11). This combined with (5.3.11), (5.3.12) and (5.3.9) finishes the proof of the theorem. □

Remark 5.3.6. Note that the constructing solution minimizes J when $z = 0$. This follows from the standard argument (see e.g. [6] for the case $\rho \equiv 1,\ b = 1/|x|^p$). Indeed,

$$\begin{aligned}\inf J & \overset{(5.3.8)}{=} \lim_{k\to\infty} J(w_k) \underset{(5.3.6)}{\overset{(5.3.8)}{=}} \lim_{k\to\infty}\left(J(w_k) - \frac{1}{p}\langle J'(w_k), w_k\rangle\right) \\ & \underset{(5.3.5),(5.3.10)}{\overset{\text{Substep 2.1}}{=}} \left(\frac{1}{p}-1\right)\langle x^*, w\rangle \underset{(5.3.6)}{\overset{(5.3.5)}{=}} J(w) - \frac{1}{p}\langle J'(w), w\rangle = J(w).\end{aligned}$$

5.4 Non-uniqueness and monotonicity

Here we discuss the non-uniqueness of solutions of the problem (5.3.3).

Remark 5.4.1. Let the assumptions of Theorem 5.3.4 be satisfied. Then we have
(i) In the linear case ($p = 2$), it is known that the solutions are unique, see e.g. [27]).
(ii) For $p > 2$ and $\lambda = 0$, the uniqueness follows form Minty-Browder Theorem, see Theorem 5.3 in [24]
(iii) For $p > 2$ and $0 < \lambda < 1/C_P$, the uniqueness of (5.3.3) may not hold. We adapt the argument from Remark 3.5.(II) in [6] to our weighted case. Let us consider the case $z = 0$ in the previous theorem. Assume that $B_{2R} \subset \Omega$ is a ball of radius $2R$ such that $0 \notin B_{2R}$. Consider $u_0 \in \mathcal{C}_0^2(\Omega)$ such that $u_0 = k > 0$ in B_R and $u_0 = 0$ in $\Omega \setminus B_{2R}$. We observe that $u_0 \in X$. Define

$$x^* = -\mathrm{div}(\rho \Phi_p(\nabla u_0)) - \lambda b\, \Phi_p(u_0) \in X^*.$$

Let v be a solution to the equation (5.3.3) given by Theorem 5.3.4 with x^* as above. We observe that $v \neq u_0$. Indeed, for any $w, \phi \in X,\ t \in \mathbb{R}$, we have from (5.3.1)

$$\begin{aligned}\langle J'(u_0 + tw), \phi \rangle = & \int_\Omega |\nabla(u_0 + tw)|^{p-2} \nabla(u_0 + tw) \cdot \nabla \phi\, \rho(x)\, dx \\ & - \lambda \int_\Omega |u_0 + tw|^{p-2}(u_0 + tw)\phi\, b(x)\, dx - \langle x^*, \phi \rangle,\end{aligned}$$

therefore

$$\langle J''(u_0), (\phi, \phi) \rangle = (p-1) \left(\int_\Omega |\nabla u_0|^{p-2} |\nabla \phi|^2 \rho(x)\, dx - \lambda \int_\Omega |u_0|^{p-2} \phi^2 b(x)\, dx \right).$$

If we consider $\phi \in \mathcal{C}_0^\infty(B_R)$, then we obtain

$$\langle J''(u_0)\phi, \phi \rangle = -\lambda \int_\Omega |u_0|^{p-2} \phi^2 b\, dx < 0,$$

which shows that u_0 is not a minimum for J, see Remarks 5.3.6.

Our further analysis contributes to the discussion on monotonicity property of the operator $\mathcal{L}_\lambda$ defined in (5.4.1) below. Clearly in the case $\lambda = 0$ it is strictly monotone.

Proposition 5.4.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $p > 2$; $b, \rho \in B_p(\Omega) \cap L^1_{\text{loc}}(\Omega)$, $\widehat{X} = W^{1,p}_{b,\rho}(\Omega)$, $X = W^{1,p}_{b,\rho,0}(\Omega), X^* = (W^{1,p}_{b,\rho,0}(\Omega))^*$. Assume further the Poncaré inequality holds:*

$$\int_\Omega |u(x)|^p b(x)\, dx \le C_P \int_\Omega |\nabla u(x)|^p \rho(x)\, dx,$$

for every $u \in X$, with constant C_P independent on u. Then for any $\lambda > 0$ the operator

$$X \ni u \mapsto \mathcal{L}_\lambda u := -\text{div}(\rho(x)\Phi_p(\nabla u + z)) - \lambda b(x)\Phi_p(u + z), \qquad \textit{where} \quad z \in \widehat{X}, \tag{5.4.1}$$

can not be monotone.

Proof. Proof follows by steps.
STEP 1. Chose $\bar{\lambda} \in (0, 1/C_P)$. Then by Theorem 5.3.4 we have a solution of $\mathcal{L}_{\bar{\lambda}} u = f \in X^*$. It follows from the definition of strictly monotone operators (i.e. $\langle \mathcal{L}_{\bar{\lambda}} u - \mathcal{L}_{\bar{\lambda}} v, u - v\rangle > 0$ for any $u, v \in X$ and $u \neq v$) that the solution is unique. This contradicts with the of Remark 5.4.1 **(iii)**.
STEP 2. We show that for $\lambda \in (0, 1/C_P)$, $\mathcal{L}_\lambda$ can not also be monotone (instead of strictly monotone). For this, we take $\lambda_0 \in (0, 1/C_P)$ and suppose that $\mathcal{L}_{\lambda_0}$ is monotone but not strictly. Consider $\lambda \in (0, \lambda_0)$. Then

$$\mathcal{L}_\lambda u = \mathcal{L}_{\lambda_0} u + (\lambda_0 - \lambda) b(x)\Phi_p(u).$$

By our assumptions and the non-uniqueness of solutions (Remark 5.4.1), for certain $f \in X^*$, there exist two different solutions $u, v (u \neq v)$, which solve

$$\mathcal{L}_{\lambda_0} u = f.$$

Then

$$\langle \mathcal{L}_\lambda u - \mathcal{L}_\lambda v, u - v\rangle \ge (\lambda_0 - \lambda) \int_\Omega b(x)(\Phi_p(u) - \Phi_p(v))(u - v)\, dx > 0,$$

which implies that $\mathcal{L}_\lambda$ is strictly monotone operator. This contradicts with STEP 1 and it proves STEP 2.
STEP 3. Let $\lambda \ge C_P$ and $\lambda_0 \in (0, C_P)$. By the non-monotonicity of $\mathcal{L}_{\lambda_0}$, we find $u, v \in X$, $u \neq v$ such that

$$\langle \mathcal{L}_{\lambda_0} u - \mathcal{L}_{\lambda_0} v, u - v\rangle < 0.$$

Then we have

$$\langle \mathcal{L}_\lambda u - \mathcal{L}_\lambda v, u - v\rangle = \langle \mathcal{L}_{\lambda_0} u - \mathcal{L}_{\lambda_0} v, u - v\rangle$$
$$+(\lambda_0 - \lambda)\int_\Omega b(x)(\Phi_p(u) - \Phi_p(v))(u - v)\, dx < 0.$$

This implies that $\mathcal{L}_\lambda$ for $\lambda \geq C_P$ can not be monotone as well. $\square$

5.5 Nonexistence of non-trivial solutions

We close our discussions by presenting certain nonexistence results based on the Derrik-Pohozaev type identity. The classical result in this direction can be found in Section 9.4.2 of [34]. Our statement generalizes the Lemma 3.7 in [6] which dealt with the case $\alpha = 0$. However, for simplicity we contribute only to the non-existence in the class of $\mathcal{C}^2(\overline{\Omega})$. We believe the further extension dealing with weak solution is also possible. Note that for $p > 1,\ \alpha < p - n$, the involved weights admit the Poincaré inequality which turns out to be Hardy inequality in $\mathbb{R}^n$, [79].

Theorem 5.5.1. *Consider the following problem*

$$\begin{cases} -\mathrm{div}\,(\rho(x)|\nabla u|^{p-2}\nabla u) - \lambda\, b(x)|u|^{p-2}u = \gamma f(u), \qquad \lambda > 0, \quad \gamma \in \mathbb{R}\setminus\{0\}, \\ \qquad\qquad u(x) = 0 \quad on \quad \partial\Omega, \end{cases} \tag{5.5.1}$$

where Ω is bounded, starshpaed with respect to the origin, $f : [0,\infty] \mapsto [0,\infty]$ is continuous, $f(s) > 0$ for $s > 0$, $f(0) = 0$, $\rho(x) = |x|^\alpha,\ b(x) = |x|^{\alpha-p},\ \alpha < p - n,\ p > 1$ and

$$\gamma\left(nF(u) - \frac{n+\alpha-p}{p}uf(u)\right) \leq 0, \quad F(u) := \int_0^u f(s)ds. \tag{5.5.2}$$

Then (5.5.1) *has no non trivial non-negative solution $u \in \mathcal{C}^2(\overline{\Omega})$.*

Proof. We use the *Derrick-Pohozaev* type identity, see e.g. [6, 34]. Assume that there is non-negative solution to (5.5.1) and multiply (5.5.1) by $\langle x, \nabla u\rangle$

and integrate by parts over Ω. We denote ν is the outward normal to $\partial\Omega$, then

$$\int_\Omega -\mathrm{div}\,(\rho(x)\Phi_p(\nabla u))\,\langle x, \nabla u\rangle\,dx = \lambda \int_\Omega b(x)|u|^{p-2}u\langle x, \nabla u\rangle\,dx + \int_\Omega \gamma f(u)\langle x, \nabla u\rangle\,dx. \tag{5.5.3}$$

Let us denote the left hand side of (5.5.3) by A. We have

$$\begin{aligned}
A :=& \sum_{i,j=1}^n \int_\Omega \rho(\Phi_p(\nabla u))_i (x_j u_{x_j})_{x_i}\,dx - \sum_{i,j=1}^n \int_{\partial\Omega} \rho(\Phi_p(\nabla u))_i \nu_i(x) x_j u_{x_j}\,d\sigma \\
&=: A_1 + A_2. \\
A_1 =& \sum_{i,j=1}^n \int_\Omega \rho(\Phi_p(\nabla u))_i \delta_{ij} u_{x_j} + \rho(\Phi_p(\nabla u))_i x_j u_{x_j x_i}\,dx \\
&= \int_\Omega |\nabla u|^p \rho\,dx + \sum_{j=1}^n \int_\Omega (|\nabla u|^p/p)_{x_j} \rho x_j\,dx, \\
=& \int_\Omega |\nabla u|^p \rho\,dx - \sum_{j=1}^n \int_\Omega (|\nabla u|^p/p)\rho - (|\nabla u|^p/p)(\rho(|x|))_{x_j} x_j\,dx \\
&+ \frac{1}{p}\int_{\partial\Omega} |\nabla u|^p \rho \nu \cdot x\,d\sigma, \\
=&(1 - n/p)\int_\Omega |\nabla u|^p \rho\,dx - \sum_{j=1}^n \int_\Omega (|\nabla u|^p/p)\rho'(|x|)(x_j/|x|)x_j\,dx \\
&+ \frac{1}{p}\int_{\partial\Omega} |\nabla u|^p \rho \nu \cdot x\,d\sigma \\
=& \left(1 - \frac{n+\alpha}{p}\right)\int_\Omega |\nabla u|^p \rho\,dx + \frac{1}{p}\int_{\partial\Omega} |\nabla u|^p \rho \nu \cdot x\,d\sigma.
\end{aligned}$$

Since $u = 0$ on $\partial\Omega$, it follows that $\nabla u \parallel \nu(x)$ at every $x \in \Omega$, which implies $\nabla u = \pm|\nabla u|\nu$. Thus

$$\begin{aligned}
-A_2 = \sum_{i,j=1}^n \int_{\partial\Omega} \rho(\Phi_p(\nabla u))_i \nu_i(x) x_j u_{x_j}\,d\sigma &= \int_{\partial\Omega} \rho(\Phi_p(|\nabla u|\nu)\cdot\nu)(x \cdot |\nabla u|\nu)\,d\sigma \\
&= \int_{\partial\Omega} \rho|\nabla u|^p \nu \cdot x\,d\sigma,
\end{aligned}$$

and so

$$A = -\left(\frac{n+\alpha}{p} - 1\right)\int_\Omega |\nabla u|^p \rho \, dx - \left(\frac{p-1}{p}\right)\int_{\partial\Omega} |\nabla u|^p \nu \cdot x \, d\sigma.$$

Similarly, computations dealing with the right hand side of (5.5.3) denoted by B, give

$$\begin{aligned}
B &:= \sum_{j=1}^n \lambda \int_\Omega b\, u^{p-1}(x_j u_{x_j})\, dx + \sum_{j=1}^n \int_\Omega \gamma f(u) x_j u_{x_j}\, dx =: \lambda B_1 + B_2,\\
B_1 &= -\sum_{j=1}^n \int_\Omega (u^{p-1} b x_j)_{x_j} u \, dx = -n \int_\Omega u^p b \, dx - \sum_{j=1}^n \int_\Omega u^p (b)_{x_j} x_j \, dx\\
&\quad - \sum_{j=1}^n (p-1)\int_\Omega u^{p-1} u_{x_j} x_j b \, dx + \int_{\partial\Omega} b|u|^p x_j \nu_j \, d\sigma\\
&= -n \int_\Omega u^p b \, dx - (\alpha - p)\int_\Omega u^p b \, dx - (p-1)B_1 + 0\\
&= \left(1 - \frac{\alpha+n}{p}\right)\int_\Omega u^p b \, dx,\\
-B_2 &= \gamma n \int_\Omega F(u)\, dx.
\end{aligned}$$

where $F(u) = \int_0^u f(s)\, ds$. Finally, we obtain

$$\begin{aligned}
&\left(\frac{n+\alpha}{p} - 1\right)\int_\Omega |\nabla u|^p \rho \, dx + \left(\frac{p-1}{p}\right)\int_{\partial\Omega} |\nabla u|^p \nu \cdot x \, d\sigma\\
&\qquad = \left(\frac{n+\alpha}{p} - 1\right)\lambda \int_\Omega u^p b \, dx + \gamma n \int_\Omega F(u)\, dx. \qquad (5.5.4)
\end{aligned}$$

Multiplying (5.5.1) by u and integrating, we obtain

$$\int_\Omega |\nabla u|^p \rho \, dx = \lambda \int_\Omega |u|^p b \, dx + \gamma \int_\Omega u f(u)\, dx \qquad (5.5.5)$$

The above identities (5.5.4), (5.5.5), together with the fact that $\langle \nu \cdot x\rangle \geq 0$

(see Lemma, Section 9.4.2, pp- 552 in [34]) give

$$\left(\frac{n+\alpha}{p}-1\right)\left\{\lambda\int_\Omega |u|^p b\,dx+\gamma\int_\Omega uf(u)\,dx\right\}\le \gamma\int_\Omega nF(u)\,dx$$
$$+\left(\frac{n+\alpha}{p}-1\right)\lambda\int_\Omega |u|^p b\,dx,$$
$$\gamma\int_\Omega \left(nF(u)-\frac{n+\alpha-p}{p}uf(u)\right)\,dx\ge 0.$$

Therefore from (5.5.2), we get $\gamma\left(nF(u)-\left(\frac{n+\alpha}{p}-1\right)uf(u)\right)=0$ everywhere. This implies $u\equiv 0$ because for $u>0$, we have

$$0<n\int_0^u f(s)\,ds=\frac{n+\alpha-p}{p}uf(u)<0.$$

□

Remark 5.5.2. Theorem 5.5.1 deals with the classical solutions because we directly used integration by parts. In the setting of weighted Sobolev spaces, for general weights which do not satisfy the socalled A_p-condition, the problem of interpretation of the condition $u(x)=0$ on $\partial\Omega$ is a difficult longstanding open problem. We refer to [23, 25, 26] for the recent approaches in that direction.

Remark 5.5.3. Large number of papers which derive various nonexistence result on the basis of Derrick-Pohozaev type identities can be found among those which quote one of the pioneering papers by Brezis and Nirenberg [14].

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